DYNAMICAL PAIRS WITH AN ABSOLUTELY CONTINUOUS BIFURCATION MEASURE

by

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Abstract. — In this article, we study algebraic dynamical pairs (f, a) parametrized by an irreducible quasi-projective curve Λ having an absolutely continuous bifurcation measure. We prove that, if f is non-isotrivial and (f, a) is unstable, this is equivalent to the fact that f is a family of Lattès maps. To do so, we prove the density of transversely prerepelling parameters in the bifucation locus of (f, a) and a similarity property, at any transversely prerepelling parameter λ_0, between the measure $\mu_{f,a}$ and the maximal entropy measure of f_{λ_0}. We also establish an equivalent result for dynamical pairs of $\mathbb{P}^k$, under an additional mild assumption.

Résumé. — Dans cet article, nous étudions les paires dynamiques algébriques (f, a) paramétrées par une courbe quasi-projective irréductible possédant une mesure de bifurcation absolument continue. Nous prouvons que, si la famille f n'est pas isotriviale et si la paire (f, a) est instable, c'est équivalent au fait que la famille f soit une famille d'exemples de Lattès flexibles. A cette fin, nous montrons la densité des paramètres transversalement prérépulsifs dans le lieu de bifurcation de la paire (f, a), ainsi qu'une propriété de similarité, en un paramètre transversalement prérépulsif λ_0, entre la mesure de bifurcation $\mu_{f,a}$ et la mesure d'entropie maximale de f_{λ_0}. Sous une hypothèse relativement générale, nous établissons également un résultat similaire pour les paires dynamiques de $\mathbb{P}^k$.

Introduction

Let Λ be an irreducible quasi-projective complex curve. An algebraic *dynamical pair* (f, a) parametrized by Λ is an algebraic family $f : \Lambda \times \mathbb{P}^1 \to \mathbb{P}^1$ of rational maps of degree $d \geq 2$, i.e. f is a morphism and f_λ is a degree d rational map for all $\lambda \in \Lambda$, together with a marked point a, i.e. a morphism $a : \Lambda \to \mathbb{P}^1$.

Recall that a dynamical pair (f, a) is *stable* if the sequence $\{\lambda \mapsto f_\lambda^n(a(\lambda))\}_{n\geq 1}$ is a normal family on Λ. Otherwise, we say that the pair (f, a) is *unstable*. Recall also that f is *isotrivial* if there exists a branched cover $X \to \Lambda$ and an algebraic family of Möbius transformations $M : X \times \mathbb{P}^1 \to \mathbb{P}^1$ so that $M_\lambda \circ f_\lambda \circ M_\lambda^{-1} : \mathbb{P}^1 \to \mathbb{P}^1$ is independent of the parameter λ and that the pair (f, a) is *isotrivial* if, in addition, $M_\lambda(a(\lambda))$ is also independent of the parameter λ. A result of DeMarco [**De**] states that any stable algebraic pair is either isotrivial or preperiodic, i.e. there exists $n > m \geq 0$ such that $f_\lambda^n(a(\lambda)) = f_\lambda^m(a(\lambda))$ for all $\lambda \in \Lambda$.

When a dynamical pair (f, a) is unstable, the *stability locus* $\mathrm{Stab}(f, a)$ is the set of points $\lambda_0 \in \Lambda$ admitting a neighborhood U on which the pair (f, a) the sequence $\{\lambda \mapsto f_\lambda^n(a(\lambda))\}_{n\geq 1}$ is a normal family. The *bifurcation locus* $\mathrm{Bif}(f, a)$ of the pair (f, a) is its complement $\mathrm{Bif}(f, a) := \Lambda \setminus \mathrm{Stab}(f, a)$. If a is the marking of a critical point, i.e. $f'_\lambda(a(\lambda)) = 0$ for all $\lambda \in \Lambda$, it is classical that the bifurcation locus $\mathrm{Bif}(f, a)$ has empty interior, [**MSS**]. However, it is not clear wether it can have non-empty interior when f is not a family of polynomials and a is not a marked critical point. For instance, if f is a trivial family, $f_\lambda = f_{\lambda'}$ for all $\lambda, \lambda' \in \Lambda$ and $J_f = \mathbb{P}^1$, then $\mathrm{Bif}(f,a)$ is either empty or the whole parameter space Λ. In fact, we can describe precisely when $\mathrm{Bif}(f,a)$ can have non-empty interior.

We say that a family $f : \Lambda \times \mathbb{P}^1 \longrightarrow \Lambda \times \mathbb{P}^1$ of degree d rational maps of $\mathbb{P}^1$ is *J-stable* if all the repelling cycles can be followed holomorphically throughout the whole family Λ, i.e. if for all $n \geq 1$, there exists $N \geq 0$ and holomorphic maps $z_1, \ldots, z_N : \Lambda \to \mathbb{P}^1$ such that $\{z_1(\lambda), \ldots, z_N(\lambda)\}$ is exactly the set of all repelling cycles of f_λ of exact period n for all $\lambda \in \Lambda$. Note that this is equivalent to the fact that all critical points are stable [**MSS**]. We prove the following:

Theorem A. — *Let (f,a) be a dynamical pair of degree d of $\mathbb{P}^1$ parametrized by a one-dimensional complex manifold Λ. Assume that* $\mathrm{Bif}(f,a) = \Lambda$*. Then f is J-stable and*
- *either f is trivial,*
- *or $J_{f_\lambda} = \mathbb{P}^1$ and f_λ carries an invariant linefield for any $\lambda \in \Lambda$.*

The bifurcation locus of a pair (f,a) is the support of natural a positive (finite) measure: the *bifurcation measure $\mu_{f,a}$* of the pair (f,a), see Section 1 for a precise definition. The properties of this measure appear to be very important for studying arithmetic and dynamical properties of the pair (f,a), see e.g. [**BD1, BD2, De, DM, DMWY, FG1, FG2, FG3**]. Note also that the entropy theory of dynamical pairs has been recently developed in [**DGV**]. In the present article, we study algebraic dynamical pairs having an absolutely continuous bifurcation measure.

Assume that for some parameter $\lambda_0 \in \Lambda$, the marked point a eventually lands on a repelling periodic point x, that is $f^n_{\lambda_0}(a(\lambda_0)) = x$. Let $x(\lambda)$ be the (local) natural continuation of x as a periodic point of f_λ. We say that a is *transversely prerepelling* at λ_0 if the graphs of $\lambda \mapsto f^n_\lambda(a(\lambda))$ and $\lambda \mapsto x(\lambda)$, as subsets of $\Lambda \times \mathbb{P}^1$, are transverse at λ_0.

Finally, recall that a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a *Lattès map* if there exists an elliptic curve E, an endomorphism $L : E \to E$ and a finite branched cover $p : E \to \mathbb{P}^1$ such that $p \circ L = f \circ p$ on E. Such a map has an absolutely continuous maximal entropy measure, see [**Z**]. On the other hand, when f is a family of Lattès maps and the pair (f,a) is unstable, then $\mathrm{Bif}(f,a) = \Lambda$, see e.g. [**DM**, §6].

Our main result is the following.

Theorem B. — *Let (f,a) be a dynamical pair of $\mathbb{P}^1$ of degree $d \geq 2$ parametrized by an irreducible quasi-projective curve Λ. Assume that f is non-isotrivial and that (f,a) is unstable. The following assertions are equivalent:*
1. *The bifurcation locus of the dynamical pair (f,a) is* $\mathrm{Bif}(f,a) = \Lambda$*,*
2. *Transversely prerepelling parameters are dense in Λ,*
3. *The bifurcation measure $\mu_{f,a}$ of the pair (f,a) is absolutely continuous with continuous Radon-Nikodym derivative,*
4. *The family f is a family of Lattès maps.*

Note that the hypothesis that f is not isotrivial is necessary to have the equivalence between 1. and 4. (see Proposition 4.2).

The first step of the proof consists in proving that transversely prerepelling parameters are dense in the support of $\mu_{f,a}$. Using properties of Polynomial-Like Maps in higher dimension and a transversality Theorem of Dujardin for laminar currents [**Duj**], under a mild assumption on Lyapunov exponents, we prove this property holds for the appropriate bifurcation current for any tuple $(f, a_1, \ldots, a_m)$, where $f : \Lambda \times \mathbb{P}^k \to \mathbb{P}^k$ is any holomorphic family of endomorphisms of $\mathbb{P}^k$ and $a_1, \ldots, a_m : \Lambda \to \mathbb{P}^k$ are any marked points (see Theorem 2.2).

As a second step, we adapt the similarity argument of Tan Lei [**T**] to show that, if λ_0 is a transversely prerepelling parameter where the bifurcation measure is absolutely continuous, the maximal entropy measure $\mu_{f_{\lambda_0}}$ of f_{λ_0} is also non-singular with respect to the Fubini-Study form on $\mathbb{P}^1$. As Zdunik [**Z**] has shown, this implies f_{λ_0} is a Lattès map.
This gives, in particular, the following.

Theorem C. — *Fix integers $d \geq 2$and let (f, a) be a holomorphic dynamical pair of degree d of $\mathbb{P}^1$ parametrized by a Kähler manifold (M, ω) of dimension* 1*. Assume the support of $\mu_{f,a}$ is* $\mathrm{supp}(\mu_{f,a}) = M$*. Then, the following are equivalent:*

1. *the measure $\mu_{f,a}$ is absolutely continuous with respect to ω and $\frac{d\mu_{f,a}}{d\omega}$ is $\mathcal{C}^0$,*
2. *the family f is a family of Lattès maps.*

We can see Theorem B as a partial parametric counterpart of Zdunik's result. However, the comparison with Zdunik's work ends there: Rational maps with $\mathbb{P}^1$ as a Julia sets are, in general, not Lattès maps. Indeed, Lattès maps form a strict subvariety of the space of all degree d rational maps, and maps with $J_f = \mathbb{P}^1$ form a set of positive volume by [**R**]. In a way, Theorem B is a stronger rigidity statement that the dynamical one.
Note also that we only use the fact that Λ is a quasi-projective curve to prove the equivalence between $\mathrm{Bif}(f, a) = \Lambda$ and the smoothness of the bifurcation measure, relying on [**Mc**]. We don't know how to get rid of this algebraicity assumption, without using the No Invariant Line Field Conjecture of McMullen, which is far from being proved.

Recall that, as in dimension 1, an endomorphism f of $\mathbb{P}^k$ is a *Lattès map* if there exists an abelian variety A, a finite branched cover $p : A \to \mathbb{P}^k$ and an isogeny $I : A \to A$ such that $p \circ I = f \circ p$ on A. Berteloot and Loeb [**BL**] and then Berteloot and Dupont [**BD3**] generalized Zdunik's work to endomorphisms of $\mathbb{P}^k$: f is a Lattès map of $\mathbb{P}^k$ if and only if the measure μ_f is is not singular with respect to $\omega^k_{\mathbb{P}^k}$, see also [**Dup**]. Recall finally that a repelling periodic point of f is *J-repelling* if it belongs to $\mathrm{supp}(\mu_f)$.
As an important part of our arguments applies in any dimension, we have the following higher dimensional counterpart to Theorem C.

Theorem D. — *Fix integers $d \geq 2$ and $k \geq 1$ and let (f, a) be any holomorphic dynamical pair of degree d of $\mathbb{P}^k$ parametrized by a Kähler manifold (M, ω) of dimension k. Assume that for all $\lambda \in M$, any J-repelling periodic point of f_λ is linearizable. Assume in addition that $\mu_{f,a} := T^k_{f,a}$ satisifies* $\mathrm{supp}(\mu_{f,a}) = M$*. Then the following are equivalent:*

1. *the measure $\mu_{f,a}$ is absolutely continuous with respect to ω^k and $\frac{d\mu_{f,a}}{d\omega^k}$ is $\mathcal{C}^0$,*
2. *the family f is a family of Lattès maps of $\mathbb{P}^k$.*

The paper is organised as follows. In section 1, we recall the construction of the biurcation currents of marked points and properties of Polynomial-Like Maps. Section 2 is dedicated to proving the density of transversely prerepelling parameters. In section 3, we establish the similarity proprety for the bifurcation and maximal entropy measures. Finally, in section 4 we prove Theorems A, B, C and D and list related questions.

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1. Dynamical preliminaries

1.1. The bifurcation current of a dynamical tuple

For this section, we follow the presentation of [**DF, Duj**]. Even though everything is presented in the case $k=1$ and for marked *critical* points, the exact same arguments give what we present below.

Let Λ be a complex manifold and let $f:\Lambda\times\mathbb{P}^k\to\mathbb{P}^k$ be a holomorphic family of endomorphisms of $\mathbb{P}^k$ of algebraic degree $d\geq 2$: f is holomorphic and $f_\lambda:=f(\lambda,\cdot):\mathbb{P}^k\to\mathbb{P}^k$ is an endomorphism of algebraic degree d.

Definition 1.1. — *Fix integers $m\geq 1$, $d\geq 2$ and let Λ be a complex manifold. A* dynamical $(m+1)$-tuple $(f,a_1,\ldots,a_m)$ *of $\mathbb{P}^k$ of degree d parametrized by Λ is a holomorphic family f of endomorphisms of $\mathbb{P}^k$ of degree d parametrized by Λ, endowed with m holomorphic maps* (marked points) $a_1,\ldots,a_m:\Lambda\to\mathbb{P}^k$.

Let $\omega_{\mathbb{P}^k}$ be the standard Fubini-Study form on $\mathbb{P}^k$ and $\pi_\Lambda:\Lambda\times\mathbb{P}^k\to\Lambda$ and $\pi_{\mathbb{P}^k}:\Lambda\times\mathbb{P}^k\to\mathbb{P}^k$ be the canonical projections. Finally, let $\widehat{\omega}:=(\pi_{\mathbb{P}^k})^*\omega_{\mathbb{P}^k}$. A family $f:\Lambda\times\mathbb{P}^k\to\mathbb{P}^k$ naturally induces a fibered dynamical system $f:\Lambda\times\mathbb{P}^k\to\Lambda\times\mathbb{P}^k$, given by $\hat{f}(\lambda,z):=(\lambda,f_\lambda(z))$. It is known that the sequence $d^{-n}(\hat{f}^n)^*\widehat{\omega}$ converges to a closed positive $(1,1)$-current $\widehat{T}$ on $\Lambda\times\mathbb{P}^k$ with continuous potential. Moreover, for any $1\leq j\leq k$,

$$\hat{f}^*\widehat{T}^j=d^j\cdot\widehat{T}^j$$

and $\widehat{T}^k|_{\{\lambda_0\}\times\mathbb{P}^1}=\mu_{\lambda_0}$ is the unique measure of maximal entropy $k\log d$ of f_{λ_0} for all $\lambda_0\in\Lambda$.

For any $n\geq 1$, we have $\widehat{T}=d^{-n}(\hat{f}^n)^*\hat{\omega}+d^{-n}dd^c\widehat{u}_n$, where $(\widehat{u}_n)_n$ is a locally uniformly bounded sequence of continuous functions.

Pick now a dynamical $(m+1)$-tuple $(f,a_1,\ldots,a_m)$ of degree d of $\mathbb{P}^k$. Let $\Gamma_{a_j}\subset\Lambda\times\mathbb{P}^k$ be the graph of the map a_j and set

$$\mathfrak{a}:=(a_1,\ldots,a_m).$$

Definition 1.2. — *For $1\leq i\leq m$, the* bifurcation current T_{f,a_i} *of the pair (f,a_i) is the closed positive $(1,1)$-current on Λ defined by*

$$T_{f,a_i}:=(\pi_\Lambda)_*\left(\widehat{T}\wedge[\Gamma_{a_j}]\right)$$

and we define the bifurcation current $T_{f,\mathfrak{a}}$ *of the $(m+1)$-tuple $(f,a_1,\ldots,a_m)$ as*

$$T_{f,\mathfrak{a}}:=T_{f,a_1}+\cdots+T_{f,a_k}.$$

For any $\ell\geq 0$, write

$$\mathfrak{a}_\ell(\lambda):=\left(f_\lambda^\ell(a_1(\lambda)),\ldots,f_\lambda^\ell(a_m(\lambda))\right),\ \lambda\in\Lambda.$$

Let now $K\Subset\Lambda$ be a compact subset of Λ and let Ω be some relatively compact neighborhood of K, then $(a_\ell)^*(\omega_{\mathbb{P}^k})$ is bounded in mass in Ω by Cd^ℓ, where C depends on Ω but not on ℓ.

Applying verbatim the proof of [**DF**, Proposition-Definition 3.1 and Theorem 3.2], we have the following.

Lemma 1.3. — *For any $1 \leq i \leq k$, the support of T_{f,a_i} is the set of parameters $\lambda_0 \in \Lambda$ such that the sequence $\{\lambda \mapsto f_\lambda^n(a_i(\lambda))\}$ is not a normal family at λ_0.*

Moreover, writing $a_{i,\ell}(\lambda) := f_\lambda^\ell(a_i(\lambda))$, there exists a locally uniformly bounded family $(u_{i,\ell})$ of continuous functions on Λ such that

$$(a_{i,\ell})^*(\omega_{\mathbb{P}^q}) = d^\ell T_{f,a_i} + dd^c u_{i,\ell} \text{ on } \Lambda.$$

As a consequence, for all $j \geq 1$, we have

$$(a_{i,\ell})^*(\omega_{\mathbb{P}^k}^j) = d^{j\ell} T_{f,a_i}^j + \sum_{s=1}^{j} \binom{j}{s} d^{\ell(j-s)} \cdot (dd^c u_{i,\ell})^s \wedge T_{f,a_i}^{j-s},$$

so that the mass of the (j,j)-current $(a_{i,\ell})^*(\omega_{\mathbb{P}^k}^j) - d^{j\ell} T_{f,a_i}^j$ is $O(d^{(j-1)\ell})$ on compact subsets of Λ. In particular, one sees that

$$T_{f,a_i}^{k+1} = 0 \text{ on } \Lambda. \tag{1}$$

Let us still denote $\pi_\Lambda : \Lambda \times (\mathbb{P}^k)^m \to \Lambda$ be the projection onto the first coordinate and for $1 \leq i \leq k$, let $\pi_i : \Lambda \times (\mathbb{P}^k)^m \to \mathbb{P}^q$ be the projection onto the i-th factor of the product $(\mathbb{P}^k)^m$. Finally, we denote by $\Gamma_{\mathfrak{a}}$ the graph of $\mathfrak{a}$:

$$\Gamma_{\mathfrak{a}} := \{(\lambda, z_1, \ldots, z_m),\ \forall j, z_j = a_j(\lambda)\} \subset \Lambda \times (\mathbb{P}^k)^m.$$

Following verbatim the proof of [**AGMV**, Lemma 2.6], we get

$$\frac{1}{(mk)!} T_{f,\mathfrak{a}}^{mk} = \bigwedge_{\ell=1}^{m} T_{f,a_\ell}^k = (\pi_\Lambda)_* \left(\bigwedge_{i=1}^{m} \pi_i^* \left(\hat{T}^k \right) \wedge [\Gamma_{\mathfrak{a}}] \right).$$

1.2. Hyperbolic sets supporting a PLB ergodic measure

Definition 1.4. — *Let $W \subset \mathbb{C}^k$ be a bounded open set. We say that a positive measure ν compactly supported on W is* PLB *if the psh functions on W are integrable with respect to ν.*

We aim here at proving the following proposition in the spirit of [**Duj**, Lemma 4.1]:

Proposition 1.5. — *Pick an endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ of degree $d \geq 2$. There exists a small ball $\mathbb{B} \subset \mathbb{P}^k$, an integer $m \geq 1$, a f^m-invariant compact set $K \Subset \mathbb{B}$ and an integer $N \geq 2$ such that*
- *$f^m|_K$ is uniformly expanding and repelling periodic points of f^m are dense in K,*
- *there exists a unique probability measure ν supported on K such that $(f^m|_K)^*\nu = N\nu$ which is PLB.*

Even though this result is considered folklore, we include a proof relying on properties of *polynomial-like map*. We refer to [**DS**] for more about polynomial-like maps. Given an complex manifold M and an open set $V \subset M$, we say that V is *S-convex* if there exists a continuous strictly plurisubharmonic function on V. In fact, this implies that there exists a smooth strictly psh function ψ, whence there exists a Kähler form $\omega := dd^c\psi$ on V.

Definition 1.6. — *Given a connected S-convex open set and a relatively compact open set $U \Subset V$, a map $f : U \to V$ is* polynomial-like *if f is holomorphic and proper.*

The *filled-Julia set* of f is the set

$$\mathcal{K}_f := \bigcap_{n\geq 0} f^{-n}(U).$$

The set $\mathcal{K}_f$ is full, compact, non-empty and it is the largest totally invariant compact subset of V, i.e. such that $f^{-1}(\mathcal{K}_f) = \mathcal{K}_f$.

The topological degree d_t of f is the number of preimages of any $z \in V$ by f, counted with multiplicity. Let $k := \dim V$. We define

$$d^*_{k-1} := \sup_{\varphi} \left\{ d_t \cdot \limsup_{n\to\infty} \|\Psi^n dd^c\varphi\|_U^{1/n} \ ; \ \varphi \text{ is psh on } V \right\},$$

where $\Psi := d_t^{-1} f_*$. According to Theorem 3.2.1 and Theorem 3.9.5 of [**DS**], we have the following.

Theorem 1.7 (Dinh-Sibony). — *Let $f : U \to V$ be a polynomial-like map of topological degree $d_t \geq 2$. There exists a unique probability measure μ supported by $\partial\mathcal{K}_f$ which is ergodic and such that*

1. *for any volume form Ω of mass 1 in $L^2(V)$, one has $d_t^{-n}(f^n)^*\Omega \to \mu$ as $n \to \infty$,*
2. *if $d^*_{k-1} < d_t$, the measure μ is PLB and repelling periodic points are dense in* $\mathrm{supp}(\mu)$.

Proof of Proposition 1.5. — The first argument is an inverse branches argument which follows Briend-Duval [**BD4**]. Let $B := \mathbb{B}(x, \epsilon)$ be a small ball around a μ_f-generic point x. Since μ_f is mixing, we have $\mu_f(f^{-n}(B) \cap B) \simeq \mu(B)^2$ for n large enough. In particular, unsing $(f^n)^*\mu_f = d^{nk}\mu_f$, we deduce there exists $C > 0$ such that f^n has $M(n) \geq Cd^{nk}$ inverse branches $g_1, \ldots, g_{M(n)}$ defined on B with

- $g_i(B) \Subset B$ and g_i is uniformly contracting on B for all i,
- $g_i(B) \cap g_j(B) = \emptyset$ for all $i \neq j$.

Fix $m \geq n_0$ large enough so that $Cd^{mk} > d^{(k-1)m} \geq 2$ and set

$$V := B, \ \ U := \bigcup_{j=1}^{M(m)} g_j(B), \ \ N := M(m) \ \text{ and } \ g := f^m|_U.$$

The map $g : U \to V$ is polynomial-like of topological degree N, whence its equilibrium measure ν is the unique probability measure which satisfies $g^*\nu = N\nu$ by the first part of Theorem 1.7. We let $K := \mathrm{supp}(\nu)$. Since the g_i's are uniformly contracting, the compact set K is f^m-hyperbolic.

To conclude, it is sufficient to verify that $N > d^*_{k-1}$. Fix $n \geq 1$ and φ psh on V. Let ω be the (normalized) restriction to V of the Fubini-Study form of $\mathbb{P}^k$. Then

$$\begin{aligned}
\|\Psi^n(dd^c\varphi)\|_U &= \int_U (\Psi^n(dd^c\varphi)) \wedge \omega^{k-1} = \int_U \frac{1}{N^n} ((g^n)_*(dd^c\varphi)) \wedge \omega^{k-1} \\
&= \frac{1}{N^n} \int_U dd^c\varphi \wedge (g^n)^*\omega^{k-1} = \frac{1}{N^n} \int_U dd^c\varphi \wedge (d^{mn}\omega + dd^c u_{nm})^{k-1}
\end{aligned}$$

where $(u_n)_n$ is a uniformly bounded sequence of continuous functions on $\mathbb{P}^k$. In particular, by the Chern-Levine-Niremberg inequality, if $U \Subset W \Subset V$, there exists a constant $C' > 0$

depending only on W such that

$$\begin{aligned}\|\Psi^n(dd^c\varphi)\|_U &= \left(\frac{d^{(k-1)m}}{N}\right)^n \int_U dd^c\varphi \wedge (\omega + d^{-nm}dd^c u_{nm})^{k-1} \\ &\leq \left(\frac{d^{(k-1)m}}{N}\right)^n C'\|dd^c\varphi\|_W.\end{aligned}$$

Taking the n-th root and passing to the limit, we get

$$\frac{d^*_{k-1}}{N} \leq \frac{d^{(k-1)m}}{N} < 1$$

by assumption. The second part of Theorem 1.7 allows us to conclude. □

2. The support of bifurcation currents

Pick a complex manifold Λ and let $m, k \geq 1$ be so that $\dim \Lambda \geq km$. Let $(f, a_1, \ldots, a_m)$ be a dynamical $(m+1)$-tuple of $\mathbb{P}^k$ of degree d parametrized by Λ.

Definition 2.1. — *We say that $a_1, \ldots, a_m$ are* transversely J-prerepelling (resp. properly J-prerepelling) *at a parameter λ_0 if there exists integers $n_1, \ldots, n_m \geq 1$ such that $f^{n_j}_{\lambda_0}(a_j(\lambda_0)) = z_j$ is a repelling periodic point of f_{λ_0} and, if $z_j(\lambda)$ is the natural continuation of z_j as a repelling periodic point of f_λ in a neighborhood U of λ_0, such that*

1. *$z_j(\lambda) \in J_\lambda$ for all $\lambda \in U$ and all $1 \leq j \leq m$,*
2. *the graphs of $A : \lambda \mapsto (f^{q_1}_\lambda(a_1(\lambda)), \ldots, f^{q_m}_\lambda(a_m(\lambda)))$ and of $Z : \lambda \mapsto (z_1(\lambda), \ldots, z_m(\lambda))$ intersect transversely (resp. properly) at λ_0.*

In this section, we prove the following:

Theorem 2.2. — *Let $(f, a_1, \ldots, a_m)$ be a dynamical $(m+1)$-tuple of $\mathbb{P}^k$ of degree d parametrized by Λ with $km \leq \dim \Lambda$.*

Then the support of $T^k_{f,a_1} \wedge \cdots \wedge T^k_{f,a_m}$ coincides with the closure of the set of parameters at which $a_1, \ldots, a_m$ are transversely J-prerepelling.

Remark. — The hypothesis on the dimension of the parameter space looks a priori artificial, but transversely J-prerepelling parameters form alanytic subsets of codimension km. In particular, it is not clear to me that you can prove the existence (and thus the Zariski density) of such parameters if $\dim \Lambda < km$.

2.1. Properly prerepelling marked points bifurcate

First, we give a quick proof of the fact that properly J-prerepelling parameters belong to the support of $T^k_{f,a_1} \wedge \cdots \wedge T^k_{f,a_m}$, without any additional assumption.

Theorem 2.3. — *Let $(f, a_1, \ldots, a_m)$ be a dynamical $(m+1)$-tuple of $\mathbb{P}^k$ of degree d parametrized by Λ with $km \leq \dim \Lambda$. Pick any parameter $\lambda_0 \in \Lambda$ such that $a_1, \ldots, a_m$ are properly J-prerepelling at λ_0. Then $\lambda_0 \in \mathrm{supp}\left(T^k_{f,a_1} \wedge \cdots \wedge T^k_{f,a_m}\right)$.*

The proof of this result is an adaptation of the strategy of Buff and Epstein [**BE**] and the strategy of Berteloot, Bianchi and Dupont [**BBD**], see also [**G, AGMV**]. Since it follows closely that of [**AGMV**, Theorem B], we shorten some parts of the proof.

Before giving the proof of Theorem 2.3, remark that our properness assumption is equivalent to saying that the local hypersurfaces

$$X_j := \{\lambda \in \Lambda\, ;\ f_\lambda^{q_j}(a_j(\lambda)) = z_j(\lambda)\}$$

intersecting at λ_0 satisfy $\mathrm{codim}\left(\bigcap_j X_j\right) = km$.

Proof of Theorem 2.3. — According to [**G**, Lemma 6.3], we can reduce to the case when Λ is an open set of $\mathbb{C}^{km}$. Take a small ball B centered at λ_0 in Λ. Up to reducing B, we can assume $z_j(\lambda)$ can be followed as a repelling periodic point of f_λ for all $\lambda \in B$. Up to reducing B, our assumption is equivalent to the fact that $\bigcap_j X_j = \{\lambda_0\}$.

We let $\mu := T^k_{f,a_1} \wedge \cdots \wedge T^k_{f,a_m}$. Our aim here is to exhibit a basis of neighborhood $\{\Omega_n\}_n$ of λ_0 in $\mathbb{B}$ with $\mu(\Omega_n) > 0$ for all n. For any m-tuple $\underline{n} := (n_1, \ldots, n_m) \in (\mathbb{N}^*)^m$, we let

$$\begin{array}{rccl} F_{\underline{n}} : \Lambda \times (\mathbb{P}^k)^m & \longrightarrow & \Lambda \times (\mathbb{P}^k)^m \\ (\lambda, z_1, \ldots, z_m) & \longmapsto & (\lambda, f_\lambda^{n_1}(z_1), \ldots, f_\lambda^{n_m}(z_m)) \ . \end{array}$$

For a m-tuple $\underline{n} = (n_1, \ldots, n_m)$ of positie integers, we set

$$|\underline{n}| := n_1 + \cdots + n_m \ .$$

We also denote

$$\mathfrak{A}_{\underline{n}}(\lambda) := \left(f_\lambda^{n_1}(a_1(\lambda)), \ldots, f_\lambda^{n_m}(a_m(\lambda))\right), \ \lambda \in \Lambda.$$

As in [**AGMV**], we have the following.

Lemma 2.4. — *For any m-tuple $\underline{n} = (n_1, \ldots, n_m)$ of positive integers, we let $\Gamma_{\underline{n}}$ be the graph in $\Lambda \times (\mathbb{P}^k)^m$ of $\mathfrak{A}_{\underline{n}}$. Then, for any Borel set $B \subset \Lambda$, we have*

$$\mu(B) \ = \ d^{-k\cdot|\underline{n}|} \int_{B \times (\mathbb{P}^k)^m} \left(\bigwedge_{j=1}^m (\pi_j)^* \left(\widehat{T}^k\right)\right) \wedge \left[\Gamma_{\underline{n}}\right] \ .$$

Suppose that the point z_j is r_j-periodic. For the sake of simplicity, we let in the sequel $\mathfrak{A}_n := \mathfrak{A}_{\underline{q}+n\underline{r}}$, where $\underline{q} = (q_1, \ldots, q_m)$, $\underline{r} = (r_1, \ldots, r_m)$ are given as above and $\underline{q} + n\underline{r} = (q_1 + nr_1, \ldots, q_m + nr_m)$. Again as above, we let Γ_n be the graph of $\mathfrak{A}_n$.

Let $z := (z_1, \ldots, z_m)$ and fix any small open neighborhood Ω of λ_0 in Λ. Set

$$I_n := \int_{\Omega \times (\mathbb{P}^k)^m} \left(\bigwedge_{j=1}^m (\pi_j)^* \left(\widehat{T}^k\right)\right) \wedge [\Gamma_n]\,,$$

and let δ be given by the expansion condition as above. Let S_n be the connected component of $\Gamma_n \cap \Lambda \times \mathbb{B}^m_\delta(z)$ containing (λ_0, z). Since $z_j(\lambda)$ is repelling and periodic for f_λ for all $\lambda \in B$ (if B has been choosen small enough), there exists a constant $K > 1$ such that

$$d_{\mathbb{P}^k}(f_\lambda^{r_j}(z), f_\lambda^{r_j}(w)) \geq K \cdot d_{\mathbb{P}^k}(z, w)$$

for all $(z, w) \in \mathbb{B}(z_j(\lambda_0), \epsilon)$ and all $\lambda \in B$ for some given $\epsilon > 0$. In particular, the current $[S_n]$ is vertical-like in in $\Lambda \times \mathbb{B}^m_\delta(z)$ and there exists $n_0 \geq 1$ and a basis of neighborhood Ω_n of λ_0 in Λ such that

$$\mathrm{supp}([S_n]) = S_n \subset \Omega_n \times \mathbb{B}^m_\delta(z),$$

for all $n \geq n_0$.

Let S be any weak limit of the sequence $[S_n]/\|[S_n]\|$. Then S is a closed positive (mk, mk)-current of mass 1 in $B \times \mathbb{B}^m_\delta(z)$ with $\mathrm{supp}(S) \subset \{\lambda_0\} \times \mathbb{B}^m_\delta(z)$. Hence $S = M \cdot [\{\lambda_0\} \times \mathbb{B}^m_\delta(z)]$, where $M^{-1} > 0$ is the volume of $\mathbb{B}^m_\delta(z)$ for the volume form $\bigwedge_j(\omega_j^k)$, where $\omega_j = (p_j)^*\omega_{\mathbb{P}^k}$ and $p_j : (\mathbb{P}^k)^m \to \mathbb{P}^k$ is the projection on the j-th coordinate.

As a consequence, $[S_n]/\|[S_n]\|$ converges weakly to S as $n \to \infty$ and, since the (mk, mk)-current $\bigwedge_{j=1}^m (\pi_j)^*(\widehat{T}^k)$ is the wedge product of $(1,1)$-currents with continuous potentials, we have

$$\bigwedge_{j=1}^m (\pi_j)^*\left(\widehat{T}^k\right) \wedge \frac{[S_n]}{\|[S_n]\|} \longrightarrow \bigwedge_{j=1}^m (\pi_j)^*\left(\widehat{T}^k\right) \wedge S$$

as $n \to +\infty$. Whence

$$\liminf_{n\to\infty} \left(\|[S_n]\|^{-1} \cdot I_n\right) \geq \liminf_{n\to\infty} \int \bigwedge_{j=1}^m (\pi_j)^*\left(\widehat{T}^k\right) \wedge \frac{[S_n]}{\|[S_n]\|} \geq \int \bigwedge_{j=1}^m (\pi_j)^*\left(\widehat{T}^k\right) \wedge S$$

By the above, this gives

$$\liminf_{k\to\infty} \left(\|[S_n]\|^{-1} \cdot I_n\right) \geq M \cdot \int [\{\lambda_0\} \times \mathbb{B}^m_\delta(z)] \wedge \bigwedge_{j=1}^m (\pi_j)^*\left(\widehat{T}^k\right) ,$$

In particular, there exists $n_2 \geq n_1$ such that for all $n \geq n_2$,

$$\|[S_n]\|^{-1} \cdot I_n \geq \frac{M}{2} \cdot \int [\{\lambda_0\} \times \mathbb{B}^m_\delta(z)] \wedge \bigwedge_{j=1}^m (\pi_j)^*\left(\widehat{T}^k\right) .$$

Finally, since $[S_n]$ is a vertical current, up to reducing $\delta > 0$, Fubini Theorem gives

$$\liminf_{n\to\infty} \|[S_n]\| \geq \prod_{j=1}^m \int_{\mathbb{B}(z_j,\delta)} \omega_{\mathrm{FS}}^k \geq \left(c \cdot \delta^{2k}\right)^m > 0 .$$

Up to increasing n_0, we may assume $\|[S_n]\| \geq (c\delta^{2k})^m/2$ for all $n \geq n_0$. Letting $\alpha = M(c\delta^{2k})^m/4 > 0$, we find

$$\int_{\Omega\times(\mathbb{P}^k)^m} \left(\bigwedge_{j=1}^m (\pi_j)^*\left(\widehat{T}^k\right)\right) \wedge [\Gamma_n] \geq \alpha \int [\{\lambda_0\} \times \mathbb{B}^m_\delta(z)] \wedge \bigwedge_{j=1}^m (\pi_j)^*\left(\widehat{T}^k\right).$$

To conclude the proof of Theorem 2.3, we rely on the following purely dynamical result, which is an immediate adaptation of [**AGMV**, Lemma 3.5].

Lemma 2.5. — *For any $\delta > 0$, and any $x = (x_1, \ldots, x_m) \in (\mathrm{supp}(\mu_{\lambda_0}))^m$, we have*

$$\int [\{\lambda_0\} \times \mathbb{B}^m_\delta(x)] \wedge \bigwedge_{j=1}^m (\pi_j)^*\left(\widehat{T}^k\right) = \prod_{j=1}^m \mu_{\lambda_0}(\mathbb{B}(x_j, \delta)) > 0.$$

We can now conclude the proof of Theorem 2.3. Pick any open neghborhood Ω of λ_0 in Λ. By the above and Lemma 2.5, we have an integer $n_0 \geq 1$ and constants $\alpha, \delta > 0$ such that for all $n \geq n_0$,

$$\mu(\Omega) \geq \alpha \cdot d^{-k\left(|\underline{q}|+n|\underline{r}|\right)} \prod_{j=1}^m \mu_f(\mathbb{D}(z_j, \delta)) > 0 .$$

In particular, this yields $\mu(\Omega) > 0$. By assumption, this holds for a basis of neighborhoods of λ_0 in Λ, whence we have $\lambda_0 \in \mathrm{supp}(\mu)$. □

2.2. Density of transversely prerepelling parameters

To finish the proof fo Theorem 2.2, it is sufficient to prove that any point of the support of $T^k_{f,a_1}\wedge\cdots\wedge T^k_{f,a_m}$ an be approximated by transversely J-prerepelling parameters. We follow the strategy of the proof of Theorem 0.1 of [**Duj**] to establish this approximation property. Precisely, we prove here the following.

Theorem 2.6. — *Let $(f,a_1,\ldots,a_m)$ be a dynamical $(m+1)$-tuple of $\mathbb{P}^k$ of degree d parametrized by Λ with $km\leq\dim\Lambda$.*

Then, any parameter $\lambda\in\Lambda$ lying in the support of the current $T^k_{f,a_1}\wedge\cdots\wedge T^k_{f,a_m}$ can be approximated by parameters at which $a_1,\ldots,a_m$ are transversely J-prerepelling.

We rely on the following property of *PLB* measures (see [**DS**]):

Lemma 2.7. — *Let ν be PLB with compact support in a bounded open set $W\subset\mathbb{C}^k$ and let ψ be a psh function on $\mathbb{C}^k$. The function G_ψ defined by*

$$G_\psi(z):=\int\psi(z-w)d\nu(w),\ z\in\mathbb{C}^k,$$

is psh and locally bounded on $\mathbb{C}^k$.

Proof of Theorem 2.6. — We follow the strategy of the proof of [**Duj**, Theorem 0.1]. Write $\mu:=T^k_{f,a_1}\wedge\cdots\wedge T^k_{f,a_m}$ and pick $\lambda_0\in\mathrm{supp}(\Omega)$.

According to Proposition 1.5, there exists an integer $m\geq 1$ and a $f^m_{\lambda_0}$-compact set $K\subset\mathbb{P}^k$ contained in a ball and $N\geq 2$ such that

- $f^m_{\lambda_0}|_K$ is uniformly hyperbolic and repelling periodic points of $f^m_{\lambda_0}$ are dense in K,
- there exisys a unique probailibty measure ν supported on K such that $(f^m_{\lambda_0}|_K)^*\nu = N\nu$ which is PLB.

Since K is hyperbolic, there exists $\epsilon>0$ and a unique holomorphic motion $h:\mathbb{B}(\lambda_0,\epsilon)\times K\to\mathbb{P}^k$ which conjugates the dynamics, i.e. h is continuous and such that

- for all $\lambda\in\mathbb{B}(\lambda_0,\epsilon)$, the map $h_\lambda:=h(\lambda,\cdot):K\to\mathbb{P}^k$ is injective and $h_{\lambda_0}=\mathrm{id}_K$,
- for all $z\in K$, the map $\lambda\in\mathbb{B}(\lambda_0,\epsilon)\mapsto h_\lambda(z)\in\mathbb{P}^k$ is holomorphic, and
- for all $(\lambda,z)\in\mathbb{B}(\lambda_0,\epsilon)\times K$, we have $h_\lambda\circ f^m_{\lambda_0}(z)=f^m_\lambda\circ h_\lambda(z)$,

see e.g. [**dMvS**, Theorem 2.3 p. 255]. For all $z:=(z_1,\ldots,z_m)\in K^m$, we denote by Γ_z the graph of the holomorphic map $\lambda\mapsto(h_\lambda(z_1),\ldots,h_\lambda(z_m))$.

We define a closed positive (km,km)-current on $\mathbb{B}(\lambda_0,\epsilon)\times(\mathbb{P}^k)^m$ by letting

$$\widehat{\nu}:=\int_{K^m}[\Gamma_z]\mathrm{d}\nu^{\otimes m}(z),$$

where $\Gamma_z=\{(\lambda,h_\lambda(z_1),\ldots,h_\lambda(z_m))\,;\ \lambda\in\mathbb{B}(\lambda_0,\epsilon)\}$ for all $z=(z_1,\ldots,z_m)\in K^m$.

Claim. — *There exists a $(km-1,km-1)$-current V on $\mathbb{B}(\lambda_0,\epsilon)\times(\mathbb{P}^k)^m$ which is locally bounded and such that $\hat{\nu}=dd^cV$.*

Recall that we have set $\mathfrak{a}_n(\lambda):=(f^n_\lambda(a_1(\lambda)),\ldots,f^n_\lambda(a_m(\lambda)))$. We define $\mathfrak{a}_n^*\hat{\nu}$ by

$$\mathfrak{a}_n^*\hat{\nu}:=(\pi_1)_*\left(\hat{\nu}\wedge[\Gamma_{\mathfrak{a}_n}]\right),$$

where $\pi_1:\mathbb{B}(t_0,\epsilon)\times(\mathbb{P}^k)^m\to\mathbb{B}(t_0,\epsilon)$ is the canonical projection onto the first coordinate. According to the claim, locally we have $\hat{\nu}=dd^cV$, for some bounded $(km-1,km-1)$-current V. In particular, we get $\mathfrak{a}_n^*\hat{\nu}=\mathfrak{a}_n^*(dd^cV)$, as expected.

Let ω be the Fubini-Study form of $\mathbb{P}^k$ and $\hat{\Omega} := (\pi_2)^*(\omega^k \otimes \cdots \otimes \omega^k)$, where $\pi_2 : \mathbb{B}(\lambda_0, \epsilon) \times (\mathbb{P}^k)^m \to (\mathbb{P}^k)^m$ is the canonical projection onto the second coordinate. Then

$$\hat{\nu} - \hat{\Omega} = dd^c V$$

where V is bounded on $\mathbb{B}(\lambda_0, \epsilon) \times (\mathbb{P}^k)^m$, hence

$$d^{-kmn}\mathfrak{a}_n^*(\hat{\nu}) - d^{-kmn}\mathfrak{a}_n^*(\hat{\Omega}) = d^{-kmn}\mathfrak{a}_n^*(dd^c V).$$

On the other hand, we have $\frac{1}{d^{km}}\hat{f}^*(\hat{\Omega}) = \hat{\Omega} + dd^c W$, where W is bounded on $\mathbb{B}(\lambda_0, \epsilon) \times (\mathbb{P}^k)^m$, hence $\frac{1}{d^{kmn}}(\hat{f^n})^*(\hat{\Omega}) = \hat{\omega} + dd^c W_n$, where $W_n - W_{n+1} = O(d^{-n})$. In particular, $\frac{1}{d^n}(\hat{f^n})^*(\hat{\Omega}) \wedge [\Gamma_a] = d^{-n}\left(\hat{\Omega} \wedge [\Gamma_{\mathfrak{a}_n}]\right) + dd^c O(d^{-n})$, hence $\mu = \lim_n d^{-kmn}\mathfrak{a}_n^*(\hat{\Omega})$. This yields

$$\lim_{n\to\infty} d^{-nkm}(\pi_1)_*\left(\hat{\nu} \wedge [\Gamma_{\mathfrak{a}_n}]\right) = \mu.$$

We now use [**Duj**, Theorem 3.1]: as $(2km, 2km)$-currents on $\mathbb{B}(\lambda_0, \epsilon) \times (\mathbb{P}^k)^m$,

$$\hat{\nu} \wedge [\Gamma_{\mathrm{p}_n}] = \int_{K^m} [\Gamma_z] \wedge [\Gamma_{\mathrm{p}_n}] \mathrm{d}\nu^{\otimes m}(z)$$

and only the geometrically transverse intersections are taken into account. In particular, this means there exists a sequence of parameters $\lambda_n \to \lambda_0$ and $z_n \in K^m$ such that the graph of $\mathfrak{a}_n$ and Γ_{z_n} intersect transversely at λ_n. Now, since repelling periodic points of $f_{\lambda_0}^m$ are dense in K, there exists $z_{n,j} \to z_n$ as $j \to \infty$, where $z_{j,n} \in K^m$ and $(f_{\lambda_0}^m, \ldots, f_{\lambda_0}^m)$-periodic repelling. Since $z_{j,n}(\lambda) := (h_\lambda, \ldots, h_\lambda)(z_{j,n})$ remains in $(h_\lambda, \ldots, h_\lambda)(K^m)$ and remains periodic, it remains repelling for all $\lambda \in \mathbb{B}(\lambda_0, \epsilon)$. By persistence of transverse intersections, for j large enough, there exists $\lambda_{j,n}$ where $\Gamma_{\mathfrak{a}_n}$ and $\Gamma_{z_{j,n}}$ intersect transversely and $\lambda_{j,n} \to \lambda_n$ as $j \to \infty$ and the proof is complete. □

To finish this section, we prove the Claim.

Proof of the Claim. — Since the compact set K is contained in a ball, we can choose an affine chart $\mathbb{C}^k$ such that $K \Subset \mathbb{C}^k$ and, up to reducing $\epsilon > 0$, we can assume $K_\lambda = h_\lambda(K) \Subset \mathbb{C}^k$ for all $\lambda \in \mathbb{B}(\lambda_0, \epsilon)$. Let $(x_1^1, \ldots, x_k^1, \ldots, x_1^m, \ldots, x_k^m) = (x^1, \ldots, x^m)$ be the coordinates of $(\mathbb{C}^k)^m$ and let $h_{\lambda,i}$ be the i-th coordinate of the function h_λ.

For all $1 \le i \le k$ and $1 \le j \le m$, we define a psh function Ψ_i^j on $\mathbb{B}(\lambda_0, \epsilon) \times (\mathbb{C}^k)^m$ by letting

$$\Psi_i^j(t, w) := \int_{K^m} \log |w_i^j - h_{t,i}(z^j)| \mathrm{d}\nu^{\otimes m}(z).$$

According to Lemma 2.7 and Proposition 1.5, we have $\Psi_i^j \in L^\infty_{\mathrm{loc}}\left(\mathbb{B}(\lambda_0, \epsilon) \times (\mathbb{C}^k)^m\right)$. Moreover, according to [**Duj**, Theorem 3.1], we have

$$\hat{\nu} = \bigwedge_{i,j} dd^c \Psi_i^j = dd^c \left(\Psi_1^1 \cdot \bigwedge_{i,j>1} dd^c \Psi_i^j\right).$$

Since the functions Ψ_i^j are locally bounded, this ends the proof. □

3. Local properties of bifurcation measures

3.1. A renormalization procedure

Pick $k,m\geq 1$ and let $\mathbb{B}(0,\epsilon)$ be the open ball centered at 0 of radius ϵ in $\mathbb{C}^{km}$ and let $(f,a_1,\ldots,a_m)$ be a dynamical $(m+1)$-tuple of degree d of $\mathbb{P}^k$ parametrized by $\mathbb{B}(0,\epsilon)$.

Assume there exists m holomorphically moving J-repelling periodic points $z_1,\ldots,z_m:\mathbb{B}(0,\epsilon)\to\mathbb{P}^k$ of respesctive period $q_j\geq 1$ with $f_0^{n_j}(a_j(0))=z_j(0)$. We also assume that $(a_1,\ldots,a_m)$ are transversely prerepelling at 0 and that $z_j(\lambda)$ is linearizable for all $\lambda\in\mathbb{B}(0,\epsilon)$ for all j. Let $q:=\mathrm{lcm}(q_1,\ldots,q_m)$ and

$$\Lambda_\lambda:=(D_{z_1(\lambda)}(f_\lambda^q),\ldots,D_{z_m(\lambda)}(f_\lambda^q)):\bigoplus_{j=1}^m T_{z_j(\lambda)}\mathbb{P}^k\longrightarrow\bigoplus_{j=1}^m T_{z_j(\lambda)}\mathbb{P}^k$$

and denote by $\phi_\lambda=(\phi_{\lambda,1},\ldots,\phi_{\lambda,m}):(\mathbb{C}^k,0)\to((\mathbb{P}^k)^m,(z_1(\lambda),\ldots,z_m(\lambda)))$, where $\phi_{\lambda,j}$ is the linearizing coordinate of f_λ^q at $z_j(\lambda)$.

Denote by $\pi_j:(\mathbb{P}^k)^m\to\mathbb{P}^k$ the projection onto the j-th factor. Up to reducing $\epsilon>0$, we can also assume there exists $r_j>0$ independent of λ such that

$$f_\lambda^q\circ\phi_{\lambda,j}(z)=\phi_{\lambda,j}\circ D_{z_j(\lambda)}(f_\lambda^{q_j})(z),\ z\in\mathbb{B}(0,r_j),$$

and $D_0\phi_{\lambda,j}:\mathbb{C}^k\to T_{z_j(\lambda)}\mathbb{P}^k$ is an invertible linear map. Up to reducing again ϵ, we can also assume $f_\lambda^{n_j}(a_j(\lambda))$ always lies in the range of $\phi_{\lambda,j}$ for all $1\leq j\leq m$. Recall that we denoted $\mathfrak{a}_{\underline{n}}(\lambda)=(f_\lambda^{n_1}(a_1(\lambda)),\ldots,f_\lambda^{n_m}(a_m(\lambda)))$, where $\underline{n}=(n_1,\ldots,n_m)$ and let

$$h(\lambda):=\phi_\lambda^{-1}\circ\mathfrak{a}_{\underline{n}}(t)=\left(\phi_{\lambda,1}^{-1}\left(f_\lambda^{n_1}(a_1(\lambda))\right),\ldots,\phi_{\lambda,m}^{-1}\left(f_\lambda^{n_m}(a_m(\lambda))\right)\right),\ \lambda\in\mathbb{B}(0,\epsilon).$$

Lemma 3.1. — *The map $h:\mathbb{B}(0,\epsilon)\to(\mathbb{C}^{km},0)$ is a local biholomorphism at* 0.

Proof. — Recall that $f_0^{n_j}(a_j(0))=z_j(0)$. Write $h=(h_1,\ldots,h_m)$ with $h_j:\mathbb{B}(0,\epsilon)\to(\mathbb{C}^k,0)$ and let $b_j(\lambda):=f_\lambda^{n_j}(a_j(\lambda))$ for all $\lambda\in\mathbb{B}(0,\epsilon)$ so that $b_j(\lambda)=\phi_{\lambda,j}\circ h_j(\lambda)$ for all $\lambda\in\mathbb{B}(0,\epsilon)$. Since $\phi_{\lambda,j}(0)=z_j(\lambda)$, differentiating and evaluating at $\lambda=0$, we find

$$D_0b_j=D_0z_j+D_0\phi_{0,j}\circ D_0h_j.$$

Now our tranversality assumption implies that

$$L:=((D_0b_1-D_0z_1),\ldots,(D_0b_m-D_0z_m)):\mathbb{C}^{km}\to\bigoplus_{j=1}^m T_{z_j(0)}\mathbb{P}^k$$

is invertible. As a consequence, the linear map

$$D_0h=(D_0h_1,\ldots,D_0h_m)=-(D_0\phi_0)^{-1}\circ L:\mathbb{C}^{km}\to\mathbb{C}^{km}$$

is invetible, ending the proof. □

Up to reducing again ϵ, we assume h is a biholomorphism onto its image and let $r:=h^{-1}:h(\mathbb{B}(0,\epsilon))\to\mathbb{B}(0,\epsilon)$. Fix $\delta_1,\ldots,\delta_m>0$ so that $\mathbb{B}_{\mathbb{C}^k}(0,\delta_1)\times\cdots\times\mathbb{B}_{\mathbb{C}^k}(0,\delta_m)\subset h(\mathbb{B}(0,\epsilon))$.

Finally, let $\Omega:=\mathbb{B}_{\mathbb{C}^k}(0,\delta_1)\times\cdots\times\mathbb{B}_{\mathbb{C}^k}(0,\delta_m)$ and, for any $n\geq 1$, let

$$r_n(x):=r\circ\Lambda_0^{-n}(x),\ x\in\mathbb{B}_{\mathbb{C}^k}(0,\delta_1)\times\cdots\times\mathbb{B}_{\mathbb{C}^k}(0,\delta_m).$$

The main goal of this paragraph is the following.

Proposition 3.2. — *In the weak sense of measures on Ω, we have*

$$\prod_{j=1}^{m} d^{k(n_j+nq)} \cdot (r_n)^* \left(T^k_{f,a_1} \wedge \cdots \wedge T^k_{f,a_m}\right) \underset{n\to\infty}{\longrightarrow} (\phi_0)^* \left(\bigwedge_{j=1}^{m} (\pi_j)^* \mu_{f_0}\right).$$

To simplify notations, we let

$$\mathfrak{a}_{(n)} := \mathfrak{a}_{\underline{n}+nq}, \text{ with } \underline{n}+nq = (n_1+nq, \ldots, n_m+nq).$$

Lemma 3.3. — *The sequence $(\mathfrak{a}_{(n)} \circ r_n)_{n\geq 1}$ converges uniformly to ϕ_0 on Ω.*

Proof. — Note first that

$$\mathfrak{a}_{(0)} \circ r(x) = \left(f^{n_1}_{r(x)}(a_1(r(x))), \ldots, f^{n_m}_{r(x)}(a_m(r(x)))\right) = \phi_{r(x)}(x),\ x \in \Omega,$$

by definition of r.

By definition, the sequence $(r_n)_{n\geq 1}$ converges uniformly and exponentially fast to 0 on Ω, since we assumed $z_1(0), \ldots, z_m(0)$ are repelling periodic points and since $r(0) = 0$. Moreover, $\Lambda_{r_n} \to \Lambda_0$ and $\phi_{r_n(x)} \to \phi_0$ exponentially fast. In particular,

$$\lim_{n\to\infty} \Lambda^n_{r_n(x)} \circ \Lambda_0^{-n}(x) = x$$

and the convergence is uniform on Ω. Fix $x \in \Omega$. Then

$$\begin{aligned}\mathfrak{a}_{(n)} \circ r_n(x) &= (f^{qn}_{r_n(x)}, \ldots, f^{qn}_{r_n(x)})\left(\mathfrak{a}_{(0)} \circ r \circ \Lambda_0^{-n}(x)\right)\\ &= (f^{qn}_{r_n(x)}, \ldots, f^{qn}_{r_n(x)}) \circ \phi_{r_n(x)}\left(\Lambda_0^{-n}(x)\right)\\ &= \phi_{r_n(x)}\left(\Lambda^n_{r_n(x)} \circ \Lambda_0^{-n}(x)\right)\end{aligned}$$

and the conclusion follows. □

Proof of Proposition 3.2. — Recall that we can assume there exists a holomorphic family of non-degenerate homogeneous polynomial maps $F_\lambda : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ of degree d such that, if $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ is the canonical projection, then

$$\pi \circ F_\lambda = f_\lambda \circ \pi \text{ on } \mathbb{C}^{k+1} \setminus \{0\}.$$

For $1 \leq j \leq m$, let $\tilde{a}_j : \mathbb{B}(0,\epsilon) \to \mathbb{C}^{k+1} \setminus \{0\}$ be a lift of a_j, i.e. $a_j = \pi \circ \tilde{a}_j$. Recall that

$$\bigwedge_{j=1}^{m} T^k_{a_j} = \bigwedge_{j=1}^{m} \left(dd^c G_\lambda(\tilde{a}_j(\lambda))\right)^k.$$

For $1 \leq j \leq m$, pick a open set $U_j \subset \mathbb{P}^k$ such that $\phi_{0,j}(B_{\mathbb{C}^k}(0,\delta_j)) \Subset U_j$ and such that there exists a section $\sigma_j : U_j \to \mathbb{C}^{k+1} \setminus \{0\}$ of π on U_j. Let $U := U_1 \times \cdots \times U_k$ and $\sigma := (\sigma_1, \ldots, \sigma_k) : U \to (\mathbb{C}^{k+1} \setminus \{0\})^m$ so that $\phi_0(\Omega) \Subset U$. According to Lemma 3.3, there exists $n_0 \geq 1$ such that

$$\mathfrak{a}_{(n)} \circ r_n(\Omega) \Subset U.$$

In other words, for any $x \in \Omega$, any $1 \leq j \leq m$ and any $n \geq n_0$,

$$a^{n,j}(x) := f^{n_j+nq}_{r_n(x)}(a_j \circ r_n(x)) \in U_j.$$

Moreover, for all $x \in \Omega$, we have

$$\begin{aligned}\pi \circ F^{n_j+nq}_{r_n(x)}(\tilde{a}_j \circ r_n(x)) &= f^{n_j+nq}_{r_n(x)} \circ \pi(\tilde{a}_j \circ r_n(x)) = f^{n_j+nq}_{r_n(x)}(a_j \circ r_n(x))\\ &= \pi \circ \sigma_j\left(a^{n,j}(x)\right).\end{aligned}$$

In particular, there exists a holomorphic function $u_{n,j} : \Omega \to \mathbb{C}^*$ such that

$$F_{r_n(x)}^{n_j+nq}(\tilde{a}_j \circ r_n(x)) = u_{n,j}(x) \cdot \sigma_j \circ a^{n,j}(x)$$

and

$$\begin{aligned} d^{nq+n_j} G_{r_n(x)}\left(\tilde{a}_j \circ r_n(x)\right) &= G_{r_n(x)}\left(F_{r_n(x)}^{n_j+nq}\left(\tilde{a}_j \circ r_n(x)\right)\right) \\ &= G_{r_n(x)}\left(\sigma_j \circ a^{n,j}(x)\right) + \log|u_{n,j}(x)|, \end{aligned}$$

for all $x \in \Omega$. Since $\log|u_{n,j}|$ is pluriharmonic on Ω, the above gives

$$d^{nq+n_j}(r_n)^* T_{f,a_j} = dd^c G_{r_n(x)}\left(\sigma_j \circ a^{n,j}(x)\right),$$

so that

$$\mu_n := \prod_{j=1}^m d^{k(n_j+nq)} \cdot (r_n)^* \left(T_{f,a_1}^k \wedge \cdots \wedge T_{f,a_m}^k\right) = \bigwedge_{j=1}^m \left(dd^c G_{r_n(x)}\left(\sigma_j \circ a^{n,j}(x)\right)\right)^k.$$

Using again Lemma 3.3 gives

$$\mu_n \underset{n\to\infty}{\longrightarrow} \bigwedge_{j=1}^m \left(dd^c G_0\left(\sigma_j \circ \phi_{0,j}(x)\right)\right)^k = \bigwedge_{j=1}^m (\phi_{0,j})^* \mu_{f_0}.$$

This ends the proof since $\phi_{0,j} = \pi_j \circ \phi_0$ by definition of ϕ_0. □

3.2. Families with an absolutely continuous bifurcation measure

Fix integers $k, m \geq 1$ and $d \geq 2$. The following is a consequence of the above renormalization process.

Proposition 3.4. — *Let $(f, a_1, \ldots, a_m)$ be a dynamical $(m+1)$-tuple of degree d of $\mathbb{P}^k$ parametrized by the unit Ball $\mathbb{B}$ of $\mathbb{C}^{km}$. Assume that $a_1, \ldots, a_m$ are transeversely J-prerepelling at 0 to a J-repelling cycle of f_0 which moves holmorphically in $\mathbb{B}$ as a J-repelling cycle of f_λ which is linearizable for all $\lambda \in \mathbb{B}$. Assume in addition that the measure $\mu := T_{f,a_1}^k \wedge \cdots \wedge T_{f,a_m}^k$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{B}$ and the Radon-Nikodym derivative $\frac{d\mu}{d\mathrm{Leb}}$ is continuous and > 0 near 0.*

Then the measure μ_{f_0} is non-singular with respect to $\omega_{\mathbb{P}^k}^k$.

Proof. — By assumption, we can write $\mu = h \cdot \mathrm{Leb}$ where $h : \mathbb{B} \to \mathbb{R}_+$ is a continuous function. Let $\Omega := \mathbb{B}_{\mathbb{C}^k}(0, \delta_1) \times \cdots \times \mathbb{B}_{\mathbb{C}^k}(0, \delta_m)$, r_n and ϕ_0 be given as in Section 3.1. We can apply Proposition 3.2:

$$\prod_{j=1}^m d^{k(n_j+nq)} h \circ r_n \cdot (r_n)^* \mathrm{Leb} = \prod_{j=1}^m d^{k(n_j+nq)} \cdot (r_n)^* \mu \underset{n\to\infty}{\longrightarrow} (\phi_0)^* \left(\bigwedge_{j=1}^m (\pi_j)^* \mu_{f_0}\right).$$

Since $\phi_0(0) = (z_1(0), \ldots, z_m(0)) \in (\mathrm{supp}(\mu_{f_0}))^k$, the measure

$$(\phi_0)^* \left(\bigwedge_{j=1}^m (\pi_j)^* \mu_{f_0}\right)$$

has (finite) strictly positive mass in Ω. In particular, the measure

$$d^{knqm} \cdot (r_n)^* (h \cdot \mathrm{Leb}) = d^{knqm} \cdot (h \circ r_n) \cdot \left(\Lambda_0^{-n}\right)^* (r^* \mathrm{Leb})$$

converges to a non-zero finite mass positive measure on Ω. As r is a local holomorphic diffeomorphism, there exists a neighborhod of 0 in $\mathbb{B}$ such that we have $r^*\mathrm{Leb} = v \cdot \mathrm{Leb}$ for some smooth function $v > 0$. Whence

$$d^{knqm} \cdot (r_n)^*\mathrm{Leb} = d^{knqm} \cdot (h \circ r_n) \cdot (v \circ \Lambda_0^{-n}) \left(\Lambda_0^{-n}\right)^* (\mathrm{Leb}).$$

By the change of variable formula and Fubini,

$$\left(\Lambda_0^{-n}\right)^* (\mathrm{Leb}) = \prod_{j=1}^{m} |\det D_{z_j(0)}(f_0^q)|^{-2nk} \cdot \mathrm{Leb}.$$

For all n, define a continuous function $\alpha_n : \mathbb{B} \to \mathbb{R}_+$ by letting

$$\alpha_n(x) := d^{knqm} \prod_{j=1}^{m} |\det D_{z_j(0)}(f_0^q)|^{-2nk} \cdot (h \circ r_n(x)) \cdot (v \circ \Lambda_0^{-n}(x)) \in \mathbb{R}_+.$$

By assumption, the measure $\alpha_n \cdot \mathrm{Leb}$ converges weakly on Ω to a non-zero finite positive measure, whence $\alpha_n \to \alpha_\infty$, as $n \to \infty$, where $\alpha_\infty : \Omega \to \mathbb{R}_+$ is not identically zero. As a consequence,

$$(\phi_0)^* \left(\bigwedge_{j=1}^{m} (\pi_j)^* \mu_{f_0} \right) = \alpha_\infty \cdot \mathrm{Leb}.$$

Using again Fubini, on Ω, we find

$$(\phi_0)^* \left(\bigwedge_{j=1}^{m} (\pi_j)^* \mu_{f_0} \right) = \alpha_\infty \cdot \mathrm{Leb}_{\mathbb{C}^k} \boxtimes \cdots \boxtimes \mathrm{Leb}_{\mathbb{C}^k}.$$

Finally, since as positive measures on $\phi_0(\Omega)$, we have

$$\bigwedge_{j=1}^{m} (\pi_j)^* \mu_{f_0} = \mu_{f_0} \boxtimes \cdots \boxtimes \mu_{f_0},$$

the measure μ_{f_0} is absolutely continuous with respect to Leb in an open set. □

We now want to deduce Theorem D from the above, using [**Z**] when $k = 1$ and [**BD3**] when $k > 1$. In fact, they prove that f is a Lattès map if and only if the sum of its Lyapunov exponents $L(f) = \int_{\mathbb{P}^k} \log|\det(Df)|\mu_f$ is equalt to $\frac{k}{2} \log d$. We use this characterization to prove Theorems C and D.

Proof of Theorem D. — Assume firt that $\mu_{f,a}$ is absolutely continuous with respect to ω^k and let T be the set of parameters $\lambda \in M$ such that a is transversely prerepelling at λ. The set T is dense in M by Theorem 2.2. Applying Proposition 3.4 at all $\lambda \in T$ gives that μ_{f_λ} is non-singular with respect to $\omega_{\mathbb{P}^k}^k$ for all $\lambda \in T$.

We then apply Zdunik or Berteloot-Dupont Theorem we have proven there exists a countable subset T which is dense in M such that the map f_λ is a Lattès map for all $\lambda \in T$. In particular, $L(f_\lambda) = \frac{k}{2} \log d$ for all $\lambda \in T$. As the function $\lambda \in M \mapsto L(f_\lambda)$ is continuous, this implies $L(f_\lambda) = \frac{k}{2} \log d$ for all $\lambda \in M$, i.e. f_λ is a Lattès map for all $\lambda \in M$.

To conclude, we assume f is a family of Lattès maps and the measure $\mu_{f,a}$ is not identically zero. Let $\omega_{\mathbb{P}^k}$ be the Fubini-Study form on $\mathbb{P}^k$. For all $\lambda \in M$, there exists a function $u_\lambda : \mathbb{P}^k \to \mathbb{R}_+$ such that

$$\mu_{f_\lambda} = u_\lambda \cdot \omega_{\mathbb{P}^k}^k.$$

Let $u(\lambda, z) := u_\lambda(z)$ for all $(\lambda, z) \in M \times \mathbb{P}^k$. The above can be expressed as

$$\widehat{T} = u \cdot \hat{\omega}^k,$$

where $\hat{\omega} = \pi_{\mathbb{P}^k}^*(\omega_{\mathbb{P}^k})$ and $\pi_{\mathbb{P}^k} : M \times \mathbb{P}^k \to \mathbb{P}^k$ is the canonical projection. By construction, the function u is continuous on $M \times \mathbb{P}^k$. Pick a local chart $U \subset M$ and a local chart $V \subset \mathbb{P}^k$ so that $a(U) \subset V$ and $\omega_{\mathbb{P}^k} = dd^c v$ on V where v is smooth. In $U \times V$, the above gives

$$(\pi_\Lambda)_* \left(\widehat{T}^k \wedge [\Gamma_a]\right) = (\pi_\Lambda)_* \left(u \cdot (dd^c_{\lambda,z} v(z))^k \wedge [\Gamma_a]\right) = u(\lambda, a(\lambda)) \left(dd^c_\lambda (v \circ a(\lambda))\right)^k .$$

Letting $h(\lambda) := u(\lambda, a(\lambda))$ and $w(\lambda) := v \circ a(\lambda)$, we find

$$\mu_{f,a} = h \cdot (dd^c w)^k \quad \text{on } U.$$

Since w is smooth, the conclusion follows. □

4. Proof of the main result and concluding remarks

4.1. J-stability and bifurcation of dynamical pairs on $\mathbb{P}^1$

Recallthat a family $f : \Lambda \times \mathbb{P}^1 \longrightarrow \Lambda \times \mathbb{P}^1$ of degree d rational maps of $\mathbb{P}^1$ is *J-stable* if all the repelling cycles can be followed holomorphically throughout the whole family Λ, i.e. if for all $n \geq 1$, there exists $N \geq 0$ and holomorphic maps $z_1, \ldots, z_N : \Lambda \to \mathbb{P}^1$ such that $\{z_1(\lambda), \ldots, z_N(\lambda)\}$ is exactly the set of all repelling cycles of f_λ of exact period n for all $\lambda \in \Lambda$.

Recall also that an endomorphism of $\mathbb{P}^1$ has a unique measure of maximal entropy μ_f and let $L(f) := \int_{\mathbb{P}^1} \log|f'|\mu_f$ be the Lyapunov exponents of f with respect to μ_f. By a classical result of Mañé, Sad and Sullivan [**MSS**], it is also locally equivalent to the existence of a unique holomorphic motion of the Julia set which is compatible with the dynamics, i.e. for $\lambda_0 \in \Lambda$, there exists $h : \Lambda \times J_{f_{\lambda_0}} \longrightarrow \Lambda \times \mathbb{P}^1$ such that

- for any $\lambda \in \Lambda$, the map $h_\lambda := h(\lambda, \cdot) : J_{f_{\lambda_0}} \longrightarrow \mathbb{P}^1$ is a homeomorphism which conjugates f_{λ_0} to f_λ, i.e. $h_\lambda \circ f_{\lambda_0} = f_\lambda \circ h_\lambda$ on $J_{f_{\lambda_0}}$,
- for any $z \in J_{f_{\lambda_0}}$, the map $\lambda \mapsto h_\lambda(z)$ is holomorphic on Λ,
- h_{λ_0} is the identity on $J_{f_{\lambda_0}}$.

Lemma 4.1. — *Let (f, a) be any dynamical pair of $\mathbb{P}^1$ of degree $d \geq 2$ parametrized by the unit disk $\mathbb{D}$. If f is J-stable and* $\mathrm{supp}(\mu_{f,a}) \neq \emptyset$*, we have*

$$\mathrm{supp}(\mu_{f,a}) = \{\lambda \in \mathbb{D}\, ;\ a(\lambda) \in J_{f_\lambda}\}.$$

Proof. — Since $\mathrm{Bif}(f, a) = \mathrm{supp}(\mu_{f,a}) \neq \emptyset$, the set D of parameters $\lambda_0 \in \mathbb{D}$ such that a is transversely prerepelling at λ_0 is a non-empty countable dense subset of $\mathrm{Bif}(f, a)$. As J-repelling points of f_{λ_0} are contained in $J_{f_{\lambda_0}}$, this gives $\mathrm{Bif}(f, a) \subset \{\lambda \in \mathbb{D}\, ;\ a(\lambda) \in J_{f_\lambda}\}$.

Pick now $\lambda_0 \in \{\lambda \in \mathbb{D}\, ;\ a(\lambda) \in J_{f_\lambda}\}$ and assume $\lambda_0 \notin \mathrm{Bif}(f, a)$. Set $a_n(\lambda) := f_\lambda^n(a(\lambda))$ for all $n \geq 0$ and all $\lambda \in \mathbb{D}$. Let $h : \mathbb{D} \times J_{f_0} \to \mathbb{P}^1$ be the unique holomorphic motion of J_{f_0} parametrized by $\mathbb{D}$ such that, if $h_\lambda := h(\lambda, \cdot)$, then

$$h_\lambda \circ f_0 = f_\lambda \circ h_\lambda \text{ on } J_{f_0}.$$

Note that for all $z \in J_{f_0}$, the sequence $\{\lambda \mapsto h_\lambda(f_0^n(z))\}_n$ is a normal family on $\mathbb{D}$.

Beware that for all periodic point $z \in J_{f_0}$ of f_0, the function $z(\lambda) := h_\lambda(z)$ is a marking of z as a periodic point of f_λ. For all $s \in \mathbb{D}$, if we let $h^s_\lambda := h_\lambda \circ h_s^{-1}$. The family $(h^s_\lambda)_\lambda$ is a holomorphic motion of J_{f_s} which satisfies

$$h^s_\lambda \circ f_s = f_\lambda \circ h^s_\lambda \text{ on } J_{f_s},$$

for all $\lambda \in \mathbb{D}(s, 1 - |s|)$. Since we assumed $\lambda_0 \notin \mathrm{Bif}(f, a)$, there exists $\epsilon > 0$ such that $\mathbb{D}(\lambda_0, \epsilon) \cap \mathrm{Bif}(f, a) = \emptyset$ and we can choose an affine chart of $\mathbb{P}^1$ such that $a_n(\lambda)$ and $h^{\lambda_0}_\lambda(a_n(\lambda_0))$ lie in this chart for all $n \geq 1$ and all $\lambda \in \mathbb{D}(\lambda_0, \epsilon)$. For all n, set

$$s_n(\lambda) := a_n(\lambda) - h^{\lambda_0}_\lambda(a_n(\lambda_0)),\ \lambda \in \mathbb{D}(\lambda_0, \epsilon).$$

Assume first $s_m \equiv 0$ on $\mathbb{D}(\lambda_0, \epsilon)$ for some $m \geq 0$. This implies $a_m(\lambda) = h_\lambda(a_m(0))$ for all $\lambda \in \mathbb{D}(\lambda_0, \epsilon)$. By the Isolated Zero Theorem, we thus have

$$a_m(\lambda) = h_\lambda(a_m(0)) \text{ for all } \lambda \in \mathbb{D}.$$

As $h_\lambda \circ f_0 = f_\lambda \circ h_\lambda$, this yields $a_n(\lambda) \equiv h_\lambda(a_n(0))$ for all $n \geq m$, and (a_n) is a normal family on $\mathbb{D}$. This is a contradiction, since we assumed $\mathrm{Bif}(f, a) \neq \emptyset$. We thus may assume $s_m \not\equiv 0$ on $\mathbb{D}(\lambda_0, \epsilon)$. In particular, up to reducing ϵ, we may assume $s_m(\lambda) \neq 0$ for all $\lambda \in \mathbb{D}(\lambda_0, \epsilon) \setminus \{\lambda_0\}$. Let $z_0 := a_m(\lambda_0)$. By Rouché Theorem, there exists $\eta > 0$ such that for any $z \in \mathbb{D}(z_0, \eta) \cap J_{f_{\lambda_0}}$, the function

$$s_{m,z}(\lambda) := a_m(\lambda) - h^{\lambda_0}_\lambda(z)$$

has finitely many isolated zeros in $\mathbb{D}(\lambda_0, \epsilon)$. As repelling periodic points are dense in $J_{f_{\lambda_0}}$, there exists $z_1 \in \mathbb{D}(z_0, \eta) \cap J_{f_{\lambda_0}}$ which is f_{λ_0}-periodic and repelling. The implies there exists $\lambda_1 \in \mathbb{D}(\lambda_0, \epsilon)$ such that a is properly prerepelling at λ_1. Finally, Theorem 2.3 (or simply Montel Theorem in this case) gives $\lambda_1 \in \mathrm{Bif}(f, a)$ ending the proof. □

Using Montel theorem, one can deduce Theorem A.

Proof of Theorem A. — Assume first f is J-stable. Note first that $J_{f_\lambda} = \mathbb{P}^1$ is true for some $\lambda \in \Lambda$ if and only if it is true for all $\lambda \in \Lambda$ in this case.

Assume now that $J_{f_\lambda} \neq \mathbb{P}^1$ for all $\lambda \in \Lambda$. By [**MS**, Theorem 7.8], there exists a countable union of proper analytic subvariety $S \subset \Lambda$ such that $\Lambda \setminus S$ is open and, for any topological disk $D \subset \Lambda \setminus S$ (centered at some λ_0), there exists a unique holomorphic motion $\phi : D \times \mathbb{P}^1 \to \mathbb{P}^1$ which conjugates f_{λ_0} to f_λ on $\mathbb{P}^1$. In particular the set $\{(\lambda, z) \in D \times \mathbb{P}^1 \ :\ z \in F_{f_\lambda}\}$ is a non-empty open subset of $D \times \mathbb{P}^1$. As we assumed $\mathrm{Bif}(f, a) = \Lambda$, the sequence $\{\lambda \longmapsto f^n_\lambda(a(\lambda))\}_{n\geq 1}$ is not a normal family on D. By Montel Theorem, there exists $n \geq 1$, $1 \leq i \leq p$ and $\lambda_1 \in D$ such that $f^n_{\lambda_1}(a(\lambda_1)) \in F_{f_{\lambda_1}}$, hence $a(\lambda_1) \in F_{f_{\lambda_1}}$. However, Lemma 4.1 gives

$$D = \mathrm{Bif}(f, a) \cap D = \{\lambda \in D \ :\ a(\lambda) \in J_{f_\lambda}\},$$

whence $a(\lambda_1) \in J_{f_{\lambda_1}}$. This is a contradiction. This implies $J_{f_\lambda} = \mathbb{P}^1$ for all $\lambda \in \Lambda$. Finally, by Lemma V.1 of [**MSS**], if f is not trivial, this implies f_λ has an invariant linefield on its Julia set for all $\lambda \in \Lambda$.

If f is not J-stable, by Montel Theorem, there exists a non-empty open set U of Λ such that $(f_\lambda)_{\lambda\in U}$ is J-stable with an attracting periodic $z_1, \ldots, z_p$ of period $p \geq 3$, and we we proceed as follows: pick a topological disk $D \subset U$. Then there exists holomorphic functions $z_1, \ldots, z_p : D \to \mathbb{P}^1$ which paramerize this attracting cycle. In particular, $z_i(\lambda) \neq z_j(\lambda)$ for all $i \neq j$ and all $\lambda \in D$. Since we assumed $\mathrm{Bif}(f, a) = \Lambda$, the sequence $\{\lambda \longmapsto f^n_\lambda(a(\lambda))\}_{n\geq 1}$

is not a normal family on D. By Montel Theorem, there exists $n \geq 1$, $1 \leq i \leq p$ and $\lambda_0 \in D$ such that

$$f^n_{\lambda_0}(a(\lambda_0)) = z_i(\lambda_0).$$

By Lemma 4.1, since $\lambda_0 \in \mathrm{Bif}(f,a)$ this implies $z_i(\lambda_0) \in J_{f_{\lambda_0}}$. This is a contradiction with the fact that z_i is attracting. □

4.2. Proof of Theorem B and the isotrivial case

Proof of Theorem B. — Remark that points 1. and 2. are equivalent by Theorem 2.2. We first prove 1. implies 4. Assume $\mathrm{Bif}(f,a) = \Lambda$. By Theorem A the family f is J-stable. As Λ is a quasi-projective manifold, by [**Mc**, Theorem 2.4], since f is not isotrivial, f is a family of Lattès maps.

We now prove 4. implies 1. We thus assume that f is a non-isotrivial family of Lattès and that $\mu_{f,a}$ is non-zero. Recall that, since f is a family of Lattès maps, it is stable. We want to prove that $\mathrm{supp}(\mu_{f,a}) = \Lambda$. Assume it is not the case, then there exists a non-empty open set $U \subset \Lambda$ such that $U \subset \Lambda \setminus \mathrm{supp}(\mu_{f,a})$. The pair (f,a) being stable in U, $a(\lambda)$ cannot be a repelling periodic point of f_λ for any $\lambda \in U$. From the uniqueness of the holomorphic motion, it follows that there exists $z_0 \in \mathbb{P}^1$ such that $a(\lambda) = h_\lambda(z_0)$ for all $\lambda \in U$. By analytic continuation, this gives $a(\lambda) = h_\lambda(z_0)$ for all $\lambda \in \Lambda$. This contradicts the fact that $\mu_{f,a}$ is non-zero.

The equivalence between 3. and 4. follows from Theorem C and the equivalence betwen 1. and 4. □

Recall that when f is isotrivial, either $J_{f_\lambda} = \mathbb{P}^1$ for all λ, or $J_{f_\lambda} \neq \mathbb{P}^1$ for all λ. We conclude this section with the following easy proposition, which clarifies the case when f is isotrivial.

Proposition 4.2. — *Let f be an isotrivial algebraic family parametrized by an irreducible quasiprojective curve Λ and let $a : \Lambda \to \mathbb{P}^1$ be such that the pair (f,a) is unstable. The following are equivalent:*

1. *the Julia set of f_λ is $J_{f_\lambda} = \mathbb{P}^1$ for all $\lambda \in \Lambda$,*
2. *the bifurcation locus of (f,a) contains a non-empty open set,*
3. *the bifurcation locus of (f,a) is $\mathrm{Bif}(f,a) = \Lambda$.*

Remark. — In fact, in the isotrivial casen we can also prove the following are equivalent:

1. the family f is an isotrivial family of Lattès maps,
2. the measure $\mu_{f,a}$ is absolutely continuous with respect to ω_Λ.

Proof. — If $\mathrm{Bif}(f,a) = \Lambda$, obviously, it contains a non-empty open subset of Λ. Now, as f is isotrivial, up to taking a finite branched cover of Λ and up to conjugating f by a family of Möbius transformantions, we can assume $f_\lambda = f_0$ for all $\lambda \in \Lambda$. In particular, it is a J-stable family and., applying Lemma 4.1 in local charts, we find

$$\mathrm{Bif}(f,a) = \{\lambda \in \Lambda\,;\ a(\lambda) \in J_{f_0}\} = a^{-1}(J_{f_0}).$$

Since $\mathrm{Bif}(f,a) \neq \varnothing$, the holomorphic map a has to be non-constant, whence it is open. In particular, if $\mathrm{Bif}(f,a)$ contains a non-empty open set, J_{f_0} has to contain a nomepty open set and $J_{f_0} = \mathbb{P}^1$. Finally, if $J_{f_0} = \mathbb{P}^1$, then we clearly have $\mathrm{Bif}(f,a) = a^{-1}(\mathbb{P}^1) = \Lambda$.

Assume first $J_{f_\lambda} = \mathbb{P}^1$ for all $\lambda \in \Lambda$. When $\mu_{f,a}$ is absolutely continuous, the conclusion follows as in the proof of Theorem B. □

4.3. Concluding remarks and questions

Dynamical pairs with a non-singular bifurcation measure. — First, when $k > 1$, the statement of Theorem D holds only if all repelling J-cycles are linearizable.

This results raises several questions:

Question 4.3. — *Can we generalize Theorem D to the cases when*

1. *There exists J-repelling cycles that are non-linearizable?*
2. *T_a^k is just* non-singular *with respect to a smooth volume form?*

The first question is very likely to have a positive answer, using Poincaré-Dulac normal forms instead of lilnear normal forms. However, it looks quite difficult to prove rigorously.

In fact, Zdunik [**Z**] completely classifies rational maps with a maximal entropy measure which is not singular with respect to a Hausdorff measure $\mathcal{H}^\alpha$: either $\alpha = 1$ and the rational map is conjugated to a monomial map $z^{\pm d}$ or to a Chebichev polynomial T_d, i.e. the only polynomial satisfying $T_d\left(z+\frac{1}{z}\right) = z^d + \frac{1}{z^d}$ for all $z \in \mathbb{C}$, or $\alpha = 2$ and the rational map is a Lattès map.

We expect the following complete parametric counterpart to [**Z**] to be true:

Question 4.4. — *Let (f, a) be any holomorphic dynamical pair of $\mathbb{P}^1$ of degree $d \geq 2$ parametrized by the unit disk $\mathbb{D}$ of $\mathbb{C}$. Assume that (f, a) is unstable. Assume also there exists $\alpha > 0$ and a function $h : \mathbb{D} \to \mathbb{R}_+$ such that $\mu_{f,a} = h \cdot \mathcal{H}^\alpha$ on $\mathbb{D}$. Can we prove that*

- *either $\alpha = 2$ and f is a family of Lattès maps,*
- *or $\alpha = 1$, f is isotrivial and all f_λ's are conjugated to $z^{\pm d}$ or a Chebichev polynomial?*

As in the case of families of Lattès maps, we can expect the proof to generalize to the case when $k > 1$. This raises the following question.

Question 4.5. — *Classify endomorphisms of $\mathbb{P}^k$ which maximal entropy measure is not singular with respect to some Hausdorff measure $\mathcal{H}^\alpha$ on $\mathbb{P}^k$ (and possible values of α).*

As seen above, the case $\alpha = 2k$ has been treated by Berteloot and Loeb [**BL**] and Berteloot and Dupont [**BD3**]. Of course, there are also easy examples where $\alpha = k$: take $f : \mathbb{P}^1 \to \mathbb{P}^1$ which maximal entropy measure has dimension 1, then the endomorphism $F : \mathbb{P}^k \longrightarrow \mathbb{P}^k$ making the following diagram commute

$$\begin{array}{ccc} (\mathbb{P}^1)^k & \xrightarrow{(f,\ldots,f)} & (\mathbb{P}^1)^k \\ \eta_k \downarrow & & \downarrow \eta_k \\ \mathbb{P}^k & \xrightarrow[F]{} & \mathbb{P}^k \end{array}$$

where η_k is te quotient map of the action by permutation of coordinates of the symmetric group $\mathfrak{S}_k$, satisfies $\dim(\mu_F) = k$ (see [**GHK**] for a study of symmetric products).

***J*-stability and dynamical pairs, when $k \geq 2$.** — We say that a family $f : \Lambda \times \mathbb{P}^k \longrightarrow \mathbb{P}^k$ of degree $d \geq 2$ endomorphisms of $\mathbb{P}^k$ is *weakly J-stable* if all the J-repelling cycles can be followed holomorphically throughout the whole family Λ, i.e. if for all $n \geq 1$, there exists $N \geq 0$ and holomorphic maps $z_1, \ldots, z_N : \Lambda \to \mathbb{P}^k$ such that $\{z_1(\lambda), \ldots, z_N(\lambda)\}$ is exactly the set of all repelling J-cycles of f_λ of exact period n for all $\lambda \in \Lambda$.

For any endomorphism f of $\mathbb{P}^k$, let $L(f) := \int_{\mathbb{P}^k} \log|\det Df|\mu_f$ be the sum of the Lyapunov exponents of f with respect to its Green measure μ_f. By a beautiful result of Berteloot, Bianchi and Dupont [**BBD**], f is J-stable if and only if $\lambda \longmapsto L(f_\lambda)$ is pluriharmonic on Λ.

A natural question is then the following:

Question 4.6. — *Given any dynamical pair (f, a) of degree d of $\mathbb{P}^k$ parametrized by the unit ball $\mathbb{B} \subset \mathbb{C}^k$ such that f is a weakly J-stable family, do we still have*

$$\mathrm{Supp}(T_a^k) = \{\lambda \in \mathbb{B}\,;\ a(\lambda) \in J_{f_\lambda}\}\ ?$$

Note that this holds for $k = 1$ by Lemma 4.1. One of the difficulties, when $k > 1$, is that the weak J-stability is equivalent to the existence of a *branched* holomorphic motion.

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