# *Higher Bifurcation Currents, Neutral Cycles, and the Mandelbrot Set*

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ABSTRACT. We prove that, given any $\theta_1, \dots, \theta_{2d-2} \in \mathbb{R} \setminus \mathbb{Z}$, the support of the bifurcation measure of the moduli space of degree $d$ rational maps coincides with the closure of classes of maps having $2d-2$ neutral cycles of respective multipliers $e^{2i\pi\theta_1}, \dots, e^{2i\pi\theta_{2d-2}}$. To this end, we generalize a famous result of McMullen, proving that homeomorphic copies of $(\partial\mathbf{M})^k$ are dense in the support of the $k^{\text{th}}$-bifurcation current $T_{\text{bif}}^k$ in general families of rational maps, where $\mathbf{M}$ is the Mandelbrot set. As a consequence, we also get sharp dimension estimates for the supports of the bifurcation currents in any family.

## 1. INTRODUCTION

Given $d \geq 2$, the *bifurcation locus* of any holomorphic family $(f_\lambda)_{\lambda\in\Lambda}$ of degree $d$ rational maps (or of the moduli space $\mathcal{M}_d$ of degree $d$ rational maps) is the closure of the set of discontinuity of the map $\lambda \mapsto \mathcal{J}_\lambda$, where $\mathcal{J}_\lambda$ is the Julia set of $f_\lambda$. DeMarco [10] has shown that the bifurcation locus of $\Lambda$ is the support of a closed positive $(1,1)$-current $T_{\text{bif}}$ which is called the *bifurcation current* of the family $(f_\lambda)_{\lambda\in\Lambda}$. When $(f_\lambda)_{\lambda\in\Lambda}$ comes with $2d-2$ marked critical points $c_1, \dots, c_{2d-2}$, the current $T_{\text{bif}}$ coincides with $\sum_i T_i$, where $T_i$ is the bifurcation current of the critical point $c_i$ (see [11]). Bassanelli and Berteloot [1] initiated the study of the self-intersections $T_{\text{bif}}^k$, $1 \leq k \leq \min(2d-2, \dim\Lambda)$, of the bifurcation current. These currents give a natural stratification of the bifurcation locus by loci of stronger bifurcations, and are well-adapted to the study of the complex geometric properties of the bifurcation locus. We refer the reader to the survey [13] or the lecture notes [5] for a report on recent results involving bifurcation currents and further references. We also refer to Section 2 for precise definitions.

Several different descriptions of the currents $T_{\mathrm{bif}}^k$ have been provided by various authors. We mention some known results. The set $\mathrm{Per}_n(w)$ of parameters $\lambda \in \Lambda$ for which $f_\lambda$ has a cycle of multiplier $w \in \mathbb{C}$ and exact period $n$ is a complex hypersurface of $\Lambda$. Bassanelli and Berteloot [2] proved that the $k^{th}$-bifurcation current $T_{\mathrm{bif}}^k$ is actually the limit of integration currents of the form

$$\frac{d^{-(s_1(n)+\cdots+s_k(n))}}{(2\pi)^m}\int_{[0,2\pi]^k}\bigwedge_{j=1}^{k}[\mathrm{Per}_{s_j(n)}(re^{i\theta_j})]d\theta_1\cdots\theta_k\ ,$$

for any $r > 0$ and a suitable choice of increasing functions $s_j : \mathbb{N} \longrightarrow \mathbb{N}$. In the family of all degree $d$ polynomials, the authors give in [3] a much stronger result when $k = 1$: they prove that the hypersurfaces $d^{-n}[\mathrm{Per}_n(re^{i\theta})]$ converge to $T_{\mathrm{bif}}$ for *fixed* $r \leq 1$ and $\theta \in \mathbb{R}$. Regarding Bassanelli and Berteloot's work, one can expect the current $T_{\mathrm{bif}}^k$ to be the limit of currents of the form $d^{-(s_1(n)+\cdots+s_k(n))}[\mathrm{Per}_{s_1(n)}(re^{i\theta_1})]\wedge\cdots\wedge[\mathrm{Per}_{s_k(n)}(re^{i\theta_k})]$ for fixed $\theta_i \in \mathbb{R}$ and $r$. Recently, Favre and the author gave [17] an affirmative answer to this problem in the case when $r < 1$ and $k = d-1$ in the family of all degree $d$ polynomials, using a Theorem of Yuan [27] concerning the equidistribution of points of small height. This problem remains wide open when $r \geq 1$.

In this paper, we focus on a different question of topological nature, namely, whether parameters possessing $k$ distinct neutral cycles of given multipliers are dense in the support of $T_{\mathrm{bif}}^k$. In the whole paper, we consider connected holomorphic families of rational maps. Our first result can be formulated as follows.

***Theorem 1.1.*** *Let* $T_{\mathrm{bif}}$ *be the bifurcation current of the moduli space* $\mathcal{M}_d$ *of degree* $d$ *rational maps. For any* $1 \leq k \leq 2d-2$ *and any* $\Theta_k = (\theta_1,\ldots,\theta_k) \in (\mathbb{R}\setminus\mathbb{Z})^k$*, we have* $\mathrm{supp}\left(T_{\mathrm{bif}}^k\right) = \overline{\mathcal{Z}_k(\Theta_k)} = \overline{\mathrm{Prerep}(k)}$*, where we have*

$$\mathrm{Prerep}(k) := \{[f]\in\mathcal{M}_d\,;\ f \text{ has } k \text{ critical points preperiodic to repelling cycles}\}$$

*and*

$$\mathcal{Z}_k(\Theta_k) := \{[f]\in\mathcal{M}_d;\ f \text{ has } k \text{ distinct cycles of resp. multipliers } e^{2i\pi\theta_1},\ldots,e^{2i\pi\theta_k}\}.$$

We mention that the equality $\mathrm{supp}(T_{\mathrm{bif}}^k) = \overline{\mathrm{Prerep}(k)}$ is known (see [8, 9, 15] for the case when $k$ is maximal). Dujardin [14, Corollary 5.3] proved it in the general case, using a transversality Theorem for laminar currents.

Let us now describe how we prove Theorem 1.1. The main point is to generalize McMullen's universality of the Mandelbrot set: McMullen [24] proved that in any one-dimensional family of rational maps, the bifurcation locus contains quasiconformal copies of the Mandelbrot set $\mathbf{M}$. We prove here that under some mild assumptions, the loci of stronger bifurcations contain also copies of products of the Mandelbrot set with itself. Relying on [24] and [18], we prove the following.

***Theorem 1.2.*** *Let $(f_\lambda)_{\lambda\in\mathbb{D}^m}$ be a holomorphic family of degree $d$ rational maps with simple marked critical points $c_1,\dots,c_k$ with $k \le m$. Assume that $c_1,\dots,c_k$ are transversely preperiodic to repelling cycles of $f_0$. Then, for any $\varepsilon > 0$, there exists a continuous embedding $\Phi : \mathbf{M}^k \times \mathbb{D}^{m-k} \hookrightarrow \mathbb{D}^m$ and integers $n_1,\dots,n_k \ge 1$ such that*

1. *For any $(\zeta_1,\dots\zeta_k,t) \in \mathbf{M}^k \times \mathbb{D}^{m-k}$, if $\lambda = \Phi(\zeta_1,\dots,\zeta_k,t)$, there exists $k$ disjoint compact sets $\mathcal{K}_1,\dots,\mathcal{K}_k \subset \mathbb{P}^1$ such that $f_\lambda^{n_i} : \mathcal{K}_i \to \mathcal{K}_i$ is hybrid conjugate to $z^2 + \zeta_i$.*
2. *The set $\Phi((\partial\mathbf{M})^k \times \mathbb{D}^{m-k})$ is contained in $\mathrm{supp}(T_1 \wedge \cdots \wedge T_k)$ and*

$$\dim_H \Phi((\partial\mathbf{M})^k \times \mathbb{D}^{m-k}) \ge 2m - \varepsilon.$$

This generalization of McMullen's Theorem is done in section 3. To prove Theorem 1.2, we use McMullen's universality for each critical point separately to produce $k$ "tubes" of Mandelbrot set homeomorphic to $\mathbf{M} \times \mathbb{D}^{m-1}$ and which are tranverse to each other. We then construct $\Phi$ as a map from $\mathbf{M}^k \times \mathbb{D}^{m-k}$ to the intersection of those tubes. Let us stress that the dimension estimate uses Shishikura's famous result [25] concerning the Hausdorff dimension of the Mandelbrot set and Hölder-regularity properties of $\Phi$ (see Theorem 3.1). Using [18, Theorem 6.2], we then prove that the copy of $(\partial\mathbf{M})^k \times \mathbb{D}^{m-k}$ given by $\Phi$ actually lies in the support of $T_1 \wedge \cdots \wedge T_k$ (see Proposition 3.4).

We also mention that Inou and Kiwi [20] and Inou [19] have already obtained strengthened versions of McMullen's unversality of the Mandelbrot set in a different setting, and have given an explicit condition for the related embedding not to be continuous. On the other hand, Buff and Henriksen [7] proved that some parameter spaces contain quasiconformal copies of Julia sets.

In [18], the author obtained sharp dimension estimates for the strong bifurcation loci of the space $\mathrm{Rat}_d$ of all degree $d$ rational maps. Using Theorem 1.2, we actually get sharp estimates for the Hausdorff dimension of the strong bifurcation loci of a general family. This is the subject of our third result.

***Theorem 1.3.*** *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of degree $d$ rational maps. Assume that there exists $\lambda_0$ such that $f_{\lambda_0}$ has simple critical points, and let $1 \le k \le 2d-2$ be such that $T_{\mathrm{bif}}^k \ne 0$. Then, $\mathrm{supp}(T_{\mathrm{bif}}^k) \setminus \mathrm{supp}(T_{\mathrm{bif}}^{k+1}) \ne \emptyset$, and for any open set $\Omega \subset \Lambda$ such that $\Omega \cap \mathrm{supp}(T_{\mathrm{bif}}^k) \setminus \mathrm{supp}(T_{\mathrm{bif}}^{k+1}) \ne \emptyset$, we have*

$$\dim_H \left(\Omega \cap \mathrm{supp}(T_{\mathrm{bif}}^k) \setminus \mathrm{supp}(T_{\mathrm{bif}}^{k+1})\right) = 2\dim_{\mathbb{C}} \Lambda.$$

Let us also remark that our results strongly rely on [18, Theorem 6.2], and that Theorems 1.1 and 1.3 also rely on [14, Theorem 0.1]. The main difference between our results and the proof of Theorem 1.1 of [18] is the transfer phenomenom which is performed. Instead of transferring directly "big" sets from the dynamical space to the parameter space, we transfer a complete "simplified" parameter space into our actual parameter space.

Section 4 is devoted to explaining how to apply results from the previous sections to the particular case of the space $\mathrm{Rat}_d^{\mathrm{cm}}$ of all critically marked degree $d$ rational maps in order to obtain Theorem 1.1. We also prove a similar result, using a simpler argument, in the case of the moduli space $\mathcal{P}_d^{\mathrm{cm}}$ of critically marked degree $d$ polynomials.

## 2. PRELIMINARIES

We begin by introducing some tools and recalling known results we will need.

***2.1. The hypersurfaces*** $\mathbf{Per}_n(w)$. To understand the geometry of the bifurcation locus of a holomorphic family of rational maps, one can investigate the geometry of the set of rational maps having a cycle of given multiplier and period. The following result describes the set of such parameters (see [26, Chapter 4]):

***Theorem 2.1 (Silverman).*** *Let* $(f_\lambda)_{\lambda\in\Lambda}$ *be a holomorphic family of degree* $d$ *rational maps. Then, for any* $n \in \mathbb{N}^*$*, there exists a holomorphic function* $p_n : \Lambda \times \mathbb{C} \longrightarrow \mathbb{C}$ *such that the following hold:*

(1) *For any* $w \in \mathbb{C} \setminus \{1\}$, $p_n(\lambda, w) = 0$ *if and only if* $f_\lambda$ *has a cycle of exact period* $n$ *and of multiplier* $w$;
(2) $p_n(\lambda, 1) = 0$ *if and only if* $f_\lambda$ *has a cycle of period* $n$ *and multiplier* $1$*, or* $f_\lambda$ *has a cycle of period* $m$ *and multiplier a* $r$*-th root of unity with* $n = mr$;
(3) *For any* $\lambda \in X$*, function* $p_n(\lambda, \cdot)$ *is a polynomial of degree* $N_d(n) \sim \frac{1}{n}d^n$.

*Moreover, if* $\Lambda$ *is a quasi-projective variety, the functions* $p_n$ *are polynomials in* $(\lambda, w)$.

For $n \geq 1$ and $w \in \mathbb{C}$, we set $\mathrm{Per}_n(w) := \{\lambda \in \Lambda \mid p_n(\lambda, w) = 0\}$. We will say that a neutral periodic point of $f_{\lambda_0}$ is *persistent* in $\Lambda$ if it can be perturbed as a neutral periodic point of $f_\lambda$ for any $\lambda$ in a neighborhood of $\lambda_0$ in $\Lambda$, that is, that $\mathrm{Per}_n(e^{i\theta}) = \Lambda$ for some $n, \theta$.

***2.2. Bifurcation current of a critical point.*** Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of degree $d$ rational maps. We say that $c$ is a *marked critical point* if $c : \Lambda \longrightarrow \mathbb{P}^1$ is a holomorphic map satisfying $f_\lambda'(c(\lambda)) = 0$ for every $\lambda \in \Lambda$. If $\deg(f_\lambda, c(\lambda)) = 2$ for any $\lambda \in \Lambda$, we will say the the marked critical point $c$ is *simple*.

***Definition 2.2.*** We say that a marked critical point $c$ is *passive* at $\lambda_0$ in $\Lambda$ if $(f_\lambda^n(c(\lambda)))_{n\geq 0}$ is a normal family in a neighborhood of $\lambda_0$. Otherwise, we say that $c$ is *active* at $\lambda_0$ in $\Lambda$.

Let $\omega$ be the Fubini-Study form on $\mathbb{P}^1$, and let $c_n(\lambda) := f_\lambda^{\circ n}(c(\lambda))$. Dujardin and Favre prove in [15, Section 3.1] that the sequence $d^{-n}c_n^*\omega$ converges to a positive closed $(1,1)$-current $T_c$ with local continuous potential, which support coincides with the activity locus of the marked critical point $c$.

***Definition 2.3.*** We denote $T_c$ the *bifurcation current* of the marked critical point $c$.

As $T_c$ has local continuous potential, the self-intersections of $T_c$ are well defined in the sense of Bedford and Taylor (see [4]). The bifurcation current of a critical point has self-intersection zero (see [15, Proposition 6.9] for polynomial families and [18, Theorem 6.1] for the general case).

***Lemma 2.4 (Dujardin-Favre, Gauthier).*** *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of degree $d$ rational maps with a marked critical point $c$. Then, $T_c \wedge T_c = 0$.*

Assume that $(f_\lambda)_{\lambda\in\Lambda}$ is with $k$ marked critical points $c_1, \dots, c_k$ and $\dim \Lambda \geq k$, and let us set $H_i(k_i, p_i) := \{\lambda \in \Lambda \mid f_\lambda^{\circ(k_i+p_i)}(c_i(\lambda)) = f_\lambda^{\circ p_i}(c_i(\lambda))$ and $f_\lambda^{\circ p_i}(c_i(\lambda))$ is repelling$\}$, for $1 \leq i \leq k$.

***Definition 2.5.*** If

$$\lambda_0 \in \bigcap_{1\leq i\leq k} H_i(k_i, p_i),$$

we say that $c_1, \dots, c_k$ *fall transversely onto repelling cycles* at $\lambda_0$ if the hypersurfaces $H_i$ are smooth at $\lambda_0$ and intersect transversely at $\lambda_0$. If they only intersect properly, we say that $c_1, \dots, c_k$ *fall properly onto repelling cycles* at $\lambda_0$.

Dujardin [14] proved the following which we will use for proving Theorems 1.1 and 1.3.

***Theorem 2.6 (Dujardin).*** *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of degree $d$ rational maps with $k$ marked critical points $c_1, \dots, c_k$, and let $T_1, \dots, T_k$ be their respective bifurcation currents. Then,*

$$\operatorname{supp}(T_1 \wedge \cdots \wedge T_k) = \overline{\{\lambda \in \Lambda \mid c_1, \dots, c_k \textit{ fall transversely onto repelling cycles}\}}.$$

***2.3. The bifurcation currents of a holomorphic family.*** Every rational map $f$ of degree $d \geq 2$ on the Riemann sphere admits a unique maximal entropy measure $\mu_f$. The Lyapunov exponent of $f$ with respect to $\mu_f$ is defined by

$$L(f) = \int_{\mathbb{P}^1} \log|f'|\mu_f\ .$$

It turns out that, for any holomorphic family $(f_\lambda)_{\lambda\in\Lambda}$ of degree $d$ rational maps, the function $L : \Lambda \longrightarrow L(f_\lambda)$ is *p.s.h* and continuous on $\Lambda$ (see [10]).

***Definition 2.7.*** The *bifurcation current* of the family $(f_\lambda)_{\lambda\in\Lambda}$ is the closed, positive $(1,1)$-current on $\Lambda$ defined by $T_{\text{bif}} := dd^c L$.

The support of $T_{\text{bif}}$ coincides with the bifurcation locus of the family $(f_\lambda)_{\lambda\in\Lambda}$ in the sense of Mañé-Sad-Sullivan, that is, the closure of the set of discontinuity of the map $\lambda \in \Lambda \longmapsto \mathcal{J}_\lambda$. This actually follows from a formula by DeMarco (see [11, Theorem 1.1] or [1, Theorem 5.2]), which, for families with $2d-2$ marked critical points $c_1, \dots, c_{2d-2}$, may be stated as follows:

$$T_{\text{bif}} = \sum_{i=1}^{2d-2} T_i.$$

***Definition 2.8.*** Let $1 \le k \le \min(2d-2, \dim \Lambda)$. The $k^{\text{th}}$-*bifurcation current* of the family $(f_\lambda)_{\lambda\in\Lambda}$ is the closed positive $(k,k)$-current defined by $T^k_{\text{bif}} := (dd^c L)^k$.

For $1 \le k \le 2d-2$, Lemma 2.4 directly gives

$$T^k_{\text{bif}} = k! \sum_{i_1<\cdots<i_k} T_{i_1} \wedge \cdots \wedge T_{i_k}. \tag{2.1}$$

The locus $\text{supp}(T^k_{\text{bif}})$ can thus be interpreted as the set of parameters for which at least $k$ critical points are active in an "independent" manner.

***2.4. Quadratic-like maps.*** Let $U, V \subset \mathbb{C}$ be topological discs with $U \Subset V$. We say that $f : U \to V$ is a *quadratic-like map* if it is a degree-2 branched cover. The *filled-in Julia set* $\mathcal{K}(f)$ of $f$ is the set

$$\mathcal{K}(f) := \bigcap_{n\ge 1} f^{-\circ n}(V)$$

of points $z \in U$ such that $f^{\circ n}(z) \in V$ for any $n \ge 1$. We say that the map $f$ is *hybrid conjugate* to a quadratic polynomial $p_\zeta(z) := z^2 + \zeta$ if there exists a quasi-conformal map $\varphi$ from a neighborhood of $\mathcal{K}_\zeta := \mathcal{K}(p_\zeta)$ to a neighborhood of $\mathcal{K}(f)$ which satisfies $\varphi \circ p_\zeta = f \circ \varphi$ and $\overline{\partial}\varphi = 0$ on $\mathcal{K}_\zeta$. Douady and Hubbard proved that, for any holomorphic family of quadratic-like maps, the Mandelbrot set plays the role of a good model. Let us summarize here the properties of quadratic-like maps established by Douady and Hubbard (see [12, Proposition 13 and Chapter IV]).

***Theorem 2.9 (Douady-Hubbard).*** *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic of quadratic-like maps parametrized by a complex manifold $\Lambda$. Let $\mathbf{M}_\Lambda := \{\lambda \in \Lambda \mid \mathcal{K}(f_\lambda)$ is connected$\}$. There exists a continuous map $\chi : \mathbf{M}_\Lambda \longrightarrow \mathbf{M}$ such that the following hold:*

*(1) $\chi$ is holomorphic from $\mathring{\mathbf{M}}_\Lambda$ to $\mathring{\mathbf{M}}$;*
*(2) For any $\lambda \in \mathbf{M}_\Lambda$, if $\chi(\lambda) = \zeta$, the map $f_\lambda$ is hybrid conjugate to $z^2 + \zeta$ on $\mathcal{K}(f_\lambda)$;*
*(3) For all $\zeta \in \mathbf{M}$, the set $\chi^{-1}\{\zeta\}$ is an analytic hypersurface;*
*(4) If $\dim \Lambda = 1$ and $\lambda_0 \in \mathbf{M}_\Lambda$, then there exists a neighborhood $V \subset \Lambda$ of $\lambda_0$ such that either $\chi$ is constant along $V$ or $\chi(V)$ contains a neighborhood of $\chi(\lambda_0)$ in $\mathbf{M}$.*

The map $\chi$ defined in Theorem 2.9 is called the *straightening map* of the family $(f_\lambda)_{\lambda\in\Lambda}$. Denote by ♥ the main cardioid of the Mandelbrot set $\mathbf{M}$. Combined with the fact that the multiplier of the non-repelling fixed point parametrizes the closure of ♥, Theorem 2.9 gives the following (see also [3, Section 3.2] for a proof based on potential theoretic arguments):

***Corollary 2.10 (Bassanelli-Berteloot, Douady-Hubbard).*** *For any $\theta \in \mathbb{R}$, the set of $\zeta \in \mathbb{C}$ for which $p_\zeta$ has a cycle of multiplier $e^{2i\pi\theta}$ is dense in $\partial\mathbf{M}$.*

Let $g_\zeta(z) := p_\zeta(z) + h(\zeta, z)$ be a holomorphic family of maps defined for $(\zeta, z) \in \mathbb{D}(0,R) \times \mathbb{D}(0,R)$, where $R > 10$ and $g'_\zeta(0) = 0$. Denote by $M_g$ the set of $\zeta \in \mathbb{D}(0,R)$ such that the orbit $(g_\zeta^{\circ n}(0))_n$ remains in $\mathbb{D}(0,R)$ for any $n > 0$. In what follows, we will use the following lemma which is due to McMullen (see [24, Lemma 4.2]).

***Lemma 2.11 (McMullen).*** *There exists $\delta > 0$ such that if $\sup_{(\zeta,z)} |h(\zeta,z)| = \varepsilon < \delta$, then there exists a homeomorphism $\varphi : \mathbf{M} \longrightarrow M_g$ such that the following hold:*

*(1) $g_{\varphi(\zeta)}$ is hybrid conjugate to $p_\zeta$ for any $\zeta \in \mathbf{M}$;*
*(2) $|\varphi(\zeta) - \zeta| < O(\varepsilon)$;*
*(3) $\varphi$ extends to a $(1 + \varepsilon/\delta)$-quasiconformal homeomorphism of $\mathbb{C}$.*

## 3. The Mandelbrot Set is Universal, Revisited

We let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of degree $d$ rational maps. In this present section, we want to prove that, under some reasonable condition on the family, the parameter space $\Lambda$ contains homeomorphically embedded copies of $\mathbf{M}^k \times \mathbb{D}^{\dim\Lambda - k}$, which generalizes the work *The Mandelbrot set is universal* [24] of McMullen. The main result of this section is the following.

***Theorem 3.1.*** *Let $(f_\lambda)_{\lambda\in\mathbb{D}^m}$ be a holomorphic family of degree $d$ rational maps with marked simple critical points $c_1, \ldots, c_k$ with $k \le m$. Assume that $c_1, \ldots, c_k$ fall transversely onto repelling cycles at $0$. Then, for any $\varepsilon > 0$, there exists a homeomorphic embedding $\Phi : \mathbf{M}^k \times \mathbb{D}^{m-k} \longrightarrow \mathbb{D}^m$ and a continuous family*

$$\{\varphi_{\zeta,t,i} : \mathbb{P}^1 \longrightarrow \mathbb{P}^1\}_{(\zeta,t)\in\mathbf{M}^k\times\mathbb{D}^{m-k}, 1\le i\le k}$$

*of $(1 + O(\varepsilon))$-quasi-conformal homeomorphisms satisfying the following properties:*

*(1) $\Phi(\zeta,\cdot) : \mathbb{D}^{m-k} \longrightarrow \mathbb{D}^m$ is holomorphic for any $\zeta \in \mathbf{M}^k$;*
*(2) $\Phi$ is holomorphic on $(\mathring{\mathbf{M}})^k \times \mathbb{D}^{m-k}$;*
*(3) $\dim_H(\Phi((\partial\mathbf{M})^k \times \mathbb{D}^{m-k})) \ge 2m - O(\varepsilon)$;*
*(4) For any $1 \le i \le k$, there exists $n_i \ge 1$ such that $\varphi_{\zeta,t,i} \circ p_{\zeta_i} = f_{\Phi(\zeta,t)}^{\circ n_i} \circ \varphi_{\zeta,t,i}$ on $\mathcal{K}_{\zeta_i}$ and the conjugacy is hybrid.*

It is the combination of this result with [18, Theorem 6.2] which will actually give Theorem 1.2 (see Section 3.3).

***3.1. Technical lemmas.*** To give the Hausdorff dimension of estimates, we need the two following lemmas. The first one is due to McMullen (see [24, Lemma 5.1]), and a proof of the second one is provided.

***Lemma 3.2.*** *Let $Y$ be a metric space, and $X \subset Y \times [0,1]^k$. Denote by $X_t$ the slice $X_t := \{y \in Y \mid (y,t) \in X\}$. If $X_t \neq \emptyset$ for almost every $t \in [0,1]^k$, then*

$$\dim_H(X) \ge k + \dim_H(X_t), \text{ for almost every } t.$$

Let us recall that a map $h : (X, d) \longrightarrow (Y, d')$ is *α-bi-Hölder* with constant $C > 0$ if $0 < \alpha \leq 1$ and

$$C^{-1}d'(x,x')^{1/\alpha} \leq d(f(x), f(x')) \leq Cd(x,x')^{\alpha}, \text{ for any } x, x' \in X.$$

***Lemma 3.3.*** *Let $E_1, \ldots, E_k \subset \mathbb{D}$ and $f : E_1 \times \cdots \times E_k \longrightarrow \mathbb{C}^k$ be a map. Assume that there exists $C > 0$ and $0 < \alpha \leq 1$ such that, for any $1 \leq j \leq k$ and any $x_i \in E_i$ with $i \neq j$, for all $x, x' \in X_j$,*

$$x \longmapsto f(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_k)$$

*is α-bi-Hölder with constant $C$, and*

$$f(\{x_1, \ldots, x_{j-1}\} \times E_j \times \{x_{j+1}, \ldots, x_k\}) \subset \{a_1, \ldots, a_{j-1}\} \times \mathbb{C} \times \{a_{j+1}, \ldots, a_k\}$$

*for some $a_i \in \mathbb{C}$, $i \neq j$ only depending on $f$, and the $x_i$, $i \neq j$. Then, $f$ is α-bi-Hölder with constant $C \cdot \max\{k, k^{1/2\alpha}\}$. In particular,*

$$\dim_H(f(E_1 \times \cdots \times E_k)) \geq \alpha \sum_{j=1}^{k} \dim_H(E_j) .$$

*Proof.* Up to taking $C' \geq C$, we can assume that $C \geq 1$. Let $E := E_1 \times \cdots \times E_k$. Let $x, x' \in E$. Then, by assumption, we have

$$\begin{aligned}\|f(x) - f(x')\| &\leq \sum_{j=1}^{k} \|f(x'_1, \ldots, x'_j, x_{j+1}, \ldots, x_k) - f(x'_1, \ldots, x'_{j-1}, x_j, \ldots, x_k)\| \\ &\leq C \sum_{j=1}^{k} |x_j - x'_j|^{\alpha} \leq C.k\|x - x'\|^{\alpha}.\end{aligned}$$

Again, by hypothesis, we have

$$\begin{aligned}\|x - x'\| &\leq \sqrt{k} \max_{1 \leq j \leq k} |x_j - x'_j| \\ &\leq C^{\alpha}\sqrt{k} \max_{1 \leq j \leq k} \|f(x) - f(x_1, \ldots, x_{j-1}, x', x_{j+1}, \ldots, x_k)\|^{\alpha}.\end{aligned}$$

By assumption, we have

$$\|f(x) - f(x_1, \ldots, x_{j-1}, x', x_{j+1}, \ldots, x_k)\| = |(f(x))_j - (f(x'))_j|,$$

and thus

$$\|x - x'\| \leq C^{\alpha}\sqrt{k} \max_{1 \leq j \leq k} |(f(x))_j - (f(x'))_j|^{\alpha} \leq C^{\alpha}\sqrt{k}\|f(x) - f(x')\|^{\alpha} .$$

The Hausdorff dimension estimate is classical (see, e.g., [16]). □

***3.2. Embeddings of $k$-fold products of* M*: proof of Theorem 3.1.*** By assumption, for any $1 \leq i \leq k$, there exist integers $p_i, k_i \geq 1$ such that

$$f_0^{\circ(k_i+p_i)}(c_i(0)) = f_0^{\circ p_i}(c_i(0)).$$

Let $a_i := f_0^{\circ k_i}(c_i(0))$. Since $a_i$ is a repelling cycle of $f_0$, by the implicit function Theorem, up to reducing $\mathbb{D}^m$, we may assume that $a_i$ can be followed holomorphically on the whole $\mathbb{D}^m$ as a $p_i$-repelling cycle $a_i(\lambda)$ of $f_\lambda$. We now let

$$\begin{aligned} \chi : \mathbb{D}^m &\longrightarrow \mathbb{C}^k \\ \lambda &\longmapsto (f_\lambda^{\circ p_1}(c_1(\lambda)) - a_1(\lambda), \ldots, f_\lambda^{\circ p_k}(c_k(\lambda)) - a_k(\lambda)). \end{aligned}$$

By assumption, up to reducing $\mathbb{D}^m$, the map $\chi$ is a submersion onto its image $\Omega$. The map $\chi$ allows us to define a system of coordinates $(x_1, \ldots, x_m)$ for which $\{\chi_i = 0\} = \{x_i = 0\}$, so that $\{\chi = 0\} = \{(0, \ldots, 0)\} \times \mathbb{D}^{m-k}$ in a neighborhood $\Omega_1$ of $0 \in \mathbb{D}^m$. We let $R = 20$, let $\delta > 0$ be given by Lemma 2.11, and let $0 < \varepsilon < \delta$. Let us also denote by $(\mathcal{H}_j)$ the following assertion:

***Claim*** $(\mathcal{H}_j)$. There exists $\rho_j > 0$, a continuous embedding $\Phi_j : \mathbf{M}^j \times \mathbb{D}_{\rho_j}^{m-j} \longrightarrow \Omega_1$, and a continuous family $\{\varphi_{\zeta,x',l} : \mathbb{P}^1 \longrightarrow \mathbb{P}^1\}_{(\zeta,x')\in\mathbf{M}^j\times\mathbb{D}_{\rho_j}^{m-j}, 1\leq l\leq j}$ of $(1+O(\varepsilon))$-quasi-conformal homeomorphisms for which the following hold:

(1) For any $1 \leq l \leq j$, $t \in \mathbb{D}_{\rho_j}^{m-j}$, $\zeta_1, \ldots, \zeta_{l-1}, \zeta_{l+1}, \ldots, \zeta_j \in \mathbf{M}^{j-1}$, the map

$$\zeta \longmapsto \Phi_j(\zeta_1, \ldots, \zeta_{l-1}, \zeta, \zeta_{l+1}, \ldots, \zeta_j, t)$$

is locally $1/(1+O(\varepsilon))$-bihölder continuous. Moreover, the hölder constants are independant of $t, \zeta_1, \ldots, \zeta_{l-1}, \zeta_{l+1}, \ldots, \zeta_j$, and

$$\Phi_j(\{\zeta_1, \ldots, \zeta_{i-1}\} \times \mathbf{M} \times \{\zeta_{i+1}, \ldots, \zeta_j\}) \subset \{a_1, \ldots, a_{i-1}\} \times \mathbb{C} \times \{a_{j+1}, \ldots, a_m\}$$

for some $a_i \in \mathbb{C}$, $i \neq l$, depending only on $\Phi_j$, $\zeta_i$, $i \neq l$ and $t \in \mathbb{D}_{\rho_j}^{m-j}$;

(2) $\Phi_j$ is holomorphic on $(\mathring{\mathbf{M}})^j \times \mathbb{D}_{\rho_j}^{m-j}$;

(3) For any $\zeta \in \mathbf{M}^j$, the set $\Phi_j(\{\zeta\} \times \mathbb{D}_{\rho_j}^{m-j})$ is a holomorphic graph of the form

$$\Phi_j(\{\zeta\} \times \mathbb{D}_{p_j}^{m-j}) = \{x_1 = u_1(x'), \ldots, x_j = u_j(x'),\ x' \in \mathbb{D}_{\rho_j}^{m-j}\};$$

(4) for $1 \leq l \leq j$, there exists $n_l \geq 1$ such that $\varphi_{\zeta,t,l} \circ p_{\zeta_l} = f_{\Phi(\zeta,t)}^{\circ n_l} \circ \varphi_{\zeta,t,l}$ on $\mathcal{K}_{\zeta_l}$ and the conjugacy is hybrid.

We want to prove Claim $(\mathcal{H}_j)$ by finite induction on $j$. Then, to conclude the proof of the Theorem, it only remains to justify assertion (3) of the Theorem. Let us begin with proving $(\mathcal{H}_1)$. To this end, we set

$$\begin{aligned} \Lambda_1 &:= \{\chi_2 = \ldots = \chi_k = 0\} \cap \{x_{k+1} = \cdots = x_m = 0\} \\ &= \{x \in \Omega_1 \mid x_2 = \ldots = x_m = 0\}. \end{aligned}$$

Since $\chi$ is a local submersion at 0, we have that $\chi_1 \not\equiv 0$ on $\Lambda_1$. By [18, Lemma 3.1], the critical point $c_1$ is thus active at 0 in $\Lambda_1$. Since $c_1(0)$ is preperiodic under iteration of $f_0$, there exists $n \geq 1$ such that $f_0^{\circ n}(c_1(0))$ is a periodic point of $f_0$. Moreover, it is a repelling periodic point. Up to multiplying $n$ by the period of $f_0^{\circ n}(c_1(0))$, we also may assume that $f_0^{\circ 2n}(c_1(0)) = f_0^{\circ n}(c_1(0))$, that is, that $f_0^{\circ n}(c_1(0))$ is a repelling fixed point for $f_0^{\circ n}$. By a Theorem of McMullen (see [24, Theorem 3.1]), there exists an integer $n_1 \geq n$ and a coordinate change on $\mathbb{P}^1$, such that in this coordinate, $c_1 \equiv 0$ on $\Lambda_1$, and

$$f_\lambda^{\circ n_1}(z) = z^2 + \zeta + h(z,\zeta), \tag{3.1}$$

whenever $z, \zeta \in \mathbb{D}(0, 2R)$, with

$$\sup |h(z,\zeta)| \leq \varepsilon/2 \quad \text{and} \quad \lambda = \psi_1(\zeta) := t_1(1+\gamma_1\zeta) \in \Lambda_1$$

and $0 < |t_1|, |\gamma_1| < \varepsilon$. Therefore, for $x' \in \mathbb{D}^{m-1}$ close enough to $0'$, in the coordinate given by Theorem 3.1 of [24], the map $f_\lambda$ satisfies (3.1) for $z, \zeta \in \mathbb{D}(0,R)$, with $\sup |h(z,\zeta)| \leq \varepsilon$ and $\lambda = \psi_1(\zeta) := t_1(1+\gamma_1\zeta) + x' \in \Lambda_1 + x'$. This means that there exists a family of quadratic-like maps $(f_\lambda^{\circ n_1})_{\lambda \in \psi_n(\mathbb{D}(0,R)) \times \mathbb{D}_{\rho_1}^{m-1}}$ for some $\rho_1 > 0$ parametrized by the open set $\psi_1(\mathbb{D}(0,R)) \times \mathbb{D}_{\rho_1}^{m-1}$ of $\mathbb{D}^m$. The existence of a surjective map

$$\phi_1 : M_{\psi_1(\mathbb{D}(0,R)) \times \mathbb{D}_{\rho_1}^{m-1}} \longrightarrow \mathbf{M}$$

follows from Theorem 2.9. Let us now set

$$\begin{aligned} \Psi_1 : M_{\psi_n(\mathbb{D}(0,R)) \times \mathbb{D}_{\rho_1}^{m-1}} &\longrightarrow \mathbf{M} \times \mathbb{D}_{\rho_1}^{m-1} \\ \lambda &\longmapsto (\phi_1(\lambda), \lambda_2, \ldots, \lambda_m). \end{aligned}$$

By Lemma 2.11, the map $\Psi_1|_{M_{\psi_1(\mathbb{D}(0,R))+x'}} : M_{\psi_1(\mathbb{D}(0,R))+x'} \longrightarrow \mathbf{M} \times \{x'\}$ is a homeomorphism which is the restriction of a $(1+O(\varepsilon))$-quasiconformal map, for any $x' \in \mathbb{D}_{\rho_1}^{m-1}$. Assertions (1)–(4) of Claim $(\mathcal{H}_j)$ are then satisfied by $\Phi_1 := \Psi_1^{-1}$, according to Theorem 2.9.

We can now assume that, for $1 \leq j \leq k-1$, we have already established Claim $(\mathcal{H}_j)$. Let us consider $\zeta^{(j)} \in (\partial \mathbf{M})^j$ to be such that the critical point of $z^2 + \zeta_i^{(j)}$ is preperiodic to a repelling cycle, and let us set

$$\Lambda_{j+1} := \Phi_j(\{\zeta^{(j)}\} \times \mathbb{D}_{\rho_j}^{m-j}) \cap \{x_{j+2} = \cdots = x_m = 0\},$$

and let $\lambda^{(j+1)} \in \Lambda_{j+1} \cap \{x_{j+1} = 0\}$. The critical points $c_{j+2}, \ldots, c_k$ are passive in the family $\Lambda_{j+1}$ and, by the assumption (4) of the induction hypothesis $(\mathcal{H}_j)$, up to reordering the critical points, we can assume that the critical points $c_1, \ldots, c_j$ are passive in the family $(f_t)_{t \in \Lambda_{j+1}}$. In addition, by assumption (3) of $(\mathcal{H}_j)$, the set $\Lambda_{j+1}$ is of the form

$$\Lambda_{j+1} = \{x_1 = u_1(x'), \ldots, x_j = u_j(x'),\ x' \in \mathbb{D}_{\rho_j}^{m-j}\} \cap \{x_{j+1} = \cdots = x_k = 0\}.$$

Therefore, the analytic sets $\Lambda_{j+1}$ and $\{x_{j+1} = 0\}$ intersect properly at $\lambda^{(j+1)}$. Therefore, by [18, Lemma 3.1], the critical point $c_{j+1}$ is active at $\lambda^{(j+1)}$ in $\Lambda_{j+1}$. Using again [24, Theorem 3.1], we find an integer $n_{j+1} \geq 1$ and a coordinate change on $\mathbb{P}^1$, such that, in this coordinate, we have $c_{j+1} \equiv 0$ on $\Lambda_{j+1}$, and $f_t^{\circ n_{j+1}}(z) = z^2 + \zeta + h(z,\zeta)$, whenever $z, \zeta \in \mathbb{D}(0, 2R)$ with $\sup |h(z,\zeta)| \leq \varepsilon/2$ and $t = \psi_{j+1}(\zeta) := t_{j+1}(1 + \gamma_{j+1}\zeta) \in \Lambda_{j+1}$ and $0 < |t_{j+1}|, |\gamma_{j+1}| < \varepsilon$. We then proceed as in the previous step to find $0 < r \leq \rho_j$ and to build a continuous injection

$$\Psi_{j+1} : \mathbb{D}_r^j \times \mathbf{M} \times \mathbb{D}_r^{m-j-1} \longrightarrow \mathbb{D}^m(\Phi_j(\zeta^{(j)}, 0'), r)$$

satisfying $(\mathcal{H}_1)$. In particular, for any $\zeta \in \mathbf{M}$, the set $\Psi_{j+1}(\mathbb{D}_r^j \times \{\zeta\} \times \mathbb{D}_r^{m-j-1})$ is a holomorphic graph of the form

$$\Psi_{j+1}(\mathbb{D}_r^j \times \{\zeta\} \times \mathbb{D}_r^{m-j-1}) = \{x_{j+1} = u_{j+1}(x', x''),\ x' \in \mathbb{D}_r^j, x'' \in \mathbb{D}_r^{m-j-1}\}.$$

We now may construct the map $\Phi_{j+1}$, using $\Phi_j$ and $\Psi_{j+1}$. By a classical result of Douady and Hubbard (see [12], also [24, Theorem 4.1]), there exist $(1+\varepsilon)$-quasiconformal embeddings $\phi_i : \mathbf{M} \longrightarrow \mathbf{M}$ whose images are, respectively, contained in arbitrary small neighborhoods of $\zeta_i^{(j)}$. Therefore, the maps $\phi_i$ can be chosen so that

$$\begin{aligned}\Phi_j(\phi_1(\mathbf{M}) \times \cdots \times \phi_j(\mathbf{M}) \times \mathbb{D}_{\rho_j}^{m-j}) &\cap \Psi_{j+1}(\mathbb{D}_r^j \times \{\zeta_{j+1}\} \times \{0\}) \\ &\Subset \Psi_{j+1}(\mathbb{D}_r^j \times \{\zeta_{j+1}\} \times \{0\})\end{aligned}$$

for any $\zeta_{j+1} \in \mathbf{M}$. By continuity of $\Psi_{j+1}$, we thus can find $0 < \rho_{j+1} \leq r$ such that

$$\begin{aligned}\Phi_j(\phi_1(\mathbf{M}) \times \cdots \times \phi_j(\mathbf{M}) \times \mathbb{D}_{\rho_j}^{m-j}) &\cap \Psi_{j+1}(\mathbb{D}_{\rho_{j+1}}^j \times \{\zeta\} \times \{x'\}) \\ &\Subset \Psi_{j+1}(\mathbb{D}_{\rho_{j+1}}^j \times \{\zeta\} \times \{x'\}),\end{aligned}$$

for any $(\zeta_{j+1}, x') \in \mathbf{M} \times \mathbb{D}_{\rho_{j+1}}^{m-j-1}$. Hypothesis (3) of Claim $(\mathcal{H}_j)$ guarantees that, for any $\zeta_1, \ldots, \zeta_{j+1} \in \mathbf{M}$ and any $x' \in \mathbb{D}_{\rho_{j+1}}^{m-j-1}$, the intersection

$$\Phi_j(\{(\phi_1(\zeta_1), \ldots, \phi_j(\zeta_j))\} \times \mathbb{D}_{\rho_j} \times \{x'\}) \cap \Psi_{j+1}(\mathbb{D}_{\rho_{j+1}}^j \times \{\zeta\} \times \{x'\})$$

is reduced to one point.

We define $\Phi_{j+1}(\zeta, x')$ as this unique intersection point. The properties of $\Phi_j$ and $\Psi_{j+1}$, respectively given by $(\mathcal{H}_j)$ and $(\mathcal{H}_1)$, directly imply that the map $\Phi_{j+1}$ satisfies assertions (2), (3), and (4) of Claim $(\mathcal{H}_j)$. To conclude, it remains to remark that, by the regularity properties of $\Phi_j$ and $\Psi_{j+1}$, the map $\Phi_{j+1}$ obviously satisfies (1).

We have shown that $\Phi$ exists and satisfies (1), (2), and (4). It remains to justify the fact that $\Phi$ satisfies (3). First, let us remark that assumption (1) of Claim $(\mathcal{H}_j)$, combined with Lemma 3.3, implies that, for any $t \in \mathbb{D}^{m-k}$, the map $\Phi(\cdot, t) : \mathbf{M}^k \longrightarrow \Omega_1$ is locally $1/(1+O(\varepsilon))$-bihölder. Let now $\zeta \in (\partial\mathbf{M})^k$, and let $\rho > 0$ be such that $\Phi(\cdot, t)$ is $1/(1+O(\varepsilon))$-bihölder on $\mathbb{D}^k(\zeta, \rho)$. Lemma 3.3 and [25, Theorem A] give

$$\begin{aligned}\dim_H\left(\Phi((\partial\mathbf{M})^k \cap \mathbb{D}^k(\zeta,\rho),t)\right) &\geq (1+O(\varepsilon))\dim_H((\partial\mathbf{M})^k \cap \mathbb{D}^k(\zeta,\rho))\\ &\geq (1+O(\varepsilon))\sum_{j=0}^{k}\dim_H((\partial\mathbf{M})\cap\mathbb{D}(\zeta_j,\rho))\\ &\geq 2k(1+O(\varepsilon)).\end{aligned}$$

Lemma 3.2 and assertion (3) of $(\mathcal{H}_k)$ then state that, for almost every $t \in \mathbb{D}^{m-k}$,

$$\begin{aligned}\dim_H\left(\Phi((\partial\mathbf{M})^k \times \mathbb{D}^{m-k})\right) &\geq 2(m-k) + \dim_H\left(\Phi((\partial\mathbf{M})^k \times \{t\})\right)\\ &\geq 2(m-k) + 2k(1+O(\varepsilon)) = 2m - O(\varepsilon),\end{aligned}$$

which ends the proof.

***3.3. A consequence: Theorem 1.2.*** We now prove that the homeomorphically embedded copies of $(\partial\mathbf{M})^k \times \mathbb{D}^{\dim\Lambda - k}$ given by Theorem 3.1 are contained in the support of the bifurcation currents. As a consequence, we obtain optimal Hausdorff dimension estimates for the supports of the bifurcation currents. Theorem 3.1 combined with [18, Theorem 6.2] yields the following key proposition.

***Proposition 3.4.*** *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of degree $d$ rational maps. Assume $c_1, \ldots, c_k$ are marked simple critical points, and denote by $T_1, \ldots, T_k$ their respective bifurcation currents. Assume $k \leq m = \dim\Lambda$ and $T_1 \wedge \cdots \wedge T_k \neq 0$. Then, for any $\varepsilon > 0$, the homeomorphic embeddings of the set $(\partial\mathbf{M})^k \times \mathbb{D}^{m-k}$ given by Theorem 3.1 are contained in* $\mathrm{supp}(T_1 \wedge \cdots \wedge T_k)$.

*Proof.* Consider a dense sequence $\zeta_j \subset \partial\mathbf{M}$ for which $0$ is preperiodic to a repelling cycle for $z^2 + \zeta_j$. Since $\partial\mathbf{M}$ is the bifurcation locus of the family $(z^2+\zeta)_{\zeta\in\mathbb{C}}$, the existence of such a sequence is just a straightforward consequence of Montel's Theorem (see, e.g., [15, Lemma 2.3] or [24, Lemma 2.1]). Set $\mathbf{j} := (j_1, \ldots, j_k)$ and $\zeta_{\mathbf{j}} := (\zeta_{j_1}, \ldots, \zeta_{j_k})$. Let $\Phi$ be the embedding given by Theorem 3.1. Since the set $\{(\zeta_{\mathbf{j}}, x') \in (\partial\mathbf{M})^k \times \mathbb{D}^{m-k} \mid \mathbf{j} \in (\mathbb{Z}_+)^k\}$ is dense in $(\partial\mathbf{M})^k \times \mathbb{D}^{m-k}$, it is sufficient to show that $\Phi(\zeta_{\mathbf{j}}, x') \in \mathrm{supp}(T_1 \wedge \cdots \wedge T_k)$ for all $\mathbf{j} \in (\mathbb{Z}_+)^k$ and all $x' \in \mathbb{D}^{m-k}$. By item (4) of Theorem 3.1, the critical points $c_1, \ldots, c_k$ fall onto repelling cycles at $\Phi(\zeta_{\mathbf{j}}, x')$ for any $x' \in \mathbb{D}^{m-k}$. Since $c_1, \ldots, c_k$ fall properly onto repelling cycles for any $\mathbf{j} \in (\mathbb{Z}_+)^k$, [18, Theorem 6.2] then states that $\Phi(\zeta_{\mathbf{j}}, x') \in \mathrm{supp}(T_1 \wedge \cdots \wedge T_k)$. □

*Proof of Theorem 1.2.* This is a direct consequence of Theorem 3.1 and Proposition 3.4. □

***3.4. Hausdorff dimension of the support of bifurcation currents.*** To end this section, we want to underline the fact that Theorem 3.1, Proposition 3.4, and the work [14] of Dujardin directly give Hausdorff dimension estimates for the support of $T_1 \wedge \cdots \wedge T_k$.

***Proposition 3.5.*** *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of degree $d$ rational maps. Assume that $c_1, \dots, c_k$ are marked simple critical points, and let $T_1, \dots, T_k$ be their respective bifurcation currents. Assume $k \le m = \dim\Lambda$ and $T_1 \wedge \cdots \wedge T_k \neq 0$. Then, for any $\varepsilon > 0$, the homeomorphic embeddings of the set $(\partial\mathbf{M})^k \times \mathbb{D}^{m-k}$ of dimension at least $2m - \varepsilon$ given by Theorem 3.1 are dense in* $\mathrm{supp}(T_1 \wedge \cdots \wedge T_k)$.

*Proof.* Let $\lambda_0 \in \mathrm{supp}(T_1 \wedge \cdots \wedge T_k)$ and $\varepsilon > 0$. By [14, Theorem 0.1], there exists a sequence $\lambda_n \to \lambda_0$ such that $c_1, \dots, c_k$ fall transversely onto repelling cyles at $\lambda_n$. Let $n \ge 1$ be such that $\lambda_n \in \mathbb{B}(\lambda_0, \varepsilon)$. Then, by Theorem 3.1 and Proposition 3.4, there exists an embedding

$$\Phi : (\partial\mathbf{M})^k \times \mathbb{D}^{m-k} \longrightarrow \mathbb{B}(\lambda_0, \varepsilon) \cap \mathrm{supp}(T_1 \wedge \cdots \wedge T_k)$$

with $\dim_H(\Phi((\partial\mathbf{M})^k \times \mathbb{D}^{m-k})) \ge 2m - \varepsilon$. □

Let $(X, d)$ be a metric space. Recall that $E \subset X$ is said to *homogeneous* if $\dim_H(E \cap U) = \dim_H(E)$ for all open sets $U \subset X$ with $U \cap E \neq \emptyset$. As a consequence of Proposition 3.5, we get the following.

***Corollary 3.6.*** *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of degree $d$ rational maps. Assume that $c_1, \dots, c_k$ are marked simple critical points, and denote by $T_1, \dots, T_k$ their respective bifurcation currents. Then, either*

- $T_1 \wedge \cdots \wedge T_k = 0$*; or,*
- $\mathrm{supp}(T_1 \wedge \cdots \wedge T_k)$ *is homogeneous and has maximal Hausdorff dimension* $2m$.

We are now in position to prove Theorem 1.3.

*Proof of Theorem 1.3.* Let $k \ge 1$ be such that $T_{\mathrm{bif}}^k \neq 0$. Up to taking a finite branched covering of the family $(f_\lambda)_{\lambda\in\Lambda}$, we can assume that it has marked critical points $c_1, \dots, c_{2d-2}$. If $T_i$ is the bifurcation current of the critical points $c_i$, then equation (2.1) gives

$$\mathrm{supp}(T_{\mathrm{bif}}^k) = \bigcup_{1\le j_1<\cdots<j_k\le 2d-2} \mathrm{supp}\left(\bigwedge_{i=1}^k T_{j_i}\right). \tag{3.2}$$

Let us now set

$$C_{i,j} := \{\lambda \in \Lambda \mid c_j(\lambda) = c_i(\lambda)\}$$

for $1 \le i \neq j \le 2d - 2$. By assumption, $C_{i,j}$ is a complex hypersurface of $\Lambda$. Let $\Lambda_1 := \Lambda \setminus \bigcup_{i\neq j} C_{i,j}$. Then, the family $(f_\lambda)_{\lambda\in\Lambda_1}$ is a family of degree $d$ rational maps with simple marked critical points. The key of the proof is the following lemma.

***Lemma 3.7.*** *Let $\mathbb{B} \subset \Lambda_1$ be an open ball, and let*

$$m := \max\{1 \le j \le 2d-2 \mid T^j_{\mathrm{bif}} \neq 0 \textit{ in } \mathbb{B}\}.$$

*Then, we have $\mathbb{B} \cap \mathrm{supp}(T^{m-1}_{\mathrm{bif}}) \setminus \mathrm{supp}(T^m_{\mathrm{bif}}) \neq \emptyset$.*

To finish the proof of Theorem 1.3, it suffices to show that $\Lambda_1 \cap \mathrm{supp}(T^k_{\mathrm{bif}}) \setminus \mathrm{supp}(T^{k+1}_{\mathrm{bif}}) \neq \emptyset$, and then to apply Corollary 3.6 in any ball $\mathbb{B} \subset \Lambda_1$ such that $\mathbb{B} \cap \mathrm{supp}(T^k_{\mathrm{bif}}) \subset \mathrm{supp}(T^k_{\mathrm{bif}}) \setminus \mathrm{supp}(T^{k+1}_{\mathrm{bif}})$.

By Lemma 3.7, if $m = \max\{j \le 2d-2 \,/\, T^j_{\mathrm{bif}} \neq 0 \text{ on } \Lambda_1\}$, there exists $\lambda_0 \in \mathrm{supp}(T^{m-1}_{\mathrm{bif}}) \setminus \mathrm{supp}(T^m_{\mathrm{bif}})$. If $\mathbb{B}_1 \subset \Lambda_1$ is a small enough ball centered at $\lambda_1$, one has $\mathrm{supp}(T^m_{\mathrm{bif}}) \cap \mathbb{B}_1 = \emptyset$; by applying again Lemma 3.7, we find that $\lambda_1 \in \mathbb{B}_1 \cap \mathrm{supp}(T^{m-2}_{\mathrm{bif}}) \setminus \mathrm{supp}(T^{m-1}_{\mathrm{bif}})$. In $m-k+1$ steps, we find that

$$\lambda_{m-k+1} \in \mathrm{supp}(T^k_{\mathrm{bif}}) \setminus \mathrm{supp}(T^{k+1}_{\mathrm{bif}}).$$ □

*Proof of Lemma 3.7.* The proof is a consequence of [14, Theorem 0.1]. Let $\lambda_0 \in \mathrm{supp}(T^m_{\mathrm{bif}}) \cap \mathbb{B}$; then, by (3.2), there exists $1 \le j_1 < \cdots < c_{j_m} \le 2d-2$ such that $\lambda_0 \in \mathrm{supp}(T_{j_1} \wedge \cdots \wedge T_{j_m})$. By Theorem 2.6, there exists $\lambda_1 \in \mathbb{B}$ such that $c_{j_1}, \ldots, c_{j_m}$ fall transversely onto repelling cycles (see Definition 2.5). Let now $n_i, k_i \ge 1$ be such that

$$\lambda_1 \in X_i := \{\lambda \in \mathbb{B} \mid f_\lambda^{\circ n_i}(c_{j_i}(\lambda)) = f_\lambda^{\circ(n_i+k_i)}(c_{j_i}(\lambda));\ \ f_\lambda^{\circ n_i}(c_{j_i}(\lambda)) \text{ is repelling}\}$$

for any $1 \le i \le m$. By [18, Lemma 3.1], the critical point $c_{j_m}$ is active at $\lambda_1$ in $X_{j_1} \cap \cdots \cap X_{j_{m-1}}$. By Montel's Theorem, there exists $\lambda_2 \in X_{j_1} \cap \cdots \cap X_{j_{m-1}}$ such that $c_{j_m}(\lambda_2)$ is a periodic point of $f_{\lambda_2}$. Therefore, there exists $\mathbb{B}_1 \Subset \mathbb{B}$ a ball centered at $\lambda_2$ such that $c_{j_m}$ is passive on $\mathbb{B}_1$ and $T_{j_1} \wedge \cdots \wedge T_{j_{m-1}} \neq 0$ on $\mathbb{B}_1$.

Assume now that $T^m_{\mathrm{bif}} \neq 0$ on $\mathbb{B}_1$. By the same procedure, we can find $j'_m \neq j_m$ and a ball $\mathbb{B}_2 \Subset \mathbb{B}_1$ such that $c_{j'_m}$ is passive on $\mathbb{B}_2$ and $T^{m-1}_{\mathrm{bif}} \neq 0$ on $\mathbb{B}_2$. In finitely many steps, we find a ball $\mathbb{B}' \Subset \mathbb{B}$ where the following hold:

(1) $2d-2-m+1$ critical points are passive on $\mathbb{B}'$;
(2) $T^{m-1}_{\mathrm{bif}} \neq 0$ on $\mathbb{B}'$, i.e., $\mathrm{supp}(T^{m-1}_{\mathrm{bif}}) \cap \mathbb{B}' \neq \emptyset$.

Since item (1) gives $\mathrm{supp}(T^{m-1}_{\mathrm{bif}}) \cap \mathbb{B}' \subset \mathrm{supp}(T^{m-1}_{\mathrm{bif}}) \setminus \mathrm{supp}(T^m_{\mathrm{bif}})$, the proof is complete. □

## 4. HIGHER BIFURCATION CURRENTS AND NEUTRAL CYCLES

One of the interesting facts provided by the work [22] of Mañé, Sad, and Sullivan, and the work [21] of Lyubich, is the existing link between the existence of a non-persistent neutral cycle and the non-persistent preperiodicity of a critical point. Namely, these authors show that, in any holomorphic family $(f_\lambda)_{\lambda\in\Lambda}$ of degree $d$ rational maps, the closure in $\Lambda$ of the set of parameters $\lambda_0$ for which $f_{\lambda_0}$ possesses a non-persistent neutral cycle coincides with the closure in $\Lambda$ of the set of parameters $\lambda_0$ for which one critical point of $f_{\lambda_0}$ is non-persistently preperiodic to a repelling cycle. In this section, we want to establish an analogous result for higher bifurcation loci.

***4.1. The space*** $\mathrm{Rat}_d^{\mathrm{cm}}$ ***of critically marked degree*** $d$ ***rational maps.*** We refer to [8, Section 1.2] for a description of the set $\mathrm{Rat}_d^{\mathrm{cm}}$ of critically marked rational maps. The space $\mathrm{Rat}_d^{\mathrm{cm}}$ is a quasiprojective variety of dimension $2d+1$, which is an algebraic finite branched cover of $\mathrm{Rat}_d$. The degree of the natural projection $\pi : \mathrm{Rat}_d^{\mathrm{cm}} \longrightarrow \mathrm{Rat}_d$ depends only on $d$. Moreover, there exist $2d-2$ holomorphic maps $c_1, \ldots, c_{2d-2} : \mathrm{Rat}_d^{\mathrm{cm}} \longrightarrow \mathbb{P}^1$ such that $C(f) = \{c_1(f), \ldots, c_{2d-2}(f)\}$, where the critical points are counted with multiplicity. Recall that a holomorphic family of rational maps is said to be *algebraic* if its parameter space is an algebraic variety, and that it is said to be *stable* if its bifurcation locus is empty. In what follows, we will need the following lemma (see [23, Lemma 2.1]).

***Lemma 4.1 (McMullen).*** *Any stable algebraic family of degree* $d$ *rational maps either is trivial or all its members are postcritically finite.*

Recall that for $n \geq 1$ and $w \in \mathbb{C} \setminus \{1\}$, we denoted by $\mathrm{Per}_n(w)$ the set of all rational maps having a cycle of multiplier $w$ and exact period $n$ (see section 2.1). In the quasiprojective variety $\mathrm{Rat}_d^{\mathrm{cm}}$, the set $\mathrm{Per}_n(w)$ is an algebraic hypersurface. Let $k \geq 2$; for $\Theta_k := (\theta_1, \ldots, \theta_k) \in (\mathbb{R} \setminus \mathbb{Z})^k$ and $N_k := (n_1, \ldots, n_k) \in (\mathbb{Z}_+)^k$, we define the set $\mathrm{Per}_{N_k}^k(\Theta_k)$ as

$$\begin{aligned}\mathrm{Per}_{N_k}^k(\Theta_k) := \{f \in \mathrm{Rat}_d^{\mathrm{cm}} \mid f &\text{ has } k \text{ distinct neutral cycles of respective} \\ &\text{multipliers } e^{2i\pi\theta_1}, \ldots, e^{2i\pi\theta_k}, \text{ and respective period } n_1, \ldots, n_k\}.\end{aligned}$$

The set $\mathrm{Per}_{N_k}^k(\Theta_k)$ is a subvariety of $\bigcap_{1\leq j\leq k} \mathrm{Per}_{n_j}(e^{2i\pi\theta_j})$.

***Lemma 4.2.*** *Let* $k \geq 2$, *let* $\Theta_k = (\theta_1, \ldots, \theta_k) \in (\mathbb{R} \setminus \mathbb{Z})^k$, *and let* $N_k = (n_1, \ldots, n_k) \in (\mathbb{Z}_+)^k$. *If* $\mathrm{Per}_{N_k}^k(\Theta_k) \neq \emptyset$, *then any irreducible component of the algebraic set* $\mathrm{Per}_{N_k}^k(\Theta_k)$ *has codimension* $k$ *in* $\mathrm{Rat}_d^{\mathrm{cm}}$.

*Proof.* Let $\Gamma$ be an irreducible component of $\mathrm{Per}_{N_k}^k(\Theta_k)$. Let us first treat the case $k = 2d-2$. If $\operatorname{codim} \Gamma < 2d-2$, the family $\Gamma$ is a non-trivial algebraic family of rational maps, since $\dim \Gamma > 3$ and is stable by the Fatou-Shishikura inequality. Lemma 4.1 asserts that the family $\Gamma$ is a family of postcritically finite rational maps. This is impossible, since postcritically finite rational maps only have repelling or attracting cycles. This implies that $\operatorname{codim} \Gamma = 2d-2$.

Assume now that $k < 2d-2$. Then, the family $\Gamma$ is not stable. Indeed, if we assume that $\Gamma$ is stable, Lemma 4.1 again implies that $\Gamma$ is a family of postcritically finie rational maps. Therefore, there exist $\theta_{k+1} \in \mathbb{R} \setminus \mathbb{Z} \cup \{\theta_1, \ldots, \theta_k\}$, an integer $n_{k+1}$, and a map $f_1 \in \Gamma \cap \mathrm{Per}_{n_{k+1}}(e^{2i\pi\theta_{k+1}})$. We thus reduce to proving that any irreducible component of $\mathrm{Per}_{N_{k+1}}^{k+1}(\Theta_{k+1})$ has codimension $k+1$, which in finitely many steps boils down to the case $k = 2d-2$. □

Let $1 \le k \le 2d-2$. For $\Theta_k = (\theta_1, \dots, \theta_k) \in (\mathbb{R} \setminus \mathbb{Z})^k$, recall that we have set

$$\mathcal{Z}_k(\Theta_k) = \bigcup_{N_k \in (\mathbb{Z}_+)^k} \mathrm{Per}^k_{N_k}(\Theta_k).$$

Recall also that we denoted by $\mathrm{Prerep}(k)$ the set of rational maps having $k$ prerepelling critical points. We still denote by $T^k_{\mathrm{bif}}$ the $k$-th bifurcation current of the family $\mathrm{Rat}^{\mathrm{cm}}_d$ which may be defined by

$$T^k_{\mathrm{bif}} := \pi^*\left((dd^c L)^k\right) = (dd^c(L \circ \pi))^k \ .$$

Our main result of the present section may be stated as follows.

***Theorem 4.3.*** *Let* $1 \le k \le 2d-2$, *and let* $\Theta_k = (\theta_1, \dots, \theta_k) \in (\mathbb{R} \setminus \mathbb{Z})^k$. *Then, in* $\mathrm{Rat}^{\mathrm{cm}}_d$, *we have*

$$\mathrm{supp}(T^k_{\mathrm{bif}}) = \overline{\mathcal{Z}_k(\Theta_k)} = \overline{\pi^{-1}(\mathrm{Prerep}(k))}.$$

*Proof.* By [14, Theorem 1], we already know that $\overline{\pi^{-1}\,\mathrm{Prerep}(k)} = \mathrm{supp}(T^k_{\mathrm{bif}})$. The first step of the proof consists in showing that $\mathcal{Z}_k(\Theta_k)$ is not empty and that $\mathrm{supp}(T^k_{\mathrm{bif}}) \subset \overline{\mathcal{Z}_k(\Theta_k)}$. After this, we show that, when $\mathrm{Per}^k_{N_k}(\Theta_k) \neq \varnothing$, it is contained in $\mathrm{supp}(T^k_{\mathrm{bif}})$.

Since $T^{2d-2}_{\mathrm{bif}} \neq 0$, the current $T_{j_1} \wedge \cdots \wedge T_{j_k}$ is non-zero for any $j_1 < \cdots < j_k$, and Proposition 3.4 implies that there exists a family of homeomorphic embeddings

$$\Phi_n : (\partial \mathbf{M})^k \times \mathbb{D}^{2d+1-k} \longrightarrow \mathrm{supp}(T_{j_1} \wedge \cdots \wedge T_{j_k})$$

whose images are dense in $\mathrm{supp}(T_{j_1} \wedge \cdots \wedge T_{j_k})$. Let $\zeta_1, \dots \zeta_k \in \partial \mathbf{M}$ be such that $z^2 + \zeta_j$ has a cycle of multiplier $e^{2i\pi\theta_j}$. Since the conjugacy given by Theorem 3.1 is hybrid, Lemma 3 of [7] thus ensures that the map $f_{\Phi_n(\zeta,0)}$ has $k$ distinct neutral cycles of respective multipliers $e^{2i\pi\theta_1}, \dots, e^{2i\pi\theta_k}$, and thus $\mathrm{Per}^k_{N_k}(\Theta_k) \neq \varnothing$ for some $N_k \in (\mathbb{Z}_+)^k$. Moreover, Corollary 2.10 asserts that, for any $1 \le j \le k$, the set of parameters $\zeta \in \partial \mathbf{M}$ for which $z^2 + \zeta$ has a cycle of multiplier $e^{2i\pi\theta_j}$ is dense in $\partial \mathbf{M}$. Therefore, $\mathrm{supp}(T_{j_1} \wedge \cdots \wedge T_{j_k}) \subset \overline{\mathcal{Z}_k(\Theta_k)}$, for any $j_1 < \cdots < j_k$. By (2.1), this implies $\mathrm{supp}(T^k_{\mathrm{bif}}) \subset \overline{\mathcal{Z}_k(\Theta_k)}$.

It thus remains to prove $\mathrm{Per}^k_{N_k}(\Theta_k) \subset \mathrm{supp}(T^k_{\mathrm{bif}})$, as soon as $\mathrm{Per}^k_{N_k}(\Theta_k) \neq \varnothing$. To this end, for $m > n \ge 1$ and $1 \le j \le 2d-2$, we set

$$\mathrm{Prerep}_j(n,m) = \{f \in \mathrm{Rat}^{\mathrm{cm}}_d \mid f^{\circ n}(c_j(f)) = f^{\circ m}(c_j(f)) \\ \text{and } f^{\circ(m-n)}(c_j(f)) \text{ is repelling}\}.$$

We proceed by induction. We let $N_k = (n_1, \dots, n_k) \in (\mathbb{Z}_+)^k$ be such that $\mathrm{Per}^k_{N_k}(\Theta_k) \neq \varnothing$, and let $f_0 \in \mathrm{Per}_{N_k}(\Theta_k)$. By Lemma 4.2, $f_0$ has a non-persistent cycle of multiplier $e^{2i\pi\theta_k}$ in the family $\mathrm{Per}^{k-1}_{N_{k-1}}(\Theta_{k-1})$. The theorem of Mañé-Sad-Sullivan asserts that $f_0$ is a bifurcation parameter in the family $\mathrm{Per}_{N_k}(\Theta_k)$. Therefore, by Montel's Theorem, there exists $f_1 \in \mathrm{Per}^{k-1}_{N_{k-1}}(\Theta_{k-1})$ arbitrarily close to $f_0$ such that $f_1$ has one critical point preperiodic to a repelling cycle, that is,

$$f_1 \in \mathrm{Per}^{k-1}_{N_{k-1}}(\Theta_{k-1}) \cap \mathrm{Prerep}_{j_1}(n_1, m_1)$$

for some $1 \leq j_1 \leq 2d-2$ and $m_1 > n_1 \geq 1$, and $\mathrm{Per}^{k-1}_{N_{k-1}}(\Theta_{k-1}) \cap \mathrm{Prerep}_j(n_1, m_1)$ has codimension $k$. Assume now that we already have found, arbitrarily close to $f_0$,

$$f_j \in \bigcap_{1\leq i\leq j} \mathrm{Prerep}_{j_i}(n_i, m_i) \cap \mathrm{Per}^{k-j}_{N_{k-j}}(\Theta_{k-j}),$$

and that $\operatorname{codim} \bigcap_{1\leq i\leq j} \mathrm{Prerep}_{j_i}(n_i, m_i) \cap \mathrm{Per}^{k-j}_{N_{k-j}}(\Theta_{k-j}) = k$. Then, the map $f_j$ has a non-persistent neutral cycle of multiplier $e^{2i\pi\theta_{k-j}}$ in the family

$$X_j := \bigcap_{1\leq i\leq j} \mathrm{Prerep}_{j_i}(n_i, m_i) \cap \mathrm{Per}^{k-j-1}_{N_{k-j-1}}(\Theta_{k-j-1}).$$

Note that the fact that a periodic point is repelling is an open condition. Thus, using again Montel's Theorem, we find integers $m_{j+1} > n_{j+1} \geq 1$ and

$$f_{j+1} \in \mathrm{Prerep}_{j_{j+1}}(n_{j+1}, m_{j+1}) \cap X_j$$

arbitrarily close to $f_j$. Moreover, $\operatorname{codim}\ \mathrm{Prerep}_{j_{j+1}}(n_{j+1}, m_{j+1}) \cap X_j = k$. Iterating this process $k$ times, we find $f_k$ arbitrarily close to $f_0$ at which $k$ critical points fall properly onto repelling cycles. Theorem 6.2 of [18] states that, under these conditions, the map $f_k$ belongs to the support of $T^k_{\mathrm{bif}}$. As $f_k$ can be taken as close to $f_0$ as we want, this concludes the proof. □

*Proof of Theorem 1.1.* Recall that we denoted by $\pi : \mathrm{Rat}^{\mathrm{cm}}_d \longrightarrow \mathrm{Rat}_d$ the natural projection, which is a finite branched covering. The projection

$$\Pi : \mathrm{Rat}^{\mathrm{cm}}_d \longrightarrow \mathcal{M}_d$$

which associates to $f$ its class of conjugacy by Möbius transformations is a principal bundle on $\mathrm{Rat}^{\mathrm{cm}}_d \setminus V$, where $V$ is a proper subvariety of $\mathrm{Rat}^{\mathrm{cm}}_d$ (see, e.g., [1], page 226). Since the function $L \circ \pi : \mathrm{Rat}^{\mathrm{cm}}_d \longrightarrow \mathbb{R}$ is continuous, the current $(dd^c(L \circ \pi))^k$ does not give mass to pluripolar sets. Therefore, Theorem 4.3 implies that the set $\mathcal{Z}_k(\Theta_k) \setminus V$ is dense in $\mathrm{supp}((dd^c(L \circ \pi))^k)$. The conclusion follows, since $\Pi(\mathrm{supp}((dd^c L \circ \pi)^k)) = \mathrm{supp}(T^k_{\mathrm{bif}})$, where $T^k_{\mathrm{bif}}$ denotes the $k^{th}$-bifurcation current of the moduli space $\mathcal{M}_d$. □

***4.2. The moduli space $\mathcal{P}_d$ of degree $d$ polynomials.*** In the present section, we give a simpler argument for the proof of Theorem 4.3 in the case of polynomial families. This argument relies on a fine control of the cluster set of the bifurcation locus at infinity. To this end, we will use the following parametrization of the moduli space $\mathcal{P}_d$ of all degree $d$ polynomials. For any $(c,a) = (c_1, \ldots, c_{d-2}, a) \in \mathbb{C}^{d-1}$, we set $c = (c_1, \ldots, c_{d-2})$ and

$$P_{(c,a)}(z) := \frac{1}{d}z^d + \sum_{j=2}^{d-1}(-1)^{d-j}\sigma_{d-j}(c)\frac{z^j}{j} + a^d,$$

where $\sigma_j(c)$ is the symmetric degree $j$ polynomial in $c_1, \ldots, c_{d-2}$. The critical points of the polynomial $P_{(c,a)}$ are $0, c_1, \ldots, c_{d-2}$ and are holomorphic functions of the parameter. This family has been introduced by Branner and Hubbard in [6] to prove the compactness of the connectedness locus of $\mathcal{P}_d$. It also has been used by Dujardin and Favre in [15] and by Bassanelli and Berteloot to study the bifurcation currents in [3]. The parameter space $\mathbb{C}^{d-1}$ can be naturally compactified as $\mathbb{P}^{d-1}$ by the following natural injection:

$$(c,a) \in \mathbb{C}^{d-1} \longrightarrow [c : a : 1] \in \mathbb{P}^{d-1}.$$

Finally, we denote by $T_i$ the bifurcation current of the marked critical point $c_i$. Let us set $C_d = \{(c,a) \in \mathbb{C}^{d-1} \mid \mathcal{J}_{c,a} \text{ is connected}\}$. We summarize the main properties of this parametrization in the following proposition (see [6], [15, §6], and [3, §4]):

***Proposition 4.4.*** *The following hold:*

*(1) The natural projection $\Pi : \mathbb{C}^{d-1} \longrightarrow \mathcal{P}_d$ is a degree-$d(d-1)$ analytic branched cover;*

*(2) The loci $\mathcal{B}_i := \{(c,a) \mid (P^{\circ n}_{(c,a)}(c_i))_{n\geq 1}$ is bounded in $\mathbb{C}\}$ accumulate at infinity of $\mathbb{C}^{d-1}$ in $\mathbb{P}^{d-1}$ on codimension $1$ algebraic sets $\Gamma_i$ of the hyperplane $\mathbb{P}_\infty = \mathbb{P}^{d-1} \setminus \mathbb{C}^{d-1}$;*

*(3) The locus $C_d$ is compact in $\mathbb{C}^{d-1}$, and, for any $0 \leq i_1 < \cdots < i_p \leq d-2$, the intersection $\Gamma_{i_1} \cap \cdots \cap \Gamma_{i_p}$ has codimension $p$ in $\mathbb{P}_\infty$;*

*(4) The bifurcation measure $\mu_{\mathrm{bif}} := T_{\mathrm{bif}}^{d-1}$ is a finite positive measure, and its support coincides with the Shilov boundary of $C_d$.*

For any $w \in \mathbb{C}$, the algebraic hypersurfaces $\mathrm{Per}_n(w)$ of $\mathbb{C}^{d-1}$ extend as algebraic hypersurfaces of $\mathbb{P}^{d-1}$. Moreover, for $w \in \overline{\mathbb{D}}$, the hypersurface $\mathrm{Per}_n(w)$ intersects the hyperplane at infinity $\mathbb{P}_\infty$ along the algebraic set $\bigcup_{0\leq j\leq d-2} \mathbb{P}_\infty \cap \mathcal{B}_j$, which has codimension 2 in $\mathbb{P}^{d-1}$.

We use the same notations as in section 4.1. Let $1 \leq k \leq d-1$; then, for $N_k = (n_1, \ldots, n_k) \in (\mathbb{Z}_+)^k$ and $\Theta_k = (\theta_1, \ldots, \theta_k) \in (\mathbb{R} \setminus \mathbb{Z})^k$, we denote by $\mathrm{Per}^k_{N_k}(\Theta_k)$ the set of parameters $(c,a) \in \mathbb{C}^{d-1}$ such that $P_{(c,a)}$ has $k$ distinct neutral cycles of respective multipliers $e^{2i\pi\theta_j}$ and period $n_j$. The set $\mathrm{Per}^k_{N_k}(\Theta_k)$ is a subvariety of $\bigcap_{1\le j\le k}\mathrm{Per}_{n_j}(e^{2i\pi\theta_j})$. We also set

$$\mathcal{Z}_k(\Theta_k) := \bigcup_{N_k\in(\mathbb{Z}_+)^k}\mathrm{Per}^k_{N_k}(\Theta_k),$$

and $\mathrm{Prerep}(k) := \{(c,a)\in\mathbb{C}^{d-1} \mid P_{(c,a)}$ has $k$ prerepelling critical points$\}$. In the present setting, Theorem 4.3 can be formulated as follows.

***Theorem 4.5.*** *Let $1\le k\le d-1$ and let $\Theta_k=(\theta_1,\dots,\theta_k)\in(\mathbb{R}\setminus\mathbb{Z})^k$. Then, in $\mathbb{C}^{d-1}$,*

$$\mathrm{supp}(T^k_{\mathrm{bif}}) = \overline{\mathcal{Z}_k(\Theta_k)} = \overline{\mathrm{Prerep}(k)},$$

The proof is the same as in the case of the space $\mathrm{Rat}^{\mathrm{cm}}_d$. The only difference is in the proof of the following lemma.

***Lemma 4.6.*** *Let $k\ge 2$, let $\Theta_k=(\theta_1,\dots,\theta_k)\in(\mathbb{R}\setminus\mathbb{Z})^k$, and let $N_k=(n_1,\dots,n_k)\in(\mathbb{Z}_+)^k$. If $\mathrm{Per}^k_{N_k}(\Theta_k)\neq\emptyset$, then any irreducible component of the algebraic set $\mathrm{Per}^k_{N_k}(\Theta_k)$ has codimension $k$ in $\mathbb{C}^{d-1}$.*

*Proof.* Let $\Gamma$ be a non-empty irreducible component of $\mathrm{Per}^k_{N_k}(\Theta_k)$. Then, there exists irreducible components $H_i$ of $\mathrm{Per}_{n_i}(e^{2i\pi\theta_i})$, such that $\Gamma$ is a Zariski open set of $H_1\cap\cdots\cap H_k$. For any $1\le i\le k$, note that $H_i\subset\bigcup_j\mathcal{B}_j$. By Proposition 4.4, this implies that $\mathbb{P}_\infty\cap\bigcap_{1\le i\le k}H_i$ has codimension $k+1$. Since $\mathbb{P}_\infty$ has codimension 1, we get $\mathrm{codim}\, H_1\cap\cdots\cap H_k = k$. ☐

***Remark.*** Dujardin and Favre proved that for a $\mu_{\mathrm{bif}}$−generic polynomial $f$, all critical orbits are dense in $\mathcal{J}_f$ (see [15, Corollary 11]). Therefore, the copies of $(\partial\mathbf{M})^{d-1}$ provided by Theorem 1.2 have zero measure for $\mu_{\mathrm{bif}}$, even though they form a homogeneous dense subset of $\mathrm{supp}(\mu_{\mathrm{bif}})$ of Hausdorff dimension $2(d-1)$.

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