Strong-bifurcation loci have maximal Hausdorff dimension

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Introduction

The *Julia set* of a degree $d \geq 2$ rational map f is the set

$$\mathcal{J}_f := \{z_0 \in \mathbb{P}^1 \ / \ (f^n) \text{ is not equicontinuous at } z_0\}.$$

The *Fatou set* of f is the complement $\mathcal{F}_f := \mathbb{P}^1 \setminus \mathcal{J}_f$.

- The set $\mathcal{J}_f$ is a non-empty compact subset of $\mathbb{P}^1$. Moreover, it is either $\mathbb{P}^1$ or has empty interior,
- The connected components of $\mathcal{F}_f$ are all eventually periodic under the action of f.

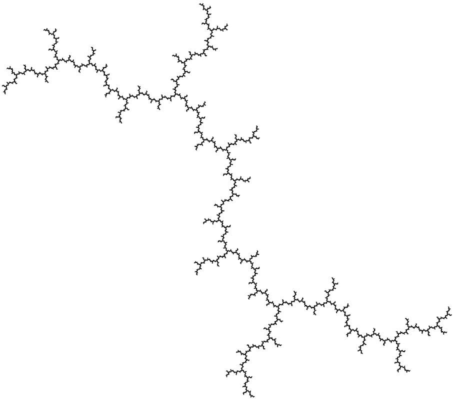

Figure: The julia set of the map $z^2 + i$.

Let $(f_\lambda)_{\lambda\in X}$ be a holomorphic family of degree d rational maps, i.e. $(\lambda, z) \longmapsto f_\lambda(z)$ is holomorphic on $X \times \mathbb{P}^1$ and $\deg(f_\lambda) = d$ for any $\lambda \in X$.

The *bifurcation locus* of $(f_\lambda)_{\lambda\in X}$ is the closure of the set of parameters λ_0 such that the map $\lambda \longmapsto \mathcal{J}_{f_\lambda}$ is not continuous at λ_0 for the Hausdorff topology of compact subsets of $\mathbb{P}^1$.

Example

Set $p_c(z) = z^2 + c$ for $c \in \mathbb{C}$. Then the bifurcation locus of the family $(p_c)_{c\in\mathbb{C}}$ is the boundary of the Mandelbrot set :

$$M := \{c \in \mathbb{C} \ / \ p_c^n(0) \text{ in bounded in } \mathbb{C}\}.$$

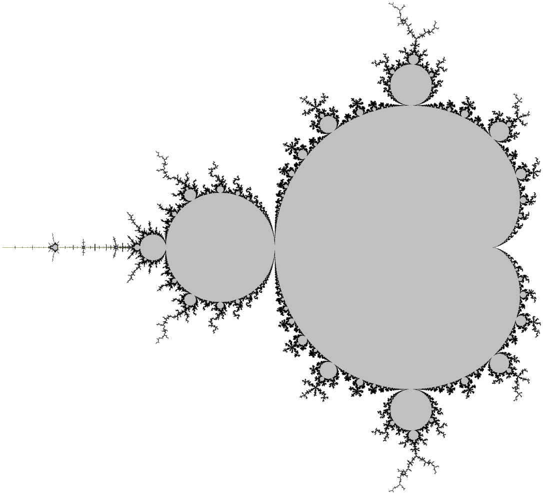

Figure: The Mandelbrot set.

Theorem (Shishikura, 1991)

$$\dim_H(\partial M) = 2.$$

Let $d \geq 2$ be an integer and denote by Rat_d the space of all the degree d rational maps.

Theorem (Tan Lei, 1998)

In Rat_d the bifurcation locus is of maximal Hausdorff dimension, i.e. $2(2d+1)$.

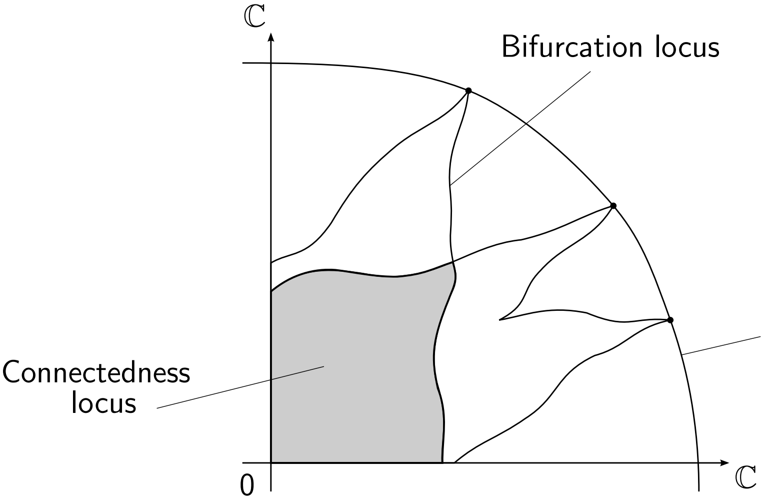

Figure: Behavior of the bifurcation locus at infinity in degree 3 polynomials (Branner-Hubbard 1992)

Recall that a n-periodic point p of f is neutral if $|(f^n)'(p)| = 1$. Set

$$\mathcal{Z}_k := \{f \in \mathrm{Rat}_d \ / \ f \text{ has at least } k \text{ distinct neutral cycles}\}$$

and

$$\mathfrak{m}_k := \{f \in \mathrm{Rat}_d \ / \ \exists c_1, \ldots, c_k \in C(f) \text{ and } n_0 \geq 1 \text{ s.t. } f \text{ is uniformly expanding on } \overline{\{f^n(c_j) \ / \ n \geq n_0\}}\}$$

Theorem (Mañé-Sad-Sullivan, 1983)

$\overline{\mathfrak{m}_1} = \overline{\mathcal{Z}_1}$ *is the bifurcation locus in* Rat_d. *Morever, it has empty interior.*

Shishikura proved, using quasiconformal surgery, that a rational map $f \in \mathrm{Rat}_d$ has at most $2d - 2$ distinct neutral cycles.

Theorem (Gauthier, 2010)

$$\dim_H(\overline{\mathcal{Z}_{2d-2}}) = 2(2d+1).$$

The bifurcation currents

The Lyapounov function $L : \mathrm{Rat}_d \longrightarrow \mathbb{R}$ of Rat_d is defined by

$$L(f) := \int_{\mathbb{P}^1} \log|f'| d\mu_f, \ \ f \in \mathrm{Rat}_d,$$

where μ_f is the maximal entropy measure of f.

Proposition

The function L is p.s.h and continuous on Rat_d.

The *bifurcation current* is defined by $T_{\mathrm{bif}} := dd^c L$. The main result about T_{bif} is the following :

Theorem (DeMarco, 2000)

The support of T_{bif} coincides with the bifurcation locus of the family Rat_d.

Recall that the k-th auto-intersection of T_{bif} is defined by induction by setting :

$$T_{\text{bif}}^k := dd^c(LT_{\text{bif}}^{k-1}),\ 2 \leq k \leq 2d-2.$$

Theorem (Bassanelli-Berteloot, 2007)

$\text{supp}(T_{\text{bif}}^k) \neq \emptyset$ *for* $1 \leq k \leq 2d-2$. *Moreover,*

$$\text{supp}(T_{\text{bif}}^k) \subset \overline{\mathcal{Z}_k}.$$

Misiurewicz maps

A rational map $f \in \mathrm{Rat}_d$ is said *Misiurewicz* if :

- $C(f) \cap \mathcal{J}_f \neq \emptyset$,
- f has no parabolic cycle (i.e. if $f^n(p) = p$, then $(f^n)'(p)$ is not a root of unity), and
- $\omega(c) \cap C(f) = \emptyset$ for every $c \in C(f) \cap \mathcal{J}_f$.

Example

A rational map is SPCF if every critical point of f is preperiodic but not periodic. Every strictly postcritically finite rational map (denoted by SPFC) is Misiurewicz.

A theorem of Mañé gives us the following proposition :

Proposition

Let $f \in \mathrm{Rat}_d$ be Misiurewicz. Then :

1. *f has no neutral cycle,*
2. *every periodic Fatou component of f is an attracting basin,*
3. *there exists an integer $k_0 \geq 1$ such that*
$$P^{k_0}(f) := \overline{\{f^n(c) \ / \ n \geq k_0, \ c \in C(f) \cap \mathcal{J}_f\}}$$
is a f-hyperbolic set, i.e. $f(P^{k_0}(f)) \subset P^{k_0}(f)$ and f is uniformly expanding on $P^{k_0}(f)$,
4. *either $\mathcal{J}_f = \mathbb{P}^1$ or $\mathrm{Leb}(\mathcal{J}_f) = 0$.*

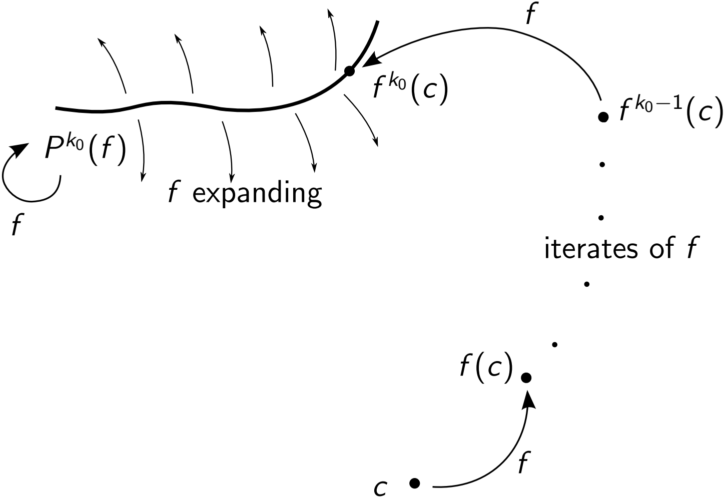

Figure: An expanding set capturing the critical point c.

We say that $X \subset \mathbb{P}^1$ is *homogeneous* if for every $x \in X$ and every neighborhood V of x in $\mathbb{P}^1$, we have

$$\dim_H(X \cap V) = \dim_H(X).$$

The *hyperbolic dimension* of $f \in \mathrm{Rat}_d$ is defined by

$$\dim_{\mathrm{hyp}}(f) := \sup\{\dim_H(X) \ / \ X \text{ is homogeneous } f\text{-hyperbolic}\}.$$

Using the thermodynamical formalism we get :

Theorem (Przytycki 1996, Urbanski 1994, McMullen 1997)

If $f \in \mathrm{Rat}_d$ *is Misiurewicz, then* $\dim_{\mathrm{hyp}}(f) = \dim_H(\mathcal{J}_f)$.

Let $1 \leq k \leq 2d - 2$ be an integer. We say that $f \in \mathrm{Rat}_d$ is *k-Misiurewicz* if f is Misiurewicz and $C(f) \cap \mathcal{J}_f$ contains k critical points of f counted with multiplicity.

We define the set $\mathfrak{M}_k$ by :

$$\mathfrak{M}_k := \{f \in \mathrm{Rat}_d \ / \ f \text{ is } k\text{-Misiurewicz and } f \text{ is not a flexible Lattès map}\}$$

Local minoration of the dimension of $\mathfrak{M}_k$

Let $f \in \mathrm{Rat}_d$. A critical point $c \in C(f)$ is *marked* at f if there exists a neigborhood $\mathbb{B}(0,r) \subset \mathrm{Rat}_d$ of $f_0 := f$ and a holomorphic map $c : \mathbb{B}(0,r) \longrightarrow \mathbb{P}^1$ such that :

- $f'_\lambda(c(\lambda)) = 0$ for all $\lambda \in \mathbb{B}(0,r)$,
- $c(0) = c$.

Let $\mathbb{B}(0,r) \subset \mathrm{Rat}_d$ be centered at $f_0 := f$ and let X be a compact f_0-hyperbolic subset of $\mathbb{P}^1$. Then there exists a holomorphic motion

$$h : \mathbb{B}(0,r) \times X \longrightarrow \mathbb{P}^1$$

which conjugates the dynamics on X, i.e.

$$h_\lambda \circ f_0 = f_\lambda \circ h_\lambda \text{ on } X.$$

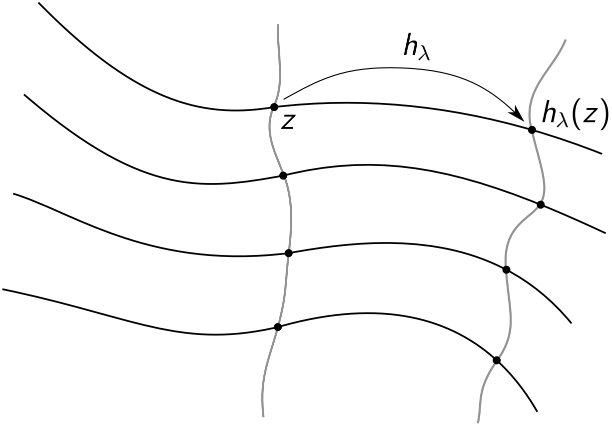

Figure: A holomorphic motion h.

Let $f \in \mathfrak{M}_k$ and $\{c_1, \ldots, c_k\} = C(f) \cap \mathcal{J}_f$. Suppose that the critical points $c_1, \ldots, c_k$ are marked at f. Let $\mathbb{B}(0,r) \subset \mathrm{Rat}_d$ be a small enough neighborhood of $f_0 := f$ and let h_λ be a holomorphic motion of $P^{k_0}(f)$ given as above. Set

$$\chi_i(\lambda) := f_\lambda^{k_0}(c_i(\lambda)) - h_\lambda(f_0^{k_0}(c_i(0))) \text{ for } \lambda \in \mathbb{B}(0,r).$$

The *activity map* of f is given by :

$$\begin{array}{rcl} \chi : \mathbb{B}(0,r) & \longmapsto & \mathbb{C}^k \\ \lambda & \longmapsto & (\chi_1(\lambda), \ldots, \chi_k(\lambda)). \end{array}$$

Theorem (Weak transversality)

The analytic set $\chi^{-1}\{0\}$ has codimension k in $\mathbb{B}(0,r)$.

Generalizing Shishikura's and Tan Lei's methods we can prove the following result :

Theorem (Gauthier, 2010)

For every $f \in \mathfrak{M}_k$ and every neighborhood $V_0 \subset \mathrm{Rat}_d$ of f :

$$\dim_H(\mathfrak{M}_k \cap V_0) \geq 2(2d+1-k) + k.\dim_{\mathrm{hyp}}(f).$$

In particular, if $k = 2d-2$ we get $\dim_H(\mathfrak{M}_{2d-2}) = 2(2d+1)$.

Misiurewicz maps and bifurcation currents

Theorem (Dujardin-Favre, 2008)

Let f be a SPCF polynomial map, then f lies in the support of the bifurcation measure $(dd^c L)^{d-1}$ of degree d polynomials.

Theorem (Buff-Epstein, 2009)

If f is SPCF and f is not a flexible Lattès map, then $f \in \mathrm{supp}(T_{\mathrm{bif}}^{2d-2})$.

Theorem (Gauthier, 2010)

For $1 \leq k \leq 2d-2$, $\mathfrak{M}_k \subset \mathrm{supp}(T_{\mathrm{bif}}^k)$.

Proof in degree 3 polynomials

- To simplify, we assume that $k = 2$ and we work in the family

$$f_\lambda(z) := \frac{1}{3}z^3 - \frac{c}{2}z^2 + a^3$$

with $\lambda = (c, a) \in \mathbb{C}^2$. Let $\mathbb{B}(\lambda_0, r) \subset \mathbb{C}^2$ be a ball centered at $f_{\lambda_0} \in \mathfrak{M}_2$. Assume that the activity map $\chi : \mathbb{B}(\lambda_0, r) \longrightarrow \mathbb{C}^2$ is a local biholomorphism at λ_0. Set

$$\mu_{\mathrm{bif}} := (dd^c L)^2.$$

Let $\mathbb{D}^2_\epsilon := \mathbb{D}(0,\epsilon) \times \mathbb{D}(0,\epsilon)$ and

$$\Omega_n := \chi^{-1}\Big(\mathbb{D}\big(0, \tfrac{\epsilon}{|(f^n)'(f^{k_0}(c_1(0)))|}\big) \times \mathbb{D}\big(0, \tfrac{\epsilon}{|(f^n)'(f^{k_0}(c_2(0)))|}\big)\Big).$$

The family $\{\Omega_n\}_{n\geq 1}$ is a neighborhood basis of λ_0 in $\mathbb{C}^2$ and there exists a biholomorphism $\chi_n : \Omega_n \longrightarrow \mathbb{D}^2_\epsilon$.

We will prove that $\mu_{\mathrm{bif}}(\Omega_n) > 0$ for all $n \geq 1$.

• Set $g_{f_\lambda}(z) := \lim_{n\to\infty} 3^{-n}\log^+|f_\lambda^n(z)|$. A formula of Przytycki (DeMarco for rational maps) asserts that

$$dd^c L(f_\lambda) = dd^c \sum_{j=1}^{2} g_{f_\lambda}(c_j(\lambda)), \qquad (1)$$

By definition of μ_{bif} and of g_λ one has

$$\begin{aligned} \mu_{\mathrm{bif}}(\Omega_n) &\gtrsim \int_{\Omega_n} dd^c g_{f_\lambda}(c_1(\lambda)) \wedge dd^c g_{f_\lambda}(c_2(\lambda)) \\ &\gtrsim 3^{-2(n+k_0)} \int_{\Omega_n} \bigwedge_{j=1}^{2} dd^c g_{f_\lambda}(f_\lambda^{n+k_0}(c_j(\lambda))). \end{aligned}$$

As $\chi_n : \Omega_n \longrightarrow \mathbb{D}_\epsilon^2$ is biholomorphism, one has

$$\begin{aligned} 3^{2(n+k_0)}\mu_{\mathrm{bif}}(\Omega_n) &\gtrsim \int_{\Omega_n} \chi_n^* \left(\bigwedge_{j=1}^{2} dd^c g_{n,j}(x) \right) \\ &\gtrsim \int_{\mathbb{D}_\epsilon^2} \bigwedge_{j=1}^{2} dd^c g_{n,j}(x) \end{aligned}$$

where $g_{n,j}(x) := g_{f_{\chi_n^{-1}(x)}}(f_{\chi_n^{-1}(x)}^{n+k_0}(c_j(\chi_n^{-1}(x))))$ is $p.s.h$ and continuous on $\mathbb{D}_\epsilon^2$.

• There exists non-constant holomorphic functions $p_j : \mathbb{D}(0,\epsilon) \longrightarrow \mathbb{C}$ s.t. the sequence $g_{n,j}(x)$ locally uniformly converges to $g_f \circ p_j(x_j)$ and $p_j(0) \in \mathcal{J}_f$.

- This and Fubini's Theorem yield :

$$\liminf_{n\to\infty} 3^{2(n+k_0)}\mu_{\text{bif}}(\Omega_n) \gtrsim \int_{\mathbb{D}_\epsilon^2} \bigwedge_{j=1}^{2} dd^c g_f \circ p_j(x_j)$$
$$\gtrsim \prod_{j=1}^{2} \int_{\mathbb{D}(0,\epsilon)} dd^c g_f \circ p_j(x_j).$$

As $\mu_f = \Delta g_f$ one can conclude that

$$\liminf_{n\to\infty} 3^{2(n+k_0)}\mu_{\text{bif}}(\Omega_n) \gtrsim \prod_{j=1}^{2} \mu_f(p_j(\mathbb{D}(0,\epsilon))) > 0. \qquad \square$$

There were two main difficulties in generalizing Buff-Epstein result :

- Their approach of transversality does not work for Misiurewicz maps which are not SPCF.
- Instead of linearizing f at repelling cycles we need to linearize f along infinite orbits.

Conclusion

For $k = 2d - 2$ we obtain :

- $\mathfrak{M}_{2d-2} \subset \overline{\{f \in \mathrm{Rat}_d \ / \ f \text{ has } 2d-2 \text{ neutral cycles}\}}$
- $\dim_H(\mathfrak{M}_{2d-2}) = 2(2d+1)$.

Thank you for your attention!