

# QUADRATIC POLYNOMIALS, MULTIPLIERS AND EQUIDISTRIBUTION

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ABSTRACT. Given a sequence of complex numbers  $\rho_n$ , we study the asymptotic distribution of the sets of parameters  $c \in \mathbb{C}$  such that the quadratic map  $z^2 + c$  has a cycle of period  $n$  and multiplier  $\rho_n$ . Assume  $\frac{1}{n} \log |\rho_n| \rightarrow L$ . If  $L \leq \log 2$ , they equidistribute on the boundary of the Mandelbrot set. If  $L > \log 2$  they equidistribute on the equipotential outside the Mandelbrot set of level  $2L - 2 \log 2$ .

## INTRODUCTION

In this article, we study equidistribution questions in the parameter space of the family of quadratic polynomials

$$f_c(z) := z^2 + c, \quad c \in \mathbb{C}.$$

We denote by  $K_c$  the filled-in Julia set of  $f_c$  and by  $J_c$  the Julia set:

$$K_c := \left\{ z \in \mathbb{C} \mid (f_c^{o n}(z))_{n \in \mathbb{N}} \text{ is bounded} \right\} \quad \text{and} \quad J_c := \partial K_c.$$

The Mandelbrot set  $M$  is the set of parameters  $c \in \mathbb{C}$  such that  $0 \in K_c$ .

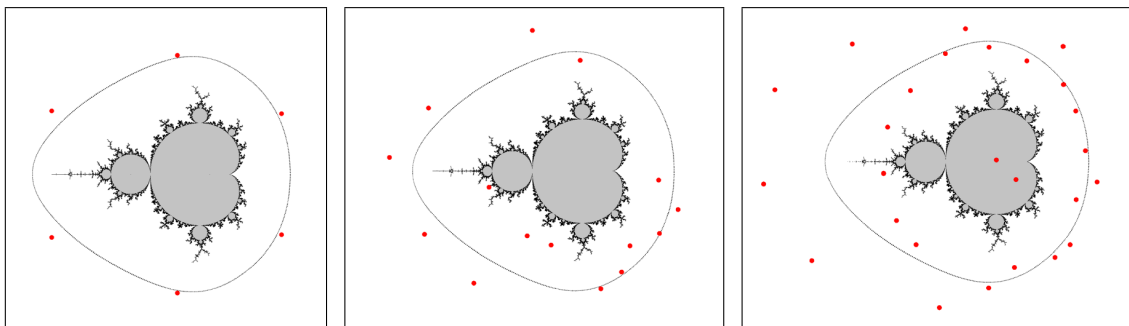


FIGURE 1. Three Mandelbrot sets. On the first picture (resp. second and third), the dots are parameters  $c \in \mathbb{C}$  such that  $f_c$  has a cycle of period 4 (resp. 5 and 6) and multiplier  $\rho^4$  (resp.  $\rho^5$  and  $\rho^6$ ) with  $\rho = \pi/(1+i)$ . The curve is  $\{c \in \mathbb{C} \mid g_M(c) = 2(\log |\rho| - \log 2)\}$ .

The Green's functions  $g_c : \mathbb{C} \rightarrow [0, +\infty)$  and  $g_M : \mathbb{C} \rightarrow [0, +\infty)$  are defined by

$$g_c := \lim_{n \rightarrow +\infty} \max \left( \frac{1}{2^n} \log |f_c^{o n}(z)|, 0 \right) \quad \text{and} \quad g_M(c) := g_c(c).$$

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The bifurcation measure  $\mu_{\text{bif}}$  is defined by

$$\mu_{\text{bif}} := \Delta g_M,$$

where  $\Delta$  stands for the generalized Laplacian. The support of the measure  $\mu_{\text{bif}}$  is the boundary of the Mandelbrot set  $M$ .

Let  $\rho$  be a complex number with  $|\rho| \leq 1$ . For  $n \geq 1$ , denote by  $X_n$  the set of parameters  $c \in \mathbb{C}$  such that the quadratic polynomial  $f_c$  has a cycle of period  $n$  with multiplier  $\rho$ . Let  $\nu_n$  be the probability measure

$$\nu_n := \frac{1}{\text{card}(X_n)} \sum_{x \in X_n} \delta_x.$$

Bassanelli and Berteloot [BB] proved that as  $n \rightarrow +\infty$ , the sequence of measures  $\nu_n$  converges to the bifurcation measure  $\mu_{\text{bif}}$  (in fact, they prove a more general result, valid in families of polynomials of arbitrary degree). In this note, we prove that their result holds without the assumption that  $|\rho| \leq 1$  and we study cases where the multiplier is not fixed.

First, the set of parameters we will consider may fail to equidistribute on the boundary of the Mandelbrot set. They may equidistribute on an equipotential, i.e. a level curve of the function  $g_M$ . For  $\eta \geq 0$ , we let  $\mu_\eta$  be the probability measure defined by

$$\mu_\eta := \Delta \max(g_M, \eta).$$

For  $\eta > 0$ , the support of  $\mu_\eta$  is exactly the equipotential  $\{c \in \mathbb{C} \mid g_M(c) = \eta\}$  and for  $\eta = 0$ , one has  $\mu_0 = \mu_{\text{bif}}$ .

Second, we are interested in parameters  $c \in \mathbb{C}$  for which  $f_c$  has a cycle of period  $n$  with multiplier  $\rho$ . When  $|\rho| > 1$ , we may have to count such parameters  $c$  with multiplicities. We proceed as follows. If  $f_c$  has no parabolic cycle, we set:

$$R_n(c, \rho) := \prod_{C \text{ cycle of } f_c \text{ of period } n} (\rho - \rho(C))$$

where  $\rho(C)$  is the multiplier of  $C$  as a cycle of  $f_c$ . According to Bassanelli and Berteloot [BB, Theorem 2.1] (see also [M, Theorem 4.2] and [S, §4]), this defines a polynomial  $R_n \in \mathbb{C}[c, \rho]$  which is therefore defined even when  $f_c$  has parabolic cycles. The set of zeros of  $R_n(\cdot, \rho)$  is exactly the set of parameters  $c$  such that  $f_c$  has a cycle of period  $n$  and multiplier  $\rho$  (except when  $\rho = 1$ , in which case the support also contains parameters of period  $k$  dividing  $n$  whose multiplier is an  $n/k$ -th root of unity).

In fact, the polynomial  $R_n$  can also be seen as follows. It can be proved that there exists a unique polynomial  $R_n(\rho, c) \in \mathbb{Z}[\rho, c]$  which is monic in  $\rho$  such that the resultant in  $z$  of the polynomial  $Q_n(z, c)$  defining the  $n$ -periodic points of  $f_c$  and the polynomial  $\rho - (f_c^{o_n})'(z)$  is  $R_n(\rho, c)^n$  (see e.g. [S, p. 225]).

**Examples.** The fixed points of  $f_c$  are the roots of  $z^2 - z + c$  and the multiplier of  $f_c$  at a fixed point  $z$  is  $\rho = 2z$ . The resultant of  $z^2 - 2z + c$  and  $\rho - 2z$  as polynomials in  $z$  is  $\rho^2 - 2\rho + 4c$ :

$$R_1(c, \rho) = \rho^2 - 2\rho + 4c.$$

The periodic points of period 2 are the roots of  $z^2 + z + c + 1$  and the multiplier at a periodic point  $z$  is  $\rho = 4z(z^2 + c)$ . The resultant of  $z^2 + z + c + 1$  and  $\rho - 4z(z^2 + c)$  as polynomials in  $z$  is  $(\rho + 4c + 4)^2$ :

$$R_2(c, \rho) = \rho + 4c + 4.$$

Similarly, one computes

$$R_3(c, \rho) = \rho^2 - 16\rho + 64 - 8\rho c + 64c + 128c^2 + 64c^3.$$

The degree  $d_n$  of  $R_n(c, \rho)$  as a polynomial of the variable  $c$  does not depend on  $\rho$ . More precisely, the term of highest degree in  $c$  of  $R_n(c, \rho)$  does not depend of  $\rho$ . Moreover,  $d_n$  is given recursively (see [M, Theorem 4.2]) by

$$d_1 = 1 \quad \text{and} \quad d_n = 2^{n-1} - \sum_{\substack{m \text{ divides } n \\ m \neq n}} d_m.$$

As  $n \rightarrow +\infty$  we have  $d_n \sim 2^{n-1}$ . For  $n \geq 1$  and  $\rho \in \mathbb{C}$ , we set

$$u_{n,\rho} := \frac{1}{d_n} \log |R_n(c, \rho)| \quad \text{and} \quad \nu_{n,\rho} := \Delta u_{n,\rho}.$$

The measure  $\nu_{n,\rho}$  is a probability measure and its support is the set of parameters  $c$  at which the polynomial  $R_n(c, \rho)$  vanishes.

Beware that when  $|\rho| \leq 1$ , since  $R_n(c, \rho)$  has only simple roots, the measure  $\nu_{n,\rho}$  coincides with the above mentioned measure  $\nu_n$ . However, when  $|\rho| > 1$ , the roots of  $R_n(c, \rho)$  might have multiplicity, which is taken into account in  $\nu_{n,\rho}$ .

Our result is the following.

**Theorem.** *Let  $(\rho_n)_{n \geq 1}$  be a sequence of complex numbers such that*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log |\rho_n| = L \in [-\infty, +\infty).$$

*As  $n \rightarrow +\infty$ , the sequence  $(\nu_{n,\rho_n})_{n \geq 1}$  converges to  $\mu_\eta$  in the weak sense of measures on  $\mathbb{C}$ , with  $\eta := \max(0, 2L - 2 \log 2)$ .*

Let us stress out that, when  $\rho_n \equiv \rho \in \mathbb{C}$  is a constant sequence, this means that  $\nu_{n,\rho} \rightarrow \mu_{\text{bif}}$  in the weak sense of measures on  $\mathbb{C}$  as  $n \rightarrow +\infty$ .

The note is organized as follows. In Section 1, we establish the Theorem assuming two Lemmas. The first one, which is proved in Section 2, is concerned with the asymptotic behavior of the sequence  $(u_{n,\rho_n})$  outside of the compact set  $\{c \in \mathbb{C} \mid g_M(c) \leq \eta\}$ . The second one is a direct application of a general comparison lemma for subharmonic functions on  $\mathbb{C}$  which we establish in Section 3.

## 1. STRATEGY OF THE PROOF OF THE THEOREM

Let us set

$$v := g_M + 2 \log 2 : \mathbb{C} \rightarrow [0, +\infty).$$

Let  $(\rho_n)_{n \geq 1}$  be a sequence of complex numbers such that

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log |\rho_n| = L \in [-\infty, +\infty)$$

and set

$$\eta := \max(0, 2L - 2 \log 2).$$

Our proof is based on the following lemmas.

**Lemma 1.** *If  $c \in \mathbb{C} - M$  satisfies  $g_M(c) > \eta$ , then*

$$\lim_{n \rightarrow +\infty} u_{n,\rho_n}(c) = v(c).$$

**Lemma 2.** *Any subharmonic function  $u : \mathbb{C} \rightarrow [-\infty, +\infty)$  which coincides with  $v$  outside  $M$  coincides with  $v$  everywhere.*

The proof is then completed as follows. Extracting a subsequence if necessary, we may assume that the sequence of probability measures  $\nu_{n_k, \rho_{n_k}}$  converges to some limit  $\nu$ . According to Lemma 1, the sequence  $u_{n_k, \rho_{n_k}}$  then converges in  $L^1_{\text{loc}}$  to a limit  $u$  which satisfies

$$\Delta u = \nu \quad \text{and} \quad u(c) = v(c) \text{ if } g_M(c) > \eta.$$

In addition, for every  $c \in \mathbb{C}$ , we have

$$u(c) \geq \limsup_{n \rightarrow +\infty} u_{n_k, \rho_{n_k}}(c)$$

with equality outside a polar set. According to Lemma 1, the subharmonic functions  $u$  and  $v$  coincide in the region  $\{c \in \mathbb{C} \mid g_M(c) > \eta\}$ . If  $L \leq \log 2$ , so that  $\eta = 0$ , Lemma 2 implies that  $u = v$  and so

$$\nu = \Delta u = \Delta(g_M + 2 \log 2) = \Delta g_M = \mu_{\text{bif}}.$$

So, let us consider the case  $L > \log 2$ , or equivalently  $\eta > 0$ . Note that for  $n \geq 2$  the cycles of  $f_0(z) = z^2$  of period  $n$  all have multiplier  $2^n$ . There are  $k_n/n$  such points, where  $k_n$  is the number of periodic points of period  $n$  which may be defined recursively by

$$k_1 := 1 \quad \text{and} \quad k_n = 2^n - \sum_{\substack{m \text{ divides } n \\ m \neq n}} k_m.$$

As  $n \rightarrow +\infty$ , we have  $k_n \sim 2^n$ . Since  $L > \log 2$ ,  $2^n = o(\rho_n)$  and so:

$$u_{n, \rho_n}(0) = \frac{k_n/n}{d_n} \log |2^n - \rho_n| \sim \frac{2^n/n}{2^{n-1}} \log |\rho_n| \rightarrow 2L.$$

As a consequence,  $u(0) \geq 2L$ . According to [R, Theorem 3.8.3],

$$\limsup_{\substack{g_M(c) \rightarrow \eta \\ g_M(c) < \eta}} u(c) = \max_{g_M(c) = \eta} u(c) = \limsup_{\substack{g_M(c) \rightarrow \eta \\ g_M(c) > \eta}} u(c) = 2L.$$

The Maximum Principle for subharmonic functions implies that  $u(c) = 2L$  as soon as  $g_M(c) \leq \eta$ . So,

$$u = \max(g_M + 2 \log 2, 2L) = \max(g_M, \eta) + 2 \log 2 \quad \text{and} \quad \nu = \mu_\eta.$$

## 2. OUTSIDE THE MANDELBROT SET

We now study the behavior of the multipliers of  $f_c$  when  $c$  is not in  $M$ . Our aim in the present section is to prove Lemma 1.

First, when  $c \in \mathbb{C} - M$ , the Julia set  $J_c$  of  $f_c$  is a Cantor set and  $f_c : J_c \rightarrow J_c$  is conjugate to the shift on 2 symbols. In addition, when  $c$  varies outside  $M$ , the Julia set moves holomorphically, but only locally. In fact, it is not true that the holomorphic motion is parameterized by  $\mathbb{C} - M$ : if we start with  $c \in (1/4, +\infty)$  and follow the Julia set as  $c$  turns around the Mandelbrot set, every point comes back to its complex conjugate. After 2 turns, the points come back to their initial location. See [BDK] for details. To avoid the monodromy problems, it is more convenient to pass to a degree 2 covering of  $\mathbb{C} - M$ .

Let  $\phi_M : \mathbb{C} - M \rightarrow \mathbb{C} - \overline{\mathbb{D}}$  be the conformal representation which sends  $(1/4, +\infty)$  to  $(1, +\infty)$  and for  $\lambda \in \mathbb{C} - \overline{\mathbb{D}}$ , set

$$\mathbf{c}(\lambda) := \phi_M^{-1}(\lambda^2).$$

Then, the holomorphic map  $\mathbf{c} : \mathbb{C} - \overline{\mathbb{D}} \rightarrow \mathbb{C} - M$  is a covering map of degree 2.

Set  $I := \{0, 1\}^{\mathbb{N}}$  and let  $\sigma : I \rightarrow I$  be the left shift. A point  $\mathbf{i} := (i_0, i_1, i_2, \dots) \in I$  is called an itinerary. There is a map  $\psi : (\mathbb{C} - \overline{\mathbb{D}}) \times I \rightarrow \mathbb{C}$  such that

- for all  $\lambda \in \mathbb{C} - \overline{\mathbb{D}}$ , the map  $\psi_\lambda : \mathbf{i} \mapsto \psi(\lambda, \mathbf{i})$  is a bijection between  $I$  and  $J_{c(\lambda)}$  conjugating  $\sigma$  to  $f_{c(\lambda)}$ :

$$\psi_\lambda \circ \sigma = f_{c(\lambda)} \circ \psi_\lambda$$

- for all  $\mathbf{i} \in I$ , the map  $\psi_{\mathbf{i}} : \lambda \mapsto \psi(\lambda, \mathbf{i})$  is holomorphic.

We may choose the map  $\psi$  so that for  $\lambda \in (1/4, +\infty)$ , the map  $\psi_\lambda$  sends itineraries for which  $i_0 = 0$  to the upper half-plane and itineraries for which  $i_0 = 1$  to the lower half-plane. Then,

$$\psi_{\mathbf{i}}(\lambda) \underset{\lambda \rightarrow +\infty}{\sim} \sqrt{-1} \cdot \lambda \text{ if } i_0 = 0 \quad \text{and} \quad \psi_{\mathbf{i}}(\lambda) \underset{\lambda \rightarrow +\infty}{\sim} -\sqrt{-1} \cdot \lambda \text{ if } i_0 = 1.$$

Next, if  $\mathbf{i}$  is periodic of period  $n$  for  $\sigma$ , let  $\rho_{\mathbf{i}}(\lambda)$  be the multiplier of  $\psi_{\mathbf{i}}(\lambda)$  as a fixed point of  $f_{c(\lambda)}^{\circ n}$ . Note that

$$\rho_{\mathbf{i}}(\lambda) := \prod_{k=0}^{n-1} 2 \cdot \psi_{\sigma^k(\mathbf{i})}(\lambda) \underset{\lambda \rightarrow +\infty}{\sim} \varepsilon_{\mathbf{i}} \cdot (2\lambda)^n \quad \text{with} \quad \varepsilon_{\mathbf{i}} = \pm \cdot (\sqrt{-1})^n.$$

Since  $\rho_{\mathbf{i}}$  has local degree  $n$  at  $\infty$ , we may write  $\rho_{\mathbf{i}} = \varepsilon_{\mathbf{i}} \cdot \sigma_{\mathbf{i}}^n$  for some holomorphic map  $\sigma_{\mathbf{i}} : \mathbb{C} - \overline{\mathbb{D}} \rightarrow \mathbb{C} - \overline{\mathbb{D}}$  which satisfies

$$\sigma_{\mathbf{i}}(\lambda) \underset{\lambda \rightarrow +\infty}{\sim} 2\lambda \quad \text{and} \quad \sigma_{\mathbf{i}}(\omega\lambda) = \omega \cdot \sigma_{\mathbf{i}}(\lambda) \text{ if } \omega^n = 1.$$

The maps  $\sigma_{\mathbf{i}}$  take their values in  $\mathbb{C} - \overline{\mathbb{D}}$  and so, form a normal family.

Let  $(\mathbf{i}_n)$  be a sequence of periodic itineraries of period  $n$ . Let  $\sigma : \mathbb{C} - \overline{\mathbb{D}} \rightarrow \mathbb{C} - \overline{\mathbb{D}}$  be a limit value of the sequence  $(\sigma_{\mathbf{i}_n} : \mathbb{C} - \overline{\mathbb{D}} \rightarrow \mathbb{C} - \overline{\mathbb{D}})$ . Then,  $\sigma$  is tangent to  $\lambda \mapsto 2\lambda$  at infinity and commutes with rotations. Thus,  $\sigma(\lambda) = 2\lambda$ . Now, if  $c \in \mathbb{C} - M$ , then  $c = \mathbf{c}(\lambda)$  with  $\log |\lambda| = \frac{1}{2}g_M(c)$ . As  $n \rightarrow +\infty$ ,

$$\log |\sigma_{\mathbf{i}_n}(\lambda)| - \frac{1}{n} \log |\rho_n| \xrightarrow{n \rightarrow +\infty} \log |2\lambda| - L = \frac{g_M(c) - \eta}{2}.$$

So, if  $g_M(c) > \eta$ , then there is an  $\varepsilon > 0$  such that for any periodic itinerary  $\mathbf{i}$  of high enough period  $n$ , we have

$$\left| \frac{\rho_n}{\rho_{\mathbf{i}}(\lambda)} \right| = \left| \frac{\rho_n}{\sigma_{\mathbf{i}}(\lambda)^n} \right| \leq e^{-n\varepsilon}.$$

In that case

$$\begin{aligned} u_{n, \rho_n}(c) &= \frac{1}{d_n} \sum_{\substack{\mathbf{i} \text{ periodic itinerary} \\ \text{of period } n}} \frac{1}{n} \log |\rho_{\mathbf{i}}(\lambda) - \rho_n| \\ &= \frac{1}{d_n} \sum_{\substack{\mathbf{i} \text{ periodic itinerary} \\ \text{of period } n}} \log |\sigma_{\mathbf{i}}(\lambda)| + \frac{1}{n} \log \left| 1 - \frac{\rho_n}{\rho_{\mathbf{i}}(\lambda)} \right| \\ &\underset{n \rightarrow +\infty}{=} \frac{k_n}{d_n} \log |2\lambda| + o\left(\frac{k_n}{d_n}\right) \sim 2 \log |2\lambda| = g_M(c) + 2 \log 2, \end{aligned}$$

which ends the proof of Lemma 1.

## 3. COMPARISON OF SUBHARMONIC FUNCTIONS

The proof of Lemma 2 follows from the following more general result.

**Lemma 3.** *Let  $K \subset \mathbb{C}$  be a compact set such that  $\mathbb{C} - K$  is connected. Let  $v$  be a subharmonic function on  $\mathbb{C}$  such that  $\Delta v$  is supported on  $\partial K$  and does not charge the boundary of the connected components of the interior of  $K$ . Then, any subharmonic function  $u$  on  $\mathbb{C}$  which coincides with  $v$  outside  $K$  coincides with  $v$  everywhere.*

Let us stress out that the condition of  $\Delta v$  not charging the boundary components of the interior of  $K$  is essential. Indeed, taking for example  $K = \overline{\mathbb{D}}$  the closed unit disk,  $u(z) = \log |z|$  and  $v(z) = \max\{\log |z|, 0\}$ , one clearly has  $\text{supp}(\Delta v) = \partial K$ ,  $u = v$  outside  $K$  and  $u \neq v$ .

*Proof of Lemma 3.* According to [R, Theorem 3.8.3], for all  $\zeta \in \partial K$ ,

$$(1) \quad u(\zeta) = \limsup_{\substack{z \in \mathbb{C} - K \\ z \rightarrow \zeta}} u(z) = \limsup_{\substack{z \in \mathbb{C} - K \\ z \rightarrow \zeta}} v(z) = v(\zeta).$$

So,  $u = v$  on  $\partial K$ . The same theorem shows that, if  $\zeta$  belongs to the boundary of a connected component  $U$  of the interior of  $K$ , then

$$(2) \quad \limsup_{\substack{z \in \overline{U} \\ z \rightarrow \zeta}} u(z) = u(\zeta) = v(\zeta) = \limsup_{\substack{z \in \overline{U} \\ z \rightarrow \zeta}} v(z).$$

First, consider the function  $w_1$  defined by

$$w_1(z) = \begin{cases} \max(u, v) & \text{on } U \\ v & \text{on } \mathbb{C} - U. \end{cases}$$

According to Equations (1) and (2),  $w_1$  is upper-semicontinuous. It is subharmonic on  $U$  and  $\mathbb{C} - \overline{U}$ . At any point  $\zeta \in \partial U$ , it satisfies the local submean inequality since  $w_1(\zeta) = v(\zeta)$  and  $v \leq w_1$  on  $\mathbb{C}$ . Thus,  $w_1$  is globally subharmonic on  $\mathbb{C}$  and coincides with  $v$  outside  $U$ . Consider a smooth test function  $\chi : \mathbb{C} \rightarrow [0, 1]$  which vanishes near  $\infty$  and is constant equal to 1 near  $K$ . On the one hand,

$$\Delta v(\mathbb{C}) = \int_{\mathbb{C}} \chi \cdot \Delta v = \int_{\mathbb{C}} \Delta \chi \cdot v = \int_{\mathbb{C}} \Delta \chi \cdot w_1 = \int_{\mathbb{C}} \chi \cdot \Delta w_1 = \Delta w_1(\mathbb{C}).$$

On the other hand,  $\Delta v = \Delta w_1$  outside  $\overline{U}$ . Therefore,

$$\Delta w_1(\overline{U}) = \Delta w_1(\mathbb{C}) - \Delta w_1(\mathbb{C} - \overline{U}) = \Delta v(\mathbb{C}) - \Delta v(\mathbb{C} - \overline{U}) = \Delta v(\overline{U}) = 0.$$

So,  $\Delta v = \Delta w_1$  on  $\mathbb{C}$ . The difference  $v - w_1$  is harmonic and vanishes outside  $U$ . Therefore  $w_1 = v$  on  $\mathbb{C}$  and so,  $\max(u, v) = v$  on  $U$ . It follows that  $u \leq v$  on  $U$ . Since this holds for any connected component  $U$  of the interior of  $K$ , we have  $u \leq v$  on  $\mathbb{C}$ .

Next, consider the function  $w_2$  defined by

$$w_2(z) = \begin{cases} u & \text{on } U \\ v & \text{on } \mathbb{C} - U. \end{cases}$$

As before,  $w_2$  is upper-semicontinuous and subharmonic on  $U$  and  $\mathbb{C} - \overline{U}$ . At any point  $\zeta \in \partial U$ , it satisfies the local submean inequality since  $w_2(\zeta) = u(\zeta)$  and  $u \leq w_2$  on  $\mathbb{C}$ . As before,  $\Delta v = \Delta w_2$  outside  $\overline{U}$  and vanishes on  $\overline{U}$ , so that  $\Delta v = \Delta w_2$  on  $\mathbb{C}$ . The function  $v - w_2$  is harmonic on  $\mathbb{C}$  and vanishes outside  $\overline{U}$ . So,  $w_2 = v$  on  $\mathbb{C}$ . In particular  $u = v$  on  $U$ .

Since this is valid for any connected component  $U$  of the interior of  $K$ , we have  $u = v$  on  $\mathbb{C}$  as required.  $\square$

Let us remark that the proof can be simplified if  $v$  is continuous, which is the case in our situation. Indeed, the inequality  $u \leq v$  follows directly from the maximum principle and doesn't require the assumption that  $\Delta v$  doesn't charge  $\partial U$ .

*Proof of Lemma 2.* According to Zakeri [Z], there is a set of parameters  $c$  which is of full measure for  $\mu_{\text{bif}}$  such that

- $J_c$  is locally connected and full, in particular  $c$  is not in the boundary of a hyperbolic component of  $M$  and
- the orbit of  $c$  is dense in  $J_c$ , in particular  $c$  is not renormalizable and so, not in the boundary of a possible queer component of  $M$ .

As a consequence,  $\mu_{\text{bif}}$  does not charge the boundary of connected components of the interior of  $M$ . So, we may apply Lemma 3 with  $K := M$  and  $v := g_M + 2 \log 2$ . This yields Lemma 2.  $\square$

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