

THE GEOMETRIC DYNAMICAL NORTHCOTT PROPERTY FOR REGULAR POLYNOMIAL AUTOMORPHISMS OF THE AFFINE PLANE

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ABSTRACT. We establish the finiteness of periodic points, that we called *Geometric Dynamical Northcott Property*, for regular polynomials automorphisms of the affine plane over a function field \mathbf{K} of characteristic zero, improving results of Ingram.

For that, we show that when \mathbf{K} is the field of rational functions of a smooth complex projective curve, the canonical height of a subvariety is the mass of an appropriate bifurcation current and that a marked point is stable if and only if its canonical height is zero. We then establish the Geometric Dynamical Northcott Property using a similarity argument.

1. INTRODUCTION

Let k be a field of characteristic zero, \mathcal{B} a normal projective k -variety, and let $\mathbf{K} := k(\mathcal{B})$ be its field of rational functions. A regular plane automorphism over the function field \mathbf{K} is a polynomial automorphism of the affine plane $\mathbb{A}^2(\mathbf{K})$ such that the unique indeterminacy point I^+ of its extension to $\mathbb{P}^2(\mathbf{K})$ is distinct to the unique indeterminacy point I^- of the extension of f^{-1} to $\mathbb{P}^2(\mathbf{K})$.

Let $h : \mathbb{P}^2(\bar{\mathbf{K}}) \rightarrow \mathbb{R}_+$ be the standard height function on $\mathbb{P}^2(\bar{\mathbf{K}})$, i.e. the height function $h = h_{\mathbb{P}^2, L}$ associated with the ample linebundle $L := \mathcal{O}_{\mathbb{P}^2}(1)$. Following Kawaguchi [K] in the number field case, one can define three different canonical heights for f :

$$\widehat{h}_f^+ := \lim_{n \rightarrow +\infty} d^{-n} h \circ f^n, \quad \widehat{h}_f^- := \lim_{n \rightarrow +\infty} d^{-n} h \circ f^{-n} \quad \text{and} \quad \widehat{h}_f := \widehat{h}_f^+ + \widehat{h}_f^-,$$

where d is the common degree of f and f^{-1} . The height function \widehat{h}_f^+ (resp. \widehat{h}_f^-) detects the arithmetic complexity of the forward orbit (resp. of the backward orbit) of a point in $\mathbb{A}^2(\bar{\mathbf{K}})$.

A particularly interesting case of regular plane automorphisms is Hénon maps, i.e. maps of the form $f(x, y) = (ay, x + p(y))$ with $a \in \mathbf{K}^*$ and $p(x) \in \mathbf{K}[x]$. In that setting, Ingram proved the following ([I, Theorem 1.2])

Theorem 1 (Ingram). *Let k be any field and let \mathbf{K} be the field of rational functions of a smooth projective k -variety. Let $f(x, y) = (y, x + p(y))$ for $p(x) \in \mathbf{K}[x]$ of degree at least 2. Then either f is isotrivial or else the set of elements $z \in \mathbb{A}^2(\mathbf{K})$ with $\widehat{h}_f(z) = 0$, is finite, bounded in size in terms of the number of places of bad reduction for f . In particular, if f is not isotrivial, then $\widehat{h}_f(z) = 0$ if and only if z is periodic for f .*

Both authors are partially supported by the ANR grant Fatou ANR-17-CE40-0002-01.

Keywords: regular plane polynomial automorphisms, canonical height, algebraic family of rational maps, arithmetic characterizations of stability.

Mathematics Subject Classification (2010): 37P15, 37P30, 37F45.

The map f is isotrivial if, after a suitable change of coordinates, the coefficients of f are constant, i.e. belong to k . Ingram ask whether one can prove a similar statement for $f(x, y) = (ay, x + p(y))$ with $a \in \mathbf{K}^*$. This is the purpose of this article in the case where k has characteristic zero (Ingram result allows positive characteristic). More precisely,

- (1) We generalize the above statement to any regular polynomial automorphism $f : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ defined over a function field of characteristic 0. More precisely, we establish the *geometric dynamical Northcott Property*: if the map f is not isotrivial, then $\widehat{h}_f(z) = 0$ if and only if z is periodic for f and they are only finitely many such points.
- (2) We replace the hypothesis that $\widehat{h}_f(P) = 0$ with the a priori weaker one $\widehat{h}_f^+(P) = 0$.
- (3) We express, in the case where \mathbf{K} is the field of rational functions of a smooth complex projective curve, the canonical height of a subvariety as the mass of an appropriate bifurcation current. Then, we show a marked point is stable, in the sense of complex dynamics, if and only if its canonical height is zero.

Consider a regular plane automorphism f over the function field \mathbf{K} . As k is a field of characteristic zero, up to replacing it with an algebraic extension, we can assume there exists an algebraically closed subfield k' of k such that f is defined over k' and the transcendental degree of k' over $\overline{\mathbb{Q}}$ is finite. In particular, k' can be embedded in the field $\mathbb{C}(\mathcal{A})$ of rational functions of a normal projective complex variety \mathcal{A} , so that we can assume $\mathbf{K} = \mathbb{C}(\mathcal{A} \times \mathcal{B})$. Hence, we can assume f is defined over $\mathbb{C}(\mathcal{B})$ for \mathcal{B} a complex normal projective variety. We thus restrict to the case $k = \mathbb{C}$ in the rest of the paper and $\mathbf{K} = \mathbb{C}(\mathcal{B})$. This will enable the use of complex methods.

To a regular automorphism $f : \mathbb{A}_{\mathbf{K}}^2 \rightarrow \mathbb{A}_{\mathbf{K}}^2$, we can associate a *model* over \mathcal{B} where $\mathbf{K} = \mathbb{C}(\mathcal{B})$, i.e. a birational map $f : \mathbb{A}^2(\mathbb{C}) \times \mathcal{B} \dashrightarrow (f_\lambda(z), \lambda) \in \mathbb{A}^2(\mathbb{C}) \times \mathcal{B}$ such that $\pi \circ f = \pi$, where $\pi : \mathbb{A}^2(\mathbb{C}) \times \mathcal{B} \rightarrow \mathcal{B}$ is the canonical projection, and such that there exists a Zariski open subset $\Lambda \subset \mathcal{B}$ for which f restricts to $\mathbb{A}^2(\mathbb{C}) \times \Lambda$ as an automorphism and such that $f_\lambda := f|_{\mathbb{A}^2(\mathbb{C}) \times \{\lambda\}}$ is a complex regular polynomial automorphism for any $\lambda \in \Lambda$. The map f can be identified with the restriction of f to the generic fiber of π . The open set Λ is the *regular part* of the family f . To any point $z \in \mathbb{A}^2(\mathbf{K})$ one can also associate a rational map $\zeta : \mathcal{B} \rightarrow \mathbb{C}$ such that ζ is defined on Λ . Such ζ is called a *marked point*. We say that ζ is *stable* if the sequence of iterates $\lambda \mapsto f_\lambda^n(\zeta(\lambda))$ is normal on compact subsets of Λ (see Remark 3.2 below).

Finally, we say that a regular polynomial automorphism $f : \mathbb{A}_{\mathbf{K}}^2 \rightarrow \mathbb{A}_{\mathbf{K}}^2$ is *isotrivial* if there exists an affine automorphism $\varphi : \mathbb{A}_{\mathbf{K}}^2 \rightarrow \mathbb{A}_{\mathbf{K}}^2$ such that $\varphi^{-1} \circ f \circ \varphi$ is defined over \mathbb{C} , or equivalently if for any model $f : \mathbb{C}^2 \times \mathcal{B} \dashrightarrow \mathbb{C}^2 \times \mathcal{B}$ with regular part Λ and for any $\lambda, \lambda' \in \Lambda$, there is an affine automorphism $\varphi_{\lambda, \lambda'} : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ such that $\varphi_{\lambda, \lambda'}^{-1} \circ f_\lambda \circ \varphi_{\lambda, \lambda'} = f_{\lambda'}$.

Our result can then be stated as

Main Theorem. *Let $f : \mathbb{C}^2 \times \mathcal{B} \dashrightarrow \mathbb{C}^2 \times \mathcal{B}$ be a non-isotrivial algebraic family of regular polynomial automorphisms of degree $d \geq 2$ parametrized by a complex projective variety \mathcal{B} with regular part Λ and let $f : \mathbb{A}_{\mathbf{K}}^2 \rightarrow \mathbb{A}_{\mathbf{K}}^2$ be the induced regular automorphism over the field $\mathbf{K} = \mathbb{C}(\mathcal{B})$ of rational functions of \mathcal{B} . Then*

- (1) *for any point $z \in \mathbb{A}^2(\mathbf{K})$ with corresponding rational map $\zeta : \mathcal{B} \dashrightarrow \mathbb{P}^2$,*
- (*) (f, ζ) *is stable* $\iff \widehat{h}_f(z) = 0 \iff \widehat{h}_f^+(z) = 0 \iff z$ *is periodic.*
- (2) *the set of marked points $\{\zeta$ such that (f, ζ) is stable* *is a finite set. In particular, a stable marked point is stably periodic.*

This generalizes Ingram's Theorem 1:

Corollary. *Let k be a field of characteristic zero and \mathbf{K} be the field of rational functions of a projective k -variety. Let $f : \mathbb{A}_{\mathbf{K}}^2 \rightarrow \mathbb{A}_{\mathbf{K}}^2$ be a regular polynomial automorphism of degree $d \geq 2$. Then either f is isotrivial or else the set of elements $P \in \mathbb{A}^2(\mathbf{K})$ with $\widehat{h}_f(P) = 0$, is finite*

In particular, if f is not isotrivial, then $\widehat{h}_f(P) = 0$ if and only if P is periodic for f .

For Hénon maps over number fields, the finiteness of periodic points is due to Silverman [S]. Constructing the canonical heights, Kawaguchi [K] proved this result over number fields for regular polynomial automorphisms using the classical Northcott property (see also Lee [L] for an alternate construction). Over a function field, such result was established for polynomials of \mathbb{A}^1 by Benedetto [Be] and rational maps of \mathbb{P}^1 by Baker [Ba] and DeMarco [D]. In higher dimension, Chatzidakis and Hrushovski gave a model-theoretic version of the statement for polarized endomorphisms in [CH]. Finally, in [GV], we extended the Northcott property to any polarized endomorphisms, giving a similar statement as that of the Main Theorem.

In order to prove the Main Theorem, we adapt the strategy of the proof of [GV, Theorem A] to the case of regular polynomial automorphisms. New difficulties appear since we need to deal with indeterminacy points and saddle periodic points instead of repelling periodic points.

Note that,

- if f is non-isotrivial and $\lambda_0 \in \Lambda$ is fixed, then the set of parameters λ such that f_λ is conjugated to f_{λ_0} is a closed subvariety of Λ ,
- the restriction of a stable marked point to a subvariety \mathcal{B}' of \mathcal{B} is still stable,
- If \mathcal{B}' is a subvariety of \mathcal{B} and $z \in \mathbb{C}(\mathcal{B})$ has height zero, then it defines a point in $\mathbb{C}(\mathcal{B}')$ whose height is again 0 by Bézout. Similarly, if for all \mathcal{B}' , the corresponding point in $\mathbb{C}(\mathcal{B}')$ has height 0, then so does z .

In particular, we can reduce to the case where $\dim \mathcal{B} = 1$ by a slicing argument. We thus restrict to the case where \mathbf{K} is the field of rational functions of a smooth complex projective curve \mathcal{B} . Finally, up to taking a branched cover of \mathcal{B} , conjugating by a suitable affine automorphism $\Phi \in \text{Aut}(\mathbb{A}_{\mathbf{K}}^2)$ and reducing Λ , we can assume that the indeterminacy point $I(f_\lambda)$ of f_λ is $[1 : 0 : 0]$ and the indeterminacy point $I(f_\lambda^{-1})$ of f_λ^{-1} is $[0 : 1 : 0]$ for every $\lambda \in \Lambda$.

Pick a point $z \in \mathbb{A}^2(\mathbf{K})$ and let \mathcal{Z}_n be the irreducible subvariety of $\mathbb{P}^2(\mathbb{C}) \times \mathcal{B}$ induced by $f^n(z)$. Using Kawaguchi's comparison result on heights [K], we show that there exists some $B_0 > 0$, independent of z , such that $\widehat{h}_f(z) = 0$ implies $\deg_m(\mathcal{Z}_n) \leq B_0$ for all $n \in \mathbb{Z}$, so they are only finitely many degrees to consider.

Then, we show that (local) stability is equivalent to having zero height for regular polynomial automorphisms: we express $\widehat{h}_f^+(z)$ in term of an appropriate bifurcation current and show that $\widehat{h}_f^+(z) = 0$ if and only if the sequence $(\deg_m(\mathcal{Z}_n))_{n \geq 0}$ is bounded by some constant $D > 0$ independent of n , hence forward stability is in fact a global notion (Propositions 6 and 7). For that, we prove a delicate degeneracy estimate of the Green function to deal with the indeterminacy point, this allows us to construct a DSH cut-off function (DSH functions, introduced by Dinh-Sibony [DS], take into account the complex structure whereas C^2 functions do not). Then we show that if \mathcal{Z} is forward stable, then it is periodic (note that in [I], one does not relate zero height with stability).

Finally, we give an application to a conjecture of Kawaguchi and Silverman in the case of regular polynomial automorphism.

2. ALGEBRAIC DYNAMICAL PAIRS OF REGULAR POLYNOMIAL AUTOMORPHISM TYPE

2.1. Definition and first properties. Let \mathcal{B} be a smooth projective complex curve and $\Lambda \subset \mathcal{B}$ a Zariski open subset. We let $f : \mathbb{C}^2 \times \Lambda \rightarrow \mathbb{C}^2 \times \Lambda$ be an algebraic family of regular polynomial automorphisms of \mathbb{C}^2 . For each $\lambda \in \Lambda$, $f_\lambda(x, y) = (p_\lambda(x, y), q_\lambda(x, y))$ where p_λ, q_λ are polynomials in (x, y) that depend holomorphically on λ with $\max \deg(p_\lambda, q_\lambda) = d$ independent of Λ (up to restricting Λ). We assume that the map f_λ extends as a birational map $f_\lambda : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with

- (1) the only indeterminacy point I^+ of f_λ is $I^+ = [1 : 0 : 0]$,
- (2) the only indeterminacy point I^- of f_λ^{-1} is $I^- = [0 : 1 : 0]$.

We call such f an *algebraic family of regular polynomial automorphisms*. From our normalization, we see that $\deg p_\lambda(x, y) < \deg q_\lambda(x, y) = \deg_y q_\lambda(x, y)$ and that I^- (resp. I^+) is a super-attracting fixed point for f_λ (resp. f_λ^{-1}).

A classical example is given by Hénon maps:

$$f(x, y, \lambda) = (a(\lambda)y, x + p_\lambda(y), \lambda)$$

where $p : \mathbb{C} \times \Lambda \rightarrow \mathbb{C}$ is an algebraic family of degree $d > 1$ polynomials in one complex variable parametrized by the quasi-projective variety Λ with $p_\lambda(y) = p(y, \lambda)$, $a \in \mathbb{C}[\Lambda]^*$ and the support of $\text{div}(a)$ is contained in the finite set $\mathcal{B} \setminus \Lambda$.

We say that the family f is *isotrivial* if there exist a finite branch cover $\rho : \Lambda' \rightarrow \Lambda$ and an algebraic family of invertible affine maps $\psi : \mathbb{C}^2 \times \Lambda' \rightarrow \mathbb{C}^2$ such that, writing $\psi((x, y), \lambda) := (\psi_\lambda(x, y), \lambda)$, there exists $\lambda_0 \in \Lambda$ such that for all $\lambda \in \Lambda'$,

$$\psi_\lambda^{-1} \circ f_{\rho(\lambda)} \circ \psi_\lambda = f_{\lambda_0}.$$

Let us recall some useful facts on regular polynomial automorphisms ([BS, BLS1, BLS2]). In what follow, a (p, p) -current is a current of bidegree (p, p) . Such objects are powerful tools in complex dynamics.

Definition 1. *The fibered Green current of f is the positive closed $(1, 1)$ -current \widehat{T}_f on $\mathbb{P}^2 \times \Lambda$ defined by*

$$\widehat{T}_f := \lim_{n \rightarrow +\infty} \frac{1}{d^n} (f^n)^*(\widehat{\omega}).$$

It is known that the convergence holds, that $f^*\widehat{T}_f := d\widehat{T}_f$ and that, for any $\lambda \in \Lambda$, the slice $\widehat{T}_f|_{\mathbb{P}^2 \times \{\lambda\}}$ of \widehat{T}_f is the forward Green current $T_{f_\lambda}^+$ of f_λ .

By definition, $f^{-1} : \mathbb{C}^2 \times \Lambda \rightarrow \mathbb{C}^2 \times \Lambda$ is also a family of regular polynomial automorphisms so we can similarly defined the backward Green current $\widehat{T}_{f^{-1}}$ as the fibered Green current of f^{-1} . Then, the current $\widehat{T}_f \wedge \widehat{T}_{f^{-1}}$ is well defined and, for any $\lambda \in \Lambda$, the slice $(\widehat{T}_f \wedge \widehat{T}_{f^{-1}})|_{\mathbb{P}^2 \times \{\lambda\}}$ is the unique maximal entropy measure μ_{f_λ} of f_λ . As the measure μ_{f_λ} gives no mass to analytic sets and is equidistributed by saddle points, it follows that saddle points are Zariski dense in \mathbb{C}^2 . So the set of points of the form $((x_0, y_0), \lambda_0)$ such that (x_0, y_0) is a saddle periodic point of f_{λ_0} is Zariski dense in $\mathbb{C}^2 \times \Lambda$.

Let $\pi : \mathbb{P}^2 \times \mathcal{B} \rightarrow \mathcal{B}$ be the projection onto the first coordinate.

Definition 2. *Let $\mathcal{Z} \subset \mathbb{P}^2 \times \mathcal{B}$ be an irreducible algebraic curve. We say that $(\Lambda, f, \mathcal{Z})$ is an algebraic dynamical pair of regular automorphism-type if $\pi|_{\mathcal{Z}} : \mathcal{Z} \rightarrow \mathcal{B}$ is a flat morphism and such that $\mathcal{Z} \cap (\mathbb{P}^2 \times \Lambda) \subset \mathbb{C}^2 \times \Lambda$.*

2.2. A degeneration lemma. We here prove the following degeneration Lemma in the spirit of [GV, Lemma 12], which is crucial in what follows. We use ideas of [DTV, Lemma 3.2.4.]. For that, we may view Λ as an affine curve: let $F := \mathcal{B} \setminus \Lambda$ and D be a very ample divisor supported by F . This defines an embedding $\iota_1 : \mathcal{B} \hookrightarrow \mathbb{P}^M$ with $\iota_1^{-1}(\mathbb{A}^M(\mathbb{C})) = \Lambda$. In what follows, λ denotes the affine coordinate $\iota(\lambda)$ on Λ .

Lemma 2. *There exist constants $C_1, C_2, C_3 > 0$ such that for all $n \geq 1$, we have*

$$\frac{1}{d^n} (f^n)^*(\widehat{\omega}) - \widehat{T}_f = dd^c \phi_n, \quad \text{on } \mathbb{P}^2 \times \Lambda,$$

with $|\phi_n(x, y, \lambda)| \leq d^{-n}(C_1 \log^+ |\lambda| + C_2 \log^+ 1/\|d((x, y), I^+)\| + C_3)$, for all $\lambda \in \Lambda$.

Proof. Pick any boundary point λ_* of Λ in \mathcal{B} and a punctured disk D^* centered at λ_* . Let t be a local coordinate in D^* centered at λ_* (i.e. $t = 0$ at λ_*). We first show that

$$(1) \quad \frac{1}{d} f^*(\widehat{\omega}) - \widehat{\omega} = dd^c \phi_1, \quad \text{on } \mathbb{P}^2 \times D^*,$$

with $|\phi_1(x, y, t)| \leq C_1 \log |t|^{-1} + C_2 \log d((x, y), I^+)^{-1} + C_3$, where the constants C_i do not depend on $t \in D^*$.

Observe for that

$$\frac{1}{d} f^*(\widehat{\omega}) - \widehat{\omega} = \frac{1}{2d} dd^c \log \frac{|p_t(x, y)|^2 + |q_t(x, y)|^2 + 1}{(|x|^2 + |y|^2 + 1)^d} \quad \text{on } \mathbb{P}^2 \times D^*.$$

Let $F_t(x, y, w) = (z^d p_t(x/w, y/w), w^d q_t(x/w, y/w), w^d)$ be the homogeneous lift of f_t to \mathbb{C}^3 and write $P := (x, y, w)$. Then, on $\mathbb{P}^2 \times D^*$, we have

$$\frac{1}{d} f^*(\widehat{\omega}) - \widehat{\omega} = \frac{1}{2d} dd^c \log \frac{|F_t|^2}{|P|^{2d}}.$$

The qpsH function $\phi_1 := (2d)^{-1} \log |F_t|^2 / |P|^{2d}$ is well defined on $\mathbb{C}^3 \times D^*$. Take some $\varepsilon > 0$ and consider the open set $U := \{1 - \varepsilon < |P| < 1 + \varepsilon\} \times D^*$. Multiplying by t^k for k large enough and shrinking D^* if necessary, we have

$$\phi_1 = c \log |t|^{-1} + d^{-1} \log |F'_t(P)|$$

where $F'_t(P)$ depends holomorphically on (P, t) and is equal to $(0, 0, 0) \in \mathbb{C}^3$ exactly when $P \in \pi^{-1}(I^+)$ (where π denotes the projection $\pi : \mathbb{C}^3 \setminus \{0\} \rightarrow \mathbb{P}^2$) or when $t = 0$. Using Łojasiewicz Theorem [Ł, Chapter IV, Proposition p.243], that provides us with two constants $\alpha > 0$ and $C > 0$ such that on $|P| = 1$ we have:

$$|F'_t(P)| \geq C(\min(\text{dist}(P, \pi^{-1}(I^+)), \text{dist}(t, 0)))^\alpha.$$

So reducing D^* , we have a constant β such that

$$\log |F'_t(P)| \geq \alpha \log |t| + \alpha \log \text{dist}(P, \pi^{-1}(I^+)) + \beta.$$

Since the function $d^{-1} \log |F'_t(P)|$ is upper semi-continuous, it is bounded from above in U (up to reducing D^*). Now from the fact that the projection $\pi : \mathbb{C}^3 \setminus \{0\} \rightarrow \mathbb{P}^2$ is Lipschitz in $|P| = 1$ and the above bound, we get constants $A > 0, B, c$ such that:

$$c \log |t|^{-1} + B \geq \phi_1 \geq c \log |t|^{-1} + A \log \text{dist}(\cdot, I^+).$$

In particular, the assertion (1) holds.

Let us now prove the inequality of the lemma in $\mathbb{P}^2 \times D^*$ which will be sufficient by a covering argument (changing the coordinates might change the constants C_1, C_2, C_3).

Observe first that we can set $\phi_n := \sum_{k \geq n} \frac{1}{d^k} \phi_1 \circ f^k$. So that, summing over k , it suffices to check that for any integer k , we have

$$|\phi_1 \circ f^k| \leq (C_1 \log |t|^{-1} + C_2 \log d((x, y), I^+)^{-1} + C_3)$$

where C_1, C_2 , and C_3 do not depend on k . For that, we remark first that

$$|\phi_1(x, y, t) \circ f^k| \leq C_1 \log |t|^{-1} + C_2 \log d(f_t^k(x, y), I^+)^{-1} + C_3.$$

Hence, it remains to control the behavior of the term $\log d(f_t^k(x, y), I^+)^{-1}$. As I^+ is a super-attracting fixed point for f_λ^{-1} , one can find $\kappa > 0$ large enough so that:

$$B^+ := \{(x, y, t), |x| \geq \max(|t|^{-\kappa}, |y|)\}$$

is stable by f^{-1} : $f^{-1}(B^+) \subset (B^+)$ (again, we shrink D^* if necessary). Furthermore, for $(x, y, t) \in B^+$, we have

$$(2) \quad d(f_t^{-1}(x, y), I^+) \leq d((x, y), I^+).$$

Indeed, this follows from the definition of B^+ and the fact that

$$d((x, y), I^+) \simeq \max\left(\frac{1}{|x|}, \frac{|y|}{|x|}\right).$$

Now, take a point $(x, y, t) \in \mathbb{C}^2 \times D^*$. Observe that for $(x, y, t) \notin B^+$, then, by definition, either $|x| \leq |t|^{-\kappa}$ or $|x| \leq |y|$. In the former case, $1/|x| \geq |t|^\kappa$ so that $d((x, y), I^+) \geq |t|^\kappa$ and in the latter case, $|y|/|x| \geq 1$ so $d((x, y), I^+) \geq 1 \geq |t|^\kappa$. In particular, we see that if $(x, y, t) \notin B^+$ then $\log d((x, y), I^+)^{-1} \leq \kappa \log |t|^{-1}$. By the above, as $(x, y, t) \notin B^+$, then, $f_t^k(x, y) \notin B^+$ for all k so that $\log d(f_t^k(x, y), I^+)^{-1} \leq \kappa \log |t|^{-1}$ for all k and the estimate is satisfied in that case.

On the other hand, for $(x, y, t) \in B^+$, take k_0 the smallest integer such that $f_t^{k_0}(x, y) \notin B^+$ (one can show that k_0 is finite but we will not need that fact). Then, by (2), we have for $k < k_0$,

$$d(f_t^k(x, y), I^+) \geq d((x, y), I^+)$$

and for $k \geq k_0$:

$$d(f_t^k(x, y), I^+) \geq |t|^\kappa.$$

Thus, for any k , we have $\log d(f_t^k(x, y), I^+)^{-1} \leq \kappa \log |t|^{-1} + \log d((x, y), I^+)^{-1}$.

The lemma follows, since the local coordinate in any disk D^* centered at a point of $\mathcal{B} \setminus \Lambda$ grows at a rate $\lambda^{-\tau}$ for some $\tau > 0$, where λ is the affine coordinate on Λ defined above, and since $\mathcal{B} \setminus \Lambda$ is finite. \square

3. HEIGHT AND STABILITY

3.1. Zero height implies uniformly bounded degree. Let $f : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ be a regular polynomial automorphism of degree d , defined over a field \mathbf{k} of characteristic 0. It is known there exists a projective surface V obtained by finitely many blow-ups of points $\pi : V \rightarrow \mathbb{P}^2$ such that $f \circ \pi$ and $f^{-1} \circ \pi$ both extend as morphisms $\psi_\pm : V \rightarrow \mathbb{P}^2$, i.e. the diagram

$$(3) \quad \begin{array}{ccccc} & & V & & \\ & \swarrow \psi_+ & \downarrow \pi & \searrow \psi_- & \\ \mathbb{P}^2 & \xleftarrow{f} & \mathbb{P}^2 & \xrightarrow{f^{-1}} & \mathbb{P}^2 \end{array}$$

commutes. We rely on the next result of Kawaguchi [K, Theorem 2.1]:

Theorem 3 (Kawaguchi [K]). *Let $f : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ be a regular polynomial automorphism of degree $d \geq 2$, let V , π and ψ_{\pm} be as in (3), and let H_{∞} be the line at infinity of \mathbb{A}^2 . Then, as a \mathbb{Q} -divisor on V ,*

$$D := \psi_+^* H_{\infty} + \psi_-^* H_{\infty} - \left(d + \frac{1}{d}\right) \pi^* H_{\infty}$$

is effective.

As in the case of number fields, this allows to prove that \widehat{h}_f is comparable (from below) with the standard height function h using the functoriality properties of height functions, see [K, Theorem 2.3 & Theorem 4.1]. We follow Kawaguchi's arguments and claim no originality in the proof.

Corollary 4. *Let $f : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ be a regular polynomial automorphism of degree $d \geq 2$, defined over a global function field of characteristic 0. Then, there is a constant $C > 0$ such that for all $z \in \mathbb{A}^2(\overline{\mathbf{K}})$, we have*

$$\widehat{h}_f(z) \geq h(z) - C.$$

Proof. We follow Kawaguchi's proof: By definition of the \mathbb{Q} -divisor D , we have

$$h_{V,D} = h_{V,\psi_+^* H_{\infty}} + h_{V,\psi_-^* H_{\infty}} - \left(d + \frac{1}{d}\right) h_{V,\pi^* H_{\infty}} + O(1).$$

According to Theorem 3, D is effective. Since $\pi(\text{supp}(D)) \subset \text{supp}(H_{\infty})$, we have $h_{V,D} \geq O(1)$ on $\mathbb{A}^2(\overline{\mathbf{K}})$. Pick now $z \in \mathbb{A}^2(\overline{\mathbf{K}})$. As π restricts as an isomorphism on $Y := \pi^{-1}(\mathbb{A}^2)$, there is a unique $\tilde{z} \in Y(\overline{\mathbf{K}})$ with $\pi(\tilde{z}) = z$ and the definition of ψ_{\pm} gives

$$h_{V,\psi_{\pm}^* H_{\infty}}(\tilde{z}) = h_{\mathbb{P}^2, H_{\infty}}(\psi_{\pm}(\tilde{z})) + O(1) = h(f^{\pm 1}(z)) + O(1).$$

Similarly, we have $h_{V,\pi^* H_{\infty}}(\tilde{z}) = h(z) + O(1)$. To summarize, we proved

$$h(f(z)) + h(f^{-1}(z)) - \left(d + \frac{1}{d}\right) h(z) \geq -C,$$

for some $C > 0$ independent of z . In particular, for any $z \in \mathbb{A}^2(\overline{\mathbf{K}})$, we have

$$(4) \quad \frac{1}{d} h(f(z)) + \frac{1}{d} h(f^{-1}(z)) - \left(1 + \frac{1}{d^2}\right) h(z) \geq -C'$$

for some $C' > 0$ independent of z . Set $h' := h - \frac{d^2}{(d-1)^2} C'$ in (4) so that it becomes:

$$\frac{1}{d} h'(f(z)) + \frac{1}{d} h'(f^{-1}(z)) \geq \left(1 + \frac{1}{d^2}\right) h'(z).$$

Applying to $f(z)$ yields

$$\frac{1}{d^2} h'(f^2(z)) + \frac{1}{d^2} h'(z) \geq \frac{1}{d} \left(1 + \frac{1}{d^2}\right) h'(f(z)).$$

Exchanging the roles of f and f^{-1} , we have the same inequality for f^{-1} and we deduce

$$\frac{1}{d^2} h'(f^2(z)) + \frac{1}{d^2} h'(f^{-2}(z)) \geq \left(1 + \frac{1}{d^4}\right) h'(z).$$

An easy induction gives

$$\frac{1}{d^{2^k}} h'(f^{2^k}(z)) + \frac{1}{d^{2^k}} h'(f^{-2^k}(z)) \geq \left(1 + \frac{1}{d^{2^{k+1}}}\right) h'(z).$$

Making $k \rightarrow +\infty$, we find $\widehat{h}_f \geq h'$, as expected. \square

3.2. Stability, bifurcation measure and the canonical height. Let us come back to the case where $\mathbf{K} = \mathbb{C}(\mathcal{B})$ is the field of rational functions over a smooth complex projective curve. We keep conventions and notations of Section 2.

Recall that we defined an embedding $\iota_1 : \mathcal{B} \hookrightarrow \mathbb{P}^M$ with $\iota_1^{-1}(\mathbb{A}^M(\mathbb{C})) = \Lambda$. Let $\omega_{\text{FS},M}$ denote the Fubini-Study form on \mathbb{P}^M and $\omega_{\mathcal{B}} := \iota_1^*(\omega_{\text{FS},M})$. Let also $\pi_{\mathbb{P}^2} : \mathbb{P}^2 \times \mathcal{B} \rightarrow \mathbb{P}^2$ be the canonical projections, $\omega_{\text{FS},2}$ the Fubini-Study form on \mathbb{P}^2 and $\widehat{\omega} := \pi_{\mathbb{P}^2}^*(\omega_{\text{FS},2})$. Note that the closed positive $(1,1)$ -form

$$\widehat{\omega} + \pi^* \omega_{\mathcal{B}}$$

is a Kähler form on $\mathbb{P}^2 \times \mathcal{B}$, which is cohomologous to the ample line bundle $\mathcal{M} := \mathcal{L} \otimes \mathcal{N}$ on $\mathbb{P}^2 \times \mathcal{B}$, where $\mathcal{L} := \pi_{\mathbb{P}^2}^* \mathcal{O}_{\mathbb{P}^2}(1)$ and $\mathcal{N} := \iota_1^* \mathcal{O}_{\mathbb{P}^N}(1)$. For any closed positive (p,p) -current S on $\mathbb{P}^2 \times \mathcal{B}$, and any Borel subset Ω of $\mathbb{P}^2 \times \mathcal{B}$, we set

$$\|S\|_{\Omega} := \int_{\Omega} S \wedge (\widehat{\omega} + \pi^* \omega_{\mathcal{B}})^{3-p}.$$

Definition 3. Consider $(\Lambda, f, \mathcal{Z})$ an algebraic dynamical pair of regular automorphism-type. We say that $(\Lambda, f, \mathcal{Z})$ is stable if for any compact subset $K \Subset \mathbb{P}^2 \times \Lambda$, we have $\|(f^n)_*[\mathcal{Z}]\|_K = O(1)$ as $n \rightarrow +\infty$.

Remark. Let us emphasize that $(f^n)_*[\mathcal{Z}] = [f^n(\mathcal{Z})]$, so that, when $(\Lambda, f, \mathcal{Z})$ is stable, the sequence $(f^n(\mathcal{Z}))_n$ of analytic subsets has locally uniformly bounded mass and, by Bishop Theorem, converges up to extraction to an analytic subset \mathcal{Z}_{∞} of $\mathbb{C}^2 \times \Lambda$. If in addition \mathcal{Z} is the graph of a morphism $a : \Lambda \rightarrow \mathbb{C}^2$, then $(\Lambda, f, \mathcal{Z})$ is stable if and only if the sequence $(\lambda \mapsto f_{\lambda}^n(a(\lambda)))_n$ converges locally uniformly to a map $a_{\infty} : \Lambda \rightarrow \mathbb{C}^2$. This justifies the definition of stability.

We now characterize stability in terms of a measure on the parameter space Λ . To do so, we let

Definition 4. Let $(\Lambda, f, \mathcal{Z})$ be an algebraic dynamical pair of regular automorphism-type. The bifurcation measure $\mu_{f, [\mathcal{Z}]}$ of $(\Lambda, f, \mathcal{Z})$ is the positive measure on Λ

$$\mu_{f, [\mathcal{Z}]} := \pi_* \left(\widehat{T}_f \wedge [\mathcal{Z}] \right).$$

Remark that, since \widehat{T}_f has continuous potentials on $(\mathbb{P}^2 \setminus \{I^+\}) \times \Lambda$, the measure $\mu_{f, [\mathcal{Z}]}$ is well-defined.

We now come to the proof of the following, which says that stability is equivalent to having bounded degree under iteration. Before starting the proof, we recall that, since \mathcal{Z} has relative dimension 0, for any integer $n \in \mathbb{Z}$, we have

$$(f^n)_*[\mathcal{Z}] = [f^n(\mathcal{Z})]$$

and that, by definition of \deg_m and of the mass of a current, we have

$$\|[f^n(\mathcal{Z})]\|_{\mathbb{C}^2 \times \Lambda} = \|[f^n(\mathcal{Z})]\|_{\mathbb{P}^2(\mathbb{C}) \times \mathcal{B}} = \deg_m(f^n(\mathcal{Z})).$$

Proposition 5. Let $(\Lambda, f, \mathcal{Z})$ be an algebraic dynamical pair of regular automorphism-type. There exists a constant $B > 0$ depending only on $(\Lambda, f, \mathcal{Z})$ such that for any $n \geq 1$,

$$\left| \deg_m(f^n(\mathcal{Z})) - d^n \int_{\Lambda} \mu_{f, [\mathcal{Z}]} \right| \leq B.$$

Proof. We adopt the same strategy as in [GV] using Lemma 2 to deal with the indeterminacy set. For any $A > 0$, we pick the following test function

$$\Psi_A(\lambda) := \frac{\log \max(\|\lambda\|, e^A) - \log \max(\|\lambda\|, e^{2A})}{A}, \quad \text{for all } \lambda \in \Lambda.$$

Then, Ψ_A is continuous and DSH on Λ , i.e. $dd^c \Psi_A = T_A^+ - T_A^-$ where T_A^\pm are some positive closed $(1, 1)$ -currents whose masses are finite with $\|T_A^\pm\| \leq C'/A$ for some $C' > 0$ depending neither on A nor on T_A^\pm . Observe also that Ψ_A is equal to -1 in $B(0, e^A)$, and 0 outside $B(0, e^{2A})$. Since $\pi \circ f^n = \pi$, we have

$$\begin{aligned} I_n^A &:= \left\langle \frac{1}{d^n} (f^n)_* [\mathcal{Z}] \wedge (\widehat{T}_f - \widehat{\omega}), \Psi_A \circ \pi \right\rangle \\ &= \left\langle [\mathcal{Z}] \wedge \frac{1}{d^n} (f^n)^* (\widehat{T}_f - \widehat{\omega}), \Psi_A \circ \pi \circ f^n \right\rangle \\ &= \left\langle [\mathcal{Z}] \wedge \left(\widehat{T}_f - \frac{1}{d^n} (f^n)^* (\widehat{\omega}) \right), \Psi_A \circ \pi \right\rangle, \end{aligned}$$

where we used $(f^n)^* \widehat{T}_f = d^n \cdot \widehat{T}_f$. By Stokes formula and using the notations of Lemma 2.

$$I_n^A = \langle \phi_n \cdot [\mathcal{Z}], dd^c(\Psi_A \circ \pi) \rangle.$$

In particular, by the properties of T_A^\pm , there exists a constant $B > 0$ such that by Bézout:

$$|I_n^A| \leq \frac{B \|[\mathcal{Z}]\|_{\mathbb{C}^2 \times \Lambda}}{Ad^n} \sup_{\pi^{-1}(B(0, e^{2A}) \cap \mathcal{Z})} |\phi_n|.$$

We now use Lemma 2. As \mathcal{C} is an algebraic curve, when A is large, we have that

$$\sup_{\pi^{-1}(B(0, e^{2A}) \cap \mathcal{Z})} d((x, y), I^+) \geq |A|^{-\zeta}$$

for a constant ζ that does not depend on A (though ζ depends on \mathcal{Z}). This implies $|I_n^A| \leq Bd^{-n}$ where B is a constant that depends on $(\Lambda, f, \mathcal{Z})$ but neither on A nor on n . We make $A \rightarrow \infty$ and multiply by d^n and find

$$\left| \int_{\mathbb{C}^2 \times \Lambda} (f^n)_* [\mathcal{Z}] \wedge \widehat{\omega} - d^n \int_{\Lambda} \pi_* \left([\mathcal{Z}] \wedge \widehat{T}_f \right) \right| \leq B.$$

On the other hand, as $\pi_\Lambda \circ f^n = \pi_\Lambda$ for all n , we see that

$$\int_{\mathbb{C}^2 \times \Lambda} (f^n)_* [\mathcal{Z}] \wedge (\pi^* \omega_B) = \int_{\mathbb{C}^2 \times \Lambda} [\mathcal{Z}] \wedge (\pi^* \omega_B) \leq B' \|[\mathcal{Z}]\|_{\mathbb{C}^2 \times \Lambda}$$

where B' is a constant that does not depend on n . This ends the proof. \square

This global estimate of mass allows us to give the following different global characterizations of stability.

Proposition 6. *Let $(\Lambda, f, \mathcal{Z})$ be an algebraic dynamical pair of regular polynomial automorphism-type. The following assertions are equivalent:*

- (1) $(\Lambda, f, \mathcal{Z})$ is stable,
- (2) the sequence $(\deg_M(f^n(\mathcal{Z})))_{n \geq 1}$ is bounded,
- (3) the function $G_{\mathcal{Z}}^+$ is constant on Λ where

$$G_{\mathcal{Z}}^+ : \lambda \in \Lambda \mapsto \sum_{(x, y) \in \mathcal{Z} \cap \mathbb{C}^2 \times \{\lambda\}} G_\lambda^+(x, y) \in \mathbb{R}_+.$$

Proof. We may view Λ as an affine curve. According to Proposition 5, we have $\mu_{f, [\mathcal{Z}]} = 0$ if and only if the sequence $(\deg_M(f^n(\mathcal{Z})))_{n \geq 1}$ is bounded, so $1 \iff 2$.

The implication $3 \Rightarrow 2$ follows from the fact that $\mu_{f, [\mathcal{Z}]} = dd^c G_{\mathcal{Z}}^+$. Indeed, the current \widehat{T}_f satisfies $\widehat{T}_f = dd^c G^+$ on $\mathbb{C}^2 \times \Lambda$ so that

$$dd^c G_{\mathcal{Z}}^+ = \widehat{T}_f \wedge [\mathcal{Z}].$$

Finally, if 1 holds, the function $G_{\mathcal{Z}}$ is harmonic on Λ . Applying Lemma 2 for $n = 1$ gives

$$0 \leq G_{\lambda}^+(x, y) \leq C_1 \log^+ |\lambda| + C_2 \log^+ \|(x, y)\| + C_3,$$

for all $((x, y), \lambda) \in \mathbb{C}^2 \times \Lambda$. As \mathcal{Z} is an algebraic curve of $\mathbb{C}^2 \times \Lambda$ such that $C_{\lambda} := \mathcal{Z} \cap (\mathbb{C}^2 \times \{\lambda\})$ is finite for all λ , this implies

$$0 \leq G_{\mathcal{Z}}^+(\lambda) \leq C \log^+ |\lambda| + C', \quad \lambda \in \Lambda.$$

Recall that \mathcal{B} is a smooth compactification of Λ . Pick a branch at infinity $\mathfrak{c} \in \mathcal{B} \setminus \Lambda$ of Λ . If t is a local coordinate at \mathfrak{c} . Then there is a constant $c_{\mathfrak{c}} \geq 0$ such that $G_{\mathcal{Z}}^+(t) = c_{\mathfrak{c}} \log |t|^{-1} + o(\log |t|^{-1})$ and Stokes Theorem implies

$$\sum_{\mathfrak{c} \in \mathcal{B} \setminus \Lambda} c_{\mathfrak{c}} = \int_{\mathcal{B}} dd^c G_{\mathcal{Z}}^+ = \int_{\mathbb{P}^2 \times \mathcal{B}} \widehat{T}_f \wedge [\mathcal{Z}] = 0.$$

Coming back to a local parametrization of Λ at some $\mathfrak{c} \in \mathcal{B} \setminus \Lambda$, we have $G_{\mathcal{Z}}^+(t) = o(\log |t|^{-1})$. Define now

$$u(\lambda) := -G_{\mathcal{Z}}^+(\lambda), \quad \lambda \in \Lambda.$$

The above implies that the function u is subharmonic on Λ with $u \leq 0$ and whose usc extension to \mathcal{B} is still subharmonic. By the maximum principle, it has to be constant. \square

3.3. Function field versus family. Recall that, as explained in the introduction, a family (resp. non-isotrivial family) $f : \mathbb{C}^2 \times \Lambda \rightarrow \mathbb{C}^2 \times \Lambda$ of regular polynomial automorphisms as above can be associated with a dynamical system over the function field $\mathbf{K} = \mathbb{C}(\mathcal{B})$: it induces a regular polynomial automorphism $f : \mathbb{A}_{\mathbf{K}}^2 \rightarrow \mathbb{A}_{\mathbf{K}}^2$ (resp. a non isotrivial regular polynomial automorphism $f : \mathbb{A}_{\mathbf{K}}^2 \rightarrow \mathbb{A}_{\mathbf{K}}^2$) which is given by

$$f(x, y) = (p(x, y), q(x, y)),$$

where $p, q \in \mathbf{K}[x, y]$.

Moreover, if $(\Lambda, f, \mathcal{Z})$ is a dynamical pair, the curve \mathcal{Z} corresponds to a finite algebraic subvariety Z of $\mathbb{A}^2(\mathbf{K})$. Similarly, if Z is a finite algebraic subvariety of $\mathbb{A}^2(\mathbf{K})$ which is defined over \mathbf{K} , let \mathcal{Z} be the Zariski closure of Z in $\mathbb{P}^2(\mathbb{C}) \times \mathcal{B}$. This is a curve $\mathcal{Z} \subset \mathbb{P}^2(\mathbb{C}) \times \mathcal{B}$ which is flat over \mathcal{B} . Finally, to $z \in \mathbb{A}^2(\mathbf{K})$, one can also associate a marked point $\mathfrak{z} : \mathcal{B} \rightarrow \mathbb{C}$ such that \mathfrak{z} is defined on Λ , in which case the subvariety \mathcal{Z} corresponds to the closure of the graph of \mathfrak{z} restricted to Λ .

As in the case of endomorphisms in [GV], we can express the canonical height of Z as the mass of the bifurcation measure.

Proposition 7. *Let $\mathbf{K} := \mathbb{C}(\mathcal{B})$ be the field of rational functions of a smooth complex projective curve and let $(\Lambda, f, \mathcal{Z})$ be an algebraic dynamical pair of regular polynomial automorphism-type with $\Lambda \subset \mathcal{B}$ and $f : \mathbb{A}_{\mathbf{K}}^2 \rightarrow \mathbb{A}_{\mathbf{K}}^2$ be the induced regular polynomial automorphism and let Z be the finite algebraic subvariety of $\mathbb{A}^2(\mathbf{K})$ induced by \mathcal{Z} . Then*

$$\widehat{h}_f^+(Z) = \int_{\mathbb{C}^2 \times \Lambda} [\mathcal{Z}] \wedge \widehat{T}_f \quad \text{and} \quad \widehat{h}_f^-(Z) = \int_{\mathbb{C}^2 \times \Lambda} [\mathcal{Z}] \wedge \widehat{T}_{f^{-1}}.$$

In particular, $\widehat{h}_f^+(Z) = 0$ if and only if $(\Lambda, f, \mathcal{Z})$ is stable.

Furthermore, there exists a constant $B_0 \geq 0$ depending only on f such that, if $z \in \mathbb{A}^2(\mathbf{K})$ corresponds to the marked point $\zeta : \mathcal{B} \dashrightarrow \mathbb{P}^2$ and \mathcal{Z} is the closure of the graph of ζ restricted to Λ , then

$$\widehat{h}_f(z) = 0 \text{ if and only if for any } n \in \mathbb{Z}, \text{ we have } \deg_m(f^n(\mathcal{Z})) \leq B_0.$$

Proof. The expression of the height is an application of Proposition 5. Indeed, for any $n \geq 1$, we have

$$\frac{1}{d^n} h(f^n(Z)) = \frac{1}{d^n} \int_{\mathbb{C}^2 \times \Lambda} [f^n(\mathcal{Z})] \wedge \widehat{\omega} = \int_{\mathbb{C}^2 \times \Lambda} [\mathcal{Z}] \wedge \widehat{T}_f + O(d^{-n})$$

by Proposition 5. We conclude letting $n \rightarrow +\infty$. Applying everything for f^{-1} instead of f gives the expected formulae. The equivalence between height zero and stability follows from Proposition 6.

Now, apply Corollary 4 to $\mathbf{K} = \mathbb{C}(\mathbb{B})$ and $z \in \mathbb{A}^2(\mathbf{K})$:

$$h(z) \leq \widehat{h}_f(z) + C,$$

where C does not depend on z . In particular, if $\widehat{h}_f(z) = 0$, we deduce for all $n \in \mathbb{Z}$

$$h(f^n(z)) \leq \widehat{h}_f(f^n(z)) + C = C.$$

Let \mathcal{N} be an ample linebundle on \mathcal{B} , be such that $\mathcal{M} := \pi_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes \pi_2^* \mathcal{N}$ is ample on $\mathbb{P}^2 \times \mathcal{B}$. By construction, for any $z \in \mathbb{A}^2(\mathbf{K})$ with Zariski closure \mathcal{Z} in $\mathbb{P}^2 \times \mathcal{B}$, we have

$$h(z) = (\mathcal{Z} \cdot c_1(\pi_1^* \mathcal{O}_{\mathbb{P}^2}(1))) = \deg_{\mathcal{M}}(\mathcal{Z}) - (\mathcal{Z} \cdot c_1(\pi_2^* \mathcal{N})) = \deg_{\mathcal{M}}(\mathcal{Z}) - 1.$$

If \mathcal{Z}_n is the Zariski closure of $f^n(z)$ for all $n \in \mathbb{Z}$, we find

$$\deg_{\mathcal{M}}(\mathcal{Z}_n) \leq C + 1, \quad \text{for all } n \in \mathbb{Z}.$$

Taking $B_0 = C + 1$ ends the proof. \square

4. THE GEOMETRIC DYNAMICAL NORTHCOTT PROPERTY

4.1. Stability implies periodicity and the Main Theorem. We prove the following rigidity theorem which is concerned with stable dynamical pairs of regular polynomial automorphism type, and then deduce the Main Theorem from it:

Theorem 8. *Let $(\Lambda, f, \mathcal{Z})$ be a non-isotrivial algebraic dynamical pair of regular polynomial automorphism-type, where \mathcal{Z} is the graph of a marked point $\Lambda \rightarrow \mathbb{C}^2$. The following assertions are equivalent:*

- (1) *the pair $(\Lambda, f, \mathcal{Z})$ is stable,*
- (2) *the pair $(\Lambda, f^{-1}, \mathcal{Z})$ is stable,*
- (3) *there exists $n > 0$ such that $f^n(\mathcal{Z}) = \mathcal{Z}$.*

Proof. Assume first that both pairs $(\Lambda, f, \mathcal{Z})$ and $(\Lambda, f^{-1}, \mathcal{Z})$ are stable and let us show that \mathcal{Z} is periodic. Assume \mathcal{Z} is not periodic. As f is a family of regular polynomial automorphisms, \mathcal{Z} is not periodic and $\deg_m(f^n(\mathcal{Z})) \leq B$ for all $n \in \mathbb{Z}$ for a given constant $B \geq 1$, by Proposition 6 applied to f and f^{-1} . Let \mathcal{Z}_B be the set of algebraic curves $\mathcal{V} \subset \mathbb{P}^2 \times \mathcal{B}$ with $\deg_m(\mathcal{V}) \leq B$ and $\mathcal{V} \cap (\mathbb{P}^2 \times \Lambda) \subset \mathbb{C}^2 \times \Lambda$. We let

$$\mathcal{W}_B := \{\mathcal{V} \in \mathcal{Z}_B : \deg_m(f^n(\mathcal{V})) \leq B, \text{ for all } n \in \mathbb{Z}\}.$$

The set \mathcal{W}_B is an algebraic subvariety of \mathcal{Z}_B which is stable under iteration of f and f^{-1} and which contains the grand orbit $\mathcal{O}(\mathcal{Z}) := \{f^n(\mathcal{Z}) : n \in \mathbb{Z}\}$ of \mathcal{Z} . Define

$$\widehat{\mathcal{W}} := \{(\mathcal{V}, (x, y), \lambda) \in \mathcal{W} \times \mathbb{C}^2 \times \Lambda : ((x, y), \lambda) \in \mathcal{V}\}$$

and let $\pi_{\mathcal{W}} : \widehat{\mathcal{W}} \rightarrow \mathcal{X}_\Lambda$ be the projection onto the second factor. By assumption, any irreducible component of $\widehat{\mathcal{W}}$ containing $f^n(\mathcal{Z})$ for some $n \in \mathbb{Z}$ has dimension at least 2 (otherwise, \mathcal{W}_B is finite and \mathcal{Z} is periodic). Assume first $\widehat{\mathcal{W}}$ has an irreducible component of dimension 2. Then it must have a periodic component of dimension 2, and for all λ outside a finite subset of Λ , there exists an algebraic curve C_λ of \mathbb{C}^2 such that $f_\lambda^m(C_\lambda) \subset C_\lambda$ for some integer m . This is a contradiction, since regular polynomial automorphisms have no invariant algebraic curve. This implies any periodic irreducible component of $\widehat{\mathcal{W}}$ has dimension at least 3 and the orbit $\mathcal{O}(\mathcal{C})$ of \mathcal{C} is Zariski dense in $\mathbb{P}^2 \times \mathcal{B}$. Let $\widehat{\mathcal{W}}_1$ be such a component. Up to replacing f by an iterate, we can assume $f(\widehat{\mathcal{W}}_1) = \widehat{\mathcal{W}}_1$.

Lemma 9. *The canonical projection $\Pi : \widehat{\mathcal{W}}_1 \rightarrow \mathbb{C}^2 \times \Lambda$ is an isomorphism and, for any $\lambda \in \Lambda$, the restriction $\Pi_\lambda : \Pi^{-1}(\mathbb{C}^2 \times \{\lambda\}) \rightarrow \mathbb{C}^2$ of Π is an isomorphism.*

We take Lemma 9 for granted for now and finish the proof. Fix $\lambda_0 \in \Lambda$. For any $\lambda \in \Lambda$, we thus can define a birational map $\phi_\lambda \in \text{Bir}(\mathbb{P}^2)$ by letting

$$\phi_\lambda = \Pi_\lambda \circ \Pi_{\lambda_0}^{-1}.$$

By construction, for any $\lambda \in \Lambda$, the indeterminacy points of ϕ_λ are contained in the line at infinity, we have $f_\lambda = \phi_\lambda \circ f_{\lambda_0} \circ \phi_\lambda^{-1}$ and, for any $\mathcal{V} \in \mathcal{W}_1$ and any $\lambda \in \Lambda$, we have

$$\phi_\lambda(\mathcal{V}_{\lambda_0}) = \mathcal{V}_\lambda.$$

We now rely on Proposition 6. By construction of the map ϕ_λ , this implies

$$\phi_\lambda(G_{\lambda_0}^\pm = \alpha) = (G_\lambda^\pm = \alpha),$$

for any $\alpha \geq 0$. Assume, by contradiction, that the map ϕ_λ does not extend as a holomorphic map on \mathbb{P}^2 , then it contracts the line at infinity L_∞ . In particular, for a point $(x, y) \in L_\infty \setminus I(\phi_\lambda)$, we have $\phi_\lambda(x, y) \in I(\phi_\lambda^{-1})$. In other word, a small neighborhood U_+ of (x, y) should be sent into a small neighborhood U_- of an indeterminacy point of ϕ_λ^{-1} . This is impossible, since any $(x', y') \in U_+$ satisfies $G_{\lambda_0}^+(x', y') = \alpha^+$ and $G_{\lambda_0}^-(x', y') = \alpha^-$ for some $\alpha^\pm > 0$ very large. Then, $G_\lambda^\pm(\phi_\lambda(x', y')) = \alpha^\pm$ and this is not close to the indeterminacy point of ϕ_λ^{-1} . This is a contradiction so ϕ_λ is in fact a biholomorphism and the family is isotrivial. We have proved that if $(\Lambda, f, \mathcal{Z})$ and $(\Lambda, f^{-1}, \mathcal{Z})$ are stable, then $f^n(\mathcal{Z}) = \mathcal{Z}$ for some $n \in \mathbb{Z}^*$.

To conclude the proof, it is sufficient to show that if the pair $(\Lambda, f, \mathcal{Z})$ is stable, then the pair $(\Lambda, f^{-1}, \mathcal{Z})$ is also stable. By assumption, there exists $M > 0$ such that $\deg_m f^n(\mathcal{Z}) \leq M$ for all n . Consider the variety $\mathcal{W}^+ := \{V \in \mathcal{C}_M : \deg(f^n(V)) \leq M, \text{ for all } n \geq 0\}$. The result follows if we show that \mathcal{W}^+ is a finite set. Assume to the contrary that $\dim \mathcal{W}^+ > 0$. As before, we may replace \mathcal{W}^+ by some of its irreducible components and f by f^n to have $f(\mathcal{W}^+) = \mathcal{W}^+$. Define

$$\widetilde{\mathcal{W}}^+ := \{(\mathcal{V}, (x, y), \lambda) \in \mathcal{W}^+ \times \mathbb{C}^2 \times \Lambda : ((x, y), \lambda) \in \mathcal{V}\}$$

and let $\pi_{\mathcal{W}^+} : \widetilde{\mathcal{W}}^+ \rightarrow \mathbb{C}^2 \times \Lambda$ be the canonical projection. Then either $\dim \widetilde{\mathcal{W}}^+ = 2$ or $\dim \widetilde{\mathcal{W}}^+ \geq 3$. Again, if $\dim \widetilde{\mathcal{W}}^+ = 2$, for all λ outside a finite subset of Λ , there exists an algebraic curve C_λ of \mathbb{C}^2 such that $f_\lambda(C_\lambda) \subset C_\lambda$, which is impossible. So we can

assume that $\dim \widetilde{\mathcal{W}}^+ \geq 3$. In particular, the map $\pi_{\mathcal{W}^+} : \widetilde{\mathcal{W}}^+ \rightarrow \mathbb{C}^2 \times \Lambda$ is dominant. Let $((x_0, y_0), \lambda_0) \in \pi_{\mathcal{W}^+}(\widetilde{\mathcal{W}}^+)$ be such that (x_0, y_0) is a saddle periodic point of f_{λ_0} and let $\mathcal{Z}' \in \mathcal{W}^+$ be such that $((x_0, y_0), \lambda_0) \in \mathcal{Z}'$ and let z' be the restriction of \mathcal{Z}' to the generic fiber of $\pi : \mathbb{P}^2 \times \Lambda \rightarrow \Lambda$. We have $\widehat{h}_f^+(z') = 0$, whence

$$\widehat{h}_f(f^n(z')) = \widehat{h}_f^-(f^n(z')) = d^{-n} \widehat{h}_f^-(z') \xrightarrow{n \rightarrow +\infty} 0.$$

As the sequence $(f^n(\mathcal{Z}'))_n$ has bounded global volume, by Bishop's Theorem, there exists a subsequence $(f^{n_k}(\mathcal{Z}'))_k$ which converges to a curve \mathcal{Z}'' with $\deg_m(\mathcal{Z}'') \leq M$. Moreover, if z'' is the generic fiber of the curve \mathcal{Z}'' , the above implies

$$\widehat{h}_f(z'') = 0.$$

By the previous step of the proof, \mathcal{Z}'' belongs to the finite set \mathcal{W} . This is a contradiction, since there are infinitely many distinct saddle periodic points. \square

Remark. (1) This proof shares many similarities with the proof of the Northcott property over number field of [K, Theorem 4.2]. Indeed, in both cases, we first establish a finiteness property for the total height \widehat{h}_f , then when z only satisfies $\widehat{h}_f^+(z) = 0$, we push it forward to produce point of small height \widehat{h}_f and we can conclude using this finiteness property for \widehat{h}_f .

(2) In fact, we have proved that for a given $B > 0$, the set of points $z \in \mathbb{A}^2(\mathbf{K})$ with $\deg_m(\mathcal{Z}_n) \leq B$ for all $n \in \mathbb{Z}$ is finite (here \mathcal{Z}_n is the Zariski closure of $f^n(z)$ in $\mathbb{P}^2 \times \mathcal{B}$).

We now conclude by the proof of Lemma 9.

Proof of Lemma 9. By construction, the map Π is dominant. First, we use a similarity argument. Pick $((x_0, y_0), \lambda_0)$ such that (x_0, y_0) is a saddle periodic point of f_{λ_0} of period $q \geq 1$ and let $\mathcal{V} \in \mathcal{W}$ with $((x_0, y_0), \lambda_0) \in \mathcal{V}$. Let (λ, V_λ) denote a local parametrization of \mathcal{V} . Let also $p(\lambda)$ be the continuation of (x_0, y_0) as a saddle periodic point of f_λ , for $\lambda \in U$.

Up to replacing f with f^q we can assume $q = 1$ for this part of the proof. Take λ in a small neighborhood U of λ_0 . We know that V_λ belong to a small neighborhood of $p(\lambda)$ where the stable and unstable manifold of $p(\lambda)$ intersects transversely. Assume, by symmetry, that V_λ does not belong to the stable manifold of $p(\lambda)$. Considering $f_\lambda^n(V_\lambda)$ for $n \gg 1$, we then deduce that the family $\lambda \mapsto (\lambda, f_\lambda^n(V_\lambda))$ is not normal, a contradiction. In particular, V_λ belong to both the stable and unstable manifold of $p(\lambda)$ so it equal to $p(\lambda)$ in a neighborhood of λ_0 : $\mathcal{V} \cap U = \{(p(\lambda), \lambda) : \lambda \in U\}$.

As the set of points of the form $((x_0, y_0), \lambda_0)$ such that (x_0, y_0) is a saddle periodic point of f_{λ_0} is Zariski dense in $\mathbb{C}^2 \times \Lambda$, the map Π is a birational morphism whose image is a Zariski open subset of $\mathbb{C}^2 \times \Lambda$. Let H be its complement. Up to removing a finite set from Λ , the set $H \cap (\mathbb{C}^2 \times \{\lambda\})$ is Zariski closed. Moreover, it is both f_λ and f_λ^{-1} invariant. Since f_λ has no invariant curve, this forces $H \cap (\mathbb{C}^2 \times \{\lambda\})$ to be finite so this is a finite union of periodic orbit. Take any sequence \mathcal{V}_n such that, for a given λ , the set $\mathcal{V}_n \cap (\mathbb{C}^2 \times \{\lambda\})$ accumulates on $H \cap (\mathbb{C}^2 \times \{\lambda\})$. By Bishop theorem, up to extraction, $(\mathcal{V}_n)_n$ converges towards $\mathcal{V}_\infty \subset H$ with $\deg(\mathcal{V}_\infty) \leq D$. This implies H is a finite union of curves $\mathcal{V} \in \mathcal{W}$. This contradicts the definition of H , so that Π is surjective.

We now need to prove it is finite. Since $\mathbb{C}^2 \times \Lambda$ is normal, this would end the proof that it is an isomorphism. As in the case of endomorphisms, if non-empty, the set $\{(x, y), \lambda\}$:

$\text{Card}(\Pi^{-1}\{(x, y), \lambda\} = +\infty)$ is totally invariant by f , whence consists of periodic points. As they are isolated, this is impossible and Π is finite.

Fix now $\lambda_0 \in \Lambda$. Note that, since C is a graph, it is irreducible and we have

$$(C \cdot (\mathbb{C}^2 \times \{\lambda\})) = 1$$

for all $\lambda \in \Lambda$, where $\mathbb{C}^2 \times \{\lambda\}$ is a general fiber of π . As all the fibers of π (resp. all varieties in \mathcal{W}) are cohomologous and as the intersection can be computed in cohomology, we deduce that

$$(\mathcal{V} \cdot (\mathbb{C}^2 \times \{\lambda\})) = 1$$

for all $\lambda \in \Lambda$ and all $\mathcal{V} \in \mathcal{W}$. In particular, $\Pi_\lambda : \Pi^{-1}(\mathbb{C}^2 \times \{\lambda\}) \rightarrow \mathbb{C}^2$ is a surjective morphism whose topological degree is exactly $(\mathcal{V} \cdot (\mathbb{C}^2 \times \{\lambda\}))$ for a general variety $\mathcal{V} \in \mathcal{W}$, the map Π_λ is a surjective finite birational morphism. By normality of \mathbb{C}^2 , it is an isomorphism. \square

We are now in position to prove our main result:

Proof of the Main Theorem. We first prove item (1). The equivalence between the four points in question is an immediate consequence of Proposition 6, Proposition 7 and Theorem 8.

The finiteness of periodic points is an immediate consequence of Proposition 7. \square

4.2. On a conjecture of Kawaguchi and Silverman for regular automorphisms. As an application, we prove the conjecture of Kawaguchi and Silverman [KS, Conjecture 6] in the case of regular polynomial automorphisms defined over a function field of characteristic zero (due to Kawaguchi and Silverman over number fields).

Recall that the first *dynamical degree* $\lambda_1(f)$ of a regular polynomial automorphism f is

$$\lambda_1(f) := \lim_{n \rightarrow +\infty} ((f^n)^* H \cdot H)^{1/n},$$

where H is any big and nef Cartier divisor on \mathbb{A}^2 , and that the *arithmetic degree* of a point $P \in \mathbb{A}^2(\bar{\mathbf{K}})$ is

$$\alpha_f(P) := \lim_{n \rightarrow +\infty} \max(h(f^n(P)), 1)^{1/n},$$

where $h = h_{\mathbb{P}^2, \mathcal{O}(1)}$ as above, when the limit exists. Actually, by Proposition 5, we have that for any point $P \in \mathbb{A}^2(\bar{\mathbf{K}})$, $\alpha_f(P)$ exists and is equal to 1 or $d = \lambda_1(f)$. In particular, points 1, 2 and 3 of the conjecture hold.

Now, as an immediate consequence of Theorem 8 and Proposition 5, either P has finite orbit, or $\widehat{h}_f(P) > 0$ and $\alpha_f(P) = d$. This actually strengthens point 4 of the conjecture, where this is supposed to hold for points of Zariski dense orbit.

We thus have proven the following

Corollary 10. *Let k be a field of characteristic zero and \mathcal{B} be a smooth projective k -curve. Let f be a non-isotrivial regular polynomial automorphism over $\mathbf{K} := k(\mathcal{B})$ of degree d .*

- (1) *For any $P \in \mathbb{A}^2(\bar{\mathbf{K}})$, the limit $\alpha_f(P)$ exists and is an integer,*
- (2) *the set $\{\alpha_f(Q) : Q \in X(\bar{\mathbf{K}})\}$ coincides with $\{1, d\} = \{1, \lambda_1(f)\}$,*
- (3) *if $P \in \mathbb{A}^2(\bar{\mathbf{K}})$ has infinite forward (or backward) orbit, we have $\alpha_f(P) = \lambda_1(f)$.*

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