Distribution of postcritically finite polynomials

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$P \in \mathbb{C}[z]$ is *postcritically finite* (PCF) if the set

$$PC(P) := \bigcup_{n\geq 1} P^n(C(P))$$

is finite, with $C(P) := \{z \in \mathbb{C} \ : P'(z) = 0\}$.

We want to give an answer to the following question :

How do postcritically finite polynomials distribute in the space of all degree d polynomials when the (pre)period of each critical point tends to ∞ *?*

Polynomial dynamics

Let $P \in \mathbb{C}[z]$ have degree $d \geq 2$. The *Filled-in Julia set* $\mathcal{K}(P)$ of P is the subset of $\mathbb{C}$ given by

$$\mathcal{K}(P) := \{z \in \mathbb{C} \ : \ \{P^n(z)\} \text{ is bounded in } \mathbb{C}\} \ .$$

- $\mathcal{K}(P)$ is non-empty and compact,
- $\mathcal{K}(P)$ is connected iff $c \in \mathcal{K}(P)$ for all $c \in C(P)$.

The *Green function* $g_P : \mathbb{C} \to \mathbb{R}$ of P is

$$g_P(z) := \lim_{n \to \infty} d^{-n} \log^+ |P^n(z)| \ .$$

g_P is subharmonic an continuous and it coincides with the Green function of the compact set $\mathcal{K}(P)$ and

$$g_P \circ P(z) = d \cdot g_P(z), \ z \in \mathbb{C}.$$

Families and bifurcations

Let $(P_\lambda)_{\lambda\in X}$ be a holomorphic family, i.e.

- X is a connected complex manifold (or variety),
- $P_\lambda(z) = a_d(\lambda)z^d + a_{d-1}(\lambda)z^{d-1} + \cdots a_1(\lambda)z + a_0(\lambda)$ with $a_i(\lambda) \in \mathcal{O}(X)$,
- $\deg(P_\lambda) = d$ for all $\lambda \in X$,
- there exists $c_1, \ldots, c_{d-1} \in \mathcal{O}(X)$ such that $C(P_\lambda) = \{c_1(\lambda), \ldots, c_{d-1}(\lambda)\}$.

Theorem (Mañé-Sad-Sullivan, 83)

1. *$\{\lambda \mapsto P_\lambda^n(c_i(\lambda))\}_n$ is normal for all i at λ_0 iff $\lambda \mapsto \mathcal{K}(P_\lambda)$ is $\mathcal{C}^0$ in a neighborhood of λ_0 (λ_0 is said stable).*
2. *The set of stable parameters is open and dense.*

The *bifurcation locus* is the complement of the satibility locus.

The quadratic case.

For $c \in \mathbb{C}$, we define $P_c : \mathbb{C} \to \mathbb{C}$ by

$$P_c(z) := z^2 + c\,.$$

In fact, one can easily show that

$$\mathbb{C} \simeq \mathcal{P}_2 := \{P \in \mathbb{C}[z] \;:\; \deg(P) = 2\}/\sim\,,$$

where $P \sim Q$ if there exists $\varphi \in \mathrm{Aut}(\mathbb{C})$ such that

$$P \circ \varphi = \varphi \circ Q\,.$$

The **Mandelbrot set** is

$$\begin{aligned} \mathcal{M}_2 &:= \{c \in \mathbb{C} : (P_c^n(0)) \text{ is a bounded sequence of } \mathbb{C}\} \\ &= \{c \in \mathbb{C} : \mathcal{K}(P_c) \text{ is connected}\}\,. \end{aligned}$$

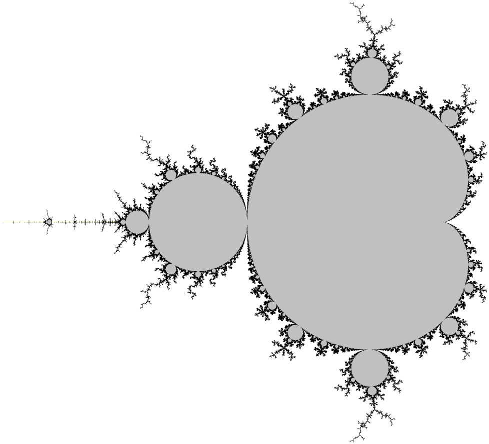

FIGURE: The Mandelbrot set

Theorem (Douday-Hubbard, 83)

$\mathcal{M}_2$ is a full compact connected subset of $\mathbb{C}$ and the set $\partial\mathcal{M}_2$ is the bifurcation locus.

Theorem (Shishikura, 91)

$\partial\mathcal{M}_2$ est homogène et $\dim_H(\partial\mathcal{M}_2) = 2$.

Let us set $g_{\mathcal{M}_2}(c) := g_{P_c}(c) = \lim_n 2^{-n} \log^+ |P_c^n(c)|, \ c \in \mathbb{C}$.

Proposition

The continuous subharmonic function $g_{\mathcal{M}_2}$ is the Green function of $\mathcal{M}_2$. In particular, $\mathrm{Supp}(\mu_{\mathrm{bif}}) = \partial\mathcal{M}_2$.

The probability measure

$$\mu_{\mathrm{bif}} := \Delta g_{\mathcal{M}_2}$$

is called **bifurcation measure**.

Complex (local) method

Let us set $\mathrm{Per}(n) := \{c \in \mathbb{C} \ : \ P_c^n(0) = 0\}$ and

$$\mu_n := \frac{1}{2^{n-1}} \sum_{c \in \mathrm{Per}(n)} \delta_c = \frac{1}{2^{n-1}} [\mathrm{Per}(n)] \ .$$

Proposition

$\partial \mathcal{M}_2 \subset \overline{\bigcup_{n\geq 1} \mathrm{Per}(n)}$.

Idea of the proof. — Montel's Theorem. □

Theorem (Levin, 91)

The sequence μ_n converges to μ_{bif} in the sense of Radon measures.

Idea of the proof. — Hartogs Lemma for subharmonic functions.

Arithmetic (global) method

This method goes back to Baker and H'sia in 2004.

Theorem (Favre-Rivera Letelier, 06)

$$\left| \int_{\mathbb{C}} \varphi \cdot d\mu_n - \int_{\mathbb{C}} \varphi \cdot d\mu_{\text{bif}} \right| \le C(\mu_{\text{bif}}) \cdot \left(\frac{n}{2^n} \right)^{\frac{1}{2}} \|\varphi\|_{\mathcal{C}^1(\mathbb{C})} ,$$

for any $\varphi \in \mathcal{C}^1(\mathbb{C})$ *with compact support and any* $n \ge 1$*.*

First important informations :

- $P_c^n(0) \in \mathbb{Z}[c]$,
- $P_c^n(0) = 0$ has only simple roots (Douady-Hubbard) and

$$\text{Card}(\text{Per}(n)) = \deg_c(P_c^n(0)) = 2^{n-1} .$$

Height functions

For $p \in \mathcal{P} = \{p \in \mathbb{Z}_+ \ : \ p \text{ is prime}\} \cup \{\infty\}$, the p-adic norm of $x \in \mathbb{Q}$ is $|x|_p = 0$ if $x = 0$. Otherwise,

$$|x|_p := \begin{cases} p^{-\mathrm{ord}_p(x)} & \text{if } p < \infty, \\ |x| & \text{if } p = \infty. \end{cases}$$

The naive height function is $h_{\mathrm{nv}} : \overline{\mathbb{Q}} \to \mathbb{R}_+$ given for $x \in \overline{\mathbb{Q}}$ by

$$h_{\mathrm{nv}}(x) := \frac{1}{\deg(x)} \sum_{p \in \mathcal{P}} \sum_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \log^+ |\sigma(x)|_p \ ,$$

where $\deg(x) = [\mathbb{Q}[x] : \mathbb{Q}]$ is the degree of minimal polynomial of x.

As in the complex case, for $p < \infty$, one can set

$$g_{\mathcal{M}_2,p}(c) := \lim_{n\to\infty} 2^{-n} \log^+ |P_c^n(c)|_p \,.$$

The **bifurcation height function** is $h_{\mathrm{bif}} : \overline{\mathbb{Q}} \to \mathbb{R}_+$ given by

$$h_{\mathrm{bif}}(x) := \frac{1}{\deg(x)} \sum_{p\in\mathcal{P}} \sum_{\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} g_{\mathcal{M}_2,p}(\sigma(c)) \,.$$

Proposition (Northcott)

- *For* $p > 2$, $g_{\mathcal{M}_2,p}(c) = \log^+ |c|_p$ *and* $|h_{\mathrm{nv}} - h_{\mathrm{bif}}| \leq C < \infty$,
- $h_{\mathrm{bif}}(c) = 0$ *iff there exists* $n > k \geq 0$ *with* $P_c^n(0) = P_c^k(0)$.

Baker-H'sia's equidistribution Theorem

The heart of the proof is an improvement of

Theorem (Baker-H'sia, 04)

Let $X_n \subset \mathbb{C}$ be a sequence of finite sets with

- $X_n \neq X_m$ *if* $n \neq m$,
- $X_n = \{P_n = 0\}$ *for some* $P_n \in \mathbb{Q}[X]$,
- $h(x_n) \to_{n\to\infty} 0$ *for all* $x_n \in X_n$.

Then, in the weak sense of measures,

$$\mu_n := \frac{1}{\mathrm{Card}(X_n)} \sum_{c \in X_n} \delta_c \to_{n\to\infty} \mu_{\mathrm{bif}} \ .$$

The cubic case

For $(c,a) \in \mathbb{C}^2$, we define $P_{c,a} : \mathbb{C} \to \mathbb{C}$ by setting

$$P_{c,a}(z) := \frac{1}{3}z^3 - \frac{c}{2}z^2 + a^3 \ .$$

This time, the map $\pi : (c,a) \in \mathbb{C} \mapsto [P_{c,a}] \in \mathcal{P}_3$ is finite to one and $C(P_{c,a}) = \{0, c\}$. Set

$$\mathcal{B}_0 := \{(c,a) \in \mathbb{C}^2 : P^n_{c,a}(0) \text{ is bounded}\},$$

$$\mathcal{B}_c := \{(c,a) \in \mathbb{C}^2 : P^n_{c,a}(c) \text{ is bounded}\}.$$

The **connectedness locus** $\mathcal{M}_3$ is

$$\mathcal{M}_3 := \{(c,a) \in \mathbb{C}^2 : \mathcal{K}(P_{c,a}) \ \text{ is connected}\} = \mathcal{B}_0 \cap \mathcal{B}_c \ .$$

Theorem (Branner-Hubbard, 93)

The set $\mathcal{M}_3$ is a compact subset of $\mathbb{C}^2$.

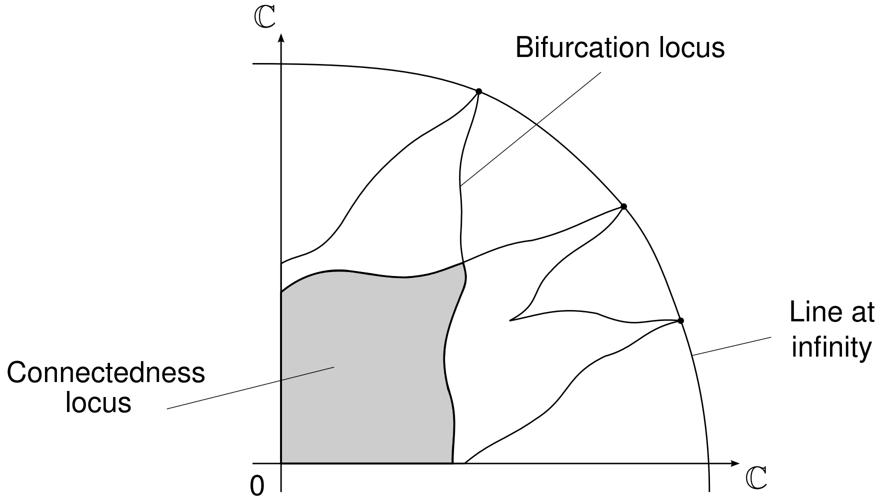

FIGURE: Behavior of the bifurcation locus at the infinity of the space of cubic polynomials

Bifurcations

In the present setting, the bifucation locus is given by

$$\mathrm{Bif} := \partial\mathcal{B}_0 \cup \partial\mathcal{B}_c.$$

The functions $g_0 : (c,a) \mapsto g_{c,a}(0)$ and $g_c(c,a) \mapsto g_{c,a}(c)$ psh and continuous and we set

$$T_0 := dd^c g_0, \quad \text{et} \quad T_c := dd^c g_c \ .$$

Theorem (DeMarco, 00 / Dujardin-Favre, 07)

- $\mathrm{Bif} = \mathrm{Supp}(T_0 + T_c)$ *(DeMarco)*,
- $\mathrm{Supp}(T_0) = \partial\mathcal{B}_0$ *and* $\mathrm{Supp}(T_c) = \partial\mathcal{B}_c$ *and* T_0 *and* T_c *have projective mass* 1 *(Dujardin-Favre).*

Questions :

- What happens on $\partial\mathcal{B}_0 \cap \partial\mathcal{B}_c$?
- Can we characterize $\mathrm{Supp}(T_0 \wedge T_c)$?

Let $G(c,a) := \max\{g_0(c,a), g_{c,a}(c)\}$ and

$$\mu_{\mathrm{bif}} := T_0 \wedge T_c = (dd^c G)^2.$$

Theorem (Dujardin-Favre, 07)

- *G is the pluricomplex Green function of $\mathcal{M}_3$,*
- *The positive measure μ_{bif} is a probability measure and*

$$\mathrm{Supp}(\mu_{\mathrm{bif}}) = \partial_S \mathcal{M}_3 \subsetneq (\partial\mathcal{B}_1 \cap \partial\mathcal{B}_1) \subsetneq \partial\mathcal{M}_3.$$

Theorem (G., 12)

$$\dim_H(\partial_S \mathcal{M}_3) = 4.$$

Complex (local) method

We set

$$\mathrm{Per}_0(n) := \{(c,a) \in \mathbb{C}^2 : P^n_{c,a}(0) = 0\},$$

and

$$\mathrm{Per}_c(m) := \{(c,a) \in \mathbb{C}^2 : P^m_{c,a}(c) = c\}.$$

Again, one has by Montel

Proposition

- $\mathrm{Supp}(T_0) \subset \overline{\bigcup_{n\geq 1} \mathrm{Per}_0(n)}$,
- $\mathrm{Supp}(T_c) \subset \overline{\bigcup_{m\geq 1} \mathrm{Per}_c(m)}$.

By Hartogs Lemma (plus real difficulties to apply it)

Theorem (Dujardin-Favre, 07)

In the weak sense of currents,

- $3^{-n}[\mathrm{Per}_0(n)] \longrightarrow_{n\to\infty} T_0$,
- $3^{-n}[\mathrm{Per}_c(n)] \longrightarrow_{n\to\infty} T_c$.

Question : What can we say about μ_{bif} ?

Theorem (Bassanelli-Berteloot, 07)

$$\mathrm{Supp}(\mu_{\mathrm{bif}}) \subset \overline{\bigcup_{n,m\geq 1} \mathrm{Per}_0(n) \cap \mathrm{Per}_c(m)}.$$

Also, as an immediate corrolary of the work of Dujardin-Favre,

$$\mu_{\mathrm{bif}} = \lim_{m\to\infty} \lim_{n\to\infty} 3^{-(n+m)}[\mathrm{Per}_0(n) \cap \mathrm{Per}_c(m)] \ .$$

Arithmetic (global) method

Theorem (Favre-G., 13)

Let $n_k, m_k \to_{k\to\infty} \infty$ *be such that* $n_k \neq m_k$ *for all* $k \geq 1$*. Then the set* $F_k := \mathrm{Per}_0(n_k) \cap \mathrm{Per}_c(m_k)$ *is finite and the sequence of measures*

$$3^{-(n_k+m_k)} \sum_{(c,a)\in F_k} \delta_{c,a}$$

converges weakly to μ_{bif} *as* $k \to \infty$*.*

- The finiteness of F_k comes from $F_k \subset \mathcal{M}_3$,
- it makes sense to try to use arithmetic methods, since

$$P_{c,a}^{n_k}(0) \in \mathbb{Q}[c,a] \text{ and } P_{c,a}^{m_k}(c) - c \in \mathbb{Q}[c,a],$$

which means that we have $F_k \subset \overline{\mathbb{Q}}^2$.

The bifurcation height function $h_{\mathrm{bif}} : \overline{\mathbb{Q}}^2 \to \mathbb{R}^+$ is given by

$$H_{\mathrm{bif}}(c,a) := \frac{1}{\deg(c,a)} \sum_{p \in \mathcal{P}} \sum_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} G_p(\sigma(c,a)) \ ,$$

where $G_p(c,a) = \lim_{n\to\infty} 3^{-n} \log^+ \max\{|P^n_{c,a}(0)|_p, |P^n_{c,a}(c)|_p\}$.

Proposition (Northcott)

$H_{\mathrm{bif}}(c,a) = 0$ iff there exists $n > k \geq 0$ and $m > r \geq 0$ such that

$$P^n_{c,a}(0) = P^k_{c,a}(0) \ \ \textit{and} \ \ P^m_{c,a}(c) = P^r_{c,a}(c) \ ,$$

i.e. $\{H_{\mathrm{bif}} = 0\} = \bigcup_{n,m\geq 1} \mathrm{Per}_0(n) \cap \mathrm{Per}_c(m)$. In particular, one has $H_{\mathrm{bif}}(c,a) = 0$ for all $(c,a) \in F_k$ for all $k \geq 1$.

Yuan's equidistribution Theorem

Theorem (Yuan, 08)

Let $X_k \subset \mathbb{C}^2$ be a sequence of finite sets with

- $X_k = \{P_k = 0\} \cap \{Q_k = 0\}$ *for some* $P_k, Q_k \in \mathbb{Q}[X]$,
- $H_{\text{bif}}(x_k) \to_{n\to\infty} 0$ *for all* $x_k \in X_k$,
- *for any irreducible curve* $Z \subset \mathbb{C}^2$,

$$\lim_{k\to\infty} \frac{\text{Card}(X_k \cap Z)}{\text{Card}(X_k)} = 0 .$$

Then, in the weak sense of measures,

$$\mu_k := \frac{1}{\text{Card}(X_k)} \sum_{(c,a)\in X_k} \delta_{c,a} \to_{n\to\infty} \mu_{\text{bif}} .$$

For a given irreducible algebraic curve $Z \subset \mathbb{C}^2$, we need

$$\mathrm{Card}(F_k \cap Z) = o(\mathrm{Card}(F_k)).$$

On the one hand, by Bezout,

$$\mathrm{Card}(F_k \cap Z) \leq 2\deg(Z) \cdot 3^{\max\{n_k, m_k\}}.$$

Theorem (Favre-G following Epstein's method, 13)

When $n \neq m$, the curves $\mathrm{Per}_0(n)$ and $\mathrm{Per}_c(m)$ are smooth and transverse at their intersection points.

In particular,

$$\mathrm{Card}(F_k) = \deg(\mathrm{Per}_0(n_k)) \cdot \deg(\mathrm{Per}_c(m_k)) = 3^{n_k + m_k}.$$

One consequence

For any $w \in \mathbb{C}$ and any $n \geq 1$, the set

$$\mathrm{Per}(n, w) \subset \mathbb{C}^2$$

of parameters $(c, a) \in \mathbb{C}^2$ for which $P_{c,a}$ has a periodic point z of exact period n and multiplier $(P^n_{c,a})'(z) = w$ is known to be an algebraic curve (Silverman and Milnor).

Theorem (Favre-G., 13)

Let $n_k, m_k \to_{k\to\infty} \infty$ be such that $n_k \neq m_k$ for all $k \geq 1$. Let also $w_1, w_2 \in \mathbb{D}$. Then the set the sequence of measures

$$\frac{3^{-(n_k+m_k)}}{4} \cdot [\mathrm{Per}(n_k, w_1) \cap \mathrm{Per}(m_k, w_2)]$$

converges weakly to μ_{bif} as $k \to \infty$.

Idea of the proof

First assume $w_1 = w_2 = 0$. For any k,

$$\mathrm{Per}_0(n_k) \cup \mathrm{Per}_c(n_k) = \bigcup_{l | n_k} \mathrm{Per}(l, 0)$$

Hence, by the Möbius inversion formula,

$$\deg(\mathrm{Per}(n_k, 0)) \sim 2 \cdot 3^{n_k}.$$

According to the transversality theorem, one has

$$\deg(\mathrm{Per}(n_k, 0) \cap \mathrm{Per}(m_k, 0)) \sim 4 \cdot 3^{n_k + m_k}.$$

The conclusion follows from the previous Theorem.

Here are the two main tools for the second step.

Theorem

Let $U \subset \mathbb{C}^2$ be a stable open set such that for $(c, a) \in U$, the polynomial $P_{c,a}$ has two attracting cycles $z_1(c, a)$ and $z_1(c, a)$ of repective exact period $n \neq m \geq 1$. Then the map

$$(c, a) \in U \longmapsto ((P^n_{c,a})'(z_1(c, a)), (P^m_{c,a})'(z_2(c, a))) \in \mathbb{D}^2$$

is a biholomorphism.

Lemma (Briend-Duval, 01)

There exists $C > 0$ such that for any holomorphic disks $\mathbb{D}_1 \Subset \mathbb{D}_2 \subset \mathbb{P}^2(\mathbb{C})$,

$$(\mathrm{diam}(\mathbb{D}_1))^2 \leq C \cdot \frac{\mathrm{Area}(\mathbb{D}_1)}{\min(1, \mathrm{mod}(\mathbb{D}_2 \setminus \mathbb{D}_1))} .$$

THANK YOU FOR YOUR ATTENTION !