THE GEOMETRIC DYNAMICAL NORTHCOTT AND BOGOMOLOV PROPERTIES

by

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Abstract. — We establish the Geometric Dynamical Northcott Property for polarized endomorphisms of a projective variety over a function field $\mathbf{K}$ of characteristic zero, and we relate this property to the notion of stability in complex dynamics. This extends previous results of Benedetto, Baker and DeMarco in dimension 1, and of Chatzidakis-Hrushovski in higher dimension. Our proof uses complex dynamics arguments and does not rely on the previous ones.

We first show that, when $\mathbf{K}$ is the field of rational functions of a normal complex projective variety, the canonical height of a subvariety is the mass of the appropriate bifurcation current and that a marked point is stable if and only if its canonical height is zero. We then establish the Geometric Dynamical Northcott Property using a similarity argument. Moving from points to subvarieties, we propose, for polarized endomorphisms, a dynamical version of the Geometric Bogomolov Conjecture, recently proved by Cantat, Gao, Habegger, and Xie in the original setting of abelian varieties.

Résumé **(Les propriétés Dynamiques Géométriques de Northocott et Bogomolov)**

Nous établissons la propriété Dynamique Géométrique de Northcott pour les endomorphismes polarisés d'une variété projective sur un corps de fonctions $\mathbf{K}$ de caractéristique nulle, et nous relions cette propriété à la notion de stabilité en dynamique complexe. Cela généralise des résultats de Benedetto, Baker, et DeMarco en dimension 1, et de Chatzidakis-Hrushovski en dimension plus grande. Notre démonstration met en jeu des arguments de dynamique complexe et ne repose pas sur ceux utilisés jusqu'alors.

Dans un premier temps nous prouvons que, lorsque $\mathbf{K}$ est le corps des fonctions rationnelles d'une variété projective complexe, la hauteur canonique d'une sous-variété coïncide avec la masse du courant de bifurcation approprié, et qu'un point marqué est stable si et seulement si sa hauteur canonique est nulle. Nous établissons alors la propriété Dynamique Géométrique de Northcott, en utilisant un argument de similarité. En passant de l'étude des points à celle des sous-variétés, nous proposons, pour les endomorphismes polarisés, une version dynamique de la Conjecture de Bogomolov Géométrique, récemment démontrée par Cantat, Gao, Habegger, et Xie dans le cas des variétés abéliennes.

1. Introduction

1.1. Polarized endomorphisms, functions fields and dynamical stability. — Fix a field $\mathbf{K}$ of characteristic zero. A *polarized endomorphism* over $\mathbf{K}$ is a triple (X, f, L) where

1. X is a projective variety defined over $\mathbf{K}$, irreducible over $\bar{\mathbf{K}}$.
2. L is an ample line bundle of X which is defined over $\mathbf{K}$, and
3. $f : X \to X$ is an endomorphism defined over $\mathbf{K}$ which is *polarized* by L, i.e. there is an integer $d \geq 2$ such that f^*L is linearly equivalent to $L^{\otimes d}$, denoted $f^*L \simeq L^{\otimes d}$.

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The integer d is the *degree* of (X, f, L). A prototypical example is an endomorphism f of a projective space of degree $d \geq 2$, since it satisfies $f^*\mathcal{O}(1) \simeq \mathcal{O}(d) \simeq \mathcal{O}(1)^{\otimes d}$.

In this article, we are interested in the case where $\mathbf{k}$ is an algebraically closed field of characteristic zero and $\mathbf{K} := \mathbf{k}(\mathcal{B})$ is the field of rational functions of an irreducible normal projective $\mathbf{k}$-variety $\mathcal{B}$ of dimension at least one (for our purpose, we can always work with the algebraic closure of $\mathbf{k}$ so we directly assume it is algebraically closed). To such a variety X endowed with the ample line bundle L, we can associate a *model* $(\mathcal{X}, \mathcal{L})$ of (X, L), i.e. a surjective morphism

$$\pi : \mathcal{X} \longrightarrow \mathcal{B}$$

between projective varieties, where $\mathcal{L}$ is a relatively ample line bundle, such that

1. the generic fiber of $\mathcal{X}$ is isomorphic to X,
2. the line bundle L is isomorphic to the restriction of $\mathcal{L}$ to the generic fiber,
3. there exists a Zariski open set Λ over which π is flat. We denote $\mathcal{X}_\Lambda := \pi^{-1}(\Lambda)$.

Note that such a model is not unique, there are infinitely many choices (see [**BLR**, Chapter 1] for generalities on models of $\mathbf{K}$-schemes). Similarly, Λ is not unique and we will shrink it if necessary. For any $\lambda \in \Lambda(\mathbf{k})$, the fiber $X_\lambda := \pi^{-1}\{\lambda\}$ is a projective $\mathbf{k}$-variety of dimension $\dim X$ and $L_\lambda := \mathcal{L}|_{X_\lambda}$ is an ample line bundle of X_λ. Furthermore, when X is normal, up to replacing $\mathcal{X}$ by its normalization, we can assume $\mathcal{X}$ is normal and X_λ is normal for all $\lambda \in \Lambda$. We call such a normal variety $\mathcal{X}$ a *normal model* of X.

Then, a polarized morphism f defined over $\mathbf{K}$ induces a dominant rational map $\mathfrak{f} : \mathcal{X} \dashrightarrow \mathcal{X}$ and we can choose a dense Zariski open subset $\Lambda \subset \mathcal{B}$ as above such that in addition

(a) the following diagram commutes

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, dashed, "\mathfrak{f}"] \arrow[dr, "\pi"'] & & \mathcal{X} \arrow[dl, "\pi"] \\
& \mathcal{B} &
\end{tikzcd}$$

(b) $\mathfrak{f}|_{\mathcal{X}_\Lambda} : \mathcal{X}_\Lambda \to \mathcal{X}_\Lambda$ is a morphism,

(c) for any $\lambda \in \Lambda$, if we set $f_\lambda := \mathfrak{f}|_{X_\lambda}$, then $(X_\lambda, f_\lambda, L_\lambda)$ is a polarized endomorphism over $\mathbf{k}$,

(d) $\mathfrak{f}$ restricted to the generic fiber of $\mathcal{X}$ can be identified with f via the isomorphism between the generic fiber of $\mathcal{X}$ and X.

Definition 1.1. — *Let (X, f, L) be a polarized endomorphism over $\mathbf{K} = \mathbf{k}(\mathcal{B})$ where $\mathbf{k}$ is an algebraically closed field of characteristic zero and $\mathbf{K}$ is the field of rational functions of a normal projective $\mathbf{k}$-variety $\mathcal{B}$. A triple $(\mathcal{X}, \mathfrak{f}, \mathcal{L})$ satisfying properties* (a)−(d) *above for some dense Zariski open set Λ is called an* algebraic family *of polarized endomorphisms; and Λ is called a regular part. Such a family is a* model *of (X, f, L).*

We will frequently shrink the open set Λ, but we will always assume it is dense and Zariski open. Note that in the article, we will use curly letters $\mathfrak{f}, \mathcal{L}, \mathcal{Z} \ldots$ for objects defined on the model $\mathcal{X}$ in order to distinguish them from their counterparts $f, L, Z \ldots$ on X. When X is not normal, letting $\mathsf{n} : \hat{X} \to X$ be its normalization, the universal property of normalization implies that f lifts to an endomorphism $\hat{f} : \hat{X} \to \hat{X}$ and $(\hat{X}, \hat{f}, \mathsf{n}^*L)$ still defines a polarized endomorphism.

A *marked point* is a rational section $a : \mathcal{B} \dashrightarrow \mathcal{X}$ whose indeterminacy locus is contained in $\mathcal{B} \backslash \Lambda$, where $\Lambda \subseteq \mathcal{B}$ is a regular part for $(\mathcal{X}, \mathfrak{f}, \mathcal{L})$ (when $\mathcal{B}$ is a curve, the indeterminacy locus is empty). In particular, a defines a regular map $a : \Lambda \to \mathcal{X}$ such that $a(\lambda) \in X_\lambda$ for all $\lambda \in \Lambda$. To any subvariety Z of X which is defined over $\mathbf{K}$, we can associate a subvariety $\mathcal{Z}$ such that $\pi|_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{B}$ is flat over a dense Zariski open subset of Λ. A point of $X(\mathbf{K})$ corresponds to such a marked point $a : \mathcal{B} \dashrightarrow \mathcal{X}$ which is regular on a suitable dense Zariski open subset of Λ. We also need the following notion of isotriviality which means, in some sense, that the family does not really depend on the parameter.

Definition 1.2. — *Let (X, f, L) be a polarized endomorphism over $\mathbf{K} = \mathbf{k}(\mathcal{B})$ where $\mathbf{k}$ is an algebraically closed field of characteristic zero and $\mathbf{K}$ is the field of rational functions of a normal projective $\mathbf{k}$-variety $\mathcal{B}$. Let $(\mathcal{X}, \mathfrak{f}, \mathcal{L})$ be a model of (X, f, L) and let Λ be a regular part. When X is normal, we say (X, f, L), or equivalently $(\mathcal{X}, \mathfrak{f}, \mathcal{L})$, is* isotrivial *over Λ if, for any $\lambda, \lambda' \in \Lambda(\mathbf{k})$, there exists an isomorphism $\phi : X_\lambda \longrightarrow X_{\lambda'}$ defined over $\mathbf{k}$ such that $\phi \circ f_\lambda = f_{\lambda'} \circ \phi$ and $\phi^* L_{\lambda'} \simeq L_\lambda$.*

In the general case, we say (X, f, L) is isotrivial *over Λ if $(\hat{X}, \hat{f}, \mathsf{n}^* L)$ is, where $\mathsf{n} : \hat{X} \to X$ is its normalization and $\hat{f} : \hat{X} \to \hat{X}$ is the lift of f.*

Remark 1.3. —
1. If (X, f, L) is isotrivial over Λ, so is the polarized variety (X, L) (i.e. in a model, fibers over λ endowed with their polarization are isomorphic) and isotriviality does not depend on the choice of the model.
2. Lemma 3.2 states that the notion of isotriviality is independent of the chosen regular part Λ, and also that a polarized endomorphism (X, f, L) is isotrivial if and only if one of its iterates (X, f^n, L) is. Isotriviality is therefore a dynamically relevant property.
3. When $\mathcal{X} = \mathbb{P}^k \times \mathcal{B}$, the last condition $\phi^* L_{\lambda'} \simeq L_\lambda$ is automatically satisfied. Indeed, we have $L = \mathcal{O}_{\mathbb{P}^k}(j)$, for some $j \in \mathbb{N}^*$, so that for any $\lambda \in \Lambda(\mathbf{k})$, we have $L_\lambda \simeq \mathcal{O}_{\mathbb{P}^k}(j)$. In this case, the isotriviality of $(\mathcal{X}, \mathfrak{f}, \mathcal{L})$ is equivalent to the existence, for any $\lambda, \lambda' \in \Lambda$, of a linear isomorphism $\phi : \mathbb{P}^k \to \mathbb{P}^k$ such that $\phi \circ f_\lambda = f_{\lambda'} \circ \phi$.
4. When $\dim X = 0$, $X(\mathbf{K})$ is a finite set and the isotriviality condition is always satisfied over the Zariski open subset Λ' over which $\mathcal{X} \to \mathcal{B}$ is unramified.

Assume furthermore that $\mathbf{K} = \mathbb{C}(\mathcal{B})$ is the field of rational functions of a normal complex projective variety $\mathcal{B}$. A marked point a is said *stable*, or equivalently the tuple $((\mathcal{X}, \mathfrak{f}, \mathcal{L}), a)$ is said *stable*, if the sequence $\{\lambda \mapsto f_\lambda^n(a(\lambda))\}_n$ is locally equicontinuous on Λ. By the Bishop Theorem [**C**, §15.5, p. 203], this means that the volume of the graph Γ_n of $\lambda \mapsto f_\lambda^n(a(\lambda))$ seen as a subvariety of $\mathcal{X}_\Lambda$ is bounded independently of n above any relatively compact open subset of Λ (i.e. locally on Λ). This notion is, by nature, a local notion.

When $\mathcal{B}$ is a curve and f is an endomorphism of degree d of $\mathbb{P}^1$ defined over $\mathbf{K}$, then $\mathfrak{f}$ defines a family of endomorphisms of degree d of $\mathbb{P}^1(\mathbb{C})$, parametrized by a smooth complex quasi-projective curve Λ, i.e. a morphism $\mathfrak{f} : (z, \lambda) \in \mathbb{P}^1 \times \Lambda \longmapsto (f_\lambda(z), \lambda) \in \mathbb{P}^1 \times \Lambda$ such that for any $\lambda \in \Lambda$, f_λ is a rational map of degree d. Let $\mathrm{Crit}(f_\lambda)$ denote the critical set of f. A fundamental result of McMullen [**Mc**] states that, if the sequence $\{\lambda \mapsto f_\lambda^n(\mathrm{Crit}(f_\lambda))\}_n$ is an equicontinuous family of correspondences (i.e. up to taking a finite branched cover of $\mathcal{B}$, each critical point can be followed as a stable marked point) then either f is isotrivial, or f is a family of Lattès maps. Dujardin and Favre [**DF**] extended the result to the case of a given marked critical point showing that stability implies preperiodicity or isotriviality. Finally, DeMarco [**DeM2**] proved such a statement for any marked point.

1.2. Canonical height and the Northcott property. — We now focus on a more arithmetic point of view. A field $\mathbf{K}$ is a *product formula field* if $\mathbf{K}$ is equipped with the possibly uncountable family $M_\mathbf{K}$ of all places of $\mathbf{K}$, a family $(|\cdot|_v)_{v \in M_\mathbf{K}}$ of non-trivial absolute values $|\cdot|_v$ representing v, and a family $(N_v)_{v \in M_\mathbf{K}}$ of positive integers satisfying the *product formula property* in that, for every $z \in \mathbf{K}^*$,

$$|z|_v = 1 \text{ for all but finitely many } v \in M_\mathbf{K}, \text{ and } \textstyle\prod_{v \in M_\mathbf{K}} |z|_v^{N_\nu} = 1.$$

A place $v \in M_{\mathbf{K}}$ is said to be finite (resp. infinite) if $|\cdot|_v$ is non-archimedean (resp. archimedean). If $M_{\mathbf{K}}$ contains an infinite place, then $\mathbf{K}$ is a *number field* and, if not, it is a *function field.* In particular, when $\mathcal{B}$ is a normal complex projective variety, $\mathbf{K} = \mathbb{C}(\mathcal{B})$ is a product formula field and the choice of the family $(N_v)_{v\in M_{\mathbf{K}}}$ is equivalent to the choice of a polarization on $\mathcal{B}$, see, e.g., [**BG**, p. 21].

Fix a polarized endomorphism (X, f, L) defined over a product formula field $\mathbf{K}$. Let $h_{X,L}$ be the standard Weil height function on $X(\bar{\mathbf{K}})$, relative to the line bundle L (and of a polarization on $\mathcal{B}$). Such a function is defined up to bounded functions and, by the functorial properties of Weil height functions, we have $h_{X,L} \circ f = dh_{X,L} + O(1)$ ([**BG**]). Call and Silverman [**CS**] defined the *canonical height* $\widehat{h}_f : X(\bar{\mathbf{K}}) \to \mathbb{R}_+$ of f as

$$\widehat{h}_f = \lim_{n\to\infty} \frac{1}{d^n} h_{X,L} \circ f^n.$$

When $\mathbf{K}$ is a number field, Northcott's Theorem [**No**] states that, for any $\varepsilon > 0$, the set $\{x \in X(\mathbf{K}),\ h_{L,X}(x) \le \varepsilon\}$ is finite. Since $\widehat{h}_f = h_{X,L} + O(1)$, this implies that the set $\{x \in X(\mathbf{K}),\ \widehat{h}_f(x) \le \varepsilon\}$ is also finite. As $\widehat{h}_f(x) \ge 0$ for all x, there is an $\varepsilon_0 > 0$ such that $\{x \in X(\mathbf{K}),\ \widehat{h}_f(x) \le \varepsilon_0\} = \{x \in X(\mathbf{K}),\ \widehat{h}_f(x) = 0\}$. Now, since $\widehat{h}_f \circ f = d\widehat{h}_f$, the finite set $\{x \in X(\mathbf{K}),\ \widehat{h}_f(x) = 0\}$ is f-invariant and all its points are preperiodic; thus, $\{x \in X(\mathbf{K}),\ \widehat{h}_f(x) \le \varepsilon_0\}$ is a finite, invariant set of preperiodic points.

However, when $\mathbf{K}$ is the function field of a projective variety over a field of characteristic zero, there are infinitely many elements of standard height 0 in $\mathbb{P}^k(\mathbf{K})$ (the elements defined over the field of constants) and all of them have uniformly bounded canonical height [**Ba**, Lemma 1.4]. In particular, to try and state a Northcott property, one first has to use the canonical height instead of the standard height and to replace "$\forall\varepsilon > 0$" by "$\exists\varepsilon_0 > 0$". Furthermore, Northcott's property fails when there is isotriviality. For example, if $f \in \mathbb{C}(t)(Z)$ is defined over the field of constants, then any element z of degree 0 is sent by f to another element of degree 0 so its canonical height is zero.

Over $\mathbb{P}^1(\mathbf{K})$, we have

Theorem 1.4 (Northcott Property [Ba]). — *Let $\mathbf{K}$ be a function field of characteristic zero and assume $f \in \mathrm{End}(\mathbb{P}^1_{\mathbf{K}})$ is non-isotrivial of degree $d \ge 2$. Then, there exists $\varepsilon_0 > 0$ such that*

$$\#\{z \in \mathbb{P}^1(\mathbf{K}), \widehat{h}_f(z) \le \varepsilon_0\} < \infty.$$

In particular, any $z \in \mathbb{P}^1(\mathbf{K})$ with $\widehat{h}_f(z) = 0$ is preperiodic under iteration of f, i.e. there exist $n > m \ge 0$ such that $f^n(z) = f^m(z)$.

This result of Baker holds in arbitrary characteristic (for a suitable definition of isotriviality) and was already proved by Benedetto [**Ben**] for polynomials. In the case of abelian varieties over a global function field endowed with the multiplication by an integer ≥ 2, the analogue of Baker's Theorem is due to Néron and Tate [**Ne, L**]. In higher dimension, Chatzidakis and Hrushovski proved a model-theoretic version of Theorem 1.4 in [**CH**] with $\varepsilon_0 = 0$ in arbitrary characteristic. They only consider dynamics which are primitive algebraic dynamics, in particular f neither has an invariant subvariety of positive dimension nor admits an invariant fibration (see also [**CL**] for a detailed exposition on this work).

Furthermore, when $\mathbf{K} = \mathbb{C}(\mathcal{B})$ for $\mathcal{B}$ a smooth complex projective curve and $f \in \mathrm{End}(\mathbb{P}^1_{\mathbf{K}})$, DeMarco [**DeM2**] gave a different proof. In addition, DeMarco showed that the (dynamical) stability of a marked point a implies that the corresponding point $\mathrm{a} \in \mathbb{P}^1(\mathbf{K})$, satisfies $\widehat{h}_f(\mathrm{a}) = 0$. In this case, the Call-Silverman equality $\widehat{h}_f = h_{X,L} + O(1)$ shows that $\widehat{h}_f(\mathrm{a}) = 0$ if and only if the volume of the Zariski closure of the graph of the marked point $f^n(a)$ in $\mathcal{X}$ is bounded (for a given polarization). Having height 0 is thus a global property and, a priori, it only implies dynamical stability.

1.3. The Geometric Dynamical Northcott Property and dynamical stability. — Our first purpose is to give a new proof of the Northcott Property on function fields of characteristic zero in higher dimension without the primitive algebraic assumption of [**CH**]. When $\mathbf{K} = \mathbb{C}(\mathcal{B})$, we also extend DeMarco's results [**DeM2**] and show that a marked point is stable if and only if its canonical height is 0.

In the sequel, if $\mathcal{Y}$ is a subvariety of $\mathcal{X}$ which is flat over a regular part Λ, we denote by $f(\mathcal{Y})$ the Zariski closure in $\mathcal{X}$ of $f(\mathcal{Y}_\Lambda)$.

Theorem A. — *Let $(\mathcal{X}, f, \mathcal{L})$ be a non-isotrivial complex algebraic family of polarized endomorphisms over a normal complex projective variety $\mathcal{B}$. Let (X, f, L) be the induced polarized endomorphism over the field $\mathbf{K} = \mathbb{C}(\mathcal{B})$. Let $\mathcal{N}$ be a very ample line bundle on $\mathcal{B}$ such that $\mathcal{M} := \mathcal{L} \otimes \pi^*(\mathcal{N})$ is ample on $\mathcal{X}$.*

1. *Let $a : \mathcal{B} \dashrightarrow \mathcal{X}$ be a marked point with corresponding point* $\mathrm{a} \in X(\mathbb{C}(\mathcal{B}))$, *and let $\mathcal{C}_n(a)$ be the Zariski closure in $\mathcal{X}$ of the rational section $a_n : \lambda \mapsto f_\lambda^n(a(\lambda))$. Then*

$$(\star) \qquad a \text{ is stable} \iff \sup_{n\geq 0} \deg_{\mathcal{M}}(\mathcal{C}_n(a)) < \infty \iff \widehat{h}_f(\mathrm{a}) = 0.$$

2. *There exist a regular part Λ, a (possibly empty) proper subvariety $\mathcal{Y}$ of $\mathcal{X}$ which is flat over Λ, with $f(\mathcal{Y}) = \mathcal{Y}$, and integers $N \geq 0$ and $D_0 \geq 1$ such that*
   (a) *for any irreducible component $\mathcal{V}$ of $\mathcal{Y}$, there exists $n \in \mathbb{N}^*$ with $f^n(\mathcal{V}) = \mathcal{V}$, and the family of polarized endomorphisms $(\mathcal{V}, f^n|_{\mathcal{V}}, \mathcal{L}|_{\mathcal{V}})$ is isotrivial over Λ.*
   (b) *For any marked point $a : \mathcal{B} \dashrightarrow \mathcal{X}$ such that $((\mathcal{X}, f, \mathcal{L}), a)$ is stable, and which is regular over Λ, then:*
      (i) *the section $a_N : \lambda \mapsto f_\lambda^N(a(\lambda))$ is in fact a section of $\pi|_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{B}$,*
      (ii) *the Zariski closure $\mathcal{C}_0(a)$ of $a(\Lambda)$ satisfies* $\deg_{\mathcal{M}}(\mathcal{C}_0(a)) \leq D_0$.

Using classical arguments, we can deduce the following corollary, which is what we call the *Geometric Dynamical Northcott Property* on arbitrary function fields of characteristic zero.

Corollary 1.5. — *Let $\mathbf{K}$ be a function field of characteristic zero. Let (X, f, L) be a non isotrivial polarized endomorphism on a normal projective variety defined over $\mathbf{K}$. Then, there exist a (possibly empty or reducible) subvariety $Y \subset X_{\mathbf{K}}$ with $f(Y) = Y$ and an integer $N \geq 0$ such that*

1. *for any irreducible component V of Y with $f^n(V) = V$, the polarized endomorphism $(V, f^n|_V, L|_V)$ is isotrivial,*
2. *for any point $z \in X(\mathbf{K})$ with $\widehat{h}_f(z) = 0$, we have $f^N(z) \in Y$.*

In particular, if $X_{\mathbf{K}}$ contains no periodic isotrivial subvariety of positive dimension, then $\{z \in X(\mathbf{K}),\ \widehat{h}_f(z) = 0\}$ is finite and consists only of preperiodic points.

Remark 1.6. — The case where Y has an irreducible component of positive dimension over $\mathbf{K}$ cannot be excluded. Consider, for example, an endomorphism of $\mathbb{P}^2_{\mathbf{K}}$ given by a non-isotrivial polynomial map of $\mathbb{A}^2_{\mathbf{K}}$, whose restriction to the line at infinity is independent of λ (e.g. $f_\lambda(z_1, z_2) := (z_1^d, z_2^d + \lambda z_1)$ defined over $\mathbf{K} := \mathbb{C}(\lambda)$). Then any non-periodic constant with value on the line at infinity has canonical height zero.

1.4. Strategy of the proof and dynamical height in term of the Green current.— We use the notation of Theorem A. Let $\omega_{\mathcal{B}}$ be a Kähler form on $\mathcal{B}$ with cohomology class equal to $c_1(\mathcal{N})$ and let $\widehat{\omega}$ be a continuous closed positive $(1,1)$-form on $\mathcal{X}$ cohomologous to $c_1(\mathcal{L})$.

To $(\mathcal{X}, f, \mathcal{L})$, in §2 we associate its *fibered Green current* $\widehat{T}_f$ (see [**Dem1**] for currents on singular varieties). This classical object in complex dynamics is a closed, positive current on $\mathcal{X}$ of bidegree $(1,1)$. It satisfies $\widehat{T}_f^{\dim X+1} = 0$, see, e.g. , [**BB**, Remark 4.7], and, for each $\lambda \in \Lambda$, the slice μ_{f_λ} of $\widehat{T}_f^{\dim X}$ above λ is the unique maximal entropy measure of f_λ and the periodic points equidistribute towards the measure μ_{f_λ}.

We now extend the notion of height from points to subvarieties. Take any irreducible subvariety Z of dimension $0 \leq \ell \leq k = \dim X$ of X defined over $\mathbf{K}$ and let $\mathcal{Z}$ be the associated subvariety of $\mathcal{X}$. We denote by $[\mathcal{Z}]$ the current of integration on $\mathcal{Z}$. Since $\mathcal{Z}$ is flat over a Zariski open subset of $\mathcal{B}$ which can be taken to be a regular part, f^n is a rational map which is well defined on a Zariski open set of $\mathcal{Z}$. That way, $f^n_*([\mathcal{Z}])$, the push-forward of $[\mathcal{Z}]$ by f^n, is well defined and coincides with a multiple of $[f^n(\mathcal{Z})]$, taking into account the multiplicity (see paragraph 1.6).

As used in [**Gub1, Gub3, Fab, CGHX**], the notion of height $h_{X,L}$ can be extended to the irreducible subvariety Z as the intersection number

$$h_{X,L}(Z) := \left(\mathcal{Z} \cdot c_1(\mathcal{L})^{\ell+1} \cdot c_1(\pi^*\mathcal{N})^{\dim \mathcal{B}-1}\right),$$

and it can be computed as

$$h_{X,L}(Z) = \int_{\mathcal{X}_\Lambda} [\mathcal{Z}] \wedge \widehat{\omega}^{\ell+1} \wedge (\pi^*\omega_{\mathcal{B}})^{\dim \mathcal{B}-1}.$$

Note again that the height function $h_{X,L}$ depends on the polarization $\mathcal{N}$ on $\mathcal{B}$. As in [**Gub1**, Theorems 10.9 & 11.14], the canonical height $\hat{h}_f(Z)$ of Z is defined by

$$\hat{h}_f(Z) = \lim_{n\to\infty} \frac{1}{d^{(\ell+1)n}} h_{X,L}((f_*)^n(Z)).$$

So, one could hope that

$$\begin{aligned}\hat{h}_f(Z) &= \lim_{n\to+\infty} \frac{1}{d^{(\ell+1)n}} \int_{\mathcal{X}_\Lambda} (f^n)_*[\mathcal{Z}] \wedge \widehat{\omega}^{\ell+1} \wedge (\pi^*\omega_{\mathcal{B}})^{\dim \mathcal{B}-1} \\ &= \int_{\mathcal{X}_\Lambda} \lim_{n\to+\infty} \frac{1}{d^{(\ell+1)n}} [\mathcal{Z}] \wedge (f^n)^*(\widehat{\omega}^{\ell+1}) \wedge (\pi^*\omega_{\mathcal{B}})^{\dim \mathcal{B}-1},\end{aligned}$$

and use the convergence $\lim_{n\to\infty} d^{-(\ell+1)n}(f^n)^*(\widehat{\omega}^{\ell+1}) \to \widehat{T}_f^{\ell+1}$ in the sense of currents in order to get the following theorem.

Theorem B. — *Let* $\mathbf{K}$ *be the field of rational functions of a normal complex projective variety* $\mathcal{B}$. *Let* (X, f, L) *be a polarized endomorphism over* $\mathbf{K}$. *Let* $(\mathcal{X}, f, \mathcal{L})$ *be an associated model with regular part* Λ. *Let* $\mathcal{N}$ *be an ample line bundle on* $\mathcal{B}$ *so that* $\mathcal{M} := \mathcal{L} \otimes \pi^*(\mathcal{N})$ *is ample. Let* Z *be any irreducible subvariety of dimension* $0 \leq \ell \leq k = \dim X$ *of* X *defined over* $\bar{\mathbf{K}}$. *Let* $\mathcal{Z}$ *be the corresponding subvariety of* $\mathcal{X}$. *Then, the canonical height satisfies*

$$\widehat{h}_f(Z) = \int_{\mathcal{X}_\Lambda} [\mathcal{Z}] \wedge \widehat{T}_f^{\ell+1} \wedge (\pi^*\omega_{\mathcal{B}})^{\dim \mathcal{B}-1},$$

where $\omega_{\mathcal{B}}$ *is any Kähler form on* $\mathcal{B}$ *cohomologous to* $c_1(\mathcal{N})$.

To make the above strategy work, one has to deal with the difficulty that some mass might be lost on the boundary $\pi^{-1}(\mathscr{B}\backslash\Lambda)$. This possible loss of mass corresponds a priori to the so-called places of bad reduction of the map f. DeMarco's idea for $X = \mathbb{P}^1$ is to do analysis at those places in a delicate way [**DeM2**, Sections 3 & 4]. In here, we look instead at what happens away from $\pi^{-1}(\mathscr{B}\backslash\Lambda)$ using an appropriate cut-off function built with a naive degeneration's estimate of potentials of the current $\widehat{T}_f$ coming from the Nullstellensatz (such an estimate is present in DeMarco's article). This gives a new proof of the existence of the canonical height, originally due to Gubler [**Gub1**], and we compute that height for any irreducible subvariety Z of X in term of the fibered Green current. By definition a current is in the dual space of smooth forms with compact support. Hence stability, the vanishing of a current, is local by nature. Theorem B shows that stability is in fact a global notion. The cut-off function we use is a DSH function, the strength of such objects, introduced by Dinh and Sibony [**DS2**], is that their definition involves the complex structure of the space (which is not the case for C^α-functions).

The formula for the canonical height in Theorem B is known in the case of rational sections of elliptic surfaces where f is the multiplication by $n \geq 2$ [**CDMZ**, Theorem 3.2] and has been investigated in other special cases where X is an abelian variety (the Betti form of a family of Abelian varieties is exactly the fibered Green current of the fiberwise multiplication by 2); see for example [**CGHX**].

Extending the notion of stability to subvarieties (Definitions 2.9 and 4.1), we shall obtain the following corollary which proves ($\star$) in Theorem A (see §1.6 for the definition of $\deg(f^n|_Z)$).

Corollary 1.7. — *Under the hypothesis of Theorem B, the pair $((\mathcal{X}, f, \mathcal{L}), [\mathcal{Z}])$ is stable if and only if $\widehat{h}_f(Z) = 0$ if and only if there exists $C > 0$ such that $d^{\ell n}/C \leq \deg_{\mathcal{M}}(f^n(\mathcal{Z}))\cdot\deg(f^n|_Z) \leq Cd^{\ell n}$.*

1.5. The Geometric Dynamical Bogomolov Property. — We say that an irreducible subvariety $Z \subset X_{\bar{\mathbf{K}}}$ comes from an *isotrivial factor* of (X, f, L) if there exist an integer $k \geq 1$ and an isotrivial polarized endomorphism (Y, g, E), a subvariety $V \subset X_{\bar{\mathbf{K}}}$ with $f^k(V) = V$, and a dominant rational map $p : V \dashrightarrow Y$ such that $p \circ (f^k|_V) = g \circ p$, and an integer $N \geq 1$ and an isotrivial subvariety $W_0 \subset Y$, such that $f^N(Z) = p^{-1}(W_0)$. We also adapt the following definition of Ghioca and Tucker [**GT**] of *special subvariety* in the number field case to the function field case.

Definition 1.8. — *With the above notations, we say that Z is f-*special *if there exist an integer n, a polarized endomorphism (X, Ψ, L) a subvariety Y with $Z \subseteq Y \subseteq X$ such that*
- *$f^n(Y) = Y = \Psi(Y)$;*
- *$f^n \circ \Psi = \Psi \circ f^n$ on Y;*
- *either Z is preperiodic under Ψ or Z comes from an isotrivial factor of (X, Ψ, L).*

We propose the following conjecture.

Conjecture 1.9 (Geometric Dynamical Bogomolov). — *Let $\mathbf{K}$ be a function field of characteristic zero. Let (X, f, L) be a polarized endomorphism defined over $\mathbf{K}$, where X is normal. Let $Z \subset X_{\bar{\mathbf{K}}}$ be an irreducible subvariety and assume that, for any $\varepsilon > 0$, the set $Z_\varepsilon := \{x \in Z(\bar{\mathbf{K}}) : \widehat{h}_f(x) < \varepsilon\}$ is Zariski dense in Z. Then, Z is f-special.*

Note that, if $X = A$ is a non-isotrivial abelian variety and if f is the multiplication by an integer $n \geq 2$, this reduces to the Geometric Bogomolov Conjecture proposed by Yamaki [**Y**, Conjecture 0.3], where the isotrivial factor is the $\mathbf{K}/\mathbf{k}$-trace of A, whence it is known to hold by [**CGHX**] (see also [**XY**] for the general case of arbitrary characteristic). Note also that the case of isotrivial factor occurs, e.g. if f preserves globally the fibers of a fibration of which Z is a fiber, we actually have $\widehat{h}_f(Z) = 0$, even though it may not be preperiodic under iteration of f, see Section 6 for more details.

As observed by Ghioca, Tucker and Zhang [**GTZ**], over a number field, if E is a CM elliptic curve and $\omega_1, \omega_2 \in \mathrm{End}(E)$ are multiplicatively independent with $|\omega_1| = |\omega_2| > 1$, then (ω_1, ω_2) defines a polarized endomorphism of $E \times E$ for which the diagonal Δ is not preperiodic but contains infinitely many points of height zero (hence its height is zero). This explains the definition of special subvariety in this setting where one can take Ψ to be $([2], [2])$ and $Y = E \times E$. As explained to us by Najmuddin Fakhruddin, one can construct similar examples over function fields using arguments of Shimura [**Shi2**]: given an imaginary quadratic field $\mathbb{K}$ and an integer $k > 2$, there exist a non-isotrivial family $\mathscr{A} \to S$ of abelian variety of relative dimension k with $\dim S > 0$ and such that the endomorphism ring of A_η is the ring of integer of $\mathbb{K}$.

Let $f : \mathbb{P}^k \times \Lambda \to \mathbb{P}^k \times \Lambda$ be an algebraic family of complex rational maps and let $f : \mathbb{P}^k_{\mathbf{K}} \to \mathbb{P}^k_{\mathbf{K}}$ be the induced endomorphism defined over $\mathbf{K} = \mathbb{C}(\Lambda)$. The fundamental work of Berteloot, Bianchi and Dupont [**BBD**] sets up a good notion of stability, J-*stability*: one way to formulate the J-stability of the family f is to require that $\widehat{h}_f(\mathrm{Crit}(f)) = 0$, where $\mathrm{Crit}(f)$ is the critical divisor of f. Can one prove a rigidity property for non isotrivial J-stable families? Apart from the dimension 1 where McMullen did characterize stable algebraic families of rational maps [**Mc**], this is largely open. Solving Conjecture 1.9 would be a first step toward proving that such a family is post-critically finite.

1.6. Notations and conventions. — In the whole article, if $\mathbf{k}$ is any field of characteristic zero, X and Y are $\mathbf{k}$-varieties, Y irreducible, and $f : X \to Y$ is a finite morphism, we let $\deg(f)$ be the topological degree of f, i.e.

$$\deg(f) = \mathrm{Card}\left(\{x \in X(\bar{\mathbf{k}}) \ : \ f(x) = y\}\right),$$

for a general point $y \in Y(\bar{\mathbf{k}})$. For every integral subvariety $Z \subseteq X$, since f is proper, the image of Z is again a subvariety we denote by $f(Z)$. We also let $f_*(Z)$ be the push-forward of Z by f. It is the cycle of Y given by

$$f_*(Z) = \deg(f|_Z) \cdot f(Z).$$

Here $\deg(f|_Z)$ is the topological degree of the map $f|_Z : Z \to f(Z)$.

When $\mathbf{k} = \mathbb{C}$ is the field of complex numbers, we also denote by $[Z]$ the integration current on the subvariety Z, i.e. for any test form ψ on X of bidegree $(\dim Z, \dim Z)$, we let

$$\int_X [Z] \wedge \psi := \int_{Z_{\mathrm{reg}}} \psi|_Z,$$

since a theorem of Lelong says it defines a positive closed current [**Dem2**, Theorem 2.7, p.140]. We also denote by $f_*[Z]$ the push-forward current, i.e. the current defined by

$$\int_Y (f_*[Z]) \wedge \phi = \int_X [Z] \wedge (f^*\phi)$$

for any test form ϕ on Y, where $f^*\phi$ is the pullback of ϕ. We can also compute that $f_*[Z] = \deg(f|_Z) \cdot [f(Z)]$. That way, $f_*[Z] = [f_*(Z)] = \deg(f|_Z)[f(Z)]$.

2. Analytic families of polarized endomorphisms

Since isotrivial subvarieties can be singular, it is natural to study the more general case of polarized endomorphisms of possibly singular projective varieties. Nevertheless, we will rely on an important result of Fakhruddin [**Fak**] which says that such polarized morphisms can always be seen as the restriction of a polarized endomorphism defined on some $\mathbb{P}^N$. This will help us build the fibered Green current of a family of polarized endomorphisms. Indeed, it will imply that we are in the settings of [**Dem1**]: we will be dealing with quasi-plurisubharmonic (qpsh for short) functions on X which are the restrictions of qpsh functions defined on a neighborhood of X in $\mathbb{P}^N(\mathbb{C})$.

2.1. On a key result of Fakhruddin. — Let X be an irreducible complex projective variety. Recall that from a very ample line bundle L on X, we can produce an embedding $\iota : X \hookrightarrow \mathbb{P}(H^0(X, L)^\vee) = \mathbb{P}^N$, where $N + 1 = \dim H^0(X, L)^\vee$ and L may then be identified with the pullback $\iota^*\mathcal{O}(1)$. We also define an hermitian metric on X by setting ω_X to be the restriction of ω_{FS} to X, where ω_{FS} is the Fubini-Study form on $\mathbb{P}^N$. In this text, when fixing a Kähler form on X, we shall always take such a Fubini-Study form ω_X.

Thanks to the next result of Fakhruddin [**Fak**, Corollary 2.2], we can find an embedding adapted to the study of the dynamics of a polarized endomorphism:

Proposition 2.1. — *Let (X, f, L) be a polarized endomorphism over an infinite field K. Then there exist $N \geq 1$, an embedding $\iota : X \hookrightarrow \mathbb{P}^N_K$ as above and an endomorphism $F : \mathbb{P}^N_K \to \mathbb{P}^N_K$ such that*

$$\iota \circ f = F \circ \iota \text{ on } X \quad \text{and} \quad L = \iota^*\mathcal{O}_{\mathbb{P}^N}(1).$$

Proof. — Let us explain briefly the construction of F, since we rely on it in the sequel. Up to replacing L with $L^{\otimes e}$ we may assume L is very ample. In this case, L induces an embedding $\iota : X \hookrightarrow \mathbb{P}(H^0(X, L)^\vee) = \mathbb{P}^N$ and, if $(s_0, \ldots, s_N)$ is a basis of $H^0(X, L)^\vee$ with no common zeros, we then have $f_j := f^*(s_j)$ which is a degree d polynomial in the s_i's. We may consider $(s_0, \ldots, s_N)$ as affine coordinates on $\mathbb{A}^{N+1}$, so that the pullback map f^* induces an endomorphism $F : \mathbb{P}^N \to \mathbb{P}^N$, which may be defined by $F([s_0 : \cdots : s_N]) = [f_0 : \cdots : f_N]$. □

In the following, the form ω_X comes from an "adapted" embedding in the sense of Proposition 2.1.

Corollary 2.2. — *Let (X, f, L) be a polarized endomorphism over $\mathbb{C}$ of degree d. There exists a continuous function $u : X \to \mathbb{R}$ such that*

$$f^*\omega_X = d \cdot \omega_X + \mathrm{dd}^c u.$$

Proof. — Let $\iota : X \hookrightarrow \mathbb{P}^N(\mathbb{C})$ and $F : \mathbb{P}^N(\mathbb{C}) \to \mathbb{P}^N(\mathbb{C})$ be given by Proposition 2.1. Then there is a smooth function $\phi : \mathbb{P}^N(\mathbb{C}) \to \mathbb{R}$ such that

$$F^*\omega_{\mathrm{FS}} = d \cdot \omega_{\mathrm{FS}} + \mathrm{dd}^c\phi.$$

Since $\omega_X = \iota^*\omega_{\mathrm{FS}}$, we just have to define u as $u := \phi \circ \iota$. □

2.2. Dynamics of polarized endomorphisms of projective varieties. — In the rest of the article, a (p, p)-current means a current of bidegree (p, p). Building on a long history of results in complex dynamics, especially ideas of Briend and Duval [**BD2**] and their adaptation to the meromorphic setting by Dinh, Nguyen, and Truong [**DNT**], we prove the following:

Proposition 2.3. — *Let (X, f, L) be a polarized endomorphism over $\mathbb{C}$ of degree d. Let $k := \dim(X)$. Then the sequence $(d^{-n}(f^n)^*\omega_X)$ converges in the sense of currents to a closed positive $(1, 1)$-current T_f. Moreover,*

1. *$T_f = \omega_X + \mathrm{dd}^c g_f$ where g_f is continuous on X and $f^*T_f = d \cdot T_f$,*
2. *the repelling periodic points of f are Zariski dense in X.*

First, we recall the definition of the dynamical degrees following [**RS, DS1**]. Let X be an irreducible complex projective variety of dimension k, L a big and nef line bundle on X and $f : X \dashrightarrow X$ be a dominant rational mapping. For any $1 \leq j \leq k$, we set

$$\deg_{j,L}(f) = \left((f^*c_1(L)^j) \cdot c_1(L)^{k-j}\right) / (c_1(L)^k).$$

Then, the j-th dynamical degree of f is the quantity

$$\lambda_j(f) := \lim_{n\to\infty} \left(\deg_{j,L}(f^n)\right)^{1/n}.$$

This limit is well-defined, the quantities $\lambda_j(f)$ are birational invariants and does not depend on the choice of the big and nef line bundle (see [**DS1**]). In particular, $\lambda_k(f)$ is the topological degree of f; we say that f has *large topological degree* if $\lambda_j(f) < \lambda_k(f)$ for all $1 \leq j < k$. When (X, f, L) is a polarized endomorphism over $\mathbb{C}$ of degree d, then for any $1 \leq j \leq k$ and any $n \geq 1$, we have

$$\deg_{j,L}(f^n) = \left(((f^n)^*c_1(L))^j \cdot c_1(L)^{k-j}\right) / (c_1(L)^k) = \left((d^n c_1(L))^j \cdot c_1(L)^{k-j}\right) / (c_1(L)^k) = d^{jn},$$

since f^n is also polarized of degree d^n. In particular, $\lambda_j(f) = d^j$ for all j and f has large topological degree d^k.

Proof of Proposition 2.3. — Let $u : X \to \mathbb{R}$ be the continuous function given by Corollary 2.2. By an induction, we find

$$\frac{1}{d^n}(f^n)^*\omega_X = \omega_X + \mathrm{dd}^c \left(\sum_{j=0}^{n-1} \frac{u \circ f^j}{d^j}\right).$$

The sequence $u_n := \sum_{j=0}^{n-1} \frac{u \circ f^j}{d^j}$ of continuous functions converges uniformly on X to a continuous function g_f and, by construction, $T_f := \omega_X + \mathrm{dd}^c g_f$ is the limit of $d^{-n}(f^n)^*\omega_X$ and $f^*T_f = d \cdot T_f$.

To prove point 2, we let $p : \tilde{X} \to X$ be a resolution of the singularities of X. Then $\tilde{X}$ is a smooth projective variety of dimension k and the morphism $f \circ p : \tilde{X} \to X$ lifts as a rational mapping $\tilde{f} : \tilde{X} \dashrightarrow \tilde{X}$. Let $\tilde{L} := p^*(L)$. As p is generically finite and L is ample, $\tilde{L}$ is big and nef, and $\tilde{f}^*\tilde{L} \simeq \tilde{L}^{\otimes d}$ by construction. The mapping $\tilde{f}$ is dominant with large topological degree, since its dynamical degrees are the same as the dynamical degrees of f. In particular, by [**DNT**][Theorem 1.1] (see also [**Gue, BD2**]), repelling periodic points of $\tilde{f}$ equidistribute towards a probability measure which does not give mass to pluripolar sets. Whence, those contained in X_{reg} are Zariski dense in X. $\square$

We will also use the next lemma in the sequel, which is proved in [**NZ**, Lemma 2.1].

Lemma 2.4. — *Let (X, f, L) be a polarized endomorphism over $\mathbb{C}$ of degree d. Let V be an invariant subvariety of dimension ℓ, i.e. $f(V) = V$ then $(V, f|_V, L|_V)$ is a polarized endomorphism of degree d. In particular, $f|_V$ is a finite endomorphism of topological degree d^ℓ.*

Consequently, for any $n \geq 1$, the set $\{z \in X \,:\, f^n(z) = z\}$ is finite.

Proof. — Take V satisfying the hypothesis of the lemma. As L is ample, so is $L|_V$, and $(V, f|_V, L|_V)$ is a polarized endomorphism, since by functoriality of pull-back $(f|_V)^*L|_V$ is isomorphic to $(f^*L)|_V$ which is isomorphic to $L^{\otimes d}|_V$. The topological degree D of $f|_V$ satisfies

$$(f|_V^* c_1(L|_V))^\ell = D \cdot c_1(L|_V)^\ell;$$

since $f|_V^*(c_1(L|_V)) = dc_1(L|_V)$ we deduce that $f|_V$ is of topological degree d^ℓ. It is finite by the arguments of Fujimoto [**Fu**, Lemma 2.3] (the proof is given in the smooth case, but adapts to the normal case).

Fix $n \geq 1$ and assume the subvariety $Y := \{z \in X \ : \ f^n(z) = z\}$ of X has an irreducible component W of dimension $\ell \geq 1$. As $f^n_{|W}$ is the identity, it has degree 1 which contradicts the above. □

2.3. The fibered Green current of a family of endomorphisms. — We extend here the definitions of [**DF**] to the case of families of endomorphisms of complex projective varieties. We are interested in local parametric aspects of such families, so we do not rule out transcendental dependence on the parameter, e.g. this section applies to the family $f_\lambda(z) = z^2 + \exp(\lambda)$, $\lambda \in \mathbb{C}$.

Fix a reduced complex analytic space $\mathscr{X}$, a complex Kähler manifold Λ and assume there is a proper surjective morphism $\pi : \mathscr{X} \to \Lambda$. Such a morphism is a *family of irreducible complex projective varieties of dimension* $k \geq 1$, if π is an analytic submersion and if $X_\lambda = \pi^{-1}\{\lambda\}$ is an irreducible complex projective variety of dimension k for all $\lambda \in \Lambda$. Fix a relatively ample line bundle $\mathscr{L}$ on $\mathscr{X}$, i.e. $\mathscr{L}$ is a line bundle on $\mathscr{X}$ such that $L_\lambda := \mathscr{L}|_{X_\lambda}$ is an ample divisor on X_λ for all λ.

Definition 2.5. — *A* family of endomorphisms *of* $\mathbb{P}^N$ *of degree* d *is a holomorphic map* $F : (z, \lambda) \in \mathbb{P}^N \times \Lambda \mapsto (F_\lambda(z), \lambda) \in \mathbb{P}^N \times \Lambda$ *such that* F_λ *is an endomorphism of degree* d *for any* $\lambda \in \Lambda$.

A triple $(\mathscr{X}, f, \mathscr{L})$ *is a (complex)* family of polarized endomorphisms *(or* holomorphic family of polarized endomorphisms*), if* $f : \mathscr{X}_\Lambda \to \mathscr{X}_\Lambda$ *is analytic and, for all* $\lambda \in \Lambda$, $f_\lambda := f|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a morphism of varieties such that there is an embedding* $\widehat{\iota} : \mathscr{X} \hookrightarrow \mathbb{P}^N \times \Lambda$ *and a family* $F : \mathbb{P}^N \times \Lambda \to \mathbb{P}^N \times \Lambda$ *of endomorphisms such that*

$$F \circ \widehat{\iota} = \widehat{\iota} \circ f \ \text{ and } \ \mathscr{L} = \widehat{\iota}^* \mathscr{O}_{\mathbb{P}^N}(1),$$

and such that $p_2 \circ \widehat{\iota} = \pi$, *where* $p_2 : \mathbb{P}^N \times \Lambda \to \Lambda$ *is the projection onto the second factor.*

Note that this implies the family f is *polarized* with respect to $\mathscr{L}$ in the sense that for any $\lambda \in \Lambda$, we have $f_\lambda^* L_\lambda \simeq L_\lambda^{\otimes d}$. We let ω be a Kähler form on $\mathbb{P}^N$, $\tilde{\omega}$ be the induced $(1,1)$-form on $\mathbb{P}^N \times \Lambda$ and we let $\widehat{\omega} := \widehat{\iota}^*(\tilde{\omega})$. Note that $\widehat{\omega}$ is a (1,1)-form on $\mathscr{X}_\Lambda$ which is cohomologous to a multiple of $c_1(\mathscr{L})$ and that $\omega_\lambda := \widehat{\omega}|_{X_\lambda}$ is a Kähler form on $(X_\lambda)_{\mathrm{reg}}$.

Proposition 2.6. — *Let* $(\mathscr{X}, f, \mathscr{L})$ *be a family of polarized endomorphisms. Then there exists a function* g, *which is the restriction to* $\mathscr{X}_\Lambda$ *of a continuous qpsh function on* $\mathbb{P}^N \times \Lambda$ *such that the sequence* $d^{-n}(f^n)^*(\widehat{\omega})$ *converges in the sense of currents towards a closed positive* $(1,1)$*-current* $\widehat{T}_f = \widehat{\omega} + \mathrm{dd}^c g$. *Moreover,*

1. $f^* \widehat{T}_f = d \cdot \widehat{T}_f$,
2. *for all* $\lambda \in \Lambda$, $T_\lambda := \omega_\lambda + \mathrm{dd}^c(g|_{X_\lambda})$ *is a well-defined positive closed* $(1,1)$ *current on* X_λ, $(f_\lambda)^* T_\lambda = d \cdot T_\lambda$ *and* $T_\lambda = c T_{f_\lambda}$ *where* $c = (\deg_{L_\lambda}(X_\lambda))^{1/k}$ *is independent of* λ.

Proof. — Let ϕ be a smooth quasi-potential of $F^* \tilde{\omega}$ (i.e. $F^* \tilde{\omega} = d \cdot \tilde{\omega} + \mathrm{dd}^c \phi$). Since $F \circ \widehat{\iota} = \widehat{\iota} \circ f$, we have

$$f^* \widehat{\omega} = \widehat{\iota}^*(F^* \tilde{\omega}) = \widehat{\iota}^*(d \cdot \tilde{\omega} + \mathrm{dd}^c \phi) = d \cdot \widehat{\omega} + \mathrm{dd}^c(\phi \circ \widehat{\iota}).$$

The function $u := \phi\circ\widehat{\iota} : \mathcal{X} \to \mathbb{R}$ is continuous and satisfies $f^*\widehat{\omega} = d\cdot\widehat{\omega} + dd^c u$. An easy induction then gives

$$\frac{1}{d^n}(f^n)^*\widehat{\omega} = \widehat{\omega} + dd^c\left(\sum_{j=0}^{n-1}\frac{u\circ f^j}{d^j}\right),$$

and $d^{-n}(f^n)^*\widehat{\omega}$ converges towards the closed positive $(1,1)$-current

$$\widehat{T}_f := \widehat{\omega} + dd^c g$$

with $g := \sum_j d^{-j} u\circ f^j$. By construction, the function g is the restriction to $\mathcal{X}_\Lambda$ of a continuous qpsh function on $\mathbb{P}^N \times \Lambda$. We have $f^*\widehat{T}_f = d\cdot\widehat{T}_f$, and $\widehat{T}_f|_{X_\lambda} := \omega_\lambda + dd^c(g|_{X_\lambda})$. We thus have proved the proposition. □

Definition 2.7. — *The current $\widehat{T}_f$ is the* fibered Green current *of the family $(\mathcal{X}, f, \mathcal{L})$.*

2.4. The bifurcation current of a closed positive current. — Let $0 \le p \le k$ be an integer. Using the notations of the previous subsection, we now define a notion of stability and a bifurcation current for a closed positive (p,p)-current on $\mathcal{X}_\Lambda$.

Definition 2.8. — *A p-*measurable dynamical pair *$((\mathcal{X}_\Lambda, f, \mathcal{L}), S)$ on $\mathcal{X}_\Lambda$ parametrized by a complex manifold Λ is the datum of a holomorphic family $(\mathcal{X}_\Lambda, f, \mathcal{L})$ as above and a positive closed (p,p)-current S on $\mathcal{X}_\Lambda$.*

The *bifurcation current* of a p-measurable dynamical pair $((\mathcal{X}_\Lambda, f, \mathcal{L}), S)$ is the positive closed $(1,1)$-current on Λ defined as

$$T_{f,S} := \pi_*\left(\widehat{T}_f^{k+1-p}\wedge S\right). \tag{1}$$

Since $\widehat{T}_f$ has locally bounded potentials, the above product is well-defined (see [**Dem1**]). The idea to study bifurcations using currents goes back to the work of DeMarco [**DeM1**] and has been intensively used since then, see e.g. [**Ber**] and references therein.

We also endow Λ with a Kähler form ω_Λ (with a possibly infinite volume), and let $\breve{\omega}_\Lambda := (\pi)^*\omega_\Lambda$. In this case, $\breve{\omega}_\Lambda + \widehat{\omega}$ is a Kähler form on $(\mathcal{X})_{\mathrm{reg}}$ and we can define the *mass* of a closed positive (p,p)-current S on a compact subset $\mathcal{K}$ of $\mathcal{X}$ as

$$\|S\|_{\mathcal{K}} := \int_{\mathcal{K}} S\wedge(\widehat{\omega}+\breve{\omega}_\Lambda)^{k+\dim\Lambda-p} < +\infty.$$

Definition 2.9. — *We say that the dynamical pair $((\mathcal{X}, f, \mathcal{L}), S)$ is* stable *if the sequence $\{(f^n)_*(S)\}_{n\ge1}$ of closed positive (p,p)-currents satisfies*

$$\|(f^n)_*(S)\wedge\breve{\omega}_\Lambda^{\dim\Lambda-1}\|_{\mathcal{K}} = o(d^{n(k+1-p)}) \text{ as } n\to\infty$$

in any compact subset $\mathcal{K}$ of $\mathcal{X}$.

Proposition 2.10. — *Let $(\mathcal{X}_\Lambda, f, \mathcal{L})$ be any holomorphic family of polarized endomorphisms as above and let S be any positive closed (p,p)-current of $\mathcal{X}_\Lambda$. Then, the following assertions are equivalent:*

1. *the pair $((\mathcal{X}_\Lambda, f, \mathcal{L}), S)$ is stable,*
2. *for any compact set $\mathcal{K}\subset\mathcal{X}_\Lambda$, we have $\|(f^n)_*(S)\wedge\breve{\omega}_\Lambda^{\dim\Lambda-1}\|_{\mathcal{K}} = O(d^{n(k-p)})$,*
3. *as a current on Λ, we have $T_{f,S}=0$.*

It is a direct consequence of the next lemma which follow the proof of [**DF**, Proposition-Definition 3.1] and of [**BBD**, Lemma 3.13].

Lemma 2.11. — *Let S be a closed positive (p,p)-current on X. For any compact sets $K_1 \Subset K_2 \Subset \Lambda$, there exists a constant $C(K_1,K_2)>0$ depending only on K_1, K_2 and on $(\mathcal{X}_\Lambda, f, \mathcal{L})$, such that for any $n \geq 0$, the quantity*

$$\left| \|(f^n)_*(S)\wedge\breve{\omega}_\Lambda^{\dim\Lambda-1}\|_{\pi^{-1}(K_1)} - d^{(k+1-p)n}\int_{\pi^{-1}(K_1)} \widehat{T}_f^{k+1-p}\wedge S\wedge\breve{\omega}_\Lambda^{\dim\Lambda-1}\right|$$

is bounded above by $C(K_1,K_2)\cdot d^{(k-p)n}\|S\wedge\breve{\omega}_\Lambda^{\dim\Lambda-1}\|_{\pi^{-1}(K_2)}$.

Proof. — Let $K_1 \Subset K_2 \Subset \Lambda$ be any compact subsets and write $S_n := (f^n)_*(S)\wedge\breve{\omega}_\Lambda^{\dim\Lambda-1}$, for any integer $n\geq 0$. Since $(\pi\circ f^n)^*\omega_\Lambda = \pi^*\omega_\Lambda = \breve{\omega}_\Lambda$, we have

$$\|S_n\|_{\pi^{-1}(K_1)} = \int_{\pi^{-1}(K_1)} S_0\wedge(\breve{\omega}_\Lambda + (f^n)^*\widehat{\omega})^{k+1-p}\,.$$

Now we use that $S_0 = S\wedge\breve{\omega}_\Lambda^{\dim\Lambda-1}$, $\omega_\Lambda^{\dim\Lambda+1}=0$ and $d^n\widehat{T}_f = (f^n)^*\widehat{\omega} + dd^c(g\circ f^n)$, where g is a continuous $\widehat{\omega}$-psh function on $\mathcal{X}_\Lambda$. So

$$\begin{aligned}\|S_n\|_{\pi^{-1}(K_1)} = &\int_{\pi^{-1}(K_1)} S\wedge\breve{\omega}_\Lambda^{\dim\Lambda-1}\wedge\Big(d^n\widehat{T}_f - dd^c(g\circ f^n)\Big)^{k+1-p}\\ &+(k+1-p)\int_{\pi^{-1}(K_1)} S\wedge\breve{\omega}_\Lambda^{\dim\Lambda}\wedge\Big(d^n\widehat{T}_f - dd^c(g\circ f^n)\Big)^{k-p}\,.\end{aligned}\tag{2}$$

Now,

$$\Big(d^n\widehat{T}_f - dd^c(g\circ f^n)\Big)^{k+1-p} = \sum_{i=0}^{k+1-p}\binom{k+1-p}{i}(-1)^{k+1-p-i}d^{ni}\widehat{T}_f^i\wedge(dd^c(g\circ f^n))^{k+1-p-i}\,,$$

$$\Big(d^n\widehat{T}_f - dd^c(g\circ f^n)\Big)^{k-p} = \sum_{j=0}^{k-p}\binom{k-p}{i}(-1)^{k-p-i}d^{ni}\widehat{T}_f^i\wedge(dd^c(g\circ f^n))^{k-p-i}\,.$$

Take $i\in\{0,\dots,k-p\}$. By the Chern-Levine-Nirenberg inequality, see [**Dem1**, Théorème 2.2] and [**Dem2**, §3.3 p. 146], there exists a constant $C_1>0$ depending only on K_1, K_2, $\|\widehat{T}_f\|_{\pi^{-1}(K_2)}$ and $\|g\|_{L^\infty(\pi^{-1}(K_2))}$ such that

$$\left|\int_{\pi^{-1}(K_1)} S\wedge\breve{\omega}_\Lambda^{\dim\Lambda-1}\wedge\widehat{T}_f^i\wedge(dd^c(g\circ f^n))^{k+1-p-i}\right| \leq C_1\|S_0\|_{\pi^{-1}(K_2)},$$

$$\left|\int_{\pi^{-1}(K_1)} S\wedge\breve{\omega}_\Lambda^{\dim\Lambda}\wedge\widehat{T}_f^i\wedge(dd^c(g\circ f^n))^{k-p-i}\right| \leq C_1\|S_0\|_{\pi^{-1}(K_2)}.$$

Now, subtracting $d^{(k+1-p)n}\int_{\pi^{-1}(K_1)}\widehat{T}_f^{k+1-p}\wedge S\wedge\breve{\omega}_\Lambda^{\dim\Lambda-1}$ from both sides of (2), we have, up to enlarging C_1:

$$\left|\|S_n\|_{\pi^{-1}(K_1)} - d^{(k+1-p)n}\int_{\pi^{-1}(K_1)}\widehat{T}_f^{k+1-p}\wedge S\wedge\breve{\omega}_\Lambda^{\dim\Lambda-1}\right| \leq C_1\|S_0\|_{\pi^{-1}(K_2)}d^{(k-p)n}$$

which ends the proof. □

Proof of Proposition 2.10. — Let $\mathcal{K}$ be any compact subset of $\mathcal{X}_\Lambda$. Remark that it is always contained in a compact subset of the form $\pi^{-1}(K)$, where K is compact in Λ, since the fibers of π are compact. By Lemma 2.11, we have $\|(f^n)_*(S)\wedge\breve{\omega}_\Lambda^{\dim\Lambda-1}\|_{\pi^{-1}(K)} = O(d^{n(k-p)})$, if and only if $T_{f,S}\wedge\omega_\Lambda^{\dim\Lambda-1}$ is zero on K.

As this holds for any compact set $K \Subset \Lambda$, this proves the equivalence between points 2 and 3, and point 2 obviously implies point 1. The proof that point 1 implies point 3 follows also from Lemma 2.11. Indeed, if $T_{f,S} \neq 0$, then for a suitable compact set K_1, $\int_{\pi^{-1}(K_1)} \widehat{T}_f^{k+1-p} \wedge S \wedge \check{\omega}_\Lambda^{\dim \Lambda - 1} := 2C \neq 0$ so that $\|(f^n)_*(S) \wedge \check{\omega}_\Lambda^{\dim \Lambda-1}\|_{\pi^{-1}(K_1)} \geq C(d^{n(k+1-p)})$ for n large enough. □

Remark 2.12. — Recall that a subvariety $\mathcal{C} \subset \mathcal{X}_\Lambda$ of pure dimension $\dim(\mathcal{B})$ is horizontal if $\mathcal{C} \cap \pi^{-1}\{\lambda\}$ is finite for all $\lambda \in \Lambda$.

When $S = [\mathcal{C}]$ is the current of integration on a horizontal variety of pure dimension $\dim(\mathcal{B})$, the definition of stability implies that, for any compact set K, $\|[f^n(\mathcal{C})]\|_K$ is bounded (Apply Proposition 2.10 (2) with $p = k$). By a famous Theorem of Bishop (see [**C**, Corollary p.205]), there exists a subsequence $(f^{n_k}(\mathcal{C}))_k$ which converges in Hausdorff topology towards an analytic set $\mathcal{C}_\infty$. In particular, if $\dim \Lambda = 1$ and $\mathcal{C}$ is the graph of a holomorphic section $\sigma : \Lambda \to \mathcal{X}_\Lambda$ of π, this is equivalent to the local uniform convergence of the sequence $\sigma_k := f^{n_k} \circ \sigma$ of sections of π to a holomorphic section. In other words, $(f^n \circ \sigma)_n$ is a normal family.

3. Algebraic families of polarized endomorphisms

Let $\mathcal{B}$ be a normal complex projective variety and let $\mathbf{K} := \mathbb{C}(\mathcal{B})$ be its field of rational functions. Let (X, f, L) be a polarized endomorphism over $\mathbf{K}$. The purpose of this section is to express the canonical height of f of a subvariety in term of the mass of bifurcation current with a precise rate of convergence. This will allow us to prove Theorem B.

3.1. Polarized endomorphisms over a function field versus algebraic families. — By Proposition 2.1 applied to (X, f, L), we have an embedding $i : X \hookrightarrow \mathbb{P}^N_{\mathbf{K}}$ with $L = i^*\mathcal{O}_{\mathbb{P}^N}(1)$ and an endomorphism $F : \mathbb{P}^N_{\mathbf{K}} \to \mathbb{P}^N_{\mathbf{K}}$ such that $i \circ F = f \circ i$. This endomorphism F gives rise to a family $(\mathbb{P}^N_{\mathbb{C}} \times \mathcal{B}, \mathcal{F}, \mathcal{O}_{\mathbb{P}^N}(1))$ of endomorphisms of $\mathbb{P}^N_{\mathbb{C}}$ parametrized by $\mathcal{B}$.

Let $\mathcal{X}$ be the Zariski closure in $\mathbb{P}^N_{\mathbb{C}} \times \mathcal{B}$ of the image of $i(X)$ by the isomorphism between $\mathbb{P}^N_{\mathbf{K}}$ and the generic fiber of $\mathbb{P}^N_{\mathbb{C}} \times \mathcal{B} \to \mathcal{B}$. Let $\mathcal{L}$ be the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ restricted to $\mathcal{X}$. Let also $\iota : \mathcal{X} \hookrightarrow \mathbb{P}^N \times \mathcal{B}$ be the inclusion and $\mathit{f} := F|_{\mathcal{X}}$. Then $(\mathcal{X}, \mathit{f}, \mathcal{L})$ is a model for (X, f, L).

The models $(\mathbb{P}^N_{\mathbb{C}} \times \mathcal{B}, \mathcal{F}, \mathcal{O}_{\mathbb{P}^N}(1))$ and $(\mathcal{X}, \mathit{f}, \mathcal{L})$ also induce analytic families of polarized endomorphisms over a common regular part $\Lambda \subset \mathcal{B}$.

Conversely, an algebraic family of polarized endomorphisms gives rise to a polarized endomorphism (X, f, L). Let X be the generic fiber of $\mathcal{X}$, and let L and f be the respective restrictions of $\mathcal{L}$ and f to the generic fiber, then (X, f, L) is a polarized endomorphism over $\mathbf{K}$.

We now illustrate the above notions in the so-called elementary Desboves family f that we will also explore in Example 4.11. We start with the action of f on several invariant sets and we explain why the family is not isotrivial (see Definition 1.2 in the introduction).

Example 3.1 (The elementary Desboves family I). — This family is already used by e.g. [**BD1, BDM**] to product attractors and in [**BT**] to construct an open set of bifurcation. For any $\lambda \in \mathbb{C}^*$, let $f_\lambda : \mathbb{P}^2 \to \mathbb{P}^2$ be the endomorphism of degree 4 given by

$$f_\lambda\left([x : y : z]\right) := [-x(x^3 + 2z^3) : y(z^3 - x^3 + \lambda(x^3 + y^3 + z^3)) : z(2x^3 + z^3)].$$

This defines an algebraic family $(\mathbb{P}^2 \times \mathbb{P}^1, \mathit{f}, \mathcal{O}_{\mathbb{P}^2}(1))$ with regular part $\mathbb{C}^* \subset \mathbb{P}^1$. This family induces an endomorphism of degree 4 of $\mathbb{P}^2_{\mathbb{C}(z)}$.

Let us now describe more precisely some aspects of the dynamics of this family. For any $\lambda \in \mathbb{C}^*$, the point $\rho_0 := [0 : 1 : 0]$ is totally invariant by f_λ, i.e. $f_\lambda^{-1}\{\rho_0\} = \{\rho_0\}$ and f_λ preserves the pencil $\mathscr{P}$ of lines of $\mathbb{P}^2$ passing through ρ_0. Furthermore, f_λ preserves the lines $X = \{x = 0\}$ and $Z = \{z = 0\}$ which belong to $\mathscr{P}$, the line $Y = \{y = 0\}$, and the Fermat curve

$$\mathcal{C} := \{[x : y : z] \in \mathbb{P}^2 \ : \ x^3 + y^3 + z^3 = 0\}.$$

Moreover, for any $\lambda \in \mathbb{C}^*$, the restriction of f_λ to each of those curves can be described:

- the restriction of f_λ to the line X is the degree 4 polynomial $p_\lambda(z) = \lambda z^4 + (1 + \lambda)z$,
- the restriction of f_λ to the line Z is the degree 4 polynomial $p_{-\lambda}$,
- the restriction of f_λ to the line Y is the Lattès map

$$g : [x : y] \in \mathbb{P}^1 \longmapsto [-x(x - 3 + 2y^3) : y^3(2x^3 + y^3)] \in \mathbb{P}^1.$$

  This defines a constant, hence isotrivial, family.
- Finally, the restriction of f_λ to the elliptic curve $\mathcal{C}$ is the isogeny $u \mapsto -2u + \beta$ for some β independent of λ so we have again isotriviality.

Note that the family $(\mathbb{P}^2 \times \mathbb{P}^1, f, \mathcal{O}_{\mathbb{P}^2}(1))$ is obviously non-isotrivial, since the restriction p_λ of f_λ to X has a fixed point with multiplier $1 + \lambda$, which is a non-constant rational function of λ.

A key fact for this section is that, when d is fixed, the space $\mathrm{End}_d(\mathbb{P}^k)$ of degree d endomorphisms of $\mathbb{P}^k$ is the complement in some projective space of an irreducible hypersurface by, e.g., [**Mac**], hence it is an affine variety.

We will regularly use the following, which in turn says that isotriviality can be read on the iterates of a family and does not depend on the chosen regular part.

Lemma 3.2. — *Let $(\mathcal{X}, f, \mathcal{L})$ be an algebraic family of polarized endomorphisms of degree d. Let Λ and Λ' be two regular parts for $(\mathcal{X}, f, \mathcal{L})$. The following are equivalent*

1. *$(\mathcal{X}, f, \mathcal{L})$ is isotrivial over Λ,*
2. *$(\mathcal{X}, f, \mathcal{L})$ is isotrivial over Λ',*
3. *there exists $n \geq 1$ such that $(\mathcal{X}, f^n, \mathcal{L})$ is isotrivial over Λ.*

Proof. — If $(\mathcal{X}, f, \mathcal{L})$ is isotrivial over Λ, obviously $(\mathcal{X}, f^n, \mathcal{L})$ is isotrivial over Λ for any $n \geq 1$. We thus prove the converse implication. Let X be the generic fiber of π, $L := \mathcal{L}|_X$ and $f := f|_X$.

Assume the family $(\mathcal{X}, f^n, \mathcal{L})$ is isotrivial over Λ, whence up to base change and conjugacy over $\mathbb{C}(\mathcal{B})$, the polarized endomorphism (X, f^n, L) is defined over $\mathbb{C}$ whence $\mathcal{X} = X_{\mathbb{C}} \times \mathcal{B}$. Fix $t_0 \in \Lambda$, replacing f^n by an iterate, we can assume that $\mathrm{Per}_1(f^n_{t_0}) := \{x, f^n_{t_0}(x) = x\}$ contains a set of points in general position and of cardinality large enough so that the data of $h : X_{\mathbb{C}} \to X_{\mathbb{C}}$, polarized of degree d, is uniquely determined by its values on that set. Observe now that for all $x \in \mathrm{Per}_1(f^n_{t_0})$ and all $t \in \Lambda$, one has $f_t(x) \in \mathrm{Per}_1(f^n_{t_0})$ since $f^n_t(f_t(x)) = f_t(x)$. In particular, there are only finitely many choices for f_t and a continuity argument implies that f is again constant.

To conclude, note that the variety X and the line bundle L have to be isotrivial themselves by assumption and that (X, f, L) is a polarized endomorphism which is isotrivial.

The proof implies the independence on the regular part. □

To finish the present discussion, we recall that to any subvariety Z of X, which is defined over $\mathbf{K}$, we can associate a subvariety $\mathcal{Z}$ of $\mathcal{X}$ (which can be defined as the Zariski closure of Z in $\mathcal{X}$) such that the restriction of π to $\mathcal{Z}$ is flat over Λ, restricting Λ if necessary. In particular, to any point $\mathrm{x} \in X(\mathbf{K})$ with corresponding subvariety x, we can associate a rational section $\sigma : \mathcal{B} \dashrightarrow \mathcal{X}$ of π, i.e. a rational map such that

1. $\pi \circ \sigma = \mathrm{id}_{\mathcal{B}}$,
2. σ is regular over Λ,

3. the Zariski closure of $\sigma(\Lambda)$ in $\mathcal{X}$ is x.

3.2. Global properties of the bifurcation current. — Pick a polarized endomorphism (X, f, L) over the field of rational functions $\mathbf{K}$ of a normal complex projective variety $\mathcal{B}$. Let $(\mathcal{X}, f, \mathcal{L})$ be a model of (X, f, L) as above.

We use the notations of Theorem B. Let $\widehat{\omega}$ be a closed positive form on $\mathcal{X}$ cohomologous to $c_1(\mathcal{L})$ and let $\check{\omega}_{\mathcal{B}} := \pi^*(\omega_{\mathcal{B}})$. The closed positive $(1,1)$-form $\widehat{\omega} + \check{\omega}_{\mathcal{B}}$ is a Kähler form on $\mathcal{X}$ cohomologous to $c_1(\mathcal{M})$.

Fix a regular part Λ and let S be a positive closed (p,p) current on $\mathcal{X}_\Lambda$. For any Borel subset Ω of $\mathcal{X}_\Lambda$, we let

$$\|S\|_\Omega := \left\langle S, \mathbf{1}_\Omega (\check{\omega}_{\mathcal{B}} + \widehat{\omega})^{k+\dim \mathcal{B} - p} \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the duality bracket between currents and forms. When $\|S\| := \|S\|_{\mathcal{X}_\Lambda} < +\infty$, the current S extends trivially to the Zariski closure $\mathcal{X}$ of $\mathcal{X}_\Lambda$ as a closed positive (p,p)-current on $\mathcal{X}$ that we still denote by S. Note that the fact that $\|S\|_{\mathcal{X}_\Lambda}$ is finite does not depend on the choice of regular part (if $\|S\|_{\mathcal{X}_{\Lambda'}} < +\infty$ for some $\Lambda' \subset \Lambda$, then so is $\|S\|_{\mathcal{X}_\Lambda}$).

We use the notations of Section 3.1. We want here to prove the following, which is a global version of Lemma 2.11:

Proposition 3.3. — *Let $(\mathcal{X}, f, \mathcal{L})$ be an algebraic family of polarized endomorphisms. Let Λ be a regular part and let $k := \dim X_\lambda$ for any $\lambda \in \Lambda$. Then there exists a constant $C > 0$ depending only on $(\mathcal{X}, f, \mathcal{L})$ such that*

1. *For any closed positive (p,p)-current S on $\mathcal{X}$ with $0 \leq p \leq \dim \mathcal{X} - 1$ and any $n \geq 1$,*

$$\left| \|\widehat{T}_f \wedge S\| - d^{-n} \|(f^n)^*(\widehat{\omega}) \wedge S\| \right| \leq C d^{-n} \|S\|.$$

2. *For any $1 \leq p \leq k$, any $0 \leq r \leq \dim \mathcal{B}$ and any closed positive (p,p)-current S on $\mathcal{X}$ and any $n \geq 1$, if $q := k + \dim \mathcal{B} - r - p$,*

$$\left| \int_{\mathcal{X}_\Lambda} (f^n)_*(S) \wedge \widehat{\omega}^q \wedge \check{\omega}_{\mathcal{B}}^r - d^{nq} \int_{\mathcal{X}_\Lambda} S \wedge \widehat{T}_f^q \wedge \check{\omega}_{\mathcal{B}}^r \right| \leq C \sum_{j<q} d^{nj} \|S \wedge \widehat{T}_f^j \wedge (f^n)^*(\widehat{\omega})^{q-j-1} \wedge \check{\omega}_{\mathcal{B}}^r\|.$$

Proof. — Let us consider first the case where Λ is affine. We thus can define an embedding $\iota_1 : \mathcal{B} \hookrightarrow \mathbb{P}^M$ with $\iota_1^{-1}(\mathbb{A}^M(\mathbb{C})) = \Lambda$. We already embedded $\mathcal{X}$ in $\mathbb{P}^N \times \mathcal{B}$ with $\mathcal{L} = \iota^*(\mathcal{O}_{\mathbb{P}^N}(1))$:

$$\mathcal{X} \stackrel{\iota}{\hookrightarrow} \mathbb{P}^N \times \mathcal{B} \stackrel{(\mathrm{id}, \iota_1)}{\hookrightarrow} \mathbb{P}^N \times \mathbb{P}^M,$$

We now write (z,t) for a point of $\mathbb{P}^N \times \mathbb{P}^M$ and let $\|\cdot\|$ be the standard Hermitian norm on $\mathbb{A}^M(\mathbb{C})$. We also identify $\lambda = \pi(x)$ with the second coordinate t of $(\mathrm{id}, \iota_1) \circ \iota(x)$ in the proof. We rely on the next lemma, which is an application of Hilbert's Nullstellensatz:

Lemma 3.4. — *There exist $C_1, C_2 > 0$ such that for all $n \geq 1$, we can write*

$$\frac{1}{d^n} (f^n)^*(\widehat{\omega}) - \widehat{T}_f = \mathrm{dd}^c \phi_n \quad \text{on } \mathcal{X}_\Lambda,$$

where $\phi_n : \mathcal{X}_\Lambda \to \mathbb{R}$ is a function such that $|\phi_n(z)| \leq d^{-n}(C_1 \log^+ \|\lambda\| + C_2)$, for all $\lambda \in \Lambda$ and all $z \in X_\lambda$.

Lemma 3.4 will be proved below. We take the lemma for granted and continue the proof. Following ideas of [**GOV**, p. 381, Proof of Theorem 1.6], we take any closed positive (p,p)-current S on $\mathcal{X}$ with $0 \leq p \leq \dim \mathcal{X} - 1$ and any $n \geq 1$. For any $A > 0$, we pick the following test function

$$\Psi_A(\lambda) := \frac{\log \max(\|\lambda\|, e^A) - \log \max(\|\lambda\|, e^{2A})}{A}.$$

For any $R > 0$, the current $dd^c \log \max(\|\lambda\|, R)$ has mass 1 on $\mathbb{C}^M$ for the Fubini-Study metric by Lelong-Poincaré, whence it restricts to Λ as a current of mass $\deg_\eta(\mathcal{B})$. Thus, Ψ_A is continuous and DSH on Λ with $dd^c\Psi_A = T_A^+ - T_A^-$ where $T_A^\pm$ are some positive closed $(1,1)$-currents whose masses are finite with $\|T_A^\pm\| \leq C'/A$ for some $C' > 0$ depending neither on A nor on $T_A^\pm$. Observe also that Ψ_A is equal to -1 in $B(0, e^A)$, and 0 outside $B(0, e^{2A})$. We first prove point 1; pick $n \geq 1$, then by Stokes

$$\begin{aligned} J_n^A &:= \left\langle \left(\widehat{T}_f - \frac{1}{d^n}(f^n)^*(\widehat{\omega}) \right) \wedge (\widehat{\omega} + \check{\omega}_{\mathcal{B}})^{k+\dim \mathcal{B} - p - 1} \wedge S, \Psi_A \circ \pi \right\rangle \\ &= \left\langle \phi_n \cdot (\widehat{\omega} + \check{\omega}_{\mathcal{B}})^{k+\dim \mathcal{B} - p - 1}, dd^c(\Psi_A \circ \pi) \wedge S \right\rangle \end{aligned}$$

and the definition of Ψ_A implies

$$\begin{aligned} |J_n^A| &\leq \int_{\iota^{-1}(\mathbb{P}^N \times B(0,e^{2A}))} |\phi_n| \, (\widehat{\omega} + \pi^*\omega_{\mathcal{B}})^{k+\dim \mathcal{B} - p - 1} \wedge \pi^*(T_A^+ + T_A^-) \wedge S \\ &\leq \frac{C_3}{A} \sup_{\mathbb{P}^N \times B(0,e^{2A})} |\phi_n| \cdot \|S\|, \end{aligned}$$

for some constant $C_3 > 0$. Lemma 3.4 then implies $|J_n^A| \leq C_4 d^{-n} \|S\|$ for some constant $C_4 > 0$ which is independent of A. Making $A \to \infty$ gives the first point of the proposition.

We now prove the second estimate of the Proposition. Let $q := k + \dim \mathcal{B} - r - p$, since $\widehat{T}_f = \frac{1}{d^n}(f^n)^*(\widehat{\omega}) - dd^c\phi_n$, a direct computation gives

$$\begin{aligned} \widehat{T}_f^q - \left(\frac{1}{d^n}(f^n)^*\widehat{\omega} \right)^q &= \left(\widehat{T}_f - \frac{1}{d^n}(f^n)^*\widehat{\omega} \right) \wedge \sum_{j=0}^{q-1} \widehat{T}_f^j \wedge \left(\frac{1}{d^n}(f^n)^*\widehat{\omega} \right)^{q-j-1} \\ &= \sum_{j=0}^{q-1} dd^c\phi_n \wedge \widehat{T}_f^j \wedge \left(\frac{1}{d^n}(f^n)^*\widehat{\omega} \right)^{q-j-1}. \end{aligned}$$

Since $\pi \circ f^n = \pi$,

$$\begin{aligned} I_n^A &:= \left\langle \frac{1}{d^{qn}}(f^n)_*(S) \wedge \left(\widehat{T}_f^q - \widehat{\omega}^q \right) \wedge \check{\omega}_{\mathcal{B}}^r, \Psi_A \circ \pi \right\rangle \\ &= \left\langle S \wedge \left(\widehat{T}_f^q - \left(\frac{1}{d^n}(f^n)^*\widehat{\omega} \right)^q \right), \Psi_A \circ \pi \circ f^n \cdot ((\pi \circ f^n)^*\omega_{\mathcal{B}})^r \right\rangle \\ &= \sum_{j=0}^{q-1} d^{-n(q-1-j)} \left\langle S \wedge (dd^c\phi_n) \wedge \widehat{T}_f^j \wedge (f^n)^*(\widehat{\omega})^{q-1-j}, \Psi_A \circ \pi \cdot \check{\omega}_{\mathcal{B}}^r \right\rangle \\ &= \sum_{j=0}^{q-1} d^{-n(q-1-j)} \int \phi_n \cdot S \wedge \widehat{T}_f^j \wedge (f^n)^*(\widehat{\omega})^{q-1-j} \wedge dd^c(\Psi_A \circ \pi) \wedge \check{\omega}_{\mathcal{B}}^r, \end{aligned}$$

where we used Stokes formula.

We now let $S_j := S \wedge \widehat{T}_f^j \wedge \check{\omega}_{\mathscr{B}}^r$ for any $0 \leq j \leq q-1$. The above implies

$$\begin{aligned} |I_n^A| &\leq \sum_{j=0}^{q-1} d^{-n(q-1-j)} \int |\phi_n| \cdot S_j \wedge (f^n)^*(\widehat{\omega})^{q-1-j} \wedge \pi^*(T_A^+ + T_A^-) \\ &\leq \sum_{j=0}^{q-1} d^{-n(q-1-j)} \frac{C_3}{A} \cdot \| S_j \wedge ((f^n)^*(\widehat{\omega}))^{q-1-j} \| \cdot \sup_{\mathbb{P}^N \times B(0,e^{2A})} |\phi_n|, \end{aligned}$$

for some universal constant $C_5 > 0$. In particular, using again Lemma 3.4, we find a constant $C_6 > 0$ such that

$$|I_n^A| \leq C_6 \sum_{j=0}^{q-1} d^{-n(q-j)} \cdot \| S_j \wedge ((f^n)^*(\widehat{\omega}))^{q-j-1} \|.$$

Since the constant does not depend on A, we can make $A \to \infty$ and multiply by d^{nq} to complete the proof.

We now explain how to deduce the case for the maximal regular part $\Lambda_{\max}$. Let $\Lambda \subset \Lambda_{\max}$ be an affine Zariski dense open subset of $\Lambda_{\max}$. Observe that $\widehat{\omega}$, $\check{\omega}_{\mathscr{B}}$ have continuous potentials, and $\widehat{T}_f$ has continuous potential on $\Lambda_{\max} \backslash \Lambda$. In particular, all the currents $(f^n)_*(S) \wedge \widehat{\omega}^q \wedge \check{\omega}_{\mathscr{B}}^r$, $\widehat{T}_f^q \wedge S \wedge \check{\omega}_{\mathscr{B}}^r$, $\widehat{T}_f^j \wedge S \wedge (f^n)^*(\widehat{\omega})^{q-j-1} \wedge \check{\omega}_{\mathscr{B}}^r$ give no mass to the analytic set $\pi^{-1}(\Lambda_{\max} \backslash \Lambda)$. The same argument now works for an arbitrary regular part. $\square$

We now give the proof of Lemma 3.4.

Proof of Lemma 3.4. — Let $F : \mathbb{P}^N \times \Lambda \to \mathbb{P}^N \times \Lambda$ be the family induced by $(\mathscr{X}, f, \mathscr{L})$. As $H^1(\mathrm{End}_d(\mathbb{P}^k), \mathbb{R}) = \{0\}$ by [**BB**, Lemma 4.9], we can write $F_\lambda(z) = [P_{0,\lambda}(z) : \cdots : P_{N,\lambda}(z)]$, $z \in \mathbb{P}^N$, where $P_{i,\lambda} \in \mathbb{C}[\Lambda][z_0, \ldots, z_N]$ are homogeneous polynomials and $\mathrm{Res}(P_{0,\lambda}, \ldots, P_{N,\lambda}) \in \mathbb{C}^*$ if and only if $\lambda \in \Lambda$. For any $\lambda \in \Lambda$, defined $\tilde{F}_\lambda = (P_{0,\lambda}, \ldots, P_{N,\lambda}) : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$. Since $p \mapsto \frac{1}{d} \log \|\tilde{F}_\lambda(p)\| - \log \|p\|$ is 0-homogeneous, it induces a function $g_0 : \mathbb{P}^N(\mathbb{C}) \times \Lambda \to \mathbb{R}$ defined, for $(z, \lambda) \in \mathbb{P}^N(\mathbb{C}) \times \Lambda$, by $g_0(z, \lambda) := \frac{1}{d} \log \|\tilde{F}_\lambda(p)\| - \log \|p\|$, for any $p = (p_0, \ldots, p_N) \in \mathbb{C}^{N+1} - \{0\}$ such that $z = [p_0 : \cdots : p_N]$. In particular, we have $d^{-1} F^* \hat{\omega}_{\mathrm{FS}} - \hat{\omega}_{\mathrm{FS}} = \mathrm{dd}^c g_0$. By construction,

$$\widehat{T}_F - \frac{1}{d^n} (F^n)^* \hat{\omega}_{\mathrm{FS}} = \mathrm{dd}^c \left(\sum_{j=n}^{\infty} \frac{1}{d^j} g_0 \circ F^j \right). \tag{3}$$

Note that the coefficient of $\tilde{F}_\lambda$ in a given system of homogeneous coordinates are regular functions on Λ and that also $\mathrm{Res}(\tilde{F}_\lambda) \in \mathbb{C}[\Lambda]$, so that

$$\Big| \log |\mathrm{Res}(\tilde{F}_\lambda)| \Big| \leq C_1 \log^+ \|\lambda\| + C_2.$$

An application of the homogeneous Hilbert's Nullstellensatz (see, e.g., [**GOV**, proof of Lemma 6.5]) implies there exist a constant $C > 0$ and an integer $N \geq 1$ such that

$$C^{-1} |\mathrm{Res}(\tilde{F}_\lambda)| \leq \frac{\|\tilde{F}_\lambda(p)\|}{\|p\|^d} \leq C \max(\|\lambda\|, 1)^N \quad \text{for any } p \in \mathbb{C}^{N+1} \setminus \{0\} \text{ and any } \lambda \in \Lambda.$$

In particular, since $g_0(z, \lambda) = \log \|\tilde{F}_\lambda(p)\| / \|p\|^d$, this gives easily $|g_0(z, \lambda)| \leq N \log^+ \|\lambda\| + \log(C)$ for any $z \in \mathbb{P}^k$ and any $\lambda \in \Lambda$.

Using that $\iota \circ f = F \circ \iota$, we deduce the lemma from the above. $\square$

3.3. Global height function versus mass of a current. — Again, pick a polarized endomorphism (X, f, L) over the field of rational functions $\mathbf{K}$ of a normal complex projective variety $\mathcal{B}$. Observe that our choice of an ample line bundle $\mathcal{N}$ on $\mathcal{B}$ provides a naive height $h_{X,L}$ on $X(\overline{\mathbf{K}})$: For any irreducible subvariety Z of dimension $0 \leq \ell \leq k = \dim X$ of X defined over $\mathbf{K}$, the height $h_{X,L}(Z)$ can be defined as the intersection number

$$h_{X,L}(Z) := \left(\mathcal{Z} \cdot c_1(\mathcal{L})^{\ell+1} \cdot c_1(\pi^*\mathcal{N})^{\dim \mathcal{B}-1}\right),$$

where $\mathcal{Z}$ is the Zariski closure of Z in $\mathcal{X}$, see, e.g., [**Gub1, Gub3, Fab, CGHX**]. As $\mathcal{Z}$ is irreducible, one has $\dim(\mathcal{Z} \cap (\mathcal{X} \setminus \mathcal{X}_\Lambda)) < \dim \mathcal{Z}$ for any regular part Λ and since $\breve{\omega}_{\mathcal{B}} = \pi^*\omega_{\mathcal{B}}$ and $\widehat{\omega}$ have continuous potentials, the height $h_{X,L}(Z)$ of Z can be computed as

$$h_{X,L}(Z) = \int_{\mathcal{X}} \widehat{\omega}^{\ell+1} \wedge \breve{\omega}_{\mathcal{B}}^{\dim \mathcal{B}-1} \wedge [\mathcal{Z}] = \int_{\mathcal{X}_\Lambda} \widehat{\omega}^{\ell+1} \wedge \breve{\omega}_{\mathcal{B}}^{\dim \mathcal{B}-1} \wedge [\mathcal{Z}],$$

independently of the regular part. When Z is defined over a finite extension $\mathbf{K}'$ of $\mathbf{K}$, we let $\rho' : \mathcal{B}' \to \mathcal{B}$ be the normalization of $\mathcal{B}$ in $\mathbf{K}'$. If $\mathcal{X}' := \mathcal{X} \times_{\mathcal{B}} \mathcal{B}'$, the projection $\rho : \mathcal{X}' \to \mathcal{X}$ onto the first factor is a finite branched cover and, if $\mathcal{Z}'$ is the Zariski closure of Z in $\mathcal{X}'$, we can set

$$h_{X,L}(Z) = \frac{1}{[\mathbf{K}' : \mathbf{K}]}\left(\mathcal{Z}' \cdot c_1(\rho^*\mathcal{L})^{\ell+1} \cdot c_1(\rho^*\pi^*\mathcal{N})^{\dim \mathcal{B}-1}\right)$$

Up to base change, we thus can assume in the sequel that Z is defined over $\mathbf{K}$.

By the classical functorial properties of Weil heights on varieties over function fields, and since $f^*L \simeq L^{\otimes d}$, we have $h_{X,L} \circ f = d \cdot h_{X,L} + O(1)$ on $X(\bar{\mathbf{K}})$ see, e.g. [**BG**]. Recall that, following [**CS**], we define the *canonical height* of f as $\widehat{h}_f := \lim_{n\to\infty} \frac{1}{d^n} h_{X,L} \circ f^n$. By [**CS**, Theorem 1.1], it is the unique function $\hat{h}_f : X(\bar{\mathbf{K}}) \to \mathbb{R}_+$ satisfying

1. $\widehat{h}_f \circ f = d \cdot \widehat{h}_f$,
2. $\widehat{h}_f = h_{X,L} + O(1)$.

Let us now prove Theorem B.

Proof of Theorem B. — Fix a regular part Λ. One can write

$$h_{X,L}(f^n(Z)) = \int_{\mathcal{X}_\Lambda} (f^n)_*[\mathcal{Z}] \wedge \widehat{\omega}^{\ell+1} \wedge \breve{\omega}_{\mathcal{B}}^{\dim \mathcal{B}-1}.$$

Applying the second point of Proposition 3.3 with $p = k - \ell$, $r = \dim(\mathcal{B}) - 1$ so that $q = k + \dim \mathcal{B} - r - p = \ell + 1$ gives

$$h_{X,L}(f^n(Z)) = d^{n(\ell+1)} \int_{\mathcal{X}_\Lambda} [\mathcal{Z}] \wedge \widehat{T}_f^{\ell+1} \wedge \breve{\omega}_{\mathcal{B}}^{\dim \mathcal{B}-1} + O(d^{n\ell}).$$

In particular,

$$\widehat{h}_f(Z) := \lim_{n\to\infty} \frac{1}{d^{n(\ell+1)}} h_{X,L}((f^n)_*(Z))$$

is well defined and we have the equality

$$\widehat{h}_f(Z) = \int_{\mathcal{X}_\Lambda} [\mathcal{Z}] \wedge \widehat{T}_f^{\ell+1} \wedge [\mathcal{Z}] \wedge \breve{\omega}_{\mathcal{B}}^{\dim \mathcal{B}-1}.$$

□

Remark 3.5. — Remark that $\widehat{h}_f(f_*(Z)) = d^{\ell+1} \cdot \widehat{h}_f(Z)$. In particular, if Z is preperiodic, i.e. if there exists $n > m \geq 0$ such that $f^n(Z) = f^m(Z)$, we have $\widehat{h}_f(Z) = 0$. Note that if $f(Z) = Z$ we have $f_*[\mathcal{Z}] = d^\ell[\mathcal{Z}]$ in the sense of currents, since above any fiber $\pi^{-1}(\{\lambda\}) \cap \mathcal{Z}$, f_λ is a polarized endomorphism of topological degree d^ℓ (see Lemma 2.4).

Even though we do not need it in the sequel, we can prove the following statement, compare with [**CGHX**, Corollary 3.3].

Corollary 3.6. — *Under the hypothesis of Theorem B, $\widehat{h}_f(Z) = 0$ if and only if for any closed positive $(1,1)$-current ν on $\mathcal{B}$ with continuous potentials we have*

$$\int_{\mathcal{X}_\Lambda} \widehat{T}_f^{\ell+1} \wedge [\mathcal{Z}] \wedge (\pi^*\nu)^{\dim \mathcal{B}-1} = 0.$$

Proof. — Theorem B implies $\widehat{h}_f(Z) = 0$ if and only if

$$\int_{\mathcal{X}_\Lambda} \widehat{T}_f^{\ell+1} \wedge [\mathcal{Z}] \wedge (\pi^*\omega_{\mathcal{B}})^{\dim \mathcal{B}-1} = 0. \tag{4}$$

It is thus sufficient to prove that if equation (4) holds, then

$$\int_{\pi^{-1}(U)} \widehat{T}_f^{\ell+1} \wedge [\mathcal{Z}] \wedge (\pi^*\nu)^{\dim \mathcal{B}-1} = 0 \tag{5}$$

for any closed positive $(1,1)$-current ν on $\mathcal{B}$ with continuous potentials and any open subset U of Λ which is relatively compact in Λ. Again, by continuity, it is sufficient to consider the case where ν is smooth since, by Richberg's theorem [**Dem2**, §5.21, p. 43], any continuous psh function can be locally approximated by smooth psh functions and we have the continuity of the wedge product in (5). Pick such a smooth current ν. As $\omega_{\mathcal{B}}$ is strictly positive on U, there exists $C > 0$ such that $C\omega_{\mathcal{B}} \geq \nu \geq 0$ in the weak sense of currents. In particular,

$$0 \leq \int_{\pi^{-1}(U)} \widehat{T}_f^{\ell+1} \wedge [\mathcal{Z}] \wedge (\pi^*\nu)^{\dim \mathcal{B}-1} \leq \int_{\mathcal{X}_\Lambda} \widehat{T}_f^{\ell+1} \wedge [\mathcal{Z}] \wedge (\pi^*\omega_{\mathcal{B}})^{\dim \mathcal{B}-1} = 0.$$

Taking the supremum over all $U \Subset \Lambda$ ends the proof. □

4. Stability of algebraic dynamical pairs

4.1. Several characterizations of stability. — Dynamical pairs (resp. stable dynamical pairs) were defined in the analytic setting in Definition 2.8 (resp. Definition 2.9). We now define the corresponding notions in the algebraic case.

Definition 4.1. — *A p-dynamical pair $((\mathcal{X}, f, \mathcal{L}), [\mathcal{Z}])$ is* algebraic *if*

($\dagger_1$) *$(\mathcal{X}, f, \mathcal{L})$ is an algebraic family of polarized endomorphisms.*

($\dagger_2$) *$\mathcal{Z}$ is an irreducible algebraic subvariety of $\mathcal{X}$ of codimension $p \leq k$, where k is the relative dimension of $\mathcal{X} \to \mathcal{B}$ which is flat over some regular part Λ of $(\mathcal{X}, f, \mathcal{L})$.*

We say that the p-dynamical pair $((\mathcal{X}, f, \mathcal{L}), [\mathcal{Z}])$ is stable *if $(\mathcal{X}_\Lambda, f, \mathcal{L}, [\mathcal{Z}])$ is stable.*

Note that the notion of stability is independent of the chosen regular part Λ. Indeed, if $(\mathcal{X}_\Lambda, f, \mathcal{L}, [\mathcal{Z}])$ is stable, then the current $T_{f,\mathcal{Z}}$ vanishes identically on Λ. In particular, its trivial extension to $\mathcal{B}$ is identically zero.

Example 4.2. — 1. Let $(\mathbb{P}^k \times \mathcal{B}, f, \mathcal{O}_{\mathbb{P}^k}(1))$ be an algebraic family of polarized endomorphisms. If in addition a is a marked point, i.e. a morphism $a : \Lambda \to \mathbb{P}^k$, the pair $(\mathbb{P}^k \times \mathcal{B}, f, \mathcal{O}_{\mathbb{P}^k}(1), [\Gamma_a])$, where Γ_a is the graph of a, is an algebraic k-dynamical pair.

Notice that up to taking a finite branched cover of $\mathcal{B}$, any k-dynamical pair of the form $(\mathbb{P}^k \times \mathcal{B}, f, \mathcal{O}_{\mathbb{P}^k}(1), [\mathcal{Z}])$ can be decomposed as a finite collection of pairs given by graphs. Nevertheless, it is convenient in what follows to allow $\mathcal{Z}$ not to be a graph.

2. Let $(\mathcal{X}, f, \mathcal{L})$ be an algebraic family of polarized endomorphisms. Set

$$\mathrm{Per}_f(n, m) := \{z_0 \in \mathcal{X} \ : \ f^n(z_0) = f^m(z_0)\}$$

for any $n > m \geq 0$. For $n - m$ large enough the set $\mathrm{Per}_f(n, m)$ is non-empty by Proposition 2.3 and defines a subvariety of $\mathcal{X}$ of pure dimension $\dim \mathcal{B}$, flat over some regular part Λ, by Lemma 2.4. As a consequence, the current $[\mathrm{Per}_f(n, m)]$ is a closed positive horizontal (k, k)-current on $\mathcal{X}_\Lambda$. As an immediate application of Theorem B and Proposition 2.3, we have the following, see also Corollary 1.7.

Corollary 4.3. — *For any $n > m \geq 0$ with $n-m$ large enough, the set $\mathrm{Per}_f(n, m)$ is non-empty. Moreover, for any irreducible component $\mathcal{C}$ of $\mathrm{Per}_f(n, m)$, the k-dynamical pair $((\mathcal{X}, f, \mathcal{L}), [\mathcal{C}])$ is stable.*

Recall that, when $\mathcal{Z}$ is a subvariety of $\mathcal{X}$ of dimension $\ell + \dim \mathcal{B}$, the degree of $\mathcal{Z}$ relatively to the ample line bundle $\mathcal{M} = \mathcal{L} \otimes \pi^*(\mathcal{N})$ (where $\mathcal{N}$ is a very ample line bundle on $\mathcal{B}$, $\widehat{\omega}$ is cohomologous to $c_1(\mathcal{L})$ and $\omega_{\mathcal{B}}$ is cohomologous to $c_1(\mathcal{N})$) is given by

$$\deg_{\mathcal{M}}(\mathcal{Z}) = \left(\mathcal{Z} \cdot c_1(\mathcal{M})^{\ell + \dim \mathcal{B}}\right) = \int_{\mathcal{Z}_\Lambda} (\widehat{\omega} + \check{\omega}_{\mathcal{B}})^{\ell + \dim \mathcal{B}} = \|[\mathcal{Z}]\|.$$

The following is a more general version of item 1. of Theorem A. Observe that, in what follows, when Z is a point (i.e. $p = k$), we have obviously $\deg(f|_Z) = 1$, and item 3 means $\deg_{\mathcal{M}}(\mathcal{Z}_n) = O(1)$, as claimed in the introduction. In particular, when $p = k$, the implication $2 \implies 3$ in the following theorem follows immediately from Theorem B and the fact that $\widehat{h}_f - h_{X,L} = O(1)$.

The bifurcation current $T_{f,[\mathcal{Z}]}$ was defined in section 2.4 and $\deg(f^n|_Z)$ in §1.6.

Theorem 4.4. — *Let $((\mathcal{X}, f, \mathcal{L}), [\mathcal{Z}])$ be an algebraic p-dynamical pair with $1 \leq p \leq k$, which is a model of (X, f, L, Z) over the field $\mathbf{K}$ of rational functions of a normal complex projective variety $\mathcal{B}$. Let $\mathcal{Z}_n$ be the Zariski closure of $f^n(\mathcal{Z}_\Lambda)$ in $\mathcal{X}$. Then the following are equivalent:*

1. *$((\mathcal{X}, f, \mathcal{L}), [\mathcal{Z}])$ is stable,*
2. *$T_{f,[\mathcal{Z}]} = 0$ as a closed positive $(1,1)$-current on Λ,*
3. *there exists $C > 0$ such that for all $n \geq 1$,*

$$C^{-1} \frac{d^{n(k-p)}}{\deg(f^n|_Z)} \leq \deg_{\mathcal{M}}(\mathcal{Z}_n) \leq C \frac{d^{n(k-p)}}{\deg(f^n|_Z)},$$

4. *$\widehat{h}_f(Z) = 0$.*

Proof. — The equivalence between points 1 and 2 is the content of Proposition 2.10 and the equivalence between 2 and 4 is an immediate consequence of Theorem B. We thus just need to prove the equivalence between 2 and 3. We now recall that, as algebraic cycles, we have

$$(f^n_\lambda)_* Z_\lambda = \deg(f^n_\lambda|_{Z_\lambda}) \cdot f^n_\lambda(Z_\lambda) = \deg(f^n|_Z) \cdot f^n_\lambda(Z_\lambda).$$

We infer that, as $\mathcal{Z} \to \Lambda$ is flat, as currents on $\mathcal{X}_\Lambda$, we also have

$$(f^n)_*[\mathcal{Z}] = \deg(f^n|_Z) \cdot [f^n(\mathcal{Z})].$$

As above, let $\ell := k-p$ be the dimension of Z and $\alpha_n := d^{n\ell}/\deg(f^n|_Z)$. Recall also that

$$\begin{aligned}\deg_{\mathcal{M}}(\mathscr{Z}_n) &= \frac{1}{\deg(f^n|_Z)}\int_{\mathcal{X}_\Lambda}(f^n)_*[\mathscr{Z}]\wedge(\widehat{\omega}+\check{\omega}_{\mathcal{B}})^{\dim\mathcal{B}+\ell}\\ &= \frac{1}{\deg(f^n|_Z)}\sum_{j=0}^{\dim\mathcal{B}}\binom{\dim\mathcal{B}+\ell}{j+\ell}\int_{\mathcal{X}_\Lambda}(f^n)_*[\mathscr{Z}]\wedge\widehat{\omega}^{j+\ell}\wedge\check{\omega}_{\mathcal{B}}^{\dim\mathcal{B}-j} \qquad (6)\\ &\geq \frac{1}{\deg(f^n|_Z)}\int_{\mathcal{X}_\Lambda}(f^n)_*[\mathscr{Z}]\wedge\widehat{\omega}^{\ell+1}\wedge\check{\omega}_{\mathcal{B}}^{\dim\mathcal{B}-1}.\end{aligned}$$

If $T_{f,[\mathscr{Z}]}$ is non-zero, we use the second point of Proposition 3.3. The above implies

$$\deg_{\mathcal{M}}(\mathscr{Z}_n)\geq d^n\alpha_n\int_{\mathcal{X}_\Lambda}\widehat{T}_f^{\ell+1}\wedge[\mathscr{Z}]\wedge\check{\omega}_{\mathcal{B}}^{\dim\mathcal{B}-1}+O(\alpha_n),$$

so there is no $C>0$ such that $\deg_{\mathcal{M}}(f^n(\mathscr{Z}))\leq C\alpha_n$. Hence 3 implies 2.

We finally assume $T_{f,[\mathscr{Z}]}=0$ and we first want to prove $\deg_{\mathcal{M}}(\mathscr{Z}_n)=O(\alpha_n)$. Since $\mathscr{Z}$ is flat over a dense Zariski open set and since $\widehat{T}_f$ has continuous potentials, this implies $\widehat{T}_f^{\ell+1}\wedge[\mathscr{Z}]=0$. Applying point 1 of Proposition 3.3 to $S=\widehat{T}_f^{\ell}\wedge[\mathscr{Z}]\wedge\check{\omega}_{\mathcal{B}}^{\tau}$ (for any τ)

$$\left|\|\widehat{T}_f\wedge\widehat{T}_f^{\ell}\wedge[\mathscr{Z}]\wedge\check{\omega}_{\mathcal{B}}^{\tau}\|-d^{-n}\|(f^n)^*(\widehat{\omega})\wedge\widehat{T}_f^{\ell}\wedge[\mathscr{Z}]\wedge\check{\omega}_{\mathcal{B}}^{\tau}\|\right|\leq Cd^{-n}\|\widehat{T}_f^{\ell}\wedge[\mathscr{Z}]\wedge\check{\omega}_{\mathcal{B}}^{\tau}\|,$$

so

$$\|(f^n)^*(\widehat{\omega})\wedge\widehat{T}_f^{\ell}\wedge[\mathscr{Z}]\wedge\check{\omega}_{\mathcal{B}}^{\tau}\|=O(1).$$

Similarly, for all $0\leq s\leq\ell$, applying point 1 of Proposition 3.3 to $S=(f^n)^*(\widehat{\omega}^{\ell-s})\wedge\widehat{T}_f^{s}\wedge[\mathscr{Z}]\wedge\check{\omega}_{\mathcal{B}}^{\tau}$ gives

$$\begin{aligned}\Big|d^n\|\widehat{T}_f\wedge(f^n)^*(\widehat{\omega}^{\ell-s})\wedge\widehat{T}_f^{s}\wedge[\mathscr{Z}]\wedge\check{\omega}_{\mathcal{B}}^{\tau}\|-\|(f^n)^*(\widehat{\omega})\wedge(f^n)^*(\widehat{\omega}^{\ell-s})\wedge\widehat{T}_f^{s}\wedge[\mathscr{Z}]\wedge\check{\omega}_{\mathcal{B}}^{\tau}\|\Big|\leq\\ C\|(f^n)^*(\widehat{\omega}^{\ell-s})\wedge\widehat{T}_f^{s}\wedge[\mathscr{Z}]\wedge\check{\omega}_{\mathcal{B}}^{\tau}\|,\end{aligned}$$

Combining the two estimates implies that for all $0\leq s\leq\ell+1$, we have

$$\|(f^n)^*(\widehat{\omega}^{\ell+1-s})\wedge\widehat{T}_f^{s}\wedge[\mathscr{Z}]\wedge\check{\omega}_{\mathcal{B}}^{\tau}\|=O(d^{n(\ell-s)}),$$

(for $s=\ell+1$, it reads as $0=O(d^{-n})$). By induction on j (initialized above for $j=1$), we show, for all $1\leq j\leq\dim\mathcal{B}$ and every s,

$$\|[\mathscr{Z}]\wedge(f^n)^*(\widehat{\omega}^{\ell+j-s})\wedge\widehat{T}_f^{s}\wedge\check{\omega}_{\mathcal{B}}^{\dim\mathcal{B}-j}\|=O(d^{n(\ell-s)}).$$

Pick j and assume this holds for $j-1$ and all s, then we prove the estimate for j by a descending induction on s. Indeed, it holds for $s=\ell+1$ since $\widehat{T}_f^{\ell+1}\wedge[\mathscr{Z}]=0$. Assume it holds for $s+1\leq\ell+1$. Point 1 of Proposition 3.3 then gives

$$\|[\mathscr{Z}]\wedge(f^n)^*(\widehat{\omega}^{\ell+j-s})\wedge\widehat{T}_f^{s}\|-d^n\|[\mathscr{Z}]\wedge(f^n)^*(\widehat{\omega}^{\ell+j-1-s})\wedge\widehat{T}_f^{s+1}\|=O(d^{n(\ell-s)}),$$

so, by the induction hypothesis for $j-1$:

$$\|[\mathscr{Z}]\wedge(f^n)^*(\widehat{\omega}^{\ell+j-s})\wedge\widehat{T}_f^{s}\|=d^nO(d^{n(\ell-(s+1))})+O(d^{n(\ell-s)}),$$

which gives the estimate.

By point 2 of Proposition 3.3, for any $1\leq j\leq\dim\mathcal{B}$, since $\widehat{T}_f^{\ell+1}\wedge[\mathscr{Z}]=0$, the quantity

$$\int_{\mathcal{X}_\Lambda}(f^n)_*[\mathscr{Z}]\wedge\widehat{\omega}^{j+\ell}\wedge\check{\omega}_{\mathcal{B}}^{\dim\mathcal{B}-j}$$

is thus bounded above by

$$\begin{aligned} C_1 \sum_{q\le \ell} d^{nq} \|\widehat{T}_f^q \wedge (f^n)^*(\widehat{\omega}^{j+\ell-1-q}) \wedge [\mathcal{Z}] \wedge \check{\omega}_{\mathcal{B}}^{\dim \mathcal{B}-j}\| \\ \le C_2 \sum_{q\le \ell} d^{nq}\cdot d^{n(\ell-q-1)} \le C_3 d^{n(\ell-1)}, \end{aligned}$$

for some constants $C_1, C_2, C_3 > 0$ independent on n and j. This summarizes as

$$0 \le \int_{\mathcal{X}_\Lambda} (f^n)_*[\mathcal{Z}] \wedge \widehat{\omega}^{j+\ell} \wedge \check{\omega}_{\mathcal{B}}^{\dim \mathcal{B}-j} \le C_4 d^{n(\ell-1)},$$

for some constant $C_4 > 0$ independent of n. Finally, for $j = 0$, we have

$$\begin{aligned} \int_{\mathcal{X}_\Lambda} (f^n)_*[\mathcal{Z}] \wedge \widehat{\omega}^{\ell} \wedge \check{\omega}_{\mathcal{B}}^{\dim \mathcal{B}} &= \int_{\mathcal{X}_\Lambda} [\mathcal{Z}] \wedge (f^n)^*(\widehat{\omega}^{\ell}) \wedge \check{\omega}_{\mathcal{B}}^{\dim \mathcal{B}} \\ &\le C_5 d^{n\ell} \|[\mathcal{Z}]\|, \end{aligned}$$

for some $C_5 > 0$, since $\pi \circ f^n = \pi$. This gives the right hand side inequality in point 3 of the proposition. To get the other inequality, we shall use that the dynamical growth of $\deg_{\mathcal{M}}(\mathcal{Z}_n)$ (i.e. above any given parameter λ), is already in $\frac{d^{n\ell}}{\deg(f^n|_Z)}$. More precisely, take some non empty open set $U \Subset \Lambda$, then we always have

$$\int_{\pi^{-1}(U)} \widehat{T}_f^{\ell} \wedge [\mathcal{Z}] \wedge \check{\omega}_{\mathcal{B}}^{\dim \mathcal{B}} > 0.$$

Indeed, for $\lambda \in U$, the slice along X_λ of $\widehat{T}_f^{\ell} \wedge [\mathcal{Z}]$ is $T_{f_\lambda}^{\ell} \wedge [Z_\lambda]$ which has mass $\deg(Z_\lambda)$ by Bézout, and

$$\int_{\pi^{-1}(U)} \widehat{T}_f^{\ell} \wedge [\mathcal{Z}] \wedge \check{\omega}_{\mathcal{B}}^{\dim \mathcal{B}} = \int_U \pi_* \left(\widehat{T}_f^{\ell} \wedge [\mathcal{Z}]\right) \cdot \omega_{\mathcal{B}}^{\dim \mathcal{B}},$$

so the global mass of $\int_{\pi^{-1}(U)} \widehat{T}_f^{\ell} \wedge [\mathcal{Z}] \wedge \check{\omega}_{\mathcal{B}}^{\dim \mathcal{B}}$ has to be positive.

Taking $j = 0$ in (6), we have

$$\deg_{\mathcal{M}}(\mathcal{Z}_n) \ge \frac{1}{\deg(f^n|_Z)} \int_{\mathcal{X}_\Lambda} (f^n)_*[\mathcal{Z}] \wedge \widehat{\omega}^{\ell} \wedge \check{\omega}_{\mathcal{B}}^{\dim \mathcal{B}} = \frac{1}{\deg(f^n|_Z)} \int_{\mathcal{X}_\Lambda} [\mathcal{Z}] \wedge (f^n)^*(\widehat{\omega}^{\ell}) \wedge \check{\omega}_{\mathcal{B}}^{\dim \mathcal{B}}.$$

By the convergence of $d^{-n}(f^n)^*(\widehat{\omega})$ towards $\widehat{T}_f$ with locally uniform convergence of the potentials, for n large enough, we deduce, taking some non empty open set $U \Subset \Lambda$

$$\deg_{\mathcal{M}}(\mathcal{Z}_n) \ge \frac{d^{n\ell}}{\deg(f^n|_Z)} \int_{\pi^{-1}(U)} [\mathcal{Z}] \wedge \widehat{T}_f^{\ell} \wedge \check{\omega}_{\mathcal{B}}^{\dim \mathcal{B}}.$$

□

Remark 4.5. — When $\dim Z > 0$, the degree $\deg(f^n|_Z)$ could as well grow as $d^{\ell n}$ or remain constant equal to 1. Consider, for example, a family of polarized endomorphisms of $\mathbb{P}^k$ given by polynomial maps of $\mathbb{C}^k$, then the hyperplane at infinity $\mathcal{H}_\infty$ defines an irreducible algebraic subvariety of $\mathcal{X}$ of codimension 1, it is periodic hence its height is zero and $\deg(f^n|\mathcal{H}_\infty) = d^{n(k-1)}$. More generally, we expect to have $\deg_{\mathcal{M}}(\mathcal{Z}_n) = O(1)$, when the pair $((\mathcal{X}, f, \mathcal{L}), [\mathcal{Z}])$ is stable unless we have some isotriviality.

4.2. A criterion for unstability: similarity. — The following is an adaptation of Tan Lei's similarity between the Julia set and the Mandelbrot set at a Misiurewicz parameter [**T**] and follows the idea of the proof of [**AGMV**, Theorem B]. Recall that a repelling periodic point z_0 of f_{λ_0} can be followed analytically as a repelling periodic point in a neighborhood of λ_0 by the Implicit Function Theorem.

Lemma 4.6. — *Let $((\mathcal{X}, f, \mathcal{L}), [\mathcal{Z}])$ be an algebraic k-dynamical pair with regular part Λ. Pick $\lambda_0 \in \Lambda$. Assume there exists a repelling periodic point $z_0 \in (X_{\lambda_0})_{\mathrm{reg}} \cap \mathcal{Z}$ for f_{λ_0}. Let $z : \lambda \in (\Lambda, \lambda_0) \mapsto z(\lambda) \in (\mathcal{X}, z_0)$ be the local analytic continuation of z_0 as a repelling periodic point of f_λ. Then*

1. *either $\lambda_0 \in \mathrm{supp}(T_{f,[\mathcal{Z}]})$, in particular $((\mathcal{X}, f, \mathcal{L}), [\mathcal{Z}])$ is not stable,*
2. *or there is a neighborhood U of λ_0 in Λ such that the graph Γ_z satisfies $\Gamma_z \subset \mathcal{Z} \cap \pi^{-1}(U)$.*

Proof. — Up to replacing f by an iterate, we can assume that z_0 is fixed. We assume $\Gamma_z \not\subset \mathcal{Z}$. First, as $T_{f,[\mathcal{Z}]}$ is a closed positive current on Λ with continuous potential, we can reduce to the case where $\dim \mathcal{B} = 1$. Indeed, for any curve $\mathcal{C}$, $\mathrm{supp}\left(T_{f,[\mathcal{Z}]} \wedge [\mathcal{C}]\right) \subset \mathrm{supp}(T_{f,[\mathcal{Z}]})$, see, e.g., [**Ga**, Lemma 6.3].

We thus assume that $\dim \mathcal{B} = 1$, and that there exists an open neighborhood U of λ_0 in $\mathcal{B}$ such that the intersection $\mathcal{Z} \cap \Gamma_z \cap \pi^{-1}(U)$ is a finite set. We want to prove that $T_{f,[\mathcal{Z}]} \neq 0$ on any neighborhood of λ_0. We assume by contradiction it is stable, so that

$$\int_{\pi^{-1}(U)} \widehat{T}_f \wedge [\mathcal{Z}] = 0.$$

As $z(\lambda)$ is repelling for all $\lambda \in \mathbb{B}(\lambda_0, \varepsilon)$, up to reducing ε, we can assume there exist $K > 1$ and $\delta > 0$ such that

$$d_{X_\lambda}(f_\lambda(x), f_\lambda(z(\lambda))) \geq K d_{X_\lambda}(x, z(\lambda)),$$

for all $\lambda \in \mathbb{B}(\lambda_0, \varepsilon)$ and all $x \in \mathbb{B}_{X_\lambda}(z(\lambda), \delta) \subset X_\lambda$. In particular, there exists a neighborhood $\Omega \Subset \mathbb{B}(\lambda_0, \varepsilon)$ of z_0 in $\mathcal{X}$ such that the analytic map $f : \Omega \to f(\Omega)$ is proper with $\Omega \subset f(\Omega)$ and $\pi(\Omega) \subset U$. In addition, if $\lambda \in \pi(\Omega)$, we have $\Omega \cap X_\lambda \Subset f(\Omega \cap X_\lambda)$. Then

$$\begin{aligned} d^n \int_\Omega \widehat{T}_f \wedge [\mathcal{Z}] &= \int_\Omega (f^n)^* \widehat{T}_f \wedge [\mathcal{Z}] = \int (f^n)^* \widehat{T}_f \wedge (\mathbf{1}_\Omega \cdot [\mathcal{Z}]) \\ &\geq \int \widehat{T}_f \wedge (f^n)_* (\mathbf{1}_\Omega \cdot [\mathcal{Z}]) \\ &\geq \int \widehat{T}_f \wedge \mathbf{1}_\Omega \cdot (f^n)_* (\mathbf{1}_\Omega \cdot [\mathcal{Z}]) . \end{aligned}$$

Let S_n be the current of integration on the connected component of $f^n(\Omega \cap \mathcal{Z}) \cap \Omega$ containing z_0. The sequence S_n defines a sequence of integration currents on analytic subsets of Ω containing z_0 and of uniformly bounded mass. By a theorem of Bishop, any weak limit of (S_n) is the integration current on an analytic curve of Ω containing z_0. Let Z_0 be such a weak limit. By construction, it is supported on $X_{\lambda_0} \cap \Omega$ whence

$$\int_\Omega \widehat{T}_f \wedge [Z_0] = \int_{X_{\lambda_0} \cap \Omega} T_{f_{\lambda_0}} \wedge [Z_0].$$

Since z_0 is repelling, the sequence of iterates $(f^n_{\lambda_0}|_{Z_0})_n$ cannot be equicontinuous, so z_0 belongs to the support of the current $T_{f_{\lambda_0}} \wedge [Z_0]$ (see the proof of Theorem 1.6.5 of [**Si**] for details),

hence
$$\int_{X_{\lambda_0}\cap\Omega} T_{f_{\lambda_0}} \wedge [Z_0] > 0.$$
This contradicts the fact that $\int_{\pi^{-1}(U)} \widehat{T}_f \wedge [\mathcal{Z}] = 0$. □

4.3. The Geometric Dynamical Northcott property. —

Theorem 4.7. — *Let $(\mathcal{X}, f, \mathcal{L})$ be a non-isotrivial algebraic family of polarized endomorphisms. Fix an integer $D \geq 1$. There exist a possibly empty subvariety $\mathcal{Y}_D$ of $\mathcal{X}$ and $N \geq 0$ such that*
1. *$f(\mathcal{Y}_D) = \mathcal{Y}_D$, and for any periodic irreducible component $\mathcal{V}$ of $\mathcal{Y}_D$ with $f^n(\mathcal{V}) = \mathcal{V}$, the family of polarized endomorphisms $(\mathcal{V}, f^n|_{\mathcal{V}}, \mathcal{L}|_{\mathcal{V}})$ is isotrivial,*
2. *let $\mathcal{C}_0(a)$ be the Zariski closure in $\mathcal{X}$ of $a(\Lambda)$ where $a : \mathcal{B} \dashrightarrow \mathcal{X}$ is a rational section, regular over Λ, assume that $\deg_{\mathcal{M}}(f^n(\mathcal{C})) \leq D$ for all $n \geq 1$, then $\mathcal{Y}_D$ contains $f^N(\mathcal{C})$.*

Remark 4.8. — 1. In the case where any irreducible component of $\mathcal{Y}_D$ has dimension $\dim \mathcal{B}$, the set $\mathcal{Y}_D$ itself is a (possibly reducible) pointwise periodic subvariety under iteration of f, i.e. there exists $n > 0$ such that $f^n(x) = x$, for all $x \in \mathcal{Y}_D$.
2. In fact we will see in Theorem 4.10 that we can make $\mathcal{Y}_D$ *independent* of D.
3. While $\mathcal{Y}_D$ can be empty for small D, Corollary 4.3 guarantees it is not empty for D large enough, up to a base change.

Proof. — First, observe that it is enough to prove the theorem in the case where $\mathcal{X}$ is normal which we assume from now on.

Recall that the Chow variety $\mathrm{Ch}_{\dim \mathcal{B}, D}(\mathcal{X})$ is the (non reduced) variety of algebraic cycles of $\mathcal{X}$ of dimension $\dim \mathcal{B}$ and degree at most D, relatively to $\mathcal{M}$. Let $\mathcal{D}$ denote the set of subvarieties of $\mathcal{X}$ of the form $\mathcal{C}_0(a)$ for some rational section $a : \mathcal{B} \dashrightarrow \mathcal{X}$. For any $n \in \mathbb{N}$, let
$$\mathcal{Z}_{n,D} := \{\mathcal{C} \in \mathcal{D} \cap \mathrm{Ch}_{\dim \mathcal{B}, D}(\mathcal{X}),\ \forall j \leq n,\ \deg_{\mathcal{M}}(f^j(\mathcal{C})) \leq D\}$$
and $\overline{\mathcal{Z}_{n,D}}$ its Zariski closure in $\mathrm{Ch}_{\dim \mathcal{B}, D}(\mathcal{X})$. Then, the decreasing intersection
$$\bigcap_{n=0}^{N} \overline{\mathcal{Z}_{n,D}}$$
is eventually constant by noetherianity so there exists a n_0 such that
$$\mathcal{Z}_D^1 := \bigcap_{n=0}^{n_0} \overline{\mathcal{Z}_{n,D}} = \bigcap_{n\geq 0} \overline{\mathcal{Z}_{n,D}},$$
is a projective variety. As $\mathcal{Z}_{n_0,D}$ contains a dense Zariski open set of any irreducible component W of $\mathcal{Z}_D^1$, f induces a rational map on W, whose image is contained in $\mathcal{Z}_D^1$. Denote by $g : \mathcal{Z}_D^1 \dashrightarrow \mathcal{Z}_D^1$ this rational map. By construction, it is well defined on a dense Zariski open set of each irreducible component, contained in $\mathcal{Z}_{n_0,D}$, and $g(\mathcal{Z}_{n_0,D}) \subset \mathcal{Z}_{n_0,D}$, whence the iterates of g are all well-defined on all irreducible components of $\mathcal{Z}_D^1$.

In particular, under iteration of g, any irreducible component of $\mathcal{Z}_D^1$ is sent to a periodic irreducible component and whose image is not fully contained in the indeterminacy set of g. Let $\mathcal{Z}_D$ be such a periodic irreducible component of $\mathcal{Z}_D^1$. Note that, since there are only finitely many possible cohomology classes of $\mathcal{V} \in \mathcal{Z}_D^1$, all the $\mathcal{V}$'s lying in a given irreducible component have to be cohomologous. As we have proved in Lemma 3.2 that $(\mathcal{X}, f, \mathcal{L})$ is isotrivial if and only if $(\mathcal{X}, f^n, \mathcal{L})$ is for some $n \geq 1$, we may assume $g(\mathcal{Z}_D) \subset \mathcal{Z}_D$. As f has finite fibers over any regular part Λ, $g(\mathcal{Z}_D)$ has the same dimension as $\mathcal{Z}_D$. If not, a general element $\mathcal{W}$ in

$g(\mathcal{Z}_D)$ would have infinitely many preimages $\mathcal{W}_i$ in $\mathcal{Z}_D$ and, for a general parameter $\lambda \in \Lambda$, $\mathcal{W}_i \cap X_\lambda \subset f_\lambda^{-1}(X_\lambda)$ (which is finite), a contradiction. As $\mathcal{Z}_D$ is irreducible, $g : \mathcal{Z}_D \dashrightarrow \mathcal{Z}_D$ is dominant.

We now define

$$\widehat{\mathcal{Z}}_D := \{(\mathcal{V}, x) \in \mathcal{Z}_D \times \mathcal{X} \ : \ x \in \mathcal{V}\}.$$

Assume that the canonical projection $\Pi : (\mathcal{V}, x) \longmapsto x$ is dominant, whence Π is surjective. We want to prove $(\mathcal{X}, f, \mathcal{L})$ is isotrivial. For a given regular part Λ, we set $\widehat{\mathcal{Z}}_D^\Lambda := \Pi^{-1}(\mathcal{X}_\Lambda)$, we have

Lemma 4.9. — *There exists a regular part Λ such that the map $\Pi|_{\widehat{\mathcal{Z}}_D^\Lambda} : \widehat{\mathcal{Z}}_D^\Lambda \to \mathcal{X}_\Lambda$ is an isomorphism and $\dim(\mathcal{Z}_D) = k$. Moreover, if $p : \widehat{\mathcal{Z}}_D^\Lambda \to \mathcal{Z}_D$ is the projection given by $p(\mathcal{V}, z) = \mathcal{V}$ then the map*

$$\Psi := p \circ \left(\Pi|_{\widehat{\mathcal{Z}}_D^\Lambda}\right)^{-1} : \mathcal{X}_\Lambda \longrightarrow \mathcal{Z}_D$$

is regular and its fibers are of the form $\mathcal{V}_\Lambda$ for some $\mathcal{V} \in \mathcal{Z}_D$. Finally, $\Psi \circ f = g \circ \Psi$.

We take the lemma for granted. Up to normalizing $\mathcal{Z}_D$, we may assume it is normal. Note that, by assumption, there is $\mathcal{C}_0 \in \mathcal{Z}_D$ with $(\mathcal{C}_0 \cdot X_\lambda) = 1$, for all λ in Λ. As all fibers of π over Λ (resp. all varieties in $\mathcal{Z}_D$) are cohomologous and as the intersection can be computed in cohomology, we deduce that

$$(\mathcal{V} \cdot X_\lambda) = 1$$

for all $\lambda \in \Lambda$ and all $\mathcal{V} \in \mathcal{Z}_D$. Let now $\psi_\lambda := \Psi|_{X_\lambda} : X_\lambda \to \mathcal{Z}_D$. As for any point $z \in X_\lambda$, (z, λ) is contained in one and only one variety $\mathcal{V} \in \mathcal{Z}_D$, ψ_λ is a morphism. Finally, since the topological degree of ψ_λ is exactly $(\mathcal{V} \cdot X_\lambda)$ for a general variety $\mathcal{V} \in \mathcal{Z}_D$, we get that ψ_λ is an isomorphism which conjugates f_λ to g. In particular, g is an endomorphism.

We have proved that for, any $\lambda \in \Lambda$, the morphism f_λ is conjugated by an isomorphism $\psi_\lambda : X_\lambda \to \mathcal{Z}_D$ to the endomorphism g. In particular, for any $\lambda, \lambda' \in \Lambda$, the map $\psi_{\lambda'}^{-1} \circ \psi_\lambda : X_\lambda \to X_{\lambda'}$ is an isomorphism which conjugates f_λ to $f_{\lambda'}$. More precisely, for any $\lambda_0 \in \Lambda$ and any small ball $\mathbb{B} \subset \Lambda$ centered at λ_0, we have built an analytic map $\phi : X_{\lambda_0} \times \mathbb{B} \to \pi^{-1}(\mathbb{B})$ such that the above isomorphism $X_{\lambda_0} \to X_\lambda$ is in fact $\phi_\lambda := \phi(\cdot, \lambda)$. In particular, $\phi_\lambda^* L_\lambda$ depends continuously on λ, whence its class in the Picard group $\mathrm{Pic}(X_{\lambda_0})$ also. As the subset of ample line bundles in the Picard group of a complex projective variety which can polarize a given morphism is discrete and $\mathbb{B}$ is connected, it must be constant, by [**NZ**, Lemma 2.1 & Lemma 2.3 (i)]. We thus have proved $(\mathcal{X}, f, \mathcal{L})$ is isotrivial.

As a consequence, when $(\mathcal{X}, f, \mathcal{L})$ is non-isotrivial, the image $\mathcal{Y}$ of $\Pi_{\widehat{\mathcal{Z}}_D} : \widehat{\mathcal{Z}}_D \to \mathcal{X}$ is a strict subvariety of $\mathcal{X}$ which is invariant by f. Moreover, as f has finite fibers, we have $f(\mathcal{Y}) = \mathcal{Y}$ and, up to taking the normalization $\mathsf{n} : \widehat{\mathcal{Y}} \to \mathcal{Y}$ of $\mathcal{Y}$, we can assume $\mathcal{Y}$ is normal. If $(\mathcal{Y}, f|_\mathcal{Y}, \mathcal{L}|_\mathcal{Y})$ is non-isotrivial, we end up with a contradiction applying the same strategy as above to $(\widehat{\mathcal{Y}}, \widehat{f}|_\mathcal{Y}, \mathsf{n}^*\mathcal{L}|_\mathcal{Y})$. In particular, $(\mathcal{Y}, f|_\mathcal{Y}, \mathcal{L}|_\mathcal{Y})$ is isotrivial (recall that, by our definition, isotriviality reads as a property on the normalization).

Finally, if $\mathcal{W}_1, \ldots, \mathcal{W}_\ell$ are all periodic irreducible components of $\mathcal{Z}_D^1$, we let $\mathcal{Y}_D$ be the union of the images of $\widehat{\mathcal{W}}_i$ under the natural projections $\Pi_{\widehat{\mathcal{W}}_i} : \widehat{\mathcal{W}}_i \to \mathcal{X}$ defined as above. By construction of the varieties $\mathcal{W}_i$, there exists an integer $N \geq 1$ such that any irreducible subvariety $\mathcal{C} \in \mathcal{D}$, with $\sup_n \deg_\mathcal{M}(f^n(\mathcal{C})) \leq D$, satisfies $f^N(\mathcal{C}) \subset \mathcal{Y}_D$. This concludes the proof. □

We now prove Lemma 4.9.

Proof of Lemma 4.9. — Take Λ to be the maximal regular part. Let S be the set of points $x \in \mathcal{X}_\Lambda$ such that $x \in (X_\lambda)_{\mathrm{reg}}$ and x is a repelling periodic point of f_λ, where $\lambda := \pi(x)$. Pick $\lambda_0 \in \Lambda$. Since repelling periodic points of f_{λ_0} are Zariski dense in X_{λ_0} by Proposition 2.3, we pick $z_0 \in S$ with $\pi(z_0) = \lambda_0$. Let $p \geq 1$ be its exact period. By the Implicit Function Theorem, there exist a neighborhood $U \subset \Lambda$ of λ_0 and a local section $z : U \to \mathcal{X}$ of $\pi : \mathcal{X} \to B$ such that

$$f_\lambda^p(z(\lambda)) = z(\lambda), \quad \lambda \in U,$$

and $z(\lambda) \in (X_\lambda)_{\mathrm{reg}}$ is repelling for f_λ for all $\lambda \in U$. By assumption, there exists $\mathcal{V} \in \mathcal{Z}_D$ such that $(z_0, \lambda_0) \in \mathcal{V}$. Lemma 4.6 states that $\mathcal{V} \cap \pi^{-1}(U) \supset z(U)$. By Proposition 2.3, S is a Zariski dense subset of $\mathcal{X}_\Lambda$ and we proved that $\Pi^{-1}\{x\}$ is a singleton for all $x \in S$. Whence there exist dense Zariski open sets $\mathcal{U} \subset \widehat{\mathcal{Z}}_D^\Lambda$ and $\mathcal{W} \subset \mathcal{X}_\Lambda$ such that $\Pi|_{\mathcal{U}} : \mathcal{U} \longrightarrow \mathcal{W}$ is a biholomorphism. Up to replacing Λ with $\Lambda \setminus G$ where G is a subvariety, we can assume $\mathcal{W} = \pi^{-1}(\Lambda) \setminus (\mathcal{X}' \cup \mathcal{X}'')$, where $\mathcal{X}'$ and $\mathcal{X}''$ are strict subvarieties of $\mathcal{X}$, flat over Λ, which are the respective Zariski closures of $\{x \in \mathcal{X}_\Lambda\,;\ 1 < \mathrm{Card}(\Pi^{-1}\{x\}) < +\infty\}$ and $\{x \in \mathcal{X}_\Lambda\,;\ \Pi^{-1}\{x\} \text{ is infinite}\}$.

First, we note that $f^{-1}(\mathcal{X}''_\Lambda) = \mathcal{X}''_\Lambda = f(\mathcal{X}''_\Lambda)$. Indeed, f is a finite morphism on $\mathcal{X}_\Lambda$ thus if $x \in \mathcal{X}_\Lambda \setminus \mathcal{X}''_\Lambda$, then any $\mathcal{V} \in \mathcal{Z}_D$ passing through x has a unique image by f and at most finitely many preimages, whence

$$f^{-1}(\mathcal{X}_\Lambda \setminus \mathcal{X}''_\Lambda) = \mathcal{X}_\Lambda \setminus \mathcal{X}''_\Lambda = f(\mathcal{X}_\Lambda \setminus \mathcal{X}''_\Lambda),$$

and thus $f^{-1}(\mathcal{X}''_\Lambda) = \mathcal{X}''_\Lambda = f(\mathcal{X}''_\Lambda)$. We thus have proved so far that the map $\Pi|_{\widehat{\mathcal{Z}}_D^\Lambda} : \widehat{\mathcal{Z}}_D^\Lambda \to \mathcal{X}_\Lambda$ restricts to $\mathcal{Z}^{(0)} := \Pi|_{\widehat{\mathcal{Z}}_D^\Lambda}^{-1}(\mathcal{X}_\Lambda \setminus \mathcal{X}''_\Lambda)$ as a finite birational morphism. Since $\mathcal{X}_\Lambda$ is normal, this implies $\Pi|_{\widehat{\mathcal{Z}}_D^\Lambda}|_{\mathcal{Z}^{(0)}}$ is an isomorphism from $\mathcal{Z}^{(0)}$ to $\mathcal{X}_\Lambda \setminus \mathcal{X}''_\Lambda$. In particular, $\mathcal{X}'_\Lambda = \varnothing$.

We finally prove that $\mathcal{X}''_\Lambda = \varnothing$. To do so, we first prove that if $\mathcal{V} \in \mathcal{Z}_D$ passes through some $x \in \mathcal{X}''_\Lambda$, then $\mathcal{V}_\Lambda \subset \mathcal{X}''_\Lambda$. Pick a general parameter $\lambda_0 \in \Lambda$. Assume that for a general parameter $\lambda \in \Lambda$, there is $x \in X_\lambda \cap \mathcal{X}''_\Lambda$ such that there is $\mathcal{V} \in \mathcal{Z}_D$ with $x \in \mathcal{V}$ and $\mathcal{V}_\Lambda \not\subset \mathcal{X}''_\Lambda$. Let

$$W_\lambda := \bigcup_{\mathcal{V}} X_{\lambda_0} \cap \mathcal{V},$$

where the union ranges over $\mathcal{V} \in \mathcal{Z}_D$ with $\mathcal{V} \cap X_\lambda \subset \mathcal{X}''_\Lambda$. By construction, the set W_λ is a Zariski closed totally invariant subset for f_{λ_0}. By hypothesis, it is non-empty. Therefore, to a general $\lambda \neq \lambda_0$, we can associate a non-empty closed subset W_λ. Since there is one and only one $\mathcal{V} \in \mathcal{Z}_D$ passing through a point in $\mathcal{W}$, for any $\lambda \neq \lambda'$, the sets W_λ and $W_{\lambda'}$ are distinct, whence f_{λ_0} has infinitely many distinct totally invariant Zariski closed subsets. This is a contradiction by [**DS3**, Theorem 1.47]. Assume now that $\dim \mathcal{X}''_\Lambda > \dim \mathcal{B}$, we are in the same situation as in the beginning of the proof but now for the family $(\mathcal{X}''_\Lambda, f|_{\mathcal{X}''_\Lambda}, \mathcal{L}|_{\mathcal{X}''_\Lambda})$. So the set $\mathcal{X}''_\Lambda$ is at worst a subvariety of dimension $\dim \mathcal{B}$ which is flat over Λ and totally invariant by f. But it is supposed to contain infinitely many distinct such subvarieties of $\mathcal{Z}_D$ which pass by points in $\mathcal{X}''_\Lambda$. This is impossible, whence $\mathcal{X}'' = \varnothing$. $\square$

4.4. Proof of the second part of Theorem A. — Now we can deduce the second part Theorem A from Theorem 4.7. Let us rephrase the second part of Theorem A.

Theorem 4.10. — *Let $(\mathcal{X}, f, \mathcal{L})$ be a non-isotrivial family of polarized endomorphisms. There exist a strict subvariety $\mathcal{Y}$ of $\mathcal{X}$ such that $\pi|_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{B}$ is surjective, whose irreducible components are periodic, and integers $N \geq 0$ and $D_0 \geq 1$ such that*

1. *$f(\mathcal{Y}) = \mathcal{Y}$ and for any periodic irreducible component $\mathcal{V}$ of $\mathcal{Y}$ with $f^n(\mathcal{V}) = \mathcal{V}$, the family of polarized endomorphisms $(\mathcal{V}, f^n|_{\mathcal{V}}, \mathcal{L}|_{\mathcal{V}})$ is isotrivial,*

2. *Pick any marked point $a : \mathcal{B} \dashrightarrow \mathcal{X}$ of π such that $((\mathcal{X}, f, \mathcal{L}), a)$ is stable. Then*
   (a) *if a_N is defined by $a_N(\lambda) := f_\lambda^N(a(\lambda))$, then a_N is a section of $\pi|_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{B}$,*
   (b) *if $\mathcal{C}_0(a)$ is the Zariski closure of the image of a regular part over which a is regular, we have $\deg_{\mathcal{M}}(\mathcal{C}_0(a)) \le D_0$.*

Proof. — By Theorem 4.7, for any integer D, if there is a rational section a of π with $\deg_{\mathcal{M}}(\mathcal{C}_n(a)) \le D$ for all $n \ge 0$, where $C_n(a)$ is the Zariski closure of $a_n(\Lambda)$ for some regular part over whic a is regular, there exist a subvariety $\mathcal{Y}_D$ of $\mathcal{X}$ such that $\pi|_{\mathcal{Y}_D} : \mathcal{Y}_D \to \mathcal{B}$ is surjective and an integer $N_D \ge 1$ such that

1. $f(\mathcal{Y}_D) = \mathcal{Y}_D$, hence all its irreducible components are periodic. For such a periodic irreducible component $\mathcal{V}$ of $\mathcal{Y}_D$ with $f^n(\mathcal{V}) = \mathcal{V}$, the family of polarized endomorphisms $(\mathcal{V}, f^n|_{\mathcal{V}}, \mathcal{L}|_{\mathcal{V}},)$ is isotrivial,
2. $\mathcal{Y}_D$ contains $\mathcal{C}_{N_D}(a)$, for any rational section $a : \mathcal{B} \dashrightarrow \mathcal{X}$with $\deg_{\mathcal{M}}(\mathcal{C}_n(a))) \le D$ for all $n \ge 0$, then $\mathcal{C}_{N_D}(a) \subset \mathcal{Y}_D$.

Moreover, if $D \le D'$, then $\mathcal{Y}_D \subset \mathcal{Y}_{D'}$. All there is left to prove is the existence of $D_0 \ge 1$ such that if $\deg_{\mathcal{M}}(\mathcal{C}_0(a)) > D_0$, then $((\mathcal{X}, f, \mathcal{L}), a)$ is not stable.

Let X be the generic fiber of π, $L := \mathcal{L}|_X$ and $f := f|_X$, so that $(\mathcal{X}, f, \mathcal{L})$ is a model for (X, f, L) with regular part Λ. Let

$$\widehat{h}_f := \lim_{n\to\infty} \frac{1}{d^n} h_{X,L} \circ f$$

be the canonical height function of (X, f, L) as above. Any rational section corresponds to some $a \in X_{\mathbf{K}}$, where $\mathbf{K} = \mathbb{C}(\mathcal{B})$ and

$$\begin{aligned} h_{X,L}(a) &= \Big(\mathcal{C}_0(a) \cdot c_1(\mathcal{L}) \cdot c_1(\pi^*\mathcal{N})^{\dim \mathcal{B}-1}\Big) = \deg_{\mathcal{M}}(\mathcal{C}_0(a)) - \Big(\mathcal{C}_0(a) \cdot c_1(\pi^*\mathcal{N})^{\dim \mathcal{B}}\Big) \\ &= \deg_{\mathcal{M}}(\mathcal{C}_0(a)) - 1, \end{aligned}$$

since $\mathcal{C}_0(a)$ is the Zariski closure of the image of a rational section of π. Indeed, the size of the field extension $\mathbf{K}(a)$ of $\mathbf{K}$ is exactly the intersection number $(\mathcal{C}_0(a) \cdot c_1(\pi^*\mathcal{N})^{\dim \mathcal{B}})$. Recall from Section 3.3 that $h_{X,L} = \widehat{h}_f + O(1)$, so there exists a constant $C > 0$ such that $\widehat{h}_f \ge h_{X,L} - C$, whence

$$\widehat{h}_f(a) \ge \deg_{\mathcal{M}}(\mathcal{C}_0(a)) - (C + 1).$$

If $\deg_{\mathcal{M}}(\mathcal{C}_0(a)) > C+1$, we thus find $\widehat{h}_f(a) > 0$. By Theorem 4.4, we deduce that $((\mathcal{X}, f, \mathcal{L}), a)$ is not stable, ending the proof. ☐

We now easily deduce Corollary 1.5 from Theorem 4.10, essentially following the argument of [**CGHX**, § 3.1].

Proof of Corollary 1.5. — Consider now the case where the polarized endomorphism (X, f, L) is defined over a function field $\mathbf{K} := \mathbf{k}(\mathcal{B})$ where $\mathbf{k}$ is a field of characteristic zero, where $\mathcal{B}$ is a normal projective $\mathbf{k}$-variety. There is an algebraically closed subfield $\mathbf{k}_0$ of $\mathbf{k}$ of finite transcendence degree over $\mathbb{Q}$ such that, via base change, we can assume $\mathcal{B}$ comes from a variety defined over $\mathbf{k}_0$. We can also assume (X, f, L) comes from a polarized endomorphism defined over $\mathbf{k}_0(\mathcal{B})$. Now, the field $\mathbf{k}_0$ can be embedded into $\mathbb{C}$, so that $\mathcal{B}$ induces a complex variety $\mathcal{B}_{\mathbb{C}}$ and (X, f, L) induces a polarized endomorphism $(X_{\mathbb{C}}, f_{\mathbb{C}}, L_{\mathbb{C}})$ defined over $\mathbf{K}' := \mathbb{C}(\mathcal{B}_{\mathbb{C}})$.

We now need to justify that if $\widehat{h}_f(z) = 0$ and if $z_{\mathbb{C}} \in X_{\mathbb{C}}(\mathbf{K}')$ is induced by z, then $\widehat{h}_{f_{\mathbb{C}}}(z_{\mathbb{C}}) = 0$. Let $\mathcal{N}$ be an ample line bundle on $\mathcal{B}$ and $\mathcal{N}_{\mathbb{C}}$ be the induced ample line bundle on $\mathcal{B}_{\mathbb{C}}$. Let $(\mathcal{X}, f, \mathcal{L})$ be a model of (X, f, L) and let $(\mathcal{X}_{\mathbb{C}}, f_{\mathbb{C}}, \mathcal{L}_{\mathbb{C}})$ be a model of $(X_{\mathbb{C}}, f_{\mathbb{C}}, L_{\mathbb{C}})$. Let finally $\mathfrak{z}$ (resp. $\mathfrak{z}_{\mathbb{C}}$) be the rational section of $\pi : \mathcal{X} \to \mathcal{B}$ induced by z (resp. of $\pi_{\mathbb{C}} : \mathcal{X}_{\mathbb{C}} \to \mathcal{B}_{\mathbb{C}}$ induced by $z_{\mathbb{C}}$). Denote as above by $\mathcal{C}_n(z)$ (resp. $\mathcal{C}_n(z_{\mathbb{C}})$) the Zariski closure of $\mathfrak{z}(\Lambda)$ in $\mathcal{X}$ (resp. of $\mathfrak{z}_{\mathbb{C}}(\Lambda_{\mathbb{C}})$ in $\mathcal{X}$). Then

$$\begin{aligned} h_{X,L}(f^n(z)) &= \left(\mathcal{C}_0(z)\cdot c_1(\mathcal{L})\cdot c_1(\pi^*\mathcal{N})^{\dim\mathcal{B}-1}\right) = \left(\mathcal{C}_0(z_{\mathbb{C}})\cdot c_1(\mathcal{L}_{\mathbb{C}})\cdot c_1(\pi^*\mathcal{N}_{\mathbb{C}})^{\dim\mathcal{B}_{\mathbb{C}}-1}\right)\\ &= h_{X_{\mathbb{C}},L_{\mathbb{C}}}(f_{\mathbb{C}}^n(z_{\mathbb{C}})), \end{aligned}$$

since the intersection numbers are invariant under extension of the field of constants. We thus find $\widehat{h}_f(z) = \widehat{h}_{f_{\mathbb{C}}}(z_{\mathbb{C}})$.

We now can assume $\mathcal{B}$ is a complex variety and $\mathbf{K} := \mathbb{C}(\mathcal{B})$. Let (X, f, L) be a polarized endomorphism over $\mathbf{K}$ and let $(\mathcal{X}, f, \mathcal{L})$ be a model of (X, f, L) over $\mathcal{B}$. Applying Theorem 4.10 gives the first point, since X is isomorphic to the generic fiber of $\pi : \mathcal{X} \to \mathcal{B}$, $L = \mathcal{L}|_X$ and $f = f|_X$. For the second point, we can remark that, up to changing model, we can assume a point $z \in X(\mathbf{K})$ induces a marked point $a : \mathcal{B} \dashrightarrow \mathcal{X}$. $\square$

Example 4.11 (The elementary Desboves family II). — We keep the notations of Example 3.1. The critical set of f_λ is the union of three lines L_j passing through $\rho_0 = [0 : 1 : 0]$ and $[1 : 0 : \alpha^j]$ with $\alpha^3 = 1$, for $j = 0, 1, 2$, each of them counted with multiplicity 2 together with the curve

$$\mathcal{C}'_\lambda := \{[x : y : z] \in \mathbb{P}^2 \ : \ -x^3 + z^3 + \lambda(4y^3 + x^3 + z^3) = 0\}.$$

Note that the lines L_j correspond to preimages of critical points of the Lattès map $g := f_\lambda|_Y$ by the fibration $p : \mathbb{P}^2 \dashrightarrow Y$ which semi-conjugates f_λ to g and whose fibers are the lines of the pencil $\mathcal{P}$.

We now make a base change for our family : Let $\mathbf{K} \supset \mathbb{C}(\lambda)$ be the splitting field of the equation $w^3 = (1+\lambda)/(1-\lambda)$. Then $\mathbf{K} = \mathbb{C}(\mathcal{B})$ for some smooth complex projective curve $\mathcal{B}$ and there exists a finite branched cover $\tau : \mathcal{B} \to \mathbb{P}^1$ of degree $[\mathbf{K} : \mathbb{C}(\lambda)]$ and changing base by τ allows to define the three intersection points of $Y = \{y = 0\}$ with $\mathcal{C}'_\lambda$ to define marked points $a_1, a_2, a_3 : \mathcal{B} \to \mathbb{P}^2$ where $a_i(\lambda) = [x_i(\lambda) : 0 : z_i(\lambda)]$ with $(1+\lambda)z_i(\lambda)^3 + (1-\lambda)x_i(\lambda)^3 = 0$. We have defined a family $(\mathbb{P}^2 \times \mathcal{B}, f_{\mathcal{B}}, \mathcal{O}_{\mathbb{P}^2}(1))$ with regular part $\Lambda := \tau^{-1}(\mathbb{C}^*)$. Let $\mathcal{Y} \subset \mathbb{P}^2 \times \mathcal{B}$ be the subvariety such that $f_{\mathcal{B}}(\mathcal{Y}) = \mathcal{Y}$ given by Theorem 4.10. As seen in Section 3.1,

$$Y \times \mathcal{B} \subset \mathcal{Y} \quad \text{and} \quad \mathcal{C} \times \mathcal{B} \subset \mathcal{Y},$$

but $X \times \mathcal{B} \not\subset \mathcal{Y}$ and $Z \times \mathcal{B} \not\subset \mathcal{Y}$. The subvariety $\mathcal{Y}$ may have other irreducible components, which would have to have period at least 2 under iteration of $f_{\mathcal{B}}$.

Any constant marked point $\alpha \in Y \cup \mathcal{C}$ is stable (but can have infinite orbit). Note also that $a_i(\lambda) \in \mathcal{Y}_\lambda$ for any $\lambda \in \mathbb{C}^*$. However a_i is *not* stable. Indeed, one has

$$\widehat{h}_f(a_i) = \int_{\mathbb{P}^2 \times \Lambda} [\Gamma_{a_i}] \wedge \widehat{T}_{f_{\mathcal{B}}} = \int_{\mathcal{B}} \tilde{a}_i{}^*(\mu_g) = \deg(\tilde{a}_i) > 0,$$

where $\tilde{a}_i(\lambda) = [x_i(\lambda) : z_i(\lambda)]$ for all $\lambda \in \mathbb{C}^*$ and μ_g is the Green measure of $g : Y \to Y$ which gives no mass to proper analytic sets. This example emphasizes that the conclusion that $\Gamma_{a_i} \subset \mathcal{Y}$ is not sufficient to have stability.

5. Applications

5.1. The case of Abelian varieties. — If A is an abelian variety defined over $\mathbf{K}$, L a symmetric ample line bundle on A and $[n]$ the morphism of multiplication by the integer n on A, the Néron-Tate height function $\widehat{h}_A : A(\bar{\mathbf{K}}) \to \mathbb{R}$ of A is defined by

$$\widehat{h}_A = \lim_{n\to\infty} \frac{1}{n^2} h_{A,L} \circ [n].$$

In particular, $\widehat{h}_A \circ [n] = n^2 \widehat{h}_A$ for any $n \geq 2$. Applying Corollary 1.5 to the polarized endomorphism $(A, [n], L)$ gives the next result due to Lang and Néron [**LN**]:

Theorem 5.1 (Lang-Néron). — *Let A be a non-isotrivial abelian variety defined over a function field of characteristic zero $\mathbf{K}$, L a symmetric ample line bundle on A and $\tau : A_0 \to A$ be the $\mathbf{K}/\mathbf{k}$-trace of A. Then, there exists an integer m such that for any $z \in A(\mathbf{K})$ with $\hat{h}_A(z) = 0$, we have $[m]z \in \tau(A_0(\mathbf{K}))$.*

Proof. — As before, we can assume $\mathbf{k} = \mathbb{C}$ without loss of generality. We apply Corollary 1.5 to the polarized endomorphism $(A, [2], L)$. We have a possibly reducible subvariety $Y = Y_1 \cup \cdots \cup Y_M$ where Y_i is periodic for all i, i.e. there exist integers $n_j > 0$ with $[2^{n_j}]Y_j = Y_j$, for all $1 \leq j \leq M$, which is isotrivial. Up to taking a finite extension of $\mathbf{K}$, we can assume $[2^{n_j}]$ has a fixed point p_j in Y_j. In particular, p_j is a torsion point of A and, if $A_j = Y_j - p_j$, then A_j is a subgroup of A, whence A_j is an abelian subvariety of A.

Note that, since A_j is isotrivial, there exist an abelian variety C_j defined over $\mathbb{C}$ and a morphism $\tau_j : C_j \to A$ such that $A_j = \tau_j(C_j)$. In particular, $A_j \subset \tau(A_0)$, which gives the sought statement. □

5.2. A conjecture of Kawaguchi and Silverman. — Finally, we remark here that Theorem A implies a conjecture of Kawaguchi and Silverman [**KS2**, Conjecture 6] in the case of polarized endomorphisms over the field $\mathbf{K}$ of rational functions of a projective variety over a field of characteristic zero.

Let X be a projective variety defined over a global $\mathbf{K}$ of characteristic zero (i.e. $\mathbf{K}$ is either a number field or a function field as above) and $f : X \dashrightarrow X$ a dominant rational map. Recall the quantities $\lambda_j(f)$ of f are defined in §2.2. Let $X_f(\bar{\mathbf{K}})$ be the set of points whose full forward orbit is well-defined. In this case, one can also define the *arithmetic degree* of a point $P \in X_f(\bar{\mathbf{K}})$ as

$$\alpha_f(P) := \lim_{n \to +\infty} \max\left(h_X(f^n(P)), 1\right)^{1/n},$$

where h_X is any given Weil height function $h_X : X(\bar{\mathbf{K}}) \to \mathbb{R}$, when the limit exists. Kawaguchi and Silverman's conjecture [**KS2**, Conjecture 6] says that:

1. the limit $\alpha_f(P)$ exists and is an algebraic integer for all $P \in X_f(\bar{\mathbf{K}})$,
2. the set $\{\alpha_f(Q) \,:\, Q \in X_f(\bar{\mathbf{K}})\}$ is finite,
3. if $P \in X_f(\bar{\mathbf{K}})$ has Zariski dense orbit, then $\alpha_f(P) = \lambda_1(f)$.

This conjecture is known, over number fields, in several cases including the case of polarized endomorphisms [**KS2**, Theorem 5], see also e.g. [**KS1, LS, MSS, Shi1, Mat**].

As a consequence of Theorem 4.7, we deduce this conjecture holds true for polarized endomorphisms over function fields of characteristic zero.

Theorem 5.2. — *Let $\mathbf{k}$ be a field of characteristic zero and $\mathcal{B}$ be a normal projective $\mathbf{k}$-variety. Let (X, f, L) be a non-isotrivial polarized endomorphism over $\mathbf{K} := \mathbf{k}(\mathcal{B})$ of degree d. Fix a point $P \in X(\bar{\mathbf{K}})$. Then*

1. *the limit $\alpha_f(P)$ exists and is an integer,*
2. *the set $\{\alpha_f(Q) \,:\, Q \in X(\bar{\mathbf{K}})\}$ coincides with $\{1, d\}$,*
3. *if $P \in X(\bar{\mathbf{K}})$ has Zariski dense orbit, we have $\alpha_f(P) = \lambda_1(f) = d$.*

Proof. — Let us first remark that, since (X, f, L) is polarized, we have $f^*L \simeq L^{\otimes d}$, so that

$$\left((f^n)^* c_1(L) \cdot c_1(L)^{\dim X - 1}\right) = \left(c_1(L)^{d^n} \cdot c_1(L)^{\dim X - 1}\right) = d^n \left(c_1(L)^{\dim X}\right),$$

which gives $\lambda_1(f) = d$.

We now take $\mathsf{x} \in X(\bar{\mathbf{K}})$. Let $\widehat{h}_f : X(\bar{\mathbf{K}}) \to \mathbb{R}_+$ be the canonical height function of f. Assume $\widehat{h}_f(\mathsf{x}) > 0$. Then

$$h_{X,L}(f^n(\mathsf{x})) = \widehat{h}_f(f^n(\mathsf{x})) + O(1) = d^n \widehat{h}_f(\mathsf{x}) + O(1),$$

so that $\alpha_f(\mathsf{x}) = d$, as sought. Note also that if $\widehat{h}_f(\mathsf{x}) = 0$, this implies

$$h_{X,L}(f^n(\mathsf{x})) = O(1),$$

so that $\alpha_f(\mathsf{x}) = 1$. We are thus left with proving that, if x has a Zariski dense orbit, then $\widehat{h}_f(\mathsf{x}) > 0$. Assume x has Zariski dense orbit, but $\widehat{h}_f(\mathsf{x}) = 0$. Let $\mathbf{K}'$ be the field of rational functions of some normal projective variety $\mathcal{B}'$ such that x is defined over $\mathbf{K}'$ and let $(\mathcal{X}, f, \mathcal{L})$ be a non-isotrivial model of (X, f, L) such that $\pi : \mathcal{X} \to \mathcal{B}'$ is a family of varieties, with regular part Λ. Up to changing model, to x, we can associate a marked point $x : \mathcal{B}' \dashrightarrow \mathcal{X}$ and a subvariety $\mathcal{C} \subset \mathcal{X}$ (which is the Zariski closure of $x(\Lambda)$ in $\mathcal{X}$ and which is flat over Λ). According to Theorem 4.4, we have

$$\sup_n \deg_{\mathcal{M}}(f^n(\mathcal{C})) = D < +\infty.$$

We now can apply Theorem 4.7, and we deduce the existence of $\mathcal{Y}_D$ such that $\mathcal{Z} := \bigcup_{n \geq 1} f^n(\mathcal{Y}_D)$ is a strict subvariety of $\mathcal{X}$ and $f^n(\mathcal{C}) \subset \mathcal{Z}$ for all $n \geq N$. If Z is the generic fiber of $\mathcal{Z} \to \mathcal{B}'$, we have $f^n(\mathsf{x}) \in Z$ for any $n \geq 1$, which is a contradiction. $\square$

We refer the reader to [**DGHLS**] for generalization of the above conjecture where Theorem B provides partial answers.

6. The Geometric Dynamical Bogomolov Conjecture

In the whole section, we let $\mathbf{k}$ be a field of characteristic 0, we also let $\mathcal{B}$ be a normal projective $\mathbf{k}$-variety and we let $\mathbf{K} := \mathbf{k}(\mathcal{B})$ be the field of rational functions of $\mathcal{B}$.

6.1. Motivation of the Conjecture. — As mentioned in the introduction, the Geometric Bogomolov conjecture has been proved in [**CGHX**]. Let us first state their result.

Theorem 6.1 ([CGHX]). — *Let A be a non-isotrivial abelian variety, L be a symmetric ample line bundle on A and $\widehat{h}_A$ be the Néron-Tate height of A relative to L. Let Z be an irreducible subvariety of $A_{\bar{\mathbf{K}}}$. Assume for all $\varepsilon > 0$, the set $Z_\varepsilon := \{x \in Z(\bar{\mathbf{K}}) : \widehat{h}_A(x) < \varepsilon\}$ is Zariski dense in Z. Then, there exist a torsion point $a \in A(\bar{\mathbf{K}})$, an abelian subvariety $C \subset A$ and a subvariety $W \subset A_0$ of the $\bar{\mathbf{K}}/\mathbf{k}$-trace (A_0, τ) of A such that*

$$Z = a + C + \tau(W \otimes_k \bar{\mathbf{K}}).$$

Let us now give a dynamical formulation of the conclusion of their result. Fix $n \geq 2$ and let $[n] : A \to A$ and $\widetilde{[n]} : A_0 \to A_0$ be the respective multiplication by n morphisms. Let $M \geq 0$ be such that $b := [n]^M(a)$ is periodic under iteration of $[n]$, i.e. there exists $k \geq 1$ with $[n]^k b = b$, and let

$$V := b + C + \tau(A_0 \otimes_{\mathbf{k}} \bar{\mathbf{K}}).$$

The variety V is fixed by $[n]^k$, i.e. $[n]^k V = V$ and, if $V_0 := \tau(A_0 \otimes_{\mathbf{k}} \bar{\mathbf{K}})$, there is a fibration $p : V \to \tau(A_0 \otimes_{\mathbf{k}} \bar{\mathbf{K}})$ which is invariant by $[n]^k$, i.e. such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "{[n]^k}"] \arrow[d, "p"'] & V \arrow[d, "p"] \\ V_0 \arrow[r, "{[n]^k|_{V_0}}"'] & V_0 \end{tikzcd}$$

and such that $(V_0, [n]^k|_{V_0}, L|_{V_0})$ is isotrivial. We thus may rephrase the conclusion as follows: There exist subvarieties $V, V_0 \subset A$ invariant by $[n]^k$ and an integer $N \geq 1$ and a surjective morphism $p : V \to V_0$ such that

1. $p \circ ([n]^k|_V) = ([n]^k|_{V_0}) \circ p$ and $(V_0, [n]|_{V_0}, L|_{V_0})$ is isotrivial,
2. $[n]^N(Z) = p^{-1}(W_0)$ where W_0 is an isotrivial subvariety of V_0.

In particular, Conjecture 1.9 in the Introduction is a generalization of the above dynamical formulation of the Geometric Bogomolov Conjecture to the case where (X, f, L) is a non-isotrivial polarized endomorphism defined over $\mathbf{K}$, X is normal and $Z \subset X_{\bar{\mathbf{K}}}$ is irreducible subvariety.

Recall that in the Introduction, we defined a subvariety $Z \subseteq X$ as *f-special* if there exist an integer k, a polarized endomorphism (X, Ψ, L) a subvariety Y with $Z \subseteq Y \subseteq X$ such that

- $f^k(Y) = Y = \Psi(Y)$;
- $f^k \circ \Psi = \Psi \circ f^k$ on Y;
- either Z is preperiodic under Ψ or Z comes from an isotrivial factor of (X, Ψ, L).

As explained in the Introduction, the existence of non-isotrivial families of CM abelian varieties in relative dimension at least 3 imposes this definition. However, the situation is simpler when studying abelian varieties A over function fields endowed with the map $[n]$: if Z is a $[n]$-special subvariety, we can take $Y = X$ and $\Psi = [n]^k$.

Recall that, by [**Fab, Gub1**], $\widehat{h}_f$ is induced by a $M_{\mathcal{B}}$-metric in the sense of Gubler [**Gub2**], and, that in this case, if $Z \subset X_{\bar{\mathbf{K}}}$ is an irreducible subvariety, we have the Zhang inequalities which imply the following are equivalent by [**Gub2**, Corollary 4.4]:

- the height of Z is zero, i.e. $\widehat{h}_f(Z) = 0$,
- the essential minimum of the metrization vanishes:
$$e_1 := \sup_Y \inf_{x \in Z(\bar{\mathbf{K}}) \setminus Y} \widehat{h}_f(x) = 0,$$
where Y ranges over all hypersurfaces of Z,
- for any $\varepsilon > 0$ the set Z_ε is Zariski-dense in Z.

In particular, it is sufficient to characterize subvarieties Z with $\widehat{h}_f(Z) = 0$.

6.2. Stable fibers of an invariant fibration. — We want here to explore basic properties of subvarieties with height 0. The first thing we want to do is to relate, when f preserves a fibration, the height of the fiber over y to the height of y.

Let us fix the context before we give a more precise statement: we let $\mathbf{K} := \mathbb{C}(\mathcal{B})$ be the field of rational function of a complex normal projective variety $\mathcal{B}$. When X and Y are projective varieties over $\mathbf{K}$ and $p : X \dashrightarrow Y$ is a dominant rational map defined over $\mathbf{K}$, for any irreducible subvariety $W \subset Y_{\bar{\mathbf{K}}}$, we let X_W be the "fiber" $p^{-1}(W)$ of p over W.

Lemma 6.2. — *Let (X, f, L) be a polarized endomorphism over $\mathbf{K}$ of degree $d > 1$. Assume there exist a polarized endomorphism (Y, g, E) over $\mathbf{K}$ of degree d with $\dim Y < \dim X$, and a dominant rational map $p : X \dashrightarrow Y$ defined over $\mathbf{K}$ with $p \circ f = g \circ p$. For any subvariety $W \subset Y_{\bar{\mathbf{K}}}$, we have*

$$\widehat{h}_f(X_W) = \widehat{h}_g(W).$$

In particular, $\widehat{h}_f(X_W) = 0$ if and only if $\widehat{h}_g(W) = 0$.

Proof. — Let $(\mathcal{X}, \mathfrak{f}, \mathcal{L})$ and $(\mathcal{Y}, \mathfrak{g}, \mathcal{E})$ be respective models for (X, f, L) and (Y, g, E) with common regular part Λ, and let $\mathfrak{p} : \mathcal{X} \dashrightarrow \mathcal{Y}$ be a model for $p : X \dashrightarrow Y$. For any subvariety $W \subset Y_{\bar{\mathbf{K}}}$, we let $\mathcal{W}$ be the Zariski closure of W in $\mathcal{Y}$. Recall that we have set $X_W := \mathbb{P}^{-1}(W)$. We let $\mathcal{X}_W := \mathfrak{p}^{-1}(\mathcal{W})$ and $\ell := \dim X - \dim Y > 0$ and $s := \dim W$. We remark that $\widehat{T}_{\mathfrak{g}}^{s+1} = \mathfrak{p}_*\left(\widehat{T}_{\mathfrak{f}}^{\ell+s+1}\right)$, so that the projection formula and Theorem B give

$$\widehat{h}_g(W) = \widehat{h}_f(X_W),$$

where we used that $\pi_{\mathcal{X}} \circ \mathfrak{p} = \pi_{\mathcal{Y}}$. □

Consider a trivial family $(g(z), \lambda)$ on $\mathbb{P}^1 \times \Lambda$ and a marked point $a : \Lambda \to \mathbb{P}^1$. Then, it is easy to see that the graph of a is stable if and only if a is constant (if not, $a(\lambda)$ is a repelling periodic point of g at some λ). More generally, when $p : X \dashrightarrow \mathbb{P}^1$ and $g : \mathbb{P}^1 \to \mathbb{P}^1$ is isotrivial, there exists an automorphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\bar{\mathbf{K}}$ such that $\phi^{-1} \circ g \circ \phi$ is defined over $\mathbb{C}$ and we say that X_z is an *isotrivial fiber* of p if $X_z = p^{-1}\{z\}$ where $\phi^{-1}(z)$ is defined over $\mathbb{C}$.

As a particular case of Lemma 6.2, we have

Corollary 6.3. — *Let (X, f, L) be a non-isotrivial polarized endomorphism over $\mathbf{K}$ of degree $d > 1$. Assume there exist a polarized endomorphism $(\mathbb{P}^1, g, \mathcal{O}_{\mathbb{P}^1}(1))$ over $\mathbf{K}$ of degree d, and a dominant rational map $p : X \dashrightarrow \mathbb{P}^1$ defined over $\mathbf{K}$ with $p \circ f = g \circ p$. For any $y \in \mathbb{P}^1(\mathbf{K})$, let Z_y be the fiber $p^{-1}\{y\}$ of p.*

1. *If $(\mathbb{P}^1, g, \mathcal{O}_{\mathbb{P}^1}(1))$ is non-isotrivial, then $\widehat{h}_f(Z_y) = 0$ if and only if Z_y is preperiodic under iteration of f,*
2. *if g is isotrivial, then $\widehat{h}_f(Z_y) = 0$ if and only if Z_y is an isotrivial fiber of p.*

Proof. — Assume first g is non-isotrivial. According to Lemma 6.2, we have $\widehat{h}_g(Z_y) = 0$ if and only if $\widehat{h}_g(y) = 0$. Since $(\mathbb{P}^1, g, E)$ is non-isotrivial, $\widehat{h}_g(y) = 0$ if and only if y is g-preperiodic. As $f(Z_y) = Z_{g(y)}$, the variety Z_y has to be preperiodic.

Assume now g is isotrivial. Then there exist a finite extension $\mathbf{K}'$ of $\mathbf{K}$ and an affine automorphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1$, defined over $\mathbf{K}'$ such that $g_0 := \phi^{-1} \circ g \circ \phi$ is defined over $\mathbb{C}$. Let $\rho : \mathcal{B}' \to \mathcal{B}$ be a finite branched cover with $\mathbf{K}' = \mathbb{C}(\mathcal{B}')$ and let $(\mathbb{P}^1 \times \mathcal{B}', \mathfrak{g}, \mathcal{O}_{\mathbb{P}^1}(1))$ be a model for $(\mathbb{P}^1, g, \mathcal{O}_{\mathbb{P}^1}(1))$ over $\mathcal{B}'$. Let $\sigma : \mathcal{B}' \dashrightarrow \mathbb{P}^1$ be the rational map induced by $\phi^{-1}(y)$ and $\mathcal{Y}$ be the Zariski closure of y in $\mathbb{P}^1 \times \mathcal{B}'$. If $\Phi : \mathbb{P}^1 \times \mathcal{B}' \dashrightarrow \mathbb{P}^1 \times \mathcal{B}'$ is a model of ϕ and Λ is a common regular part for all above models, then

$$\begin{aligned}\widehat{h}_g(y) &= \int_{\mathbb{P}^1 \times \Lambda} [\mathcal{Y}] \wedge \widehat{T}_{\mathfrak{g}} \wedge (\pi_{\mathcal{B}'}^* \omega_{\mathcal{B}'})^{\dim \mathcal{B}' - 1} = \int_{\mathbb{P}^1 \times \Lambda} [\mathcal{Y}] \wedge \Phi_*\left(\pi_{\mathbb{P}^1}^*(\mu_{g_0})\right) \wedge (\pi_{\mathcal{B}'}^* \omega_{\mathcal{B}'})^{\dim \mathcal{B}' - 1} \\ &= \int_{\mathbb{P}^1 \times \Lambda} [\Gamma_\sigma] \wedge \pi_{\mathbb{P}^1}^*(\mu_{g_0}) \wedge (\pi_{\mathcal{B}'}^* \omega_{\mathcal{B}'})^{\dim \mathcal{B}' - 1} = \int_\Lambda \sigma^*(\mu_{g_0}) \wedge \omega_{\mathcal{B}'}^{\dim \mathcal{B}' - 1} = \deg_{\eta}(\sigma).\end{aligned}$$

In particular, $\widehat{h}_g(y) = 0$ if and only if σ is constant, i.e. $\phi^{-1}(y) \in \mathbb{C}$. This concludes the proof. □

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