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THE ARITHMETIC OF POLYNOMIAL DYNAMICAL PAIRS

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INTRODUCTION

This book is intented as an exploration of the moduli space Poly_d of complex polynomials of degree $d \geq 2$ in one variable using tools primarily coming from arithmetic geometry.

The Mandelbrot set in Poly_2 has undoubtedly been the focus of the most comprehensive set of studies, and its local geometry is still an active research field in connection with the Fatou conjecture, see [**19**] and the references therein. In their seminal work, Branner and Hubbard [**30, 31**] gave a topological description of the space of cubic polynomials with disconnected Julia sets using combinatorial tools. In any degree, Poly_d is a complex orbifold of dimension $d-1$, and is therefore naturally amenable to complex analysis and in particular to pluripotential theory. This observation has been particularly fruitful to describe the locus of instability, and to investigate the boundary of the connectedness locus. DeMarco [**49**] constructed a positive closed $(1,1)$ current whose support is precisely the set of unstable parameters. Dujardin and the first author [**68**] then noticed that the Monge-Ampère measure of this current defines a probability measure μ_{bif} whose support is in a way the right generalization of the boundary of the Mandelbrot set in higher degree, capturing the part of the moduli space where the dynamics is the most unstable (see also [**11**] for the case of rational maps). The support of μ_{bif} has a very intricate structure: it was proved by Shishikura [**152**] in degree 2 and later generalized in higher degree by the second author [**87**] that the Hausdorff dimension of the support of μ_{bif} is maximal equal to $2(d-1)$.

A polynomial is said to be post-critically finite (or PCF) if all its critical points have a finite orbit. The Julia set of a PCF polynomial is connected, of zero measure, and the dynamics on it is hyperbolic off the post-critical set. PCF polynomials form a countable subset of larger classes of polynomials (such as Misiurewicz, or Collet-Eckmann) for which the thermodynamical formalism is well understood, [**141, 142**]. They also play a pivotal role in the study of the connectedness locus of Poly_d: their distribution was described in a series of papers [**76, 90, 91**] and proved to represent the bifurcation measure μ_{bif}.

Any PCF polynomial is the solution of a system of $d-1$ equations of the form $P^n(c) = P^m(c)$ where c denotes a critical point and n, m are two distinct integers. In the moduli space, these equations are algebraic with integral coefficients, so that any PCF polynomial is in fact defined over a number field. Ingram [**109**] pushed this remark further and built a natural height $h_{\mathrm{bif}}\colon \mathrm{Poly}_d(\bar{\mathbb{Q}}) \to \mathbb{R}_+$ for which the set of PCF polynomials coincides with $\{h_{\mathrm{bif}} = 0\}$.

Height theory yields interesting new perspectives on the geometry of Poly_d, and more specifically on the distribution of PCF polynomials. We will be mostly interested here in the so-called dynamical André-Oort conjecture which appeared in [**6**], see also [**156**].

This remarkable conjecture was set out by Baker and DeMarco who were motivated by deep analogies between PCF dynamics and CM points in Shimura varieties, and more specifically by works by Masser-Zannier [**27, 123, 171**] on torsion points in elliptic curves. An historical account on the introduction of these ideas in arithmetic dynamics is given in [**6**, §1.2], and [**5**, §1.2], see also [**93**]. We note that this analogy goes far beyond the problems considered in this book, and applies to various conjectures described in [**155, 52**]. We refer to the book by Zannier [**171**] for a beautiful discussion of unlikely intersection problems in arithmetic geometry.

Baker and DeMarco proposed to characterize irreducible subvarieties of Poly_d (or more generally of the moduli space of rational maps) containing a Zariski dense subset of PCF polynomials, and conjectured that such varieties were defined by critical relations. This conjecture was proven for unicritical polynomials in [**97**] and [**98**], and in degree 3 in [**77**] and [**103**].

It is our aim to give a proof of that conjecture for *curves* in Poly_d for any $d \geq 2$, and based on this result to attempt a classification of these curves in terms of combinatorial data encoding critical relations.

Our proof roughly follows the line of arguments devised in the original paper of Baker and DeMarco, and relies on equidistribution theorems of points of small height by Thuillier [**160**] and Yuan [**168**]; on the expansion of the Böttcher coordinates; and on Ritt's theory characterizing equalities of composition of polynomials.

We needed, though, to overcome several important technical difficulties, such as proving the continuity of metrics naturally attached to families of polynomials. We also had to inject new ingredients, most notably some dynamical rigidity results concerning families of polynomials with a marked point whose bifurcation locus is real-analytic.

For the most part of the memoir, we shall work in the more general context of polynomial dynamical pairs (P, a) parameterized by a complex affine curve C, postponing the proof of the dynamical André-Oort conjecture to the last chapter. We investigate quite generally the problem of unlikely intersection that was promoted in the context of torsion points on elliptic curves by Zannier and his co-authors [**123, 171**], and later studied by Baker and DeMarco [**5, 6**] in our context. This problem amounts to understanding when two polynomial dynamical pairs (P, a) and (Q, b) parameterized by the same curve C have an infinite set of common parameters for which the marked points are preperiodic. We obtain quite definite answers for polynomial pairs, and we prove finiteness theorems that we feel are of some interest for further exploration.

We have tried to review all the necessary material for the proof of the dynamical André-Oort conjecture, but we have omitted some technical proofs that are already available in the literature in an optimal form. On the other hand, we have made some efforts to clarify some proofs which we felt too sketchy in the literature. The group of dynamical symmetries of a polynomial play a very important role in unlikely intersection problems, and we have thus included a detailed discussion of this notion.

Let us now describe in more detail the content of the book.

Polynomial dynamical pairs. — In this paragraph we present the main players of our memoir. The central notion is that of a POLYNOMIAL DYNAMICAL PAIR parameterized by a curve. Such a pair (P, a) is by definition an algebraic family of polynomials P_t parameterized by an irreducible affine curve C defined over a field K, accompanied by a regular function $a \in K[C]$ which defines an algebraically varying marked point. Most of the time, these objects will be defined over the field of complex numbers $K = \mathbb{C}$, but it will also be important to consider polynomial dynamical pairs over other fields like number fields, p-adic fields, or finite fields.

Any polynomial dynamical pair leaves a "trace" on the parameter space C, which may take different forms. Suppose first that K is an arbitrary field, and let $\bar{K}$ be an algebraic closure of K. The first basic object to consider is the set

Preper(P, a) of (closed) points $t \in C(\bar{K})$ such that $a(t)$ is preperiodic under P_t. This set is either equal to C or at most countable.

A slightly more complicated but equally important object one can attach to (P, a) is the following divisor. Let $\bar{C}$ be the completion of C, that is the unique projective algebraic curve containing C as a Zariski dense open subset, and smooth at all points $\bar{C} \setminus C$. Points in $\bar{C} \setminus C$ are called branches at infinity of C. Any pair (P, a) induces an effective divisor $\mathsf{D}_{P,a}$ on $\bar{C}$, which is obtained by setting

$$\text{(1)} \qquad \operatorname{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) := \lim_{n\to\infty} -\frac{1}{d^n} \min\{0, \operatorname{ord}_{\mathfrak{c}}(P^n(a))\},$$

for any branch $\mathfrak{c}$ at infinity. The limit is known to exist and is always a rational number, see §4.2.2.

When $K = \mathbb{C}$, one can associate various topological objects to a polynomial dynamical pair. One can consider the locus of stability of the pair (P, a) which consists of the open set in which the family of holomorphic maps $\{P^n(a)\}_{n\geq 0}$ is normal. Its complement is the BIFURCATION LOCUS which we denote by Bif(P, a). This set can be characterized using potential theory as follows. Recall the definition of the Green function of a polynomial P of degree d:

$$g_P(z) := \lim_{n\to\infty} \frac{1}{d^n} \max\{\log |P^n(z)|, 0\},$$

so that $\{g_P = 0\}$ is the filled-in Julia set of P consisting of those points having bounded orbits. On the parameter space C, we then define the function

$$g_{P,a}(t) = g_{P_t}(a(t)).$$

It is a non-negative continuous subharmonic function on C, and the support of the measure $\mu_{\text{bif}} = \Delta g_{P,a}$ is precisely equal to Bif(P, a). Of crucial technical importance is the following result from [**78**] which relates the function $g_{P,a}$ to the divisor defined above.

Theorem 1. — *In a neighborhood of any branch at infinity $\mathfrak{c} \in \bar{C}$, one has the expansion*

$$g_{P,a}(t) = \operatorname{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) \log |t|^{-1} + \tilde{g}(t)$$

where t is a local parameter centered at $\mathfrak{c}$ and $\tilde{g}$ is continuous at 0.

This result can be interpreted in the langage of complex geometry by saying that $g_{P,a}$ induces a continuous semi-positive metrization on the $\mathbb{Q}$-line bundle $\mathcal{O}_{\bar{C}}(\mathsf{D}_{P,a})$. This fact is the key to apply techniques from arithmetic geometry.

Let us now suppose that $K = \mathbb{K}$ is a number field. For any place v of $\mathbb{K}$, denote by $\mathbb{K}_v$ the completion of $\mathbb{K}$, and by $\mathbb{C}_v$ the completion of its algebraic closure. It is then possible to mimic the previous constructions at any (finite or infinite) place v of $\mathbb{K}$ to obtain functions $g_{P,a,v}\colon C_v^{\mathrm{an}} \to \mathbb{R}_+$ on the analytification (in the sense of Berkovich) C_v^{an} of the curve C over $\mathbb{C}_v$. Summing all these functions yield a height function $h_{P,a}\colon C(\bar{\mathbb{K}}) \to \mathbb{R}_+$.

Alternatively, we may start from the standard Weil height $h_{\mathrm{st}}\colon \mathbb{P}^1(\bar{\mathbb{K}}) \to \mathbb{R}_+$, see e.g. [**105**]. Then for any polynomial with algebraic coefficients, we define its canonical height [**36**] to be:

$$h_P(z) := \lim_{n\to\infty} \frac{1}{d^n} h_{\mathrm{st}}(P^n(z)),$$

and finally we set $h_{P,a}(t) := h_{P_t}(a(t))$. Using the Northcott theorem, one obtains that $\{h_{P,a} = 0\}$ coincides with the set $\mathrm{Preper}(P,a)$ of parameters $t \in C(\bar{\mathbb{K}})$ for which $a(t)$ is a preperiodic point of P_t.

It is an amazing fact that all the objects attached to a polynomial dynamical pair (P,a) we have seen so far are tightly interrelated, as the next theorem due to DeMarco [**51**] shows.

An isotrivial pair (P,a) is a pair which is conjugated to a constant polynomial and a constant marked point, possibly after a base change. A marked point is STABLY PREPERIODIC when there exist two integers $n > m$ such that $P_t^n(a(t)) = P_t^m(a(t))$.

Theorem 2. — *Let (P,a) be a polynomial dynamical pair of degree $d \geq 2$ parametrized by an affine irreducible curve C defined over a number field $\mathbb{K}$. If the pair is not isotrivial, then the following assertions are equivalent:*

1. *the set* $\mathrm{Preper}(P,a)$ *is equal to* $C(\bar{\mathbb{K}})$*;*
2. *the marked point is stably preperiodic;*
3. *the divisor $\mathsf{D}_{P,a}$ of the pair (P,a) vanishes;*
4. *for any Archimedean place v, the bifurcation measure $\mu_{P,a,v} := \Delta g_{P,a,v}$ vanishes;*
5. *the height $h_{P,a}$ is identically zero.*

A pair (P,a) which satisfies either one of the previous conditions is said to be passive, otherwise it is called an ACTIVE PAIR. For an active pair, $\mathrm{Preper}(P,a)$ is countable, the bifurcation measure $\mu_{P,a}$ is non trivial, and the height $h_{P,a}$ is non zero.

Holomorphic rigidity for polynomial dynamical pairs. — Rigidity results are pervasive in (holomorphic) dynamics. One of the most famous rigidity result was obtained by Zdunik [**172**] and asserts that the measure of maximal entropy of a polynomial P is absolutely continuous with respect to the Hausdorff measure of its Julia set iff P is conjugated by an affine transformation to either a monomial map $M_d(z) = z^d$, or to a Chebyshev polynomial $\pm T_d$ where $T_d(z + z^{-1}) = z^d + z^{-d}$. In particular, these two families of examples are the only ones having a smooth Julia set, a theorem due to Fatou [**74**].

The following analog of Zdunik's result for polynomial dynamical pairs is our first main result.

Theorem A. — *Let (P, a) be a polynomial dynamical pair of degree $d \geq 2$ parametrized by a connected Riemann surface S. Assume that* $\mathrm{Bif}(P, a)$ *is non-empty and included in a smooth real curve. Then one of the following holds:*

- *either P_t is conjugated to M_d or $\pm T_d$ for all $t \in S$;*
- *or there exists a univalent map $\imath\colon \mathbb{D} \to S$ such that $\imath^{-1}(\mathrm{Bif}(P, a))$ is a non-empty closed and totally disconnected perfect subset of the real line and the pair $(P \circ \imath, a \circ \imath)$ is conjugated to a real family over $\mathbb{D}$.*

We say that a polynomial dynamical pair (P, a) parameterized by the unit disk is a real family whenever the power series defining the coefficients of P and the marked point have all real coefficients.

The previous theorem is a crucial ingredient for handling the unlikely intersection problem that we will describe later. Its proof builds on a transfer principle from the parameter space to the dynamical plane which can be decomposed into two parts.

The first step is to find a parameter t_0 at which $a(t_0)$ is preperiodic to a repelling orbit of P_{t_0} and such that $t \mapsto a(t)$ is transversal at t_0 to the preperiodic orbit degenerating to $a(t_0)$. This step builds on an argument of Dujardin [**67**]. The second step relies on Tan Lei's similarity theorem [**159**] which shows that the bifurcation locus $\mathrm{Bif}(P, a)$ near t_0 is conformally equivalent at small scales to the Julia set of P_{t_0}.

Combining these two ingredients, we see that if $\mathrm{Bif}(P, a)$ is connected, then Zdunik's theorem implies that P_t is isotrivial conjugated to M_d or $\pm T_d$ for all $t \in C$. When $\mathrm{Bif}(P, a)$ is disconnected, then we prove that all multipliers of P_{t_0} are real and we conclude that P_t is real for all nearby parameters using an argument of Eremenko and Van Strien [**73**].

In many results that we present below, we shall exclude all polynomials that are affinely conjugated to either M_d or $\pm T_d$. These dynamical systems carry different names in the literature: Zdunik [**172**] name them maps with parabolic orbifolds; they are called special in [**55, 136**]; and Medvedev and Scanlon call them non-disintegrated polynomials, see the discussion on [**126**, p.16]. We shall refer them to as INTEGRABLE polynomials by analogy with the notion of integrable system in hamiltonian dynamics (see [**40, 164**]). A family of polynomials $\{P_t\}_{t\in C}$ will be called non-integrable whenever there exists a dense open set $U \subset C$ such that P_t is not integrable for any $t \in U$.

Unlikely intersections for polynomial dynamical pairs. — Our next objective is to investigate the problem of characterizing when two polynomial dynamical pairs (P, a) and (Q, b) parameterized by the same algebraic curve C leave the same "trace" on C.

Analogies with arithmetic geometry suggested that the quite weak condition of $\mathrm{Preper}(P, a) \cap \mathrm{Preper}(Q, b)$ being infinite in fact implies very strong relations between the two pairs. This phenomenon was first observed for Lattès maps by Masser and Zannier [**123**], and later for unicritical polynomials by Baker and DeMarco [**5**], and for more general families of polynomials parameterized by the affine line by Ghioca, Hsia and Tucker [**95**]. We refer to the surveys [**93**], [**52**] and [**14**] where this problem is also addressed.

A precise conjecture was formulated by DeMarco in [**53**, Conjecture 4.8]: up to symmetries and taking iterates the two families P and Q are actually equal, and the marked points belong to the same grand orbit. In other words, the existence of unlikely intersections forces some algebraic rigidity between the dynamical pairs.

We prove here DeMarco's conjecture for polynomial dynamical pairs defined over a number field.

Theorem B. — *Let (P, a) and (Q, b) be active non-integrable polynomial dynamical pairs parametrized by an irreducible algebraic curve C of respective degree $d, \delta \geq 2$. Assume that the two pairs are defined over a number field $\mathbb{K}$. Then, the following are equivalent:*

1. *the set* $\mathrm{Preper}(P, a) \cap \mathrm{Preper}(Q, b)$ *is an infinite subset of $C(\bar{\mathbb{K}})$;*
2. *the two height functions $h_{P,a}, h_{Q,b} \colon C(\bar{\mathbb{K}}) \to \mathbb{R}_+$ are proportional;*
3. *there exist integers $N, M \geq 1$, $r, s \geq 0$, and families R, τ and π of polynomials of degree ≥ 1 parametrized by C such that*

$$(\dagger) \qquad \tau \circ P^N = R \circ \tau \quad \textit{and} \quad \pi \circ Q^M = R \circ \pi,$$

and $\tau(P^r(a)) = \pi(Q^s(b))$.

It is not difficult to see that (3)⇒(2)⇒(1) so that the main content of the theorem is the implication (1)⇒(3). To obtain (1)⇒(2), we first apply Yuan-Thuillier's equidistribution result [**160, 168**] for points of small height: it is precisely at this step that the continuity of $\tilde{g}$ in Theorem 1 is crucial.

This allows one to prove that the bifurcation measures $\mu_{P,a,v}$ and $\mu_{Q,b,v}$ are proportional at any place v of $\mathbb{K}$. From there, one infers the proportionality of height functions i.e. (2) using our above rigidity result (Theorem A).

The implication (2)⇒(3) is more involved. We first prove that $\deg(P)$ and $\deg(Q)$ are multiplicatively dependent using an argument taken from [**69**] which relies on computing the Hölder constants of continuity of the potentials of the bifurcation measures at a complex place. From this, we obtain (3) by combining in a quite subtle way several ingredients including:

- a precise understanding of the expansion at infinity of the Böttcher coordinate;
- an algebraization result of germs of curves defined by adelic series due to Xie [**165**];
- and the classification of invariant curves by product maps $(z, w) \mapsto (R(z), R(w))$.

The latter result is due to Medvedev and Scanlon [**126**] whose proof elaborates on Ritt's theory [**144**]. This theory aims at describing all possible ways a polynomial can be written as the composition of lower degree polynomials. It is very combinatorial in nature and was treated by several authors, by Zannier [**170**], by Müller-Zieve [**175**], see also the references therein. Of particular relevance for us are the series of papers by Pakovich [**134, 135, 136**], and by Ghioca, Nguyen and Ye [**99, 101**].

As mentioned above, the line of arguments for proving Theorem B is mostly taken from the seminal paper of Baker and DeMarco, but with considerably more technical issues. The core of the proof takes about 8 pages and is the content of §5.4.

It would be desirable to extend Theorem B to families defined over an arbitrary field of characteristic zero. Reducing to a family defined over a number field typically uses a specialization argument. We faced an essential difficulty in the course of this argument, and thus had to require an additional assumption.

Theorem C. — *Pick any irreducible algebraic curve C defined over a field of characteristic 0. Let (P, a) and (Q, b) be active non-integrable polynomial dynamical pairs parametrized by C of respective degree $d, \delta \geq 2$. Assume that*

(△) *any branch at infinity $\mathfrak{c}$ of C belongs to the support of the divisor $\mathsf{D}_{P,a}$.*

Then, the following are equivalent:

1. *the set* $\mathrm{Preper}(P, a) \cap \mathrm{Preper}(Q, b)$ *is an infinite subset of C;*

2. *there exist integers* $N, M \geq 1$, $r, s \geq 0$, *and families* R, τ *and* π *of polynomials parametrized by* C *such that*

$$\tau \circ P^N = R \circ \tau \quad \textit{and} \quad \pi \circ Q^M = R \circ \pi,$$

and $\tau(P^r(a)) = \pi(Q^s(b))$.

Note that although $(\triangle)$ may not hold in general, it is always satisfied when C admits a unique branch at infinity, e.g. when C is the affine line. In particular, our result yields a far-reaching generalization of [**5**, Theorem 1.1].

In the sequel, we call two active polynomial dynamical pairs (P, a) and (Q, b) ENTANGLED when $\mathrm{Preper}(P, a) \cap \mathrm{Preper}(Q, b)$ is Zariski dense. This terminology inspired by quantum theory reflects the fact the two pairs are dynamically strongly correlated.

Description of all pairs entangled to a fixed pair. — Let us fix a polynomial dynamical pair (P, a) parameterized by an algebraic curve C and for which the previous theorems apply (i.e. either the field of definition of the pair is a number field, or condition $(\triangle)$ holds). We would like now to determine *all* pairs that are entangled to (P, a).

In principle this problem is solvable by Ritt's theory. Given a polynomial P, it is, however, very delicate to describe all polynomials Q for which $(\dagger)$ holds, in particular because there is no a priori bounds on the degrees of τ and π. Much progress have been made by Pakovich [**136**] but it remains unclear whether one can design an algorithm to solve this problem.

To get around this, we consider a more restrictive question which is to determine all pairs (P, b) that are entangled with (P, a). In this problem, the notion of symmetries of a polynomial plays a crucial role, and most of Chapter 3 is devoted to the study of this notion from the algebraic, topological and adelic perspectives. The group $\Sigma(P)$ of symmetries of a complex polynomial P is the group of affine transformations preserving its Julia set. Over an arbitrary field, the definition is less satisfactory. Any monic centered polynomial can be written under the form $P(z) = z^\mu Q(z^m)$ with $\deg(Q)$ minimal, and when P is not integrable we set $\Sigma(P)$ to be the cyclic group of order m. It is also the maximal finite group of affine transformations such that $P(g \cdot z) = \rho(g) \cdot P(z)$ for some morphism $\rho\colon \Sigma(P) \to \Sigma(P)$.

We then prove the following more intrinsic characterization of the symmetry group:

Theorem 3. — *For any field K of characteristic zero and any $P \in K[z]$ of degree $d \geq 2$, the group $\Sigma(P)$ coincides with the set of $g \in \mathrm{Aff}(K)$ such that $g(\mathrm{Preper}(P, \bar{K})) \cap \mathrm{Preper}(P, \bar{K})$ is infinite.*

Of importance in the latter discussion is the subgroup $\Sigma_0(P)$ of affine maps $g \in \Sigma(P)$ such that $P^n(g \cdot z) = P^n(z)$ for some $n \in \mathbb{N}^*$.

We also introduce the notion of PRIMITIVE polynomials. A polynomial P is primitive if any equality $P = g \cdot Q^n$ with $g \in \Sigma(P)$ implies $n = 1$.

These notions of symmetries and primitivity allow us to obtain the following neat statement.

Theorem D. — *Let (P, a) be any active primitive non-integrable polynomial dynamical pair parameterized by an algebraic curve defined over a field K of characteristic 0. Assume that K is a number field, or that ($\triangle$) is satisfied.*

For any marked point $b \in K[C]$ such that (P, b) is active, the following assertions are equivalent:

1. *the set* $\mathrm{Preper}(P, a) \cap \mathrm{Preper}(P, b)$ *is infinite (i.e. a and b are entangled),*
2. *there exist $g \in \Sigma(P)$ and integers $r, s \geq 0$ such that $P^r(b) = g \cdot P^s(a)$.*

Note that this gives a positive answer to [**95**, Question 1.3] for polynomials.

Suppose that $s = 0$ and r is sufficiently large. Then solutions b to the equation $P^r(b) = a$ are not necessarily regular functions on C: they belong to a finite extension of $K(C)$, and their degree is expected to tend to infinity as $r \to \infty$. The next result gives a more detailed description on all marked points parameterized by C which are entangled with (P, a).

Theorem E. — *Let (P, a) be any active primitive non-integrable polynomial dynamical pair parameterized by an irreducible affine curve C defined over $\bar{\mathbb{Q}}$.*

The set of marked points in $\bar{\mathbb{Q}}[C]$ that are entangled with a is the union of $\{g \cdot P^n(a)\,; n \geq 0$ and $g \in \Sigma_0(P)\}$ and a finite set.

This result seems to be new even for the unicritical family.[(1)]

It would obviously be more natural to assume the pair to be defined over an algebraically closed field of characteristic 0, but we use at a crucial step the assumption that (P, a) is defined over $\bar{\mathbb{Q}}$.

[(1)]Suppose $\Sigma(P)$ is trivial. Then Theorem E states that if $b \in \bar{\mathbb{Q}}[C]$ and $P^r(b) = P^s(a)$ for some $r, s \geq 0$ then in fact $b = P^n(a)$ for some n except for finitely many exceptions. This result bears strong ties with the recent work of Bell, Matsusawa and Satriano on the cancellation conjecture for endomorphisms of algebraic varieties defined over number fields [**13**, Conjecture 1.4].

Interestingly enough, the proof of this finiteness theorem relies on the same ingredients as Theorem C, namely the expansion of the Böttcher coordinate, an algebraization result of adelic curves, and Ritt's theory. The proof in fact shows that one may only suppose $b \in \bar{\mathbb{Q}}(C)$.

Unicritical polynomials. — In the short Chapter 7, we discuss in more depth some aspects of unlikely intersection problems for unicritical polynomials.

Recall that in their seminal paper, Baker and DeMarco obtained the following striking result: for any $d \geq 2$, and any $a, b \in \mathbb{C}$, the pairs $\mathrm{Preper}(z^d + t, a)$, $\mathrm{Preper}(z^d + t, b)$ are entangled iff $a^d = b^d$. This result was further expanded by Ghioca, Hsia, and Tucker to more general families of polynomials and not necessarily constant marked points, see [**95**, Theorem 2.3].

Building on our previous results, we obtain the following statement which slightly generalizes op.cit.

Theorem F. — *Let $d, \delta \geq 2$. If a, b are polynomials of the same degree and such that $\mathrm{Preper}(z^d + t, a(t)) \cap \mathrm{Preper}(z^\delta + t, b(t))$ is infinite, then $d = \delta$ and $a(t)^d = b(t)^d$.*

After proving this theorem, we make some preliminary exploration of the set $\mathbb{M}$ of complex numbers $\lambda \in \mathbb{C}^*$ such that the bifurcation locus ∂M_λ is connected, where we have set $M_\lambda := \{t,\ \lambda^{-1} t \in K(z^d + t)\}$, $K(z^d + t)$ being the filled-in Julia set of $z^d + t$. We observe that $\lambda \in \mathbb{M}$ iff $M_\lambda \subset \mathcal{M}(d, 0)$, and prove that $\mathbb{M}$ is the union of finitely isolated points with a closed set of $\mathbb{C}^*$ included in the unit disk, and containing the punctured disk $\mathbb{D}^*(0, 1/8)$. We also include a series of pictures obtained by A. Chéritat suggesting that the core of $\mathbb{M}$ is a topological punctured disk.

Special curves in the parameter space of polynomials. — We finally come back in Chapter 8 to our original objective, namely the classification of curves in Poly_d which contain an infinite subset of PCF polynomials, and the proof of Baker and DeMarco's conjecture claiming that these curves are cut out by critical relations.

A first answer to Baker and DeMarco's question is given by the next result.

Theorem G. — *Pick any non-isotrivial complex family P of polynomials of degree $d \geq 2$ with marked critical points, parameterized by an algebraic curve C, and containing infinitely many PCF parameters.*

If the family is primitive, then possibly after a base change, there exists a subset A *of the set of critical points of* P *such that for any pair* $c_i, c_j \in \mathsf{A}$*, there exists a symmetry* $\sigma \in \Sigma(P)$ *and integers* $n, m \geq 0$ *such that*

$$P^n(c_i) = \sigma \cdot P^m(c_j) \ ; \tag{2}$$

and for any $c_i \notin \mathsf{A}$ *there exist integers* $n_i > m_i \geq 0$ *such that*

$$P^{n_i}(c_i) = P^{m_i}(c_i) \ . \tag{3}$$

When the family is not primitive the statement is not true because the family may exhibit symmetries of degree ≥ 2, as examplified by Baker and Demarco [**6**, Example 4]. After base change, we may write $P = \sigma \cdot P_0^n$ with $\sigma \in \Sigma(P)$ and P_0 primitive and apply our result.

Following the terminology of [**6**, §1.4] inspired from arithmetic geometry, we call SPECIAL any curve in Poly_d containing infinitely many PCF polynomials.

Our theorem says that a special curve in the moduli space of polynomials of degree d either arises as the image under the composition map of a special curve in a lower degree moduli space, or is defined by critical relations (including symmetries) such that all active critical points are entangled.

This result opens up the possibility to give a combinatorial classification of all special curves in the moduli space of polynomials of a fixed degree Poly_d. Recall that a combinatorial classification of PCF polynomials in terms of Hubbard trees has been developed by Douady and Hubbard [**61, 62**], Bielefeld-Fisher-Hubbard [**25**] and further expanded by Poirier [**140**], and Kiwi [**112**]. We make here a first step towards the ambitious goal of classification of special curves using a combinatorial gadget: THE CRITICALLY MARKED DYNAMICAL GRAPH.

We refer to §8.2 for a precise definition of critically marked dynamical graph. It is a (possibly infinite) graph $\Gamma(P)$ with a dynamics that encodes precisely all dynamical critical relations (up to symmetry) of a given polynomial P. We show that to any irreducible curve C in the moduli space of (critically marked) polynomials, one can attach a marked dynamical graph $\Gamma(C)$ such that $\Gamma(P) = \Gamma(C)$ for all but countably many $P \in C$. We then identify a class of marked graphs that we call special which arise from special curves. Under the assumption that the special graph Γ has no symmetry and that its marked points are not periodic, we conversely prove that the set of polynomials such that $\Gamma(P) = \Gamma$ defines a (possibly reducible) special curve.

Our precise statement is quite technical, see Theorem 8.30. To give a sample of the results we obtain, let us describe the situation for cubic polynomials in which case the picture is quite complete. Recall first that the space of cubic polynomials with marked critical points $\mathrm{MPoly}^3_{\mathrm{crit}}$ is two-dimensional and that any cubic polynomial has two critical points (counted with multiplicity). Cubic polynomials having a non trivial symmetry group are either unicritical ($P_t(z) = z^3 + t$, $\Sigma(P_t) = \mathbb{U}_3$), or of the form $P_t(z) = z(z^2 + t)$ with $\Sigma(P_t) = \mathbb{U}_3$. We obtain our first two special curves in $\mathrm{MPoly}^3_{\mathrm{crit}}$ that we denote by $\Sigma(3, 3, 0)$ and $\Sigma(3, 2, 1)$. Let C be any special curve different from these two curves. By Theorem G, either one critical point c_1 is persistently preperiodic on C; or there is a persistent collision between the two critical points c_1 and c_2. In the former case, the graph $\Gamma(C)$ is a union of a straight half-line and a finite connected graph having a cycle with n vertices together with a segment with m vertices attached which encodes the fact that m is the smallest integer such that $P_t^m(c_1)$ is periodic of exact period n for all $t \in C$ (this graph is depicted in the upper left of Figure 4). Denote by $\Gamma_1(n, m)$ this graph. In the latter case, $\Gamma(C)$ is a infinite tree obtained by attaching by their extremities two segments with n_1 and n_2 vertices respectively to the origin of a half-line (see the upper right of Figure 4). Here n_1 and n_2 correspond to the least integers such that $P_t^{n_1}(c_1) = P_t^{n_2}(c_2)$ for all $t \in C$ (note that in this case we cannot have $n_1 = n_2 = 1$ for degree issues). Denote by $\Gamma_2(n_1, n_2)$ this graph.

Theorem 4. — *For any pair of integers (n, m) with $n > m \geq 0$ (resp. $(n_1, n_2) \neq (1, 1)$), there exists a special curve C in $\mathrm{MPoly}^3_{\mathrm{crit}}$ such that $\Gamma(C) = \Gamma_1(n, m)$ (resp. $= \Gamma_2(n_1, n_2)$).*

We don't know whether the curve C is unique (see our question (SC3) on p.233).

The proof of Theorem 8.30 (which in turn implies the previous statement) builds on two constructions of polynomials with special combinatorics, one by Floyd, Kim, Koch, Parry, and Saenz on the realization of PCF combinatorics [**84**], and one by McMullen and DeMarco of dynamical trees [**57**]. Binding together these two results was quite challenging. In arbitrary degree, we have been able to prove only a partial correspondence under simplifying additional assumptions (e.g. the family should have no symmetry).

Some technical details that we have worked out and hopefully clarified! — Beside presenting a set of new results, we have made special efforts to clarify some technical aspects of the standard approach to the unlikely intersection problem for polynomials. We emphasize some of them below.

- We include a self-contained proof (by J. Xie) of his algebraization result for adelic curves (Theorem 1.17).

- We give the full expansion of the Böttcher coordinates for polynomials over a field of characteristic 0 without assuming it to be centered or monic (§2.5).
- We study over an arbitrary field the group of symmetries of a polynomial. In particular, we give a purely algebraic characterization of this group (Theorem 3.18).
- We introduce the notion of primitivity in §3.4, which seems appropriate to exclude tricky examples of entangled pairs.
- We give a detailed proof of the fact that the height $h_{P,a}(t) = h_{P_t}(a(t))$ attached to any polynomial dynamical pair is adelic (Proposition 4.35).
- For a family of polynomials $\{P_t\}$ parameterized by an algebraic variety Λ, we consider the preperiodic locus in $\Lambda \times \mathbb{A}^1$ which is a union of countably many algebraic subvarieties. We study the set of points which are included in an infinite collection of irreducible components of the preperiodic locus (Theorem 2.35). This result is crucial to our specialization argument to obtain Theorem C and clarifies some arguments used in [**100**].

Open questions and perspectives. — There are many directions in which our results could find natural generalizations.

Let us indicate first why the restriction to families of polynomials is crucial for us. Given a family of rational maps R_t parameterized by an algebraic curve C, and given any marked point a, then one can attach to the pair (R, a) a natural height by setting $h_{R,a}(t) = h_{R_t}(a(t))$ and a divisor at infinity $\mathsf{D}_{R,a}$ generalizing the definition (1) above. It is not completely clear, however, whether $\mathsf{D}_{R,a}$ has rational coefficients. Some cases have been worked out by DeMarco and Ghioca [**54**] but the general case remains elusive. It is not completely clear either if this height is a Weil height associated to $\mathsf{D}_{R,a}$ (in the sense of Moriwaki [**131**]). There are instances (see [**60**]) where this height is not adelic, but a recent result by Demarco and Mavraki [**58**] proves $h_{R,a}(t)$ to be a height associated to a continuous semi-positive metrization under quite general assumptions.

Yuan-Zhang [**169**] and the second author [**89**] have however singled out a class of height functions on quasi-projective varieties for which equidistribution of small points holds unconditionnally. It turns out that all height functions of the form $h_{R,a}$ fall into these classes. In particular, the first step of the general strategy developed in the current text can be now adapted in general rational maps.

We also note that Ritt's theory is much less powerful for rational maps leading to weaker classification of curves left invariant by product maps (see [**137**]). We also refer to [**166**] for a characterization of rational maps having the same maximal entropy measure; and to [**130**] for a version of Theorem B for constant families of rational maps (but varying marked point).

It would be extremely interesting to look at polynomial dynamical tuples parameterized by higher dimensional algebraic variety Λ and prove unlikely intersection statements. The obstacles to surmount are also formidable. It is already unclear whether the canonical height is a Weil height on a suitable compactification of Λ. Also in this case, the bifurcation measure is naturally defined as a Monge-Ampère measure of some psh function on Λ, and dealing with a non-linear operator makes things much more intricate. We refer, though, to the papers by Ghioca, Hsia & Tucker [**96**] and by Ghioca, Hsia & Nguyen [**94**] for attempts to handle higher dimensional parameter spaces using one-dimensional slices.

Let us list a couple of questions that are directly connected to our work.

(Q1) Prove the following purely archimedean rigidity statement: two *complex* polynomial dynamical pairs (P, a) and (Q, b) having identical bifurcation measures are necessarily entangled. One of the problem to get such a statement is to prove the multiplicative dependence of the degrees in this context. Observe that Theorem 5.10 yields this dependence but at the cost of a much stronger assumption.

(Q2) Is it possible to remove condition $(\triangle)$ and obtain Theorem C over any field of characteristic 0?

(Q3) Can one extend Theorem E to any field of characteristic 0?

(Q4) Give a classification of special (irreducible) curves in the moduli space of critically marked polynomials in terms of suitable combinatorial data. Ideally, one would like to attach to each special irreducible curve a combinatorial object (like a family of decorated graphs) and prove a one-to-one correspondence between special curves and these objects. It would also be interesting to study the distribution (as currents) of special curves whose associated combinatorics has complexity increasing to infinity.

Further more specific open problems can be found in the three sections §3.7 (related to Ritt's theory), §5.6 (on extensions of Theorem B) and §8.8 (on special curves).

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Notations

- $\bar{K}$ the algebraic closure of a field K
- K° the ring of integers of a non-Archimedean metrized field
- $\tilde{K}$ the residue field of a non-Archimedean metrized field
- $\mathbb{C}_K$ the completion of an algebraic closure of K
- $\mathbb{K}$ a number field
- $\mathsf{O}(z)$ the set of Galois conjugates over an algebraic closure of $\mathbb{K}$ of a point $z \in \mathbb{K}$
- $M_{\mathbb{K}}$ the set of places of a number field, i.e. of norms extending the usual p-adic and real norms on $\mathbb{Q}$
- $\mathbb{K}_v$ the completion of $\mathbb{K}$ w.r.t $v \in M_{\mathbb{K}}$
- $\mathbb{C}_v$ the completion of an algebraic closure of $\mathbb{K}_v$

- X an algebraic variety
- $\mathcal{O}_X$ the structure sheaf of X
- X^{an} the analytification in the sense of Berkovich of X
- X_v^{an} the analytification of $X_{\mathbb{C}_v}$ when X is defined over a number field $\mathbb{K}$ and $v \in M_{\mathbb{K}}$
- C an algebraic curve
- $\bar{C}$ a projective compactification of a curve C such that $\bar{C} \setminus C$ is smooth
- $\hat{C}$ the normalization of $\bar{C}$, and $\mathsf{n}\colon \hat{C} \to \bar{C}$ the normalization map

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- $\log^+ = \max\{0, \log\}$
- $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$ the complex open unit disk
- S^1 the unit circle in $\mathbb{C}$
- $\mathbb{U}$ the group of complex numbers of modulus 1
- $\mathbb{U}_m = \{z \in \mathbb{C}, z^m = 1\}$ the group of complex m-th root of unity
- $\mathbb{U}_\infty$ the group of all roots of unity
- $\mathbb{D}_K(z, r) = \{|w - z| < r\}$ the open disk of center z and radius r (as a subset of either K or the Berkovich affine line $\mathbb{A}_K^{1,\mathrm{an}}$)
- $\mathbb{D}_K^*(z, r) = \{0 < |w - z| < r\}$ the punctured open disk of center z and radius r (as a subset of either K or the Berkovich affine line $\mathbb{A}_K^{1,\mathrm{an}}$)
- $\overline{\mathbb{D}_K(z, r)} = \{|w - z| \leq r\}$ the closed disk of center z and radius r (as a subset of either K or the Berkovich affine line $\mathbb{A}_K^{1,\mathrm{an}}$)
- $\mathbb{D}_K^N(z, r)$ the open polydisk of center z and polyradius $r = (r_1, \cdots, r_N)$ (as a subset of either K^N or the Berkovich affine space $\mathbb{A}_K^{N,\mathrm{an}}$)
- $\overline{\mathbb{D}_K^N(z, r)}$ the closed polydisk of center z and polyradius $r = (r_1, \cdots, r_N)$ (as a subset of either K^N or the Berkovich affine space $\mathbb{A}_K^{N,\mathrm{an}}$)

- $\mathbb{D}_v(z,r), \overline{\mathbb{D}_v(z,r)}, \mathbb{D}_v^N(z,r), \overline{\mathbb{D}_v^N(z,r)}$ the respective open and closed disks in $\mathbb{K}_v$ if v is a place on a number field $\mathbb{K}$
- $\mathbb{M}_K$: the ring of analytic functions on the punctured unit disk $\mathbb{D}_K^*(0,1)$ that are meromorphic at 0

- $\mathcal{C}_c^0(X)$ the space of compactly supported continuous functions on X
- $\mathcal{D}(U)$ the space of smooth (resp. model) functions on U
- Δu the Laplacian of u
- u, g subharmonic functions
- h harmonic functions
- $o(1), O(1)$: Landau notations

- g_P the Green function associated to a polynomial P
- $G(P)$ the critical local height of a polynomial (the maximum of g_P at all critical points)
- φ_P the Böttcher coordinate of a polynomial P at infinity
- $J(P)$ the Julia set of a polynomial P
- $K(P)$ the filled-in Julia set of a polynomial P
- μ_P the equilibrium measure of a polynomial P
- $\mathrm{Crit}(P)$ the critical set of a polynomial P
- $\Sigma(P)$ the group of dynamical symmetries of a polynomial P
- $\mathrm{Aut}(P)$ the group of affine transformations commuting with a polynomial P
- $\mathrm{Aut}(J)$ the group of affine transformations fixing a compact subset J of the complex plane
- $\mathrm{Preper}(P,K)$ the set of preperiodic points *lying in* K of a polynomial $P \in K[z]$

- Poly^d the space of polynomials of degree d
- $\mathrm{Poly}^d_{\mathrm{mc}}$ the space of monic and centered polynomials of degree d
- MPoly^d the space of polynomials of degree d modulo conjugacy
- $\mathrm{MPoly}^d_{\mathrm{crit}}$ the moduli space of critically marked polynomials of degree d modulo conjugacy
- MPair^d the moduli space of dynamical pairs of degree d modulo conjugacy
- $\mathrm{Stab}(P)$ the stability locus of a holomorphic family of polynomials
- T_d the Chebyshev polynomial of degree d
- M_d the monomial map of degree d

- (P,a) a dynamical pair (either holomorphic or algebraic)

- $\mathrm{Bif}(P, a)$ the bifurcation locus of a holomorphic polynomial pair
- $g_{P,a} = g_{P_t}(a(t))$ the Green function associated to a holomorphic polynomial pair
- $\mu_{P,a}$ the bifurcation measure of a dynamical pair defined over a metrized field
- $\mathrm{Preper}(P, a)$ the set of parameters t such that the marked point $a(t)$ is preperiodic for P_t

CHAPTER 1

GEOMETRIC BACKGROUND

We briefly review some material from analytic and arithmetic geometry. This includes the notion of subharmonic functions on analytic curves defined over a non-Archimedean or an Archimedean field; the construction of the Laplace operator on the space of subharmonic functions; a discussion of the notion of semi-positive and adelic metrics on a line bundle over a curve; and the definition of heights attached to an adelic metrized line bundle.

We then give a proof of an algebraization theorem of Xie [**165**] for germs of curves defined by adelic series.

This section will be mainly used as a reference for the rest of the book.

1.1. Analytic geometry

In this book, we shall exclusively work with complex analytic spaces and with Berkovich analytic spaces in the non-Archimedean case as developped in [**20**]. Other theories of analytic spaces over a non-Archimedean fields have been developped, see e.g. [**47**]. But Berkovich spaces enjoy specific topological properties (such as local compactness, or local contractibility) that are essential in order to do analysis (either under the form of harmonic analysis as explained in §1.2, or when exploring dynamics of analytic maps, see §2.2 below).

1.1.1. Analytic varieties

Let $(K, |\cdot|)$ be any complete metrized field. When the norm is non-Archimedean, we let $K^\circ = \{|z| \le 1\}$ be its ring of integers with maximal ideal $K^{\circ\circ} = \{|z| < 1\}$. We write $\tilde{K} = K^\circ/K^{\circ\circ}$ for its residue field, and $|K^*| = \{|z|, z \in K^*\} \subset \mathbb{R}_+^*$ for its value group.

If $X = \operatorname{Spec}(A)$ is an affine K-variety, we denote by X^{an} its Berkovich analytification. As a set, it is given by the Berkovich spectrum of A, i.e. the set of all multiplicative semi-norms on A whose restriction to K is equal to the field norm. We endow it with the topology of the pointwise convergence for which it becomes a locally compact and locally arcwise connected space. The space X^{an} is also endowed with a structural sheaf of analytic functions.

When $K = \mathbb{C}$, we recover the complex analytification of X with its standard euclidean topology and the structural sheaf is the sheaf of complex analytic functions.

When K is non-Archimedean, we refer to [**20**] for a proper definition of the structural sheaf. Any point $x \in X^{\mathrm{an}}$ defines a semi-norm $|\cdot|_x$ on A, and it is customary to write $|f(x)| = |f|_x$ for any $f \in A$. The kernel $\ker(x)$ is a prime ideal of A which may or may not be trivial. The quotient ring $A/\ker(x)$ is a field and one denotes by $\mathcal{H}(x)$ its completion with respect to the norm induced by $|\cdot|_x$. It is the complete residue field of x. When $\mathcal{H}(x)$ is a finite extension of K, then we say that x is a rigid point.

One can naturally extend the analytification functor to the category of algebraic varieties over K using a standard patching procedure. In this context, the GAGA principle remains valid, see [**20**, Chapter 3].

1.1.2. The non-Archimedean affine and projective lines

Suppose $(K, |\cdot|)$ is a complete metrized non-Archimedean field. The analytification of the affine line $\mathbb{A}^1$ is the space of multiplicative semi-norms on $K[T]$ restricting to $|\cdot|$ on K. It is topologically an $\mathbb{R}$-tree in the sense that it is uniquely pathwise connected, see [**111**, §2] for precise definitions. In particular for any pair of points $x, y \in \mathbb{A}^{1,\mathrm{an}}_K$ there is a well-defined segment $[x, y]$.

We define closed balls as usual $\bar{B}(z_0, r) = \{z \in K, |z - z_0| \leq r\}$. To any closed ball is attached a point $x_B \in \mathbb{A}^{1,\mathrm{an}}_K$ defined by the relation $|P(x_B)| = \sup_B |P|$ for any $P \in K[T]$. The Gauß point x_g is the point associated to the closed unit ball $x_g = x_{\bar{B}(0,1)}$. It induces the Gauß norm on the ring of polynomials: $|P(x_g)| = \max\{|a_i|\}$ with $P(T) = \sum a_i T^i$.

When K is algebraically closed, points in $\mathbb{A}^{1,\mathrm{an}}_K$ fall into one of the following four categories. If the kernel of x is non-trivial, then x is rigid and $x = x_{B(z,0)}$ for some $z \in K$. We also say that x is of type 1. When $x = x_{B(z,r)}$ with $r \in |K^*|$ (resp. $r \notin |K^*|$), we say that x is of type 2 (resp. 3). If x is not of one of the preceding types, then it is of type 4: one can then show that there exists a decreasing sequence of balls B_n with empty intersection such that $|P(x)| = \lim_{n\to\infty} |P(x_{B_n})|$ for all $P \in K[z]$, see [**10**, §1.2].

When K is not algebraically closed, and K'/K is any complete field extension, the inclusion $K[Z] \subset K'[Z]$ yields by restriction a canonical surjective and continuous map $\pi_{K/k}\colon \mathbb{A}^{1,\mathrm{an}}_{K'} \to \mathbb{A}^{1,\mathrm{an}}_{K}$, and the Galois group $\mathrm{Gal}(K'/K)$ acts continuously on $\mathbb{A}^{1,\mathrm{an}}_{K'}$.

Let $\mathbb{C}_K$ be the completion of an algebraic closure of K. Then $\mathbb{A}^{1,\mathrm{an}}_{K}$ is homeomorphic to the quotient of $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_K}$ by $\mathrm{Gal}(\mathbb{C}_K/K)$, see [**20**, Corollary 1.3.6]. The group $\mathrm{Gal}(\mathbb{C}_K/K)$ preserves the types of points in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_K}$, so that we may define the type of a point $x \in \mathbb{A}^{1,\mathrm{an}}_{K}$ as the type of any of its preimage by $\pi_{\mathbb{C}_K/K}$ in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_K}$. Note that since the field extension $\mathbb{C}_K/K$ is not algebraic in general, it may happen that some type 1 points in $\mathbb{A}^{1,\mathrm{an}}_{K}$ have trivial kernel hence are not rigid.

Any open subset of the affine line carries a canonical analytic structure in the sense of Berkovich. We shall refrain from defining this notion precisely and discuss only the case of balls and annuli.

The Berkovich analytic open unit ball $\mathbb{D}_K(0,1)$ is defined as the space of semi-norms $|\cdot|_x \in \mathbb{A}^{1,\mathrm{an}}_{K}$ such that $|x| := |T|_x < 1$. Its structure sheaf is the restriction of the analytic sheaf of $\mathbb{A}^{1,\mathrm{an}}_{K}$. Any analytic isomorphism of $\mathbb{D}_K(0,1)$ is given by a power series of the form $\sum_{n\geq 0} a_n T^n$ with $|a_0| < 1$, $|a_1| = 1$, and $|a_n| < 1$ for all $n \geq 2$.

For any $\rho > 1$, the standard open annulus $A = A(\rho)$ of modulus ρ is the analytic subset of $\mathbb{A}^{1,\mathrm{an}}_{K}$ defined by $\{1 < |x| < e^{\rho}\}$. Any analytic isomorphism of A is given by a Laurent series of the form $\sum_{n\in\mathbb{Z}} a_n T^n$ with $|a_1| = 1$ and $|a_1| r > |a_n| r^n$ for all $n \neq 1$ and all $1 < r < e^{\rho}$, possibly composed with the inversion a/T with $e^{\rho} = |a|$[(1)].

The skeleton on A is the set of points $\Sigma(A) = \{x_{B(0,e^t)},\, 0 < t < \rho\}$. Any automorphism of A leaves $\Sigma(A)$ invariant so that the skeleton only depends on the analytic structure of A.

The projective line $\mathbb{P}^{1,\mathrm{an}}_{K}$ is homeomorphic to the one-point compactification of $\mathbb{A}^{1,\mathrm{an}}_{K}$. The point at infinity in $\mathbb{P}^{1,\mathrm{an}}_{K}$ is rigid/of type 1.

1.1.3. Non-Archimedean Berkovich curves

Let $(K, |\cdot|)$ be any algebraically closed and complete non-Archimedean metrized field. Let C be any smooth connected projective curve defined over K. Berkovich [**20**, Chapter 4] proved that the geometry of the Berkovich analytification C^{an} can be completely understood using the semi-stable reduction theorem. Berkovich's results were further expanded by Baker, Payne and Rabinoff in [**7, 8**]. We refer the interested reader to the unpublished monograph of Ducros [**64**] for a detailed account on the geometry of any

(1) The latter automorphism exists only if $e^{\rho} \in |K^*|$.

analytic Berkovich curve (not necessarily algebraic). Our presentation follows [**7**].

Models. — A model of C is a normal K°-scheme $\mathfrak{C}$ that is projective and flat over $\mathrm{Spec}(K^\circ)$, together with an isomorphism of its generic fiber with C. Denote by $\mathfrak{C}_s$ its special fiber: it is a proper scheme defined over the residue field $\tilde{K}$. The valuative criterion of properness implies the existence of a reduction map $\mathrm{red}\colon C^{\mathrm{an}} \to \mathfrak{C}_s$ which is anti-continuous, see [**7**, 1.3].

By a theorem of Berkovich [**20**, Proposition 2.4.4], the preimage by red of the generic point η_E of any irreducible component E of $\mathfrak{C}_s$ is a single point in C^{an}, which we denote by x_E. Such a point is called of type 2. This terminology is compatible with the case $C = \mathbb{P}^1$. Indeed any point of the form $\mathrm{red}^{-1}(\eta_E) \in \mathbb{P}^{1,\mathrm{an}}_K$ is of type 2. Conversely, for any type 2 point $x \in \mathbb{P}^{1,\mathrm{an}}_K$, there exists a model $\mathfrak{P}$ of $\mathbb{P}^1$ and a component E of $\mathfrak{P}_s$ such that $x = x_E$.

If $\bar{x}$ is a closed point in $\mathfrak{C}_s$, then $\mathrm{red}^{-1}(\bar{x})$ is an open subset of C^{an} whose boundary is finite and consists of those points x_E where E is an irreducible component of the central fiber containing $\bar{x}$, see [**20**, Theorem 4.3.1].

A model with simple normal singularities (or simply an snc model) is a smooth model for which the special fiber is a curve with only ordinary double point singularities, i.e. $\mathfrak{C}$ admits a covering formally isomorphic to $\mathrm{Spf}(K^\circ\langle x, y\rangle/(xy - a))$ for some $a \in K^\circ - \{0\}$ near any of its closed point, see [**7**, Proposition 4.3]. A fundamental theorem of Bosch and Lütkebohmert [**29**, Propositions 3.2 & 3.3] implies the following result.

Theorem 1.1. —
1. *If $\bar{x}$ is a smooth point of $\mathfrak{C}_s$ lying in a component E, then* $\mathrm{red}^{-1}(\bar{x})$ *is analytically isomorphic to the (Berkovich) open unit ball and its boundary in C^{an} is equal to x_E.*
2. *If $\bar{x}$ is an ordinary double singularity of $\mathfrak{C}_s$ and belongs to the two components E and E', then* $\mathrm{red}^{-1}(\bar{x})$ *is analytically isomorphic to a (Berkovich) open annulus of the form $\{1 < |z| < r\}$ for some $r \in |K^*|$, and its boundary is equal to $\{x_E, x_{E'}\}$.*

Skeleta. — The skeleton $\Sigma(\mathfrak{C})$ of an snc model is the union of all points x_E for all components E of the special fiber, together with the union of all skeleta of the annuli $\mathrm{red}^{-1}(\bar{x})$ for all singular points of $\mathfrak{C}_s$. The skeleton contains no rigid points.

Since $\mathfrak{C}_s$ is a curve with only ordinary singularities, we may define its dual graph $\Delta(\mathfrak{C})$ whose vertices (resp. of edges) are in bijection with the irreducible components of $\mathfrak{C}_s$ (resp. with the singular points of $\mathfrak{C}_s$). The skeleton $\Sigma(\mathfrak{C})$ is a geometric realization of the graph $\Delta(\mathfrak{C})$ in C^{an}.

There a canonical continuous map $\tau_{\mathfrak{C}}\colon C^{\mathrm{an}} \to \Sigma(\mathfrak{C})$, [**7**, Definition 3.7]. For any irreducible component E of $\mathfrak{C}_s$, $\tau_{\mathfrak{C}}(x_E)$ is equal to the vertex $[E]$ associated to E. More generally when $\mathrm{red}(x)$ is a smooth point lying in E then $\tau_{\mathfrak{C}}(x) = [E]$. Finally when $\mathrm{red}(x)$ is the intersection of two components, then $\tau_{\mathfrak{C}}(x)$ belongs to the edge joining these two components.

Since open subsets of the affine line are retractible, it follows that $\tau_{\mathfrak{C}}$ is a retraction, and that C^{an} is locally modeled on an $\mathbb{R}$-tree. In particular, for any two points $x \neq y$ there exists a continuous injective map $\gamma\colon [0,1] \to \mathbb{C}^{\mathrm{an}}$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Up to reparameterization, there are only finitely many such maps, and when C^{an} is a tree this map is unique. The latter occurs iff the dual of some (or any) snc model of C is a graph having no loop.

Metrics. — Using the $\mathbb{R}$-tree structure of the affine line, one can endow the complement of its set of rigid point $\mathbb{H}(\mathbb{A}^1) = \mathbb{A}^{1,\mathrm{an}} \setminus \mathbb{A}^1(K)$ with a complete metric $d_{\mathbb{H}}$ as follows.

Suppose first that $x_0, x_1 \in \mathbb{A}^{1,\mathrm{an}}$ are associated to closed balls $x_i = x_{B_i}$ as in the previous section. If B_0 and B_1 are not disjoint, then one is contained into the other say $B_0 \subset B_1$, and one sets $d_{\mathbb{H}}(x_0, x_1) = \log(\mathrm{diam}(B_1)/\,\mathrm{diam}(B_0))$. When the balls are disjoint, we consider the closed ball of smallest radius B containing B_0 and B_1, and set

$$d_{\mathbb{H}}(x_0, x_1) = \log(\mathrm{diam}(B)/\,\mathrm{diam}(B_0)) + \log(\mathrm{diam}(B)/\,\mathrm{diam}(B_1)).$$

It is a fact that this distance extends to $\mathbb{H}(\mathbb{P}^1) = \mathbb{H}(\mathbb{A}^1)$ as a complete metric [**10**, §2.7].

Lemma 1.2. — *Any injective analytic map from the open unit ball or from an annulus to the affine line induces an isometry for $d_{\mathbb{H}}$.*

Sketch of proof. — Let us treat the case of the open unit ball and take any injective analytic map $f\colon \mathbb{D}_K(0,1) \to \mathbb{A}^{1,\mathrm{an}}_K$. The image of f is an open ball, since it has at most one boundary point. Since any affine automorphism is an isometry for $d_{\mathbb{H}}$, we may suppose that $f(\mathbb{D}_K(0,1)) = \mathbb{D}_K(0,1)$. From the explicit description of the automorphism groups of the ball given in §1.1.2, we get that $f(T) = \sum_n a_n T^n$ with $|a_0| < |a_1| = 1$ and $|a_n| < 1$ for all $n \geq 2$. The lemma then follows from the following estimation of the diameter of a ball:

$$\mathrm{diam}(f(B(0,r))) = \max_{n\geq 1} |a_n| r^n\ .$$

The case of the annulus is treated analogously. □

Observe that the metric spaces $(\mathbb{D}_K(0,1)\cap\mathbb{H}(\mathbb{A}^1), d_{\mathbb{H}})$ and $(A(\rho)\cap\mathbb{H}(\mathbb{A}^1), d_{\mathbb{H}})$ are not complete. Their completions are respectively equal to $(\mathbb{D}_K(0,1) \cup \{x_g\}, d_{\mathbb{H}})$ and $(A(\rho) \cup \{x_g, x_{\bar{B}(0,e^\rho)}\}, d_{\mathbb{H}})$.

Remark 1.3. — The preceding observation combined with Lemma 1.2 show that $\mathbb{D}_K(0,1)\cap\mathbb{H}(\mathbb{A}^1)\cup\{x_g\}$ and $(A(\rho)\cap\mathbb{H}(\mathbb{A}^1)\cup\{x_g, x_{\bar{B}(0,e^\rho)}\})$ are canonically endowed with a complete metric that we shall again denote by $d_{\mathbb{H}}$.

Let now C be any connected projective smooth curve, and denote by $\mathbb{H}(C) = C^{\mathrm{an}} \setminus C(K)$ the complement of the set of rigid points.

Proposition-Definition 1.4. — *There exists a unique complete metric $d_{\mathbb{H},C}$ on $\mathbb{H}(C)$ such that the following holds. There exists $\epsilon > 0$ such that any analytic embedding $f\colon U \to C$ where U is either the open unit ball or an annulus of modulus $\leq \epsilon$ induces an isometry from $(U, d_{\mathbb{H}})$ onto its image $(f(U), d_{\mathbb{H},C})$.*

When the context is clear we shall drop the index C and simply write $d_{\mathbb{H}}$ for the metric on $\mathbb{H}(C)$.

Sketch of proof. — Choose an snc model $\mathfrak{C}$. We shall build a metric $d_{\mathfrak{C}}$ on $\mathbb{H}(C)$ and latter show that this metric does not depend on the choice of the model.

By Bosch and Lütkebohmert's theorem (Theorem 1.1) the Berkovich curve C^{an} is the disjoint union of finitely many type 2 points (those of the form x_E for some irreducible component E of the special fiber), of finitely many annuli A_i (the preimages under the reduction map of the singular points of $\mathfrak{C}_s$), and (possibly infinitely many) disjoints open balls B_j.

Observe that the closures $\bar{A}_i, \bar{B}_j$ forms a compact covering of C^{an} and that the intersection of any two of these pieces is either empty or equal to a point x_E for some E as before. By Remark 1.3, we may endow each piece $\bar{A}_i\cap\mathbb{H}(C)$, $\bar{B}_j \cap \mathbb{H}$ with a canonical metric $d_{\mathbb{H}}$ for which they become an $\mathbb{R}$-tree. We then extend the metric to $\mathbb{H}(C)$ by setting for any two points $x, y \in \mathbb{H}(C)$

$$d_{\mathfrak{C}}(x,y) = \inf\{\mathrm{Length}(\gamma)\}$$

where γ ranges over all injective continuous maps $\gamma\colon [o,t] \to \mathbb{H}(C)$ such that $\gamma(o) = x$ and $\gamma(t) = y$. Since each piece of the covering is an $\mathbb{R}$-tree, one can take the infimum over paths which are parameterized by length, hence this infimum is taken over finitely many maps and is thus attained. The metric $d_{\mathfrak{C}}$ is complete since each piece is.

Let us now verify that $d_{\mathfrak{C}}$ satisfies the expected property that any embedding of a ball or an annulus is isometric. Note that this will prove that distance does actually not depend on the choice of the model. Take any embedding of the open unit ball $f\colon \mathbb{D}_K(0,1) \to C$. For any $s,t \in \mathbb{D}_K(0,1))$ note that any path γ minimizing $d_{\mathfrak{C}}(f(s),f(t))$ is actually included in $f(\mathbb{D}_K(0,1))$ since the latter has only one boundary point. And it follows from Lemma 1.2 that $d_{\mathfrak{C}}(f(s),f(t)) = d_{\mathbb{H}}(s,t)$ as required.

Let ϵ be half of the minimum of the length of loops included in the skeleton $\Delta(\mathfrak{C})$. If $f\colon A \to C$ is an injective analytic map, and A is an annulus of modulus $\leq \epsilon$, then again any path γ minimizing $d_{\mathfrak{C}}(f(s),f(t))$ has to be included in A. We conclude the proof as before. □

Remark 1.5. — Another construction of this metric is given in [**7**, §5.3] for details. We refer also to [**64**] for a construction of the metric in the case of an arbitrary analytic curve.

The metric topology $(\mathbb{H}(C), d_{\mathbb{H}})$ is stronger than the restriction of the compact topology C^{an} to $\mathbb{H}(C)$. However for any snc model $\mathfrak{C}$, the restriction of both topologies to $\Delta(\mathfrak{C})$ coincide.

1.2. Potential theory

1.2.1. Pluripotential theory on complex manifolds

We shall mainly use potential theory on Riemann surfaces. However we need to pass to higher dimensional complex varieties at a few places (most notably to prove Theorem 4.7 in Chapter 4). We also review basic properties of plurisubharmonic (psh) functions, see [**107**] for a general reference.

Let C be any Riemann surface. We let Δ be the usual Laplace operator defined on $\mathcal{C}^2$ functions in any holomorphic chart by $\Delta\varphi = \frac{i}{\pi}\partial_z\partial_{\bar z}\varphi(z,\bar z)$. This operator extends to any L^1_{loc} function in which case $\Delta\varphi$ is merely a distribution on C. Harmonic functions are those $\mathcal{C}^2$ functions $h\colon C \to \mathbb{R}$ such that $\Delta h = 0$. A subharmonic function $u\colon C \to \mathbb{R}\cup\{-\infty\}$ is a upper semicontinuous function such that for any harmonic function h on a holomorphic disk $D \subset C$ such that $u \leq h$ on ∂D then $u \leq h$ on D. For any subharmonic function, Δu is a positive measure. Conversely, any L^1_{loc} function whose Laplacian is a positive measure is equal a.e. to a (unique) subharmonic function, see [**106**, §1.6].

For any holomorphic function $f\colon C \to \mathbb{C}$, the function $\log|f|$ is subharmonic. Subharmonic functions are stable by sum, multiplication by a positive constant, by taking maximum. Any decreasing limit of a sequence of subharmonic functions remains subharmonic (or is identically $-\infty$).

Let M be any connected complex manifold. To simplify the discussion we shall assume that the dimension of M is 2 (it covers all our needs for this book). A function $u\colon M \to \mathbb{R} \cup \{-\infty\}$ is said to be psh if it is upper semicontinuous and for any holomorphic map $\imath\colon \mathbb{D} \to M$ the function $u \circ \imath$ is subharmonic. Again for any holomorphic function $f\colon M \to \mathbb{C}$, the function $\log|f|$ is psh, and psh functions are stable by sum, multiplication by a positive constant, by taking maximum, by composition by holomorphic functions and by decreasing limits.

In a holomorphic chart (z_1, z_2), define the operator

$$dd^c\varphi = \sum_{\alpha,\beta} \frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_\beta} \frac{i}{\pi} dz_\alpha \wedge d\bar{z}_\beta$$

on $\mathcal{C}^2$ functions $\varphi\colon M \to \mathbb{R}$. The form $dd^c\varphi$ is then a real closed form of type $(1,1)$. We may extend this operator to any L^1_{loc} functions in which case $dd^c\varphi$ becomes a current of bidegree $(1,1)$, i.e. a linear form on the space of $(1,1)$ forms with compact support. For any smooth form ω, we thus have a pairing $\langle dd^c\varphi, \omega\rangle \in \mathbb{C}$.

Recall that a smooth real $(1,1)$ form $\omega = \sum_{\alpha,\beta} \omega_{ij} \frac{i}{2} dz_\alpha \wedge d\bar{z}_\beta$ is positive if the hermitian matrix $(\omega_{\alpha,\beta})_{\alpha,\beta}$ is positive at any point. When u is psh, then $dd^c u$ is a positive current in the sense that for any smooth positive real $(1,1)$ form ω, we have $\langle dd^c u, \omega\rangle \geq 0$. Conversely, any L^1_{loc} function whose dd^c is a positive current is equal a.e. to a subharmonic function, see [**107**, Theorem 4.3.5.2]. In a local chart $dd^c u$ can be expressed as $dd^c u = \sum_{\alpha,\beta} T_{\alpha,\beta} \frac{i}{2} dz_\alpha \wedge d\bar{z}_\beta$ where $T_{\alpha,\beta}$ are signed Borel measures such that $\frac{i}{2} \sum_{\alpha,\beta} T_{\alpha,\beta} \lambda_i \bar{\lambda}_j \geq 0$ for any choice of complex numbers (λ_1, λ_2).

If N is a closed analytic subvariety in M of pure dimension 1 (e.g. when N is a closed Riemann surface in M), then we can define the following current of integration

$$\langle [N], \omega\rangle = \int_{\mathrm{Reg}(N)} \omega$$

for any smooth $(1,1)$-form ω. Here $\mathrm{Reg}(N)$ denotes the set of smooth points of N, and it follows from a theorem of Lelong that $[N]$ is a well-defined closed positive $(1,1)$-current. When N is defined by the vanishing of some holomorphic function $f\colon M \to \mathbb{C}$ then the Poincaré-Lelong formula states that $[N] = dd^c \log|f|$.

Let now u be any psh function such that $u|_N$ is not identically $-\infty$ on any of the irreducible component of N. Then we may define a positive measure supported on N by setting

$$dd^c u \wedge [N] := \sum_i dd^c(u|_{\mathrm{Reg}(N_i)})$$

where N_i denotes the irreducible components of N.

It is a very delicate issue to prove the convergence $dd^c u_n \wedge [N] \to dd^c u \wedge [N]$ when u_n is a sequence of psh functions converging in L^1_{loc} to u. This is for instance true when u_n, u are continuous and the convergence is uniform.

1.2.2. Potential theory on Berkovich analytic curves

Let $(K, |\cdot|)$ be any algebraically closed complete non-Archimedean metrized field, and let C be any smooth connected projective curve defined over K. For any open subset $U \subset C$, we define the notions of harmonic and subharmonic functions on U and explain how to construct a natural Laplace operator from the latter space of functions to the space of positive measures. Potential theory on arbitrary Berkovich curves was fully developed in Thuillier's PhD [**160**], and we refer to this monograph and to [**10**] in the case of the projective line for more details.

Model functions. — Let $\mathfrak{C}$ be any snc model. Recall that the skeleton $\Sigma(\mathfrak{C})$ is a finite graph included in $\mathbb{H}(C) \subset C^{\mathrm{an}}$, and can thus be endowed with a natural distance $d_{\mathbb{H}}$. Any segment of the skeleton comes with a unique (up to translation and change of direction) parameterization by a segment of the real line. We may thus define the space $\mathrm{PL}(\mathfrak{C})$ of piecewise affine functions on $\Sigma(\mathfrak{C})$ as the space of continuous real-valued functions $h\colon \Sigma(\mathfrak{C}) \to \mathbb{R}$ whose restriction to any segment is affine. Any piecewise affine function is thus determined by its values on the set of vertices of $\Sigma(\mathfrak{C})$, hence $\mathrm{PL}(\mathfrak{C})$ is isomorphic to $\mathbb{R}^N$ where N is the number of irreducible components of the special fiber. Define a model function $\varphi\colon C^{\mathrm{an}} \to \mathbb{R}$ as a function of the form $\varphi = h \circ \tau_{\mathfrak{C}}$ where $\mathfrak{C}$ is any snc model, and $h \in \mathrm{PL}(\mathfrak{C})$.

If U is an open subset of C^{an}, we say that $\varphi\colon U \to \mathbb{R}$ is a model function when it has compact support in U and its trivial extension to C^{an} is a model function. We denote by $\mathcal{D}(U)$ the space of all model functions: it is an $\mathbb{R}$-algebra which is stable by max. It follows from Stone-Weierstrass theorem that $\mathcal{D}(U)$ is dense in the space of continuous function on U for the topology of the uniform convergence on compact subsets.

Subharmonic functions. — Pick any open subset V of $\Sigma(\mathfrak{C})$. Note that V is a countable union of finite metrized graphs having a finite number of branched and boundary points. We say that a function $h\colon V \to \mathbb{R}$ is subharmonic when it is convex and continuous, and for any branched and end point $v \in V$, we have

$$\sum_{\vec{v} \in Tv} D_{\vec{v}} h \geq 0. \tag{4}$$

Some explanations are in order here. If v is any point in V, we let Tv be the set of branches at v: when v is an endpoint, then Tv is reduced to a singleton whereas v is a branched point precisely when Tv has at least three points.

For any $\vec{v} \in Tv$, we may fix an isometric embedding $\phi\colon [0, \epsilon) \to V$ such that $\phi(0) = v$ and $\phi(t)$ belongs to the branch determined by $\vec{v}$ for t small. The isometry condition ensures $d_{\mathbb{H}}(\phi(t), \phi(t')) = |t - t'|$ where $d_{\mathbb{H}}$ is the metric constructed in §1.1.3. In particular, any two parameterizations coincide on a small neighborhood of 0. Our assumption on h to be convex is equivalent to say that $h \circ \phi$ is convex so that we may define the directional derivative

$$D_{\vec{v}} h := \left.\frac{d}{dt}\right|_{t=0^+} h \circ \phi \in \mathbb{R} \cup \{-\infty\}.$$

Observe that (4) actually implies $D_{\vec{v}} h$ to be finite for all $\vec{v}$.

Definition 1.6. — *Let U be any open subset of C^{an}. A function $u\colon U \to \mathbb{R} \cup \{-\infty\}$ is said to be subharmonic when it is upper semi-continuous and for all snc models $\mathfrak{C}$, the function $u|_{\Sigma(\mathfrak{C}) \cap U}$ is subharmonic.*

The set $\mathrm{SH}(U)$ of all subharmonic functions on U satisfies the same properties as its complex analog: it is stable by scaling by positive constants, by taking sums and maxima, by decreasing limits, by composition by analytic functions and by restriction to smaller open subsets. Subharmonic functions satisfy the maximum principle. When f is analytic on U, then $\log |f|$ is subharmonic.

A function h is harmonic when $+h$ and $-h$ are both subharmonic.

The Laplace operator. — For any locally compact topological space X, let $\mathcal{M}(X)$ be the set of positive Radon measures on X, that is positive linear functional on the space $\mathcal{C}^0_c(X)$ of continuous functions with compact support on X. We now define a linear operator $\Delta\colon \mathrm{SH}(U) \to \mathcal{M}(U)$.

To that end we first define for any $\varphi \in \mathcal{D}(U)$

$$\Delta\varphi := \sum_v \left(\sum_{\vec{v} \in Tv} D_{\vec{v}} \varphi \right) \delta_v.$$

If $\varphi = h \circ \tau_{\mathfrak{C}}$ and $h \in \mathrm{PL}(\mathfrak{C})$ as above, then Δg is a signed atomic measure supported on type 2 points associated to the irreducible components of the special fiber of $\mathfrak{C}$.

Proposition-Definition 1.7. — *Let U be any open subset of C^{an} and pick any function $u \in \mathrm{SH}(U)$.*

Then there exists a unique positive Radon measure Δu such that for any $\varphi \in \mathcal{D}(U)$ one has

$$\int_U \varphi\, \Delta u = \int_U u\, \Delta\varphi. \tag{5}$$

Sketch of proof. — We refer to [**160**] for a careful construction of Δu. Note that the uniqueness immediately follows from the density of model functions in $\mathcal{C}^0_c(U)$. Here is one way to proceed for the construction of Δu.

We first suppose that U is an open subset of C^{an} which has finitely many boundary points of type 2 and is an $\mathbb{R}$-tree. Pick any $x_0 \in \partial U$. Define the Gromov product $\langle x, y\rangle_{x_0} \in \mathbb{R}_+$ as the distance for the metric $d_{\mathbb{H}}$ between the segment $[x, y]$ and x_0. For any positive measure ρ on U, set $g_\rho(x) = \int \langle x, y\rangle_{x_0}\, d\rho(y)$. One can then show using [**79**, Theorem 7.50] that the map $\rho \mapsto g_\rho$ is a bijection between the set of positive measures of finite mass on $\bar U$ and the set of subharmonic functions on U which extends continuously to $\bar U$ and have value 0 at x_0. Denote by $\mathrm{SH}(\bar U)_0$ this space and by $\mathrm{SH}(\bar U)$ the set of sums of a function in $\mathrm{SH}(\bar U)_0$ and a constant. For any $u \in \mathrm{SH}(\bar U)$, we define Δu to be the unique measure such that $g_{\Delta u} - u$ is a constant. One can then check that this measure satisfies (5). In particular, the operator we have constructed is local in the sense if U and V are two open sets as above, and if we pick $u \in \mathrm{SH}(\bar U)$ and $v \in \mathrm{SH}(\bar V)$ such that $u = v$ on $U \cap V$ then $\Delta u = \Delta v$ on $U \cap V$.

Now pick any open set U. We cover it by a countable family U_i of open sets satisfying the above condition, and observe that $u|_{U_i} \in \mathrm{SH}(\bar U_i)$ for all i so that one may define

$$(\Delta u)|_{U_i} := \Delta \left(u|_{U_i}\right).$$

By the previous discussion, this measure is well-defined and satisfies (5). □

Let us list a couple of properties of Δu without proof.

Proposition 1.8. — *Let u be any subharmonic function on an open subset U of C^{an}.*

1. *For any connected open and relatively compact subset* $V \subset U$ *such that* ∂V *is finite, we have*
$$\Delta u(V) = \sum_{v \in \partial V} D_{\vec{v}} u$$
where $\vec{v}$ *denotes the unique direction at* v *pointing towards* V*.*
2. *Let* $\mathfrak{C}$ *be any snc model, and suppose that* $u = h \circ \tau_{\mathfrak{C}}$ *for some convex and continuous function* $h \colon \Sigma(\mathfrak{C}) \cap U \to \mathbb{R}$*. Then* Δu *is supported on the graph* $\Sigma(\mathfrak{C}) \cap U$*, and*
$$\Delta u = \sum_{v \in E} \left(\sum_{\vec{v} \in Tv} D_{\vec{v}} h \right) \delta_v + \sum_j (\phi_j)_* \frac{d^2(h \circ \phi_j(t))}{dt^2}$$
where E *denotes the set of end and branched points of* $\Sigma(\mathfrak{C}) \cap U$*, and* $\phi_j \colon I_j \to$ *is a collection of isometries with* $I_j \subset \mathbb{R}$ *such that* $\phi_j(I_j) \cap \phi_i(I_i) = \varnothing$ *for all* $i \neq j$ *and* $\bigcup_j \phi_j(I_j) = \Sigma(\mathfrak{C}) \setminus E$*.*

Note that since h is convex, the function $h \circ \phi_j$ is also convex so that $\frac{d^2(h \circ \phi_j(t))}{dt^2}$ is a well-defined positive measure.

The Laplace operator defined above is natural in the sense that it satisfies the Poincaré-Lelong formula
$$\Delta \log |f| = \sum_{f(p)=0} \mathrm{ord}_p(f)\, \delta_p$$
for any analytic function f on U.

It is also continuous in the following sense. If $u_n, u \in \mathrm{SH}(U)$ are subharmonic functions such that $u_n|_{\Sigma(\mathfrak{C})} \to u|_{\Sigma(\mathfrak{C})}$ for any snc model $\mathfrak{C}$, then we have $\Delta u_n \to \Delta g$. Finally if $h \in \mathrm{SH}(U)$, then h is harmonic iff $\Delta h = 0$.

Pull-back of measures. — Let $f \colon C \to C'$ be any regular surjective map between smooth projective irreducible curves C and C' defined over a complete metrized field K (which may be Archimedean or non-Archimedean). Then f is a finite map, and for any $x \in C^{\mathrm{an}}$ the local ring $\mathcal{O}_x$ with maximal ideal $\mathfrak{m}_x$ is a module of finite type over $\mathcal{O}_{f(x)}$. We may thus define the local degree of f at any point x by setting
$$\deg_f(x) = \dim_{\kappa(x)} \left(\mathcal{O}_x / \mathfrak{m}_{f(x)} \cdot \mathcal{O}_x \right) ,$$
where $\kappa(x)$ denotes the residue field $\mathcal{O}_x/\mathfrak{m}_x$.

When x is a type 1 point and K is non-Archimedean, or for any x when K is Archimedean, then one may find local coordinates at x and $f(x)$ so that

f is determined by a power series $\sum_{i\geq 0} a_i z^i$. In this case, one has $\deg_f(x) = \min\{i \geq 1, a_i \neq 0\}$.

For any open subset U of $(C')^{\mathrm{an}}$, one can show that the integer-valued function $y \mapsto \sum_{x\in f^{-1}(y)\cap U} \deg_f(x)$ is constant. When $U = (C')^{\mathrm{an}}$, one usually writes $\deg(f) = \deg_f((C')^{\mathrm{an}})$ and call it the degree of f[(2)]. We refer to [**21**, §6.3] for a more precise discussion of this notion in the case of Berkovich non-Archimedean analytic curves.

For any function $\varphi \colon C \to \mathbb{R}$, set

$$f_*\varphi(y) = \sum_{x\in f^{-1}(y)} \deg_f(x)\, \varphi(x) \ .$$

It is a fact that if φ is continuous, then $f_*\varphi$ is also continuous and $\sup |f_*\varphi| \leq \deg(f) \times \sup |\varphi|$. The proof of this fact is purely local, and the arguments of [**82**, Proposition 2.4] apply verbatim over any complete metrized field.

One can thus define by duality the pull-back of any Radon measure μ on C' as the unique Radon measure such that

$$\int_C \varphi\, d(f^*\mu) = \int_{C'} (f_*\varphi)\, d(\mu) \ .$$

The pull-back measure is positive when μ is, and the total mass of $f^*\mu$ is equal to $\deg(f) \times \mathrm{Mass}(\mu)$.

Finally, if U is any open subset of $(C')^{\mathrm{an}}$ over which $\mu|_U = \Delta u$ for some subharmonic function $u \colon U \to \mathbb{R} \cup \{-\infty\}$, then we have $f^*\mu|_{f^{-1}(U)} = \Delta(u \circ f)$. This identity follows from the Poincaré-Lelong formula when μ is an atomic measure supported at type 1 points, and we get the general case by continuity and by density of these measures in the space of positive Radon measures.

1.2.3. Subharmonic functions on singular curves

We shall also work on arbitrary singular curves. In this context, one can define the notion of subharmonic functions and its Laplacian. We restrict ourselves to the notion of continuous subharmonic functions for which the theory is better behaved, and which will be sufficient for our purposes.

Let C be any complex algebraic curve (possibly with some singularities) defined over a metrized field $(K, |\cdot|)$. Let $\mathrm{Reg}(C)$ be its regular locus, and $\mathsf{n} \colon \hat{C} \to C$ be its normalization. A continuous function $g \colon C \to \mathbb{R}$ is said to be subharmonic whenever its restriction to $\mathrm{Reg}(C)$ is subharmonic. Since

[(2)] When K is algebraically closed and f is separable, it is the number of preimages of a generic closed point in C'.

any bounded subharmonic function on the punctured disk extends through the origin (see e.g. [**77**, Lemma 3.7] and the reference therein), it follows that a continuous function $g\colon C \to \mathbb{R}$ is subharmonic iff $g \circ \mathsf{n}$ is subharmonic.

Let $g\colon C \to \mathbb{R}$ be any continuous subharmonic function. Then Δg is defined as the trivial extension to C of the positive measure $\Delta(g|_{\mathrm{Reg}(C)})$. Since the Laplacian of a bounded subharmonic function does not charge closed points, see e.g. [**81**, Lemme 2.3 & Lemme 4.2], Δg is also equal to $\mathsf{n}_*\Delta(g \circ \mathsf{n})$.

1.3. Line bundles on curves

1.3.1. Metrizations of line bundles

We refer to [**44**] for more details.

Let C be any algebraic curve defined over a complete metrized field $(K, |\cdot|)$. A line bundle $L \to C$ is an invertible sheaf on C. Since C is a curve, one can always find a divisor D such that $L = \mathcal{O}_C(D)$. When C is complete, we define the degree of L as the degree of any of its defining divisor $\deg_C(L) = \deg_C(D)$. To simplify notation, we still denote by L the analytification of the line bundle over C^{an}.

A continuous metrization $\|\cdot\|$ on $L \to C$ is the data for each local analytic section σ defined over an open subset $U \subset C^{\mathrm{an}}$ of a continuous function $\|\sigma\|_U\colon U \to \mathbb{R}_+$ such that:

- $\|\sigma\|_U$ vanishes only at the zeroes of σ;
- the restriction of $\|\sigma\|_U$ to any open subset $V \subset U$ is equal to $\|\sigma\|_V$;
- for any analytic function f on U, one has $\|f\,\sigma\|_U = |f| \times \|\sigma\|_U$.

A local frame on U is a section σ of the line bundle over U which does not vanish. Any local frame induces a local trivialization, and the identity $\|f\,\sigma\|_U = |f| \times \|\sigma\|_U$ implies that one can write the metrization over U under the form $|\cdot|e^{-\varphi}$ for some continuous function φ. In particular, two metrizations $\|\cdot\|_1, \|\cdot\|_2$ of the same line bundle L differ by a multiplicative function $\|\cdot\|_1 = \|\cdot\|_2 e^{-\varphi}$ with $\varphi\colon C \to \mathbb{R}$.

Let $f\colon C' \to C$ be any morphism between two algebraic curves. If $L \to C$ is a line bundle, recall that one may define $f^*L \to C'$ as the line bundle whose local sections over $f^{-1}(U)$ are given by sections of L over U so that for any $\sigma \in H^0(f^{-1}(U), f^*L)$ there exists $\sigma' \in H^0(U, L)$ such that $\sigma = \sigma' \circ f$. We may thus transport any metric $|\cdot|_L$ on L, by imposing $|\sigma|_{f^*L} := |\sigma'|_L \circ f$.

When K is Archimedean and C is smooth, one can make sense of smooth (resp. $\mathcal{C}^k$, Hölder) metrics. In a local chart there are given by $|\cdot|e^{-\varphi}$ with φ smooth (resp. $\mathcal{C}^k$, Hölder).

In the non-Archimedean case, the notion of smooth metrics is not really relevant. Following Zhang [**174**], one defines instead the notion of model metrics. A model of the line bundle $L \to C$ is the choice of a model $\mathfrak{C}$ of C together with a line bundle $\mathfrak{L} \to \mathfrak{C}$ whose restriction to the generic fiber is equal to L. When $L = \mathcal{O}_C(D)$ is determined by a divisor D on C, then $\mathfrak{L}$ is determined by a divisor $\mathfrak{D}$ on $\mathfrak{C}$ whose restriction to the generic fiber is equal to D.

Any model $\mathfrak{L} \to \mathfrak{C}$ gives rise to a metrization of L as follows. Cover $\mathfrak{C}$ by affine charts $\mathfrak{U}_i = \operatorname{Spec}(B_i)$ for some finitely generated K°-algebras B_i. For each i, the space $\bar{U}_i$ of bounded multiplicative semi-norms on $B_i \otimes_{K^\circ} K$ that restrict to $|\cdot|$ on K is compact, and the $\bar{U}_i$'s form a compact cover of C^{an}.

Choose any invertible section σ of $\mathfrak{L}$ on $\mathfrak{U}_i$. Observe that any other local frame of $\mathfrak{L}$ over $\mathfrak{U}_i$ can be written as $\sigma' = \sigma \times h$ with $h \in B_i$ being invertible so that $|h| = 1$ on $\bar{U}_i$. One can thus define a continuous metric on L by imposing $\|\sigma\| = 1$ on $\bar{U}_i$.

When $|\cdot|_1$ and $|\cdot|_2$ are two model metrics of the same line bundle, then $|\cdot|_1 = |\cdot|_2 e^{-\varphi}$ for some model function φ in the sense of §1.2.2.

Model metrics arise in practice by the following token. Let $\mathcal{F} = \{f_1, \cdots, f_N\}$ by any non-empty finite set of non-constant meromorphic functions on C. Let $D_{\mathcal{F}}$ be the effective divisor on C such that

$$\operatorname{ord}_p(D_{\mathcal{F}}) = \max\{0, -\operatorname{ord}_p(f_1), \cdots, -\operatorname{ord}_p(f_N)\}$$

for any $p \in C$. Observe that any f_i induces a section of $L_{\mathcal{F}} := \mathcal{O}_C(D_{\mathcal{F}})$, so that C can be covered by a family of charts U_i such that f_i is invertible on U_i, and $L_{\mathcal{F}}$ is globally generated. Define the function $g_{\mathcal{F}} \colon C^{\mathrm{an}} \to \mathbb{R} \cup \{+\infty\}$ by

$$g_{\mathcal{F}} = \log^+ \max\{|f_1|, \cdots, |f_N|\}.$$

Any section of $L_{\mathcal{F}}$ is determined by a rational function σ on C such that $\operatorname{div}(\sigma) + D_{\mathcal{F}} \geq 0$ so that

$$|\sigma|_{\mathcal{F}} := |\sigma| \times e^{-g_{\mathcal{F}}}$$

defines a continuous metric on $L_{\mathcal{F}}$.

Lemma 1.9. — *Any metric of the form $|\cdot|_{\mathcal{F}}$ is a model metric.*

Proof. — The sections induce a map $\Phi \colon C \setminus \operatorname{supp}(D_{\mathcal{F}}) \to \mathbb{P}^N$ given in homogenous coordinates by $p \mapsto [1 : f_1(p) : \cdots : f_N(p)]$ which extends to a regular map through the punctures, and such that $L_{\mathcal{F}} = \Phi^* \mathcal{O}_{\mathbb{P}^N}(1)$. Recall that sections of $\mathcal{O}_{\mathbb{P}^N}(1)$ are in bijection with linear forms in $(N+1)$ variables $Z_0, \ldots, Z_N$ so that each f_i corresponds to Z_i. It follows that $|\cdot|_{\mathcal{F}}$ is the metrization obtained by pulling-back the standard metrization on $\mathcal{O}_{\mathbb{P}^N}(1)$ given by

$$|\sigma_u| = \frac{u(Z_0, \cdots, Z_N)}{\max\{|Z_0|, \cdots, |Z_N|\}}$$

where σ_u is the section associated to the linear form $u = u(Z_0, \ldots, Z_N)$. The latter metric is a model metric arising from the standard model $\mathbb{P}^N_{K^\circ} = \mathrm{Proj}(K^\circ[Z_0, \cdots, Z_N])$. □

1.3.2. Positive line bundles

We refer again to [**44**] for details.

Let C be any complete algebraic curve defined over a complete metrized field $(K, |\cdot|)$, and let $L \to C$ be any line bundle.

A metrization $(L, \|\cdot\|)$ is said to be semi-positive iff in any local chart the metric can be written under the form $|\cdot|e^{-\varphi}$ with φ subharmonic. Since $\log|f|$ is harmonic for any invertible analytic function f, this notion of positivity is independent on the choice of trivialization.

Let $\mathbb{C}_K$ be the completion of an algebraic closure of K. Observe that the notion of semi-positive (resp. model) metric is stable by base change. One may thus define the curvature form $c_1(L, \|\cdot\|) \in \mathcal{M}(C^{\mathrm{an}}_{\mathbb{C}_K})$ of a semi-positive metrization by setting

$$c_1(L, \|\cdot\|)|_U := \Delta\varphi$$

in any open set of trivialization U where the metric writes $\|\cdot\| = |\cdot|e^{-\varphi}$. The curvature form is a positive measure of total mass $\deg_C(L)$ (the proof of this fact follows from the Poincaré-Lelong formula).

In the Archimedean case, $c_1(L, \|\cdot\|)$ is a smooth measure when the metric is smooth. In the non-Archimedean case, it is an atomic measure supported at type 2 points when $\|\cdot\|$ is a model metric.

Lemma 1.10. — *Any metric of the form $|\cdot|_{\mathcal{F}}$ is semi-positive.*

Proof. — Indeed, any function of the form $\log\max\{|f_0|, \cdots, |f_N|\}$ is subharmonic off its poles, and bounded subharmonic functions on a punctured disk extend through the puncture, see e.g. [**77**, Lemma 3.7]. □

Let $f\colon C' \to C$ be any finite morphism between two algebraic curves. Since subharmonic functions are stable by composition by analytic maps, the metric $|\sigma|_{f^*L}$ is semi-positive as soon as $|\sigma|_L$ is positive, and the curvature forms satisfy $f^*c_1(L, |\cdot|_L) = c_1(f^*L, |\cdot|_{f^*L})$.

Finally in the non-Archimedean case, model metrics are preserved by pull-back since for any model $\mathfrak{C}$ there exists a model $\mathfrak{C}'$ and a regular map $\mathfrak{f}\colon \mathfrak{C}' \to \mathfrak{C}$ which is equal to f on the generic fiber (to build $\mathfrak{C}'$ start with any model $\mathfrak{C}'_0$ of C' and take the graph of the induced rational map $\mathfrak{C}'_0 \dashrightarrow \mathfrak{C}$).

1.4. Adelic metrics, Arakelov heights and equidistribution

A general reference for this section is [**44**].

1.4.1. Number fields

Fix any number field $\mathbb{K}$, and denote by $M_{\mathbb{K}}$ its set of places, that is the set of multiplicative norms on $\mathbb{K}$ whose restriction to $\mathbb{Q}$ is equal to either the standard euclidean norm $|\cdot|_\infty$ or to one of the p-adic norms $|\cdot|_p$ normalized by $|p|_p = \frac{1}{p}$.

Given $v \in M_{\mathbb{K}}$, we write $\mathbb{K}_v$ for the completion of $\mathbb{K}$ w.r.t. $|\cdot|_v$, and we let $\mathbb{C}_v$ be the completion of an algebraic closure of $\mathbb{K}_v$. We also let $\mathbb{Q}_v$ be the completion of the restriction of $|\cdot|_v$ to the prime field. Then for any $x \in \mathbb{K}$, the following product formula holds:

$$\prod_{v \in M_{\mathbb{K}}} |x|_v^{n_v} = 1$$

where $n_v = [\mathbb{K}_v : \mathbb{Q}_v]$.

1.4.2. Adelic metrics

Any line bundle $L \to C$ over an algebraic curve C defined over $\mathbb{K}$ determines a line bundle over the base change of C by $\mathbb{K}_v$. To simplify notation, we let C_v be the Berkovich analytification of C over $\mathbb{K}_v$, and denote by $L_v \to C_v$ the induced line bundle.

Recall from §1.3.1 the definition of $\mathbb{K}^\circ$-model of the line bundle $L \to C$. Observe that any $\mathbb{K}^\circ$-model determines a $\mathbb{K}_v^\circ$-model of the line bundle L_v for any v, hence a continuous metric $|\cdot|_{\mathfrak{L},v}$ over L_v.

A semi-positive adelic metric on an ample line bundle $L \to C$ defined over $\mathbb{K}$ is the data for each place $v \in M_{\mathbb{K}}$ of a semipositive continuous metric $\|\cdot\|_v$ on $L_v \to C_v$ such that there exists a model $\mathfrak{L} \to \mathfrak{C}$ of $L \to C$ over $\mathbb{K}^\circ$ satisfying $|\cdot|_v = |\cdot|_{\mathfrak{L},v}$ for all but finitely many places.

A simple adaptation of the proof of Lemma 1.9 together with Lemma 1.10 yields

Lemma 1.11. — *Any metric of the form* $|\cdot|_{\mathcal{F}}$ *is semi-positive and adelic.*

We simply write $\bar{L}$ to indicate that we have fixed a semi-positive adelic metric on an ample line bundle $L \to C$.

If $\bar{L}$ is a semi-positive adelic metric on $L \to C$, and $f\colon C' \to C$ is a finite map then the pull-back metrized line bundle $f^*\bar{L}$ is also adelic and semi-positive.

1.4.3. Heights

Let $\bar{\mathbb{K}}$ be an algebraic closure of $\mathbb{K}$, and suppose that C is projective. Then a semi-positive adelic metric $\bar{L}$ induces a height function $h_{\bar{L}} : C(\bar{\mathbb{K}}) \to \mathbb{R}$ as follows.

For any point $t \in C(\bar{\mathbb{K}})$, we denote by $\mathsf{O}(t) \subset C(\bar{\mathbb{K}})$ its orbit under the absolute Galois group of $\mathbb{K}$, and write $\deg(t) := \mathrm{Card}(\mathsf{O}(t))$. We then choose any rational section σ of L which has neither a zero nor a pole at t, and we set

$$h_{\bar{L}}(t) := \frac{1}{\deg(t)} \sum_{t' \in \mathsf{O}(t)} \sum_{v \in M_{\mathbb{K}}} -\log|\sigma|_v(t') \ .$$

Since the metrization is adelic, for all but finitely many terms $|\sigma|_v(t') = 1$, hence the sum is well-defined. It follows from the product formula that the definition does not depend on the choice of σ.

Look at the affine space $\mathbb{A}^1 = \mathrm{Spec}(K[x])$ and consider its completion $\mathbb{P}^1$. Endow $\mathcal{O}(1) \to \mathbb{P}^1$ with its canonical metrics given by $|\cdot|\max\{1, |x|\}^{-1}$ at all places. The induced metric is adelic and semi-positive, and the associated height is the standard height on $\mathbb{P}^1$ so that for any $x \in \bar{\mathbb{Q}}$ we have

$$h_{\mathrm{st}}(x) := \frac{1}{\deg(x)} \sum_{y \in \mathsf{O}(x)} \sum_{v \in M_{\mathbb{Q}}} \log^+|y|_v \ . \tag{6}$$

One can alternatively define the height of x by fixing a number field $\mathbb{K} \ni x$ and set

$$h_{\mathrm{st}}(x) := \frac{1}{[\mathbb{K} : \mathbb{Q}]} \sum_{v \in M_{\mathbb{K}}} n_v \log^+|x|_v. \tag{7}$$

Let us return to our general context with $\bar{L}$ a semi-positive adelic metric on a line bundle $L \to C$. The function $h_{\bar{L}}$ lies in the class of functions associated to L and given by Weil's machinery (see [**105**, Theorem B.3.2]). In the sequel, we shall call any such height an Arakelov height. In particular, $h_{\bar{L}}$ is always bounded from below, and satisfies the *Northcott property*: for any integer $d \geq 1$, and for any real number A, the set of points $t \in C(\bar{\mathbb{K}})$ such that $\deg(t) \leq d$ and $h_{\bar{L}}(t) \leq A$ is finite.

The height of the total curve is defined as the following quantity

$$h_{\bar{L}}(\bar{C}) = \sum_{v\in M_{\mathbb{K}}} \sum_{p\in C(\bar{\mathbb{K}})} \operatorname{ord}_p(\sigma_1) \log \|\sigma_0(p)\|_v^{-1} + \int_{\hat{C}} \log \|\sigma_1\|_v^{-1} c_1(L, \|\cdot\|_v)$$

where σ_0 and σ_1 are two sections of L having disjoint sets of zeroes and poles. Again by the product formula this definition does not depend on the choice of sections.

It follows from the arithmetic Hilbert-Samuel theorem the following fundamental estimate [**160**, Théorème 4.3.6], [**2**, Proposition 3.3.3], [**173**, Theorem 5.2], or [**45**, Lemme 5.1].

Theorem 1.12. — *Let $\bar{L}$ be any adelic semi-positive continuous metrization on $L \to C$. Then for any sequence of distinct points $x_n \in C(\bar{\mathbb{K}})$ we have*

$$\liminf_n h_{\bar{L}}(x_n) \geq \frac{h_{\bar{L}}(C)}{2\deg(L)} \ .$$

1.4.4. Equidistribution

In some situation, it is possible to understand the repartition of those points whose height tends to the limit $h_{\bar{L}}(C)$. This is the content of the following result which plays a crucial role in any approach to the dynamical André-Oort conjecture.

If $\bar{L}$ is a semi-positive adelic metrization of L, and v is any place on $\mathbb{K}$, then the line bundle L_v is endowed with a continuous semi-positive metrization $|\cdot|_v$, and we may look at the curvature of $(L_v, |\cdot|_v)$ in the sense of the previous section. It is a positive measure on C_v of total mass $\deg_C(L)$ which we denote by $c_1(\bar{L})_v$ to simplify notations. Observe that if a line bundle carries a semi-positive metric of non-zero curvature then it is automatically ample.

Theorem 1.13 **(Equidistribution of points of small height)**

Let $\bar{L}$ be any semi-positive adelic metrization of a line bundle $L \to C$ over an irreducible projective curve defined over a number field $\mathbb{K}$. Pick any sequence of distinct points $x_n \in C(\bar{\mathbb{K}})$ such that $h_{\bar{L}}(x_n) \to h_{\bar{L}}(C)$. Then, for any place $v \in M_{\mathbb{K}}$, we have

$$\frac{1}{\deg(x_n)} \sum_{y\in \mathsf{O}(x_n)} \delta_y \longrightarrow \frac{1}{\deg_C(L)} c_1(\bar{L})_v$$

on C_v in the weak topology on the space of probability measures.

This theorem originated in the work of Szpiro-Ullmo-Zhang on the Manin-Mumford conjecture [**158**] and was first proved in the case of abelian varieties, see also [**26**] for tori. It was successively extended to the case of semi-abelian varieties [**41**] and to the case of curves by Autissier [**2**], Baker-Rumely [**9**], Favre-Rivera-Letelier [**81**] and the statement above was finally obtained by Thuillier [**160**]. A far-reaching generalization of the previous theorem was later proved by Yuan [**168**] in any dimension, see also [**43, 45**].

More recently, several authors were able to relax the hypothesis on the metrization, when considering the case of quasi-projective varieties: it was first done by Kühne in the case of semi-abelian varieties [**113**] and in the case of families of abelian varieties [**114**]. It was generalized by Yuan-Zhang [**169**] and by the second author [**89**] to a more general setting.

1.5. Adelic series and Xie's algebraization theorem

Let $\mathbb{K}$ be a number field, and S a finite set of places of $\mathbb{K}$ containing all archimedean ones. We denote by $\mathcal{O}_{\mathbb{K},S}$ the ring of S-integers of $\mathbb{K}$, i.e. the set of $x \in \mathbb{K}$ such that $|x|_v \leq 1$ for all $v \notin S$.

Adelic series. — A power series $a(t) = \sum_{j\geq 1} a_j t^j$ is said to be adelic if $a_j \in \mathcal{O}_{\mathbb{K},S}$ for all j, and the radius of convergence ρ_v of $a(t)$ is positive for each place $v \in M_{\mathbb{K}}$. It is sufficient to impose $\rho_v > 0$ for all $v \in S$ since $a(t)$ is analytic in the open unit disk for all $v \notin S$.

The set $\mathcal{O}_{\mathbb{K},S}\{t\}$ of all adelic series is a $\mathcal{O}_{\mathbb{K},S}$-module which is stable by products (hence is a ring), by quotients by an adelic series $a(t)$ satisfying $a(0) \neq 0$, and by composition.

For any adelic series $a(t) = \sum_{j\geq k} a_j t^j$ with $k \geq 1$ and $a_k \neq 0$, there exist two adelic series ρ and θ such that $a(t) = \rho(t)^k$, and $a(\theta(t)) = t^k$, see [**77**, Lemmas 3.2 & 3.3]. In particular, any adelic series with $a_1 \neq 0$ is invertible.

Adelic arcs and branches. — Let X be any projective variety defined over $\mathbb{K}$. Choose any projective model $\mathfrak{X}$ of X over $\mathcal{O}_{\mathbb{K},S}$. An *adelic arc* on X is a $\mathcal{O}_{\mathbb{K},S}\{t\}$-point in $\mathfrak{X}$.

When X is embedded into $\mathbb{P}^N$ and $\mathfrak{X}$ is the closure of X in the standard model of $\mathbb{P}^N$, so that $X = \cap_{i\in I}(P_i = 0)$ for a collection of homogeneous polynomials in $(N+1)$ variables with coefficients in $\mathcal{O}_{\mathbb{K},S}$, then an adelic arc γ on X is determined in homogeneous coordinates by $N+1$ adelic series $x_i \in \mathcal{O}_{\mathbb{K},S}\{t\}$ such that $P_I(x_0(t), \cdots, x_N(t)) \equiv 0$ for all i, and $(x_0(0), \cdots, x_N(0)) \neq$

(0). The point $\gamma(0) = [x_1(0) : \cdots : x_N(0)]$ is the origin of the arc and belongs to $X(\mathbb{K})$.

Pick any place $v \in M_{\mathbb{K}}$. An adelic arc γ is defined by convergent power series, so it induces a natural analytic map from $\gamma_v \colon \mathbb{D}_v(0, R_v(\gamma)) \to X_v^{\mathrm{an}}$ where $R_v(\gamma)$ is the minimum of the radii of convergence of the series $x_i(t)$ determining γ. In particular, we have $R_v(\gamma) > 0$ for all v, and $R_v(\gamma) = 1$ for all but finitely many places. Observe that the series converge only in open disks in general.

Let γ be any adelic arc, and θ be an invertible adelic series such that $\theta(0) = 0$. Then $\gamma \circ \theta$ is an adelic arc, and we say that it is obtained by reparameterizing γ. An adelic branch $\mathfrak{s}$ is an equivalent class of adelic arcs modulo reparameterization. Note that the origin of a branch is well-defined.

Adelic arcs on curves. — Suppose C is an algebraic curve defined over $\mathbb{K}$. Any adelic arc γ on C centered at a point p defines a formal arc $\gamma \in C(\mathbb{K}[[t]])$, and induces a morphism of local rings $\widehat{\mathcal{O}_{C,p}} \to \mathbb{K}[[t]]$.

Lemma 1.14. — *For any smooth algebraic curve C defined over $\mathbb{K}$, and any $p \in C(\mathbb{K})$, there exists an adelic arc γ originated at p inducing an isomorphism of local rings $\widehat{\mathcal{O}_{C,p}} \simeq \mathbb{K}[[t]]$.*

Proof. — An argument is given in [**110**, Lemma 7]. An alternative proof goes as follows. We may pick an immersion of C into $\mathbb{P}^2$ such that the image of p is a smooth point. Locally in affine coordinates (x, y) the curve can be defined by a polynomial of the form $f(x, y) = y + h(x, y)$ where $h = O(x, y)^2$. By the analytic implicit function theorem, one can find a (unique) power series $\theta(t) = \sum_{k \geq 2} \theta_k t^k$ such that $f(t, \theta(t)) = 0$. If $h(x, y) = \sum a_I x^i y^j$, then the coefficients of θ are determined recursively and one sees that θ_k is a polynomial with integral coefficients in the variables $(a_I, \theta_2, \ldots, \theta_{k-1})$. It follows that $\theta \in \mathcal{O}_{\mathbb{K},S}\{t\}$ for any set of places S such that $a_I \in \mathcal{O}_{\mathbb{K},S}$. It is clear that the formal arc $\gamma(t) = (t, \theta(t))$ induces an isomorphism of local rings. □

Remark 1.15. — In fact γ induces an analytic isomorphism from the Berkovich unit open disk $\mathbb{D}_v(0, 1)$ onto its image for all but finitely many places v.

Remark 1.16. — Suppose C is singular, and consider $\mathsf{n} \colon \hat{C} \to C$ its normalization. Given any point $p \in C$, we may apply the preceding lemma to each point in $\mathsf{n}^{-1}(p)$. In this way we obtain adelic arcs parameterizing each branch of C at p.

Algebraization of adelic branches in the affine plane. — We say that an arc in $\mathbb{P}^2$ sits at infinity if its origin lies on the line at infinity $L_\infty = \{[x : y : 0]\} \subset \mathbb{P}^2$. One can always find an affine chart (z, w) centered at the origin of the arc such that the arc is actually given by two adelic series $(z(t), w(t))$ with $z(0) = w(0) = 0$.

Suppose $\mathfrak{s}$ is an adelic branch at infinity whose origin is not the point $[0 : 1 : 0]$. Then $\mathfrak{s}$ is determined by an adelic arc of the form $\gamma(t) = [1 : y(t) : t^k]$ for some $k \geq 1$ and $y \in \mathcal{O}_{\mathbb{K},S}$. The integer k is uniquely determined, since it is the order of vanishing at 0 of $h \circ \gamma$ where h is a local equation of L_∞ at the origin of $\mathfrak{s}$. The adelic series y is not uniquely defined but any other adelic series defining $\mathfrak{s}$ is of the form $y(\zeta t)$ with $\zeta^k = 1$.

We set $R_v(\mathfrak{s}) = R_v(y)$, and define

$$C_v(\mathfrak{s}) := \{(\tau^{-k}, \tau^{-k} y(\tau)),\ |\tau|_v > R_v(\mathfrak{s})\}$$

which is a closed analytic irreducible curve in the open set $\max\{|x|_v, |y|_v\} > R_v(\mathfrak{s})$ of $\mathbb{A}^2_{\mathbb{K}_v}$. When the origin of $\mathfrak{s}$ is the point $[0 : 1 : 0]$, then $\mathfrak{s}$ is determined by an arc $\gamma(t) = [x(t) : 1 : t^k]$ and we define analogously $R_v(\mathfrak{s}) = R_v(x)$, and $C_v(\mathfrak{s}) = \{(\tau^{-k} x(\tau), \tau^{-k}),\ |\tau|_v > R_v(\mathfrak{s})\}$.

***Theorem 1.17* (Xie [165]).** — *Suppose $\mathfrak{s}_1, \ldots, \mathfrak{s}_l$ is a finite set of adelic branches at infinity, and $\{B_v\}_{v \in M_{\mathbb{K},S}}$ is a collection of positive real numbers $B_v \geq 1$ such that $B_v = 1$ for all but finitely many places.*

Let $p_n = (x_n, y_n)$ be an infinite sequence of points in $\mathbb{K}^2$ such that for each place $v \in M_\mathbb{K}$ we have either $p \in \cup_{i=1}^l C_v(\mathfrak{s}_i)$, or $\max\{|x_n|_v, |y_n|_v\} \leq B_v$.

Then there exists an algebraic curve $C \subset \mathbb{A}^2_\mathbb{K}$ such that any branch of C at infinity is contained in the set $\{\mathfrak{s}_1, \ldots \mathfrak{s}_l\}$ and $p_n \in C(\mathbb{K})$ for all n sufficiently large.

Remark 1.18. — Under the assumption of the theorem the genus of any resolution of singularities of the completion of C in $\mathbb{P}^2$ is a most 1 by Faltings' theorem.

Remark 1.19. — This algebraization result bears strong similarities with the Borel-Dwork-Pólya-Carlson-Bertrandias' criterion ensuring the algebraicity of formal power series with coefficients in a number field. Compare with [**38**, §5.2], see also the discussion following [**42**, Théorème 2.6] (We thank Antoine Chambert-Loir for pointing this out to us).

The proof below is due to Junyi Xie.

Proof. — For any branch at infinity $\mathfrak{s}$ determined by an arc γ and for any polynomial $P \in \overline{\mathbb{K}}[x, y]$, define the order of vanishing of P along $\mathfrak{s}$ by

$$v_{\mathfrak{s}}(P) := \mathrm{ord}_t(P \circ \gamma(t)) \in \mathbb{Z} \cup \{+\infty\}$$

with the convention that $v_{\mathfrak{s}}(P) = +\infty$ if $P \circ \gamma$ is identically 0.

Lemma 1.20. — *There exists a polynomial $P \in \bar{\mathbb{K}}[x, y]$ such that $v_{\mathfrak{s}_i}(P) > 0$ for all i.*

For each branch $\mathfrak{s}_i$, we fix an adelic arc $\gamma_i(t) = [1 : y_i(t) : t^{k_i}]$ defining $\mathfrak{s}_i$, and write $\varphi_i(t) := P(t^{-k_i}, t^{-k_i} y_i(t))$.

Proof. — The space of all polynomials of degree $\leq d$ such that $v_{\mathfrak{s}_i}(P) > 0$ is an algebraic subvariety of $\mathbb{A}^{d+1}$ given by the vanishing of all coefficients of non-positive powers in the expansion of φ_i. Note that the number of conditions on P grows linearly with the degree since $\varphi_i(t) = t^{-dk_i} O(1)$. On the other hand, the dimension of the space of all polynomials of degree d is quadratic in d. It follows that for $d \gg 0$, a generic polynomial satisfies $v_{\mathfrak{s}_i}(P) > 0$ for all $\mathfrak{s}_i$. $\square$

Fix any polynomial P as in the previous lemma. Replacing $\mathbb{K}$ by a finite extension, we may suppose that $P \in \mathbb{K}[x, y]$. Observe that $\varphi_i \in t \cdot \mathcal{O}_{\mathbb{K},S}\{t\}$ for all i since $v_{\mathfrak{s}_i}(P) > 0$.

Lemma 1.21. — *There exists a collection of positive real numbers $B'_v \geq 1$ such that $B'_v = 1$ for all but finitely many places, and $|P(p_n)|_v \leq B'_v$ for all n.*

Proof. — Set $C_v = \sup\{|P(x, y)|_v,\ |(x, y)|_v \leq B_v\}$, and

$$D_v = \sup\left\{|P(t^{-k_i}, t^{-k_i} y_i(t))|_v, |t|_v < R_v(\mathfrak{s})\right\}.$$

Note that y_i is an adelic series so that for all but finitely places v its coefficients lie in $\mathbb{K}_v^\circ$. It follows that for all but finitely many places, $R_v(\mathfrak{s}) = 1$ and $D_v = 1$.

The proof is complete by taking $B'_v = \max\{C_v, D_v\}$. $\square$

Since all p_n belongs to the same number field, the previous lemma gives the following height estimate:

$$h_{\mathrm{st}}(P(p_n)) = \frac{1}{[\mathbb{K} : \mathbb{Q}]} \sum_{v \in M_{\mathbb{K}}} \log \max\{1, |P(p_n)|_v\} < \infty$$

so that by Northcott, $P(p_n)$ belongs to a finite set $T \subset \mathbb{K}$, and $\{p_n\} \subset D := \{\prod_{\lambda \in T}(P - \lambda) = 0\}$. Let C be the Zariski closure of $\{p_n\}$ in D.

To conclude the proof, we need to show that any branch at infinity of C lies in $\{\mathfrak{s}_1, \ldots, \mathfrak{s}_l\}$. Suppose by contradiction that $\mathfrak{s}$ is a branch at infinity of C different from the $\mathfrak{s}_i$'s. Let C_0 be the irreducible component of C containing $\mathfrak{s}$.

It may happen that several branches at infinity of C_0 meet at the same point in $\mathbb{P}^2$. To avoid that, we blow-up finitely many points on the line at infinity, and build a smooth projective compactification X of $\mathbb{A}^2$ such that the closure of C_0 intersects transversally the divisor at infinity H of X. Let $\bar{C}_0$ be the closure of C_0 in X.

Choose an effective divisor D supported on $\mathfrak{s}$ such that $L = \mathcal{O}_{\bar{C}_0}(D)$ is globally generated. Pick any finite set of generating sections: we get rational functions in two variables $Q_1, \ldots, Q_N \in \mathbb{K}(x, y)$ whose restrictions to $\bar{C}_0$ have only a pole at $\mathfrak{s}$, and such that $\min_i\{\mathrm{ord}_p(Q_i)\} = 0$ for any $p \neq \mathfrak{s}$ and $\min_i\{\mathrm{ord}_{\mathfrak{s}}(Q_i)\} = \mathrm{ord}_{\mathfrak{s}}(D)$. It follows that the collection of functions $\max\{|Q_0|_v, \ldots, |Q_N|_v\}$ is continuous on C_0 and defines an adelic semi-positive metrization $\bar{L}$ of L.

We now estimate the induced height $h_{\bar{L}}(p_n)$ for all $p_n \in C_0$. The same proof as for Lemma 1.21 applies and we get uniform upper bounds on $\max\{|Q_0|_v, \cdots, |Q_N|_v\}(p_n)$, so that $h_{\bar{L}}(p_n)$ is bounded from above. By Northcott property, the set of p_n's lying in C_0 is finite which is absurd. □

CHAPTER 2

POLYNOMIAL DYNAMICS

This chapter is mainly expository. We first define various moduli spaces of polynomials of interest in §2.1. The following sections §2.2 and §2.3 contain brief discussions of basic aspects of the iteration of complex and non-Archimedean polynomials in one variable (the Fatou-Julia theory and the construction of the canonical invariant measure). We look at a few important examples of polynomial dynamics in §2.4. The fourth section is devoted to the detailed study of the expansion of the Böttcher coordinates. Section 2.6 builds on the previous chapter and discuss the notion of canonical height. In Section 2.7, we review the Mañé-Sad-Sullivan theory of bifurcation of holomorphic dynamical systems in the context of polynomials. We conclude this chapter by a discussion of the locus of preperiodic points in an arbitrary family of polynomials. Our main result (Theorem 2.35) will play a key role in one of our specialization argument in §5.5.

2.1. The parameter space of polynomials

In this section we assume the defining field K has characteristic zero. Recall that a polynomial $P(z) = a_0 z^d + \cdots + a_d$ of degree d is monic (resp. centered) if $a_0 = 1$ (resp. $a_1 = 0$).

Polynomials modulo affine conjugacy. — A polynomial of degree $d \geq 2$ is determined by $(d+1)$ coefficients $P(z) = a_0 z^d + \cdots + a_d$ with a_0 invertible so that the space Poly^d of all polynomials of degree $d \geq 2$ is canonically endowed with a structure of affine variety which is isomorphic to $(\mathbb{A}^1)^* \times \mathbb{A}^d$. The group $\mathrm{Aff} = \{az + b,\, a \neq 0\}$ of affine transformations of the affine line acts by conjugacy on Poly^d by $\phi \cdot P = \phi \circ P \circ \phi^{-1}$.

In characteristic zero, any polynomial is conjugated by a unique translation to a centered polynomial so that over $\mathbb{Q}$ the quotient of Poly^d by Aff is isomorphic to the quotient $(\mathbb{A}^1)^* \times \mathbb{A}^{d-1}$ by the multiplicative group $\mathbb{G}_m$ under the action

$$\lambda \cdot (a_0, a_2, \ldots, a_{d-1}, a_d) = (\lambda^{1-d} a_0, \lambda^{3-d} a_2, \ldots, a_{d-1}, \lambda a_d).$$

Over an algebraically closed field a polynomial is conjugated by a suitable dilatation to a monic polynomial. It follows that the quotient of Poly^d by Aff is isomorphic to the space of monic and centered polynomials (which is isomorphic to $\mathbb{A}^{d-1}$) quotiented by the finite cyclic group $\mathbb{U}_{d-1}$ of $(d-1)$-th root of unity acting diagonally on $\mathbb{A}^{d-1}$ by $\zeta \cdot (a_2, \ldots, a_{d-1}, a_d) = (\zeta^{3-d} a_2, \ldots, a_{d-1}, \zeta a_d)$.

The moduli space of polynomials MPoly^d thus exists a geometric group quotient and is an affine variety over $\mathbb{Q}$. It can in fact be identified with the product of $\mathbb{A}^1$ with an affine open subset of the weighted projective space $\mathbb{P}(1, \ldots, d-1)$. In particular, it is an affine variety of dimension $(d-1)$ which is rational and has only cyclic quotient singularities[(1)].

Example 2.1. — We have the isomorphisms $\mathrm{MPoly}^2 \simeq \mathbb{A}^1$; and $\mathrm{MPoly}^3 \simeq \mathbb{A}^2$. However for any $d \geq 4$, the space MPoly^d admits singularities.

The space MPoly^4 is isomorphic to $\mathbb{A}^3$ modulo the action of $\mathbb{U}_3$ given by $\zeta \cdot (a_2, a_3, a_4) = (\zeta^{-1} a_2, a_3, \zeta a_4)$ which is the product of $\mathbb{A}^1$ by the cone $xy = t^3$. Its singular locus is the image under the quotient map of the set of polynomials of the form $z^4 + a_3 z$.

Remark 2.2. — When the characteristic of the field say $p > 0$ divides the degree d, the discussion above does not apply since a polynomial is no longer conjugated to a centered polynomial. In fact the action of the affine group becomes quite wild. When $p = 2$ the stabilizer of *any* separable quadratic polynomial $a_0 z^2 + a_1 z + a_2$ is equal to the group of translations $z + \beta$ with $a_0 \beta^2 - a_1 \beta = \beta$ which is always non-trivial.

Lemma 2.3. — *Let $(K, |\cdot|)$ be any complete metrized field of characteristic* 0*, and $\{P_t\}_{t \in \mathbb{D}^*_K(0,1)}$ be an analytic family of monic and centered polynomials defined over the punctured disk that is meromorphic at* 0*. Suppose that we can find a meromorphic family of affine transformations A_t such that $A_t^{-1} \circ P_t \circ A_t$ extends analytically through* 0*.*

Then the family P_t is analytic at 0*.*

[(1)] The corresponding statement for the moduli space of rational maps is due to Silverman and Levy, see [**154, 118**].

Proof. — Write $P_t(z) = z^d + a_2(t)z^{d-2} + \cdots + a_d(t)$, and $A(t) = a(t)z + b(t)$, with a, b, a_i analytic on $\mathbb{D}^*_K(0,1)$ and meromorphic at 0. Then $A_t^{-1} \circ P_t \circ A_t(z) = a(t)^{d-1}z^d + da(t)^{d-2}b(t)z^{d-1} + \text{l.o.t}$ so that the result follows. □

Critically marked polynomials. — Even though MPoly^d is the most natural parameter space to consider, it is also important to work with polynomials with additional structures. A critically marked polynomials is a d-tuple $(P, c_0, \ldots, c_{d-2})$ with $P \in \mathrm{Poly}^d$ and where $c_0, \ldots, c_{d-2}$ ranges over all critical points of P (written with repetitions taking into account their multiplicities). The space of critically marked polynomials MPcrit^d is the quotient of this space by the natural action of the group of affine transformations. A brief discussion of the geometry of this space is given in [**68**, §5].

It is convenient to work in a finite ramified cover of MPcrit^d which is isomorphic to the affine space, i.e. with an "orbifold" parametrization of MPcrit^d. For any field K, and any $(c, a) \in K^{d-1}$, we let

$$P_{c,a}(z) := \frac{1}{d}z^d + \sum_{j=2}^{d-1}(-1)^{d-j}\sigma_{d-j}(c)\frac{z^j}{j} + a^d, \tag{8}$$

where $\sigma_j(c)$ is the monic symmetric polynomial in $(c_1, \ldots, c_{d-2})$ of degree j. Observe that the critical points of $P_{c,a}$ are exactly $c_1, \ldots, c_{d-2}$, c_{d-1} with the convention that $c_{d-1} = 0$. One obtains a canonical projection $\pi\colon \mathbb{A}^{d-1} \to \mathrm{MPcrit}^d$ defined over $\mathbb{Q}$ which maps $(c_1, \ldots, c_{d-2}, a) \in \mathbb{A}^{d-1}$ to the class of $P_{c,a}$ in MPcrit^d which is $d(d-1)$-to-one.

Polynomial dynamical pairs. — One can also look at the space Pair^d of polynomial dynamical pairs (P, a) with $P \in \mathrm{Poly}^d$ and $a \in \mathbb{A}^1$ modulo the natural action of Aff given by $\phi\cdot(P, a) = (\phi \circ P \circ \phi^{-1}, \phi(a))$. The structure of the quotient space MPair^d is similar to MPoly^d: it is the product of $\mathbb{A}^1$ by an open affine subspace of $\mathbb{P}(1, 2, \ldots, d-1, d-1)$, and therefore is a d-dimensional affine variety defined over $\mathbb{Q}$. We have a natural submersion $\pi\colon \mathrm{MPair}^d \to \mathrm{MPoly}^d$. In the complex analytic category this map is an orbifold line bundle in the sense of [**146**, §2].

2.2. Fatou-Julia theory

We fix any algebraically closed complete metrized field $(K, |\cdot|)$ of characteristic zero. When the norm $|\cdot|$ is Archimedean, then $K = \mathbb{C}$ and $|\cdot|$ is the standard Euclidean norm. Basic references in the complex case include [**39, 128**]. Over a non-Archimedean field, the use of Berkovich analytic spaces is essential to develop the theory in depth: locally compactness and locally contractibility play an important role in that matter. Our basic reference is [**18**].

The filled-in Julia set. — See [**128**, p.95] (K Archimedean), or [**18**, p.110 & p.194] (K non-Archimedean). Pick any polynomial $P(z) = a_0 z^d + \ldots + a_d$ of degree $d \geq 2$ with coefficients in K. It induces a continuous map $P : \mathbb{A}^{1,\mathrm{an}}_K \to \mathbb{A}^{1,\mathrm{an}}_K$ on the Berkovich analytification of the affine line, and we define the filled-in Julia set as

$$K(P) = \{z \in \mathbb{A}^{1,\mathrm{an}}_K,\ |P^n(z)| \text{ is bounded as } n \to \infty\}\ .$$

Observe that for some $\epsilon > 0$ small enough and some $R > 1$ big enough we have

$$|P(z)| \geq \epsilon |z|^d \geq 2|z|$$

for all $|z| \geq R$. It follows that the basin of attraction of infinity

$$\Omega(P) = \{z \in \mathbb{A}^{1,\mathrm{an}}_K,\ |P^n(z)| \to \infty \text{ as } n \to \infty\}\ ,$$

is an open set whose complement is equal to $K(P)$, and that the latter set is a non-empty compact set. Both sets $K(P)$ and $\Omega(P)$ are totally invariant by P.

The Julia set $J(P)$ of P is defined to be the boundary of $K(P)$: it is also a compact set which is totally invariant. The Fatou $F(P)$ is the complement of the Julia set: it is the disjoint union of $\Omega(P)$ and the interior of $K(P)$. It can be characterized as the open set where the sequence of iterates $\{P^n\}$ are normal (in the usual sense in the Archimedean case, and in the sense of [**80**] in the non-Archimedean case).

Periodic points. — See [**128**, §8 – 11] (K Archimedean). In the non-Archimedean case, the discussion in [**18**, p.71] contains many references but no proof.

Let $p \in \mathbb{A}^1(K)$ be a fixed point for P. Then p is repelling (resp. neutral, attracting, or super-attracting) when $|P'(p)| > 1$ (resp. $|P'(p)| = 1$, $|P'(p)| < 1$, or $|P'(p)| = 0$).

When p is repelling or attracting, it is always possible to find an analytic change of coordinates ϕ locally at p such that $\phi^{-1} \circ P \circ \phi(z) = \lambda z$ with $\lambda = P'(p)$. When p is super-attracting[(2)] then we can find ϕ such that $\phi^{-1} \circ P \circ \phi(z) = z^k$ where $k = \mathrm{ord}_p(P)$.

(2) In positive characteristic this result is no longer true, see [**148**].

Observe that the polynomial $P(z) = a_0 z^d + \text{l.o.t}$ also defines a continuous map on $\mathbb{P}^1_K$ for which the point at infinity is totally invariant, and super-attracting. It follows that one can find an analytic function $\varphi_P(z) = az + \sum a_j z^{-j}$ with $a \neq 0$ converging in $|z| \geq R$ for R sufficiently large and such that $\phi_P \circ P = (\phi_P)^d$. This analytic function is uniquely determined by the previous equation and the choice of a such that $a^{d-1} = a_0$, and is called the Böttcher coordinates. We shall discuss extensively the expansion of φ_P in §2.5.

When p is neutral, then the situation is quite delicate.

- When the multiplier $P'(p)$ is a root of unity(3), then one can never linearize P near p (otherwise we would have $P^r = \mathrm{id}$ for some $r > 1$ which contradicts $d \geq 2$). The dynamics is described by the Fatou-Leau theory in the complex case, see [**127**, §10]. In the non-Archimedean case, the dynamics depends on the residue characteristic of K. When the residue characteristic is positive, the dynamics can be investigated using the iterative logarithm, see [**18**, §10.2] or [**145**, §3.2].
- When the multiplier $P'(p)$ is not a root of unity, then P is always linearizable when $(K, |\cdot|)$ is a non-Archimedean metrized field of characteristic zero, see [**145**, §3.3]. The linearizability of P near p in the Archimedean case depends in a subtle way on the continued fraction expansion of the argument of the multiplier. We refer to [**127**, §11] for a thorough discussion of this very intricate problem.

We conclude this section by the following observation.

Lemma 2.4. — *Let $p \in \mathbb{A}^1(K)$ be a fixed point for P.*

- *When $K = \mathbb{C}$ is Archimedean, then p belongs to the Fatou set iff it is attracting either $|P'(p)| < 1$, or neutral $|P'(p)| = 1$ and P is linearizable at p.*
- *When K is non-Archimedean, then p belongs to the Fatou set iff it is non-repelling, i.e. $|P'(p)| \leq 1$.*

Proof. — Assume $K = \mathbb{C}$. One direction is clear. For the converse suppose first that p is repelling. Then the sequence of iterates $\{P^n\}$ cannot be normal at p since its derivative explode. When p is neutral and belongs to the Fatou set, then a simple argument for linearizability goes as follows, see [**127**, Corollary 5.3]. Let $U \subset F(P)$ be the Fatou component of P containing p. By the maximum principle, for all Jordan curve $\gamma \subset U$, the bounded component of $\mathbb{C} \setminus \gamma$ is contained in U, hence U is simply connected. Let $\psi : U \to \mathbb{D}$ be a

(3)In this case p is said to be a parabolic fixed point.

conformal isomorphism with $\psi(p) = 0$, then $g := \psi \circ f \circ \psi^{-1}$ satisfies $g : \mathbb{D} \to \mathbb{D}$, $g(0) = 0$ and $g'(0) = P'(p)$. By Schwarz lemma, this gives $g(z) = P'(p) \cdot z$.

Suppose now that K is non-Archimedean and write $\lambda = P'(p)$. Choose an affine coordinate z such that $p = 0$. When $|\lambda| \leq 1$ we have $|P(z) - \lambda z| \leq C|z|^2$ for all $|z|$ small enough so that $|P(z)| = |\lambda z|$ is a neighborhood of 0. It follows that $|P^n(z)|$ is bounded for all n and p belongs to the Fatou set. Conversely, when p belongs to the Fatou set, then we can find a disk D around 0 such that $P^n(D) \subset B(0, 1)$ for all n. It follows $|(P^n)'(0)| \leq \mathrm{diam}(D)$ for all n, hence p is not repelling. □

Non-rigid periodic points (non-Archimedean case). — See [**18**, §8.2]. Suppose K is a non-Archimedean metrized field. Recall that we wrote $K^\circ := \{z \in K : |z| \leq 1\}$, $K^{\circ\circ} := \{z \in K : |z| < 1\}$, and $\tilde{K} := K^\circ / K^{\circ\circ}$. For any $z \in K^\circ$, we let $\tilde{z} \in \tilde{K}$ be the image of z under the reduction map $K^\circ \to \tilde{K}$.

Given any polynomial $P \in K[T]$ of degree $d \geq 2$, then it may appear that P fixes some points in the Berkovich affine line that are not rigid.

Proposition 2.5. — *Suppose x is a non-rigid point in $\mathbb{A}^{1,\mathrm{an}}_K$ which is fixed by P.*

1. *If $x \in J(P)$, then there exists a finite extension L/K and an affine map ϕ defined over L such that $\phi(x)$ is the Gauß point, and the reduction of $\phi \circ P \circ \phi^{-1}$ is a polynomial with coefficients in $\tilde{K}$ of degree at least 2.*
2. *If x lies in the Fatou set, there exists a neighborhood of x on which P induces an analytic isomorphism.*

Definition 2.6. — *Any point satisfying Condition (1) of the previous proposition is called a non-rigid repelling fixed point.*

Proof. — Suppose that K is algebraically closed. If x is of type 2, then there exists an affine map ϕ defined over K such that $\phi(x)$ is the Gauß point, in which case the polynomial P can be decomposed as the sum of two polynomials: $Q_1 \in \mathfrak{m}[T]$ and Q_2 having coefficients in $K^\circ \setminus \mathfrak{m}$. When $\deg(Q_2) \geq 2$, then we are in case (1) of the Proposition. When $\deg(Q_2) = 1$, then we can find $r > 1$ such that P induces an analytic isomorphism on the disk centered at 0 of radius r, and we fall into case (2).

When x is of type 3 or of type 4, we are always in the second case, see [**10**, Lemma 10.80 & Theorem 10.81] or [**18**]. □

Dynamics in the Fatou set. — See [**128**, §15 – 16] (K Archimedean), or [**18**, Chapter 9] (K non-Archimedean). A Fatou component is a connected component of $F(P)$. It is either bounded, or equal to $\Omega(P)$. The image by P of a Fatou component remains a Fatou component.

When $K = \mathbb{C}$, then any Fatou component is pre-periodic by the famous non-wandering theorem of Sullivan, see, e.g., [**127**, Theorem 16.4]. One is thus reduced to consider periodic (and even fixed) Fatou component to understand the dynamics of P on $F(P)$. Any fixed Fatou component U is either the basin of attraction of a fixed attracting or super-atttracting point; or a parabolic domain so that any point in U converges under iteration toward a parabolic fixed point; or a Siegel domain, i.e. a disk on which P is conjugated to an irrational rotation.

When K is non-Archimedean, the situation is more delicate and highly depends on the residual characteristic of K. The classification of periodic components is due to Rivera-Letelier. We refer to [**18**, Theorem 9.14] for the following result. Any fixed Fatou component U is either the basin of attraction of a fixed attracting or super-atttracting orbit; or it is an affinoid domain whose boundary consist of periodic type 2 repelling points, and $P : U \to U$ is an analytic isomorphism. In the latter case, we say that U is an indifferent component.

An analogue of Sullivan's non-wandering theorem was proved by Benedetto [**16**] when the residual characteristic of K is 0. A Fatou component U is either pre-periodic, or it is a ball and its (unique) boundary point x is pre-periodic (and replacing P by an iterate we have $P^n(z) \to P(x)$ for all $z \in U$). When the residual characteristic of K is positive, then this result is no longer true: there exist wandering Fatou components which are disks and whose boundary point has an infinite orbit, see [**17**]. There is no general conjecture explaining the appearance of wandering domains over an arbitrary field.

2.3. Green functions and equilibrium measure

2.3.1. Basic definitions

Let $(K, |\cdot|)$ be any complete and algebraically closed metrized field of characteristic 0. For any polynomial $P(z) = Az^d + a_1 z^{d-1} + \ldots + a_d \in K[z]$ of degree $d \geq 2$, there exists a constant $C \geq 0$ such that

$$\left| \frac{1}{d} \log^+ |P(z)| - \log^+ |z| \right| \leq C$$

Indeed $|P(z)| \leq \max\{|A|, |a_i|\} \max\{1, |z|\}^d$, and for any $\epsilon \ll 1$ small enough such that $A > \epsilon \sum |a_i|$, we have $|P(z)| \geq (A - \epsilon \sum |a_i|)|z|^d$ when $|z| \geq \epsilon^{-1}$.

It follows that the sequence of functions $\frac{1}{d^n}\log^+|P^n|$ converges uniformly on $\mathbb{A}^{1,\mathrm{an}}_K$ to a continuous function g_P.

Definition 2.7. — *The function g_P is called the Green function of P.*

The proof of the next result is left to the reader, see [**10**, §10.8] in the non-Archimedean case (in the complex case, we refer to [**24**] which contains lots of informations on the dynamics of rational maps).

Proposition 2.8. — *The Green function of P satisfies the following properties:*

1. $g_P \circ P = d g_P$ *on* $\mathbb{A}^{1,\mathrm{an}}_K$;
2. $g_P(z) = \log|z| + \frac{1}{d-1}\log|A| + o(1)$ *as* $|z| \to \infty$;
3. *the set* $\{g_P = 0\}$ *is the filled-in Julia set* $K(P)$ *of* P;
4. *the function* g_P *is harmonic outside* $J(P)$;
5. *the set of functions* $\frac{1}{d^n}\log^+|P^n(z)|$ *converges uniformly on* $L \times \mathbb{A}^{1,\mathrm{an}}_K$ *for any compact subset* L *of* $\mathbb{A}^{d+1,\mathrm{an}}_K$ *so that the function* $(P,z) \mapsto g_P(z)$ *is continuous on* $\mathbb{A}^{d+2,\mathrm{an}}_K$.

As g_P is a subharmonic function of on $\mathbb{A}^{1,\mathrm{an}}_K$ with $g_P(z) = \log^+|z| + O(1)$ as $|z| \to \infty$, its Laplacian is a probability measure.

Definition 2.9. — *The equilibrium measure μ_P of P on $\mathbb{A}^{1,\mathrm{an}}_K$ is $\mu_P := \Delta g_P$.*

By the above properties of the Green function, the measure μ_P has the following properties (see [**32**] when K is Archimedean, and [**82**] when K is non-Archimedean):

- the measure μ_P is a probability measure supported on $J(P)$;
- $P^*\mu_P = d \cdot \mu_P$ and $P_*\mu_P = \mu_P$;
- μ_P is ergodic and mixing;
- the entropy of μ_P is at most $\log d$ (with equality when $K = \mathbb{C}$).

One also defines the Lyapunov exponent of P as the quantity

$$\mathrm{Lyap}(P) := \lim_{n\to\infty} \frac{1}{n} \int \log|(P^n)'|\, d\mu_P \ .$$

The Misiurewicz-Przytycki's formula states that

$$\mathrm{Lyap}(P') = \log|d| + \sum_{P'(c)=0} g_P(c) \ . \tag{9}$$

Observe that $|d| = d$ if the norm is Archimedean, and $|d| \le 1$ when it is non-Archimedean so that $\mathrm{Lyap}(P)$ may be negative in the latter case. A proof is given over any metrized field of characteristic zero by Y. Okuyama in [**132**, §5].

Remark 2.10. — When a polynomial P is defined over a number field $\mathbb{K}$, for each place $v \in M_{\mathbb{K}}$, there is a Green function $g_{P,v}$ (and an equilibrium measure $\mu_{P,v}$) for the polynomial P. The function $g_{P,v}$ and the measure $\mu_{P,v}$ depend on the place v (see §2.6).

2.3.2. Estimates on the Green function

Recall that if $(c,a) \in K^{d-1}$, we set

$$P_{c,a}(z) := \frac{1}{d}z^d + \sum_{j=2}^{d-1}(-1)^{d-j}\sigma_{d-j}(c)\frac{z^j}{j} + a^d \ . \tag{10}$$

This parameterization is particulary adapted to estimate the variation of the Green function in terms of the polynomial. We refer to [**76**, §2] for a more detailed exposition.

Proposition 2.11. — *There exist a constant $\theta \geq 0$ and $C \geq 1$ such that the following holds for all $c, a \in K^{d-1}$:*

1. *for all $z \in \mathbb{A}^{1,\mathrm{an}}_K$, we have*

$$g_{P_{c,a}}(z) \leq \log^+ \max\{|z|, |c|, |a|\} + \theta \ ,$$

2. *for all $z \in \mathbb{A}^{1,\mathrm{an}}_K$ with $|z| > C \cdot \max\{1, |c|, |a|\}$, we have*

$$\begin{cases} g_{P_{c,a}}(z) = \log^+|z| - \frac{1}{d-1}\log|d|, & \text{when } K \text{ is non-Archimedean,} \\ g_{P_{c,a}}(z) \geq \log^+|z| - \log 8, & \text{when } K \text{ is Archimedean.} \end{cases}$$

Furthermore, when K is non-Archimedean then C, θ only depend on the residual characteristic, and they equal $C = 1$ and $\theta = 0$ when the residual characteristic of $(K, |\cdot|)$ is 0 or at least $d+1$.

Definition 2.12. — *For any polynomial $P \in K[T]$, we set*

$$G(P) = \max\{g_P(c), P'(c) = 0\} \in \mathbb{R}_+. \tag{11}$$

The next result follows from Proposition 2.11 and the Nullstellensatz.

Proposition 2.13. — *The function $(c,a) \mapsto G(P_{c,a})$ extends continuously to $\mathbb{A}^{d-1,\mathrm{an}}_K$, and there exists a constant $C \geq 0$ such that*

$$\sup_{\mathbb{A}^{d-1,\mathrm{an}}_K} \left|G(P_{c,a}) - \log^+ \max\{|c|, |a|\}\right| \leq C \ .$$

Furthermore, when K is non-Archimedean then C only depends on the residual characteristic, and it equals 0 when the residual characteristic is 0 or $\geq d+1$.

Proof of Proposition 2.11. — Let us argue first in the non-Archimedean case. Let p be the residue characteristic of K. By the strong triangle inequality, there exists $C_p \geq 1$ depending only on p with $C_p = 1$ when $p = 0$ or $p \geq d+1$, such that

$$|P_{c,a}(z)| \leq C_p \max\{1, |z|, |c|, |a|\}^d$$

hence $g_{P_{c,a}}(z) \leq \log^+ \max\{|z|, |c|, |a|\} + \frac{1}{d-1} \log C_p$ by induction. On the other hand, again by the strong triangle inequality, we have

$$|P_{c,a}(z)| = \left| \frac{z^d}{d} \right| \geq |z|^d,$$

when $|z| > C'_p \max\{1, |c|, |a|\}$ where $C'_p = 1$ when $p = 0$ or $p \geq d+1$. Again, an easy induction gives

$$g_{P_{c,a}}(z) = -\frac{1}{d-1} \log |d| + \log^+ |z|,$$

when $|z| > C'_p \max\{1, |c|, |a|\}$.

For the Archimedean case, we refer to [**68**, §6.1] for details. By the triangle inequality and the maximum principle, there exists $A \geq 1$ which depends only on d such that

$$|P_{c,a}(z)| \leq A \cdot \max\{1, |z|, |c|, |a|\}^d$$

and as above, we find $g_{P_{c,a}}(z) \leq \log^+ \max\{|z|, |c|, |a|\} + \frac{1}{d-1} \log A$ by induction. In particular, $G(P_{c,a}) \leq \log^+ \max\{|c|, |a|\} + \frac{1}{d-1} \log A$. For the second inequality, we use [**68**, Lemma 6.5]:

$$\max\{g_{P_{c,a}}(z), G(P_{c,a})\} \geq \log |z - \delta| - \log 4$$

where $\delta = \sum_j c_j/(d-1)$. Let $\tilde{A} := \max\{A^{1/(d-1)}, 2\}$, so that

$$\log |z - \delta| \geq \log |z| - \log 2$$

if $|z| \geq \tilde{A} \cdot \max\{1, |c|, |a|\}$. The conclusion follows taking $C := 8\tilde{A}$. □

Lemma 2.14. — *There exists a constant $B \leq 1$ such that for all $(c, a) \in K^{d-1}$,*

$$\max_{0 \leq j \leq d-2} |P_{c,a}(c_j)| \geq B \cdot \max\{|c|, |a|\}^d.$$

Moreover, when K is non-Archimedean, B depends only on the residual characteristic of K and $B = 1$ if it is 0 or $\geq d+1$.

Proof. — Let I be the ideal generated by $(a^d, P_{c,a}(c_1), \ldots, P_{c,a}(c_{d-2}))$ in $K[c, a]$. Observe that the generators of I have no common zero other than $(0, \ldots, 0)$. Let $R := \mathbb{Z}[1/2, \ldots, 1/d]$. As R is an integral domain and as

$P_{c,a}(c_j) \in R[c,a]$ are homogeneous of degree d, a standard fact from elimination theory asserts that there exists $m \geq d$ and homogeneous polynomials $Q_{j,k} \in R[c,a]$ of degree $m-d$ such that for $1 \leq k \leq d-2$,

$$c_k^m = \sum_j Q_{j,k}(c,a) P_{c,a}(c_j).$$

Let A be the maximum of the norms of coefficients appearing in one of the $Q_{j,k}$'s. and let $B > 0$ be such that $B^{-1} := \max\{1, A\}$. Then $|c_k|^m \leq B^{-1} \cdot \max\{|c|,|a|\}^{m-d} \cdot \max_{0\leq j\leq d-2} |P_{c,a}(c_j)|$. Since $a^d = P_{c,a}(c_0)$, this gives the lemma. □

Proof of Proposition 2.13. — The inequality

$$G(P_{c,a}) \leq \log^+ \max\{|c|,|a|\} + C$$

is an immediate consequence of Proposition 2.11 (1). We also see that $C = 0$ when K is non-Archimedean and its residue characteristic is 0 or $\geq d+1$.

Assume now K is non-Archimedean and its residual characteristic is 0 or at least $d+1$. Assume first that $\max\{|c|,|a|\} \leq 1$. Then $G(P_{c,a}) = 0$, again by Proposition 2.11 (1). If $\max\{|c|,|a|\} > 1$, then either there is an index i such that $|c_i| = \max\{|c|,|a|\}$ and $g_{P_{c,a}}(c_i) \geq \log^+ |c_i| = \log^+ \max\{|c|,|a|\}$ by the second point of Proposition 2.11; or $|a| = \max\{|c|,|a|\}$ and we easily see that $|P^n_{c,a}(0)| = |a|^{d^n}$ for $n \geq 1$. The conclusion follows.

Assume now the residual characteristic p of K satisfies $0 < p \leq d$. Let $C \geq 1$ be given by Proposition 2.11. When $|P_{c,a}(c_j)| \leq C \max\{1,|c|,|a|\}$ for all j, then Lemma 2.14 yields $B \cdot \max\{|c|,|a|\}^{d-1} \leq C$, so that $G(P_{c,a}) \geq 0 \geq \frac{1}{d-1}\log(B/C) + \log^+ \max\{|c|,|a|\}$. When there exists j such that $|P_{c,a}(c_j)| > C \max\{1,|c|,|a|\}$, the second point of Proposition 2.11 and Lemma 2.14 give

$$\begin{aligned} dG(P_{c,a}) &\geq dg_{P_{c,a}}(c_j) = g_{P_{c,a}}(P_{c,a}(c_j)) \\ &\geq \log^+ |P_{c,a}(c_j)| \geq d \log^+ \max\{|c|,|a|\} + \log B \end{aligned}$$

and the conclusion follows.

To conclude, we assume K is Archimedean. If $\max\{|c|,|a|\} \leq 2$, we have $G(P_{c,a}) \leq \log 2 + \theta$. We may thus assume $A := \max\{|c|,|a|\} \geq 2$. As in the proof of Proposition 2.11, let $\delta := \frac{1}{d-1}\sum_j c_j$. If $|c_j| \leq A/2$ for all j, we also have $|\delta| \leq A/2$ and $A = |a|$. By [**68**, Lemma 6.5],

$$dG(P_{c,a}) \geq \max\{g_{P_{c,a}}(a^d), G(P_{c,a})\} \geq \log|a^d - \delta| - \log 4 \geq d\log A - \log 8.$$

In the opposite case, either $|\delta| \geq A/2$ or $|c_j - \delta| \geq A/2$ and [**68**, Lemma 6.5] directly gives $G(P_{c,a}) \geq \log A - \log 8$, as required. □

2.4. Examples

2.4.1. Integrable polynomials

Fix an integer $d \geq 2$. The degree d Chebychev polynomial is the unique polynomial $T_d \in \mathbb{Z}[z]$ satisfying

$$T_d(z + z^{-1}) = z^d + z^{-d}$$

in the field $\mathbb{Q}(z)$. It is a monic and centered polynomial of degree d, and for all $d, k \geq 2$, we have

$$T_d \circ T_k = T_{kd} \text{ and } T_d(-z) = (-1)^d T_d(z).$$

In particular, $T_d^n = T_{d^n}$ for all $n \geq 1$.

Observe that $\pi(z) = z + z^{-1}$ defines a Galois cover of degree 2 from $\mathbb{A}^1 \setminus \{0\}$ onto $\mathbb{A}^1$, with two ramified points at ± 1, and that we have $\pi(M_d(z)) = T_d(\pi(z))$ where $M_d(z) = z^d$. In particular the critical points of T_d are $0, \pm 1$, and are pre-periodic.

When $K = \mathbb{C}$, the Julia set of T_d is the image under π of the unit circle so that $K(T_d) = J(T_d) = [-2, 2]$. The equilibrium measure μ_P is absolutely continuous with respect to the normalized Lebesgue measure on $[-2, 2]$.

When K is non-Archimedean, then the filled-in Julia set of T_d coincides with the closed unit ball so that its Julia set is reduced to a singleton, namely to the Gauß point.

Definition 2.15. — *Let K be a field of characterisic 0 and let $P \in K[z]$ be a degree $d \geq 2$ polynomial. We say P is integrable if P is affine conjugated (in a finite extension of K) to either M_d or $\pm T_d$.*

This terminology is taken from [**164**] and inspired from hamiltonian dynamics.

Observe that when d is odd, then $-T_d$ is conjugated to T_d. There are many characterizations of integrable polynomials in algebraic or analytic terms. Over the complex numbers, we have the following famous theorem of Zdunik [**172**].

Theorem 2.16. — *Let P be a complex polynomial of degree at least 2. If its equilibrium measure is absolutely continuous with respect to the Hausdorff measure of its Julia set, then P is integrable. In particular a complex polynomial P is integrable iff its Julia set is smooth near one of its point.*

One can in fact push a little further the analysis and get

Corollary 2.17. — *Let P be a complex polynomial of degree at least 2. Then P is integrable iff $J(P)$ is either a circle or a closed segment.*

We refer to Theorem 3.8 below for a characterization of integrable complex polynomials in terms of the symmetries of their Julia set.

2.4.2. Potential good reduction

We assume here that the metrized field $(K, |\cdot|)$ is non-Archimedean. Recall that given $z \in K^\circ$ we let $\tilde{z} \in \tilde{K}$ be the image of z under the reduction map $K^\circ \to \tilde{K}$.

Fix $d \geq 2$ and pick any polynomial P of degree d with coefficients in K°. Then P induces a polynomial

$$\tilde{P} \colon \tilde{K} \to \tilde{K}$$

by taking the residue class of its coefficients. We say that P has *good reduction* if $\deg(\tilde{P}) = \deg(P)$. More generally, we say that P has potential good reduction if there exists a finite extension L/K and an affine transformation $\phi(z) = az + b$ with $a, b \in L$ such that $Q := \phi^{-1} \circ P \circ \phi \in L^\circ[z]$ has good reduction.

Observe that when P has good reduction, then we have $g_P(z) = \log^+ |z|$ so that the filled-in Julia set of P is the closed unit ball of $(K, |\cdot|)$ and $J(P)$ is reduced to the Gauß point.

Proposition 2.18. — *Let $P \in K[T]$ be any polynomial of degree $d \geq 2$. The following are equivalent.*

1. *P has potential good reduction;*
2. *the Julia set of P is reduced to a point;*
3. *the filled-in Julia set is a closed ball.*

Several other characterizations of potential good reduction polynomials are given in [**82**, Théorème E].

Proof. — The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are easy. When the filled-in Julia set is a closed ball, then the Julia set is reduced to a singleton $\{x\}$. The point x is periodic of type 2 or 3 and belongs to the Julia set. It follows that x is a type 2 repelling point, and up to conjugacy we may suppose that it is the Gauß point. Any polynomial for which the Gauß point is totally invariant has coefficients in K° and has good reduction, which concludes the proof. □

2.4.3. PCF maps

Suppose K is a metrized field of characteristic 0. A polynomial P of degree $d \geq 2$ is said to be post-critically finite (PCF) when all its critical points have a finite orbit.

Integrable maps are PCF. We shall see that any PCF map is conjugated to a polynomial with coefficients in $\bar{\mathbb{Q}}$ (see Corollary 2.25 below), and that the set of PCF maps forms a set of bounded height in the parameter space.

When $K = \mathbb{C}$ and all critical points are periodic then the dynamics of P on its Julia set is hyperbolic in the sense that it expands some Riemannian metric, see [**39**, §V.2]. When P is PCF but none of its critical point is periodic, then we say that P is strictly PCF (these polynomials are sometimes called Misiurewicz).

When K is non-Archimedean, then any PCF polynomial has good reduction since all its critical points have a bounded orbit.

2.5. Böttcher coordinates & Green functions

2.5.1. Expansion of the Böttcher coordinate

Let R be any integral domain whose fraction field K has characteristic zero, and $P \in R[z]$ be any polynomial of degree $d \geq 2$ given by

$$P(z) = Az^d + a_1 z^{d-1} + a_2 z^{d-2} + \cdots + a_d,$$

with $a_i, A \in R^{d+1}$. We fix any element α of an algebraic closure $\bar{K}$ of K such that $\alpha^{d-1} = A$.

The following result holds, see [**77**] in degree 3, [**6**, §5.5] or [**55**, §6].

Proposition-Definition 2.19. — *There exists a unique formal Laurent series φ_P of the form*

$$\varphi_P(z) = \alpha\left(z + \frac{a_1}{dA}\right) + \sum_{j\geq 1} \alpha_j z^{-j} \in R\left[\frac{1}{dA}, \alpha\right][[z^{-1}]][z] \tag{12}$$

such that

$$\varphi_P \circ P = (\varphi_P)^d\ . \tag{13}$$

It is called the Böttcher coordinate of P at infinity.

Note that $R[\frac{1}{dA},\alpha]$ is a subring a $\bar{K}$ and the proof will show that we can take more precisely

$$\alpha_j \in \mathbb{Z}\left[\frac{1}{dA},\alpha,a_1,\ldots,a_d\right] \text{ for all } j\ .$$

Proof. — We look for coefficients $\alpha_j \in R[\frac{1}{dA},\alpha]$ such that the power series

$$\varphi(z) = \alpha\left(z+\frac{a_1}{dA}\right)+\sum_{j\geq 1}\alpha_j z^{-j}$$

solves the equation (13). Observe that

$$\begin{aligned}\varphi_P\circ P(z) &= \alpha\left(P(z)+\frac{a_1}{dA}\right)+\sum_{j\geq 1}\frac{\alpha_j}{(Az^d)^j}\left(1+\frac{a_1}{Az}+\ldots+\frac{a_d}{Az^d}\right)^{-j}\\ &= \alpha\left(P(z)+\frac{a_1}{dA}\right)+\sum_{l\geq 1}\frac{1}{z^l}\left(\sum_{l/d\geq j\geq 1}\alpha_j Q_{j,l}(a_1,\ldots,a_d)\right)\end{aligned}$$

with $Q_{j,l}\in\mathbb{Z}[\frac{1}{A},a_1,\ldots,a_d]$, and

$$\begin{aligned}\varphi_P(z)^d &= \left(\alpha\left(z+\frac{a_1}{dA}\right)+\sum_{j\geq 1}\alpha_j z^{-j}\right)^d\\ &=\alpha^d z^d+\alpha a_1 z^{d-1}+\left(dA\alpha_1+\frac{(d-1)\alpha a_1^2}{2dA}\right)z^{d-2}+\\ &\quad+\sum_{l\leq d-3} z^l\left(dA\alpha_{d-1-l}+Q_l(\alpha_1,\ldots,\alpha_{d-2-l})\right)\end{aligned}$$

where Q_l is a polynomial having coefficients in $\mathbb{Z}[\frac{1}{dA},a_1,\alpha]$. The terms in z^d and z^{d-1} of both series agree by our choice of α. Identifying the terms in z^l successively for $l=d-2,d-3,\ldots,1,0,-1,\ldots$, we see that the coefficients α_s are uniquely determined by the relations:

$$dA\alpha_1 = -\frac{(d-1)\alpha a_1^2}{2dA}+\alpha a_2; \tag{14}$$

$$dA\alpha_s = -Q_{d-1-s}(\alpha_1,\ldots,\alpha_{s-1})+\alpha a_{s+1}; \tag{15}$$

when $2\leq s\leq d-2$,

$$dA\alpha_{d-1} = -Q_0(\alpha_1,\ldots,\alpha_{d-2})+\alpha\left(a_d+\frac{a_1}{dA}\right); \tag{16}$$

and

$$dA\alpha_s = -Q_{d-1-s}(\alpha_1,\ldots,\alpha_{s-1})+\sum_{j=1}^{(s-d+1)/d}\alpha_j Q_{j,s-d+1}(a_1,\ldots,a_d) \tag{17}$$

for all $s \geq d$. This shows that for all $s \geq 1$ the coefficient α_s can be expressed as a polynomial in the variables $a_1, \ldots, a_d$ with coefficients in $\mathbb{Z}[\frac{1}{dA}, \alpha]$. □

We shall need some precise informations on the dependence of the coefficients of the Böttcher coordinates in terms of the coefficients of the polynomials. Compare with [**77**, Lemma 2.2] and [**55**, Theorem 6.5]. We use the same notation as in the previous proposition.

Define the weighted degree of a polynomial $P = \sum c_I a^I$ in the variables $a_1, \ldots, a_d$ as $\widetilde{\deg}(P) := \min\{\sum j i_j,\, c_I \neq 0\}$ so that $\widetilde{\deg}(a_i) = i$.

Proposition 2.20. — *For any $k \geq 1$, one has the following expansion*

$$(\varphi_P(z))^k = \hat{P}_k(z) + \sum_{j=1}^{+\infty} \frac{\alpha_{k,j}}{z^j}, \tag{18}$$

where $\hat{P}_k$ is a polynomial of degree k with leading coefficient α^k such that

$$(2dA)^{2k} \hat{P}_k \in \mathbb{Z}[\alpha, a_1, \ldots, a_d][z] \ ;$$

and for any $k, j \geq 1$,

$$(2dA)^{2(k+j-1)} \alpha_{k,j} \in \mathbb{Z}[\alpha, a_1, \ldots, a_d]$$

is a polynomial of degree $\leq 3(d-1)j + (3d-2)k - 2d + 1$ in α, and of weighted degree $\leq k + j$ in the variables a_i.

Proof. — Observe that $\alpha_j = \alpha_{1,j}$ for all j, and recall that α_j is a polynomial in the coefficients $\alpha, a_1, \ldots, a_d$. Denote by $\deg_\alpha(\alpha_j)$ the degree of this polynomial in α. We claim that

$$(2dA)^{2j} \alpha_j \in \mathbb{Z}[\alpha, a_1, \ldots, a_d] \ . \tag{19}$$

$$\widetilde{\deg}(\alpha_j) \leq j + 1, \ \text{and} \tag{20}$$

$$\deg_\alpha(A^{2j} \alpha_j) \leq 3(d-1)j, \tag{21}$$

Grant this claim. The proof then goes as follows.

Fix $k \in \mathbb{N}^*$. We first have

$$\varphi_P^k(z) = \left(\alpha \left(z + \frac{a_1}{dA} \right) + \sum_{j \geq 1} \alpha_j z^{-j} \right)^k = \left(\alpha \left(z + \frac{a_1}{dA} \right) + \sum_{j=1}^{k} \alpha_j z^{-j} \right)^k + O\left(\frac{1}{z}\right)$$
$$= \alpha^k z^k + \sum_{j=0}^{k-1} \beta_j z^j + O\left(\frac{1}{z}\right)$$

where β_j is a sum of terms of the form

$$\alpha^\kappa \left(\frac{\alpha a_1}{dA} \right)^\tau \alpha_{j_1} \cdots \alpha_{j_\mu}$$

with $\kappa, \tau, j_1, \ldots, j_\mu \in \mathbb{N}$, $\kappa + \tau + \mu = k$, and $\kappa - (j_1 + \cdots + j_\mu) = j$. Observe that $\tau + 2(j_1 + \cdots + j_\mu) = \tau + 2\kappa - 2j \le 2k$. By (19), we infer that $(2dA)^{2k}\beta_j \in \mathbb{Z}[\alpha, a_1, \ldots, a_d]$ as required.

Now we focus on the coefficient $\alpha_{k,j}$ in the power series expansion (18). As above $\alpha_{k,j}$ is a sum of terms of the form

$$\alpha^\kappa \left(\frac{\alpha a_1}{dA}\right)^\tau \alpha_{j_1} \cdots \alpha_{j_\mu}$$

where $\mu \ge 1$, $\kappa+\tau+\mu = k$, and $(j_1+\cdots+j_\mu)-\kappa = j$. Since $\tau+2(j_1+\cdots+j_\mu) = \tau+2(\kappa+j) \le 2(k+j-1)$, (19) implies $(dA)^{2(k+j-1)}\alpha_{k,j} \in \mathbb{Z}[\alpha, a_1, \ldots, a_d]$. We have the following estimates:

$$\begin{aligned}
&\kappa + \tau + 3(d-1)(j_1 + \ldots + j_\mu) + (2(k+j-1) - \tau - 2(j_1 + \ldots + j_\mu))(d-1) \\
&\quad \le k - \mu + 3(d-1)(j+\kappa) + (2(k+j-1) - \tau - 2(j+\kappa))(d-1) \\
&\quad = 3(d-1)j + (d-1)(2k-2) + k + (d-1)(\kappa - \tau) - \mu \\
&\quad \le 3(d-1)j + (3d-2)k - 2d + 1
\end{aligned}$$

which implies $\deg_\alpha(\alpha_{k,j}) \le 3(d-1)j + (3d-2)k - 2d + 1$. On the other hand $\tau + (j_1 + 1 + \cdots + j_\mu + 1) = \tau + \kappa + j + \mu = k + j$, hence the weighted degree of $\alpha_{k,j}$ as a polynomial in the variables a_i's is at most $k + j$ by (20).

Let us now prove the claim. We proceed by induction building on the formulas defining α_j obtained in the proof of the previous proposition. The case $j = 1$ follows from the equality $2(dA)^2\alpha_1 = -(d-1)\alpha a_1^2 + 2dA\alpha a_2 = -(d-1)\alpha a_1^2 + 2d\alpha^d a_2$. Assume the claim has been proved for all $j = 0, \ldots, s-1$.

We first observe that the polynomial Q_{d-1-s} is a sum of terms of the form

$$\alpha^\kappa \left(\frac{\alpha a_1}{dA}\right)^\tau \alpha_{j_1} \cdots \alpha_{j_\mu}$$

with $\tau \in \mathbb{N}$, $d-2 \ge \kappa \in \mathbb{N}$, $\mu \ge 1$, $j_1, \ldots, j_\mu \in \{1, \ldots, s-1\}$, $\kappa + \tau + \mu = d$, and $\kappa - (j_1 + \cdots + j_\mu) = d - 1 - s$. Since

$$\tau + (j_1 + 1 + \cdots + j_\mu + 1) = \tau + \kappa - d + 1 + s + \mu = s + 1,$$

we get from the induction hypothesis that $\widetilde{\deg}(Q_{d-1-s}) \le s + 1$. Similarly

$$\tau + 2(j_1 + \cdots + j_\mu) = \tau + 2(\kappa - d + 1 + s) \le \begin{cases} 2(s-1) & \text{if } \tau = 0 \\ 2s - \tau & \text{otherwise} \end{cases},$$

hence $(2dA)^{2s-1}Q_{d-1-s} \in \mathbb{Z}[\alpha, a_1, \ldots, a_d]$, and $(2dA)^{2s}\alpha_s \in \mathbb{Z}[\alpha, a_1, \ldots, a_d]$. Estimating the degree in α is a bit more involved. One needs to estimate

$\deg_\alpha(A^{2s-1}\,Q_{d-1-s})$. This quantity can be bounded from above by:

$$\begin{aligned}
&\kappa+\tau+3(d-1)(j_1+\cdots+j_\mu)+(d-1)(2s-1-\tau-2(j_1+\cdots+j_\mu))\\
&\quad= d-\mu+3(d-1)(\kappa-d+1+s)+(d-1)(2s-1-\tau-2(\kappa-d+1+s))\\
&\quad= d-\mu+3s(d-1)+\kappa(d-1)+(d-1)(-d-3-\tau)\\
&\quad= 3s(d-1)-\mu d-2\tau(d-1)-2d+3\le 3s(d-1)
\end{aligned}$$

so that $\deg_\alpha(Q_{d-1-s})\le 3(d-1)s$. By (15) and (16), this proves the induction step when $s\le d-1$.

For $s\ge d$, we also need to control the polynomials $Q_{j,s-d+1}$ with $1\le j\le (s-d+1)/d$. The polynomial $Q_{j,s-d+1}$ is a sum of terms of the form

$$\frac{1}{A^j}\frac{a_{j_1}}{A}\cdots\frac{a_{j_\mu}}{A}$$

with $j_1,\ldots,j_\mu\ge 1$, $dj+(j_1+\ldots+j_\mu)=s-d+1$, and $\mu\le j$. We have $s-j\ge(d-1)j+(j_1+\ldots+j_\mu)+d-1\ge(j_1+\ldots+j_\mu)$, hence $\widetilde{\deg}(Q_{j,s-d+1})\le s-j$. Similarly, one has

$$j+\mu\le 2j\le 2(s-j)-1 \tag{22}$$

since $j\le(s-d+1)/d$. It follows that $(2dA)^{2(s-j)-1}Q_{j,s-d+1}\in\mathbb{Z}[a_1,\ldots,a_d]$, hence $(2dA)^{2s}\alpha_s\in\mathbb{Z}[\alpha,a_1,\ldots,a_d]$.

We have:

$$\begin{aligned}
\deg_\alpha(A^{2s-1}Q_{j,s-d+1}) &\le (d-1)(2s-1-2j)+3(d-1)j-(j+\mu)(d-1)\\
&\le 3s(d-1)-(2+\mu)(d-1)\le 3s(d-1)\ .
\end{aligned}$$

Using (17), we can now conclude the induction step for $s\ge d$ which proves our claim. This concludes the proof of the proposition. □

Proposition 2.21. — *For any $k\ge 1$ we have*

$$\hat{P}_k\circ P=\hat{P}_{kd}\ .$$

Proof. — Observe that $\varphi_P(P(z))^k=\varphi_P(z)^{dk}$. The result follows from (18) and identifying the polar parts of both members. □

2.5.2. Böttcher coordinate and Green function

We now fix an algebraically closed complete metrized field $(K,|\cdot|)$ of characteristic 0 and pick an integer $d\ge 2$. Recall that, if X is an algebraic variety over K, we denote by X^{an} its Berkovich analytification.

Recall that if $(c,a) \in K^{d-1}$, we let

$$P_{c,a}(z) := \frac{1}{d}z^d + \sum_{j=2}^{d-1}(-1)^{d-j}\sigma_{d-j}(c)\frac{z^j}{j} + a^d,$$

see (8); and that the critical set of $P_{c,a}$ is given by $c_1, \ldots, c_{d-2}, c_{d-1} = 0$. Recall that $G(P_{c,a}) = \max_{1\leq i\leq d-1}\{g_{P_{c,a}}(c_i)\}$, and that the map $(z,c,a) \mapsto g_{P_{c,a}}(z)$ is continuous, see §2.3. In particular, for any $\tau > 0$ the set $\{(c,a,z) \in \mathbb{A}^{d,\mathrm{an}}_K ;\ g_{P_{c,a}}(z) > G(P_{c,a}) + \tau\}$ is open.

We shall rely on the following estimates, see [**77**, Proposition 2.3] in the cubic case.

Proposition 2.22. —

1. *There exists a constant $\sigma = \sigma(K) \geq 0$ such that the Böttcher coordinate $\varphi_{P_{c,a}}$ is a convergent power series in the neighborhood of infinity $\{z \in \mathbb{A}^{1,\mathrm{an}}_K ;\ \log|z| > G(P_{c,a}) + C\}$ for any $(c,a) \in K^{d-1}$.*
2. *There exists a constant $\rho = \rho(K) \geq 0$ such that the map $(c,a,z) \longmapsto \varphi_{P_{c,a}}(z)$ extends analytically to the open set*
$$\{(c,a,z) \in \mathbb{A}^{d,\mathrm{an}}_K ;\ g_{P_{c,a}}(z) > G(P_{c,a}) + \rho\}\ ,$$
and we have the relations
$$g_{P_{c,a}} = \log|\varphi_{P_{c,a}}| \text{ and } \varphi_{P_{c,a}} \circ P_{c,a} = \varphi^d_{P_{c,a}} \text{ on } U_{c,a} = \{g_{P_{c,a}} > G(P_{c,a}) + \rho\}.$$
3. *There exists a constant $\tau = \tau(K) \geq 0$ such that*
$$\varphi_{P_{c,a}} : \{g_{P_{c,a}} > G(P_{c,a}) + \tau\} \longrightarrow \mathbb{A}^{1,\mathrm{an}}_K \setminus \overline{\mathbb{D}(0, \exp(G(P_{c,a}) + \tau))}$$
induces an analytic isomorphism.
4. *Finally, when K is Archimedean, or when the residual characteristic of K is zero, or larger than $d+1$ we can take $C = \tau = \rho = 0$.*

Proof. — Let us treat first the case K is Archimedean. In that case most of the statements are proved in [**61**] (see also [**30**, §1]). In particular, $\varphi_{c,a}(z)$ is analytic in a neighborhood of ∞ and extends to $U_{c,a} := \{g_{P_{c,a}} > G(P_{c,a})\}$ by invariance (so that $\rho = 0$), and it defines an isomorphism between the claimed domains with $\tau = 0$. It is moreover analytic in c, a, z, the relation $\varphi_{P_{c,a}} \circ P_{c,a} = \varphi^d_{P_{c,a}}$ holds on $U_{c,a}$ since the latter set is connected and the relation is satisfied at a formal level. Since $\varphi_{P_{c,a}}(z) - \alpha z$ is bounded near infinity, we have

(23)
$$g_{P_{c,a}}(z) = \lim_{n\to\infty} \frac{1}{d^n}\log|P^n_{c,a}(z)| = \lim_{n\to\infty}\frac{1}{d^n}\log|\varphi_{P_{c,a}}(P^n_{c,a}(z))| = \log|\varphi_{P_{c,a}}(z)|$$

on $U_{c,a}$.

To estimate more precisely the radius of convergence of the power series (12), we rely on [**30**, §4] as formulated in [**68**, §6]. Write $\delta = \sum_{i=1}^{d-1} c_i/(d-1)$. First choose $C = C_K > 0$ such that $G(P_{c,a}) > \log^+ \max\{|a|,|c|\} - C$, and set $\sigma := C + \log 5$. Suppose $\log|z| > G(P_{c,a}) + \sigma$. We first infer

$$|z-\delta| > 5\max\{1,|a|,|c|\} - |\delta| \geq (4 + \frac{1}{d-1})\max\{1,|a|,|c|\}$$

hence $\log|z-\delta| > G(P_{c,a}) + \log(4+(d-1)^{-1})$, so that

$$g_{P_{c,a}}(z) \geq \log|z-\delta| - \log 4 > G(P_{c,a})$$

by [**68**, Lemma 6.5]. We have shown that φ is analytic in $\{z,\ \log|z| > G(P_{c,a}) + \sigma\}$ hence converges in this domain.

From now on, we assume that the norm on K is non-Archimedean. Choose any α such that $\alpha^{d-1} = \frac{1}{d}$. Recall that by Proposition 2.20, we have

$$\varphi_{P_{c,a}}(z) = \alpha\left(z - \frac{\sigma_1(c)}{d-1}\right) + \sum_{j\geq 1} \alpha_j z^{-j} \in \mathbb{Z}\left[\alpha, \frac{\sigma_1(c)}{d-1}, \ldots, \frac{\sigma_{d-2}(c)}{2}, a^d\right][\![z^{-1}]\!][z]$$

where

$$2^{2j}\alpha_j \in \mathbb{Z}\left[\alpha, \frac{\sigma_1(c)}{d-1}, \ldots, \frac{\sigma_{d-2}(c)}{2}, a^d\right] \tag{24}$$

is a polynomial in the variables c and a of degree $\leq j+1$, and in the variable α of degree $\leq 3(d-1)j$.

Suppose first that the residual characteristic of K is either zero, or larger than $d+1$. Then (24) implies $|\alpha_j| \leq \max\{1,|c|,|a|\}^{j+1}$ so that $\varphi_{P_{c,a}}$ converges in $U_{c,a} := \{|z| > \max\{1,|c|,|a|\}\}$, and $\log|\varphi_{P_{c,a}}(z)| = \log|z|$ in $U_{c,a}$. It is easy to check that $U_{c,a}$ is invariant by the dynamics, hence (23) applies and $g_{P_{c,a}} = \log|\varphi_{P_{c,a}}(z)|$. Recall that we have $G(P_{c,a}) = \log^+\max\{|c|,|a|\}$ by Proposition 2.13. It follows that $\varphi_{P_{c,a}}$ is a well-defined analytic function on $\{g_{P_{c,a}} > G(P_{c,a})\} = \{|z| > \max\{1,|c|,|a|\}\}$. It induces an isomorphism between $U_{c,a}$ and $\mathbb{A}^{1,\mathrm{an}}_K \setminus \overline{\mathbb{D}_K(0, e^{G(P_{c,a})})}$ since $\log|\varphi_{P_{c,a}}(z)| = \log|z|$. The proposition is thus proved in this case with $C = \rho = \tau = 0$.

In residual characteristic $0 < p \leq d$, the estimates are more delicate. Let us set

$$\tilde{B}_p := \max\left\{|j|_p^{-1/j}, j = 1, \ldots, d-2\right\}.$$

Then (24) shows that

$$|\alpha_j| \leq (\tilde{B}_p \max\{1,|c|,|a|\})^{j+1} |4|_p^{-j} |d|_p^{-j}$$

hence $\varphi_{P_{c,a}}$ converges for $|z| > B_p \max\{1, |c|, |a|\}$, with $B_p = \tilde{B}_p \max\{|4|_p^{-1}, |d|_p^{-1}\}$, and we have $g_{P_{c,a}} = \log|\varphi_{P_{c,a}}| = \log|\alpha z|$ in that range.

Recall the definition of the constants θ and C from Propositions 2.11 and 2.13. Define

$$U_{c,a} = \{g_{P_{c,a}} > G(P_{c,a}) + \tau\}$$

with $\tau = C + \theta + \log B_p$, and pick any $z \in U_{c,a}$. Since $G(P_{c,a}) \geq \log^+ \max\{|c|, |a|\} - C$, we have

$$\log^+ \max\{|z|, |c|, |a|\} \geq g_{P_{c,a}}(z) - \theta > G(P_{c,a}) + \tau - \theta \geq \log^+ \max\{|c|, |a|\} + \log B_p$$

so that $|z| > B_p \max\{1, |c|, |a|\}$. It follows that the series $\varphi_{P_{c,a}}$ converges on $U_{c,a}$. Since $g_{P_{c,a}} = \log|\varphi_{P_{c,a}}| = \log|\alpha z|$ on $U_{c,a}$, the map $\varphi_{c,a}$ induces an isomorphism onto $\mathbb{A}^{1,\mathrm{an}}_K \setminus \overline{\mathbb{D}_K(0, e^{G(P_{c,a})+\tau})}$ with $\rho = \tau$, and the proof is complete. $\square$

2.6. Polynomial dynamics over a global field

For any number field $\mathbb{K}$, recall from §1.4 that $M_{\mathbb{K}}$ is the set of places of $\mathbb{K}$. For any $v \in M_{\mathbb{K}}$, we write $\mathbb{K}_v$ for the completion $(\mathbb{K}, |\cdot|_v)$, and we shall also let $\mathbb{C}_v$ be the completion of the algebraic closure $\bar{\mathbb{K}}_v$ of $\mathbb{K}_v$. Recall that $n_v = [\mathbb{K}_v : \mathbb{Q}_v]$, and that the standard height of $x \in \mathbb{K}$ is defined by

$$h_{\mathrm{st}}(x) = \frac{1}{[\mathbb{K} : \mathbb{Q}]} \sum_{v \in M_{\mathbb{K}}} n_v \log^+ |x|_v = \frac{1}{\deg(x)} \sum_{y \in \mathsf{O}(x)} \sum_{v \in M_{\mathbb{Q}}} \log^+ |y|_v \ .$$

Pick $P \in \mathbb{K}[z]$ of degree $d \geq 2$. Following Call and Silverman [**36**], we define the canonical height h_P of P as the limit

$$h_P(x) = \lim_{n \to \infty} \frac{1}{d^n} h_{\mathrm{st}}(P^n(x)).$$

This limit always exists and the height function h_P satisfies $h_P \circ P = d h_P$ and $\sup_{\mathbb{K}} |h_P - h_{\mathrm{st}}| < +\infty$. Furthermore, the Northcott property holds: for all $x \in \bar{\mathbb{K}}$, we have $h_P(x) \geq 0$ and $h_P(x) = 0$ if and only if x is preperiodic under iteration of P, i.e. if there exists $n > m \geq 0$ such that $P^n(x) = P^m(x)$.

Just like the standard height function, the height function h_P can also be decomposed as a sum of local contributions either under the form (6) or (7). For any $v \in M_{\mathbb{K}}$, denote by $g_{P,v}$ the Green function of P at the place v:

$$g_{P,v}(z) := \lim_{n \to \infty} \frac{1}{d^n} \log^+ |P^n(z)|_v, \ z \in \mathbb{C}_v.$$

Fix $x \in \bar{\mathbb{Q}}$. Then

$$h_P(x) = \frac{1}{[\mathbb{K}:\mathbb{Q}]} \sum_{v \in M_{\mathbb{K}}} n_v g_{P,v}(x) = \frac{1}{\deg(x)} \sum_{y \in \mathsf{O}(x)} \sum_{v \in M_{\mathbb{Q}}} g_{P,v}(y).$$

These relations reflect the fact that h_P is induced from the (unique P^*-invariant) semi-positive adelic metric on $\mathcal{O}(1)$.

Definition 2.23. — *Let P be any polynomial defined over a number field $\mathbb{K}$. Its bifurcation height is by definition $h_{\mathrm{bif}}(P) := \sum_{P'(c)=0} h_P(c)$.*

It was proved by Ingram [**109**] that the function $P \mapsto h_{\mathrm{bif}}(P)$ defines a Weil height on the parameter space MPoly^d of polynomials of a fixed degree d. It was further noticed in [**76**] that if one views $\mathbb{A}^{d-1}$ as an open subset of the projective space, then using the orbifold parameterization $(P_{c,a})_{c,a}$ by critically marked polynomials the height function

$$\widetilde{h_{\mathrm{bif}}}(c,a) := \frac{1}{\deg(c,a)} \sum_{c'a' \in \mathsf{O}(c,a)} \sum_{v \in M_{\mathbb{K}}} G_v(P_{c',a'})$$

with

$$G_v(P_{c,a}) = \max_{1 \le i \le d-1} \{g_{P_{c,a},v}(c_i)\}$$

is in fact determined by a semi-positive continuous adelic metrization on the line bundle $\mathcal{O}(1) \to \mathbb{P}^{d-1}$, and satisfies $\frac{1}{d-1} h_{\mathrm{bif}}(c,a) \le \widetilde{h_{\mathrm{bif}}}(c,a) \le h_{\mathrm{bif}}(c,a)$. In fact, one has the following.

Proposition 2.24. — *There exists a constant $C > 0$ such that for all $(c,a) \in \bar{\mathbb{Q}}^{d-1}$,*

$$|\widetilde{h_{\mathrm{bif}}}(P_{c,a}) - h_{\mathrm{st}}(c,a)| \le C \ .$$

Proof. — Pick any $(c,a) \in \bar{\mathbb{Q}}^{d-1}$, and choose any place $v \in M_{\mathbb{Q}}$. By Proposition 2.13, for any $(c',a') \in \bar{\mathbb{Q}}^{d-1}$, we have

$$|G_v(P_{c',a'}) - \log^+ \max\{|c'|,|a'|\}| \le C_v \ ,$$

and the set S of places for which $C_v \neq 0$ is finite. Summing up all contributions over $\mathsf{O}(c,a)$ we get

$$\begin{aligned} |h_{\mathrm{bif}}(P_{c,a}) - h_{\mathrm{st}}(c,a)| \le & \frac{1}{\deg(c,a)} \sum_{c'a' \in \mathsf{O}(c,a)} \sum_{v \in S} \left| G_v(P_{c',a'}) - \log^+ \max\{|c'|,|a'|\} \right| \\ \le & \sum_{v \in S} C_v < \infty \ , \end{aligned}$$

as required. □

Corollary 2.25. — *The set of complex polynomials $P_{c,a}$ of degree $d \geq 2$ which are PCF is a countable subset of $\bar{\mathbb{Q}}^{d-1}$ and forms a set of bounded (standard) height.*

Proof. — The set $\mathcal{P}$ of PCF polynomials of the form $P_{c,a}$ is defined as the union of countably many algebraic subvarieties defined by equations with coefficients in $\mathbb{Q}$ of the form $\cap_{i=0}^{d-1}\{P_{c,a}^{n_i}(c_i) = P_{c,a}^{m_i}(c_i)\}$ with $n_i > m_i$. By the preceding result, $\mathcal{P} \cap \bar{\mathbb{Q}}^{d-1}$ forms a set of bounded height, hence $\mathcal{P}$ cannot contain a positive dimensional subvariety. This proves the result. □

2.7. Bifurcations in holomorphic dynamics

We review briefly Mañé-Sad-Sullivan's theory of bifurcation of holomorphic dynamical systems in the context of polynomials. We refer to either the original papers [**122, 121**] or to the survey [**23**] by Berteloot for a more detailed account.

Let Λ be any connected complex manifold. A holomorphic family $(P_\lambda)_{\lambda\in\Lambda}$ of polynomials of degree $d \geq 2$ parameterized by Λ is by definition a holomorphic map $(\lambda, z) \mapsto P_\lambda(z)$ from $\Lambda \times \mathbb{C} \to \mathbb{C}$ such that for all $\lambda \in \Lambda$, the map $z \mapsto P_\lambda(z)$ is a polynomial of degree d.

A critically marked holomorphic family of polynomials is a holomorphic family $(P_\lambda)_{\lambda\in\Lambda}$ together with $d-1$ holomorphic functions $c_1, \ldots, c_{d-1} : \Lambda \to \mathbb{C}$ such that $\mathrm{Crit}(P_\lambda) = \{c_1(\lambda), \ldots, c_{d-1}(\lambda)\}$ for all λ.

Definition 2.26. — *Let $(P_\lambda)_{\lambda\in\Lambda}$ be any critically marked holomorphic family of polynomials of degree d. The stability locus* Stab(P) *of a holomorphic family is the union of all open subsets $U \subset \Lambda$ over which the families $\{\lambda \mapsto P_\lambda^n(c_i(\lambda))\}_n$ are normal on U for all $i = 1, \ldots, d-1$.*

When $(P_\lambda)_{\lambda\in\Lambda}$ is an arbitrary holomorphic family of polynomials, then one can define the stability locus as follows.

The behaviour of the Julia set on the stability locus is governed by holomorphic motions. We recall briefly this crucial notion here. Fix any $\lambda_0 \in \Lambda$.

Definition 2.27. — *A holomorphic motion of a subset $X \subset \mathbb{C}$ parameterized by (Λ, λ_0) is a family of holomorphic maps $\{\lambda \mapsto x(\lambda)\}_{x\in X}$ such that $x(\lambda_0) = x$, and for all λ, $x \mapsto x(\lambda)$ is injective from X to $\mathbb{C}$.*

A holomorphic motion can also be viewed as a map $h\colon \Lambda \times X \to \mathbb{C}$ such that $h(\lambda_0, \cdot) = \mathrm{id}$, $h(\lambda, \cdot)$ is injective and $h(\cdot, x)$ is holomorphic for all x.

Using Montel's and Hurwitz's theorems, one can show that any holomorphic motion of a set X extends canonically to its closure $\bar{X}$. Moreover for any fixed parameter λ, the injective map $X \to \mathbb{C}$ sending x to $x(\lambda)$ is quasi-symmetric, see [**108**, §5.2].

Theorem 2.28. — *The stability locus* $\mathrm{Stab}(P)$ *of a holomorphic family of polynomials is the union of all connected pointed open subsets* $(U, \lambda_0) \subset \Lambda$ *on which there exists a holomorphic motion* $h\colon U \times J(P) \to \mathbb{C}$ *such that* $P_\lambda(h_\lambda(z)) = h_\lambda(P_{\lambda_0}(z))$.

In particular the dynamics of any two polynomials P_{λ_0} *and* P_{λ_1} *for which* λ_0 *and* λ_1 *lie in the same connected component of* $\mathrm{Stab}(P)$ *are topologically conjugated on their Julia sets.*

Because of this theorem it is of common use to refer to the stability locus as the J-stability locus. Beware that in general however, the stability locus of P is not connected (this phenomenon already appears in degree 2).

The previous result relies on the characterization of the stability locus in terms of the stability of periodic points.

Definition 2.29. — *A parameter* λ *is said to have an unstable periodic point* z*, iff there exist two sequences of parameters* $\lambda_n^\pm$ *such that* $\lambda_n^\pm \to \lambda$*, and a sequence* z_n^+ *(resp.* z_n^-*) of repelling (resp. attracting) periodic points for* $P_{\lambda_n^+}$ *(resp.* $P_{\lambda_n^-}$*) such that* $z_n^\pm \to z$ *as* $n \to \infty$.

Observe that any unstable periodic point is necessarily neutral.

When λ_0 has an unstable periodic point at z_0, then one can find a finite map $r\colon (B, 0) \to (\Lambda, \lambda_0)$ defined on a small open ball B centered at the origin in $\mathbb{C}^{\dim(\Lambda)}$, and a holomorphic map $p\colon (B, 0) \to \mathbb{C}$ such that $p(r(t))$ is a periodic point of some fixed period n for all $t \in \mathbb{B}$, $p(r(0)) = z_0$ and the multiplier of $P_{r(t)}$ at $p(r(t))$ is a non-constant holomorphic function whose value at 0 equals $|(P_{\lambda_0}^n)'(z_0)| = 1$.

Theorem 2.30. — *The complement of stability locus of a holomorphic family of polynomials coincides with the closure of the set of parameters having an unstable periodic point.*

Again using Montel's theorem, one can infer the following crucial density statement.

Theorem 2.31. — *The stability locus of any holomorphic family of polynomials is open and dense.*

Remark 2.32. — We shall return to the more general notion of stability of a pair in Chapter 4 where a characterization of the stability locus will be given in terms of the variation of the Green function, see Proposition 4.2 and Theorem 4.30.

The previous theorem was made more precise by McMullen and Sullivan [**125**, Theorem 7.1].

Theorem 2.33. — *Let* $(P_\lambda)_{\lambda\in\Lambda}$ *be any analytic family of polynomials. Then there exists an open and dense subset* $C(P) \subset \mathrm{Stab}(P)$ *and, for any simply connected pointed domain* $(U, \lambda_0) \subset C(P)$*, a holomorphic motion* $h\colon U \times \mathbb{C} \to \mathbb{C}$ *that conjugates* P_λ *to* $P_{\lambda'}$ *on* $\mathbb{C}$ *for any pair of parameters* $\lambda, \lambda' \in U$*.*

Remark 2.34. — The previous theorem is valid for any family of rational maps of the projective line.

2.8. Components of preperiodic points

In order to develop a specialization argument in §5.5 we shall need a detailed analysis of the locus of preperiodic points in arbitrary families of polynomials. We are indebted to R. Roeder to have informed us about this delicate issue.

Let K be any algebraically closed field of characteristic 0, and Λ be any algebraic variety defined over K. Let $P : (\lambda, z) \in \Lambda\times\mathbb{A}^1 \mapsto (\lambda, P_\lambda(z)) \in \Lambda\times\mathbb{A}^1$ be any algebraic family of degree d polynomials, and define

$$\mathrm{Preper} := \{(\lambda, z) \in \Lambda \times \mathbb{C} \ \text{ s.t. } \{P^n_\lambda(z)\}_{n\geq1} \text{ is finite}\}\,.$$

Observe that the set Preper is a countable union of irreducible analytic hypersurfaces of $\Lambda \times \mathbb{A}^1$.

Theorem 2.35. — *Let* (λ_0, z_0) *be any point in* $\Lambda\times\mathbb{A}^1$ *which belongs to an infinite sequence* $\{Z_i\}_{i\geq0}$ *of distinct irreducible hypersurfaces included in* Preper*. Then after a base change, conjugating the family by suitably affine transformations, and possibly replacing* P *by some iterate we are in the following situation.*

- *The point* z_0 *is fixed for any parameter* λ*. Further it is super-attracting for* P_{λ_0} *and the local degree function* $\lambda \mapsto \mathrm{ord}_{z_0}(P_\lambda)$ *is not locally constant near* λ_0*.*
- *One can find a sequence of integers* $n_i \to \infty$ *such that* Z_i *is a component of* $\{(\lambda, z), P^{n_i}_\lambda(z) = z_0\}$*.*

Remark 2.36. — The statement above is actually true in the analytic category, for any holomorphic family of endomorphisms of degree at least 2 of the projective space of any dimension. It also explains the complexity of the locus of preperiodic parameters in stable families, see Theorem 4.5 below. It is an important (yet sometimes overlooked) source of difficulty in the course of specialization arguments. It ultimately explains our introduction of the condition ($\triangle$), see the proof of Theorem C in §5.5.

Example 2.37. — Consider the family $P_\lambda(z) = \lambda z + z^2$ parameterized by the affine line $\Lambda = \mathbb{A}^1_K$. Define the hypersurfaces $Z_1 = \{z + \lambda = 0\}$, and $Z_{n+1} = \{P^n_\lambda(z) + \lambda = 0\}$ for all $n \geq 1$. Since $P^n_\lambda(z)$ is a polynomial of degree 2^n of dominant term z^{2^n} with no linear terms in λ, z, these varieties are all smooth at $(0, 0)$ and tangent to $\{\lambda = 0\}$ up to order 2^n. It follows that for each n the variety Z_n is irreducible and included in Preper. When $\lambda = 0$, then $\{Z_n(0)\}_{n\in\mathbb{N}^*} = \{(0, 0)\}$, and since $-\lambda$ is strictly prefixed, $\{Z_n(\lambda)\}_{n\in\mathbb{N}^*}$ is infinite when $\lambda \neq 0$.

Proof of Theorem 2.35. — Since K has characteristic 0, we can resolve Λ, and suppose it is a smooth variety. Replacing P by some iterate, we may suppose that z_0 is preperiodic to a fixed point $z_\star$ for P_{λ_0}. This fixed point also belongs to an infinite set of irreducible hypersurfaces W_i included in Preper since P is finite. Up to an affine change of coordinates and base change, $z_\star = 0$ is a persistent fixed point i.e. $P(0) = 0$. Moreover, since the problem is local, we shall also argue in a formal neighborhood of $\lambda_0 = 0$ in Λ. Denote by $\mu := (P_0)'(0)$ the multiplier of the fixed point 0.

We shall prove that $\mu = 0$.

We first claim that Preper is locally an irreducible subvariety near $(0, 0)$ when μ is non-zero and not a root of unity. Observe that $P_\lambda(z) = \mu z + O(\lambda z, z^2)$ so that

$$P^m_\lambda(z) = \mu^m z + O(\lambda z, z^2) \text{ for all } m.$$

It follows that for all l, m we have

$$P^m_\lambda(z) - P^{m+l}_\lambda(z) = \left(\mu^m - \mu^{l+m}\right) z + O(\lambda z, z^2),$$

Since $\mu^m - \mu^{l+m} \neq 0$, we conclude that

$$W_{m,l} := \{P^m_\lambda(z) = P^{m+l}_\lambda(z)\}$$

is locally at the origin given by the equation $\{z = 0\}$, ending the proof in this case.

Next suppose that μ is a q-th root of unity. Then we claim that at most $(d - 1)q + 2$ irreducible hypersurfaces included in Preper may contain $(0, 0)$.

By [**37**, §1], we may formally conjugate P_0 to the following normal form:

$$P_0(z) = \mu\left(z + z^{\nu q+1}\right) + O(z^{\nu q+2}),\ \text{for some } \nu \geq 1\ .$$

By [**61**, Proposition 6 p. 72], we have[(4)] $\nu \leq d-1$. For any $m \geq 1$, we find

$$P_\lambda^m(z) = \mu^m\left(z + mz^{\nu q+1}\right) + O(\lambda z, z^{\nu q+2}),$$

so that

$$P_\lambda^m(z) - P_\lambda^{m+l}(z) = (\mu^m - \mu^{m+l})z + (m\mu^m - (m+l)\mu^{m+l})z^{\nu q+1} + O(\lambda z, z^{\nu q+2}).$$

Observe that when $\mu^l \neq 1$, then the unique component of $W_{m,l}$ passing through the origin is given by $z = 0$. Otherwise q divides l, and $P_\lambda^m(z) - P_\lambda^{m+l}(z)$ is equivalent to a Weierstraß polynomial in z of degree $\nu q+1$. Since this degree is independent on m and l and $W_{0,q} \subset W_{m,l}$ we conclude that $W_{0,q} = W_{m,l}$. The maximal number of irreducible components of $W_{0,q}$ is $\nu q + 1$ which concludes the proof of the claim.

At this point we have proved that 0 is a super-attracting fixed point so that we can write

$$P_0(z) = z^k + O(z^{k+1}),\ \text{for some } 2 \leq k \leq d\ ,$$

and

$$P_\lambda(z) = \sum_{i=i_0}^{k} a_i(\lambda) z^i + O_\lambda(z^{k+1}),$$

with $a_{i_0} \not\equiv 0$, $a_i(0) = 0$ for $i \leq k-1$, and $a_k(0) = 1$. If the local degree of 0 is locally constant, we have $a_i \equiv 0$ for all $i \leq k-1$ so that $i_0 = k$, and we can write

$$P_\lambda^m(z) - P_\lambda^{m+l}(z) = a_{m,l}(\lambda) z^{k^m} + O_\lambda(z^{k^m+1}),$$

with $a_{m,l}(0) = 1$. As above, we infer that there exists a unique irreducible hypersurface passing through $(0,0)$ and included in Preper, namely $\{z = 0\}$.

It remains to show that any component of $W_{m,l}$ passing through 0 is included in $\{P_\lambda^m(z) = 0\}$. We observe that the image under $(\lambda, z) \mapsto (\lambda, P_\lambda^m(z))$ of $W_{m,l}$ is included in $W_{0,l}$ which is defined by the equation

$$0 = z - P_\lambda^l(z) = z + O_\lambda(z^2),$$

hence locally equal to $z = 0$. This concludes the proof of the theorem. □

The proof of the previous theorem works verbatim for any family of rational maps of the projective line. It implies in particular the following statement.

(4) The arguments of Douady and Hubbard only works when $K = \mathbb{C}$, but we may invoke Lefschetz principle over an arbitrary field of characteristic zero.

Theorem 2.38. — *Let* $\{R_t\}_{t\in\mathbb{D}}$ *be any holomorphic family of rational maps of degree* $d \geq 2$ *parameterized by the unit disk. Let* $Z_n \subset \mathbb{P}^1_{\mathbb{C}} \times \mathbb{D}$ *be any sequence of distinct irreducible closed curves such that for all* t*, the point* $Z_n(t) := Z_n \cap (\mathbb{P}^1_{\mathbb{C}} \times \{t\})$ *is preperiodic for* R_t*.*

Then there exists an open subset $U \subset \Lambda$ *such that* $\{Z_n(t)\}_{n\geq 0}$ *is infinite for all* $t \in U$*.*

Proof. — By Theorem 2.33 we may find a connected open set $U \subset \mathbb{D}$ over which the family is stable and the dynamics of R_t and $R_{t'}$ are topologically conjugated for any $t, t' \in U$, see Remark 2.34. In particular, the number of super-attracting orbits remains the same with the same multiplicity since two super-attracting periodic points are topologically conjugated iff their local degree is the same. The result then follows from Theorem 2.35. □

Remark 2.39. — The previous result may be used to clarify the specialization argument of [**100**, p.16].

CHAPTER 3

DYNAMICAL SYMMETRIES

Symmetries of Julia sets play an important role in problems of unlikely intersections. In this chapter we study various notions of dynamical symmetries for polynomials.

In §3.1 we define the group of dynamical symmetries $\Sigma(P)$ of a single polynomial P and present various characterizations of it especially in the Archimedean case. We investigate the variation of this symmetry group when P belongs to an algebraic family in §3.2. We then prove (Theorem 3.18) that $\Sigma(P)$ consists of those affine transformations mapping the set of preperiodic points of P onto itself.

In §3.4 we introduce the notion of primitive polynomials, which are polynomials that cannot be written as iterates of polynomials of lower degree up to symmetries. Families of non primitive polynomials are sources of undesirable examples of unlikely intersections. We show that any family of non primitive polynomials is induced by a family of lower degree polynomials (Proposition 3.26).

It was progressively realized that a polynomial might have symmetries induced by polynomials of degree ≥ 2, see [**6, 95**]. We investigate this phenomenon in §3.5 building on Ritt's theory which aims at describing all compositional factors of a given polynomial. We introduce the notion of intertwined polynomials. Two polynomials P and Q of the same degree are intertwined if the map $(z, w) \mapsto (P(z), Q(w))$ fixes a non trivial curve. Building on works by Müller-Zieve [**175**], Medvedev-Scalon [**126**], Ghioca-Nguyen-Ye [**99, 101**], and Pakovich [**134, 135, 136**], we explore this equivalence relation in the moduli space of polynomials.

We conclude this chapter by a description in §3.6 of the stratification of the moduli space of polynomials in degree $d \leq 6$ induced by the presence of symmetries.

3.1. The group of dynamical symmetries of a polynomial

Let K be any algebraically closed field of characteristic zero.

A reduced presentation of a monic and centered polynomial $P \in K[z]$ is the choice of two integers μ and m and a polynomial P_0 such that $P_0(0) \neq 0$, $P(z) = z^\mu P_0(z^m)$, and P_0 cannot be further written as $Q(z^l)$ for some polynomial Q and some integer $l \geq 2$. Such a presentation is unique, P_0 is again monic and it is centered when $m = 1$.

Definition 3.1. — *The group $\Sigma(P)$ of dynamical symmetries of a monic and centered polynomial having a reduced presentation $P(z) = z^\mu P_0(z^m)$ is the cyclic group $\mathbb{U}_m$ when $P_0 \neq 1$ and equals the group of all roots of unity when P is a monomial map.*

When P is not a monomial map, the order of $\Sigma(P)$ is always less than d, and it equals d iff $P(z) = z^d + c$ for some $c \in K^*$, i.e. P is unicritical and non monomial (whence non-integrable if $d \geq 3$).

Since two monic and centered polynomials can be conjugated only by mutiplication by a root of unity, it follows that $\Sigma(P)$ does *not* depend on the conjugacy class of P in the set of monic and centered polynomials. In fact since the group of roots of unity is abelian, $\Sigma(P)$ is *canonically* isomorphic to $\mathbb{U}_m$.

We can thus define the group of dynamical symmetries of any polynomial P by setting

$$\Sigma(P) = \{A^{-1} \circ \sigma \circ A, \, \sigma \in \Sigma(A^{-1}PA)\}$$

for any affine map A such that $A^{-1} \circ P \circ A$ is monic and centered.

Remark 3.2. — Our definition of the dynamical symmetries of a polynomial is quite indirect and is not satisfactory as such. But it is a convenient way to present this group so as to get as quickly as possible its main properties. We shall later prove that this group can be characterized in a purely algebraic way in terms of preservation of preperiodic points, see Theorem 3.18 below.

Remark 3.3. — When K is not algebraically closed, we embed it into some algebraically closed field L. If P is a polynomial with coefficients in K, we may view it as a polynomial in L and consider $\Sigma_L(P) \subset \mathrm{Aff}(L)$ its group of dynamical symmetries. We shall set $\Sigma(P) := \Sigma_L(P) \cap \mathrm{Aff}(K)$. In general $\Sigma(P)$ is much smaller than $\Sigma_L(P)$.

Remark 3.4. — Symmetries of rational maps have been considered in several papers, see e.g. [**116, 133, 138, 166**]. Most of the results presented in this section remains however open in the rational case.

Recall the definition of the automorphism group of the polynomial P:

$$\mathrm{Aut}(P) = \{g \in \mathrm{Aff}(K),\ P(g \cdot z) = g \cdot P(z)\}\ .$$

Example 3.5. — The group of symmetries and the group of automorphisms of the Chebyshev polynomials T_d are equal to $\Sigma(T_d) = \mathbb{U}_2$ and

$$\mathrm{Aut}(T_d) = \begin{cases} \{\mathrm{id}\} & \text{if } k \text{ is even,} \\ \mathbb{U}_2 & \text{if } k \text{ is odd.} \end{cases}$$

Proposition 3.6. — *Let P be any polynomial of degree $d \geq 2$. The group $\Sigma(P)$ is the union of all finite subgroups G of $\mathrm{Aff}(K)$ such that there exists a morphism $\rho : G \to G$ satisfying $P(g \cdot z) = \rho(g) \cdot P(z)$.*

Remark 3.7. — In particular, we get $\mathrm{Aut}(P) \subset \Sigma(P)$, so that $\mathrm{Aut}(P^n) \subset \Sigma(P)$ for all $n \geq 1$. In general the inclusion $\cup_{n\in\mathbb{N}}\mathrm{Aut}(P^n) \subset \Sigma(P)$ is strict, for instance for any quadratic polynomial not conjugated to the square map.

Proof. — One may suppose that P is monic and centered and has a reduced presentation $P(z) = z^\mu P_0(z^m)$. The case P is monomial (i.e. $P_0 = 1$) is easy to deal with so that we assume $\deg(P_0) \geq 1$ during the whole proof.

Take any finite subgroup G of $\mathrm{Aff}(K)$ with a morphism $\rho : G \to G$ satisfying $P(g{\cdot}z) = \rho(g){\cdot}P(z)$. Any finite group is of the form $G = \{\zeta z + a(\zeta-1),\ \zeta \in \mathbb{U}_r\}$ for some r and some $a \in \mathbb{C}$. For any ζ one can thus find another root of unity ξ such that

$$(\zeta z + a(\zeta - 1))^\mu P_0((\zeta z + a(\zeta - 1))^m) = \xi z^\mu P_0(z^m) + a(\xi - 1)$$

Since P is centered the term of degree $\deg(P) - 1$ is zero on the right hand side. As P_0 is centered when $m = 1$, the left hand side can be written as

$$(\zeta^\mu z^\mu + \mu a(\zeta-1)\zeta^{\mu-1}z^{\mu-1} + \mathrm{l.o.t.})(\zeta^{d-\mu}z^{d-\mu} + (d-\mu)\zeta^{d-\mu-1}a(\zeta-1)z^{d-\mu-1} + \mathrm{l.o.t.})$$

so that comparing both sides we get $da(\zeta - 1)\zeta^{d-1} = 0$ hence $a = 0$. We thus have

$$(\zeta z)^\mu P_0((\zeta z)^m) = \xi z^\mu P_0(z^m)$$

Since P_0 is monic, we have $\zeta^\mu = \xi$ and therefore $P_0((\zeta z)^m) = P_0(z^m)$ which implies $\zeta^m = 1$. This proves $G \subset \Sigma(P)$. Since $\Sigma(P)$ is a finite group, the conclusion follows. □

For any polynomial P, we define the following subgroup of $\Sigma(P)$ by letting

$$\Sigma_0(P) := \bigcup_{n\geq 1} \ker(\rho^n)$$

where ρ is the morphism given by Proposition 3.6. When P is a monic and centered polynomial having reduced presentation $z^\mu P_0(z^m)$ then the group $\Sigma_0(P)$ satisfies

$$\Sigma_0(P) = \bigcup_{n\geq 1} \{\zeta \in \mathbb{U}_m,\, \zeta^{\mu^n} = 1\}.$$

It is thus clear that

1. $\Sigma_0(P)$ is trivial if and only if $\mu \neq 0$ and μ and m are coprime,
2. $\Sigma_0(P) = \Sigma(P) = \mathbb{U}_m$ if and only if $\mu = 0$, or all prime divisors of m divide μ.

Over the field $K = \mathbb{C}$ of complex numbers, the dynamical symmetries have been studied in details. We have the following result by [**3, 151**]. For any compact set J of the complex plane, denote by $\mathrm{Aut}(J)$ the subgroup of affine transformations $g \in \mathrm{Aff}(\mathbb{C})$ fixing J.

Theorem 3.8. — *Suppose J is the Julia set of a complex polynomial of degree at least 2 which is not integrable.*

Then there exists a polynomial Q such that any polynomial P with $J(P) = J$ is of the form $P = g \circ Q^n$ for some integer n, and some $g \in \mathrm{Aut}(J)$.

From this theorem, we obtain

Proposition 3.9. — *Pick any complex polynomial $P \in \mathbb{C}[z]$ of degree at least 2.*

1. *The group $\mathrm{Aut}(J(P))$ of affine transformations fixing $J(P)$ coincides with the group of all $g \in \mathrm{Aff}(\mathbb{C})$ such that the Green function satisfies $g_P(g \cdot z) = g_P(z)$.*
2. *When P is not conjugated to a monomial, then $\Sigma(P) = \mathrm{Aut}(J(P))$; otherwise $\mathrm{Aut}(J(M_d)) = S^1$ and $\Sigma(M_d) = \mathbb{U}_\infty$ is the set of torsion elements of $\mathrm{Aut}(J(M_d))$.*

Remark 3.10. — It would be interesting to generalize the second item to any metrized non-archimedean field $(K, |\cdot|)$. Note however that when P has good reduction, then $g_P = \log^+|z|$ and the whole group $\mathrm{Aff}(K^\circ)$ preserves the Green function. It is likely that for a polynomial P not having potential good reduction then $\Sigma(P)$ coincides with the set of $g \in \mathrm{Aff}(K)$ such that $g_P(g \cdot z) = g_P(z)$.

Proof. — We assume P is monic and centered. Denote by G be the group of all $g \in \mathrm{Aff}(\mathbb{C})$ such that $g_P(g \cdot z) = g_P(z)$. Any $g \in G$ leaves the filled-in Julia set of P invariant, hence belongs to $\mathrm{Aut}(J(P))$. Conversely, any $g \in \mathrm{Aut}(J(P))$ preserves the Green function with a logarithmic pole at infinity of $J(P)$ hence belongs to G. We thus have $G = \mathrm{Aut}(J(P))$.

In the case P is a monomial, $\mathrm{Aut}(J(P)) = S^1$ and $\Sigma(P)$ is the set of roots of unity, hence the result follows. We suppose from now on that P is not conjugated to a monomial map. Observe that for any element g in the group $\Sigma(P)$ then $P^n(g \cdot z) = \rho^n(g) \cdot P^n(z)$ for all n hence

$$g_P(g \cdot z) = \lim_n \frac{1}{d^n} \log^+ |\rho^n(g) \cdot P^n(z)| = g_P(z) \ ,$$

and $\Sigma(P) \subset \mathrm{Aut}(J(P))$.

Since $J(P)$ is compact, the group $\mathrm{Aut}(J(P))$ is also compact hence finite, since otherwise $J(P)$ would be a circle and P is monomial by Corollary 2.17. Assume $J(P)$ is not a segment. For any $g \in \mathrm{Aut}(J(P))$ the polynomial $P(g \cdot z)$ fixes $J(P)$ hence by Theorem 3.8 there exists $\rho(g) \in \mathrm{Aut}(J(P))$ such that $P(g \cdot z) = \rho(g) \cdot P(z)$, and Proposition 3.6 (1) implies $\mathrm{Aut}(J(P)) \subset \Sigma(P)$.

When $J(P)$ is a segment, then P is conjugated to $\pm T_d$ by Corollary 2.17 so that we may assume that $J(P) = [-2, 2]$ in which case we have $\mathrm{Aut}(J(P)) = \{\pm \mathrm{id}\}$ which is included in $\Sigma(P)$. □

Corollary 3.11. — *Let K be any field of characteristic zero and pick any $P \in K[z]$ of degree $d \geq 2$. Then $\Sigma(P^n) = \Sigma(P)$ for all $n \in \mathbb{N}^*$.*

Proof. — Replacing K by the field generated by the coefficients of P over $\bar{\mathbb{Q}}$ we may suppose that K is finitely generated over $\bar{\mathbb{Q}}$, and further fix an embedding $K \subset \mathbb{C}$. It is sufficient to treat the case P is not conjugated to a monomial map. By the previous proposition, we obtain that

$$\Sigma(P^n) = \mathrm{Aut}(J(P^n)) = \mathrm{Aut}(J(P)) = \Sigma(P) \ .$$

This concludes the proof. □

Remark 3.12. — It is not clear how to get a purely algebraic proof of the previous fact which could also be applied in positive characteristic.

3.2. Symmetry groups in family

As in the previous section, K is any algebraically closed field of characteristic zero. Let V be a K-affine variety. Recall that an algebraic family P of polynomials of degree $d \geq 2$ parameterized by V is a map $P : V \times \mathbb{A}^1_K \to V \times \mathbb{A}^1_K$ such that $P(t, z) = (t, P_t(z))$ and P_t is a polynomial of degree d for all $t \in K$.

Given any regular morphism $\pi : W \to V$, one can lift an algebraic family parameterized by V to an algebraic family Q parameterized by W by setting $Q(t, z) = (\pi(t), P_{\pi(t)}(z))$.

Observe that in the case $\Sigma(P_t)$ is finite, it is a cyclic group hence its isomorphism class is determined by its cardinality. The following explains the variation of the dynamical symmetry group in a family.

Proposition 3.13. — *Let V be an irreducible affine variety defined over K, and let P be an algebraic family parametrized by V. Then the function $t \mapsto \mathrm{Card}(\Sigma(P_t)) \in \mathbb{N}^* \cup \{+\infty\}$ is upper semi-continuous with respect to the Zariski topology.*

In particular there exists a Zariski dense open subset of V on which all group of dynamical symmetries are isomorphic.

Proof. — Write $P_t(z) = \alpha_0(t)z^d + \alpha_1(t)z^{d-1} + \text{l.o.t.}$ and consider the affine subvariety

$$W = \{(a, b, t) \in \mathbb{A}^2_K \times V,\ \alpha_0(t)a^{d-1} = 1 \text{ and } d\alpha_0(t)a^{d-1}b + \alpha_1(t) = 0\}\ .$$

The second projection $\pi : W \to V$ is a finite ramified cover, and the lift of the family P by π is a family of polynomials of degree d which is conjugated to a family of monic and centered polynomials $\tilde{P}_t(z) = z^d + \sum_{i=0}^{d-2} \tilde{\alpha}_{d-i}(t)z^i$ parametrized by W. The result then follows from the observation that the set of $t \in W$ such that $\mathrm{Card}(\Sigma(\tilde{P}_t))$ is divisible by m is equal to the union over all integers $\mu \leq d$ such that $m | d - \mu$ of the sets of $t \in W$ such that $\tilde{\alpha}_{d-i}(t) = 0$ when $i - \mu$ is not divisible by m. □

Remark 3.14. — Any polynomial family P parameterized by an algebraic variety V can be viewed as a polynomial $P \in K(V)[T]$ over the field $K(V)$. The above implies there exists a finite field extension $L|K(V)$ such that $\Sigma(P_L) \simeq \Sigma(P_t)$ for all t in a dense Zariski open set of V.

Proposition 3.15. — *For any degree $d \geq 3$, there exists a non-empty Zariski open subset of $U \subset \mathrm{MPoly}^d$ such that $\Sigma(P) = \{\mathrm{id}\}$ for all polynomials P of degree d whose conjugacy class lies in U.*

Remark 3.16. — Any quadratic polynomial P is conjugated to $z^2 + c$ for some $c \in K$, hence $\Sigma(P)$ has cardinality equal to 2 except if P is conjugated to a quadratic monomial map.

Proof. — The set of monic and centered polynomials $P(z) = z^d + a_2 z^{d-2} + \ldots + a_d$ having a trivial group of dynamical symmetries is Zariski open, since it contains all polynomials for which $\prod_{i=2}^d a_i \neq 0$. The proposition follows from Proposition 3.13, and the fact that the projection map from the space of monic and centered polynomials to MPoly^d is finite. □

3.3. Algebraic characterization of the group of dynamical symmetries

Let us first treat the case of a polynomial P defined over a number field. Recall the definition of the canonical height h_P from §2.6.

Proposition 3.17. — *Let $\mathbb{K}$ be a number field and pick $P \in \mathbb{K}[z]$ of degree $d \geq 2$ and $g \in \mathrm{Aff}(\mathbb{K})$. The following assertions are equivalent:*

1. *$g \in \Sigma(P)$,*
2. *$h_P(g \cdot z) = h_P(z)$ for all $z \in \bar{\mathbb{K}}$,*
3. *$g(\mathrm{Preper}(P, \bar{\mathbb{K}})) = \mathrm{Preper}(P, \bar{\mathbb{K}})$,*
4. *$g(\mathrm{Preper}(P, \bar{\mathbb{K}})) \cap \mathrm{Preper}(P, \bar{\mathbb{K}})$ is infinite.*

Proof. — Assume first $g \in \Sigma(P)$. Recall that $P^n(g \cdot z) = \rho^n(g) \cdot P^n(z)$ for all n. Pick any place $v \in M_{\mathbb{K}}$ and $g \in \Sigma(P)$. For any $z \in \bar{\mathbb{K}}$, we have

$$g_{P,v}(g \cdot z) = \lim_n \frac{1}{d^n} \log^+ |\rho^n(g) \cdot P^n(z)|_v = g_{P,v}(z) ,$$

so that the canonical heights are equal $h_P(g \cdot z) = h_P(z)$. This implies $g(\mathrm{Preper}(P, \bar{\mathbb{K}})) = \mathrm{Preper}(P, \bar{\mathbb{K}})$ and shows the sequence of implications (1) ⇒ (2) ⇒ (3) ⇒ (4).

Assume now that $g(\mathrm{Preper}(P, \bar{\mathbb{K}})) \cap \mathrm{Preper}(P, \bar{\mathbb{K}})$ is infinite and pick an infinite sequence of distinct preperiodic points z_n and $g \in \mathrm{Aff}(\mathbb{K})$ such that $g \cdot z_n$ is still preperiodic. Then the canonical heights of both z_n and $g \cdot z_n$ are zero and by Theorem 1.13 we get $g_* \mu_{v,P} = \mu_{v,P}$ for any place v. Choose any archimedean place v. The previous invariance implies that $g_{P,v}(g \cdot z) - g_{P,v}(z)$ is a harmonic function on $\mathbb{C}$ with at most logarithmic growth, hence is a constant. Since it is zero at all points z_n we get $g_{P,v}(g \cdot z) = g_{P,v}(z)$. We conclude that g belongs to $\Sigma(P)$ using the first point of Proposition 3.9. □

Using a specialization argument, we now extend the previous result to any field of characteristic zero (Theorem 3 from the Introduction).

Theorem 3.18. — *For any field K of characteristic zero and any $P \in K[z]$ of degree $d \geq 2$, the group $\Sigma(P)$ coincides with the set of $g \in \mathrm{Aff}(K)$ such that $g(\mathrm{Preper}(P, \bar{K})) \cap \mathrm{Preper}(P, \bar{K})$ is infinite.*

Remark 3.19. — Observe that [**5**, Theorem 5.7] is very similar to our theorem. It states that if P is defined over a number field, then $g \in \Sigma(P)$ iff P and $g \circ P$ have an infinite number of common preperiodic points.

Proof of Theorem 3.18. — One direction is clear since $P^n(g \cdot z) = \rho^n(g) \cdot P^n(z)$ so that $g(\mathrm{Preper}(P, \bar{K})) = \mathrm{Preper}(P, \bar{K})$.

Let us now focus on the converse implication. If K is a number field, this is part of the statement of Proposition 3.17, so we may assume that K has transcendence degree ≥ 1 over $\mathbb{Q}$. Let R be the $\bar{\mathbb{Q}}$-algebra of finite type generated over $\bar{\mathbb{Q}}$ by all coefficients of P. The spectrum of R is an affine variety X of positive dimension defined over $\bar{\mathbb{Q}}$ and the coefficients of P can be viewed as regular functions on X, so that P defines an algebraic family of degree d polynomials parameterized by X. Observe that any maximal ideal t of R corresponds to a point in $X(\bar{\mathbb{Q}})$, and that you can associate to t a polynomial P_t with coefficients in $\bar{\mathbb{Q}}$.

Pick any $g \in \mathbb{G}_a(K)$ such that $g(\mathrm{Preper}(P, \bar{K})) \cap \mathrm{Preper}(P, \bar{K})$ is infinite. We claim that g has finite order, and that there exists a unique automorphism $\rho(g) \in \mathrm{Aff}(K)$ such that

$$P(g \cdot z) = \rho(g) \cdot P(z) \ .$$

This claim implies that the group $G := \langle g \rangle \subset \mathrm{Aff}(K)$ generated by g is finite, and that the map $\rho(g^j) := \rho(g)^j$, $j \geq 0$ is a well-defined group homomorphism $\rho : G \to \mathrm{Aff}(K)$. One concludes $g \in \Sigma(P)$ by Proposition 3.6.

Let us prove this claim, and pick $g \in \mathrm{Aff}(K)$ and a sequence of distinct points $z_n \in \mathrm{Preper}(P, \bar{K})$ such that $g \cdot z_n$ is preperiodic. We may suppose that z_n and z_m are not $\mathrm{Gal}(\bar{K}/K)$-conjugated for any $n \neq m$.

The polynomial P induces an algebraic family of polynomials $(t, z) \in X \times \mathbb{A}^1 \mapsto (t, P_t(z)) \in X \times \mathbb{A}^1$. According to Proposition 3.13, and possibly replacing X by a Zariski open dense subset we have $\Sigma(P_t) \simeq \Sigma(P)$ for all $t \in X(\bar{\mathbb{Q}})$. Restricting again X if necessary, the affine automorphism g can also be identified with a family of affine maps

$$(t, z) \in X \times \mathbb{A}^1 \mapsto (t, g_t \cdot z) \in X \times \mathbb{A}^1$$

with $g_t \cdot z = \alpha(t)z + \beta(t)$, $\alpha, \beta \in \bar{\mathbb{Q}}[X]$ and α is not identically zero.

We fix any embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}$ so that P now induces a holomorphic family of complex polynomials parameterized by $X(\mathbb{C})$. By Theorem 2.33, we may find an open polydisk $\Delta \subset X(\mathbb{C})$ and a holomorphic motion $h\colon \Delta \times \mathbb{C} \to \Delta \times \mathbb{C}$ conjugating the dynamics, i.e. such that $P_t(h_t(z)) = h_t(P_{t_0}(z))$, and $h_{t_0} = \mathrm{id}$ (where $t_0 \in \Delta$ is any marked point).

Fix some integer n. The point z_n is defined over some finite extension of $K = \bar{\mathbb{Q}}(X)$, hence the field $\bar{\mathbb{Q}}(X)[z_n]$ is isomorphic to $\bar{\mathbb{Q}}(X)[Z]/(R_n)$ where $R_n \in \bar{\mathbb{Q}}(X)[Z]$ is the minimal polynomial of z_n. In particular z_n can be viewed as a rational function on the graph Y_n of R_n in $X \times \mathbb{P}^1$. Observe that the natural projection map $\pi_n\colon Y_n \dashrightarrow X$ defines a generically finite rational map so that one can find Zariski dense open subsets $V_n \subset X$, and $W_n \subset Y_n$ over which $\pi_n\colon W_n \to V_n$ defines a finite (proper) unramified cover. With these notations, $\pi_n(z_n(\tau))$ is a preperiodic point for $P_{\pi_n(\tau)}$ for all $\tau \in W_n$. Since V_n is Zariski dense, one can find a simply connected and connected open dense set $\Delta_n \subset \Delta \cap V_n$ (see [**22**]) and finitely many holomorphic functions $z_n^{(i)}\colon \Delta_n \to \mathbb{C}$ such that

$$F_n(t) := \{z_n(\tau),\, \tau \in \pi_n^{-1}(t)\} = \{z_n^{(i)}(t)\} \subset \mathrm{Preper}(P_t, \mathbb{C})$$

for all $t \in \Delta_n$.

Changing the base point if necessary, one can assume that $t_0 \in \Delta_n$. Since the holomorphic motion on Δ conjugates the dynamics of P_{t_0} with P_t and $z_n^{(i)}(t)$ is preperiodic, it follows that the holomorphic function $t \in \Delta \mapsto h_t(z_n^{(i)}(t_0))$ equals $z_n^{(i)}(t)$ on Δ_n. In other words, $z_n^{(i)}(t)$ can be continued analytically over Δ for all i.

Lemma 3.20. — *For any $n \neq m$, for any $t \in \Delta$ and for any i, j we have $z_n^{(i)}(t) \neq z_m^{(j)}(t)$.*

Proof. — Pick any parameter $t_* \in \Delta$ such that $z_n^{(i)}(t_*) = z_m^{(j)}(t_*)$ for some i, j. Since there is a holomorphic motion of the complex plane conjugating the dynamics on Δ, and the set of points whose orbit has a fixed cardinality is finite, it follows that $z_n^{(i)} = z_m^{(j)}$ in a neighborhood of t_*.

The Zariski closure of the set of points $\{(t, z_n^{(i)}(t)), t \in \Delta\}$ in the complex algebraic variety $X \times \mathbb{A}^1$ is equal to $\{R_n = 0\}$ since the latter is irreducible. It follows that $\{R_n = 0\} = \{R_m = 0\}$, hence the two points z_n and z_m are Galois conjugated in $\bar{K}$. This contradicts our standing assumption. □

Fix any $t \in \Delta$. The preceding lemma shows that the set $\{z_n^{(i)}(t)\}$ is infinite. Since $g_t \cdot z_n^{(i)}(t)$ is preperiodic by assumption, we conclude from Proposition 3.17

that g_t belongs to $\Sigma(P_t)$. By Proposition 3.6, g_t has finite order, and there exists an affine map $\rho_t(g_t) \cdot z = a(t)z + b(t)$ such that

$$P_t(g_t \cdot z) = \rho_t(g_t) \cdot P_t(z).$$

If we write $P_t(z) = \sum_{i=0}^{d} a_i(t)z^i$, then the preceding equation yields

$$a(t) = \alpha(t)^d \quad \text{and} \quad b(t) = P_t(\beta(t)) - \alpha(t)^d a_0(t) \ . \tag{25}$$

Now observe that α, β, a_i all define regular functions on X. We may thus set $a := \alpha^d$, and $b := P(\beta) - \alpha^d a_0$ which both belong to $\bar{\mathbb{Q}}[X] \subset K$, and we set $\rho(g) = az + b \in \mathrm{Aff}(K)$.

Since for all $t \in \Delta$, g_t has finite order, and Δ is a complex open subset of X, it follows that g has finite order in $\mathrm{Aff}(K)$. Similarly $P_t(g_t \cdot z) - \rho_t(g_t) \cdot P_t(z)$ vanishes on the complex open subset of points $(t, z) \in \Delta \times \mathbb{C} \subset X \times \mathbb{A}^1$ hence $P(g \cdot z) = \rho(g) \cdot P(z)$. The uniqueness of $\rho(g)$ follows from (25), and the proof of the Claim is complete. □

3.4. Primitive families of polynomials

In this section, we fix any field K of characteristic 0 (not necessarily algebraically closed).

Definition 3.21. — *A polynomial P of degree $d \geq 2$ defined over a field K is said to be* primitive *when the following holds. For any polynomial Q defined over an algebraic extension L of K, and for any $\sigma \in \Sigma_L(P)$ such that $P = \sigma \circ Q^{\circ n}$ for some integer $n \geq 1$, we have $n = 1$.*

Remark 3.22. — A polynomial P whose degree is not a power of an integer is primitive.

We also introduce the notion of weak primitivity. A polynomial P is weakly primitive iff $P = Q^{\circ n}$ implies $n = 1$. Any primitive polynomial is weakly primitive. However the two notions do not coincide in general as the next example shows.

Example 3.23. — Pick any $a \neq 0$, and set $Q(z) = z(z^2 + a)$. Observe that $\Sigma(Q) = \{\pm \mathrm{id}\}$. The polynomial $P_1 = Q^{\circ 2}$ is not primitive, but $P_2 = -Q^{\circ 2}$ is weakly primitive. Indeed if $P_2 = (Q_1)^{\circ 2}$ then $J(Q_1) = J(Q)$ hence $Q_1 = \pm Q$ by [**3**] which is absurd.

Example 3.24. — Observe that any polynomial of the form $P(z) = (z^2 + c)^2 + c = z^4 + 2cz^2 + c^2 + c$ for some $c \in K$ is not weakly primitive. Any other centered quartic polynomial is weakly primitive. Here is the complete list of monic and centered quartic polynomials which are not primitive:

– $P(z) = z^2(z^2 + c)$ with $c = 0$ or $c = -2\zeta$ and $\zeta^3 = -1$;
– $P(z) = z^4 + az^2 + c$ with $4c = a^2 - 2a\zeta$ and $\zeta^3 = -1$.

Definition 3.25. — *An algebraic family P parameterized by an irreducible algebraic variety V defined over a field K is primitive when the induced polynomial $P \in K(V)[z]$ is primitive over the field $K(V)$. A family parameterized by an arbitrary algebraic variety is primitive when its restriction to any irreducible components is primitive.*

Proposition 3.26. — *Let P be any algebraic family of polynomials parameterized by an irreducible algebraic variety Z defined over a field K.*

1. *Either the family is primitive, and the set of parameters $t \in Z$ such that P_t is primitive forms a Zariski open subset of V.*
2. *Or there exist an integer $n \geq 2$, a finite proper and surjective map $\pi\colon W \to Z$, a primitive family of polynomials Q and an algebraic family $\sigma_t \in \Sigma(P_{\pi(t)})$ both parameterized by W such that $P_{\pi(t)} = \sigma_t \circ Q_t^{\circ n}$ for all $t \in W$.*

This result relies on the properness of the composition map.

Proposition 3.27. — *The composition map $(P, Q) \mapsto P \circ Q$ induces a regular map $\Phi_{d,l}\colon \mathrm{Poly}^d_{\mathrm{mc}} \times \mathrm{Poly}^l_{\mathrm{mc}} \to \mathrm{Poly}^{dl}_{\mathrm{mc}}$ which is proper when $d, l \geq 2$.*

Proof. — We first need to check that the composition of two monic and centered polynomials remains monic and centered. This easy fact follows from the computation $P(z) = z^d + O(z^{d-2})$, $Q(z) = z^l + O(z^{l-2})$, $P \circ Q(z) = (z^d + O(z^{d-2}))^l + O(z^{l(d-2)}) = z^{dl} + O(z^{dl-2})$. To check the properness of the composition map, we first suppose $K = \mathbb{C}$. Write

$$P(z) = z^d + a_2 z^{d-2} + \cdots + a_d,\ Q(z) = z^l + b_2 z^{l-2} + \cdots + b_d$$

so that

$$\begin{aligned} P \circ Q(z) &:= z^{dl} + c_2 z^{dl-2} + \cdots + c_{dl} \\ &= (z^l + b_2 z^{l-2} + \cdots + b_d)^d + a_2 (z^l + b_2 z^{l-2} + \cdots + b_d)^{d-2} + \cdots + a_d\ . \end{aligned}$$

Choose any constant $C > 0$. Suppose by contradiction that we have a sequence of polynomials with coefficients $a_i^{(n)}, b_j^{(n)}$ such that $|c_l^{(n)}| \leq C$ for all n, but $\max\{|a_i^{(n)}|, |b_j^{(n)}|\} \to \infty$. If $\max\{|b_j^{(n)}|\} \to \infty$, choose i_0 minimal such that

$|b_{j_0}^{(n)}| \to \infty$. Since $l(d-1)+j_0 \geq l(d-2)$, we have

$$c_{l(d-1)+j_0}^{(n)} = d b_{j_0}^{(n)} + \text{ Polynomial in } b_2^{(n)}, \dots, b_{j_0-1}^{(n)}$$

which gives a contradiction. When $\sup_n \max\{|b_j^{(n)}|\} < \infty$, and $\max\{|a_i^{(n)}|\} \to \infty$, choose i_0 minimal with $|a_{i_0}^{(n)}| \to \infty$. This time

$$c_{l(d-i_0)}^{(n)} = a_{i_0}^{(n)} + \text{ Polynomial in } b_2^{(n)}, \dots, b_d^{(n)}, a_2^{(n)}, \dots, a_{i_0-1}^{(n)}$$

which again gives a contradiction.

Since the properness of a map is preserved by faithfully flat descent, see [**157**, Lemma 35.20.14], it follows that the composition map is proper when $K = \mathbb{Q}$. It follows that it remains proper over any field of characteristic zero by base change, see [**157**, Lemma 29.39.5]. □

Corollary 3.28. — *For any finite collection of integers $d_1, d_2, \dots, d_n \geq 2$, the composition map*

$$\Phi_{d_1,\dots,d_n} \colon \prod_i \mathrm{Poly}_{\mathrm{mc}}^{d_i} \to \mathrm{Poly}_{\mathrm{mc}}^{d_1 \cdots d_n},$$

sending $(P_1, \dots, P_n)$ to $\Phi_{d_1,\dots,d_n}(P_1, \dots, P_n) := P_1 \circ \cdots \circ P_n$ is finite (hence proper).

Proof. — Since $\mathrm{Poly}_{\mathrm{mc}}^d$ is affine, it is sufficient to prove that the map $\Phi_{d_1,\dots,d_n}$ is finite. We proceed by induction on n. The case $n = 2$ follows from the previous proposition. The induction step can be proved using the observation

$$\Phi_{d_1,\dots,d_n}(P_1, \dots, P_n) = \Phi_{d_1 \cdots d_{n-1}, d_n}\left(\Phi_{d_1,\dots,d_{n-1}}(P_1, \dots, P_{n-1}), P_n\right) .$$

and the fact that a composition of two proper maps remains proper. □

Remark 3.29. — It is *not* the case that the composition map $(P, Q) \mapsto P \circ Q$ is proper on $\mathrm{Poly}^d \times \mathrm{Poly}^l$. Indeed if α_t is a one-parameter subgroup of $\mathbb{G}_m$, e.g. $\alpha_t(z) = tz$, then $(P \circ \alpha_t^{-1}, \alpha_t \circ Q)$ is sent to the polynomial $P \circ Q$. However if P is monic and centered and $P'(0) \neq 0$, then over the complex $P \circ \alpha_t^{-1}$ diverges in the moduli space when $t \to 0$.

Remark 3.30. — Corollary 3.28 implies the iteration map $\Phi^n \colon \mathrm{MPoly}^d \to \mathrm{MPoly}^{d^n}$ defined by $\Phi^n(P) := P^{\circ n}$ to be finite. This iteration map is in fact proper (hence finite) on the moduli space of complex rational maps of a fixed degree by [**50**, Corollary 0.3]. A short argument in the moduli space of complex polynomials bypassing the computations above goes as follows. Recall that the critical Green function $G(P) = \max\{g_P(c),\ P'(c) = 0\}$ defines a proper function on Poly^d, see Proposition 2.13. Suppose $P^{\circ n}$ lies in a compact set A of Poly^{d^n}. Then we have $G(P) = G(P^{\circ n}) \leq \sup_A G =: C' < \infty$ so that P

belongs to $G^{-1}([0, C'])$ which is compact. This proves $\Phi^n\colon \mathrm{Poly}^d \to \mathrm{Poly}^{d^n}$ is proper.

Proof of Proposition 3.26. — For any l, we introduce the space $\widetilde{\mathrm{Poly}}^l_{\mathrm{mc}}$ of all centered polynomials of degree l whose dominant term is a root of unity of order $\leq d$. It is a disjoint union of copies of $\mathrm{Poly}^l_{\mathrm{mc}}$, and the n-th composition map $\Phi_n\colon \widetilde{\mathrm{Poly}}^l_{\mathrm{mc}} \to \widetilde{\mathrm{Poly}}^{l^n}_{\mathrm{mc}}$ is proper by Corollary 3.28.

Pick any irreducible subvariety Z of $\widetilde{\mathrm{Poly}}^d_{\mathrm{mc}}$ for some $d \geq 1$. We get a polynomial P_Z defined over the field $K(Z)$. Since P_Z is centered any of its symmetry is a multiplication by a root of unity of order $\leq d$. Moreover, given any $\sigma \in \Sigma(P_Z)$, the composition $\sigma \circ P_Z$ determines a family of centered polynomials hence an irreducible subvariety $\sigma(Z)$ in $\widetilde{\mathrm{Poly}}^d_{\mathrm{mc}}$.

For any (possibly reducible) subvariety V of $\widetilde{\mathrm{Poly}}^d_{\mathrm{mc}}$, we define

$$S(V) := \bigcup_{\sigma \in \Sigma(P_W)} \sigma(W)$$

where W ranges over all irreducible subvarieties of V.

Lemma 3.31. — *The set $S(V)$ is Zariski-closed.*

Proof. — Observe that the symmetry group of any polynomial $P \in \widetilde{\mathrm{Poly}}^d_{\mathrm{mc}}$ is a subgroup of $\mathbb{U}_{d!}$ except if $P = \zeta M_d$ in which case it is equal to $\mathbb{U}_\infty$. For each subgroup G of $\mathbb{U}_{d!}$, the space $W_G \subset \widetilde{\mathrm{Poly}}^d_{\mathrm{mc}}$ of polynomials such that $G \subset \sigma(P)$ is a closed subvariety, so that

$$S(V) = \bigcup_{G \subset \mathbb{U}_{d!}} \bigcup_{\sigma \in G} \sigma(W_G \cap Z).$$

This implies the claim. ☐

Observe that if a monic and centered polynomial P of degree $d \geq 2$ equals $\sigma \circ Q^{\circ n}$ for some $\sigma \in \Sigma(P)$ then $Q(T) = \zeta T^l + O(T^{l-2})$ where $l^n = d$, and ζ is a root of unity of order $\leq d$. It follows that the set of polynomials in $\widetilde{\mathrm{Poly}}^d_{\mathrm{mc}}$ which are not primitive is equal to

$$\bigcup_{l^n = d} S\left(\Phi^n(\widetilde{\mathrm{Poly}}^l_{\mathrm{mc}})\right) .$$

It is thus Zariski closed since the image of a closed set by a proper morphism remains closed. In particular the subset $\mathrm{Prim} \subset \mathrm{Poly}^d_{\mathrm{mc}}$ of primitive polynomials is Zariski open and dense in $\mathrm{Poly}^d_{\mathrm{mc}}$.

Pick now any family of polynomials of degree d parameterized by an irreducible algebraic variety V. By base change, we may suppose that P_t is monic and centered for all $t \in V$, so that we have an induced map $\pi\colon V \to \mathrm{Poly}^d_{\mathrm{mc}}$.

When $\pi(V)$ intersects Prim, then the set of parameters $t \in V$ such that P_t is primitive is equal to $\pi^{-1}(\mathrm{Prim})$ which is Zariski dense since V is irreducible. We claim that the family is primitive which implies (1). We argue by contradiction, and pick an algebraic extension $L/K(V)$ and a family of centered polynomials $Q \in L[z]$ whose dominant term is a root of unity such that $P = \sigma \circ Q^{\circ n}$ for some root of unity $\sigma \in \Sigma(P)$ and some $n \geq 2$. Since Q has finitely many coefficients, we can assume that $L/K(V)$ is finite, and find an irreducible algebraic variety W with a generically finite rational map $\pi\colon W \dashrightarrow V$ such that $L = K(W)$ and the field extension is induced by π. Reducing V and W to suitable Zariski open subset, we may assume that π is regular and surjective. We get a contradiction since for any $t \in V$ there exists $\tau \in W$ such that $P_t = \sigma \circ Q_\tau^{\circ n}$.

When $\pi(V)$ is disjoint from Prim, one can find a maximal integer $n \geq 2$ such that there exists $l \geq 2$ with $\pi(V) \subset \sigma \circ \Phi^n(\widetilde{\mathrm{Poly}}^l_{\mathrm{mc}})$ for some root of unity $\sigma \in \Sigma(P_{\pi(V)})$, and we can form the fiber product:

$$\begin{array}{ccc} W & \overset{\varpi}{\dashrightarrow} & \widetilde{\mathrm{Poly}}^l_{\mathrm{mc}} \\ {\scriptstyle \Psi^n}\Big\downarrow & & \Big\downarrow{\scriptstyle \sigma\circ\Phi^n} \\ V & \xrightarrow{\pi} & \mathrm{Poly}^d_{\mathrm{mc}} \end{array}$$

Concretely when K is algebraically closed the set of K-points of W is obtained as the set of pairs $\tau = (t, Q) \in V(K) \times \widetilde{\mathrm{Poly}}^l_{\mathrm{mc}}(K)$ such that $P_t = \sigma Q^n$.

In any case, since Φ^n is finite, the map $\Psi\colon W \to V$ is also finite and we get a family Q_τ of monic and centered polynomials of degree l parameterized by W such that $Q_\tau^n = \sigma \circ P_{\Psi(\tau)}$. Observe that $\varpi(W)$ cannot be included in

$$\bigcup_{\substack{j^m = l \\ m \geq 2}} S\left(\Phi^m(\mathrm{Poly}^j_{\mathrm{mc}})\right) \subset \mathrm{Poly}^l_{\mathrm{mc}}$$

since we chose n to be maximal. It follows from our previous arguments that there exists at least one irreducible component W' of W for which Q_τ is primitive for a Zariski dense open subset of $\tau \in W'$, and this family is primitive which proves (2). □

3.5. Ritt's theory of composite polynomials

In this section we review some aspects of Ritt's theory of decomposition of polynomials extended by Medvedev and Scanlon in [**126**] and further developed by Ghioca, Nguyen and his co-authors [**99, 101**], and Pakovich in a series of papers [**134, 135, 136**]. A modern account on the original approach of Ritt is described by Müller and Zieve in [**175**]. Ritt's theorems are proved over a field of arbitrary characteristic by Zannier in [**170**] (see also [**150**]).

As in the previous section the base field K is any field of characteristic zero.

3.5.1. Decomposability

We start with the following basic definition

Definition 3.32. — *A polynomial P of degree $d \geq 2$ is said to be* decomposable *if it may be written $P = Q \circ R$ with* $\deg(Q), \deg(R) \geq 2$*, and* indecomposable *otherwise.*

Remark 3.33. — If the degree of P is prime, then P is indecomposable. If P is indecomposable, then it is primitive in the sense of the previous section. Observe that an integrable map (i.e. $P = M_d$ or $\pm T_d$) is indecomposable iff d is prime.

It is easy to see that any polynomial admits a complete decomposition, i.e. can be written $P = P_1 \circ \cdots \circ P_s$ with $P_1, \ldots, P_s$ indecomposable. Complete decompositions are not unique, but Ritt described how to pass from one decomposition to another, see [**144**].

Let P and Q be two indecomposable polynomials. A Ritt move for (P, Q) is a pair of two indecomposable polynomials $(\bar{P}, \bar{Q})$ such that $P \circ Q = \bar{P} \circ \bar{Q}$. There is a short list of possible Ritt moves:

(M1) P and Q arbitrary and $\bar{P} = P \circ \sigma^{-1}$, $\bar{Q} = \sigma \circ Q$ for some affine map σ;

(M2) $P = \nu \circ z^s R^n(z) \circ \sigma_1^{-1}$, $Q = \sigma_1 \circ z^n \circ \mu$, and $\bar{P} = \nu \circ z^n \circ \sigma_2^{-1}$, $\bar{Q} = \sigma_2 \circ z^s R(z^n) \circ \mu$, where ν, σ_1, σ_2 and μ are affine, R is a polynomial, $n \geq 1$ and $s \geq 0$ are coprime;

(M3) $P = \nu \circ \pm T_m \circ \sigma_1^{-1}$, $Q = \sigma_1 \circ \pm T_n \circ \mu$, and $\bar{P} = \nu \circ \pm T_n \circ \sigma_2^{-1}$, $\bar{Q} = \sigma_2 \circ \pm T_m \circ \mu$ where n and m are coprime, and ν, σ_1, σ_2 and μ are affine.

Remark 3.34. — In (M2), the notation R^n denotes the n-th power of R not its n-th iterate.

Theorem 3.35. — *Any polynomial $P \in K[z]$ of degree ≥ 2 admits a complete decomposition $P = P_1 \circ \cdots \circ P_s$ with $P_1, \ldots, P_s$ indecomposable.*

Any other complete decomposition $P = Q_1 \circ \cdots \circ Q_s$ has the same cardinality and there exists a sequence of complete decompositions $P = P_1^{(i)} \circ \cdots \circ P_s^{(i)}$ such that $P_j^{(0)} = P_j$, $P_j^{(n)} = Q_j$ and the decomposition at step $(i+1)$ is obtained by applying a Ritt move to a pair of consecutive polynomials $P_{j_i}^{(i)}, P_{j_i+1}^{(i)}$ at step i.

Put it broadly, any two complete decompositions are connected by a sequence of Ritt moves. We shall call the number of factors in any complete decomposition of P its *complexity*.

Theorem 3.36. — *The complexity function is lower-semicontinuous for the Zariski topology in any algebraic family of polynomials. Moreover the set of conjugacy classes of indecomposable polynomials is an open and dense Zariski subset in* MPoly^d.

Proof. — It is only necessary to prove that the complexity function is lower semicontinuous on the space of monic and centered polynomials $\mathrm{Poly}^d_{\mathrm{mc}}$.

Lemma 3.37. — *Any monic and centered polynomial P admits a complete decomposition $P = P_1 \circ \ldots \circ P_s$ where $P_1, \ldots, P_s$ are again monic and centered.*

It follows that the set of polynomials $P \in \mathrm{Poly}^d_{\mathrm{mc}}$ whose complexity is larger than a fixed integer k is the union of the images under the composition map

$$\Phi_{d_1,\ldots,d_k}\left(\mathrm{Poly}^{d_1}_{\mathrm{mc}} \times \ldots \times \mathrm{Poly}^{d_k}_{\mathrm{mc}}\right) \subset \mathrm{Poly}^d_{\mathrm{mc}}$$

over all integers $d_1, \ldots, d_k \geq 2$, such that $d_1 \cdots d_k = d$. By Corollary 3.28 these images are Zariski closed in $\mathrm{Poly}^d_{\mathrm{mc}}$. This shows the lower semicontinuity of the complexity function.

Observe that $\mathrm{Poly}^d_{\mathrm{mc}}$ is an affine variety of dimension $d-1$ whereas for all k as above the dimension of $\Phi_{d_1,\ldots,d_k}\left(\mathrm{Poly}^{d_1}_{\mathrm{mc}} \times \ldots \times \mathrm{Poly}^{d_k}_{\mathrm{mc}}\right)$ is at most $\sum(d_i - 1) \leq kd/2^{k-1} - k \leq d - k < d - 1$. It follows that the set of monic and centered decomposable polynomials forms a strict algebraic subvariety of $\mathrm{Poly}^d_{\mathrm{mc}}$. This ends the proof. □

Proof of Lemma 3.37. — Let P be any monic an centered polynomial and choose an arbitrary complete decomposition $P = P_1 \circ \ldots \circ P_s$. Write

$$P_i(z) = a_i z^{d_i} + b_i z^{d_i - 1} + \mathrm{l.o.t.}$$

with $a_i \neq 0$. We first choose inductively dilatations $\nu_s(z) = a_s^{-1} z$, $\nu_{s-1}(z) = (a_{s-1} a_s^{d_{s-1}})^{-1} z$, etc, such that $\bar{P}_s = \nu_s \circ P_s$, and $\bar{P}_i := \nu_i \circ P_i \circ \nu_{i+1}^{-1}$ becomes monic for all $2 \leq i$. In this way, we obtain a complete decomposition $P = \bar{P}_1 \circ \cdots \circ \bar{P}_s$ for which all polynomials $\bar{P}_2, \ldots, \bar{P}_s$ are monic. A direct computation shows that the leading term of $\bar{P}_1$ should also be equal to 1 since P is monic.

Replacing P_i by $\bar{P}_i$ we may thus assume that $a_i = 1$ for all i in the expansion above. Next we choose inductively translations $\tau_1(z) = z + \gamma_1, \tau_2 = z + \gamma_2$, etc, so that $\bar{P}_1 = P_1 \circ \tau_1$, $\bar{P}_i = \tau_{i-1}^{-1} \circ P_i \circ \tau_i$ are centered for all $i \leq s-1$. If we write $\bar{P}_s = \tau_{s-1}^{-1} \circ P_s = z^{d_s} + \alpha z^{d_s - 1} + \text{l.o.t.}$, then we obtain

$$P(z) = \bar{P}_1 \circ \cdots \circ \bar{P}_s(z) = z^{d_1 \cdots d_s} + (d_1 \cdots d_{s-1})\alpha z^{d_1 \cdots d_s - 1} + \text{l.o.t.}$$

which implies $\alpha = 0$ since P is centered.

This concludes the proof of the lemma. □

3.5.2. Intertwined polynomials

We introduce the following terminology.

Definition 3.38. — *Let P and Q be two polynomials of the same degree.*

1. *We say that P and Q are semi-conjugate if there exists a polynomial π (possibly of degree 1) such that $\pi \circ P = Q \circ \pi$. We write $P \geq Q$, or $P \geq_\pi Q$ if we want to emphasize the semi-conjugacy.*
2. *We say that P and Q are strictly intertwined iff there exists a polynomial R such that $R \geq P$ and $R \geq Q$.*
3. *We say that P and Q are intertwined iff there exists a polynomial R and $n \geq 1$ such that $R \geq P^n$ and $R \geq Q^n$.*

We have the following basic observations (see [**136**, Theorem 4.4] for 2. and 3.).

Theorem 3.39. —
1. *Semi-conjugacy implies strict intertwining which implies intertwining.*
2. *The polynomial P is intertwined with M_d iff it is conjugated to M_d.*
3. *The polynomial P is intertwined with $\pm T_d$ iff it is conjugated to $\pm T_d$.*
4. *Two polynomials P and Q of the same degree are intertwined (resp. strictly intertwined) iff there exists an algebraic subvariety $Z \subset \mathbb{A}^2$ (resp. an irreducible subvariety $Z \subset \mathbb{A}^2$) whose projections to both axis are onto, and which is fixed by the map $(z, w) \mapsto (P(z), Q(w))$.*
5. *Intertwining defines an equivalence relation in the moduli space of polynomials of a fixed degree.*

As in [**99, 126**], we shall write $P \approx Q$ when P and Q are intertwined.

Remark 3.40. — There exists a semi-conjugacy $\pi(z) = z + \frac{1}{z}$ between M_d and T_d but this semi-conjugacy is not given by a polynomial.

Proof. — The first item is obvious.

Let P_* be an integrable polynomial, and let P be any polynomial satisfying $P_* \geq P$ so that $\pi \circ P_* = P \circ \pi$ for some polynomial π. We embed the defining field of the coefficients of P and π in the field of complex numbers. By the previous lemma, $\mathrm{Preper}(P, \mathbb{C}) = \pi(\mathrm{Preper}(P_*, \mathbb{C}))$ hence $J(P) \subset \mathbb{C}$ is smooth near any point outside finitely many exceptions, and P is integrable thanks to Theorem 2.16.

If $P_* = \pm T_d$, then $\pi^{-1}(K(P)) = [-2, +2]$ hence P is also equal to $\pm T_d$. If $P_* = M_d$, then P cannot be a Chebyshev polynomial since $K(M_d)$ has non-empty interior whereas $K(P) = J(P)$, hence $P = M_d$.

Suppose now that $P \geq P_*$ with P_* integrable. The same argument applies and show that P is monomial (resp. Chebyshev) when P_* is.

For the fourth item, one direction is clear. Indeed if $P \leq_\pi R$ and $Q \leq_\varpi R$ then the curve $Z = \{(\pi(\tau), \varpi(\tau)), \tau \in \mathbb{A}^1\}$ is fixed by (P, Q).

Suppose that Z is irreducible. Let $\bar{Z}$ be the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of Z, and $\mathsf{n}\colon \hat{Z} \to \bar{Z}$ its normalization. The restriction of map f to Z lifts to $\hat{Z}$ and defines a non-invertible map $g\colon \hat{Z} \to \hat{Z}$. Observe that $\mathcal{E} = \mathsf{n}^{-1}(\bar{Z} \setminus Z) \subset \hat{Z}$ is a finite totally invariant set of cardinality 1 or 2 hence $\hat{Z}$ is isomorphic to $\mathbb{P}^1$. When $\mathcal{E}$ is reduced to one point, the restriction of g to $\hat{Z} \setminus \mathcal{E} = \mathbb{A}^1$ is a polynomial say R, and the composition of n with the first (resp. the second) projection semi-conjugates R to P (resp. to Q) so that P and Q are strictly intertwined. When $\mathcal{E}$ has 2 points, g is a monomial map and (2) and (3) show that both P and Q are integrable and strictly intertwined.

The fifth item can be treated as follows. Suppose $S \geq_{\pi_1} P$, $S \geq_{\pi_2} Q$, and $T \geq_{\varpi_1} Q$, $T \geq_{\varpi_2} R$. The curve $C := \{\pi_2(y) = \varpi_1(z)\}$ in $\mathbb{A}^2$ is fixed by $\phi = (S, T)$. The projections $\pi_1, \varpi_2\colon C \to \mathbb{A}^1$ satisfy $\phi \geq_{\pi_1} P$ and $\phi \geq_{\varpi_2} R$, hence the result. □

Proposition 3.41. — 1. *The polynomials $P \circ g$ and P are intertwined for any $g \in \Sigma(P)$.*
2. *Two commuting polynomials are intertwined.*
3. *Let A and B be two arbitrary polynomials. Then $P = A \circ B$ and $Q = B \circ A$ are strictly intertwined.*

Lemma 3.42. — *Suppose $\pi \circ P = Q \circ \pi$ for some non-constant π. Then* $\mathrm{Preper}(P, \bar{K}) = \pi^{-1}(\mathrm{Preper}(Q, \bar{K}))$.

Proof. — Indeed one has $\pi \circ P^n(z) = Q^n \circ \pi(z)$ for all n so that z has a finite P-orbit iff $\pi(z)$ has a finite Q-orbit. The result follows from the fact that π is necessarily finite.[(1)] □

Proof of Proposition 3.41. — Let us prove 1. We may suppose that P is not integrable. Pick $g \in \Sigma(P)$. By Proposition 3.6 for any $n \geq 1$ there exists $g_n \in \Sigma(P)$ such that $(P \circ g)^n = g_n \circ P^n$. Since $\Sigma(P)$ is finite, it follows that we can find $n_1 > n_0$ such that $g_{n_1} = g_{n_0}$. Let Δ be the diagonal in $\mathbb{A}^2$, and define the map $f(z,w) = (P(z), P \circ \sigma(w))$. We have

$$f^n(\Delta) = \{(P^n(z), g_n \circ P^n(z)), z \in \mathbb{A}^1\}$$

so that $f^{n_1}(\Delta) = f^{n_0}(\Delta)$. In particular $\cup_{n\geq 0} f^n(\Delta)$ is an algebraic subvariety of $\mathbb{A}^2$ which is f-invariant, and Theorem 3.39 proves (1).

To prove (2), suppose that Q is a polynomial commuting with P (and of the same degree). As above, we may work over the field of complex numbers. The set of preperiodic points of P and Q are then equal which implies the Julia set of P and Q to coincide. By Theorem 3.8, and using the fact that P and Q are supposed to have the same degree, we conclude to the existence of $\sigma \in \mathrm{Aut}(J(P)) = \Sigma(P)$ such that $Q = \sigma \circ P$. It follows that Q is conjugated to $P \circ \sigma$ which is intertwined with P by (1).

Finally $B \circ P = Q \circ B$ so that $P \geq Q$ which proves (3). □

3.5.3. Uniform bounds and invariant subvarieties

It is a striking fact that one may obtain uniform bounds in the context of Ritt's theory when degrees are fixed. A first example of such bounds was given in [**175**, Theorem 1.4]. In this book, we shall use the next two results.

Theorem 3.43 **([99]).** — *For any two polynomials P, Q of the same degree $d \geq 2$ such that $P \approx Q$, there exists an integer $n \leq 2d^4$ such that P^n and Q^n are strictly intertwined.*

Theorem 3.44 **([136]).** — *For any integer d, there exists a constant $c(d)$ such that the following holds.*

For any polynomial P of degree $d \geq 2$, there exist a polynomial $P_{\min}$ of degree d and $\pi_{\min}$ of degree $\leq c(d)$, such that for any $Q \leq P$ there exist polynomials π, ϖ such that $P \geq_{\varpi} Q \geq_{\pi} P_{\min}$ and $\pi_{\min} = \pi \circ \varpi$.

Moreover the set of monic and centered polynomials $Q \leq P$ is finite of cardinality $\leq c_2(d)$ for some constant depending only on d.

(1) Two such polynomials are called congruent, see page 95 below.

This result implies the following characterization of invariant curves by product maps which is due to [**126**]. We sketch the proof given in [**136**] thereafter.

Theorem 3.45. — *Let P and Q be two non-integrable polynomials of the same degree $d \geq 2$.*

Let C be any algebraic irreducible curve in $\mathbb{A}^2$ which is invariant by the map $(x, y) \mapsto (P(x), Q(y))$. Then we can find two polynomials u, v whose degrees are coprime such that $C = \{u(x) = v(y)\}$ and which satisfy

$$P \circ u = u \circ R$$
$$Q \circ v = v \circ R$$

for some polynomial R.

Sketch of proof. — The normalization $\tilde{C}$ of C is a smooth affine curve over which $\phi(x, y) := (P(x), Q(y))$ induces a non-invertible finite surjective map. It follows that $\tilde{C}$ is either the affine line and ϕ is a polynomial, or $\tilde{C}$ is the punctured affine line and ϕ is a monomial map. The composition of the normalization map and the first projection semi-conjugate ϕ to P hence the latter case cannot appear, see Theorem 3.39.

We now apply Theorem 3.44 and set $R := \phi_{\min}$. We get the existence of two polynomials u and v such that $P \geq_u R$ and $Q \geq_v R$ such that $C = \{u(x) = v(y)\}$. Using Ritt's theorem (Theorem 3.54 below), one can argue that the degrees of u and v are coprime. □

3.5.4. Intertwining classes

Given any polynomial P of degree $d \geq 2$, we are interested in the description of the set of polynomials that are intertwined with P. Note that since any two conjugated polynomials are intertwined, it makes sense to consider the set of conjugacy classes in Poly_d that are intertwined with P.

More precisely, for any integer D we define $\mathrm{Inter}_D(P)$ to be the set of monic and centered polynomials of degree D such that $Q^m \approx P^n$ for some $n, m \geq 1$. Observe that $\mathrm{Inter}_D(P) = \varnothing$ whenever $D^m \neq d^n$ for all $n, m \geq 1$. We also set $\mathrm{Inter}(P) = \bigcup_{D \geq 1} \mathrm{Inter}_D(P)$.

Theorem 3.46. — *There exists a constant $C = C(d)$ such that for any polynomial P of degree d, we have $\# \mathrm{Inter}_D(P) \leq C$ for all D.*

Remark 3.47. — The proof actually shows that if P has coefficients in $\bar{\mathbb{Q}}$ then any $Q \approx P$ does. Using the critical height h_{bif} defined on p.66, and

Misiurewicz-Prytycky's formula (9), it also implies that if $Q \approx P$ then Q is PCF iff P is.

Proof. — We begin with two lemmas. Recall the definition of the Lyapunov exponent from §2.3.

Lemma 3.48. — *If $P \approx Q$, and both polynomials are defined over some metrized field K, then* $\mathrm{Lyap}(P) = \mathrm{Lyap}(Q)$.

Proof. — Indeed suppose $\pi \circ Q = P \circ \pi$ for some polynomial π. Since $\pi^*\mu_P = \Delta(g_P \circ \pi) = \deg(\pi) \cdot \Delta(g_Q) = \deg(\pi) \cdot \mu_Q$, we have $\pi_*\mu_Q = \mu_P$. It follows that

$$|(P^n)' \circ \pi| = |\pi' \circ Q^n| \cdot |(Q^n)'| \cdot |\pi'|^{-1}$$

for all n, hence

$$\mathrm{Lyap}(P) = \lim_n \frac{1}{n} \int \log |(P^n)' \circ \pi| \, d\mu_Q =$$

$$\lim_n \frac{1}{n} \int \log |\pi' \circ Q^n| \, d\mu_Q + \lim_n \frac{1}{n} \int \log |(Q^n)'| \, d\mu_Q + \lim_n \frac{1}{n} \int \log |\pi'|^{-1} \, d\mu_Q$$

and the result follows. □

Lemma 3.49. — *For any integer $N \geq 2$, there exists a constant $C_1(d, N)$ (independent on P) such that the set of monic and centered polynomials Q such that $\pi \circ Q = P \circ \pi$ for some polynomial π with* $\deg(\pi) \leq N$ *is finite of cardinality $\leq C_1(d, N)$.*

Proof. — Write $P(T) = T^d + \sum_{j=0}^{d-2} p_{d-j} T^j$. We look for polynomials $Q(T) = T^d + \sum_{j=0}^{d-2} q_{d-j} T^j$ such that there exists $\pi = \zeta T^N + \pi_2 T^{N-2} + \cdots + \pi_N$ with $\zeta^{d-1} = 1$, and $\pi \circ Q(T) = P \circ \pi(T)$. Identifying the terms in T^{Nd-j} with $j = 2, 3, \ldots, N$ yields equations of the form $\zeta^{d-1} d \pi_j = L_j(\pi_2, \cdots, \pi_{j-1}, q_2, \cdots, q_d)$ with L_j a polynomial with integral coefficients so that $\pi_2, \ldots, \pi_N$ can be expressed as polynomials in the variables $q_2, \ldots, q_N$. It follows that the equation $\pi \circ Q(T) = P \circ \pi(T)$ is equivalent to the vanishing of the coefficients in T^j from $j = 0$ to $j = Nd - N - 1$, hence of $Nd - N$ polynomials in the variables $q_2, \ldots, q_N$ of degree depending only on d and N. We get that $\{Q, \pi \circ Q = P \circ \pi, \deg(\pi) \leq N\}$ is an algebraic subvariety defined by the intersection of C_1 hypersurfaces of degree $\leq C_2$ where C_1, C_2 are constants depending only on d and N. But Lemma 3.48 shows that over the field of complex numbers, this variety is bounded, since the Lyapunov function is proper on the space of monic and centered polynomials by (9) and Proposition 2.13. It is hence finite, and the result follows from [**86**, Theorem 12.3]. □

Now fix any monic and centered polynomial P of degree d and choose an embedding of the defining field of P in $\mathbb{C}$. Pick any monic and centered polynomial $Q \approx P$. By Theorem 3.43, there exists an integer $n \leq 2d^4$ and a polynomial R such that $R \geq P^n$ and $R \geq Q^n$.

By Theorem 3.44, there exists S of degree $\leq c(d^{2d^4})$ such that $S \leq R$ and S is universal for R, so that we may find polynomials π, ϖ with $\deg(\pi), \deg(\varpi) \leq c(d^{2d^4})$ such that $R \geq_\pi P^n$ and $R \geq_\varpi Q^n$.

By Lemma 3.49, there are at most $C_1(d^{2d^4}, c(d^{2d^4}))$ possibilities for R, and for each R at most $c_2(d^{2d^4})$ for Q by Theorem 3.44. This ends the proof. □

3.5.5. Intertwining classes of a generic polynomial

For convenience, we say that P is pseudo-integrable if it is of the form $P = T_d, M_d, T^s R(T^n)$ or $T^s R^n(T)$ for some $n \geq 2$. Also write $P \sim Q$ if there exist two affine maps such that $P = \sigma \circ Q \circ \tau$.

We say that P has an integrable (resp. pseudo-integrable) factor if it admits a complete decomposition $P = P_1 \circ \cdots \circ P_s$ for which one of the factor satisfies $P_i \sim Q$ where Q is integrable (resp. pseudo-integrable).

Proposition 3.50. — *– If P has an (pseudo)-integrable factor, then any complete decomposition contains an (pseudo)-integrable polynomial (up to composition by affine maps).*

- *A polynomial P has a (pseudo)-integrable factor iff one of its iterate has a (pseudo)-integrable factor.*
- *A polynomial P without any pseudo-integrable factor has a trivial group of dynamical symmetries.*

Proof. — Recall that any two complete decompositions are related by Ritt's moves. The first item follows from the observation that any non trivial move (not of the form (M1)) involves an integrable polynomial.

Observe that if $P = P_1 \circ \cdots \circ P_s$ is a complete decomposition, then $P^n = (P_1 \circ \cdots \circ P_s)^{\circ n}$ is also a complete decomposition. Indeed if it were not, one of the polynomial P_i would be decomposable. It follows that P has no (pseudo)-integrable factor iff P^n does.

Take any complete decomposition $P = P_1 \circ \cdots \circ P_s$ and suppose P has no pseudo-integrable factors. If $\Sigma(P)$ is non-trivial, then there exist affine maps g, g' such that $(g' \circ P_1) \circ \cdots \circ P_s = P_1 \circ \cdots \circ (P_s \circ g)$. By Ritt's theorem, these two decompositions are connected by a sequence of Ritt moves which are necessarily of the type (M1) since P has no pseudo-integrable factors. It follows that $g'' \circ P_s = P_s \circ g$ for some affine g'' hence P_s has a non-trivial group of symmetries, and P_s is pseudo-integrable. This yields a contradiction. □

In degree 2, any intertwining class is trivial. This was shown by Ghioca, Nguyen, and Ye [**101**, Theorem 1.4].

Theorem 3.51 **([101]).** — *Two quadratic polynomials $T^2 + c$ and $T^2 + c'$ are intertwined iff $c = c'$.*

In higher degree, we have the following result.

Theorem 3.52. — *Suppose that P is an indecomposable polynomial with no pseudo-integrable factor. Then any $Q \in \mathrm{Inter}(P)$ is conjugate to an iterate of P.*

In particular for any $d \geq 3$, the set of polynomials P such that $\mathrm{Inter}_d(P)$ is reduced to a single conjugacy class is a Zariski open dense subset of Poly_d.

Proof. — We rely on the following lemma.

Lemma 3.53. — *Suppose that P has no pseudo-integrable factors. If $Q \geq P$ or $Q \leq P$, then there exists some integer n such that $Q^n = U \circ V$ and $P^n = V \circ U$ for some polynomials U and V.*

Following the terminology of [**136**], we shall say that P and Q are congruent when the conclusion of the lemma is satisfied.

Suppose that $Q \in \mathrm{Inter}(P)$. By definition, one can find a polynomial R such that $R \geq Q^m$ and $R \geq P^n$ for some $n, m \geq 1$. By the previous lemma, R and P^n are congruent (maybe after increasing n) so that $R = U \circ V$ and $P^n = V \circ U$. Since P is indecomposable and has no pseudo-integrable factor, any complete decomposition of P^n is trivial. It follows that $V = P^k \circ \sigma$ and $U = \sigma^{-1} \circ P^{n-k}$ with $k \geq 0$ and σ affine, hence R is conjugated to P^n, and $P^n \geq Q^m$. The same argument implies that Q^m and P^n are conjugated.

Since P has no non-trivial symmetries and is indecomposable, it is also primitive, and it follows from Theorem 3.8 that P and Q are conjugated.

The last statement follows from Theorem 3.36. □

The proof of Lemma 3.53 relies on two fundamental theorems in Ritt's theory that we now recall. Fix four polynomials A, B, C, D such that $D \circ B = A \circ C$, i.e. the following diagram is commutative

$$\begin{array}{ccc} \mathbb{A}^1 & \xrightarrow{B} & \mathbb{A}^1 \\ {\scriptstyle C}\downarrow & & \downarrow{\scriptstyle D} \\ \mathbb{A}^1 & \xrightarrow[A]{} & \mathbb{A}^1 \end{array}$$

Theorem 3.54 **(Reduction, [72]).** — *There exist two polynomials* U *and* V *such that* $A = U \circ \tilde{A}$, $D = U \circ \tilde{D}$, $B = \tilde{B} \circ V$, *and* $C = \tilde{C} \circ V$,

$$\deg(U) = \gcd\{\deg(A), \deg(D)\} \text{ and } \deg(V) = \gcd\{\deg(B), \deg(C)\}$$

and $\tilde{D} \circ \tilde{B} = \tilde{A} \circ \tilde{C}$.

Theorem 3.55 **(Solution in the primitive case, [144])**
Suppose that $\gcd\{\deg(A), \deg(D)\} = \gcd\{\deg(B), \deg(C)\} = 1$. *Then we are in one of the following cases:*

1. $B \sim z^s R(z^n)$, $A \sim z^s R^n(z)$, $C \sim D \sim z^n$ *with* $\gcd\{n, s\} = 1$;
2. $C \sim z^s R(z^n)$, $D \sim z^s R^n(z)$, $A \sim B \sim z^n$ *with* $\gcd\{n, s\} = 1$;
3. $A \sim B \sim T_n$, $C \sim D \sim T_m$ *with* $\gcd\{n, m\} = 1$.

Proof of Lemma 3.53. — Suppose that $Q \geq_\pi P$. Observe that if $\pi = \pi' \circ Q$ for some polynomial π', then we may replace π by π'. We may thus assume that π is not a polynomial in Q.

Write $\deg(\pi) = l \times b$ where $l = \prod_{p \wedge d = 1} p^{v_p(\deg(\pi))}$, and pick n minimal such that b divides d^n. By the reduction Theorem 3.54, we can write

$$P^n = U \circ P_0, \pi = U \circ \pi_0$$
$$Q^n = Q_1 \circ V, \pi = \pi_1 \circ V$$

with $\deg(U) = \deg(V) = \gcd\{\deg(P)^n, \deg(\pi)\} = \gcd\{d^n, l \times b\} = b$, and $\pi_0 \circ Q_1 = P_0 \circ \pi_1$. Note that $\deg(P_0) \geq 2$ since otherwise we would have $\deg(Q_1) = \deg(P_0) = 1$, and π would be a polynomial in Q^n.

Observe that $\gcd\{\deg(P_0), \deg(\pi_0)\} = \gcd\{\deg(Q_1), \deg(\pi_1)\} = 1$. Apply Theorem 3.55 to the quadruple P_0, Q_1, π_0, π_1. Since P_0 is a factor of P^n, and P^n has no pseudo-integrable factor, we fall into case 1: π_0 and π_1 have thus degree 1, and P^n and Q^n are congruent.

The same argument applies when $Q \leq_\pi P$. □

3.6. Stratification of the parameter space in low degree

The five tables below summarize the stratification of the space of monic and centered polynomials of degree ≤ 6 in terms of the size of its group of dynamical symmetries. All computations are done over $\mathbb{C}$ (but the results are valid over any algebraically closed field of characteristic 0).

For any triple (d,k,μ) with $k \geq 2$ and $\mu \leq d-2$, we let $\Sigma(d,k,\mu)$ be the set of monic and centered polynomial of degree d which can be written under the form $z^\mu Q(z^k)$ with $Q(0) \neq 0$ and k maximal. It is an open Zariski-dense subset of a linear subspace of $\mathbb{C}^{d-1}$ of dimension $\frac{d-\mu}{k}$.

Recall that

$$\begin{aligned} \mathrm{Aut}(P) &= \{g \in \Sigma(P), gP = Pg\}; \\ \Sigma_0(P) &= \{g \in \Sigma(P), P^n g = P^n \text{ for some } n\}. \end{aligned}$$

The numbers is the colums "complexity" corresponds to the number of polynomials appearing in some/any decomposition $P = P_1 \circ \cdots \circ P_s$ with P_s indecomposable, see Theorem 3.35. This number is always 1 when the degree of P is prime.

$P(z) = z^2 + c$				
Range	Domain	$\mathrm{Aut}(P)$	$\Sigma(P)$	$\Sigma_0(P)$
$c \neq 0$	$\mathbb{C}^*$	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_2$
z^2	$\{0\}$	$\mathbb{U}_1$	$\mathbb{U}_\infty$	$\mathbb{U}_{2^\infty}$

$P(z) = z^3 + az + b$				
Range	Domain	$\mathrm{Aut}(P)$	$\Sigma(P)$	$\Sigma_0(P)$
$ab \neq 0$	$(\mathbb{C}^*)^2$	$\mathbb{U}_1$	$\mathbb{U}_1$	$\mathbb{U}_1$
$\Sigma(3,3,0)$: $z^3 + b$	$\mathbb{C}^*$	$\mathbb{U}_1$	$\mathbb{U}_3$	$\mathbb{U}_3$
$\Sigma(3,2,1)$: $z(z^2 + a)$	$\mathbb{C}^*$	$\mathbb{U}_2$	$\mathbb{U}_2$	$\mathbb{U}_1$
z^3	$\{0\}$	$\mathbb{U}_2$	$\mathbb{U}_\infty$	$\mathbb{U}_{3^\infty}$

$P(z) = z^4 + az^2 + bz + c$						
Range	Domain	$\mathrm{Aut}(P)$	$\Sigma(P)$	$\Sigma_0(P)$	Complexity	Primitivity
No symmetry	U_4	$\mathbb{U}_1$	$\mathbb{U}_1$	$\mathbb{U}_1$	1	Yes
$\Sigma(4,2,2)$: $z^2(z^2 + a)$	$\mathbb{C}^*$	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_2$	2	iff $a \neq -2\zeta$, $\zeta^3 = -1$
$\Sigma(4,3,1)$: $z(z^3 + b)$	$\mathbb{C}^*$	$\mathbb{U}_3$	$\mathbb{U}_3$	$\mathbb{U}_1$	1	Yes
$\Sigma(4,2,0)$: $z^4 + az^2 + c$	$(\mathbb{C}^*)^2$	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_2$	2	iff $4c \neq a^2 - 2\zeta a$, $\zeta^3 = -1$
$\Sigma(4,4,0)$: $z^4 + c$	$\mathbb{C}^*$	$\mathbb{U}_1$	$\mathbb{U}_4$	$\mathbb{U}_4$	2	Yes
z^4	$\{0\}$	$\mathbb{U}_3$	$\mathbb{U}_\infty$	$\mathbb{U}_{4^\infty}$	2	No

$$U_4 = \mathbb{C}^3 \setminus \{b = 0\} \cup \{a = c = 0\}$$

$P(z) = z^5 + az^3 + bz^2 + cz + d$				
Range	Domain	Aut(P)	$\Sigma(P)$	$\Sigma_0(P)$
No symmetry	U_5	$\mathbb{U}_1$	$\mathbb{U}_1$	$\mathbb{U}_1$
$\Sigma(5,2,3)$: $z^3(z^2+a)$	$\mathbb{C}^*$	$\mathbb{U}_2$	$\mathbb{U}_2$	$\mathbb{U}_1$
$\Sigma(5,3,2)$: $z^2(z^3+b)$	$\mathbb{C}^*$	$\mathbb{U}_1$	$\mathbb{U}_3$	$\mathbb{U}_1$
$\Sigma(5,2,1)$: $z(z^4+az^2+c)$	$(\mathbb{C}^*)^2$	$\mathbb{U}_2$	$\mathbb{U}_2$	$\mathbb{U}_1$
$\Sigma(5,4,1)$: $z(z^4+c)$	$\mathbb{C}^*$	$\mathbb{U}_4$	$\mathbb{U}_4$	$\mathbb{U}_1$
$\Sigma(5,5,0)$: z^5+d	$\mathbb{C}^*$	$\mathbb{U}_1$	$\mathbb{U}_5$	$\mathbb{U}_5$
z^5	$\{0\}$	$\mathbb{U}_4$	$\mathbb{U}_\infty$	$\mathbb{U}_{5^\infty}$

$$U_5 = \mathbb{C}^4 \setminus \{b=d=0\} \cup \{a=c=d=0\} \cup \{a=b=c=0\}$$

$P(z) = z^6 + az^4 + bz^3 + cz^2 + dz + e$					
Range	Domain	Aut(P)	$\Sigma(P)$	$\Sigma_0(P)$	Complexity
No symmetry	U_6	$\mathbb{U}_1$	$\mathbb{U}_1$	$\mathbb{U}_1$	2 iff $4c=a^2$ and $ab=2d$
$\Sigma(6,2,4)$: $z^4(z^2+a)$	$\mathbb{C}^*$	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_2$	2
$\Sigma(6,3,3)$: $z^3(z^3+b)$	$(\mathbb{C}^*)^2$	$\mathbb{U}_1$	$\mathbb{U}_3$	$\mathbb{U}_3$	2
$\Sigma(6,4,2)$: $z^2(z^4+c)$	$\mathbb{C}^*$	$\mathbb{U}_1$	$\mathbb{U}_4$	$\mathbb{U}_4$	2
$\Sigma(6,2,2)$: $z^2(z^4+az^2+c)$	$(\mathbb{C}^*)^2$	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_2$	2
$\Sigma(6,5,1)$: $z(z^5+d)$	$\mathbb{C}^*$	$\mathbb{U}_5$	$\mathbb{U}_5$	$\mathbb{U}_1$	1
$\Sigma(6,3,0)$: z^6+bz^3+e	$(\mathbb{C}^*)^2$	$\mathbb{U}_1$	$\mathbb{U}_3$	$\mathbb{U}_3$	2
$\Sigma(6,2,0)$: $z^6+az^4+cz^2+e$	$U_{6,2,0}$	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_2$	2
$\Sigma(6,6,0)$: z^6+e	$\mathbb{C}^*$	$\mathbb{U}_1$	$\mathbb{U}_6$	$\mathbb{U}_6$	2
z^6	$\{0\}$	$\mathbb{U}$	$\mathbb{U}_\infty$	$\mathbb{U}_{6^\infty}$	2

$$U_6 = \mathbb{C}^5 \setminus \{b=d=0\} \cup \{a=c=d=e=0\} \cup \{a=b=c=e=0\}$$
$$U_{6,2,0} = \mathbb{C}^3 \setminus \{a=0\} \cup \{a=c=d=e=0\} \cup \{a=b=c=e=0\}$$

3.7. Open problems

Many questions about the intertwining relation remain unclear. We have selected a few below.

(TW1) Prove that strict intertwining does not define an equivalence relation. In other words, there exist polynomials of the same degree P, Q, R such that $\phi \geq P, \phi \geq R$ and $\varphi \geq R, \varphi \geq Q$ for some ϕ, φ but no polynomial Φ satisfies $\Phi \geq P, \Phi \geq Q$.

(TW2) Is it true that the smallest equivalence relation generated by strict intertwining is strictly weaker than the intertwining relation? Equivalently, does there exist two polynomials $P \approx Q$ such that there exists no sequence of polynomials P_i, ϕ_i with $P = P_0$, $Q = P_N$ such that $\phi_i \geq P_i$, and $\phi_i \geq P_{i-1}$ for all i.

(TW3) Suppose that $P \circ g$ and P are intertwined, and P is not integrable. Is it true that $g \in \Sigma(P)$?

(TW4) Fix any polynomial P. Design an algorithm to determine all monic and centered polynomial of a fixed degree such that $Q \approx P$.

(TW5) Describe all intertwining classes for any polynomial of low degree, say $d = 3, 4, 5$.

(TW6) Is the intertwining class of any unicritical polynomial trivial? Observe that [**101**, Theorem 1.4] proves that two unicritical polynomials are intertwined iff they are conjugated. Is the intertwining class of any PCF polynomial trivial?

We would like to ask for a uniform boundedness result.

Conjecture 1. — *For any $d \geq 2$, there exists a constant $c(d)$ such that for any polynomial P of degree d, one can find $N \leq c(d)$ polynomials $Q_1, \cdots, Q_N$ such that $Q \in \mathrm{Inter}(P)$ implies Q to be conjugated to $g \circ Q_i^n$ for some integer $n \geq 1$ and some $g \in \Sigma(Q_i)$.*

The problem is open even for $d = 2$.

Finally observe that for any two complex polynomials such that $P \approx Q$, there exists a local biholomorphism σ defined on an open disk U such that $\sigma(J(P) \cap U) = J(Q) \cap \sigma(U)$. Since Julia sets determine polynomials up to symmetry (by Theorem 3.8) and since by invariance under the dynamics the shape of Julia set is locally the same near any of its points, it is natural to expect the following to be true.

Conjecture 2. — *Suppose that there exists a univalent map $\sigma\colon U \to \mathbb{C}$ such that $\sigma(J(P) \cap U) = J(Q) \cap \sigma(U) \neq \varnothing$. Then $P \approx Q$.*

Several partial results in that direction have been already obtained, see [**119, 34**]. Recently, together with Dujardin, the authors proved Conjecture 2 for semi-hyperbolic polynomials [**70**]. The general case remains elusive.

CHAPTER 4

POLYNOMIAL DYNAMICAL PAIRS

A polynomial dynamical pair (P, a) (or simply a dynamical pair) is a family of polynomials together with a marked point. We first review basic notions of bifurcation and activity for holomorphic dynamical pairs, and prove the following important rigidity property when the bifurcation locus is included in a smooth real curve.

Theorem A. — *Let (P, a) be a polynomial dynamical pair of degree $d \geq 2$ parametrized by a connected Riemann surface S. Assume that $\mathrm{Bif}(P, a)$ is non-empty and included in a smooth real curve. Then one of the following holds:*

- *either P_t is conjugated to M_d or $\pm T_d$ for all $t \in S$;*
- *or there exists a univalent map $\imath \colon \mathbb{D} \to S$ such that $\imath^{-1}(\mathrm{Bif}(P, a))$ is a non-empty closed and totally disconnected perfect subset of the real line and the pair $(P \circ \imath, a \circ \imath)$ is conjugated to a real family over $\mathbb{D}$.*

In §4.2, we turn to algebraic dynamical pairs. We first explain how to attach a canonical line bundle to such a pair, and discuss the continuity of the Green function associated to a non-isotrivial pair (Theorem 4.22) which turns out to be a key technical point for applications. We recall DeMarco's theorem (Theorem 4.30) stating that a non isotrivial complex algebraic dynamical pair admits bifurcations.

We conclude this chaper by discussing in §4.4 dynamical pairs defined over a number field and prove that they induce a natural height arising from an adelic semi-positive metrization of a suitable divisor. This allows us to characterize isotrivial adelic pairs in terms of their height function.

4.1. Holomorphic dynamical pairs and proof of Theorem A

In this section, we prove a rigidity property for holomorphic dynamical pairs parametrized by the unit disk whose bifurcation locus is included in a smooth curve (Theorem 4.10). This result plays an important role in the proof of our main results.

4.1.1. Basics on holomorphic dynamical pairs

Let V be any complex manifold.

Definition 4.1. — *A* holomorphic polynomial dynamical pair (P, a) *of degree $d \geq 2$ parametrized by V is a holomorphic family $P : V \times \mathbb{C} \to V \times \mathbb{C}$ of degree d polynomials together with a holomorphic map $a \colon V \to \mathbb{C}$.*

We shall be concerned only with families of polynomials in the rest of the book. When we refer to a dynamical pair, this always means a polynomial dynamical pair.

We say that a family P parametrized by V is *isotrivial* if it has dimension 0 in moduli, i.e. if there exists a finite branched cover $\pi \colon \tilde{V} \to V$ and a holomorphic family of affine transformations $\phi_t \in \mathrm{Aut}(\mathbb{C})$ parametrized by $\tilde{V}$ such that $\phi_t^{-1} \circ P_{\pi(t)} \circ \phi_t$ is independent of t.

We also say that a dynamical pair (P, a) parametrized by V is *isotrivial* if there exists a finite branched cover $\pi \colon \tilde{V} \to V$ and a holomorphic family of affine transformations $\phi_t \in \mathrm{Aut}(\mathbb{C})$ parametrized by $\tilde{V}$ such that $\phi_t \circ P_{\pi(t)} \circ \phi_t^{-1}$ is independent of t and $\phi_t(a(\pi(t)))$ is constant.

Observe that when the pair (P, a) is isotrivial, then P is too, but the converse is not true. Recall that $\mathrm{Preper}(P, a)$ be the set of parameters $t \in V$ such that $a(t)$ is preperiodic for P_t.

We shall mostly be interested in the case where the parameter space is a Riemann surface. We thus pick any (connected) Riemann surface S and let (P, a) be a dynamical pair of degree d parametrized by S.

Observe that the set $\mathrm{Preper}(P, a) = \bigcup_{n > m \geq 0} \{t \in S,\ P_t^n(a(t)) = P_t^m(a(t))\}$ is either equal to S, or is at most countable.

We say the pair (P, a) is *stable* at a parameter $t_0 \in S$ if there exists an open set $U \subset S$ with $t_0 \in S$ and such that the sequence of holomorphic maps $t \in U \longmapsto P_t^n(a(t))$ forms a normal family on U. The set of stable parameters is an open set called the *stability locus* whose complement is the *bifurcation locus* denoted by $\mathrm{Bif}(P, a)$.

Proposition 4.2. — *For any any holomorphic dynamical pair* (P, a) *parametrized by a Riemann surface* S*, the function*

$$g_{P,a}(t) := g_{P_t}(a(t)),\ t \in S$$

is continuous and subharmonic.

Moreover the boundary of $\{g_{P,a} = 0\}$ *coincides with the support of the positive measure* $\Delta g_{P,a}$*, which in turn equals* $\mathrm{Bif}(P, a)$*. In particular,* $\mathrm{Bif}(P, a)$ *is closed and perfect, i.e. has no isolated point, and has empty interior.*

Remark 4.3. — In particular, the stability locus of any holomorphic dynamical pair is open and dense.

Definition 4.4. — *The* bifurcation measure *of the dynamical pair* (P, a) *is the positive measure* $\mu_{P,a} := \Delta g_{P,a}$ *on* S*.*

Proof. — Recall that $g_t(z)$ is the uniform limit on the product of any compact subset of S with the complex plane of the sequence of continuous psh functions $\frac{1}{d^n}\log^+|P_t^n(z)|$, so that $g_{P,a}(t)$ is continuous and psh.

We have $g_{P,a}(t) \geq 0$ and $g_{P,a}(t) > 0$ iff $a(t) \notin K(P_t)$ so that $g_{P,a}(t) = \lim_n \frac{1}{d^n}\log|P_t^n(z)|$ when $g_{P,a}(t) > 0$ which implies the harmonicity of $g_{P,a}$ on $\{g_{P,a} > 0\}$. It follows from the maximum principle that the support of $\mu_{P,a}$ is equal to the boundary of $\{g_{P,a} = 0\}$.

Pick a point t_0 in the support of $\mu_{P,a}$. Then the sequence of functions $t \mapsto P_t^n(a(t))$ cannot be normal near t_0 since $g_{P,a}(t) > 0$ for some point close to t_0 hence $P_t^n(a(t)) \to \infty$ whereas $P_t^n(a(t_0))$ remains bounded. When t_0 is not in the support of $\mu_{P,a}$, then either $g_{P,a}(t) > 0$ and $P^n(a(t)) \to \infty$ uniformly in a neighborhood of t_0; or $g_{P,a}(t) = 0$ on a small disk containing t_0 and $P^n(a(t))$ takes its value in a fixed compact set thereby being normal by Montel's theorem.

Finally, since $\mathrm{Bif}(P, a) = \partial\{g_{P,a} = 0\}$ it is closed and has empty interior. If $t_0 \in \mathrm{Bif}(P, a)$ is isolated, as $g_{P,a} \geq 0$, there exists a neighborhood U of t_0 with $\mathrm{supp}(\Delta g_{P,a}) \cap U = \{t_0\}$, hence $\Delta g_{P,a}$ gives mass to $\{t_0\}$. In particular, $\Delta g_{P,a}(\{t_0\}) > 0$. This is impossible since $g_{P,a}$ is locally bounded near t_0. □

Let us include here for completeness the following

Theorem 4.5. — *Let* (P, a) *be any holomorphic dynamical pair parametrized by the unit disk. If* (P, a) *is stable and* a *is not stably preperiodic, then the accumulation points of the set* $\mathrm{Preper}(P, a)$ *is included in the analytic subset* Z *of* $\mathbb{D}$ *where there exists a super-attracting periodic point at which the local degree is not locally constant.*

This result is a direct consequence of [**68**, Theorem 1.1], which is a refinement of [**120**], see [**46**, §2.3] for this precise formulation. Recall from Example 2.37 that $\mathrm{Preper}(P, a)$ may not be isolated even if the family is stable, and that the obstruction comes from the existence of a super-attracting periodic point at which the local degree is not locally constant, see Theorem 2.35.

4.1.2. Density of transversely prerepelling parameters

Given any holomorphic dynamical pair (P, a), we say that the marked point is *prerepelling* at $t_0 \in S$ if $a(t_0)$ eventually lands on a repelling periodic point z_0. In such a situation there exists $m \in \mathbb{N}$ such that $z_0 = P_{t_0}^m(a(t_0))$ is a repelling periodic point of P_{t_0} of exact period k. By the Implicit Function Theorem, there exists $\epsilon > 0$ and an analytic function $z : \mathbb{D}(t_0, \epsilon) \to \mathbb{C}$ such that $P_t^k(z(t)) = z(t)$ for all t and $z(t_0) = z_0$.

Definition 4.6. — *The marked point a is said to be properly (resp. transversally) prerepelling at t_0, if the two graphs $\{(t, P_t^m(a(t))) \in \mathbb{D}(t_0, \epsilon) \times \mathbb{C}\}$ and $\{(t, z(t)) \in \mathbb{D}(t_0, \epsilon) \times \mathbb{C}\}$ intersect properly (resp. transversally) at the point (t_0, z_0) in $\mathbb{D}(t_0, \epsilon) \times \mathbb{C}$.*

Our aim is to prove the following characterization of the bifurcation locus. This result is essentially due to Dujardin, see [**67**, Theorem 0.1] except that the latter reference only deals with marked critical point.

Theorem 4.7. — *Let (P, a) be any dynamical pair of degree d parametrized by a Riemann surface S. Then the bifurcation locus $\mathrm{Bif}(P, a)$ of the pair (P, a) is the closure of the set of parameters $t \in S$ at which the marked point a is transversely prerepelling.*

Remark 4.8. — The second author obtained a version of this theorem for holomorphic dynamical pairs in arbitrary dimension, see [**88**] for details. We present here an argument that can be adapted to treat *properly* prerepelling parameters.

We begin with the following

Lemma 4.9. — *Assume the marked point a is properly prerepelling at $t_0 \in S$. Then $t_0 \in \mathrm{Bif}(P, a) = \mathrm{supp}(\mu_{P,a})$.*

Proof. — Pick $t_0 \in S$ such that a is properly prerepelling at t_0. We use the notation above: k is the period of z_0 and ϵ is a sufficiently small positive number.

We proceed by contradiction assuming that the family $(P_t^n(a(t)))_n$ is normal at t_0. Let $K > 1$ and $\delta > 0$ be small enough so that $|(P_t^k)'(z)| \geq K > 1$ for all $(z,t) \in \mathbb{D}(z_0,\delta) \times \mathbb{D}(t_0,\epsilon)$. Reducing ϵ if necessary, we may assume that $z(t)$ and $P_t^{m+kn}(a(t))$ belong to $\mathbb{D}(z_0,\delta/2)$ for all $t \in \mathbb{D}(t_0,\epsilon)$ and for all $k \geq 0$.

For any integer $n \geq 0$, and for every $t \in \mathbb{D}(t_0,\epsilon)$ set $\varepsilon_n(t) := P_t^{m+kn}(a(t)) - z(t)$. Differentiating the quantity $\varepsilon_{n+1}(t) = P_t^k(P_t^{m+kn}(a(t))) - P_t^k(z(t))$, we get

$$\begin{aligned}\varepsilon'_{n+1}(t) = (P_t^k)'(P_t^{m+kn}(a(t))) \cdot \varepsilon'_n(t) - \\ \big((P_t^k)'(z(t)) - (P_t^k)'(P_t^{m+kn}(a(t)))\big) \cdot z'(t) + \\ \frac{\partial P_t^k}{\partial t}\big(P_t^{m+kn}(a(t))\big) - \frac{\partial P_t^k}{\partial t}(z(t)).\end{aligned}$$

Pick now $0 < \tau \ll 1$. Since $\{P_t^{m+kn}(a(t))\}_n$ forms a normal family whose value at $t = t_0$ equals $z(t_0)$, we may again reduce ϵ to obtain

$$|\varepsilon'_{n+1}(t)| \geq K|\varepsilon'_n(t)| - \tau$$

for all $t \in \mathbb{D}(0,\epsilon)$. By induction, we infer

$$|\varepsilon'_n(t)| \geq K^n \left(|\varepsilon'_0(t)| - \frac{\tau}{(K-1)}\right) .$$

By assumption a is properly repelling hence ε_0 cannot be identically zero and we get a contradiction if τ is small enough. □

Proof of Theorem 4.7. — According to Lemma 4.9, it is sufficient to prove the density of transversely prerepelling parameters in $\mathrm{supp}(\mu_{P,a})$. We follow closely the proof of [**67**, Theorem 0.1].

Pick $t_0 \in \mathrm{supp}(\mu_{P,a})$. According to [**67**, Lemma 4.1], there exists an integer $m \geq 1$ and a $P_{t_0}^m$-compact set $K \subset \mathbb{C}$ such that

- $P_{t_0}^m|_K$ is uniformly hyperbolic and conjugated to the one-sided shift on two symbols,
- the unique invariant measure ν on K satisfying $(P_{t_0}^m)^*\nu = 2\nu$ has continuous potential.

Moreover, by [**152**, §2], there exists $\epsilon > 0$ and a holomorphic motion $h : \mathbb{D}(t_0,\epsilon) \times K \to \mathbb{C}$ which conjugates the dynamics, i.e. satisfying

$$h_t \circ P_{t_0}^m(z) = P_t^m \circ h_t(z), \text{ for all } (t,z) \in \mathbb{D}(t_0,\epsilon) \times K .$$

The function $\hat{h}$ defined by

$$\hat{h}(t,z) := \int_K \log|z - h_t(w)| \mathrm{d}\nu(w)$$

is psh and it is continuous on $\mathbb{D}(t_0,\epsilon)\times\mathbb{C}$ by [**65**, Lemma 6.4]. Observe also that

$$\hat{h}(t,z)=\log^+|z|+O(1) \text{ as } |z|\to\infty$$

where the $O(1)$ is locally uniform in t since K_t is included in a fixed closed ball for all t. Approximating ν by Dirac masses, we see that

$$dd^c\hat{h}=\int_K[\Delta_z]\mathrm{d}\nu(z), \text{ with } \Delta_z:=\{(t,h_t(z))\in\mathbb{D}(t_0,\epsilon)\times\mathbb{C}\}\ .$$

Write $u_n(t):=\hat{h}(t,P_t^n(a(t)))$ for all $t\in\mathbb{D}(t_0,\epsilon)$. We claim that $d^{-n}\Delta u_n\to\mu_{P,a}$ as $n\to\infty$. Indeed for any compact set $E\subset\mathbb{D}(t_0,\epsilon)$, there exists $C>0$ such that

$$\left|\log^+|z|-\hat{h}(t,z)\right|\leq C$$

for all $t\in E$ and all $z\in\mathbb{C}$. In particular, for all $n\geq 0$ and all $t\in E$, we have

$$\left|\frac{1}{d^n}\log^+|P_t^n(a(t))|-\frac{1}{d^n}u_n(t)\right|\leq\frac{C}{d^n},$$

and the claim follows by taking the Laplacian of the left hand side and letting $n\to\infty$.

To conclude the proof we interpret the bifurcation measure as the image of the intersection of the graph $\Gamma_n=\{(t,P_t^n(a(t)))\in\mathbb{D}(t_0,\epsilon)\times\mathbb{C}\}$ with the positive closed $(1,1)$ current $dd^c\hat{h}$.

For each $z\in K$ such that $\Gamma_n\cap\Delta_z$ is discrete, the intersection of the two positive closed currents

$$[\Gamma_n]\wedge[\Delta_z]=dd^c_{t,w}(\log|w-P_t^n(a(t))|[\Delta_z])$$

is well-defined by [**48**, Proposition 4.12, page 156], and equals to the atomic measure supported on the set of intersection points of Γ_n and Δ_z with weight given by the multiplicity of intersection. If $\Gamma_n\cap\Delta_z$ is not discrete then $\Gamma_n=\Delta_z$ (so that z is uniquely determined) and we set by convention $[\Gamma_n]\wedge[\Delta_z]:=0$. We get

$$\mu_n:=[\Gamma_n]\wedge dd^c\hat{h}=\int_K[\Gamma_n]\wedge[\Delta_z]\mathrm{d}\nu(z)\ . \tag{26}$$

Indeed, for any $\varphi\in\mathcal{C}_c^\infty(\mathbb{D}(t_0,\epsilon)\times\mathbb{C})$, Fubini gives

$$\begin{aligned}\langle\mu_n,\varphi\rangle&=\int\log|w-P_t^n(a(t))|dd^c\hat{h}\wedge dd^c\varphi\\&=\int_K\left(\int_{\Delta_z}\log|w-P_t^n(a(t))|dd^c_{w,t}\varphi\right)\mathrm{d}\nu(z)\\&=\int_K\langle[\Gamma_n]\wedge[\Delta_z],\varphi\rangle\,\mathrm{d}\nu(z).\end{aligned}$$

Since $(\pi_1)_*([\Gamma_n] \wedge dd^c\hat{h}) = \Delta u_n$ and $t_0 \in \operatorname{supp}(\mu_{P,a})$, by the claim above, we get a point t_* arbitrarily close to t_0 such that $q_* := \pi_1^{-1}(t_*) \cap \Gamma_n$ lies in the support of μ_n for some large n.

By (26), Γ_n intersects Δ_{z_*} for some $z_* \in K$ near the point q_*. Since the set of repelling periodic points of $P_{t_0}^m$ is dense in K, we may find a sequence $z_p \in K$ such that $z_p \to z$, and $h_t(z_p)$ is P_t-periodic for all t. We conclude by [**12**, Lemma 6.4] which implies that Γ_n and Δ_{z_p} must intersect transversally near q_* for all p sufficiently large. □

4.1.3. Rigidity of the bifurcation locus

Our next (somehow technical) result gives a precise description of all situations in which the bifurcation locus of a holomorphic dynamical pair is included in a smooth real curve.

We say that a holomorphic family of polynomials P parametrized by the unit disk is *real* when the coefficients of P are defined by power series with real coefficients converging in the segment $]-1, +1[$. We call a *real dynamical pair* is a dynamical pair (P, a) for which P is a real family and the marked point is also defined by a real power series.

Theorem 4.10. — *Let (P, a) be any dynamical pair of degree d parametrized by the unit disk $\mathbb{D}$, and suppose that* $\operatorname{Bif}(P, a)$ *is non-empty and included in a smooth real curve.*

Then $\operatorname{Bif}(P, a)$ *is included in a real-analytic curve. Moreover if the family is not conjugated to a constant integrable polynomial, then* $\operatorname{Bif}(P, a)$ *is a closed, totally disconnected, perfect set, and the following holds.*

Any point $t_0 \in \operatorname{Bif}(P, a)$, possibly outside a discrete subset of $\mathbb{D}$ admits a small neighborhood U such that :

1. *for any $t \in U$, a critical point c of P_t either escapes to ∞ or satisfies $P_t^4(c) = P_t^2(c)$,*
2. *$J(P_t) = K(P_t)$ is totally disconnected for all $t \in U$,*
3. *the family $(P_t)_{t\in U}$ is J-stable.*

Moreover there exists a reparametrization of $(P_t)_{t\in U}$ for which the family is conjugated to a real family on U, and $\operatorname{Bif}(P, a)$ *is included in the real line.*

Example 4.11. — Suppose that $P_t \equiv P_*$ is a constant family, that P_* is real and satisfies (1) and (2). Since we have $\operatorname{Bif}(P, a) = \{t,\, a(t) \in J(P_*)\}$ by Proposition 4.14 below, the bifurcation locus is then included in the real-analytic curve $a^{-1}(\mathbb{R})$.

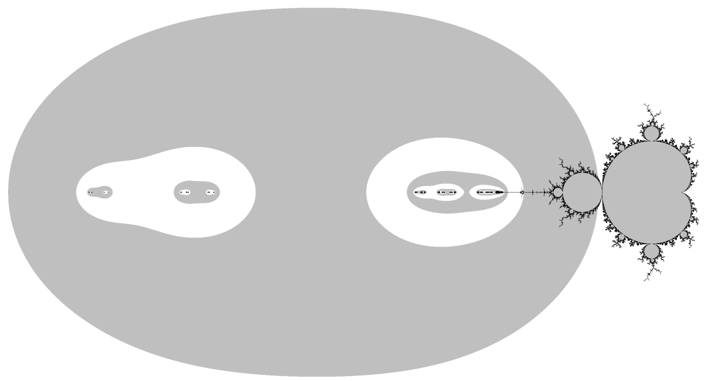


FIGURE 1. The superposition of the bifurcation locus of the pair $(z^2 + 8t, -2)$ (which is a real Cantor set); and the bifurcation of the family $z^2 + 8t$ (which is a homothetical image of the Mandelbrot set).

Example 4.12. — Take $P_t(z) = z^2 + 8t$ with $t \in \mathbb{D}$ and $a(t) = -2$. Then $\mathrm{Bif}(P, a)$ is a Cantor set included in the real line but the family P_t is not J-stable on the homothetical image of the boundary of the Mandelbrot set (see the figure above).

Proof of Theorem A. — Pick a dynamical pair (P, a) parametrized by some connected Riemann surface S and a holomorphic disk $U \subset S$, such that $\mathrm{Bif}(P, a) \cap U \neq \varnothing$ is included in a smooth curve. Theorem 4.10 implies that we fall into two cases. Either the family $(P_t)_{t \in U}$ is isotrivial and P_t is integrable for all t which implies the family is also isotrivial over S. Or we may find another holomorphic disk $\imath\colon \mathbb{D} \to U$ intersecting the bifurcation locus over which the family is conjugated to a real family and the bifurcation locus is included in the real line. □

The proof of Theorem 4.10 ranges over the next three subsections.

4.1.4. A renormalization procedure

We expose here how we reinterpret the similarity argument of Tan Lei [**159**] in our setting. Note that we need a more general and more precise statement than that in [**88**].

Let $(P_t)_{t \in \mathbb{D}}$ be a holomorphic family of degree d polynomials parametrized by the unit disk and $a : \mathbb{D} \to \mathbb{C}$ be a marked point. We assume there exists a holomorphically moving repelling periodic point $z : \mathbb{D} \to \mathbb{C}$ of period k with $P_0^m(a(0)) = z(0)$. We say a is properly prerepelling of *order* $q \geq 1$ if

$$P_t^m(a(t)) - z(t) = \alpha \cdot t^q + O(t^{q+1})$$

for some $\alpha \in \mathbb{C}^*$. Let $\rho(t) := (P_t^k)'(z(t))$ and denote by ϕ_t the linearizing coordinate of P_t^k at $z(t)$ which is tangent to the identity, i.e. such that $P_t^k \circ \phi_t(z) = \phi_t(\rho(t) \cdot z)$, and $\phi_t'(0) = 1$ for all $t \in \mathbb{D}$ and all $z \in \mathbb{D}(0, r)$ for some $r > 0$.

The next result is standard, and reflects the similarity between the Julia set of P_t and the bifurcation locus at t. We give here a complete proof for sake of completeness, adapting the proof by Buff and Epstein [**35**].

Proposition 4.13. — *Assume that the marked point a is properly prerepelling at 0 with order $q \geq 1$ and pick any q-th root λ of $\rho(0)$. Then*

$$g_{P_0} \circ \phi_0(t^q) = \lim_{n\to\infty} d^{m+kn} g_{P,a}(\lambda^{-n} t) ,$$

and the convergence is uniform on some small disk containing 0.

Proof. — For $n \geq 1$, we set $r_n(t) := \lambda^{-n} t$ and

$$a_n(t) := P_{r_n(t)}^{m+kn} (a \circ r_n(t)) .$$

First prove that, there exist a constant $C > 0$ and $\epsilon > 0$ small enough such that for all $t \in \mathbb{D}(0, \epsilon)$ and all $n \geq 1$, we have

$$|a_n(t) - \phi_0(t^q)| \leq C \frac{n|t|}{|\rho(0)|^{n/q}}. \tag{27}$$

We fix $\epsilon > 0$ small enough such that $P_t^m(a(t))$ lies in the range of ϕ_t for all $t \in \mathbb{D}(0, \epsilon)$. We may thus define $h(t) := \phi_t^{-1}(P_t^m(a(t)))$ for all $t \in \mathbb{D}(0, \epsilon)$. As $\phi_t(z)$ depends analytically on (t, z), the map h is holomorphic and

$$\begin{aligned} h(t) &= \phi_t^{-1}\left(z(t) + P_t^m(a(t)) - z(t)\right) \\ &= \phi_t^{-1}\left(z(t) + \alpha t^q + O(t^{q+1})\right) = \alpha t^q + O(t^{q+1}), \end{aligned}$$

where we used that $\phi_t(0) = z(t)$, $\phi_t'(0) = 1$ so that $\phi_t^{-1}(z(t) + w) = w + O(w^2)$.

To simplify notation we reparametrize the unit disk and assume $\alpha = 1$. In particular, there exists a constant $M > 0$ such that $|h(t) - t^q| \leq M|t|^{q+1}$. Again as $\phi_t(z)$ is analytic, there exist $C_1, C_2 > 0$ such that $|\phi_t(z) - \phi_s(w)| \leq C_1|z - w| + C_2|t - s|$, for all $z, w \in \mathbb{D}(0, r)$ and all $s, t \in \mathbb{D}(0, \epsilon)$. In particular, for all $t \in \mathbb{D}(0, \epsilon)$, all $n \geq 1$ and all $z, w \in \mathbb{D}(0, r)$, we find

$$\left|\phi_{r_n(t)}(z) - \phi_0(w)\right| \leq C_1|z - w| + C_2 \frac{|t|}{|\rho(0)|^{n/q}}.$$

Similarly there exists a constant $C_3 \geq 1$ such that $|\rho(t)/\rho(0) - 1| \leq C_3|t|$ for all $t \in \mathbb{D}(0,\epsilon)$, and for $|t|$ small enough we get

$$\left|\left(\frac{\rho(r_n(t))}{\rho(0)}\right)^n - 1\right| \leq \left|\frac{\rho(r_n(t))}{\rho(0)} - 1\right| \times (2n) \leq 2nC_3\frac{|t|}{|\rho(0)|^{n/q}} \ .$$

Observe that

$$\begin{aligned} a_n(t) &= P^{kn}_{r_n(t)}\left(P^m_{r_n(t)}(a \circ r_n(t))\right) = P^{kn}_{r_n(t)}\left(\phi_{r_n(t)}\left(h \circ r_n(t)\right)\right) \\ &= \phi_{r_n(t)}\Big(\rho(r_n(t))^n\left(h \circ r_n(t)\right)\Big). \end{aligned}$$

Putting all the above together, for all $t \in \mathbb{D}(0,\epsilon)$, we find

$$\begin{aligned} |a_n(t) - \phi_0(t^q)| \leq & C_2\frac{|t|}{|\rho(0)|^{n/q}} + C_1\left|\rho(r_n(t))^n\left(h \circ r_n(t)\right) - t^q\right| \\ \leq & C_2\frac{|t|}{|\rho(0)|^{n/q}} + C_1\left|\left(\frac{\rho(r_n(t))}{\rho(0)}\right)^n - 1\right||t|^q \\ & \qquad + C_1|\rho(r_n(t))|^n|h \circ r_n(t) - r_n(t)^q| \\ \leq & C_2\frac{|t|}{|\rho(0)|^{n/q}} + 2nC_3\frac{|t|^{q+1}}{|\rho(0)|^{n/q}} + C_1M\left|\rho(r_n(t))\right|^n|r_n(t)|^{q+1} , \end{aligned}$$

and

$$|\rho(r_n(t))|^n\,|r_n(t)|^{q+1} = \left|\frac{\rho(r_n(t))}{\rho(0)}\right|^n\frac{|t|^{q+1}}{|\rho(0)|^{n/q}} \leq 2\frac{|t|^{q+1}}{|\rho(0)|^{n/q}},$$

for $|t|$ small enough which implies (27).

In particular, the sequence $a_n(t)$ converges uniformly on $\mathbb{D}(0,\epsilon)$ to $\phi_0(t^q)$ and, if

$$g_n(t) := g_{P_{r_n(t)}}(a_n(t))\,, \ t \in \mathbb{D}(0,\epsilon),$$

since $r_n \to 0$ uniformly on $\mathbb{D}(0,\epsilon)$, the above implies $g_n(t) \to g_0 \circ \phi_0(t^q)$ uniformly on $\mathbb{D}(0,\epsilon)$. Using that $g_{P_t} \circ P_t = dg_{P_t}$, we get the wanted convergence. □

4.1.5. Bifurcation locus of a dynamical pair and J-stability

Proposition 4.14. — *Let (P,a) be any dynamical pair parametrized by the unit disk $\mathbb{D}$. If P is J-stable and $\mathrm{Bif}(P,a) \neq \varnothing$, we have*

$$\mathrm{Bif}(P,a) = \{t \in \mathbb{D}\,;\ a(t) \in J(P_t)\}.$$

Proof. — Let $h : \mathbb{D} \times J(P_0) \to \mathbb{P}^1$ be the unique holomorphic motion of $J(P_0)$ such that

$$h_t \circ P_0 = P_t \circ h_t \text{ on } J(P_0).$$

Observe that the set $\{(t, z), t \in \mathbb{D}, z \in J(P_t)\}$ is equal to $\{(t, h_t(z)), t \in \mathbb{D}, z \in J(P_0)\}$ hence is closed. Since $\{t \in \mathbb{D}\,;\ a(t) \in J(P_t)\}$ is the image under the first projection of the intersection of this set with the graph of a, it is also closed in $\mathbb{D}$.

By Theorem 4.7, the set of parameters $t_0 \in \mathbb{D}$ such that a is transversely prerepelling at t_0 is a dense subset of $\mathrm{Bif}(P, a)$. As repelling points of P_{t_0} are contained in $J(P_{t_0})$ and $\{t \in \mathbb{D}\,;\ a(t) \in J(P_t)\}$ is closed, we get $\mathrm{Bif}(P, a) \subset \{t \in \mathbb{D}\,;\ a(t) \in J(P_t)\}$.

Suppose conversely that $t_* \in \{t \in \mathbb{D}\,;\ a(t) \in J(P_t)\}$ so that $a(t) = h_t(z_*)$ for some $z_* \in J(P_0)$. Since the bifurcation locus is assumed to be non-empty, the two curves $\Gamma := \{(t, a(t)), t \in \mathbb{D}\}$ and $\{(t, h_t(z_*)), t \in \mathbb{D}\}$ cannot coincide. Choose any sequence of repelling periodic point z_n accumulating z_*. Then by [**12**, Lemma 6.4] the curves $\{(t, h_t(z_n)), t \in \mathbb{D}\}$ intersect transversally Γ near t_* for all n sufficiently large. We conclude by Lemma 4.9 that t_0 is accumulated by points in the bifurcation locus, hence $\{t \in \mathbb{D}\,;\ a(t) \in J(P_t)\} \subset \mathrm{Bif}(P, a)$. □

4.1.6. Proof of Theorem 4.10

Our first objective is to prove the following fact, whose proof is essentially contained in [**73**].

Proposition 4.15. — *Suppose* $\mathrm{Bif}(P, a)$ *is included in a smooth curve* γ*. Suppose* t_0 *is a transversally prerepelling parameter such that* P_{t_0} *is not integrable. Then* $\{t_0\}$ *is a connected component of* $\mathrm{Bif}(P, a)$*, and* P_{t_0} *is conjugated to a real polynomial whose critical points are either escaping to* ∞*, or satisfy* $P^4_{t_0}(c) = P^2_{t_0}(c)$*, whose Julia set is totally disconnected and included in* $\mathbb{R}$*.*

Proof. — Let m and k be two integers such that $P^m_{t_0}(a(t_0))$ is a repelling periodic point p_* of period k of multiplier λ. We let $\phi\colon \mathbb{C} \to \mathbb{C}$ be the unique linearization map at this periodic point which is tangent to the identity and so that $\phi(\lambda z) = P^k_{t_0}(\phi(z))$. By Proposition 4.13, we have

$$\Delta g_{P_{t_0}}(\phi(t)) = \lim_{n\to\infty} \Delta(d^{m+kn} g_{P,a}(\lambda^{-n}t)) \ .$$

Let L be the real line tangent to γ at the point t_0. In a fixed disk D centered at t_0, observe that $\lambda^n\gamma \cap D$ becomes C^∞-close to L. Since the support of the measure $\Delta(g_{P,a}(\lambda^{-n}t))$ is included in $\lambda^n\gamma$ we conclude that

$$J_* := \phi^{-1}(J(P_{t_0})) = \mathrm{supp}(\phi^*\Delta g_{P_{t_0}}) \subset L \ .$$

Since J_* is a closed subset of a real line, it is either totally disconnected or it contains a segment. In the latter case, $J(P_{t_0})$ is locally real-analytic at some point, and P_{t_0} is integrable by Theorem 2.16 which contradicts our assumption. In particular, $\{t_0\}$ is a connected component of $\mathrm{Bif}(P, a)$.

Observe that J_* is invariant by the dilatation by the dynamics hence $\lambda \in \mathbb{R}$, and conjugating P_{t_0} by a suitable homothety we may assume that $J_* \subset \mathbb{R}$.

Pick any periodic point p for P_{t_0} of period N which does not lie in its post-critical set so that ϕ is a local diffeomorphism at any preimage of $\phi^{-1}(p)$. Then we obtain that locally at p, $J(P_{t_0}) \subset \phi(J_*)$ which is included in a smooth (in fact real-analytic) curve. We may now repeat the argument with the linearization map ϕ_p at p, and we conclude that the multiplier λ_p of p is real, and that $\phi_p^{-1}(J(P_{t_0}))$ is real and totally disconnected.

Now we follow the arguments of Eremenko and Van Strien. We only sketch the main ideas referring to the original paper for details. Recall that the order of the entire function ϕ_p is defined by

$$\rho(p) := \limsup_{r\to\infty} \frac{\log\left(\log \sup_{|z|\le r} |\phi_p(z)|\right)}{\log r} \ .$$

Let $C = \sup_{\mathbb{D}(0,|\lambda_p|)} \max\{|\phi_p|, 1\}$, and pick any constant $C' \geq 1$ such that $|P_{t_0}(z)| \leq C' \max\{|z|, 1\}^d$. Since $\phi_p(\lambda_p z) = P_{t_0}^N(\phi_p(z))$ we get

$$|\phi_p(z)| = |P_{t_0}^{nN}(\phi_p(\lambda_p^{-n} z))| \leq \left(C' \max\{|\phi_p(\lambda_p^{-n} z)|, 1\}\right)^{d^{nN}} \ .$$

If n is chosen so that $|\lambda_p| \geq |\lambda_p^{-n} z| \geq 1$, then we have

$$\log|\phi_p(z)| \leq (\log C'C)|z|^{\frac{N\log d}{\log|\lambda_p|}}$$

so that $\rho(p) \leq \frac{N \log d}{\log|\lambda_p|}$.

Since $J(P_{t_0})$ is totally disconnected, at least one critical point escapes, and by the Misiurewicz-Przytycki's formula (9), the Lyapunov exponent χ of P satisfies $\chi > \log d$. By e.g. [**23**, proof of Theorem 30] we can find a sequence of periodic points accumulating p_* whose multiplier is arbitrarily close to χ. In particular, we may find p of period N having a multiplier $|\lambda_p| > d^N$ so that $\rho(p) < 1$. Conjugate P_{t_0} by a real translation in order to have $p = 0$. Denote by $z_j \neq 0$ the zeroes of ϕ_p. Since p belongs to $J(P_{t_0})$, these zeroes are all real. By Hadamard's theorem [**1**, Theorem 8 p.209], we may write

$$\phi_p(z) = z \prod_j \left(1 - \frac{z}{z_j}\right) e^{\frac{z}{z_j} + \frac{1}{2}\left(\frac{z}{z_j}\right)^2 + \cdots + \frac{1}{m_j}\left(\frac{z}{z_j}\right)^{m_j}}$$

for some $m_j \in \mathbb{N}^*$.

We infer that $\phi_p(\mathbb{R}) \subset \mathbb{R}$ which shows that $J(P_{t_0})$ is real. Let I be the convex hull of $J(P_{t_0})$: it is a compact segment whose end points $b_- < b_+$ are either pre-fixed or form a 2-cycles. Since all preimages of $b_\pm$ belongs to I, the intermediate value theorem implies that all critical points of P_{t_0} belong to I, and all critical values lie outside the interior of I. Since $J(P_{t_0})$ is totally disconnected at least one critical escapes and any non-escaping critical point c is mapped by P_{t_0} to one of the boundary point of I, hence $P_{t_0}^4(c) = P_{t_0}^2(c)$.

This concludes the proof of the proposition. □

Remark 4.16. — Eremenko and Van Strien's argument implies that if $\mathrm{Bif}(P, a)$ is included in the real line, then the pair (P, a) is real.

We now come back to the proof of Theorem 4.10. When the family is isotrivial, we may assume that $P_t = P_*$ is a fixed polynomial, and the previous proposition implies that P_* is real, that all its critical points are either escaping or satisfy $P_*^4(c) = P_*^2(c)$, and $J(P_*) = K(P_*)$ is a totally disconnected subset of $\mathbb{R}$. The statements (1), (2) and (3) are thus clear in this case so that the theorem is proved in this case.

From now on, we assume that the family is not isotrivial. We reparametrize the family and assume that P_t is monic and centered for all t, and all critical points are marked. We denote by $\mathcal{E}$ the set of parameters for which P_t is integrable. Since the set of integrable polynomials is finite in the moduli space of critically marked polynomials, and the family P_t is not isotrivial, it follows from the principle of isolated zeroes that $\mathcal{E}$ is discrete.

Since the bifurcation locus is non-empty, it follows from Theorem 4.7 and the previous Proposition 4.15 that for a set $\mathcal{D}$ of parameters t that is dense in $\mathrm{Bif}(P, a)$, there exists $\alpha_t \in \mathbb{C}^*$ and $\beta_t \in \mathbb{C}$ s.t. $\alpha_t^{-1} P_t(\alpha_t z + \beta_t) - \beta_t$ has real coefficients. Pre-composing by the dilatation of factor $1/|\alpha_t|$, we may suppose that $|\alpha_t| = 1$, and since P_t is monic we get $\alpha_t^{d-1} = \pm 1$. Pre-composing by a real translation of vector B replaces β_t by $B\alpha_t + \beta_t$. As P_t is centered we have $\beta_t \in \alpha_t \mathbb{R}$ so that we may actually choose $\beta_t = 0$. It follows that $\alpha^{-1} P_t(\alpha z) \in \mathbb{R}[z]$ for at least one $(2d-2)$-th root of unity α.

For each $\alpha \in \mathbb{U}_{2(d-1)}$, and for any t write

$$P_{t,\alpha}(z) = \alpha^{-1} P_t(\alpha z) = z^d + \sum a_{i,\alpha}(t) z^i,$$

and set

$$\Gamma_\alpha := \bigcap_{i=0}^{d-1} a_{i,\alpha}^{-1}(\mathbb{R}), \text{ and } \Gamma = \bigcup_{\mathbb{U}_{2(d-1)}} \Gamma_\alpha.$$

Then Γ is a real-analytic subvariety of $\mathbb{D}$ containing $\mathcal{D}$ which is dense in $\mathrm{Bif}(P, a)$, so that $\mathrm{Bif}(P, a) \subset \Gamma$. Observe that $\mathrm{Bif}(P, a)$ is totally disconnected otherwise we could find a segment in the bifurcation locus containing a transversally repelling parameter and this would contradict our standing assumption by Proposition 4.15.

We consider the set of points $t_0 \in \Gamma$ which admits a neighborhood U such that $\Gamma \cap U = \Gamma_\alpha$ for some $\alpha \in \mathbb{U}_{2(d-1)}$. The complement of this set is a discrete subset of Γ that we adjoin to $\mathcal{E}$ together with all singular and all isolated points of Γ.

For any $t_0 \in \mathrm{Bif}(P,a) \setminus \mathcal{E}$, we may thus replace the family (P_t) by $(P_{t,\alpha})$ on U. Reducing U if necessary we also suppose that $\Gamma \cap U$ is a segment, and reparametrizing the family in U, we have $\Gamma \cap U = \mathbb{R} \cap U$. For a dense subset of $\mathrm{Bif}(P,a)$ we get $P_t \in \mathbb{R}[z]$, thus $P_t \in \mathbb{R}[z]$ for all $t \in \Gamma \cap U$ which implies the family to be real on U.

Recall that all critical points are marked. Let $c_1, \ldots, c_r$ be the critical points satisfying $P_t^4(c_i(t)) = P_t^2(c_i(t))$ persistently in U, and let $\tilde{c}_1, \ldots, \tilde{c}_s$ be the other critical points.

For any $t \in \mathcal{D}$, we have $J(P_t) = K(P_t) \subset \mathbb{R}$ and $\tilde{c}_j(t)$ escape for all j. Since the measure μ_{P_t} varies continuously, it follows that $J(P_t)$ remains included in the real line for all $t \in \mathrm{Bif}(P,a)$ hence $J(P_{t_0}) = K(P_{t_0})$.

Lemma 4.17. — *If $P_{t_0}^4(\tilde{c}_j(t_0)) \neq P_{t_0}^2(\tilde{c}_j(t_0))$ for all j, then $\tilde{c}_1(t_0), \ldots, \tilde{c}_s(t_0)$ escape and $J(P_{t_0})$ is totally disconnected.*

Since for $t \in \mathcal{D}$ all critical points $\tilde{c}_j(t)$ escape under P_t, the set of parameters t having a critical point $c = \tilde{c}_j$ satisfying $P_t^4(c) = P_t^2(c)$ is discrete and we may add it to $\mathcal{E}$. Our assumption $t_0 \notin \mathcal{E}$ implies that $J(P_{t_0}) = K(P_{t_0})$ is real, totally disconnected and that all critical points $\tilde{c}_j(t_0)$ escape. The latter property implies the J-stability of the family (P_t) in a neighborhood of t_0, and this concludes the proof of the theorem.

Proof of Lemma 4.17. — Recall that t_0 is a limit of (real) parameters $t_n \in \mathcal{D}$ for which all critical points are real and all critical points $\tilde{c}_j(t_n)$ escape. More precisely, we know that the convex hull I_t of $J(P_t)$ is a segment containing all critical points, and $P_{t_n}(\tilde{c}_j(t_n)) \notin I_{t_n}$ for all j.

By continuity, either this property remains true for P_{t_0} and $J(P_{t_0})$ is totally disconnected; or an image of one critical point $\tilde{c}_j(t_0)$ is equal to a boundary point of I_{t_0}. □

4.2. Algebraic dynamical pairs

4.2.1. Algebraic dynamical pairs

Fix any algebraically closed field K of characteristic 0. Assume V is an irreducible affine K-variety and fix an integer $d \geq 2$.

Definition 4.18. — *An (algebraic)* dynamical pair (P, a) *of degree d parametrized by V is an algebraic family $P : V \times \mathbb{A}^1_K \to V \times \mathbb{A}^1_K$ of degree d polynomials together with a marked point $a \in K[V]$.*

Given a dynamical pair, we let $\mathrm{Preper}(P, a)$ be the subset of parameters $t \in V(K)$ for which $a(t)$ is preperiodic for P_t. It is a countable union of algebraic subvarieties.

One defines the notion of isotriviality of an algebraic family of map and of a dynamical pair exactly as in the holomorphic case.

Definition 4.19. — *We say that a dynamical pair (P, a) parametrized by an irreducible algebraic variety V is* active *if the set* $\mathrm{Preper}(P, a)$ *is a proper Zariski dense subset of V. Otherwise, we say the pair (P, a) is* passive*.*

Suppose that (P, a) is an algebraic dynamical pair parametrized by a complete algebraic curve C.

Since any regular function on C is a constant, it follows that the pair (P, a) is constant, hence passive. It follows that if (P, a) is active, then the curve C is not complete hence it is affine[(1)]. Observe also that (P, a) is active when $\mathrm{Preper}(P, a)$ is infinite countable, Zariski dense, and its complement is nonempty. We shall see below that in this case $C(K) \setminus \mathrm{Preper}(P, a)$ is actually Zariski dense too, see Remark 4.31. When C is reducible, we declare that (P, a) is active when its restriction to one of its irreducible component is.

We shall see below a characterization of passive dynamical pairs that was obtained by DeMarco (see Theorem 2 from the Introduction).

4.2.2. The divisor of a dynamical pair

Let C be any irreducible affine curve defined over an algebraically closed field K of characteristic 0. We let $\bar{C}$ be any projective compactification of C such that $\bar{C} \setminus C$ is a finite set of smooth points on $\bar{C}$. Let $\mathsf{n} \colon \hat{C} \to \bar{C}$ be the normalization map. A branch $\mathfrak{c}$ of C is by definition a (closed) point in $\hat{C}$.[(2)]

(1)since a finite set of points on a complete curve always support an ample divisor.

(2)Note that this definition is consistent with our definition of adelic branch in §1.5.

We say the branch lies at infinity if $\mathfrak{c}$ belong to $\mathsf{n}^{-1}(\hat{C} \setminus C)$. We may (and shall) identify the set of branches at infinity with $\bar{C} \setminus C$.

Lemma 4.20. — *Let (P, a) be any dynamical pair parametrized by C. For any branch $\mathfrak{c}$ of C, the sequence $-\frac{1}{d^n}\min\{\mathrm{ord}_{\mathfrak{c}}(P^n(a)), 0\}$ converges to a non-negative rational number $q_{\mathfrak{c}}(P, a) \in \mathbb{Q}_+$.*

Moreover, when $q_{\mathfrak{c}}(P, a) > 0$, then we have $q_{\mathfrak{c}}(P, a) = -\frac{1}{d^n}\mathrm{ord}_{\mathfrak{c}}(P^n(a))$ for all n large enough. Connversely when $q_{\mathfrak{c}}(P, a) = 0$, then the sequence $-\min\{\mathrm{ord}_{\mathfrak{c}}(P^n(a)), 0\}$ is in fact bounded.

Proof. — Consider the local ring $R = \mathcal{O}_{\hat{C},\mathfrak{c}}$ endowed with the unique $\mathfrak{m}_{\hat{C},\mathfrak{c}}$-norm. In other words, one choose a (formal) parametrization $t \mapsto \theta(t)$ of $\hat{C}$ at $\mathfrak{c}$ and set $|\phi| = \exp(-\mathrm{ord}_t(\phi \circ \theta))$ for any $\phi \in R$. The completion $\hat{R}$ of $(R, |\cdot|)$ is isomorphic to $K[[t]]$ with its usual t-adic norm, and we denote by L the fraction field of $\hat{R}$.

The pair (P, a) induces a polynomial $P_{\mathfrak{c}} \in L[z]$ and a point $a \in L$. Unwinding definitions, we get

$$\begin{aligned} q_n &:= \frac{1}{d^n}\log^+|P_{\mathfrak{c}}^n(a)|_L = \frac{1}{d^n}\max\{0, -\mathrm{ord}_t(P^n \circ a \circ \mathsf{n})\} \\ &= -\frac{1}{d^n}\min\{\mathrm{ord}_{\mathfrak{c}}(P^n(a)), 0\} \ . \end{aligned}$$

It follows from §2.3 that q_n converges which justifies the existence of $q_{\mathfrak{c}}(P, a)$. More precisely, when $q_{\mathfrak{c}}(P, a) > 0$ then for n sufficiently large we can apply Proposition 2.11 (2) to $z = P_{\mathfrak{c}}^n(a)$ and we infer $q_{\mathfrak{c}}(P, a) = \frac{1}{d^n}\log^+|P_{\mathfrak{c}}^n(a)|_L \in \mathbb{Q}_+$.

When $q_{\mathfrak{c}}(P, a) = 0$, then the point a lies in the filled-in Julia set of $P_{\mathfrak{c}}$ and $|P_{\mathfrak{c}}^n(a)|_L$ is bounded. This implies the lemma. □

Since P and a are determined by regular functions on C, it follows that $q_{\mathfrak{c}}(P, a) = 0$ for every branch $\mathfrak{c}$ mapped by n to a point in C. By the previous lemma, we can set the following

***Definition 4.21* (The divisor of a dynamical pair)**

The divisor of the dynamical pair (P, a) is by definition

$$\mathsf{D}_{P,a} := \sum_{\mathfrak{c} \in \bar{C}} q_{\mathfrak{c}}(P, a)[\mathfrak{c}] \ .$$

It is an effective rational divisor on $\bar{C}$ whose support is included in the set of branches at infinity of C.

4.2.3. Meromorphic dynamical pairs parametrized by the punctured disk

In order to relate Green functions to the divisor defined above we need to do a short détour through meromorphic families and review the main result of [**78**].

Let us fix any complete metrized field $(K, |\cdot|)$ of characteristic zero. Let us introduce the ring $\mathbb{M}_K$ of analytic functions on the punctured unit disk $\mathbb{D}^*_K(0,1)$ that are meromorphic at 0. When K is non-Archimedean, $\mathbb{M}_K$ is equal to the set of Laurent series $\sum a_n T^n$ such that $a_n = 0$ for all n sufficiently negative, and $\sup_n |a_n| < \infty$.

By convention, a meromorphic family of degree d is a polynomial $P \in \mathbb{M}_K[z]$ such that P_t is a polynomial with coefficients in K of degree d for all $t \neq 0$. In other words, we assume the leading coefficient of P to be invertible in $\mathbb{M}_K$ (i.e. to have no zero on the punctured disk).

The next result is the key step in the proof of Theorem 1 from the Introduction.

Theorem 4.22. — *For any meromorphic family $P \in \mathbb{M}_K[z]$ of polynomials of degree $d \geq 2$ and for any function $a(t) \in \mathbb{M}_K$, there exists a nonnegative rational number $q(P,a) \in \mathbb{Q}_+$ such that the function*

$$\hat{g}(t) := g_{P_t}(a(t)) - q(P,a)\, \log |t|^{-1}$$

on $\mathbb{D}^*_K(0,1)$ extends continuously across the origin. Moreover, one of the three possibilities following occurs.

1. *There exists an affine change of coordinates depending analytically on t conjugating P_t to Q_t such that the function a and the family Q are analytic, $\deg(Q_0) = d$, and the constant $q(P,a)$ vanishes.*
2. *The constant $q(P,a)$ is strictly positive and $\hat{g}$ is harmonic in a neighborhood of 0.*
3. *The constant $q(P,a)$ vanishes and $\hat{g}(0) = 0$.*

Example 4.23. — In the case $P_t(z) = z^d + t^{-1}$ and $a(t) = 0$, then we fall into case 2 with $q(P,a) = \frac{1}{d}$.

Example 4.24. — This example is due to DeMarco. Set $P_t(z) = -\frac{4}{3}tz^3 + z^2 + (1+t)z$, and $a_1(t) = -\frac{1}{2}$, $a_2(t) = \frac{t+1}{2t}$. Then P_t is a family that diverges in the moduli space of cubic polynomials, and admits 0 as a fixed point with multiplier $1+t$. The circle $|1+t| = 1$ lies in the bifurcation locus of the family, since for any such t, the polynomial P_t has a fixed point at 0 with multiplier $1+t$ of modulus 1, and the multiplier varies with the parameter. The points a_1 and a_2 are the critical points of P_t.

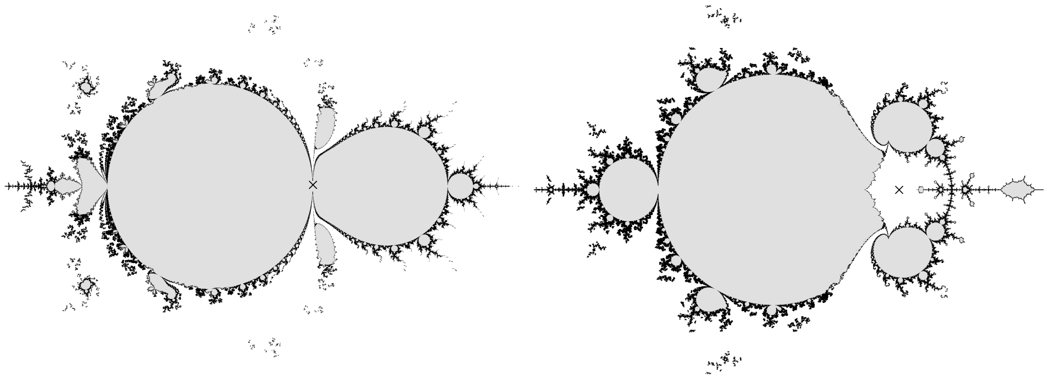


FIGURE 2. The bifurcation loci of the pair $(P_t, a_1(t))$ (right figure) and $(P_t, a_2(t))$ (left figure) from Example 4.24. The cross denotes the origin is both cases.

One can show that $q(P, a_2) = \frac{1}{2}$. In particular, we are in case 2 of the theorem for a_2. As the bifurcation locus of family accumulates at 0, the function $g_{P_t}(a_1(t))$ can not be harmonic in any neighborhood of 0. In particular, we are in case 3 of the the theorem for a_1, $q(P, a_1) = 0$ and $g_{P_t}(a_1(t))$ extends continuously through the origin. See Figure 2 below.

The proof of the theorem is a good illustration of the interplay between various metrized fields namely K with its given norm and $K(\!(t)\!)$ with the t-adic norm. It relies on delicate estimates inspired from a work by Ghioca and Ye [**103**]. We give here a sketch of proof refereeing to [**78**] for more details. We begin with the following lemma.

Lemma 4.25. — *For any meromorphic family $P \in \mathbb{M}_K[z]$ of polynomials of degree $d \geq 2$ there exists a positive constant $C > 0$ such that*

$$\left| \frac{1}{d} \log \max\{1, |P_t(z)|\} - \log \max\{1, |z|\} \right| \leq C \log |t|^{-1} \tag{28}$$

for all $t \in \mathbb{D}_K^*(0, \frac{1}{2})$ and for all z in the affine line.

Note that this lemma holds for any endomorphism of the projective space in any dimension, see [**51**, Lemma 3.3] and [**75**, Proposition 4.4]. Let $\mathbb{M}_K^*$ be the set of invertible elements in $\mathbb{M}_K$.

Proof. — Observe that for any $a \in \mathbb{M}_K$, there exist constant $A > 1$, $\alpha \in \mathbb{N}^*$ such that $|a(t)| \leq A|t|^{-\alpha}$ for all $t \in \mathbb{D}_K^*(0, \frac{1}{2})$, and that we may further assume $|a(t)| \geq A^{-1}|t|^{\alpha}$ when $a \in \mathbb{M}_K^*$.

Since $P_t(z) = a_0(t)z^d + \cdots + a_d(t)$ for some $a_i \in \mathbb{M}_K$, we get $|P_t(z)| \leq B|t|^{-\beta} \max\{1, |z|^d\}$ for some $B, \beta > 0$ which implies the upper bound

$$\frac{1}{d} \log \max\{1, |P_t(z)|\} \leq \log \max\{1, |z|\} + C \log |t|^{-1} .$$

For the lower bound we write $P_t(z) = a_0(t)z^d \left(1 + \frac{b_1(t)}{z} + \cdots + \frac{b_d(t)}{z^d}\right)$, where $a_0 \in \mathbb{M}_K^*$, and $b_i \in \mathbb{M}_K$. When $|z| \geq 3 \max\{|b_i(t)|, 1\}$, we have $|P_t(z)| \geq \frac{1}{2}A|t|^{\alpha}|z|^d$, hence

$$\frac{1}{d} \log \max\{1, |P_t(z)|\} \geq \log \max\{1, |z|\} - C \log |t|^{-1} , \tag{29}$$

for some $C > 0$, whereas when $|z| \leq 3 \max\{|b_i(t)|, 1\}$ we may increase $C > 0$ so that $\log \max\{1, |z|\} \leq C \log |t|^{-1}$ and (29) holds trivially. $\square$

We now give a sketch of proof of Theorem 4.22:

Proof. — To simplify notation write $L := K(\!(t)\!)$. This is a complete metrized field with the t-adic norm $|\cdot|_t := \exp(-\mathrm{ord}_0)$. The polynomial $P \in \mathbb{M}_K[z]$ induces a polynomial over L which we denote by P_L to clarify the discussion. We may then consider the Green function

$$g_{P,L} := \lim_{n\to\infty} \frac{1}{d^n} \log \max\{1, |P_L^n|_t\} \colon \mathbb{A}_L^{1,\mathrm{an}} \to \mathbb{R} .$$

Suppose first that $g_{P,L}(a) > 0$. Up to conjugacy and to a base change, we may assume that $P_t(z) = z^d + a_1(t)z^{d-1} + \cdots + a_d(t)$ is monic, and that $a(t) = t^{-\ell}(1 + O(t))$.

By assumption, $-\mathrm{ord}_0(P^n(a)) \to \infty$ and, up to replacing a by one of its iterate, we may suppose that $\mathrm{ord}_0(a) = \ell$ is as large as we want, and $-\mathrm{ord}_0(P^n(a)) = \ell d^n$ for all $n \geq 0$. We claim that for $0 < |t| \leq \varepsilon$ with $\varepsilon > 0$ small enough

$$g_{P_t}(a(t)) = \lim_{n\to\infty} \frac{1}{d^n} \log |P_t^n(a(t))| .$$

Indeed, Lemma 4.25 implies the existence of a constant $C' > 0$ such that for all n we have

$$\begin{aligned}\frac{1}{d^n} \log \max\{1, |P_t^n(a(t))|\} &\geq \log \max\{1, |a(t)|\} - C' \log |t|^{-1} \\ &\geq (\ell - C') \log |t|^{-1} + O(1) > 0,\end{aligned}$$

for all t in a uniform punctured disk centered at 0, and this implies the claim.

It follows that $\varphi_n(t) := \frac{1}{d^n} \log |P_t^n(a(t))| - \ell \log |t|^{-1}$ forms a sequence of harmonic functions on $\mathbb{D}_K(0, \varepsilon)$ which converges uniformly on any compact subset of $\mathbb{D}_K^*(0, \varepsilon)$ to $g_{P_t}(a(t)) - \ell \log |t|^{-1}$. By the maximum principle, the convergence is uniform in the disk $\mathbb{D}_K(0, \varepsilon)$ and we can write $g_{P_t}(a(t)) =$

$\ell \log|t|^{-1} + \hat{g}(t)$, where $\hat{g}$ is harmonic on $\mathbb{D}_K(0,\varepsilon)$, so that we are in case 2 of Theorem 4.22.

Suppose next that $g_{P,L}(a) = 0$ and the point a lies in the Fatou set of $P_L\colon \mathbb{A}^{1,\mathrm{an}}_L \to \mathbb{A}^{1,\mathrm{an}}_L$. By [**161**, Corollary B], we may find a closed ball $B \subset (L, |\cdot|_t)$ whose radius lies in the value group $|L^*|$ which is preperiodic under P_L and contains the point a.

Up to conjugating P, replacing P by an iterate and replacing a by $P^m(a)$, we may assume B is the closed unit ball, $P(B) = B$ and $a \in B$. In this case, we can write $P_t = Q + tR_t$ where $Q \in K[z]$ with $1 \le \deg(Q) = \delta \le d$ and $R_t \in K[\![t]\!][z]$ with $\deg(R_t) \le d$. When $\delta = d$, we are in the case 1 of Theorem 4.22. When $\delta < d$, there is a constant $C_1 \ge 1$ such that

$$\max\{1, |P_t(z)|\} \le C_1 \cdot \max\{1, |t|\,|z|^d, |z|^\delta\},$$

for all $t \in \mathbb{D}_K(0,1/2)$ and all $z \in K$. Note that a defines a holomorphic function on the unit disk. We claim the existence of a constant $C_2 \ge 1$ independent of n such that

$$\max\{1, |P^n_t(a(t))|\} \le C_1^{1+\cdots+\delta^{n-1}} \cdot C_2^{\delta^n} \tag{30}$$

if $|t| = \left(C_1^{1+\cdots+\delta^{n-2}} \cdot C_2^{\delta^{n-1}}\right)^{\delta-d}$.

This estimate follows from a direct induction and the following series of inequalities:

$$\begin{aligned}\max\{1, |P^{n+1}_t(a(t))|\} &\le C_1 \cdot \max\{1, |t|\,|P^n_t(a(t))|^d, |P^n_t(a(t))|^\delta\}\\ &\le C_1 \cdot \max\{1, |t|\,(C_1^{1+\cdots+\delta^{n-1}} \cdot C_2^{\delta^n})^d, (C_1^{1+\cdots+\delta^{n-1}} \cdot C_2^{\delta^n})^\delta\}\\ &\le C_1^{1+\cdots+\delta^n} \cdot C_2^{\delta^{n+1}}\end{aligned}$$

Fix now $N \ge 2$. Using (30) and again Lemma 4.25, we infer

$$0 \le g_{P_t}(a(t)) \le \frac{1}{d^N} \log\max\{1, |P^N_t(a(t))|\} - \frac{C'}{d^N}\log|t| \le \left(\frac{\delta}{d}\right)^N \cdot C_3$$

if $|t| = R_N := \left(C_1^{1+\cdots+\delta^{N-2}} \cdot C_2^{\delta^{N-1}}\right)^{\delta-d}$, where C_3 is a constant independent of N. Thus $g_{P_t}(a(t))$ extends as a subharmonic function on $\mathbb{D}_K(0,1)$ and we have

$$0 \le g_{P_t}(a(t)) \le \left(\frac{\delta}{d}\right)^N \cdot C_3$$

for all $t \in \mathbb{D}(0, R_N)$. Fix $\varepsilon > 0$, take $N \ge 1$ such that $\left(\frac{\delta}{d}\right)^N \cdot C_3 \le \varepsilon$ and let $\eta := R_N$. Then for any $t \in \mathbb{D}_K(0,\eta)$, we have $0 \le g_{P_t}(a(t)) \le \varepsilon$, so that we are in case 3 of Theorem 4.22.

Finally suppose that $g_{P,L}(a) = 0$, and the point a lies in the Julia set of $P_L : \mathbb{A}^{1,\mathrm{an}}_L \to \mathbb{A}^{1,\mathrm{an}}_L$. The key observation is that the orbit of a is then compact in $(L, |\cdot|_t)$. A proof of this fact is given in [**78**, Theorem 3]. It also follows from [**161**, Proposition 6.7].

Up to conjugacy, we may assume the orbit of a under P_L lies in the closed unit ball B and up to replacing P by an iterate, we may assume $d \geq 3$. In this case, for any $m \geq 0$, the function

$$g_m(t) := \frac{1}{d^m} \log \max\{1, |P_t^m(a(t))|\}$$

extends as a subharmonic function to $\mathbb{D}_K(0,1)$. The sequence $(g_m)_m$ converges uniformly locally on $\mathbb{D}^*_K(0,1)$ to $g_{P_t}(a(t))$. By the maximum principle, $g_{P_t}(a(t))$ also extends as a subharmonic function to $\mathbb{D}_K(0,1)$, and there is a constant $M \geq 1$ such that

$$\sup_m \sup_{|t|\leq 1/2} \{g_m(t), g_{P_t}(a(t))\} \leq M.$$

We use once again Lemma 4.25, and find $C_4 > 0$ independent of n such that

$$g_{n+j}(t) \leq g_n(t) + \frac{C_4}{d^n} \log |t|^{-1} \tag{31}$$

for all $t \in \mathbb{D}^*_K(0,1/2)$ and all $n, j \geq 1$.

Fix now $l \geq C_4 \cdot d$. Since the closure of the orbit of a is compact in L, we may cover it by finitely many balls of radius e^{-l} and centers $Q_1, \ldots, Q_N \in K[t]$, so that for any $n \geq 1$, there is $1 \leq i_n \leq N$ such that $P_t^n(a(t)) - Q_{i_n}(t) = O(t^l)$.

Let $A := \max_j \{\sup_{|t|<1} |Q_j(t)| + 2\}$ and fix once and for all a large integer n_0. For $r_0 > 0$ small enough, we have

$$\sup_{|t|<r_0} |P_t^{n_0}(a(t))| \leq A,$$

and for $j \geq 0$, let $r_j := r_0^{2^j}$ so that r_j decreases to 0 as $j \to \infty$.

Lemma 4.26. — *There exists a constant $C_5 > 0$ depending only on d and A such that*

$$\sup_{|t|<r_j} g_{n_0+j}(t) \leq \frac{C_5}{d^{n_0}},$$

for all $j \geq 0$.

Combining this lemma with (30) and the maximum principle, we obtain

$$0 \leq g_{n_0+j+k}(t) \leq \sup_{|s|<r_j} g_{n_0+j}(s) - \frac{C_4}{d^{n_0+j}} \log r_j \leq \frac{C_6}{d^{n_0}} \left(1 - \left(\frac{2}{d}\right)^j \log r_0\right),$$

for all $j, k \geq 0$ and all $|t| < r_j$, where $C_6 > 0$ is independent of n_0 and j. Fix now $\varepsilon > 0$. Up to increasing n_0, we may assume $2C_6 \leq d^{n_0}\varepsilon$ and we can fix $j \geq 0$ such that $1 - (2/d)^j \log r_0 \leq 2$. By the above, we thus have

$$0 \leq g_n(t) \leq \varepsilon$$

for all $n \geq n_0 + j$ and all $|t| < r_j$. Making $n \to \infty$, we find $0 \leq g_{P_t}(a(t)) \leq \varepsilon$ for all $|t| < r_j$. This proves $g_{P_t}(a(t))$ extends continuously through 0 and we are again in case 3 of Theorem 4.22. □

Proof of Lemma 4.26. — The proof of this lemma is taken from [**78**]. We included it here for the convenience of the reader. To simplify notation we write $u_n(t) = P_t^n(a(t))$.

It is sufficient to show by induction on $j \geq 0$ that

$$\sup_{|t|<r_{j+1}} g_{n_0+j+1}(t) \leq \frac{\log(3A)}{d^{n_0+j+1}} + \sup_{|s|<r_j} g_{n_0+j}(s).$$

By assumption, for all $j \geq 1$, there exists $1 \leq i_j \leq N$ such that the function

$$\frac{u_{n_0+j}(t) - Q_{i_j}(t)}{t^l}$$

is analytic on $\mathbb{D}_K(0,1)$, and for $0 < r < 1$ the maximum principle gives

$$\left|\frac{u_{n_0+j}(t) - Q_{i_j}(t)}{t^l}\right| \leq \left(A + \sup_{|s|<r} |u_{n_0+j}(s)|\right) \cdot \frac{1}{r^l} \quad \text{for all } |t| < r.$$

In particular, we find

$$\begin{aligned} \sup_{|t|<r_{j+1}} |u_{n_0+j+1}(t)| &\leq A + \left(A + \sup_{|s|<r_j} |u_{n_0+j+1}(s)|\right) \cdot \left(\frac{r_{j+1}}{r_j}\right)^l \\ &\leq 2A + \left(\sup_{|s|<r_j} |u_{n_0+j+1}(s)|\right) r_j^l, \end{aligned}$$

hence

$$\sup_{|t|<r_{j+1}} \max\{1, |u_{n_0+j+1}(t)|\} \leq (3A) \sup_{|s|<r_j} \max\{1, |u_{n_0+j+1}(s)|\, r_j^l\}\ .$$

When $\sup_{|s|<r_j} |u_{n_0+j+1}(s)|\, r_j^l \leq 1$, we get

$$\sup_{|t|<r_{j+1}} g_{n_0+j+1}(t) \leq \frac{\log(3A)}{d^{n_0+j+1}} \leq \frac{\log(3A)}{d^{n_0+j+1}} + \sup_{|s|<r_j} g_{n_0+j}(s)\ ,$$

as required. Otherwise, we have

$$\begin{aligned}\sup_{|t|<r_{j+1}} g_{n_0+j+1}(t) &\leq \frac{\log(3A)}{d^{n_0+j+1}} + \frac{l}{d^{n_0+j+1}}\log r_j + \sup_{|s|<r_j} g_{n_0+j+1}(s)\\ &\overset{\text{by (28)}}{\leq} \frac{\log(3A)}{d^{n_0+j+1}} + \left(\frac{l}{d^{n_0+j+1}} - \frac{C}{d^{n_0+j}}\right)\log r_j + \sup_{|s|<r_j} g_{n_0+j}(s)\\ &\leq \frac{\log(3A)}{d^{n_0+j+1}} + \sup_{|s|<r_j} g_{n_0+j}(s)\ ,\end{aligned}$$

since $r_j < 1$ and $l \geq C''d$. The lemma follows. □

4.2.4. Metrizations and dynamical pairs

Let $(K, |\cdot|)$ be any algebraically closed complete metrized field of characteristic zero. We fix any dynamical pair (P, a) parametrized by an affine curve C defined over K. Set $g_{P,a}(t) := g_{P_t}(a(t))$ for all $t \in C^{\mathrm{an}}$. In this section, we relate the behaviour of $g_{P,a}$ near the branches at infinity of C to the divisor $\mathsf{D}_{P,a}$.

By a local parametrization of $\mathfrak{c}$, we mean an analytic map from the open unit disk $\mathbb{D}_K(0,1)$ to $\hat{C}^{\mathrm{an}}$ which is an isomorphism onto its image and sends the origin to $\mathfrak{c}$.

The next result is Theorem 1 from the Introduction.

Theorem 4.27. — *Let (P, a) be any dynamical pair parametrized by an affine curve C. Then $g_{P,a}$ defines a non-negative continuous subharmonic function on C^{an}.*

Moreover, for any branch $\mathfrak{c}$ of C at infinity, and for any local parametrization of a punctured neighborhood of $\mathfrak{c}$ one can write

$$g_{P,a}(t) = q_{\mathfrak{c}}(P,a)\,\log|t|^{-1} + \tilde{g}(t)$$

where $\tilde{g}$ is continuous and subharmonic.

Proof. — By Proposition 2.8 (5), $g_{P,a}$ is locally a uniform limit of continuous and subharmonic functions, hence it is continuous and subharmonic. From Theorem 4.22, we may write $g_{P,a}(t) = q(P,a)\,\log|t|^{-1} + \tilde{g}(t)$ with $\tilde{g}$ continuous, and Lemma 4.25 implies

$$\left| g_{P,a}(t) - \frac{1}{d^n}\log^+|P_t^n(a(t))| \right| \leq \frac{C}{d^n}\log|t|^{-1}$$

for some positive constant C. Since

$$\frac{1}{d^n}\log^+|P_t^n(a(t)| = -\frac{1}{d^n}\min\{\mathrm{ord}_{\mathfrak{c}}(P^n(a)), 0\}\log|t|^{-1} + \tilde{g}_n(t)$$

with $\tilde{g}_n$ continuous, we deduce that

$$q_{\mathfrak{c}}(P,a) = \lim_{n\to\infty} -\frac{1}{d^n}\min\{\mathrm{ord}_{\mathfrak{c}}(P^n(a)),0\} = q(P,a)$$

as required. □

Recall how a function on C induces a metrization on a suitable line bundle on $\bar{C}$. Let D be any integral divisor on $\bar{C}$ supported on the set of branches at infinity of C, and let $g\colon C \to \mathbb{R}$ be any continuous function.

Any local section $\sigma \in \mathcal{O}_{\bar{C}}(\mathsf{D})(U)$ on an open subset U of $\bar{C}$ is determined by a rational function on $\bar{C}$ satisfying $\mathrm{ord}_{\mathfrak{c}}(\sigma)+\mathrm{ord}_{\mathfrak{c}}(\mathsf{D}) \geq 0$ for all $\mathfrak{c} \in U$. We say that g defines a continuous metrization $|\cdot|_g$ on $\mathcal{O}_{\bar{C}}(\mathsf{D})$ whenever the continuous function $t \mapsto |\sigma(t)|e^{-g(t)}$ defined on $U \setminus \mathrm{supp}(\mathsf{D})$ extends continuously to U for any local section σ.

This condition can be rephrased as follows. Pick any branch at infinity $\mathfrak{c}$, and any local parametrization $\theta\colon \mathbb{D}_K(0,1) \to \bar{C}$ mapping 0 to $\mathfrak{c}$. This parametrization induces a meromorphic map $N_{\mathfrak{c}}\colon \mathbb{D}_K(0,1) \to C$ which is analytic on $\mathbb{D}^*_K(0,1)$. Through the map $\sigma \mapsto \sigma \circ \theta$, sections defined in a local (analytic) neighborhood of $\mathfrak{c}$ are identified with meromorphic functions with at most one pole at 0 of order $\leq \mathrm{ord}_{\mathfrak{c}}(\mathsf{D})$. It follows that g defines a continuous metrization on $\mathcal{O}_{\bar{C}}(\mathsf{D})(U)$ iff one can write

$$g \circ N_{\mathfrak{c}}(t) = \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}) \log |t|^{-1} + g_{\mathfrak{c}}(t) \ ,$$

for some *continuous* function $g_{\mathfrak{c}}$.

One can translate Theorem 4.27 into the language of metrizations as follows.

Theorem 4.28. — *The function $g_{P,a}$ is continuous and subharmonic on C^{an}. It induces a continuous semi-positive metrization on the ($\mathbb{Q}$-)line bundle $\mathcal{O}_{\bar{C}}(\mathsf{D}_{P,a})$. In particular, we have*

$$\int_{C^{\mathrm{an}}} \Delta g_{P,a} = \deg(\mathsf{D}_{P,a}) \ .$$

We call $\mu_{P,a} = \Delta g_{P,a}$ the bifurcation measure of the pair (P,a). It is a positive measure on $\bar{C}^{\mathrm{an}}$ of total mass $\deg(\mathsf{D}_{P,a})$ which has no atoms.

Remark 4.29. — The right hand side can be interpreted as the canonical height of the point a with respect to P viewed as a polynomial over the global field $K(C)$, see e.g. [**15**, §4]. When $K = \mathbb{C}$, the second author and Vigny [**92**] have extended this fact to arbitrary families of polarized endomorphisms and arbitrary marked subvarieties.

4.2.5. Characterization of passivity

The next result is well-known and was proved by DeMarco [**51**] in the more general case of families of rational maps. We include a proof for sake of completeness.

For any non-constant rational function $f \in K(C)$, let $\deg(f) \in \mathbb{N}^*$ be the number of poles (or zeroes) counted with multiplicities.

Also given any family P parametrized by C, we say that a marked point a is *stably preperiodic* if there exists $n > m \geq 0$ such that $P_t^n(a(t)) = P_t^m(a(t))$ for all $t \in C$.

Theorem 4.30. — *Let (P, a) be a dynamical pair of degree $d \geq 2$ parametrized by an affine irreducible curve C defined over an algebraically closed field K of characteristic 0. Then the following assertions are equivalent:*

(1) *the pair (P, a) is passive on C,*
(2) *the pair (P, a) is either isotrivial or the marked point is stably preperiodic,*
(3) *the divisor $\mathsf{D}_{P,a}$ of the pair (P, a) vanishes,*
(4) *there exists a constant $M > 0$ such that $\deg(P_t^n(a(t))) \leq M$ for all $n \geq 1$.*

When K is a complete metrized field, these assertions are equivalent to:

(5) *the bifurcation measure $\mu_{P,a}$ vanishes.*

When $K = \mathbb{C}$, this is further equivalent to:

(6) $\mathrm{Bif}(P, a)$ *is empty.*

Observe that this implies all characterizations stated in Theorem 2 from the Introduction except for (5) which will be dealt with in §4.4.

Proof. — The implication (2) $\Rightarrow$ (3) is easy. When (P, a) is isotrivial, then $t \mapsto P^n(a(t))$ is constant for all n, and when a is stably preperiodic, then we have $P^n(a) = P^m(a)$ for some $n > m$. In both cases, $\mathrm{ord}_{\mathfrak{c}}(P^n(a))$ is bounded for all branches of C at infinity.

Assume (4). Since $\deg(P^n(a)) = \sum_{\mathfrak{c}} \mathrm{ord}_{\mathfrak{c}}(P^n(a))$, we the sum ranges over all branches at infinity of C, the sequence $\mathrm{ord}_{\mathfrak{c}}(P^n(a))$ is bounded for all branches of C at infinity therefore (3) holds. Conversely, if (3) holds, then Lemma 4.20 implies that $\mathrm{ord}_{\mathfrak{c}}(P^n(a))$ is bounded for all branches of C at infinity so that $\deg(P^n(a))$ is bounded. This shows (3) $\Leftrightarrow$ (4).

Suppose (4) holds and K is a metrized field. Then by Theorem 4.28 we get that $g_{P,a}$ is a continuous semi-positive metrization on the trivial line bundle on $\hat{C}$ which implies $g_{P,a}$ to be constant, and $\mu_{P,a}$ to be equal to 0. This shows (4) $\Rightarrow$ (5). When $K = \mathbb{C}$, the previous argument and Proposition 4.2 shows

(4) $\Rightarrow$ (6). Conversely when either (5) or (6) holds, the divisor $\mathsf{D}_{P,a} = 0$ by Theorem 4.28 which implies (3).

Suppose that $\deg(P^n(a))$ is unbounded. We shall prove that (1) cannot hold. Since the family is algebraic we may replace K by an algebraically closed field which is finitely generated over $\bar{\mathbb{Q}}$, and fix an embedding $K \subset \mathbb{C}$. Let C^{an} be the Riemann surface defined by this embedding. By what precedes, the bifurcation measure $\mu_{P,a}$ is non-zero, hence $\mathrm{Preper}(P, a)$ is dense (for the complex topology) in $\mathrm{Bif}(P, a)$. Since $\mathrm{Bif}(P, a)$ is the support of the Laplacian of a continuous subharmonic function, it is Zariski-dense and we conclude that $\mathrm{Preper}(P, a)$ is Zariski-dense in C^{an}. But $\mathrm{Preper}(P, a)$ is a subset of $C(K)$, hence it is also Zariski-dense in C (viewed as a curve over K). We claim that $C(K) \setminus \mathrm{Preper}(P, a)$ is non-empty showing that (P, a) is active. In C^{an} the complement of $\mathrm{Bif}(P, a)$ is the stability locus which is open and dense, see Remark 4.3. Take any connected component U of the stability locus. Then a cannot be stably periodic on U, otherwise it would be stably periodic on C. The set of $t \in U$ for which a is not preperiodic is thus discrete by Theorem 4.5. Since K is algebraically closed, $C(K)$ is dense in C^{an}, and we get parameters $t \in C(K) \cap U$ which do not belong to $\mathrm{Preper}(P, a)$. We have proved (1) $\Rightarrow$ (4).

Suppose finally that $\deg(P^n(a))$ is bounded. As in the previous argument, we may suppose that $K \subset \mathbb{C}$ so that the bifurcation locus of the pair is empty. By Lemma 4.9, there exists no properly prerepelling parameter. When a is stably preperiodic, then the dynamical pair is passive. When it is not stably preperiodic, we fix any $t_0 \in C$. Conjugating P_{t_0} if necessary, and replacing P by a suitable iterate, we may suppose that 0 and 1 are fixed and repelling. By base change, we may also suppose that the fixed points 0 and 1 can be followed over C: we get two regular functions $p_0, p_1 \colon C \to \mathbb{A}^1$ such that $P_t(p_i(t)) = p_i(t)$ for all t and $p_0(t_0) = 0$, and $p_1(t_0) = 1$. Let C^* be the open Zariski dense subset of C where $p_0 \neq p_1$. Replacing $\phi_t \circ P_t \circ \phi_t^{-1}$ with $\phi_t(z) = \frac{z-p_0}{p_1-p_0}$, we get a family $(P_t)_{t\in C^*}$ such that 0 and 1 are fixed and repelling for P_{t_0}. It follows that $P^n(a)$ forms a sequence of regular functions from C^* to $\mathbb{A}^1 \setminus \{0, 1\}$. By De Franchis theorem, see e.g. [**163**], there exists only finitely many such non-constant maps $C^* \to \mathbb{A}^1 \setminus \{0, 1\}$, hence $P^n(a)$ is constant for all n sufficiently large. This implies $t \mapsto P_t(b)$ to be constant for infinitely many $b \in \mathbb{A}^1$ hence $P_t(b)$ is constant for all b. And we conclude that $(P, P^n(a))$ is isotrivial for some $n \gg 1$ hence (P, a) is isotrivial. This concludes the proof (4) $\Rightarrow$ (1). $\square$

Remark 4.31. — Observe that our proof of (1) $\Rightarrow$ (4) implies that for any active dynamical pair (P, a) both sets $\mathrm{Preper}(P, a)$ and $C(K) \setminus \mathrm{Preper}(P, a)$ are Zariski-dense.

4.3. Family of polynomials and Green functions

We explain how the results of the previous section applies to study the bifurcation locus of a family of polynomials. Recall that $G(P) = \max\{g_P(c),\ P'(c) = 0\}$. As usual we write $\bar{C}$ is a projective compactification of C such that $\bar{C} \setminus C$ is a finite set of smooth points.

Theorem 4.32. — *Let P be any family of polynomials parametrized by a curve C defined over a field K of characteristic 0. For each branch $\mathfrak{c}$ of C at infinity, there exists a constant $q_{\mathfrak{c}}(P) \in \mathbb{Q}_+$ such that for any norm $|\cdot|$ on K, and for any analytic parametrization $t \mapsto \theta(t)$ of a neighborhood of $\mathfrak{c}$ in $\hat{C}^{\mathrm{an}}$ we have*

$$G(P_{\theta(t)}) = q_{\mathfrak{c}}(P) \log |t|^{-1} + \tilde{G}(t)$$

where $\tilde{G}$ extends continuously through the origin.

Remark 4.33. — This result is completely analogous to [**78**, Corollary 1] where $G(P) = \max\{g_P(c),\ P'(c) = 0\}$ is replaced by the Lyapunov exponent of P with respect to the equilibrium measure which satisfies the Misiurewicz-Prytycky's formula $\lambda(P) = \log |d| + \sum_{P'(c)=0} g_P(c)$, see (9).

Proof. — Fix a branch and take a finite cover $\bar{D} \to \bar{C}$ to follow all critical points $c_1, \ldots, c_{d-1}$. Apply Theorem 4.27 to each c_i, and write

$$g_{P,c_i}(t) = q_{\mathfrak{c}}(P, c_i) \log |t|^{-1} + \tilde{g}_i(t)$$

with $q_{\mathfrak{c}}(P, c_i) \in \mathbb{Q}_+$ and $\tilde{g}_i$ continuous and subharmonic. We conclude by setting $q_{\mathfrak{c}}(P) = \max_i\{q_{\mathfrak{c}}(P, c_i)\}$ and $\tilde{G} = \max_{i \in I} \tilde{g}_i$ with I denotes the set of all i such that $q_{\mathfrak{c}}(P) = q_{\mathfrak{c}}(P, c_i)$. □

Assume P is parametrized by a curve C which is defined over a complete and algebraically closed field $(K, |\cdot|)$ of characteristic 0 and that P is critically marked, i.e. there exist $c_1, \ldots, c_{d-1} \in K[C]$ that follow the critical points of P.

Definition 4.34. — *The bifurcation measure of the family P is the probability measure μ_{bif} on C^{an} which is proportional to the positive measure $\sum_{i=1}^{d-1} \mu_{P,c_i}$.*

4.4. Arithmetic polynomial dynamical pairs

Assume C is an algebraic curve defined over a number field $\mathbb{K}$, and (P, a) is an algebraic dynamical pair parametrized by C and defined over $\mathbb{K}$. Recall that $\bar{C}$ is a projective compactification of C such that $\bar{C} \setminus C$ is a finite set of smooth points.

If $t \in C(\bar{\mathbb{K}})$, then P_t is defined over a number field, and we attach to it a canonical height h_t, see §2.6.

The canonical height of a dynamical pair. — Ingram [**110**] proved that the function $t \mapsto \hat{h}_{P_t}(a(t))$ defines a Weil height on $\bar{C}$. We improve his result and show that it is an Arakelov height.

Proposition 4.35. — *Assume (P, a) is active. There exists a positive integer $n \geq 1$ such that $n \cdot \mathsf{D}_{P,a}$ is a positive integral divisor on $\bar{C}$ and the collection of subharmonic functions $\{n \cdot g_{P,a,v}\}_{v \in M_{\mathbb{K}}}$ induces a semi-positive adelic metrization on the ample line bundle $\mathcal{L} := \mathcal{O}_{\bar{C}}(n \cdot \mathsf{D}_{P,a})$. The induced height function $h_{\bar{\mathcal{L}}}$ on $\bar{C}$ satisfies*

1. *$h_{\bar{\mathcal{L}}}(t) = n \cdot \hat{h}_{P_t}(a(t))$ for all $t \in C(\bar{\mathbb{K}})$, i.e.*
$$h_{\bar{\mathcal{L}}}(t) = \frac{n}{\deg(t)} \sum_{v \in M_{\mathbb{K}}} \sum_{t' \in \mathsf{O}(t)} g_{P_{t'},v}(a(t')),$$
where $\mathsf{O}(t)$ is the $\mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$-orbit of t and $\deg(t) = \mathrm{Card}(\mathsf{O}(t))$;
2. *$h_{\bar{\mathcal{L}}}(t) = 0$ if and only if $t \in \mathrm{Preper}(P, a)$;*
3. *the global height of $\bar{C}$ is equal to $h_{\bar{\mathcal{L}}}(\bar{C}) = 0$;*
4. *at any place $v \in M_{\mathbb{K}}$, the associated measure $c_1(\bar{\mathcal{L}})_v$ on $\bar{C}_v^{\mathrm{an}}$ is $n \cdot \Delta g_{P,a,v}$.*

Proof. — Let n be any integer such that $n \cdot \mathsf{D}_{P,a}$ has integral coefficients. For any place $v \in M_{\mathbb{K}}$, Theorem 4.28 implies that $n \cdot g_{P,a,v}$ defines a continuous and semi-positive metrization $|\cdot|_{a,v}$ on $\mathcal{O}_{\bar{C}}(n \cdot \mathsf{D}_{P,a})$. Let us justify that the collection $\{|\cdot|_{a,v}\}_{v \in M_{\mathbb{K}}}$ is adelic.

We make some preliminary comments. By a suitable base change over C we may (and shall) assume that the family $\{P_t\}$ is of the form (10). To avoid conflict of notation we write $P_t = P_{c(t),\alpha(t)}$ with $c, \alpha \in \mathbb{K}[C]$.

We fix an embedding $C \subset \mathbb{A}^N$, so that the completion $\bar{C}$ of C in $\mathbb{P}^N$ is smooth at infinity. For any non-Archimedean place v, let $\mathrm{red}_v \colon \mathbb{P}^N(\mathbb{C}_v) \to \mathbb{P}^N(\widetilde{\mathbb{C}_v})$ be the canonical reduction map, defined by sending a point $[z_0 : \cdots : z_N]$ to $[\widetilde{\lambda z_0} : \cdots : \widetilde{\lambda z_N}]$ where $\lambda = \max\{|z_0|, \cdots, |z_N|\}^{-1}$. We let $\tilde{C} \subset \mathbb{P}^N(\widetilde{\mathbb{C}_v})$ be the image of C under red_v. We shall make a systematic use of the following results.

Lemma 4.36. — *Let $f \in \mathbb{K}(C)$. Then for all but finitely many places $v \in M_{\mathbb{K}}$, we have*

$$\{t \in C_v^{\mathrm{an}},\, |f(t)| < 1\} = \mathrm{red}_v^{-1}(\mathrm{red}_v(f^{-1}(0)))\ .$$

Observe that replacing f by its inverse implies $\{t \in C_v^{\mathrm{an}},\, |f(t)| > 1\} = \mathrm{red}_v^{-1}(\mathrm{red}_v(f^{-1}(\infty)))$.

Lemma 4.37. — *Let $f, g \in \mathbb{K}[C]$ such that $-\mathrm{ord}_{\mathfrak{c}}(f) \geq -\mathrm{ord}_{\mathfrak{c}}(g)$ for all branches at infinity of C. Then for all but finitely many places we have* $\max\{1, |f|_v\} \geq |g|_v$.

During the proof S will be a set of places on $\mathbb{K}$ containing all archimedean ones which may vary from line to line but shall remain finite.

Let us now prove that $\{|\cdot|_{a,v}\}_{v\in M_{\mathbb{K}}}$ is adelic. Denote by $\mathcal{B}$ the set of branches at infinity that do not lie in the support of $\mathsf{D}_{P,a}$.

Suppose first that $\mathcal{B} = \varnothing$. By Lemma 4.20, we can find a sufficiently large integer $q \in \mathbb{N}^*$ such that

$$-d^q \cdot \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) > -\max\{\mathrm{ord}_{\mathfrak{c}}(c), \mathrm{ord}_{\mathfrak{c}}(\alpha)\} \text{ and } \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) = -\frac{1}{d^q}\mathrm{ord}_{\mathfrak{c}}(P^q(a))$$

for all branches at infinity $\mathfrak{c}$. It follows from Lemma 4.37 that

$$\max\{1, |P_t^q(a(t))|_v\} \geq \max\{|c(t)|_v, |\alpha(t)|_v\}\ ,$$

for all $v \notin S$, so that Proposition 2.11 yields $g_{P,a,v}(t) = \frac{1}{d^q} g_{P,v}(P^q(a)) = \frac{1}{d^q}\log^+|P^q(a)|_v$. The metric is thus adelic by Lemma 1.11.

When $\mathcal{B}$ is non-empty, one proceeds as follows. Replacing $\mathsf{D}_{P,a}$ by a sufficiently high multiple, we may suppose that it is very ample so that we can find a rational function $h \in \mathbb{K}(C)$ whose divisor of poles is equal to $\mathsf{D}_{P,a}$. We need to prove that $g_{P,a,v} = \log^+|h|_v$ for all but finitely many places $v \notin S$.

As above, we pick a sufficiently large integer $q \in \mathbb{N}^*$ such that

$$-d^q \cdot \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) > -\max\{\mathrm{ord}_{\mathfrak{c}}(c), \mathrm{ord}_{\mathfrak{c}}(\alpha)\} \text{ and } \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) = -\frac{1}{d^q}\mathrm{ord}_{\mathfrak{c}}(P^q(a))$$

for all $\mathfrak{c} \in \mathrm{supp}(\mathsf{D}_{P,a})$ (observe that the latter set is now strictly included in the set of branches at infinity of C).

Since any pole of h is a pole of $P^q(a)$, Lemma 4.36 shows that

$$U := \{|h|_v > 1\} \subset \{|P^q(a)|_v > 1\}\ .$$

Similarly the function $h^{d^q}/P^q(a)$ does not vanish at any point in $\mathrm{supp}(\mathsf{D}_{P,a})$, hence we have $|h|_v^{d^q} = |P^q(a)|_v$ on U.

We fix an auxiliary rational function $\eta \in \mathbb{K}(C)$ whose zero locus is equal to the support of $\mathsf{D}_{P,a}$, whose set of poles is equal to $\mathcal{B}$ and which satisfies

$$-\operatorname{ord}_{\mathfrak{c}}(\eta \times P^q(a)) \geq \max\{-\operatorname{ord}_{\mathfrak{c}}(c), -\operatorname{ord}_{\mathfrak{c}}(\alpha)\}$$

for all branches at infinity (one may need to increase q).

Applying again Lemma 4.36, we get $\{|\eta|_v < 1\} = U$. On the other hand Lemma 4.37 implies

$$\max\{1, |P_t^q(a(t))|_v \times |\eta(t)|\} \geq \max\{|c(t)|_v, |\alpha(t)|_v\} ,$$

so that $|P_t^q(a(t))| > \max\{|c(t)|_v, |\alpha(t)|_v\}$ for all $t \in U$. We infer by Proposition 2.11 that

$$g_{P,a,v} = \frac{1}{d^q} \log |P^q(a)|_v = \log |h|_v$$

on U.

Observe now that $g_{P,a,v}$ extends to $C_0 := \bar{C} \setminus \operatorname{supp}(\mathsf{D}_{P,a})$ as a non-negative continuous subharmonic function by Theorem 4.27. The set $\{|h| \leq 1\}$ is compact in C_0^{an} and its boundary is equal to ∂U, so that $g_{P,a,v} \equiv 0$ on $\partial\{|h| \leq 1\}$. Note that the latter set consists of finitely many type 2 points corresponding to the irreducible components of the reduction of C containing a pole of $\tilde{h}$ (see the discussion on models in §1.1.3). We may now apply the maximum principle to any connected component of the interior of $\{|h| \leq 1\}$, and we obtain that $g_{P,a,v} \equiv 0$ on $\{|h| \leq 1\}$.

This concludes the proof that $g_{P,a,v} = \log |h|_v$ everywhere.

Properties (1), (2) and (4) follow from the definitions. To prove (3), we use [**44**, (1.2.6) & (1.3.10)]. Choose any two meromorphic functions ϕ_0, ϕ_1 on $\hat{C}$ such that $\operatorname{div}(\phi_0) + n\mathsf{D}_{P,a}$ and $\operatorname{div}(\phi_1) + n\mathsf{D}_{P,a}$ are both effective with disjoint support included in C. Let σ_0 and σ_1 be the associated sections of $\mathcal{O}_{\hat{C}}(n\mathsf{D}_{P,a})$. Let $\sum n_i[t_i]$ be the divisor of zeroes of σ_0, and $\sum n'_j[t'_j]$ be the divisor of zeroes of σ_1. Then

$$\begin{aligned} h_{\bar{\mathcal{L}}}(\hat{C}) &= \sum_{v \in M_{\mathbb{K}}} (\widehat{\operatorname{div}}(\sigma_0) \cdot \widehat{\operatorname{div}}(\sigma_1) | \hat{C})_v \\ &= \sum_i n_i \cdot n \cdot \hat{h}_{P_{t_i}}(a(t_i)) - \sum_{v \in M_{\mathbb{K}}} \int_{\bar{C}} \log |\sigma_0|_{a,v} \, \Delta(n \cdot g_{P,a,v}) \\ &= \sum_{v \in M_{\mathbb{K}}} \int_{\bar{C}} n \cdot g_{P,a,v} \, \Delta(n \cdot g_{P,a,v}) \geq 0 , \end{aligned}$$

where the third equality follows from Poincaré-Lelong formula and writing $\log |\sigma_0|_{a,v} = \log |\phi_0|_v - n \cdot g_{P,a,v}$.

Pick any archimedean place v_0. The total mass on $\hat{C}$ of the positive measure $\Delta g_{P,a,v_0}$ is the degree of $\mathcal{L}$, hence is non-zero. It follows from e.g. [**68**, Lemma 2.3] that any point t_0 in the support of $\Delta g_{P,a,v_0}$ is accumulated by parameters $t_* \in C(\bar{\mathbb{K}})$ such that $P^n_{t_*}(a(t_*)) = P^m_{t_*}(a(t_*))$ for some $n > m \geq 0$. For any such point, the point (1) implies $h_{\bar{\mathcal{L}}}(t_*) = 0$. The result follows from Theorem 1.12. □

Proof of Lemma 4.36. — In some affine chart, C is given by the vanishing of polynomials with coefficients in $\mathcal{O}_{\mathbb{K},S}$ and f is the restriction of a polynomial with coefficients in the same ring of integers. Enlarging S if necessary, for any place $v \notin S$, the reduction $\tilde{C}_v$ of C over $\mathbb{C}_v$ is an affine curve over $\widetilde{\mathbb{C}}_v$, and f induces a regular function $\tilde{f}$ on $\tilde{C}_v$. We have the following commutative diagram:

$$\begin{array}{ccc} C(\mathbb{C}_v) & \xrightarrow{\text{red}} & \tilde{C}_v \\ \downarrow f & & \downarrow \tilde{f} \\ \mathbb{P}^1(\mathbb{C}_v) & \xrightarrow{\text{red}} & \mathbb{P}^1(\widetilde{\mathbb{C}_v}) \end{array}$$

from which the result follows. □

Proof of Lemma 4.37. — We first embed $\mathbb{K}$ into $\mathbb{C}$ and argue analytically. The function field $K(C)$ is finitely generated over $K(f)$, hence we may find rational functions $R_0, \cdots, R_n$ such that $g^n + g^{n-1}R_1(f) + \cdots + gR_{n-1}(f) + R_n(f) = 0$. In fact $R_j(z)$ is the symmetric polynomial of degree j of the collection of points $g(w_j)$ where w_j are the roots of $f(w) = z$. Since the set of poles of g is included in the set of poles of f, all R_j are regular on $\mathbb{A}^1$ and are hence polynomials. By Galois invariance, they are defined over a finite extension of $\mathbb{K}$. Fix a local coordinate w at a branch at infinity $\mathfrak{c}$. We may then write $|f(w)| \asymp |w|^{d_\mathfrak{c}}$ and $|g(w)| \asymp |w|^{c_\mathfrak{c}}$ for some integers $c_\mathfrak{c} \leq d_\mathfrak{c}$ so that

$$\log |R_j(z)| \lesssim \left(\sum_{|I|=j} \prod_{i \in I} \frac{c_{\mathfrak{c}_i}}{d_{\mathfrak{c}_i}} \right) \log |z|,$$

near infinity, and R_j has degree $\leq j$.

Take a place $v \in M_\mathbb{K}$ for which all coefficients of R_j have norm ≤ 1. For each $w \in C$, we get $|g(w)|^n \leq \max\{|g^j(w)| \cdot |R_{n-j}(f(w))|\} \leq \max\{|g^j(w)| \cdot \max\{1, |f(w)|\}^{n-j}\}$ hence $|g(w)| \leq \max\{1, |f(w)|\}$. □

Canonical dynamical height and Weil heights. — Since $h_{P,a}$ is a height induced by a semi-positive adelic metrization, it differs from any Weil height attached to the ample line bundle $\mathcal{L}$ by a bounded function, a result that was first proved by Ingram [**110**]. We then obtain from Northcott's theorem the following

Corollary 4.38. — *Let (P, a) be an active dynamical pair parametrized by an affine curve defined over a number field $\mathbb{K}$. For any integer N, and any $B > 0$, then there exists a constant $C = C(N, B)$ such that the set of parameters $t \in C(\bar{\mathbb{K}})$ defined over a field of degree at most $\leq N$ over $\mathbb{K}$ for which $h_{P,a}(t) \leq B$ is finite.*

In particular $\mathrm{Preper}(P, a) \cap C(\mathbb{K})$ *is finite.*

The following characterization of activity of an arithmetic pair then follows quickly from Theorem 4.30 (note that the next statement implies Theorem 2).

Theorem 4.39. — *Let (P, a) be a dynamical pair parametrized by an affine irreducible curve C defined over $\mathbb{K}$. If the pair (P, a) is not isotrivial, then the following assertions are equivalent:*

1. *the pair (P, a) is passive on C,*
2. *the height function $h_{P,a}$ satisfies $h_{P,a}(t) = 0$ for all $t \in C(\bar{\mathbb{K}})$,*
3. *for any place $v \in M_{\mathbb{K}}$, we have $g_{P,a,v} \equiv 0$ on C_v^{an},*
4. *there exists a place $v \in M_{\mathbb{K}}$ such that $g_{P,a,v} \equiv 0$ on C_v^{an},*
5. *for any place $v \in M_{\mathbb{K}}$, we have $\Delta g_{P,a,v} = 0$ on C_v^{an},*
6. *there exists a place $v \in M_{\mathbb{K}}$ such that $\Delta g_{P,a,v} = 0$ on C_v^{an}.*

Proof. — If (P, a) is passive, then Theorem 4.30 implies that (P, a) is stably preperiodic so that (2) – (6) are clearly satisfied. The next diagram of implications is also clear:

$$\begin{array}{ccccccc} (2) & \Longrightarrow & (3) & \Longrightarrow & (4) & \Longrightarrow & (6) \\ & & & \searrow & & \nearrow & \\ & & & & (5) & & \end{array}$$

So assume $\Delta g_{P,a,v} = 0$ on C_v^{an} for some $v \in M_{\mathbb{K}}$. Then Statement (5) of Theorem 4.30 is satisfied, and we conclude that (P, a) is passive. □

CHAPTER 5

ENTANGLEMENT OF DYNAMICAL PAIRS

We introduce the notion of entanglement of dynamical pairs and prove that for any two entangled pairs (P, a) and (Q, b) there exist iterates of P and Q, that have the same degree, and are intertwined in the sense of §3.5.2. This result leads to the proof of Theorem B in the number field case. We then deduce Theorem C over an arbitrary field of characteristic zero using a specialization argument.

5.1. Dynamical entanglement

5.1.1. Definition

We fix a field K of characteristic 0, and let $\bar{K}$ be an algebraic closure of K. We introduce the following terminology.

Definition 5.1. — *Let (P, a) and (Q, b) be two active dynamical pairs parametrized by an affine curve C defined over K. We say that (P, a) and (Q, b) are dynamically entangled when the set of parameters $t \in C(\bar{K})$ for which $a(t)$ and $b(t)$ are preperiodic for both P_t and Q_t is Zariski dense, i.e.* $\mathrm{Preper}(P, a, \bar{K}) \cap \mathrm{Preper}(Q, b, \bar{K})$ *is infinite.*

Being entangled defines an equivalence relation on the set of active dynamical pairs parametrized by a fixed affine curve.

Remark 5.2. — When K is algebraically closed, the entanglement is equivalent to say that $\mathrm{Preper}(P, a) \cap \mathrm{Preper}(Q, b)$ is Zariski dense. When K is not algebraically closed, e.g. when K is a number field, $\mathrm{Preper}(P, a) \subset C(K)$ can be finite, and one needs to look at all parameters in $C(\bar{K})$ to test the entanglement.

Remark 5.3. — The dynamical entanglement is a rigidity property that links two dynamical pairs. Since passive pairs are already rigid, the entanglement property is only relevant for active pairs.

Remark 5.4. — One can extend the notion of entanglement to a dynamical pair defined over a base of arbitrary dimension.

Recall the definition of intertwining from §3.5.2 and its geometric characterization (Theorem 3.39).

Proposition 5.5. — *Let P, Q be any two active families of polynomials of the same degree $d \geq 2$ parametrized by an irreducible affine curve C. Assume that they are intertwined as polynomials with coefficients in $K(C)$, and pick any algebraic subvariety $Z \subset \mathbb{A}^1_C \times \mathbb{A}^1_C$ which projects onto each factor and which is invariant by the map $(z, w) \mapsto (P(z), Q(w))$.*

For any $a, b \in K(C)$ such that $(a, b) \in Z$, there exists a Zariski open dense subset $U \subset C$ over which (P, a) and (Q, b) define dynamical pairs parametrized by U that are dynamically entangled.

Proof. — Pick any Zariski open set $U \subset C$ such that a, b define regular functions on U, and the restriction of the two natural projections $Z' = Z \cap \mathbb{A}^2_U \to \mathbb{A}^1_U$ are finite.

Then (P, a) and (Q, b) define active dynamical pairs and by definition $\mathrm{Preper}(P, a)$ is Zariski dense in U. Pick any point $t \in \mathrm{Preper}(P, a) \subset U$. Then for all $n \in \mathbb{N}$, we have $(P^n(a(t)), Q^n(b(t))) \in Z$. Since the set $\{P^n(a(t))\}_{n\in\mathbb{N}}$ is finite, and the second projection $Z_U \to U$ is finite, it follows that the set $\{Q^n(b(t))\}_{n\in\mathbb{N}}$ is also finite which proves that $\mathrm{Preper}(Q, b) \cap U = \mathrm{Preper}(P, a) \cap U$ is Zariski-dense. The two pairs are thus entangled. □

As a corollary of the previous result, one obtains:

Corollary 5.6. — *Let (P, a) be an active dynamical pair parametrized by an affine curve C.*

- *For any $n \in \mathbb{N}$, and for any $g \in \Sigma(P)$, the pairs (P, a), $(P, g \cdot a)$, $(g \cdot P, a)$, and $(P, P^n(a))$ are entangled.*
- *For any polynomial family Q parametrized by C and commuting with P, the pairs (P, a), $(P, Q(a))$ and (Q, a) are entangled.*

Proof. — When $\Sigma(P)$ is infinite, then we may suppose $P = M_d$ and the result is easy. Otherwise the curve $Z = \cup_{g' \in \Sigma(P)}\{(z, g' \cdot z)\} \subset \mathbb{A}^1 \times \mathbb{A}^1$ is invariant by (P, P) and $(P, g \cdot P)$, hence (P, a), $(P, g \cdot a)$, and $(g \cdot P, a)$ are entangled by Proposition 5.5. Also the graph $\{(z, P^n(z))$ is invariant by (P, P) so that (P, a), and $(P, P^n(a))$ are entangled.

If Q commutes with P, then the graph $\{(z, Q(z))\}$ is (P, P)-invariant hence (P, a), and $(P, Q(a))$ are entangled. Finally if $a(t)$ has a finite orbit for P of length $\leq N$, then $Q^n(a(t))$ has also a finite orbit of length $\leq N$ for any $n \geq 0$, hence a is also preperiodic for Q. It follows that (P, a) and (Q, a) are entangled. □

5.1.2. Characterization of entanglement

Let us recall the following theorem from the introduction.

Theorem B. — *Let (P, a) and (Q, b) be active non-integrable polynomial dynamical pairs parametrized by an irreducible algebraic curve C of respective degree $d, \delta \geq 2$. Assume that the two pairs are defined over a number field $\mathbb{K}$. Then, the following are equivalent:*

1. *the set* $\mathrm{Preper}(P, a) \cap \mathrm{Preper}(Q, b)$ *is an infinite subset of* $C(\bar{\mathbb{K}})$*;*
2. *the two height functions* $h_{P,a}, h_{Q,b} \colon C(\bar{\mathbb{K}}) \to \mathbb{R}_+$ *are proportional;*
3. *there exist integers $N, M \geq 1$, $r, s \geq 0$, and families R, τ and π of polynomials of degree ≥ 1 parametrized by C such that*

$$\tau \circ P^N = R \circ \tau \quad \textit{and} \quad \pi \circ Q^M = R \circ \pi, \tag{†}$$

and $\tau(P^r(a)) = \pi(Q^s(b))$.

The most interesting implication is (1) ⇒ (3) which is an arithmetic rigidity result. In plain words it implies that two entangled pairs define families of intertwined polynomials. The proof of Theorem B will occupy most of this chapter including all sections from §5.2 to §5.4. For the convenience of the reader, we give an overview of this technically involved proof in the next paragraph. The proof of Theorem C is given in §5.5.

5.1.3. Overview of the proof of Theorem B

Assume (3), and pick any parameter t for which $\{P_t^n(a(t))\}_{n\in\mathbb{N}}$ is a finite set. Since

$$\pi \circ Q^{nM}(Q^s(b)) = R^n \tau(P^r(a)) = \tau \circ P^{nN}(a)$$

and the two maps τ, π are finite, it follows that $\{Q_t^n(b(t))\}_{n\in\mathbb{N}}$ is also a finite set, hence $t \in \mathrm{Preper}(Q, b)$. This implies (1).

The proof of the implication (1) ⇒ (2) is given in §5.2. By Proposition 4.35, we may apply Thuillier-Yuan's theorem (Theorem 1.13) to both height functions, which gives equality of bifurcation measures $n \cdot \mu_{P,a,v} = m \cdot \mu_{Q,b,v}$ at all places (for suitable integers n and m). The equality $n \cdot h_{P,a} = m \cdot h_{Q,b}$ is equivalent to the equality of Green functions $n \cdot g_{P,a,v} = m \cdot g_{Q,b,v}$ at all place v, which we obtain in Theorem 5.7 using our rigidity result in the parameter space (Theorem A).

The implication (2) ⇒ (3) is substantially harder. We first prove that (2) implies that $\deg(P)$ and $\deg(Q)$ are multiplicatively dependent, see Theorem 5.10. We argue by contradiction. We compute the Hölder constant of continuity of the Green function (at a fixed Archimedean place) of the two dynamical pairs at a parameter $t \in C$ which is prerepelling (for the two marked points). Since the Green functions are proportional, the Hölder constants are equal so that the multiplier of P_t at $a(t)$ and Q_t at $b(t)$ are multiplicatively dependent. Using ideas from Levin [**117**], we build a real-analytic flow commuting with the dynamics which implies P and Q to be integrable.

The core of the argument is the content of §5.4. The main ideas are already present in Baker and DeMarco's original paper [**5**] and relies on the expansion of the Böttcher coordinate and Ritt's theory. Note however that C may have several places at infinity, and this makes things considerably more technical.

We have divided the proof into four steps indicated (I) to (IV). Step (IV) is not essential. In this last step, we explain how to reduce the problem to families of monic and centered polynomial. We thus focus on the description of the first three steps.

Step (I) consists in the analysis of the Böttcher coordinate near a suitable branch at infinity of C. More precisely, we fix such a branch $\mathfrak{c}$ lying in the support of the divisor $\mathsf{D}_{P,a}$ (i.e. of $\mathsf{D}_{Q,b}$ since these divisors are proportional). Let us take an adelic parametrization $t \mapsto \theta(t)$ of $\mathfrak{c}$. To simplify notation, we write P_t, $a(t)$, instead of $P_{\theta(t)}$, $a(\theta(t))$, etc.

Using the expansion of the Böttcher coordinates explained in §2.5, we next show that $\varphi_{P_t}(P_t^{lN}(a(t)))$ and $\varphi_{Q_t}(Q_t^{lM}(b(t)))$ are well-defined as adelic series, and satisfies

$$\left(\varphi_{P_t}\left(P_t^{lN}(a(t))\right)\right)^n = \zeta\ \left(\varphi_{Q_t}\left(Q_t^{lM}(b(t))\right)\right)^m \text{ for all } l\ ,$$

for some root of unity ζ that may depend on l and on $\mathfrak{c}$. We get some control on ζ by observing that it always belongs to a fixed finite extension of our field of definition $\mathbb{K}$.

During the whole duration of Step (II) we work with a suitable fixed parameter t sufficiently close (at some place fixed Archimedean place v_0) to $\mathfrak{c}$. We consider the analytic curve $\mathfrak{s}_\zeta = \{(x,y),\, \varphi_{P_t}(x)^n = \zeta\varphi_{Q_t}(y)^m\}$ defined in an open set of the form $\{|x| \geq R_v, |y| \geq R_v\}$ inside $\mathbb{A}_v^{2,\mathrm{an}}$. We observe that this analytic curve contains many algebraic points of the form $(P_t^{lN}(a(t)), Q_t^{lM}(b(t)))$. This allows us to apply the algebricity result of Xie (Theorem 1.17 from Chapter 1), which proves that the Zariski closure of $\mathfrak{s}_\zeta$ is in fact an algebraic curve Z_t. Using a result from Pakovich relying on Ritt's theory, we may also assume that Z_t is of the form $\{u(x) = v(y)\}$ for some polynomials u and v (depending on t). Unwinding definitions, this implies that P_t and Q_t are intertwined.

Finally we exhibit in Step (III) a series of conditions $(\mathcal{P}_1)$ – $(\mathcal{P}_3)$ on the polynomials P_t, Q_t and the points $a(t), b(t)$ which imply P_t and Q_t to be intertwined and that depend algebraically on t. The second step furnishes an infinite set of parameters t satisfying these conditions, hence they are satisfied for all $t \in C$. This implies the families P and Q to be intertwined and concludes the proof.

5.2. Dynamical pairs with identical measures

In this section, we focus on dynamical pairs having identical bifurcation measures and prove the implication (1) ⇒ (2) of Theorem B.

5.2.1. Equality at an Archimedean place

In this section we assume $K = \mathbb{C}$.

Theorem 5.7. — *Let (P,a) and (Q,b) be two active and non-integrable dynamical pairs parametrized by an irreducible affine complex curve C. Assume there exist two integers $n, m \geq 1$ such that $n \cdot \Delta g_{P,a} = m \cdot \Delta g_{Q,b}$.*

Then we have equality of Green functions $n \cdot g_{P,a} = m \cdot g_{Q,b}$ on C.

Proof. — Recall from §1.2.3 that a continuous function is subharmonic iff its pull-back on the normalization $\tilde{C}$ of C is subharmonic, and that its Laplacian is given by the push-forward by the normalization map of the Laplacian on $\tilde{C}$. We may therefore replace C by its normalization, and assume the curve to be smooth.

By assumption, the function

$$h(t) := n \cdot g_{P,a}(t) - m \cdot g_{Q,b}(t)$$

is harmonic on C. By Proposition 4.2, we have

$$\mathrm{Bif}(P,a) = \mathrm{supp}(\Delta g_{P,a}) = \partial\{g_{P,a}=0\} \text{ and}$$
$$\mathrm{Bif}(Q,b) = \mathrm{supp}(\Delta g_{Q,b}) = \partial\{g_{Q,b}=0\}\ ,$$

so that

$$\mathrm{Bif}(P,a) = \mathrm{Bif}(Q,b) \subset \{h=0\}.$$

Since the pairs are active the bifurcation loci are non-empty. Observe that there is nothing to prove when h is identically zero, so that we may assume that $\{h=0\}$ is a non-empty real-analytic curve. If $\mathrm{Bif}(P,a)$ is not totally disconnected, then we can find a holomorphic disk U for which $\mathrm{Bif}(P,a)\cap U$ is a smooth curve, and Theorem A implies P and Q to be integrable which contradicts our assumption.

When $\mathrm{Bif}(P,a)$ is totally disconnected, the open set $U := C\setminus \mathrm{Bif}(P,a)$ is connected, the functions $g_{P,a}$ and $g_{Q,b}$ are harmonic on U, and we have

$$U = \{g_{P,a}>0\} = \{g_{Q,b}>0\}.$$

For any $\varepsilon>0$, we set $U_\varepsilon := \{g_{P,a}>\varepsilon\}\cap\{g_{Q,b}>\varepsilon\}\subset C$. Observe that $\cup_{\varepsilon>0}U_\varepsilon = U$.

Proposition 5.8. — *For any $\varepsilon>0$ small enough, there exist two integers $N,M\geq 1$ depending on ε such that both functions $\varphi_{P_t}(P_t^N(a(t)))$ and $\varphi_{Q_t}(Q_t^M(b(t)))$ are well-defined on U_ε, and*

$$U_\varepsilon \ni t \mapsto \frac{\varphi_{P_t}\left(P_t^N(a(t))\right)^{n\delta^M}}{\varphi_{Q_t}\left(Q_t^M(b(t))\right)^{md^N}} \tag{32}$$

extends as a holomorphic nowhere vanishing function f on C satisfying

$$d^N\delta^M h = \log|f|\ . \tag{33}$$

Let us take this proposition for granted. By Theorem A, we may find a univalent holomorphic map $\imath\colon \mathbb{D}\to C$ whose image intersects $\mathrm{Bif}(P,a)$ so that the dynamical pair $(\bar P,\bar a) = (P\circ\imath, a\circ\imath)$ induced on $\mathbb{D}$ is real in the sense of §4.1.3, and $\mathcal{B} := \mathrm{Bif}(\bar P,\bar a) = \mathrm{Bif}(\bar Q,\bar b)$ is a closed perfect totally disconnected subset of $]-1,1[$. By Remark 4.16, we infer that $(\bar Q,\bar b) = (Q\circ\imath, b\circ\imath)$ is also real.

We may thus choose $\varepsilon>0$ so small that the open set $I :=]-1,1[\,\cap\,\imath^{-1}(U_\varepsilon)$ is non-empty. Proposition 5.8 furnishes a holomorphic map $f\colon\mathbb{D}\to\mathbb{C}$ satisfying (32) and (33). By Proposition-Definition 2.19, $\varphi_{\bar P_t}$ and $\varphi_{\bar Q_t}$ are also formal Laurent series with real coefficients, hence the real-analytic map $f|_I$ is real, which implies $f(]-1,1[)\subset\mathbb{R}$. Recall that the harmonic function h is vanishing on $\mathcal{B}$, hence on $]-1,1[$, and we conclude from (33) that $f(t)\in\{-1,1\}$ for all

t real. This implies f to be a constant, hence h too. This contradicts our standing assumption. □

Proof of Proposition 5.8. — Recall that we assumed C to be smooth. Let $\bar{C}$ be any smooth projective compactification C. As above, write $\mathcal{B} := \mathrm{Bif}(P, a) = \mathrm{Bif}(Q, b)$.

Let $\mathfrak{c}$ be any branch at infinity of C. Suppose $\mathfrak{c} \in \bar{\mathcal{B}}$. Apply Theorem 4.22 to the dynamical pair (P, a) restricted to a punctured disk centered at $\mathfrak{c}$. Then $q_{\mathfrak{c}}(P, a) = 0$, hence we fall in case (1) or (3) of Theorem 4.22. It follows that $g_{P,a}$ extends through $\mathfrak{c}$ as a continuous subharmonic function. The same argument shows that $g_{Q,b}$ also extends continuously through $\mathfrak{c}$. It follows that h also extends continuously, hence defines a harmonic function in a neighborhoof of $\mathfrak{c}$. Suppose that $\mathfrak{c} \notin \bar{\mathcal{B}}$. Then $g_{P,a}$ is harmonic and positive in a punctured neighborhood of $\mathfrak{c}$, hence either $q_{\mathfrak{c}}(P, a) > 0$ or $q_{\mathfrak{c}}(P, a) = 0$ and $g_{P,a}$ extends continuously through $\mathfrak{c}$. In the latter case, by the maximum princple $g_{P,a}(\mathfrak{c}) > 0$.

Let C_+ be the union of C together with all branches at infinity of C lying in $\bar{\mathcal{B}}$. Then $g_{P,a}$ and $g_{Q,b}$ are continuous and subharmonic and h is harmonic on C_+. And for $\epsilon > 0$ small enough, the set $K_\epsilon := C_+ \setminus U_\epsilon$ is a compact neighborhood of $\bar{\mathcal{B}}$ of C_+.

Fix $\epsilon > 0$ small. Recall that for any t close enough to a puncture $\mathfrak{c}$ one has $g_{P,a}(t) \geq q_{\mathfrak{c}}(P, a) \log |t|^{-1} + O(1)$, whereas $G(P_t) \leq \kappa \log |t|^{-1} + O(1)$ for some constant $\kappa \geq 0$ by Proposition 2.13. It follows that $g_{P,a}(t) \geq \eta \cdot G(P_t)$ for some $\eta > 0$ and for all $t \notin K_\epsilon$. We conclude that for a sufficiently large integer $N \geq 1$, we have $g_{P_t}(P^N(a(t))) = d^N g_{P,a}(t) > G(P_t)$ for all $t \in U_\varepsilon$. Similarly we get the existence of an integer $M \geq 1$ such that $g_{Q_t}(Q^M(b(t))) > G(Q_t)$ for all $t \in U_\varepsilon$.

Proposition 2.22 shows that for any $t \in U_\varepsilon$ the points $P_t^N(a(t))$ and $Q_t^M(b(t))$ fall into the range of the Böttcher coordinates of P_t and Q_t respectively, and we have

$$\begin{aligned}
d^N \delta^M h(t) &= d^N \delta^M \Big(n g_{P_t}(a(t)) - m g_{Q_t}(b(t)) \Big) \\
&= n \delta^M g_{P_t}(P_t^N(a(t))) - m d^N g_{Q_t}(Q_t^M(b(t))) \\
&= n \delta^M \log \left| \varphi_{P_t}\left(P_t^N(a(t))\right) \right| - m d^N \log \left| \varphi_{Q_t}\left(Q_t^M(b(t))\right) \right| \\
&= \log \left| \frac{\varphi_{P_t}\left(P_t^N(a(t))\right)^{n\delta^M}}{\varphi_{Q_t}\left(Q_t^M(b(t))\right)^{md^N}} \right|
\end{aligned}$$

for all $t \in U_\varepsilon$.

Recall that $\mathcal{B}$ is included in $\{h = 0\}$ where h is harmonic on C. Being the difference of two continuous functions on C_+, the function h extends as a harmonic function to C_+. It follows that $\bar{\mathcal{B}}$ is a totally disconnected compact subset of a closed real analytic curve γ of C_+.

We may thus find a finite collection of segments $I_j \subset \gamma$ that covers $\bar{\mathcal{B}}$. Cutting them into pieces, we may suppose their closures are disjoint, and by thickening them we find finitely many disjoint topological disks $\bar{D}_1, \cdots, \bar{D}_l$ covering $\bar{\mathcal{B}}$. Up to reduce $\varepsilon > 0$ if necessary we have $K_\varepsilon \Subset \bigcup_j D_j$.

As D_j is simply connected and $d^N h$ is harmonic, there exists a nowhere vanishing holomorphic function $f_j \colon D_j \to \mathbb{C}$ such that $d^N \delta^M h = \log|f_j|$ on $D(p_j)$. We infer that

$$|f_j(t)| = \exp(d^N \delta^M h(t)) = \left| \frac{\varphi_{P_t}\left(P_t^N(a(t))\right)^{n\delta^M}}{\varphi_{Q_t}\left(Q_t^M(b(t))\right)^{md^N}} \right|$$

for any t in an annulus A_j included in $U_\varepsilon \cap D_j$ whose boundary contains the boundary of D_j. Multiplying f_j by a suitable complex number of modulus 1, we may suppose that the two holomorphic functions f_j and $\frac{\varphi_{P_t}\left(P_t^N(a(t))\right)^{n\delta^M}}{\varphi_{Q_t}\left(Q_t^M(b(t))\right)^{md^N}}$ coincide on A_j so that the function

$$f = \begin{cases} f_j \text{ on } D_j \\ \frac{\varphi_{P_t}\left(P_t^N(a(t))\right)^{n\delta^M}}{\varphi_{Q_t}\left(Q_t^M(b(t))\right)^{md^N}} \text{ on } U_\varepsilon \setminus \bigcup_i (D_i \setminus A_i) \end{cases}$$

is a well-defined holomorphic function on C_+. By construction it satisfies $d^N \delta^M h = \log|f|$ everywhere. Finally observe that U_ε is connected by the maximum principle, hence

$$f = \frac{\varphi_{P_t}\left(P_t^N(a(t))\right)^{n\delta^M}}{\varphi_{Q_t}\left(Q_t^M(b(t))\right)^{md^N}}$$

everywhere on U_ε. This concludes the proof. □

5.2.2. The implication (1) ⇒ (2) of Theorem B

Let (P, a) and (Q, b) be active non-integrable dynamical pairs of respective degree $d, \delta \geq 2$ parametrized by an affine curve C defined over a number field $\mathbb{K}$. By assumption one can find a sequence of distinct points $p_l \in C(\bar{\mathbb{K}})$ such that $p_l \in \mathrm{Preper}(P, a) \cap \mathrm{Preper}(Q, b)$ for all l.

Since both dynamical pairs are assumed to be active, Proposition 4.35 implies that the two height functions $h_{P,a}$ and $h_{Q,b}$ are induced by semi-positive adelic metrizations on two (a priori different) ample line bundles $L_{P,a} = \mathcal{O}_{\bar{C}}(n \cdot \mathsf{D}_{P,a})$ and $L_{Q,b} = \mathcal{O}_{\bar{C}}(m \cdot \mathsf{D}_{Q,b})$ where n and m are integers. Recall also that the curvatures of $L_{P,a}$ and $L_{Q,b}$ in $C^{v,\mathrm{an}}$ are the positive measures $n \cdot \Delta g_{P,a,v}$ and $m \cdot \Delta g_{Q,b,v}$ respectively by Proposition 4.35 (4).

Since $h_{P,a}(p_l) = h_{Q,b}(p_l) = 0$ for all l, we may apply Thuillier-Yuan's theorem, see Theorem 1.13, and we infer that for any place $v \in M_{\mathbb{K}}$, we have

$$\frac{1}{\deg(p_l)} \sum_{q \in \mathsf{O}(p_l)} \delta_q \longrightarrow n \cdot \Delta g_{P,a,v} = m \cdot \Delta g_{Q,b,v} \text{ as } l \to \infty\ .$$

Theorem 5.7 gives the equality $n g_{P,a,v} = m g_{Q,b,v}$ at all Archimedean places, hence $n \cdot \mathsf{D}_{P,a} = m \cdot \mathsf{D}_{Q,b}$ by Proposition 4.27. Let v be any non-Archimedean place. The function $n g_{P,a,v} - m g_{Q,b,v}$ is harmonic on C_v^{an} and can be extended continuously through all branches of C at infinity. By the maximum principle, it is thus a constant, and this constant must be zero since $g_{P,a,v}(p_l) = g_{Q,b,v}(p_l) = 0$ for all l.

From Proposition 4.35 (1) we finally infer $n h_{P,a} = m h_{Q,b}$ which concludes the proof.

For further reference we note that we have actually proved the following

Theorem 5.9. — *Let (P, a) and (Q, b) be active non-integrable dynamical pairs parametrized by an affine curve C defined over a number field $\mathbb{K}$. The following are equivalent:*

1. $\mathrm{Preper}(P, a, \bar{\mathbb{K}}) \cap \mathrm{Preper}(Q, b, \bar{\mathbb{K}})$ *is infinite.*
2. *there exist $n, m \in \mathbb{N}^*$ such that $n \cdot g_{P,a,v} = m \cdot g_{Q,b,v}$ for all places $v \in M_{\mathbb{K}}$;*
3. *there exist $n, m \in \mathbb{N}^*$ such that $n \cdot h_{P,a} = m \cdot h_{Q,b}$;*
4. $\mathrm{Preper}(P, a, \bar{\mathbb{K}}) = \mathrm{Preper}(Q, b, \bar{\mathbb{K}})$.

Proof. — The arguments above show (1)⇒(2) and (3). Then (4) follows from (3) since $\mathrm{Preper}(P, a, \bar{\mathbb{K}}) = \{h_{P,a} = 0\} = \mathrm{Preper}(Q, b, \bar{\mathbb{K}}) = \{h_{Q,b} = 0\}$. And (4) implies trivially (1). □

5.3. Multiplicative dependence of the degrees

In this section, we prove an important step of the proof of Theorem B: the degrees of any two entangled dynamical pairs (P, a), (Q, b) are multiplicatively dependent.

Throughout this section, we assume $K = \mathbb{C}$.

Theorem 5.10. — *Let (P, a) and (Q, b) be two active non-integrable complex dynamical pairs of respective degree $d, \delta \geq 2$ parametrized by an irreducible affine curve C. Assume that* $\mathrm{Preper}(P, a) = \mathrm{Preper}(Q, b)$*, and that $g_{P,a}$ and $g_{Q,b}$ are proportional on C.*

Then one may find two positive integers $N, M \geq 1$ such that $d^N = \delta^M$.

Remark 5.11. — The assumptions can probably be weakened. For instance, when both pairs are defined over a number field, the assumption $\mathrm{Preper}(P, a) = \mathrm{Preper}(Q, b)$ implies $g_{P,a}$ and $g_{Q,b}$ to be proportional. It is also likely that when the complex Green functions are proportional then the pairs have common preperiodic parameters.

We refer to [**101**, Theorem 5.1] for a statement implying equality of degrees in a related context.

The idea of the proof is to look at the expansion of the Green functions $g_{P,a}$ and $g_{Q,b}$ near a common prerepelling parameter. We proceed in two steps, first proving that such a parameter exists (Proposition 5.12), and then analyzing the Green functions near that parameter (Proposition 5.13). In the second step, we compute the Hölder exponents of the Green functions and relate them to the degrees of the families: this argument is inspired from [**69**, Proposition 3.1].

Proposition 5.12. — *Assume (P, a) and (Q, b) are two dynamical pairs satisfying the assumptions of Theorem 5.10. Then for any parameter $t_0 \in$* $\mathrm{Bif}(P, a)$ *at which a is transversally prerepelling, the point b is properly prerepelling for Q at t_0.*

Proof. — Pick any transversely prerepelling parameter t_0 for the pair (P, a). This parameter automatically belongs to $\mathrm{Bif}(P, a)$ by Lemma 4.9, hence $t_0 \in \mathrm{Bif}(Q, b) \cap \mathrm{Preper}(Q, b)$ by our assumptions and Proposition 4.2. In particular $b(t_0)$ cannot be preperiodic to an attractive cycle. We need to argue that $b(t_0)$ is preperiodic to a *repelling* cycle.

To simplify notation write $t_0 = 0$. We let m be any integer such that $P_0^m(a(0))$ is periodic, and let k be the period of that point. Denote by $\lambda = (P_0^k)'(P_0^m(a(0)))$ the multiplier of this cycle. Let also ϕ be the linearizing coordinate at $P_0^m(a(0))$, i.e. the univalent map $\phi : \mathbb{D}(0, r) \to \mathbb{C}$ such that $\phi'(0) = 1$, $\phi(0) = P_0^k(a(0))$ and $\phi(\lambda z) = P_0^k(\phi(z))$ for all $z \in \mathbb{D}(0, r)$ (for some $r \ll 1$). By Lemma 4.13 we have

$$d^{m+kn} g_{P_{t/\lambda^n}}(a(t/\lambda^n)) \to g_{P_0}(\phi(t)),$$

uniformly on $\mathbb{D}(0, r)$ as $n \to \infty$.

We may assume that $Q_0^m(b(0))$ is also periodic of period l, and let μ be its multiplier. We suppose by contradiction that $|\mu| = 1$. Up to translation, we can assume $b(0) = 0$. By base change, we may also follow locally the periodic orbit we shall also suppose that $Q_t^l(0) = 0$ for all $t \in \mathbb{D}(0, r)$. Recall that $\deg(Q) = \delta$.

Claim. — *For any $|t|, |z| > 0$ small enough, one has*

$$g_{Q_t}(z) \leq C\delta^{-l\cdot\min\left\{|t|^{-1/2}, |z|^{-1}\right\}}$$

for some constant $C > 0$.

Since $|Q_t^m(b(t))| \leq C'|t|$ for some constant $C' > 0$ and for all $|t| \leq r$, and for all n, we find

$$g_{Q_{t/\lambda^n}}(b(t/\lambda^n)) = \frac{1}{\delta^m} g_{Q_{t/\lambda^n}}\left(Q^m_{t/\lambda^n}(b(t/\lambda^n))\right) \leq C\delta^{-m}\delta^{-C''\sqrt{|\lambda^n/t|}}$$

By assumption, we have $g_{P_t}(a(t)) = cg_{Q_t}(b(t))$ for some positive $c > 0$, hence the above gives

$$d^{kn+m}\, g_{P_{t/\lambda^n}}(a(t/\lambda^n)) = c \cdot d^{kn+m} g_{Q_{t/\lambda^n}}(b(t/\lambda^n)) \lesssim d^{kn} \cdot \delta^{-C''\sqrt{|\lambda^n/t|}},$$

so that $\overline{\lim}_n\, d^{kn+m}\, g_{P_{t/\lambda^n}}(a(t/\lambda^n)) \leq 0$. This implies

$$g_{P_0}(\phi(t)) = \lim_{n\to\infty} \frac{1}{d^{kn+m}} g_{P_{t/\lambda^n}}(a(t/\lambda^n)) = 0,$$

which contradicts the fact that $a(0)$ lies in the Julia set of P_0. $\square$

Proof of the Claim. — Set $f_t := Q_t^l$. Since $f_t(0) = 0$ for all t, and $f_0'(0) = 1$, we may write

$$|f_t(z)| \leq |z| + C(|t| + |z|^2)$$

for some $C > 0$ and for all t and z small enough, say $\leq r$ with $1/C \leq r$. Let us prove by induction on n that for all $|t| \leq \frac{1}{(4Cn)^2}$ and all $|z| \leq \frac{1}{16Cn}$, then

$$|f_t^n(z)| \leq |z| + 4Cn(|t| + |z|^2) \leq \frac{1}{Cn} \leq r\ . \tag{34}$$

Only the first inequality needs an argument. For $n = 1$, this is obvious. Assume $|t| \leq \frac{1}{(4C(n+1))^2}$ and $|z| \leq \frac{1}{16C(n+1)}$. Then we have

$$\begin{aligned}|f_t^n(z)|^2 &\leq (|z| + 4Cn(|t| + |z|^2))^2 \leq (2|z| + 4Cn|t|)^2\\ &\leq 4|z|^2 + 16Cn|z|\cdot|t| + 16C^2n^2|t|^2\\ &\leq 4|z|^2 + 2|t|\end{aligned}$$

so that

$$\begin{aligned}|f_t^{n+1}(z)| &\le |f_t^n(z)| + C(|t| + |f_t^n(z)|^2)\\ &\le |z| + 4Cn(|t| + |z|^2) + C(3|t| + 4|z|^2)\\ &\le |z| + 4C(n+1)(|t| + |z|^2).\end{aligned}$$

ending the proof of (34).

Now pick any t, z small enough, and set $n := \min\{(4C\sqrt{|t|})^{-1}, (16C|z|)^{-1}\}$ so that $|f_t^n(z)| \le r$ by (34). We conclude that

$$\delta^{ln} g_{Q_t}(z) = g_{Q_t}(Q_t^{ln}(z)) = g_{Q_t}(f_t^n(z)) \le C := \sup_{|t|,|z|\le r} g_Q \ ,$$

which proves the claim, observing that C is a large constant which may be assumed to be ≥ 1. □

Proposition 5.13. — *Assume (P, a) and (Q, b) are two dynamical pairs satisfying the assumptions of Theorem 5.10. Suppose further that there exist infinitely many parameters $t \in C$ such that a is transversely prerepelling and b is simultaneously properly prerepelling at t.*

Then one may find two positive integers $N, M \ge 1$ such that $d^N = \delta^M$.

Proof. — Assume by contradiction that

$$\theta := \frac{\log d}{\log \delta} \notin \mathbb{Q} \ .$$

We shall prove that for any parameter t at which a is transversely prerepelling and b is simultaneously properly prerepelling, then P_t is integrable. If this occurs at infinitely many parameters, then P is isotrivial and integrable which contradicts the assumptions of Theorem 5.10.

So let us pick a parameter t_0 such that a is transversely prerepelling and b is properly prerepelling. Write $t_0 = 0$, suppose $P_0^m(a(0))$ and $Q_0^m(b(0))$ are both preperiodic of period k and l respectively. Denote by $\lambda := (P_0^k)'(P_0^m(a(0)))$ and $\mu := (Q_0^l)'(Q_0^m(b(0)))$ the multipliers of these periodic points so that $|\lambda| > 1$ and $|\mu| > 1$.

Lemma 5.14. —

$$\lim_{r\to 0} \frac{1}{\log r} \log\left(\sup_{|t|\le r} g_{Q,b}(t)\right) = \frac{l \log \delta}{\log |\mu_1|}$$

where q is the order of vanishing of $Q^{l+m}(b) - Q^m(b)$ at 0 and μ_1 is is a q-th root of μ.

Since $g_{P,a}$ and $g_{Q,b}$ are proportional, the previous lemma implies that

$$\frac{l \log \delta}{\log |\mu_1|} = \frac{k \log d}{\log |\lambda|}.$$

Lemma 5.15. — *For any $\alpha \in \mathbb{R}$, there exists sequences $n_j, m_j \to \infty$ such that*

$$\lambda^{n_j} \mu_1^{-m_j} \to e^{\alpha} .$$

Proof. — Our assumption implies $\beta := \log |\lambda| / \log |\mu_1|$ to be irrational so that the abelian group generated by β and 1 is dense in $\mathbb{R}$. We conclude that there exist sequences of integers $n_j, m_j \to \infty$ such that $n_j \log |\lambda| - m_j \log |\mu_1| \to \alpha$, and up to extracting a subsequence we have $\lambda^{n_j} \mu_1^{-m_j} \to e^{\alpha} e^{i\theta}$ for some $\theta \in \mathbb{R}$. Pick $r_j \to \infty$ with $e^{i r_j \theta} \to 1$, so that we get $\lambda^{r_j n_j} \mu_1^{-r_j m_j} \to e^{\alpha}$, as required. □

Fix any real number α, and choose sequences of integers n_j, m_j as in the previous lemma. Observe that $d^{k n_j} \delta^{-l m_j} \to e^{C\alpha}$ for some positive constant $C > 0$. Write $g_{P,a} / g_{Q,b} = \kappa > 0$, so that

$$\begin{aligned} g_{P_0} \circ \phi(t) &= \lim_j d^{m + k n_j} g_{P,a}(t/\lambda^{n_j}) = \left(\kappa \frac{d^m}{\delta^m} e^{C\alpha} \right) \lim_j \delta^{m + l n_j} g_{Q,b}(t/\lambda^{n_j}) \\ &= (\kappa' e^{C\alpha}) \lim_j \delta^{m + l m_j} g_{Q,b}(e^{-\alpha} t / \mu^{m_j}) = (\kappa' e^{C\alpha})\, g_{Q_0} \circ \psi(e^{-q\alpha} t^q) \end{aligned}$$

for some $\kappa' > 0$. For each α, let us introduce the real-analytic flow of local analytic isomorphisms: $\sigma_\alpha(t) = \psi(e^{q\alpha} \psi^{-1}(t)) = e^{\alpha} t + O(t^2)$. Our previous computations used twice imply

$$e^{C\alpha} g_{Q_0} \circ \psi(e^{-q\alpha} t^q) = g_{Q_0} \circ \psi(t^q)$$

so that

$$g_{Q_0} = e^{C\alpha} g_{Q_0} \circ \sigma_\alpha$$

for all α.

It follows that the flow σ_α preserves locally the Julia set of Q_0 near 0. By an argument due to Eremenko, see [**117**, Remark 3], $J(Q_0)$ cannot be homeomorphic to the product of a Cantor set and an interval. It has furthermore empty interior, hence $J(Q_0)$ is necessarily smooth near at least one of its point. We conclude by Theorem 2.16 that Q_0 is integrable. □

Proof of Lemma 5.14. — By Lemma 4.13 we have uniform convergence for all t small enough

$$\delta^{m+ln} g_{Q_{t/\mu_1^n}}(b(t/\mu_1^n)) \longrightarrow g_{Q_0} \circ \psi(t^q)$$

where ψ is the linearizing coordinate of Q_0^l at $Q_0^m(b(0))$. In particular $\delta^{ln} g_{Q,b}(t/\mu_1^n)$ converges to a continuous subharmonic function h which vanishes at the origin and is not identically zero. In particular $\rho \mapsto H(\rho) = \sup_{|t|=\rho} h(t)$

is a continuous function which is positive for any $\rho > 0$, see, e.g., [**143**, Theorem 2.6.8].

For any $r \leq \rho$, there exists a unique integer n such that $\rho/|\mu_1| \leq r|\mu_1|^n \leq \rho$ and we get

$$\begin{aligned}\sup_{|t|\leq r} g_{Q,b}(t) &= \sup_{|\tau|\leq r|\mu_1|^n} g_{Q,b}(\tau/\mu_1^n) \\ &= \frac{1}{\delta^{ln}} \sup_{|\tau|\leq r|\mu_1|^n} \delta^{ln} g_{Q,b}(\tau/\mu_1^n) \begin{cases} \leq & \delta^{-ln} 2H(\rho) \\ \geq & \delta^{-ln} \frac{H(\rho/|\mu_1|)}{2} \end{cases}\end{aligned}$$

which implies

$$\frac{1}{\log r} \log \left(\sup_{|t|\leq r} g_{Q,b}(t) \right) = \frac{-ln \log \delta + O(1)}{-n \log|\mu_1| + O(1)} \longrightarrow \frac{l \log \delta}{\log|\mu_1|} \ .$$

when $r \to 0$. □

5.4. Proof of the implication (2) ⇒ (3) of Theorem B

The setting is as follows. We let (P, a) and (Q, b) be two dynamical pairs of degree d and δ respectively, parametrized by an affine curve C defined over a number field $\mathbb{K}$. We assume that they are both active and non-integrable and that $h_{P,a}$ and $h_{Q,b}$ are proportional.

We shall first work under the following assumption (⋄): there exist two regular invertible functions $\alpha, \beta \in \mathbb{K}[C]$ such that

$$P_t(z) = \alpha(t)^{d-1} z^d + o_t(z^d) \text{ and } Q_t(z) = \beta(t)^{\delta-1} z^\delta + o_t(z^\delta) \ .$$

Under this assumption, one can define the Böttcher coordinate of P_t as the unique formal Laurent series in z^{-1} of the form

$$\varphi_{P_t} = \alpha(t) z + O(z^{-1})$$

such that $\varphi_{P_t} \circ P_t = (\varphi_{P_t})^d$, see Proposition-Definition 2.19. One defines analoguously the Böttcher coordinate φ_{Q_t}.

(I) Behaviour of the Green functions near the branches at infinity. — Let $\bar{C}$ be a projective compactification of C such that $\bar{C} \setminus C$ is a finite set of smooth points. Up to taking a finite extension of $\mathbb{K}$, we can assume all branches at infinity of C are defined over $\mathbb{K}$. We let $U_{\mathbb{K}}$ be the number of roots of unity lying in $\mathbb{K}$.

Recall from Theorem 4.30 that the divisor $\mathsf{D}_{P,a}$ is effective and non-zero since (P,a) is active. We also consider the effective divisor $\mathsf{D}_P = \sum q_{\mathfrak{c}}(P)[\mathfrak{c}]$ where $q_{\mathfrak{c}}(P)$ is the constant defined by Theorem 4.32.

The set of branches at infinity of C can be decomposed into two disjoints sets: the set $\mathcal{B}$ of branches $\mathfrak{c}$ such that $\mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) > 0$; and the set $\mathcal{G}$ of branches $\mathfrak{c}$ such that $\mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) = 0$.

Pick any branch $\mathfrak{c}$ of C at infinity, and choose a local adelic parametrization $t \mapsto \theta(t)$ of that branch, see §1.5 for a precise definition. To simplify notation we abuse notation and write $P_t = P_{\theta(t)}$ and $a(t) = a(\theta(t))$. If $\mathfrak{c} \in \mathcal{B}$, then we have

$$g_{P,a,v}(t) = \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) \log |t|_v^{-1} + O(1) \to \infty$$

for all places $v \in M_{\mathbb{K}}$, by Proposition 4.27. Otherwise $\mathfrak{c} \in \mathcal{G}$ and $g_{P,a,v}(t)$ extends continuously at $t = 0$ for all places.

The same discussion applies to the dynamical pair (Q, b).

By Theorem 5.9 we have $\mathrm{Preper}(P, a, \bar{\mathbb{K}}) = \mathrm{Preper}(Q, b, \bar{\mathbb{K}})$, and there exist two integers $n, m > 0$ such that $n \cdot \mathsf{D}_{P,a} = m \cdot \mathsf{D}_{Q,b}$, and $n \cdot g_{P,a,v} = m \cdot g_{Q,b,v}$ for all places $v \in M_{\mathbb{K}}$. We may and shall assume that n and m are *coprime* (in particular they are uniquely determined).

Since these equalities hold in particular at an Archimedean place v, Theorem 5.10 applies and we may thus find two positive integers $N, M \in \mathbb{N}^*$ such that $d^N = \delta^M$. We take N and M to be minimal over all integers satisfying this equality: again these integers are uniquely determined. In the sequel we write

$$\Delta = d^N = \delta^M \ .$$

Observe that the set of branches $\mathcal{B}$ coincides with $\{\mathfrak{c}, \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{Q,b}) > 0\}$. For each branch $\mathfrak{c} \in \mathcal{B}$, we let $l(\mathfrak{c})$ be the least integer such that

$$\Delta^{l(\mathfrak{c})} \times \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) > \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_P) \text{ and } \Delta^{l(\mathfrak{c})} \times \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{Q,b}) > \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_Q) \ .$$

Lemma 5.16. — *For any branch at infinity $\mathfrak{c} \in \mathcal{B}$, and for any integer $l \geq l(\mathfrak{c})$ there exists a root of unity $\zeta = \zeta(\mathfrak{c}, l) \in \mathbb{K}$ such that $\varphi_{P_t}\left(P_t^{lN}(a(t))\right)$ and $\varphi_{Q_t}\left(Q_t^{lM}(b(t))\right)$ are well-defined adelic series and*

$$\varphi_{P_t}\left(P_t^{lN}(a(t))\right)^n = \zeta \times \varphi_{Q_t}\left(Q_t^{lM}(b(t))\right)^m \ .$$

Proof. — Fix any integer $l \geq 1$ such that $\Delta^l \cdot \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) > \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_P)$, and fix any place $v \in M_{\mathbb{K}}$.

1. Our first objective is to show that $P_t^{lN}(a(t))$ belongs to the domain of definition of the Böttcher coordinate $\varphi_{P_t,v}$. Recall from Definition 4.21 that

$\mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) = q_{\mathfrak{c}}(P,a)$, and similarly $\mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_P) = q_{\mathfrak{c}}(P)$. By Proposition 4.27 and Theorem 4.32 respectively, we have

$$g_{P,a,v}(t) = \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) \cdot \log|t|^{-1} + O(1) \ ,$$
$$G_v(P_t) = \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_P) \cdot \log|t|^{-1} + O(1) \ ,$$

so that for any t small enough, $g_{P_t,v}(P_t^{lN}(a(t))) - G_v(P_t)$ is very large. It follows from Proposition 2.22 (2) that $P_t^{lN}(a(t))$ belongs to the domain of definition of the Böttcher coordinate.

2. By the previous step, the function $\Phi_v(t) := \varphi_{P_t}\left(P_t^{lN}(a(t))\right)^n$ is well-defined and analytic in some punctured disk $\mathbb{D}^*_{\mathbb{K}}(0, r_v)$. Since we have

$$\log|\Phi_v(t)|_v = n\Delta^l \cdot g_{P,a}(t) = c\log|t|^{-1} + O(1)$$

for some $c > 0$, it follows that Φ is meromorphic at 0.

In the next two steps, we argue that the Laurent series associated to Φ_v is in fact adelic (in particular we shall see that this series is independent of v).

Observe that for any $L > l$, we have

$$\Phi_v(t)^{\Delta^{L-l}} = \varphi_{P_t}\left(P_t^{lN}(a(t))\right)^{n\Delta^{L-l}} = \varphi_{P_t}\left(P_t^{NL}(a(t))\right)^n \ .$$

Since any root of an adelic series remains adelic by [**77**, Lemma 3.2] (but possibly over a finite extension of $\mathbb{K}$), we may suppose that l is arbitrarily large.

3. We now identify the Laurent series associated to Φ_v.

From Proposition 2.11 (1) and Proposition 2.13, we have

$$\Delta^l \cdot g_{P,a,v}(t) = g_{P_t,v}(P_t^{lN}(a(t))) \le \log^+|P_t^{lN}(a(t))| + G_v(P_t) + O(1) \ .$$

Combining this with the previous two estimates, we get

$$|P_t^{lN}(a(t))|_v = \beta|t|^{-b_l}(1 + o(1))$$

for some $\beta > 0$, with $b_l := \Delta^l \cdot \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) - \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_P) > 0$. In other words, we may expand as a formal Laurent series $P_t^{lN}(a(t)) = t^{-b_l}(\sum_{n\ge0}\beta_n t^n)$ with $\beta_0 = \beta \neq 0$.

We now apply Proposition 2.20 to the degree d polynomial $P_t(z) = A(t)z^d + a_1(t)z^{d-1} + \cdots + a_d(t) \in \mathbb{K}(\!(t)\!)[z]$, and write

$$\varphi_{P_t}(z) = \alpha\left(z + \frac{a_1}{dA}\right) + \sum_{j\ge1}\frac{\alpha_j}{z^j}$$

where $\alpha^{d-1} = A$, and $A^{2j}\alpha_j$ is a polynomial in α and the coefficients of P_t. In fact thanks to (19)–(21), the degrees of these polynomials grow at most linearly in j so that we may find an integer $\mu > 0$ (a careful examination shows that setting $\mu := 2d\max\{0, -\mathrm{ord}_0(A), -\mathrm{ord}_0(a_j)\}$ works) such that $\mathrm{ord}_0(\alpha_j) \geq -j\mu$ for any integer j.

From now on, we assume that l is large enough so that $b_l > \mu$. Then for each j, $\alpha_j(t)/(P_t^{lN}(a(t)))^j$ is a power series vanishing at 0 up to order $\geq (b_l - \mu)j$, hence the series $\sum_{j\geq 1} \frac{\alpha_j(t)}{(P_t^{NL}(a(t)))^j}$ converges formally in $\mathbb{K}[\![t]\!]$.

Let $\tilde{\Phi}$ be the formal power series obtained by summing $\sum_{j\geq 1} \frac{\alpha_j(t)}{(P_t^{lN}(a(t)))^j}$ with $\alpha\left(P_t^{lN}(a(t)) + \frac{a_1}{dA}\right)$.

4. We show that $\tilde{\Phi}$ is the Laurent series expansion of Φ_v at 0. Observe that since all coefficients of $\tilde{\Phi}$ lie in a finite extension of $\mathbb{K}$, and v is an arbitrary place of $\mathbb{K}$, this will show that $\tilde{\Phi}$ is an adelic series as required.

We further increase l so that $b_l > \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_P)$, and $\log|P_t^{lN}(a(t))|_v \gg G_v(P_t)$ for all t small enough. In this range, $P_t^{lN}(a(t))$ thus belongs to the domain of convergence of the series $\sum_{j\geq 1} \frac{\alpha_j(t)}{z^j}$ by Proposition 2.22 (1).

It follows that $f_p(t) := \sum_{j=1}^{p} \frac{\alpha_j(t)}{(P_t^{lN}(a(t)))^j}$ forms a sequence of analytic functions which converges uniformly on compact subsets of some punctured disk $\mathbb{D}_{\mathbb{K}}^*(0, r_v')$. By the maximum principle, this sequence of analytic functions actually converges uniformly on the disk $\mathbb{D}_{\mathbb{K}}(0, r_v')$, and its power series expansion is precisely the series $\sum_{j\geq 1} \frac{\alpha_j(t)}{(P_t^{lN}(a(t)))^j}$ considered above. Adding the terms $\alpha\left(P_t^{lN}(a(t)) + \frac{a_1}{dA}\right)$, we conclude that $\tilde{\Phi}$ is the Laurent series expansion of Φ_v, hence is an adelic series.

5. Similarly, the function $\Psi(t) = \varphi_{Q_t}\left(Q_t^{lM}(b(t))\right)^m$ is an adelic series. Let us consider an arbitrary Archimedean place $v_0 \in M_{\mathbb{K}}$. Since $n \cdot g_{P,a,v_0} = m \cdot g_{Q,b,v_0}$, we have $\log|\Phi/\Psi|_{v_0} = 0$, hence the meromorphic function Φ/Ψ has a constant modulus equal to 1 in some punctured disk $\mathbb{D}_K^*(0, r_{v_0}'')$. It is therefore equal to a constant $\zeta \in \mathbb{U}$. In particular we have an equality of adelic series $\Phi = \zeta \cdot \Psi$, and ζ is necessarily an algebraic number. Now for any other place $v \in M_{\mathbb{K}}$, $n \cdot g_{P,a,v} = m \cdot g_{Q,b,v}$ holds, which implies $|\zeta|_v = 1$. We conclude by Kronecker's theorem that ζ is a root of unity. □

Lemma 5.17. — *There exist two constants L_0, L depending only on $\mathbb{K}, d$ and δ such that*

$$\zeta(\mathfrak{c}) := \zeta(\mathfrak{c}, l(\mathfrak{c}) + L_0) = \zeta(\mathfrak{c}, l(\mathfrak{c}) + L_0 + l \cdot L)$$

for all $\mathfrak{c} \in \mathcal{B}$ and all $l \in \mathbb{N}$.

Proof. — Observe that we have $\zeta(\mathfrak{c}, l') = \zeta(\mathfrak{c}, l)^{\Delta^{l'-l}}$ for all $l' \geq l$, where $\Delta = d^N = \delta^M$. The proof then follows since $\zeta(\mathfrak{c}, l)$ is a root of unity belonging to the field $\mathbb{K}$ of definition of our families. □

To simplify notation, we shall write $\ell(\mathfrak{c}) = l(\mathfrak{c}) + L_0$ in the sequel.

Lemma 5.18. — *Pick any* $t \in C(\bar{\mathbb{K}})$ *and any place* $v \in M_{\mathbb{K}}$ *such that* $g_{P,a,v}(t) > 0$. *Then for all* l *large enough, the Böttcher coordinates* $\varphi_{P_t,v}$ *and* $\varphi_{Q_t,v}$ *are well-defined at* $P_t^{lN}(a(t))$ *and* $Q_t^{lM}(b(t))$ *respectively. Moreover there exists a branch at infinity* $\mathfrak{c}$ *of* C *such that*

$$\varphi_{P_t,v}\left(P_t^{lN}(a(t))\right)^n = \zeta(\mathfrak{c}, l) \times \varphi_{Q_t,v}\left(Q_t^{lM}(b(t))\right)^m$$

for all $l \gg 1$.

Proof. — For each branch $\mathfrak{c} \in \mathcal{B}$, fix a point $t_{\mathfrak{c}}$ which is sufficiently close to $\mathfrak{c}$ such that both adelic series $\varphi_{P_t}\left(P_t^{lN}(a(t))\right)$ and $\varphi_{Q_t}\left(Q_t^{lM}(b(t))\right)^m$ are well-defined in a neighborhood of $t_{\mathfrak{c}}$ and their quotient is equal to $\zeta(\mathfrak{c}, l)$ as in the previous lemma.

Let U be the connected component of $\{g_{P,a,v} > 0\} \subset C^{\mathrm{an}}$ containing t. By the maximum principle, U is unbounded hence contains some point $t_{\mathfrak{c}}$. For each l, let

$$\Omega_l := \{\tau, d^l \cdot g_{P,a,v}(\tau) > G_v(P_\tau)\} \ .$$

Then $\{U \cap \Omega_l\}_l$ forms an increasing sequence of open sets which cover U. It follows from a purely topological argument that t and $t_{\mathfrak{c}}$ belong to the same connected component V of $U \cap \Omega_l$ for l large enough. The analytic function

$$\Psi(t) = \varphi_{P_t,v}\left(P_t^{lN}(a(t))\right)^n - \zeta(\mathfrak{c}, l) \times \varphi_{Q_t,v}\left(Q_t^{lM}(b(t))\right)^m$$

is well-defined on V and constant equal to 0 near $\mathfrak{c}$. It follows from the identity principle that $\Psi = 0$ on V, hence $\Psi(t) = 0$. □

(II) Construction of the semi-conjugacy at a fixed parameter. — We fix any Archimedean place $v_0 \in M_{\mathbb{K}}$. For each $\mathfrak{c} \in \mathcal{B}$, we also choose a connected neighborhood $U_{v_0}(\mathfrak{c})$ of $\mathfrak{c}$ in $C_{v_0}^{\mathrm{an}}$ such that $d^{l(\mathfrak{c})} \cdot g_{P,a,v}(\tau) > G_v(P_\tau)$, and $\delta^{l(\mathfrak{c})} \cdot g_{Q,b,v}(\tau) > G_v(Q_\tau)$ for all $\tau \in U_{v_0}(\mathfrak{c})$.

Fix any parameter $t \in C(\bar{\mathbb{K}})$ which belongs to $U_{v_0}(\mathfrak{c})$ for some $\mathfrak{c} \in \mathcal{B}$, and such that neither P_t nor Q_t are integrable. Since t is fixed, we drop all references to t to simplify notations in this paragraph. We also fix a finite extension $\mathbb{K}'$ of $\mathbb{K}$ such that $t \in C(\mathbb{K}')$.

Consider the family of adelic branches at infinity $\mathfrak{s}_\zeta$ indexed by all roots of unity $\zeta \in \mathbb{K}$ defined by the equation

$$\varphi_P(x)^n = \zeta \varphi_Q(y)^m \ .$$

We claim that we may apply Theorem 1.17 to the sequence of points

$$(a_l, b_l) = (P^{lN}(a), Q^{lM}(b)) \ .$$

Indeed, observe first that (a_l, b_l) all belong to $\mathbb{K}'$. Fix a place $v \in M_{\mathbb{K}'}$. If $g_{P,v}(a) = g_{Q,v}(b) = 0$ then both sequences $|a_l|_v$ and $|b_l|_v$ are bounded so that

$$B_v = \sup_l \max\{|a_l|_v, |b_l|_v\} < \infty \ .$$

Note also that $B_v = 1$ at any place where both P and Q have good reduction. Now consider any place v at which $g_{P,v}(a) > 0$ and $g_{Q,v}(b) > 0$, and pick $R_v > 0$ such that both series φ_P and φ_Q converge in the domain $\{|x|_v > R_v\}$. Define

$$C_v(\mathfrak{s}_\zeta) := \left\{(x, y) \in \mathbb{C}_v^2,\ \min\{|x|_v, |y|_v\} > R_v,\ \varphi_P(x)^n = \zeta\varphi_Q(y)^m\right\} \ .$$

It follows from Lemma 5.18, that $(a_l, b_l) \in \cup_\zeta C_v(\mathfrak{s}_\zeta)$ so that Theorem 1.17 applies as claimed.

We infer the existence of an algebraic curve $Z' \subset \mathbb{A}^2$ such that $(a_l, b_l) \in Z'$ for all l large enough, and any branch at infinity of Z' is contained in the set $\{\mathfrak{s}_\zeta\}_{\zeta\in\mathbb{U}_\infty\cap\mathbb{K}}$.

Observe that at the place v_0, we have

$$(a_{\ell(\mathfrak{c})+l\cdot L}, b_{\ell(\mathfrak{c})+l\cdot L}) \in C_{v_0}(\mathfrak{s}_{\zeta(\mathfrak{c})})$$

for all $l \geq 0$. The Zariski closure of $\mathfrak{s}_{\zeta(\mathfrak{c})}$ is thus an irreducible component Z of Z' containing $(a_{\ell(\mathfrak{c})+l\cdot L}, b_{\ell(\mathfrak{c})+l\cdot L})$ for all $l \geq 0$ and any branch at infinity of Z is also contained in the set $\{\mathfrak{s}_\zeta\}_{\zeta\in\mathbb{U}_\infty\cap\mathbb{K}}$.

Consider now the map $\Phi(x, y) = (P^N(x), Q^M(y))$. Its iterate Φ^L stabilizes the set $\{(a_{\ell(\mathfrak{c})+l\cdot L}, b_{\ell(\mathfrak{c})+l\cdot L})\}_{l\geq 0}$ hence fixes $\mathfrak{s}_{\zeta(\mathfrak{c})}$ and Z.

Recall that by assumption neither P nor Q are integrable. By Theorem 3.45, one can thus find two semi-conjugacies $u, v \in \mathbb{K}'[T]$ whose degrees are *coprime*, and a polynomial $R \in \mathbb{K}'[T]$ such that $Z = \{u(x) = v(y)\}$ and $u \circ P^{NL} = R \circ u$, $v \circ Q^{ML} = R \circ v$. Since the degrees of u and v are coprime, Z has a unique branch at infinity which is necessarily $\mathfrak{s}_{\zeta(\mathfrak{c})}$.

Now by Proposition 2.20, there exist polynomials $\hat{P}_n$ and $\hat{Q}_m$ such that $\varphi_P(x)^n = \hat{P}_n(x) + o(1)$, and $\varphi_Q(y)^m = \hat{Q}_m(y) + o(1)$. It follows that the restriction to Z of the polynomial $\hat{P}_n(x) - \zeta(\mathfrak{c}) \cdot \hat{Q}_m(y)$ actually vanishes at the branch at infinity $\mathfrak{s}_{\zeta(\mathfrak{c})}$, hence vanishes identically on Z so that $Z = \{\hat{P}_n(x) = \zeta \cdot \hat{Q}_m(y)\}$.

Possibly conjugating R by a suitable dilatation, we have thus obtained:

Lemma 5.19. — *Fix any parameter $t \in C(\bar{\mathbb{K}}) \cap U_{v_0}(\mathfrak{c})$ such that neither P_t nor Q_t are integrable.*

Then the curve $\hat{P}_n(x) = \zeta(\mathfrak{c}) \cdot \hat{Q}_m(y)$ *is irreducible in* $\mathbb{A}^2$*, and has a unique branch at infinity given by*

$$\varphi_P(x)^n = \zeta(\mathfrak{c}) \cdot \varphi_Q(y)^m \ .$$

Moreover, one can find a polynomial R_t *of degree* Δ^L *such that*

$$\hat{P}_{t,n} \circ P_t^{NL} = R_t \circ \hat{P}_{t,n} \ ; \ \text{and} \tag{35}$$

$$(\zeta(\mathfrak{c}) \cdot \hat{Q}_{t,m}) \circ Q_t^{ML} = R_t \circ (\zeta(\mathfrak{c}) \cdot \hat{Q}_{t,m}) \ . \tag{36}$$

Moreover we have $\hat{P}_{t,n}(a_{\ell(\mathfrak{c})}(t)) = \zeta(\mathfrak{c}) \cdot \hat{Q}_{t,m}(b_{\ell(\mathfrak{c})}(t))$.

(III) End of proof when (◇) is satisfied. — Recall that n and m are determined by our data as the minimal integers such that $n \cdot \mathsf{D}_{P,a} = m \cdot \mathsf{D}_{Q,b}$. Also N and M are the least integers such that $\Delta := d^N = \delta^M$.

We fix any branch at infinity $\mathfrak{c} \in \mathcal{B}$, and any Archimedean place $v_0 \in M_{\mathbb{K}}$. Recall the definitions of $L, \zeta(\mathfrak{c})$ and $\ell(\mathfrak{c})$ from Lemma 5.17.

Consider the set $\mathcal{S}$ of all parameters $t \in C(\mathbb{K}_{v_0})$ such that the following conditions hold:

($\mathcal{P}_1$) there exists a polynomial $R_t \in \mathbb{K}_{v_0}[T]$ of degree Δ^L satisfying (35) and (36) above;

($\mathcal{P}_2$) the curve $\hat{P}_{t,n}(x) = \zeta(\mathfrak{c}) \cdot \hat{Q}_{t,m}(y)$ is irreducible in $\mathbb{A}^2$ and has a unique branch at infinity given by $\varphi_{P_t}(x)^n = \zeta(\mathfrak{c}) \cdot \varphi_{Q_t}(y)^m$;

($\mathcal{P}_3$) we have $\hat{P}_{t,n}(a_{\ell(\mathfrak{c})}(t)) = \zeta(\mathfrak{c}) \cdot \hat{Q}_{t,m}(b_{\ell(\mathfrak{c})}(t))$.

Observe that Lemma 5.19 implies that $\mathcal{S}$ contains all points in $C(\bar{\mathbb{K}}) \cap U_{v_0}(\mathfrak{c})$. We shall prove that the set of $t \in C(\mathbb{K}_{v_0})$ satisfying the three conditions ($\mathcal{P}_1$), ($\mathcal{P}_2$), and ($\mathcal{P}_3$) is a Zariski-closed subset.

Condition ($\mathcal{P}_3$) states the equality of two regular functions on C. Since $C(\bar{\mathbb{K}}) \cap U_{v_0}(\mathfrak{c})$ is infinite, Condition ($\mathcal{P}_3$) is satisfied for all $t \in C(\mathbb{K}_{v_0})$ hence for all parameters $t \in C(\bar{\mathbb{K}})$.

Let us deal next with Condition ($\mathcal{P}_2$). Pick any $t \in C'(\mathbb{K}_{v_0})$. Observe that the affine curve $Z = \{\hat{P}_{t,n}(x) = \zeta(\mathfrak{c}) \cdot \hat{Q}_{t,m}(y)\}$ is always irreducible, and has a unique branch at infinity because $n = \deg(\hat{P}_{t,n})$ and $m = \deg(\hat{Q}_{t,m})$ are coprime. The fact that the branch at infinity of Z is given by $\varphi_{P_t}(x)^n = \zeta(\mathfrak{c}) \cdot \varphi_{Q_t}(y)^m$ is equivalent to the vanishing of the formal Laurent series $\hat{P}_{t,n}(\varphi_{P_t}^{-1}(\xi \cdot t^m)) - \zeta \hat{Q}_{t,m}(\varphi_{Q_t}^{-1}(t^n)))$ where ξ is any root of unity satisfying $\xi^n = \zeta(\mathfrak{c})$. Since $\varphi_{P_t}^{-1}$ and $\varphi_{Q_t}^{-1}$ are formal Laurent series in z^{-1} whose coefficients are regular functions on C by (12), the set of $t \in C(\mathbb{K}_{v_0})$ for which Condition ($\mathcal{P}_2$) holds is a Zariski closed subset of C.

To understand Condition ($\mathcal{P}_1$), recall that we wrote $P_t(z) = \alpha(t)^{d-1}z^d + o_t(z^d)$. To simplify notation we assume $NL = 1$. We remark that the equation (35) given by $\hat{P}_n \circ P = R \circ \hat{P}_n$ is equivalent to resolution of $n \times d + 1$ linear equations $\mathcal{L}_i$, $i = 0, \ldots, n \times d$ obtained by identifying the coefficients in z^i of both sides.

Look at the equations $\mathcal{L}_{nd}, \mathcal{L}_{n(d-1)}, \ldots, \mathcal{L}_0$. We get a linear system of the form

$$\begin{cases} a_0\alpha^{dn} & = A_0 \\ a_1\alpha^{(d-1)n} + a_0 B_{10} & = A_1 \\ a_2\alpha^{(d-2)n} + a_1 B_{21} + a_0 B_{20} & = A_2 \\ \ldots & \\ a_d + a_{d-1}B_{d,d-1} + \cdots + a_0 B_{d0} & = A_d \end{cases}$$

where B_{ij} and A_i are regular functions on C. Since α is invertible, we get a *unique* polynomial $R_{P,n}$ satisfying all these equations whose coefficients are regular functions $a_i \in K[C]$.

It follows from this discussion that (35) is solvable iff the coefficients of $R_{P,n}$ satisfy the linear equations $\mathcal{L}_i$, $i = 0, \ldots, n \times d$. In other words, the set of parameters t such that (35) is solvable is the intersection of the zero locus of finitely many regular functions on C hence is Zariski closed. A similar argument applies to (36), and this concludes the proof.

We have proved that the set of parameters for which the three conditions ($\mathcal{P}_1$), ($\mathcal{P}_2$), and ($\mathcal{P}_3$) hold is equal to $C(\bar{\mathbb{K}})$. We also observe that the coefficients to the polynomial R_t depends algebraically on t, since all coefficients α, A_i and B_{ij} are regular functions on C in the linear system above. This completes the proof of (2) ⇒ (3) of Theorem B.

(IV) Base change and Condition (◇). — Let us assume now that P and Q are arbitrary families parametrized by C. Pick any base change $C' \to C$ such that the induced families on C' satisfy the condition (◇). By what precedes, there exists a branch at infinity $\mathfrak{c}$ of C', integers $N, M, n, m, L, l(\mathfrak{c})$, and a root of unity $\zeta(\mathfrak{c})$ such that for all $t \in C'(\bar{\mathbb{K}})$, there exists a polynomial R_t satisfying the three Conditions ($\mathcal{P}_1$), ($\mathcal{P}_2$), and ($\mathcal{P}_3$).

Since R_t is uniquely determined (for instance by (35)), we have $R_t = R_{t'}$ whenever $\pi(t) = \pi(t')$. It follows that the coefficients of R_t are regular functions on C' that are pull-back of regular functions on C, and we obtain a family of polynomial $\tilde{R}$ parametrized by C satisfying the three Conditions ($\mathcal{P}_1$), ($\mathcal{P}_2$), and ($\mathcal{P}_3$) for the dynamical pairs (P, a) and (Q, b) as required.

Another approach in the case of a single branch. — The previous proof is quite intricate. One difficulty lies in dealing with the existence of several branches of C at infinity for which the roots of unity $\zeta(\mathfrak{c}, l)$ might a priori be different.

Assume that we may find an integer l and ξ such that $\xi = \zeta(\mathfrak{c}, l)$ for all $\mathfrak{c}$. Note that this is in particular the case when C has a single place at infinity.

The function $\hat{P}_n(P^{Nl}(a)) - \xi\hat{Q}_m(Q^{Ml}(b))$ is regular on C and vanishes at all branches at infinity for l large enough by Lemma 5.18. It is hence equal to 0.

This argument simplifies Step II and avoids to rely on Xie's theorem.

5.4.1. More precise forms of Theorem B

The proof that we have developed for Theorem B actually gives more. We first observe that the arguments of the previous section yield

Theorem 5.20. — *Pick any irreducible affine curve C which is defined over a number field $\mathbb{K}$. Assume that all its branches at infinity are also defined over $\mathbb{K}$.*

Let (P, a) and (Q, b) be active non-integrable dynamical entangled *pairs parametrized by C of respective degree $d, \delta \geq 2$.*

Then there exist coprime integers n and m such that $n \cdot \mathsf{D}_{P,a} = m \cdot \mathsf{D}_{Q,b}$ and coprime integers N, M such that $\Delta := d^N = \delta^M$.

Furthermore, let ℓ and L be any integers such that

- $\Delta^\ell \times \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) > \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_P)$ *for all branch at infinity of C;*
- $M_\Delta^{\ell+L}(\xi) = M_\Delta^\ell(\xi)$ *for any root of unity in $\xi \in \mathbb{K}$.*

Then one can find a root of unity $\zeta \in \mathbb{K}$, and a family of polynomial R_t of degree Δ^L such that

$$\hat{P}_{t,n} \circ P_t^{NL} = R_t \circ \hat{P}_{t,n} \ ; \ \text{and} \tag{37}$$

$$(\zeta \cdot \hat{Q}_{t,m}) \circ Q_t^{ML} = R_t \circ (\zeta \cdot \hat{Q}_{t,m}) \ . \tag{38}$$

Moreover we have $\hat{P}_{t,n}(P^{\ell N}(a(t))) = \zeta \cdot \hat{Q}_{t,m}(Q^{\ell M}(b(t)))$.

In the statement we wrote $M_\Delta(z) = z^\Delta$.

Remark 5.21. — Observe that the integers n, m, N and M are uniquely determined by the condition to be coprime, and that L can be bounded from above by a constant depending only on $[\mathbb{K} : \mathbb{Q}]$.

We also have the next result, compare with [**6**, Theorem 1.3].

Theorem 5.22. — *Pick any irreducible affine curve C defined over a number field $\mathbb{K}$. Let (P, a) and (Q, b) be active, non-integrable monic centered dynamical pairs parametrized by C of respective degree $d, \delta \geq 2$. Then, the following are equivalent:*

1. $\mathrm{Preper}(P, a, \bar{\mathbb{K}}) \cap \mathrm{Preper}(Q, b, \bar{\mathbb{K}})$ *is an infinite subset of* $C(\bar{\mathbb{K}})$*;*
2. $\mathrm{Preper}(P, a, \bar{\mathbb{K}}) = \mathrm{Preper}(Q, b, \bar{\mathbb{K}})$*;*
3. *the height functions $h_{P,a}$ and $h_{Q,b}$ are proportional;*
4. *there exists a non-repeating sequence $t_n \in C(\bar{\mathbb{K}})$ such that*
$$\lim_{n\to\infty} h_{P,a}(t_n) = \lim_{n\to\infty} h_{Q,b}(t_n) = 0;$$
5. *there exist integers $n, m \geq 1$ such that for any place $v \in M_{\mathbb{K}}$,*
$$n \cdot \Delta g_{P,a,v} = m \cdot \Delta g_{Q,b,v},$$
as positive measures on $C^{v,\mathrm{an}}$;
6. *there exist integers $n, m \geq 1$ such that for all places $v \in M_{\mathbb{K}}$, we have*
$$n \cdot g_{P,a,v} = m \cdot g_{Q,b,v} \quad \text{on } C_v^{\mathrm{an}};$$
7. *there exist integers $n, m \geq 1$, $N, M \geq 1$, $r, s \geq 0$, and families R, τ and π of polynomials parametrized by C such that*
$$\tau \circ P^N = R \circ \tau \quad \text{and} \quad \pi \circ Q^M = R \circ \pi,$$
and $\tau(P^r(a)) = \pi(Q^s(b))$.

Proof. — The fact that (1) ⇔ (2) ⇔ (3) ⇔ (6) is the content of Theorem 5.9. The implication (3) ⇒ (7) is the content of §5.4, and (7) ⇒ (2) is clear. The implication (1) ⇒ (4) is clear as we may take $t_n \in \mathrm{Preper}(P, a, \bar{\mathbb{K}}) \cap \mathrm{Preper}(Q, b, \bar{\mathbb{K}})$ and $h_{P,a}(t_n) = h_{Q,b}(t_n) = 0$. The implication (4) ⇒ (5) is a direct application of Thuillier-Yuan's Theorem 1.13, see §5.2.2.

Let us prove (5) ⇒ (6). The argument is contained in §5.2.2, but we repeat it for the convenience of the reader. Suppose $n \cdot \Delta g_{P,a,v} = m \cdot \Delta g_{Q,b,v}$ for all places $v \in M_{\mathbb{K}}$. If v is archimedean, then $n \cdot g_{P,a,v} = m \cdot g_{Q,b,v}$ by Theorem 5.7, and $n \cdot \mathsf{D}_{P,a} = m \cdot \mathsf{D}_{Q,b}$ by Theorem 5.7. Pick any place $v \in M_{\mathbb{K}}$, then $n \cdot g_{P,a,v} - m \cdot g_{Q,b,v}$ is a harmonic function which extends continuously to $\bar{C}_v^{\mathrm{an}}$ hence is constant by the maximum principle. Since it vanishes on $\mathrm{Preper}(P, a, \bar{\mathbb{K}})$ we get $n \cdot g_{P,a,v} = m \cdot g_{Q,b,v}$ as required. □

5.5. Proof of Theorem C

Let us recall its statement.

Theorem C. — *Pick any irreducible algebraic curve C defined over a field of characteristic* 0*. Let (P, a) and (Q, b) be active non-integrable polynomial dynamical pairs parametrized by C of respective degree $d, \delta \geq 2$. Assume that*

($\triangle$) *any branch at infinity $\mathfrak{c}$ of C belongs to the support of the divisor $\mathsf{D}_{P,a}$.*

Then, the following are equivalent:

1. *the set* $\mathrm{Preper}(P, a) \cap \mathrm{Preper}(Q, b)$ *is an infinite subset of C;*
2. *there exist integers $N, M \geq 1$, $r, s \geq 0$, and families R, τ and π of polynomials parametrized by C such that*

$$\tau \circ P^N = R \circ \tau \quad \textit{and} \quad \pi \circ Q^M = R \circ \pi,$$

and $\tau(P^r(a)) = \pi(Q^s(b))$.

We let K be a field of characteristic 0 over which C, (P, a) and (Q, b) are defined. As in §5.1.3 the implication (2) ⇒ (1) is easy. We thus assume that $\mathrm{Preper}(P, a, \bar{K}) \cap \mathrm{Preper}(Q, b, \bar{K})$ is infinite. To prove (2), we shall use a specialization argument to reduce to the case K is a number field which was treated in the previous section. We proceed in four steps.

1. Fix any embedding $C \subset \mathbb{A}^N$ and pick polynomials $a_i, b_j \in K[z_1, \ldots, z_N]$, and $V_1, \ldots, V_M \in K[z_1, \ldots, z_N]$ such that C is the scheme theoretic intersection of the hypersurfaces $\{V_i = 0\}$, $P(T) = \sum_i a_i|_C T^i$, and $Q(T) = \sum b_j|_C T^j$. The completion $\bar{C}$ of C in $\mathbb{P}^N$ is defined as the vanishing of the homogeneous polynomials $\bar{V}_i(z_0, \cdots, z_N) = z_0^{\deg(V_i)} V_i(\frac{z_1}{z_0}, \cdots, \frac{z_N}{z_0})$. By choosing an adequate embedding we may suppose that $\bar{C}$ is smooth near any of its point lying on the hyperplane at infinity.

Choose any finitely generated $\mathbb{Q}$-algebra R contained in K that contains all coefficients defining our data so that the polynomials a_i, b_j and V_l belong to $R[z_1, \cdots, z_N]$. We may assume that $K = \mathrm{Frac}(R)$. Then $\Lambda := \mathrm{Spec}(R)$ is an affine variety defined over $\mathbb{Q}$ which is in general neither reduced nor irreducible. We let $\mathfrak{C}$ (resp. $\bar{\mathfrak{C}}$) be the affine (resp. projective) Λ-scheme defined by the vanishing of all polynomials V_i (resp. $\bar{V}_i$). Up to replacing Λ by a Zariski open dense subset (i.e. to replace R by a larger ring), we may assume by generic flatness and [**104**, Theorem 9.9] that Λ is irreducible, and $\bar{\mathfrak{C}} \to \Lambda$ is a flat family of curves such that $\bar{C} = \bar{\mathfrak{C}}_K$ and $C = \mathfrak{C}_K$.

The dynamical pairs (P, a) and (Q, b) induce regular maps $P_{\mathfrak{C}}, Q_{\mathfrak{C}} \colon \mathbb{P}^1_{\mathfrak{C}} \to \mathbb{P}^1_{\mathfrak{C}}$ of degree d and δ respectively, and regular functions $\mathfrak{a}, \mathfrak{b} \colon \mathfrak{C} \to \mathbb{P}^1_{\mathfrak{C}}$.

2. Any maximal ideal s of R corresponds to a closed point in Λ. We denote by $\bar{C}_s$, C_s, P_s, a_s, Q_s and b_s the specializations of the corresponding objects at s.

These are all defined over the residue field $\kappa(s)$ of s which is a finite extension of $\mathbb{Q}$.

Note that the intersection of $\bar{C}$ with the hyperplane at infinity is a finite set of points defined over an algebraic extension of K. Enlarging R if necessary, we may suppose that all these points are defined over R. It follows that the Zariski closure in $\bar{\mathfrak{C}}$ of a branch $\mathfrak{c}$ at infinity of C defines a branch at infinity $\mathfrak{c}_s$ of $\mathfrak{C}_s$ for all s.

To clarify notation write $D_{P,a} = D(P,a)$. In the next lemma, we do not assume Condition ($\triangle$).

Lemma 5.23. — *There exists a Zariski open dense subset U of Λ such that $D(P,a)_s = D(P_s,a_s)$ for all $s \in U$.*

Proof. — We may suppose that $D(P,a)$ is non zero and the family is given under the form (10) so that $P = P_{c,\alpha}$ for some $c,\alpha \in R$.

Pick any branch at infinity of C lying in the support of $D(P,a)$. By Proposition 2.11, for any q such that

$$-\operatorname{ord}_{\mathfrak{c}}(P^q(a)) > -\operatorname{ord}_{\mathfrak{c}}(c,\alpha)\ ,$$

we have $\operatorname{ord}_{\mathfrak{c}}(D(P,a)) = \frac{1}{d^q}\operatorname{ord}_{\mathfrak{c}}(P^q(a))$. Since for any $\phi \in K(C)$, one has $\operatorname{ord}_{\mathfrak{c}}(\phi) = \operatorname{ord}_{\mathfrak{c}_s}(\phi_s)$ for all s in a Zariski open dense subset of Λ, we conclude that

$$\operatorname{ord}_{\mathfrak{c}}(D(P,a)) = \frac{1}{d^q}\operatorname{ord}_{\mathfrak{c}}(P^q(a)) = \frac{1}{d^q}\operatorname{ord}_{\mathfrak{c}_s}(P^q_s(a_s)) = \operatorname{ord}_{\mathfrak{c}_s}(D(P_s,a_s))$$

for all s in a Zariski open dense subset.

This concludes the proof when all branches at infinity lie in the support of $D(P,a)$ (that is when Condition ($\triangle$) holds). Otherwise, let $\mathfrak{c}_1,\dots,\mathfrak{c}_k$ be the set of branches at infinity of C such that $\operatorname{ord}_{\mathfrak{c}_j}(D(P,a)) = 0$. By Lemma 4.20, we may fix an integer $N \geq 1$, such that for any local parametrization θ of a branch $\mathfrak{c}_j$ the series $t^N \times P^q_{\theta(t)}(a(\theta(t)))$ has no pole at 0.

Pick any regular function $A\colon C \to \mathbb{A}^1$ whose poles are contained in $\mathfrak{c}_1,\dots,\mathfrak{c}_k$ and have order $\geq N$, and whose zeroes are included in the support of $D(P,a)$. Up to replacing N by a larger integer, such a function always exists. In the sequel, we identify this function to a family of dilatations $z \mapsto Az$ on $\mathbb{A}^1_C$. Observe that the conjugated pair is defined by the family $\tilde{P} = A^{-1}\circ P \circ A$ which is still parametrized by C, and by the new marked point $\tilde{a} = A_t^{-1}(a(t))$. By construction, for any parametrization θ of a branch $\mathfrak{c}_j$, we have

$$\tilde{P}^q_{\theta(t)}(\tilde{a}(\theta(t))) = A^{-1}_{\theta(t)} \circ P^q_{\theta(t)}(a(\theta(t)))$$

which is regular near 0 for all q. Observe that this remains true by specialization at any point s so that

$$0 \leq \mathrm{ord}_{\mathfrak{c}_j}(\tilde{P}^q(\tilde{a})) \leq \mathrm{ord}_{\mathfrak{c}_{j,s}}(\tilde{P}^q_s(\tilde{a}_s))$$

for all q , hence

$$0 = \mathrm{ord}_{\mathfrak{c}_j}(D(P,a)) = \mathrm{ord}_{\mathfrak{c}_{j,s}}(D(P_s,a_s))$$

for all s. □

3. The key point of the proof is contained in the next result.

Proposition 5.24. — *If Condition (△) is satisfied, for any archimedean place v, the set*

$$\{s \in \Lambda^{\mathrm{an}}_v, \mathrm{Preper}(P_s,a_s) \cap \mathrm{Preper}(Q_s,b_s) \textit{ is infinite}\}$$

contains a non-empty open subset of Λ^{an}_v.

Proof. — In the whole proof, we argue in the analytifications with respect to a fixed archimedean place. To simplify notation, we drop the reference to this place. We view $\pi\colon \mathfrak{C} \to \Lambda$ as a flat family of complex algebraic curves. The closure in $\mathfrak{C}$ of any closed point $a \in C$ determines an irreducible hypersurface $Z(a)$ such that $Z(a) \to \Lambda$ is finite-to-one onto a Zariski dense open subset of Λ. In general this projection is not proper.

Let x_i be a sequence of distincts points in $\mathrm{Preper}(P,a,\bar{K}) \cap \mathrm{Preper}(Q,b,\bar{K})$. Then $Z_i := Z(x_i)$ is a sequence of distinct irreducible hypersurfaces of $\mathfrak{C}$ included in the space $\mathrm{Preper}(P_{\mathfrak{C}},\mathfrak{a}) \cap \mathrm{Preper}(Q_{\mathfrak{C}},\mathfrak{b})$.

Lemma 5.25. — *For each i, the projection map $Z_i \to \Lambda$ is finite-to-one, proper and surjective.*

Proof of Lemma 5.25. — It is sufficient to prove that $Z_i \to \Lambda$ is proper. We fix any archimedean place v, and prove that $Z^{\mathrm{an}}_{i,v} \to \Lambda^{\mathrm{an}}_v$ is proper, see [**20**, Proposition 3.4.7]. Pick any point $\mathfrak{c} \in \bar{\mathfrak{C}}$ which defines a branch at infinity of some curve of C_{s_0}.

Observe that near the point $\mathfrak{c}$, we have

$$\frac{1}{d^q}\log^+|P^q(a)| = \frac{1}{d^q}\mathrm{ord}_{\mathfrak{c}}(P^q(a))\log|z|^{-1} + O(1).$$

The condition (△) thus implies $\frac{1}{d^q}\log^+|P^q(a)| > G(P)$ in a neighborhood U of $\mathfrak{c}$. This implies $g_{P,a} > 0$ on U by Proposition 2.22, hence $Z_i \cap U = \varnothing$. □

Consider the set $\mathcal{F} := \{s \in \Lambda^{\mathrm{an}}, \mathrm{Preper}(P_s, a_s) \cap \mathrm{Preper}(Q_s, b_s)$ is finite$\}$. We need to exhibit an open euclidean subset of Λ^{an}_v which does not intersect the countable set $\mathcal{F}$. Pick any $s_0 \in \mathcal{F}$. By the previous lemma, each Z_i intersects the fiber C_{s_0} so that we may find a closed point $t_0 \in C_{s_0}$ which belongs to infinitely many Z_i's.

By Theorem 2.35, we may thus suppose that 0 is a fixed point for the family P which is super-attracting at t_0, whose local degree is not locally constant, and Z_i is an irreducible component of $\{P^{n_i}(a) = 0\}$ with $n_i \to \infty$.

The local degree function $t \mapsto \deg_0(P_t)$ is upper-semi continuous with respect to the Zariski topology on $\mathfrak{C}$. Restricting Λ if necessary, we thus suppose that the locus $\mathcal{E} \subset \mathfrak{C}$ where $\deg_0(P_t)$ is not locally constant is a smooth hypersurface, which intersects each curve C_s transversally, and whose projection onto Λ is proper. We may also assume that $\deg_0(P_t) = \nu$ if t belongs to $\mathcal{E}$, and $\deg_0(P_t) = \mu < \nu$ otherwise.

Lemma 5.26. — *Any point $t_1 \in \mathcal{E}$ admits a (euclidean) neighborhood V such that $Z_i \cap \mathcal{E} \cap V = \{t_1\}$ for all i.*

By definition $\mathcal{E} \cap C_{s_0}$ is finite, and we may apply the previous lemma to each point in this set. By properness of $\mathcal{E} \to \Lambda$, we get a euclidean neighborhood W of s_0 such that for any $s \in W$ distinct from s_0, for any i the intersection $Z_i \cap C_s$ does not lie in $\mathcal{E}$. By Theorem 2.35, no point in C_s may thus contain infinitely many hypersurfaces Z_i. This implies that $C_s \cap \{Z_i\}$ is infinite which shows that the euclidean open set $W \setminus \{s_0\}$ does not intersect $\mathcal{F}$. □

Proof of Lemma 5.26. — Locally near t_1, choose local analytic coordinates (λ, t) such that $\mathcal{E} = \{t = 0\}$, $\pi(\lambda, t) = \lambda$, and

$$P_{\lambda,t}(z) = b_\mu(\lambda, t) z^\mu + \cdots + b_{\nu-1}(\lambda, t) z^{\nu-1} + z^\nu (1 + h(\lambda, t, z))$$

where $t | b_j$ for all $j = \mu, \cdots, \nu - 1$, and $h(0) = 0$.

The restriction of the family P to $\mathcal{E}$ is thus equal to $P_{\lambda,0}(z) = z^\nu(1 + h(\lambda, 0, z))$ and we may find an analytic change of coordinates $w = z + O_\lambda(z)$ such that $P_{\lambda,0}(w) = w^\nu$, see e.g. [**147**, Theorem 1.3]. In an open neighborhood V of t_1, for any integer $n \geq 1$ the intersection of the hypersurface $\{P^n(a) = 0\}$ with $\mathcal{E}$ is thus determined by the equation $\{a^\nu = 0\}$. By reducing V if necessary, we thus have $Z_i \cap \mathcal{E} \cap V \subset \{a = 0\} = \{t_1\}$. □

4. One can now conclude the proof as follows. Fix any archimedean place v. By Proposition 5.24 there exists an open set $U \subset \Lambda^{\mathrm{an}}_v$ such that any closed point $s \in \Lambda \cap U$ satisfies $\mathrm{Preper}(P_s, a_s) \cap \mathrm{Preper}(Q_s, b_s)$ is infinite, and $D(P, a)_s = D(P_s, a_s)$.

For any $s \in \Lambda \cap U$, Theorem 5.20 shows that for any $t \in C_s$ we have

$$\hat{P}_{t,n} \circ P_t^{NL} = R_t \circ \hat{P}_{t,n} \text{ ; and} \tag{39}$$

$$(\zeta \cdot \hat{Q}_{t,m}) \circ Q_t^{ML} = R_t \circ (\zeta \cdot \hat{Q}_{t,m}) \tag{40}$$

$$\hat{P}_{t,n}(P^{\ell N}(a(t))) = \zeta \cdot \hat{Q}_{t,m}(Q^{\ell M}(b(t))) \text{ for all } l \geq 1, \tag{41}$$

for some polynomial R_t, for some integers n, m, N, M, ℓ, L and for a root of unity ζ.

By Lemma 5.23, the four integers n, m, N and M are actually independent on s so that $\Delta := d^N = \delta^M$ too. We need to argue that ζ may be chosen uniformly in s.

To see this, we go back to Lemma 5.16. For each branch at infinity $\mathfrak{c}$ such that $\mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) > 0$ we pick an integer $l(\mathfrak{c}) > 0$ such that

$$\Delta^{l(\mathfrak{c})} \times \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) > \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_P) \text{ and } \Delta^{l(\mathfrak{c})} \times \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{Q,b}) > \mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_Q) \ .$$

Lemma 5.16 then yields for each $l \geq l(\mathfrak{c})$ a root of unity $\zeta(\mathfrak{c}, l, U)$ such that

$$\varphi_{P_t}\left(P_t^{lN}(a(t))\right)^n = \zeta(\mathfrak{c}, l, U) \times \varphi_{Q_t}\left(Q_t^{lM}(b(t))\right)^m$$

for all $s \in U$, and all t in the branch $\mathfrak{c}$ on C_s. In particular Lemma 5.17 holds uniformly on U: we can find two integers ℓ and L such that $\zeta(\mathfrak{c}, \ell, U) = \zeta(\mathfrak{c}, \ell + l \cdot L, U)$ for all $l \geq 0$.

It follows that for a Zariski dense set of points $s \in \Lambda \cap U$, there exist ℓ, L and ζ (independent on s) satisfying (39), (40), and (41). Since these equations are linear in the coefficients of R_t, one may choose these coefficients in the fraction field of Λ, and we conclude that these relations are actually satisfied for all $s \in \Lambda$.

This implies the next theorem and concludes the proof of Theorem C.

Theorem 5.27. — *Pick any irreducible affine curve C defined over an algebraically closed field K of characteristic* 0. *Let (P, a) and (Q, b) be active non-integrable dynamical* intricated *pairs parametrized by C of respective degree $d, \delta \geq 2$ such that (P, a) satisfies the condition ($\triangle$).*

Then there exist coprime integers n and m such that $n \cdot \mathsf{D}_{P,a} = m \cdot \mathsf{D}_{Q,b}$ and coprime integers N, M such that $\Delta := d^N = \delta^M$.

Furthermore, there exists root of unity ζ satisfying $M_\Delta^{\ell+L}(\zeta) = M_\Delta^{\ell}(\zeta)$ for some integers ℓ and L; and a family of polynomial R_t of degree Δ^L such that

$$\hat{P}_{t,n} \circ P_t^{NL} = R_t \circ \hat{P}_{t,n} \ ; \tag{42}$$

$$(\zeta \cdot \hat{Q}_{t,m}) \circ Q_t^{ML} = R_t \circ (\zeta \cdot \hat{Q}_{t,m}) \ ; \text{ and} \tag{43}$$

$$\hat{P}_{t,n}(P^{\ell N}(a(t)) = \zeta \cdot \hat{Q}_{t,m}(Q^{\ell M}(b(t)). \tag{44}$$

5.6. Further results and open problems

5.6.1. Effective versions of the theorem

Suppose that (P, a) and (Q, b) are algebraic dynamical pairs parametrized by a curve C defined over a number field $\mathbb{K}$. If the two pairs are entangled then we have seen that the two heights $h_{P,a}$ and $h_{Q,b}$ are proportional so that $\mathrm{Preper}(P, a) \cap \mathrm{Preper}(Q, b) = \{h_{P,a} = 0\} = \{h_{Q,b} = 0\}$.

When the two pairs are not entangled then $H := h_{P,a} + h_{Q,b}$ is an Arakelov height whose essential minimum is positive (apply Theorem 5.22 (4)). In particular, there exists $\epsilon > 0$ such that $\mathrm{Preper}(P, a) \cap \mathrm{Preper}(Q, b) \cap \{H \leq \epsilon\}$ is a finite subset of $C(\bar{\mathbb{K}})$.

It would be very interesting to explore whether one can obtain uniform bounds on $\mathrm{Preper}(P, a) \cap \mathrm{Preper}(Q, b)$ when P, Q, a, b vary in families.

Conjecture 3. — *Let P be any non-integrable family of polynomials of degree $d \geq 2$. Then there exists a constant $C = C(d, N)$ such that for any a, b of degree $\leq N$ the set* $\mathrm{Preper}(P, a) \cap \mathrm{Preper}(P, b)$ *is either infinite or its cardinality is $\leq C$.*

Using Zhang's pairing for heights, Fili [**83**] gave an upper bound on the cardinality of $\mathrm{Preper}(z^2 + c, 0) \cap \mathrm{Preper}(z^2 + c, 1)$. The latter set was later computed exactly by Buff [**33**]. DeMarco, Krieger and Ye [**56**] have recently obtained uniform bounds for a similar problems: by estimating Zhang's pairings, they managed to prove the existence of a constant $B > 1$ such that, for any two complex parameters $c_1 \neq c_2$, the set $\mathrm{Preper}(z^2 + c_1) \cap \mathrm{Preper}(z^2 + c_2)$ has cardinality $\leq B$.

Recently, Fu [**85**] proved the above conjecture when P is the unicritical family $z^d + t$ and a, b are constant marked points, i.e. when $N = 0$.

5.6.2. The integrable case

In all statements above, we have supposed that the families were not integrable. Suppose that (P, a) is an active integrable pair parametrized by a curve C, and pick any any active pair (Q, b) which is entangled with it. We shall describe all such possibilities up to a base change.

1. The family Q is also integrable.

Observe that the bifurcation locus of P is equal to $\{t, a(t) \in J(P)\}$ by Proposition 4.14. It thus contains a smooth curve, and Q is integrable by Theorem A.

2. Reduction to the monomial case.

Replacing C by an open affine subset and doing a base change if necessary, one may conjugate the two families to constant families, so that $(P,Q) = (M_d, M_\delta), (M_d, \pm T_\delta)$, or $= (\pm T_d, \pm T_\delta)$ (up to a permutation). Since T_d and M_d are semi-conjugate for all d, and $-T_d$ and $-M_d$ are semi-conjugate for all odd d, we may also make a base change to reduce the situation to $(P,Q) = (M_d, M_\delta)$.

3. Apply Lang's result.

The entanglement of the pairs (M_d, a) and (M_δ, b) says that the set of $t \in C$ such that both $a(t)$ and $b(t)$ are roots of unity is infinite. Fix any t_0 in this set, and write $\zeta = a(t_0)^{-1}, \xi = b(t_0)^{-1}$ so that $\zeta, \xi \in \mathbb{U}_\infty$.

Consider the algebraic curve $D = \{(\zeta a(t), \xi b(t)),\, t \in C\}$ inside $\mathbb{A}^2$. Then D contains the point $(1,1)$ and infinitely many other points for which both coordinates are roots of unity. By [**115**], the curve D is given by the equation $D = \{x^n y^m = 1\}$ for some non-zero integers $n, m \in \mathbb{Z}$. We conclude that $a(t)^n b(t)^m$ is a constant equal to some root of unity.

5.6.3. Algorithm

In this section we discuss briefly the possibility of constructing an algorithm deciding whether two dynamical pairs (P,a) and (Q,b) are entangled. Suppose these two pairs are defined over an algebraic curve C defined over a number field and have respective degree d and δ.

1. Decide whether the pairs are active or not.

To see whether (P,a) is active, we first make a base change and conjugate the family to polynomials of the form (8). Under this parametrization, (P,a) is passive iff P is a constant family and a is a constant.

If either (P,a) or (Q,b) is passive, then we stop. Otherwise we proceed to the next step.

2. Compare the divisors $\mathsf{D}_{P,a}$ and $\mathsf{D}_{Q,b}$.

We first compute $\mathsf{D}_{P,a}$. Pick any branch $\mathfrak{c}$ of C at infinity. If $\mathrm{ord}_\mathfrak{c}(\mathsf{D}_{P,a}) > 0$, then it is posible to compute $\mathrm{ord}_\mathfrak{c}(\mathsf{D}_{P,a})$ in finitely many steps. Indeed, for all $q \geq 1$ sufficiently large, we have $\mathrm{ord}_\mathfrak{c}(\mathsf{D}_{P,a}) = -d^{-q}\,\mathrm{ord}_\mathfrak{c}(P^q(a))$ in this case.

We do not know however whether one can decide if $\mathfrak{c}$ belongs to the support of $\mathsf{D}_{P,a}$ or not. To continue our algorithm, we shall thus suppose that C has a single place at infinity(1).

If $\mathsf{D}_{P,a}$ and $\mathsf{D}_{Q,b}$ are not proportional, then we stop. Otherwise we find the smallest integers n and m such that $n \cdot \mathsf{D}_{P,a} = m \cdot \mathsf{D}_{Q,b}$.

(1) This works when $C = \mathbb{A}^1$ for instance.

3. Solving a linear system.

We first determine a field of definition $\mathbb{K}$ for the two pairs. We then decompose d and δ into prime factors, and pick coprime integers N, M so that $\Delta := d^N = \delta^M$ (if there are no such integer then we stop). We next determine ℓ and l such that the conditions

– $\Delta^\ell \times \mathrm{ord}_\mathfrak{c}(\mathsf{D}_{P,a}) > \mathrm{ord}_\mathfrak{c}(\mathsf{D}_P)$ for all branch at infinity of C;
– $M_\Delta^{\ell+L}(\xi) = M_\Delta^\ell(\xi)$ for any root of unity in $\xi \in \mathbb{K}$.

are both satisfied. Note that ℓ and l are bounded solely in terms of $\mathsf{D}_{P,a}$, D_P (which can be computed as in Step 2), and $[\mathbb{K} : \mathbb{Q}]$.

Then we solve the equations (37) and (38) with the family of polynomials R_t being the unknowns. Note that these equations are *linear*: if they are not solvable then we stop our algorithm. Otherwise, we produce R_t. If we have $\hat{P}_{t,n}(P^{\ell N}(a(t))) = \zeta \cdot \hat{Q}_{t,m}(Q^{\ell M}(b(t)))$, then we conclude that the two pairs are entangled.

5.6.4. Application to Manin-Mumford's problem

Let $f : X \to X$ be a surjective endomorphism of a projective variety (of dimension ≥ 2). The Manin-Mumford problem for f concerns the classification of all f-invariant subvarieties of X, and asks whether an irreducible subvariety Z containing a Zariski dense subset of $\mathrm{Preper}(f)$ is itself preperiodic.

This problem has been explored for a polarized endomorphisms[(2)], see [**102, 139, 101, 100**]; and for Hénon maps [**69**].

Let (P, a) and (Q, b) be two dynamical pairs with *fixed* polynomials and marked points parametrized by the same affine curve C. Observe that $f(z, w) = (P(z), Q(w))$ induces an endomorphism[(3)] of $\mathbb{P}^1 \times \mathbb{P}^1$. Denote by Z the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of the image of C by the map $t \mapsto (a(t), b(t))$, so that Z contains infinitely many preperiodic points iff (P, a) and (Q, b) are entangled. Ghioca, Nguyen, and Ye [**101**] have proved that this occurs iff $\deg(P)^N = \deg(Q)^M$ for some integers N, M and Z is preperiodic for the endomorphism (P^N, Q^M).[(4)].

Using our results, we obtain the following extension of their theorem.

Theorem 5.28. — *Let P and Q be two algebraic families of non-integrable polynomials of degree ≥ 2 defined over a number field $\mathbb{K}$ parametrized by a smooth affine curve C.*

(2) i.e. whose action on the Néron-Severi space admits an ample class as an eigenvector
(3) It is polarized iff $\deg(P) = \deg(Q)$.
(4) The rational case was explored by Mimar [**130**].

Suppose that Z is an irreducible curve of $\mathbb{A}^2 \times C$ which projects surjectively onto C, and contains infinitely many preperiodic points of the endomorphism $(z, w, t) \mapsto (P_t(z), Q_t(w), t)$.

Either Z is preperiodic under $(z, w, t) \mapsto (P_t(z), Q_t(w), t)$, or Z is contained in an algebraic surface S which is preperiodic under $(z, w, t) \mapsto (P_t^N(z), Q_t^M(w), t)$ for some integers $N, M \geq 1$.

Remark 5.29. — When $\deg(P) > \deg(Q)$, the only irreducible surfaces S which are invariant under $(z, w, t) \mapsto (P_t(z), Q_t(w), t)$ and project onto C are of the form $\mathbb{A}^1 \times \{w_0\} \times C$ or $\{z_0\} \times \mathbb{A}^1 \times C$. Indeed for a generic point t_0, the curve $S \cap \mathbb{A}^2 \times \{t_0\}$ is fixed by the endomorphism (P_{t_0}, Q_{t_0}) which admits only the vertical and horizontal fibers of $\mathbb{P}^1 \times \mathbb{P}^1$ as invariant classes.

In particular if a surface S which is not of the form $\mathbb{A}^1 \times \{w_0\} \times C$ or $\{z_0\} \times \mathbb{A}^1 \times C$ is invariant under $(z, w, t) \mapsto (P_t^N(z), Q_t^M(w), t)$, then $\deg(P^N) = \deg(Q^M)$.

Remark 5.30. — Suppose conversely that Z is included in an algebraic surface S which projects onto C, and is fixed by F. We obtain a family of affine curves $S_t, t \in C$ with an algebraic dynamical system $F_t : S_t \to S_t$. The latter cannot be invertible since $\deg(P), \deg(Q) \geq 2$. It follows that F_t defines a family of polynomials and Z determines a marked point. If the corresponding marked point is active, then Z contains infinitely many preperiodic points for F. The same is true when Z is stably preperiodic. By Theorem 4.30, the only remaining case is when the pair (F_t, Z_t) is isotrivial. The projections $S \to \mathbb{A}^1 \times C$ induce semi-conjugacies between F_t and the families P_t and Q_t hence the latter are both isotrivial. We have thus shown that the converse to Theorem 5.28 holds except if P and Q are isotrivial and both semi-conjugated to the same polynomial.

Proof. — Let $\mathsf{n} : \hat{Z} \to Z$ be the normalization of Z and let $a = \pi_1 \circ \mathsf{n}$ and $b = \pi_2 \circ \mathsf{n}$. Observe that we may assume by base change $\hat{Z} \to C$ that $Z = \{(a(t), b(t), t), t \in \hat{Z}\}$, and that the two dynamical pairs (P, a) and (Q, b) are parametrized by $\hat{Z}$.

If both pairs (P, a) and (Q, b) are passive, then $a(t)$ is P_t-preperiodic, and $b(t)$ is Q_t-preperiodic, so that Z is preperiodic under $F(z, w, t) := (P_t(z), Q_t(w), t)$. If only one of the two pairs is passive, say (P, a), then $a(t)$ is P_t-preperiodic for all t, and Z is included in the surface $S = \{(a(t), w, t), t \in C, w \in \mathbb{A}^1\}$ which is again preperiodic under $F(z, w, t) := (P_t(z), Q_t(w), t)$.

Otherwise, (P, a) and (Q, b) are active and our assumption implies that (P, a) and (Q, b) are entangled. By Theorem B, there exist integers $N, M \geq 1$,

$r, s \geq 0$, and families R, τ and π of polynomials parametrized by C such that

$$\tau \circ P^N = R \circ \tau \quad \text{and} \quad \pi \circ Q^M = R \circ \pi,$$

and $\tau(P^r(a)) = \pi(Q^s(b))$.

We conclude observing that if $p = (a(t), b(t), t) \in Z$ for some $t \in \hat{Z}$, then for any integer l, the point $F^l(p) = (P^{Nl}(a(t)), Q^{Ml}(b(t)), t)$ is included in the surface $\{(z, w, t),\, \tau(P^r(z)) = \pi(Q^s(w))\}$ which is fixed by F. $\square$

CHAPTER 6

ENTANGLEMENT OF MARKED POINTS

We specialize the results of the previous chapters to a single family of polynomials. Let P be any algebraic family of polynomials of degree d parameterized by a curve C defined over an algebraically closed field K of characteristic 0. Given any marked point $a \in K[C]$, we first give strong restrictions on dynamical pairs (P, b) that are entangled with (P, a), and obtain Theorem D from the introduction that we recall for the convenience of the reader.

Theorem D. — *Let (P, a) be any active primitive non-integrable polynomial dynamical pair parameterized by an algebraic curve defined over a field K of characteristic 0. Assume that K is a number field, or that ($\triangle$) is satisfied.*

For any marked point $b \in K[C]$ such that (P, b) is active, the following assertions are equivalent:

1. *the set* $\mathrm{Preper}(P, a) \cap \mathrm{Preper}(P, b)$ *is infinite (i.e. a and b are entangled),*
2. *there exist $g \in \Sigma(P)$ and integers $r, s \geq 0$ such that $P^r(b) = g \cdot P^s(a)$.*

As mentioned in the introduction, the previous result is not satisfactory in the sense that for most $r, s \in \mathbb{N}$ and $g \in \Sigma(P)$ there exists no $b \in K[C]$ such that $P^r(b) = g \cdot P^s(a)$ (solutions to the previous equations are defined over a finite field extension of $K(C)$). The next result describes in more detail the set of pairs (P, b) entangled with (P, a) when our data are defined over a number field (Theorem E from the introduction).

Theorem E. — *Let (P, a) be any active primitive non-integrable polynomial dynamical pair parameterized by an irreducible affine curve C defined over $\bar{\mathbb{Q}}$.*

The set of marked points in $\bar{\mathbb{Q}}[C]$ that are entangled with a is the union of $\{g \cdot P^n(a) \,;\, n \geq 0$ and $g \in \Sigma_0(P)\}$ and a finite set.

6.1. Proof of Theorem D

Suppose first that there exists $g \in \Sigma(P)$ and integers $r, s \geq 0$ such that

$$P^r(b) = g \cdot P^s(a),$$

as regular functions on C. Since P is not integrable, $\Sigma(P)$ is a finite group and there exists a morphism $\rho\colon \Sigma(P) \to \Sigma(P)$ such that $P(g \cdot x) = \rho(g) \cdot P(x)$ for all x. In particular, for any parameter t such that $P_t^s(a(t))$ has a finite orbit, there exists $g_n \in \Sigma(P)$ such that $P_t^{rn}(b(t)) = g_n \cdot P_t^{sn}(a(t))$ so that $b(t)$ is also preperiodic. We have proved that $\mathrm{Preper}(P, a, \bar{K}) \subset \mathrm{Preper}(P, b, \bar{K})$.

Since a is active, Remark 4.31 implies $\mathrm{Preper}(P, a, \bar{K})$ to be infinite, hence $\mathrm{Preper}(P, a, \bar{K}) \cap \mathrm{Preper}(P, b, \bar{K})$ is infinite.

Suppose conversely that $\mathrm{Preper}(P, a, \bar{K}) \cap \mathrm{Preper}(P, b, \bar{K})$ is infinite. Application of Theorems B and C do not quite imply what we are looking for, so that we go back to Section 5.4 and follow the details of the proof of (3) $\Rightarrow$ (2).

Let us suppose that $K = \mathbb{K}$ is a number field. Following the arguments of §5.2.2 we get two coprime integers say $n \leq m$ such that $n \cdot h_{P,a} = m \cdot h_{P,b}$.

We now follow carefully the arguments in Section 5.4 (I) & (II) using the fact that $N = M = 1$. Assume ($\diamond$), i.e. the leading term of P admits a $(d-1)$-th root. We get a constant $L \geq 1$ such that the following holds. For any fixed Archimedean place $v_0 \in M_{\mathbb{K}}$, and for each branch at infinity $\mathfrak{c}$ with $\mathrm{ord}_{\mathfrak{c}}(\mathsf{D}_{P,a}) > 0$, there exists a connected neighborhood $U_{v_0}(\mathfrak{c})$ of the branch in C^{an,v_0} such that for any parameter $t \in C(\bar{\mathbb{K}}) \cap U_{v_0}(\mathfrak{c})$, the map $\Phi'_t(x, y) = (P_t^L(x), P_t^L(y))$ fixes the Zariski closure Z of

$$C_{v_0}(\mathfrak{s}_\zeta) = \left\{(x, y) \in \mathbb{C}^2_{v_0},\ \min\{|x|_{v_0}, |y|_{v_0}\} > R_{v_0},\ \varphi_{P_t}(x)^n = \zeta \varphi_{P_t}(y)^m\right\} .$$

Recall that Z is an irreducible algebraic curve of $\mathbb{A}^2$. By [**126**, Theorem 6.24] (see also [**136**, Theorem 4.9]), Z is necessarily the graph or the cograph of a polynomial v_t which commutes with P_t. As $n \leq m$, this gives $n = 1$ and $Z = \{x = v_t(y)\}$, and arguing as in the paragraph preceding the statement of Lemma 5.19, we get the existence of a root of unity $\zeta(\mathfrak{c}) \in \mathbb{K}$, and of an integer $\ell(\mathfrak{c})$ satisfying (35) and (36), i.e.

$$\zeta(\mathfrak{c}) \cdot \hat{P}_{m,t} \circ P_t^L = P_t^L \circ (\zeta(\mathfrak{c}) \hat{P}_{m,t})$$
$$a_l = \zeta(\mathfrak{c}) \hat{P}_{m,t}(b_l),\ \text{for } l \geq \ell(\mathfrak{c}) .$$

The Main Theorem of [**151**] applied to P_t and $\zeta(\mathfrak{c})\hat{P}_{m,t}$ now implies the existence of a polynomial $R_t \in \mathbb{C}[z]$ and $\sigma_1, \sigma_2 \in \Sigma(R_t)$ such that

$$\zeta(\mathfrak{c}) \cdot \hat{P}_{m,t} = \sigma_1 \cdot R_t^{k_1} \text{ and } P_t = \sigma_2 \cdot R_t^{k_2} ,$$

for some integers k_1, k_2.

Since the family P is primitive, we may reduce the neighborhood $U_{v_0}(\mathfrak{c})$ so that P_t is primitive for all $t \in U_{v_0}(\mathfrak{c})$, and assume $R_t = P_t$. Observe that $k_2 = 1$ and $k_1 = m$. We have obtained

Lemma 6.1. — *Fix any parameter $t \in C(\bar{\mathbb{K}}) \cap U_{v_0}(\mathfrak{c})$ such that P_t is primitive, and not integrable. There exists $\sigma_t \in \Sigma(P_t)$ such that*

$$a_l = \sigma_t(b_{m+l})$$

for all $l \geq \ell(\mathfrak{c})$, where $a_l = P_t^l(a(t))$ and $b_{m+l} = P_t^{m+l}(b(t))$.

We continue arguing as in Section 5.4 (III). It is clear that the set of parameters t in C such that there exists $\sigma_t \in \Sigma(P_t)$ satisfying

$$a_{\ell(\mathfrak{c})} = \sigma_t(b_{m+\ell(\mathfrak{c})})$$

is Zariski closed. It follows that this set is to C by the preceding Lemma. Since σ_t is uniquely determined by the data (except for those finitely many parameters for which b_t is the fixed point of a non-trivial symmetry of P_t), we conclude to the existence of $\sigma \in \Sigma(P)$ such that $a_{\ell(\mathfrak{c})} = \sigma(b_{m+\ell(\mathfrak{c})})$ as required.

The case when ($\diamond$) is not satisfied is taken care of as in Section 5.4 (IV), and the same specialization argument as in Section 5.5 yields the theorem over any field of characteristic zero if condition ($\triangle$) holds.

6.2. Proof of Theorem E

We assume that P is a non-integrable and primitive algebraic family of degree $d \geq 2$ polynomials parameterized by an affine curve which is defined over a number field K. We also fix a marked point $a \in K[C]$ such that (P, a) is active.

Up to an affine transformation, we shall assume that P has a reduced presentation $P(z) = z^\mu P_0(z^m)$ so that the center of P is 0. Recall that the group of dynamical symmetries $\Sigma(P) = \mathbb{U}_m$ of a degree d polynomial P comes equipped with a morphism $\rho : \Sigma(P) \to \Sigma(P)$ such that $P(g \cdot z) = \rho(g) \cdot P(z)$. Recall that we denoted by $\Sigma_0(P)$ the union of the kernels of ρ^n for all $n \geq 1$.

Lemma 6.2. — *Fix any $g \in \Sigma(P)$ and any integer $n \geq 0$.*

Then the point $g \cdot P^n(a)$ belongs to the grand orbit of a iff either $g \in \Sigma_0(P)$, or $P^m(a)$ is the center of P for some $m \geq n$.

Proof. — If g belongs to $\Sigma_0(P)$ and $\rho^m(g) = \mathrm{id}$, then we have $P^m(g \cdot P^n(a)) = \rho^m(g) \cdot P^{nm}(a) = P^{nm}(a)$.

If $P^m(a) = 0$ with $m \geq n$, then $P^{m-n}(g \cdot P^n(a)) = \rho^{m-n}(g) \cdot P^m(a) = 0$ hence $g \cdot P^n(a)$ lies in the grand orbit of a.

This proves one implication.

Suppose conversely that $g \cdot P^n(a)$ belongs to the grand orbit of a. Then there exist two integers l and q such that $P^l(a) = P^q(g \cdot P^n(a)) = \rho^q(g) \cdot P^{qn}(a)$. Since a is active the divisor at infinity $\mathsf{D} := \mathsf{D}_{P,a}$ is non zero, and $\mathsf{D}_{P,P^l(a)} = d^l\mathsf{D}$ whereas $\mathsf{D}_{P,\rho^q(g)\cdot P^{qn}(a)} = d^{qn}\mathsf{D}$. Indeed we may always assume that K is a subfield of $\mathbb{C}$, and apply Propositions 3.9 and 4.27. We conclude that $l = qn$ hence $P^l(a) = \rho^q(g) \cdot P^l(a)$. We thus have either $P^l(a) = 0$ with $l \geq n$, or $g \in \Sigma_0(P)$ as required. □

Denote by $\mathcal{E}(a)$ the set of marked points $b \in \bar{K}[C]$ such that the pairs (P, a) and (P, b) are entangled. Observe that

$$\mathcal{E}(a) \supset \mathcal{A}_0 := \{g \cdot P^n(a) \text{ with } n \geq 0 \text{ and } g \in \Sigma_0(P)\} \ .$$

Our objective is to show that the complement of the right hand side in $\mathcal{E}(a)$ is a finite set. Since by Theorem D we have

$$\mathcal{E}(a) := \{b \in \bar{K}[C], \text{ there exists } n, m \geq 0 \text{ and } g \in \Sigma(P) \text{ s.t. } P^n(b) = g{\cdot}P^m(a)\} \ ,$$

we are reduced to proving that

Theorem 6.3. — *The set $\mathcal{E}(a) \setminus \mathcal{A}_0$ is finite.*

We begin with the following

Proposition 6.4. — *For any integer $D \geq 1$, the set of points $b \in \bar{K}[C]$ such that $P^n(b) = P^m(a)$ for some integers n, m satisfying $n \geq m - D$ is finite.*

Remark 6.5. — By a theorem of Benedetto [**15**], (see also [**4**] for the case of rational maps), the set of points on $K(C)$ of sufficiently small height is finite when P is not isotrivial. This implies that one can assume $|n - m|$ bounded when the family is not isotrivial. Beware that it is not true that for all $c > 0$ the set of points of height $\leq c$ is bounded.

Indeed, as remarked in [**51**], if $P(z) = z^2 + t \in \mathbb{C}(t)[z]$, then, over the field $\mathbb{C}(t)$ of rational functions on $\mathbb{P}^1$, any point $b \in \mathbb{C}$ has canonical height $h_P(b) = 1/2$.

Proof. — We proceed by contradiction and pick an infinite sequence $b_l \in \bar{K}[C]$ in the grand orbit of a such that $P^{n_l}(b_l) = P^{m_l}(a)$ with $n_l \geq m_l - D$. Replacing a by $P^D(a)$, and b_l by a suitable iterate, we may always assume that $n_l = m_l$.

1. We may interpret the family P as a degree d polynomial

$$\mathsf{P} : \mathbb{A}^1_{\bar{K}(C)} \to \mathbb{A}^1_{\bar{K}(C)}$$

and a and b_l as points $\mathsf{a}, \mathsf{b}_l \in \mathbb{A}^1_{\bar{K}(C)}$.

Fix any branch $\mathfrak{c}$ of C at infinity. To this branch, we can associate a non-Archimedean norm $|\cdot|_{\mathfrak{c}}$ on $\bar{K}(C)$ by setting

$$|f|_{\mathfrak{c}} := \exp(-\mathrm{ord}_{\mathfrak{c}}(f)), \text{ for all } \ f \in \bar{K}(C).$$

We infer

$$0 \le g_{\mathsf{P},\mathfrak{c}}(\mathsf{b}_l) = d^{m_l - n_l} g_{\mathsf{P},\mathfrak{c}}(\mathsf{a}) \le d^D g_{\mathsf{P},\mathfrak{c}}(\mathsf{a}),$$

so that

$$\max\{0, -\mathrm{ord}_{\mathfrak{c}}(\mathsf{b}_l)\} = \log^+ |\mathsf{b}_l|_{\mathfrak{c}} < d^D g_{\mathsf{P},\mathfrak{c}}(\mathsf{a}) + \sup |g_{\mathsf{P},\mathfrak{c}}(z) - \log^+ |z|| < \infty \ .$$

In particular the sequence $\mathrm{ord}_{\mathfrak{c}}(\mathsf{b}_l)$ is bounded from below.

Since the degree of the rational map $b_l \colon C \to \mathbb{P}^1$ is equal to

$$\deg(b_l) = \sum_{\mathfrak{c}} \max\{0, -\mathrm{ord}_{\mathfrak{c}}(\mathsf{b}_l)\},$$

we conclude that the graph Γ_l of b_l belongs to a family of curves of bounded degree in the projective surface $\bar{C} \times \mathbb{P}^1$. In other words there exists an irreducible variety W, a regular map $\Gamma \colon W \times C \to \mathbb{A}^1$, and a Zariski-dense subset w_l of W such that $\Gamma(w_l, \cdot) = b_l$ for all l.

2. We fix any embedding of $\bar{K}$ into $\mathbb{C}$, and work with the euclidean topology on C^{an}. Pick any connected component Ω of $\{g_{P,a} > 0\}$ in C^{an}.

By the maximum principle one can find a branch at infinity $\mathfrak{c}$ lying in the closure of Ω (in $\bar{C}$), as well as in the support of $\mathsf{D}_{P,a}$. As usual we fix a local parameterization $t \mapsto \theta(t)$ of $\mathfrak{c}$ in $\bar{C}$, and drop the reference to θ to simplify notation. Replacing a by an iterate if necessary, we can evaluate the Böttcher coordinate of P_t at $a(t)$ for any t sufficiently small. Since $P^{n_l}(b_l) = P^{n_l}(a)$, we have $g_{P,a} = g_{P,b_l}$ so that the Böttcher coordinate is also defined at $b_l(t)$ for t small, and there exists a root of unity ζ_l such that

$$\varphi_t(b_l(t)) = \zeta_l \varphi_t(a(t)) \ .$$

We claim that for all $w \in W$ there exists a constant $\zeta(w) \in \mathbb{C}$ such that

$$\varphi_t(\Gamma(w,t)) = \zeta(w) \varphi_t(a(t)) \ ,$$

for all t small enough.

To see this recall from §2.5 that we have an expansion of the Böttcher coordinate of the form

$$\varphi_t(z) = z + \sum_{k \ge 1} \frac{\alpha_k(t)}{z^k}$$

where $\alpha_k(t)$ are analytic so that $\varphi_t(a(t)) = \sum_{k\geq -k_0} a_k t^k$ for some $a_k \in \mathbb{C}$ (we may take a further iterate of a as in Step 3 on p.148 so that the series formally converges).

Observe that we can write $\Gamma(w,t) = \sum_{k\geq -k_0} h_k(w)t^k$ where h_k is a regular function on W. It follows that the equation $\varphi_t(\Gamma(w,t)) = c\varphi_t(a(t))$ is equivalent to a series of equations of the form

$$H_k := h_k + P_k(h_{-n}, \cdots, h_{k-1}) = ca_k \ ,$$

where P_k is a polynomial in $n+k$ variables. For each integer $N \geq -n$, we obtain

$$[H_{-n}(w) : \cdots : H_N(w)] = [a_{-n} : \cdots : a_N] \in \mathbb{P}^{n+N-1}$$

for all $w = w_l$. Since $\{w_l\}$ is Zariski-dense the equality holds for all $w \in W$. Letting $N \to \infty$, we get our claim.

3. Note that by construction the map $w \mapsto \zeta(w)$ is algebraic. We claim that it is not constant. Indeed, for all t close enough to the branch at infinity $\mathfrak{c}$, the Böttcher coordinate φ_t is a isomorphism on $\{g_t > \frac{1}{2}g_t(a(t))\}$ so that $\zeta_l = \zeta_{l'}$ implies $\varphi_t(b_l(t)) = \varphi_t(b_{l'}(t))$, hence $b_l(t) = b_{l'}(t)$, from which we infer $b_l = b_{l'}$.

Since W is irreducible, we may pick $w \in W$ such that $|\zeta(w)| < 1$. Recall from the previous step that Ω is a connected component of $\{g_{P,a} > 0\}$, and that $\mathfrak{c}$ is a branch at infinity lying in its closure. Near that branch, we have $g_{P,\Gamma(w,\cdot)} = g_{P,a} - \log|\zeta(w)|$ hence by analytic continuation $g_{P,\Gamma(w,\cdot)} = g_{P,a} - \log|\zeta(w)|$ on Ω.

Pick any point t_* on the boundary of Ω. Then $g_{P,a}(t_*) = 0$, and $g_{P,\Gamma(w,\cdot)}$ is harmonic near t_*. It follows that in a neighborhood of t_* the boundary of Ω is contained in $g_{P,\Gamma(w,\cdot)} = -\log|\zeta(w)|$ hence is locally real-analytic.

Since Ω was an arbitrary component of $\{g_{P,a} > 0\}$, we have proved that the bifurcation locus of the pair (P,a) is real-analytic. By Theorem A, the family is integrable which contradicts our assumption. □

Lemma 6.6. — *Suppose there exists an integer $n \geq 0$, and a sequence $b_l \in \bar{K}[C]$ such that $P^n(b_l) = P^{m_l}(a)$ with $m_l \to \infty$.*

Then for all l large enough, we have $b_l \in \mathcal{A}_0 = \{g\cdot P^n(a),\ g \in \Sigma_0(P),\ n \geq 0\}$.

Remark 6.7. — The previous proposition was valid for any field of characteristic zero. But our proof of Lemma 6.6 uses our standing assumption that K is a number field.

Proof. — The Böttcher coordinates $\varphi_P(z)$ is a formal Laurent series in z^{-1} whose coefficients belong to $K[C]$ by Proposition 2.19. Since a is active and $m_l - n \to \infty$, we may suppose that for some branch $\mathfrak{c}$ at infinity of C (any branch in the support of $\mathsf{D}_{P,a}$ works), the series $\varphi_t(P_t^{m_l-n}(a(t)))$ is well-defined in a neighborhood of $\mathfrak{c}$ for all l. It follows that for any l one can write

$$\varphi_t(b_l(t)) = \zeta_l \varphi_t(P_t^{m_l-n}(a(t)))$$

for some d^n-th root of unity ζ_l and all $|t| \ll 1$.

Let us fix some d^n-th root of unity ζ. We claim that the set of indices l such that $\zeta_l = \zeta$ and $b_l \notin \mathcal{A}_0$ is finite. Note that this implies the lemma.

To simplify notation, we shall assume that $\zeta_l = \zeta$ for all l. Fix any Archimedean place v_0 and a sufficiently small (euclidean) neighborhood $U_{v_0}(\mathfrak{c}) \subset C^{\mathrm{an},v_0}$ of the branch at infinity $\mathfrak{c}$, as in the first paragraph of (II) on p.150.

Pick any closed point $t \in U_{v_0}(\mathfrak{c}) \cap C(\bar{K})$. Observe that $a(t)$ is not preperiodic, and choose any finite extension L/K such that $t \in C(L)$. For any l we have $P_t^n(b_l(t)) = P_t^{m_l}(a(t)) \in L$ hence $b_l(t)$ belongs to a fixed finite extension of L (the one in which P_t^n splits).

Consider the adelic series at infinity $\mathfrak{s}_\zeta$ given by

$$\mathfrak{s}_\zeta = \{(x, y) \in \mathbb{A}^1 \times \mathbb{A}^1,\ \varphi_t(x) = \zeta\, \varphi_t(y)\},$$

and observe that, by our assumption, for each integer l and for each place $v \in M_L$, either the point $(b_l(t), P_t^{m_l-n}(a(t)))$ belongs to $Z^v(\mathfrak{s}_\zeta)$ (e.g. when $v = v_0$) or has bounded norm.

By Theorem 1.17, one can find thus an irreducible algebraic curve $Z \subset \mathbb{A}^1 \times \mathbb{A}^1$ such that Z has a single branch at infinity included in $\mathfrak{s}_\zeta$, and $(b_l(t), P_t^{m_l-n}(a(t))) \in Z$ for all l.

Since Z has a single branch which is smooth and transverse to the fibrations induced by the two projections $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$, it is the graph of an automorphism of $\mathbb{A}^1$, say $z \mapsto g(z)$ with g affine. It follows that for all z large enough one has

$$\varphi_t(P_t^n(g(z))) = \varphi_t(g(z))^{d^n} = (\zeta\, \varphi_t(z))^{d^n} = \varphi_t(P_t^n(z)))$$

hence g belongs to $\Sigma_0(P_t^n) = \Sigma_0(P_t)$. Because P_t is monic and centered, g is necessarily linear, and since φ_t is tangent to the identity at infinity, we conclude that $g(z) = \zeta z$.

It follows that $b_l(t) = \zeta P^{m_l-n}(a(t))$ for all l and infinitely many parameters t, so that $b_l = \zeta P^{m_l-n}(a)$, and $b_l \in \mathcal{A}_0$ as was to be shown. □

We may now prove Theorem 6.3.

Proof of Theorem 6.3. — Suppose by contradiction that we can find a sequence of distinct points $b_l \in \bar{K}[C]$ such that $P^{n_l}(b_l) = P^{m_l}(a)$, $P^{n_l-1}(b_l) \neq P^{m_l-1}(a)$ and $b_l \notin \mathcal{A}_0$.

By Proposition 6.4, we may suppose that $m_l - n_l \to \infty$. When n_l does not tend to ∞, then we can extract a subsequence such that $n_l = n$ and apply the preceding lemma. This shows that $n_l \to \infty$.

Observe that since P is not integrable, the group of dynamical symmetries of P is finite, and there exists an integer N such that for any $g \in \Sigma_0(P)$ we have $\rho^N(g) = 1$ (where $\rho\colon \Sigma(P) \to \Sigma(P)$ is the morphism arising in Proposition 3.6).

Pick any integer $n > N$. We can then apply Lemma 6.6 to the sequence of points $P^{n_l - n}(b_l)$, and we obtain that $P^{n_l - n}(b_l)$ belongs to $\mathcal{A}_0$ for all l large enough. But then we get $P^{n_l - n}(b_l) = g \cdot P^{m'_l}(a)$ for some integer $m'_l \geq 0$ and some $g \in \Sigma_0(P)$, hence $P^{n_l - n + N}(b_l) = P^{m'_l + N}(a)$ which contradicts the minimality of n_l. $\square$

CHAPTER 7

THE UNICRITICAL FAMILY

The original papers by Baker-DeMarco were mainly focused on the unicritical family $P_t(z) = z^d + t$. In this short chapter, we propose to extend some of their results, and to illustrate the theorems proved in the previous chapters on this special family.

We also introduce the set $\mathbb{M}$ of those $\lambda \in \mathbb{C}^*$ such that the bifurcation locus of the pair $(P_t, \lambda^{-1}t)$ is connected. We prove this set is compact and perfect. We finally provide some images of bifurcation loci that was obtained by A. Chéritat.

7.1. General facts

In this section, we gather several facts about the unicritical family and make some computations that will be useful in the sequel. To simplify the discussion we work over the field of complex numbers.

We fix $d \geq 2$ and consider the family $P_t(z) = z^d + t$ parametrized by the affine line $t \in \mathbb{A}^1$. Observe that P_t is integrable iff $t = 0$ or $d = 2$ and $t = -2$, and that $\Sigma(P_t) = \mathbb{U}_d$ when $t \neq 0$. The family is not isotrivial, and primitive[(1)]. Beware though that P_t is decomposable as soon as d is not a prime.

We pick any marked point $a \in \mathbb{C}[t]$, which we write as $a(t) = \alpha t^\kappa + o(t^\kappa) \in \mathbb{C}[t]$, for some $\alpha \neq 0$, $\kappa \geq 0$. We do not exclude the case a is a constant. Define

$$\mathcal{M}(d, a) = \{t \in \mathbb{C},\, a(t) \in K(P_t)\} = \{t \in \mathbb{C},\, g_{P_t}(a(t)) = 0\},$$

so that the bifurcation locus of (P, a) is equal to the boundary of $\mathcal{M}(d, a)$. When $a = 0$, the boundary of $\mathcal{M}(d, 0)$ is the bifurcation locus of the unicritical family, and $\mathcal{M}(2, 0)$ is the Mandelbrot set.

(1) If $z^d + t = \sigma \circ Q^n(z)$ with $n \geq 2$, then $\bigcup_{0 \leq j \leq n-1} Q^{-j}(\mathrm{Crit}(Q)) = \{0\}$ which implies $\mathrm{Crit}(Q)$ to be totally invariant, a contradiction.

Recall the definition of the (logarithmic) capacity of a compact set K in the plane, see [**143, 162**] or [**153**, §A.8]. First one defines the Green function g_K of K as the upper-semi-continuous regularization of the supremum of all subharmonic functions u of the plane such that $u|_K \le 0$, and $u(z) = \log|z| + O(1)$ at infinity. When g_K is not identically $+\infty$, it is subharmonic and harmonic on $\mathbb{C} \setminus K$. Note also that if g is a continuous subharmonic function on $\mathbb{C}$ which is harmonic on $\mathbb{C} \setminus K$ and 0 on K, then it equals the Green function of K.

The measure $\mu_K := \Delta g_K$ is the harmonic measure of K. Near infinity, we have the expansion $g_K(z) = \log|z| + V + o(1)$, and the constant $\mathrm{cap}(K) := e^{-V} > 0$ is called the capacity of K. When g_K is identically $+\infty$, we set $\mathrm{cap}(K) = 0$. Recall that one can define the energy of any probability measure μ which is compactly supported by

$$\mathcal{E}(\mu) := \int \log|x-y|^{-1}\, d\mu(x) d\mu(y) \in (-\infty, +\infty]\ .$$

When $\mathrm{cap}(K) > 0$, the harmonic measure of K is the unique measure such that $\mathcal{E}(\mu_K) = \inf\{\mathcal{E}(\mu), \mathrm{supp}(\mu) \subset K\}$, and $\mathcal{E}(\mu_K) = -\log \mathrm{cap}(K)$.

To simplify notations, we write $g_t = g_{P_t} = \lim_n \frac{1}{d^n} \log^+ |P_t^n|$ and $g_a(t) = g_{P_t}(a(t))$, and we let $\varphi_{d,t}$ be the Böttcher coordinate of P_t which is defined in the open set $\{g_t > g_t(0)\}$, see Proposition 2.22.

An easy induction shows

$$P_t^n(z) = z^{d^n} + d^{n-1} t z^{d^n - d} + O_t(z^{d^n-d-1}) \text{ for all } n \ge 1, \tag{45}$$

and by Proposition 2.20, we have

$$\varphi_{d,t}(z) = z + \frac{t}{dz^{d-1}} + \sum_{j=0}^{\infty} \frac{\alpha_j(t)}{z^{d+j}}\ , \tag{46}$$

where $\alpha_j \in \mathbb{Z}[t]$ satisfies $\deg_t(\alpha_j) \le (1+j)/d$.

Let us first treat the case when the marked point is constant.

Proposition 7.1. — *Suppose a is a constant function. Then for all $n \in \mathbb{N}^*$ we have*

$$P_t^{n+1}(a) = t^{d^n} + a^d t^{d^n - 1} + o(t^{d^n-1})\ , \text{ and}$$
$$g_a(t) = \frac{1}{d} \log|t| + o(1),$$

so that dg_a is the Green function of $\mathcal{M}(d,a)$, and $\mathrm{cap}(\mathcal{M}(d,a)) = 1$. Moreover, the inequality $g_t(a^d + t) > g_{d,t}(0)$ holds for all t large enough, and

$$g_a(t) = \frac{1}{d} \log|\varphi_{d,t}(a^d + t)|\ . \tag{47}$$

This result is proved in [**5**, Lemma 3.2 & Proposition 3.3].

Proof. — The first equality is obtained by induction on n. The second follows from Proposition 2.11 which[2] gives

$$\log^+|z| - C_1 \leq g_t(z) \leq \log^+\max\{|z|, |t|^d\} + C_1$$

for some constant C_1, and for all $|z| \geq C_2 \max\{1, |t|^d\}$. For all n, and for t large enough we get

$$\left| g_a(t) - \frac{1}{d^n}\log^+|P_t^n(a)| \right| \leq \frac{C_3}{d^n}$$

which implies $g_a(t) = \frac{1}{d}\log|t| + o(1)$ by the previous computations. The Green function of $\mathcal{M}(d,a)$ is $d \times g_a$ since the latter is subharmonic, is equal to 0 exactly on $\mathcal{M}(d,a)$ and has the expansion at infinity $= \log|t| + o(1)$. We also get $\mathrm{cap}(\mathcal{M}(d,a)) = 1$.

Since $g_t(a^d + t) = \log|t| + o(1)$ and $g_t(0) = g_0(t) = \frac{1}{d}\log|t| + o(1)$, we get $g_t(a^d + t) > g_{d,t}(0)$ holds for all t large enough so that (47) holds. □

When the marked point is not constant, the previous proposition needs to be modified as follows.

Proposition 7.2. — *Suppose a is a complex polynomial of degree $\kappa \geq 1$ whose dominant term is equal to α. Then for all $n \in \mathbb{N}^*$ we have*

$$P_t^n(a(t)) = \alpha^{d^n} t^{\kappa d^n} + o(t^{\kappa d^n}) \text{ and}$$
$$g_a(t) = \kappa \log|t| + \log|\alpha| + o(1),$$

so that $\frac{1}{\kappa}g_a$ is the Green function of $\mathcal{M}(d,a)$, and $\mathrm{cap}(\mathcal{M}(d,a)) = |\alpha|^{-1/\kappa}$. Moreover for all t large enough, we have

$$g_a(t) = \frac{1}{d\kappa}\log|\varphi_{d,t}(a^d(t) + t)| \tag{48}$$

Remark 7.3. — When $\kappa \geq 2$, or $\kappa = 1$ and $|\alpha| \geq 1$, then $g_a(t) > g_0(t)$ for all t large enough, hence $\varphi_{d,t}$ is well-defined at the point t and in that case we have $g_a(t) = \frac{1}{\kappa}\log|\varphi_{d,t}(a(t))|$.

The proof is identical to the previous one and is left to the reader.

When the marked point is 0 (or any of its image under P_t), the set $\mathcal{M}(d,0)$ is the classical Multibrot set [**71**], and its boundary is the bifurcation locus of the unicritical family. The central hyperbolic component of $\mathcal{M}(d,0)$ consists of those parameters $t \in \mathbb{C}$ for which $z^d + t$ admits an attracting fixed point, see [**71**, §4].

(2) Beware of the change of parametrization.

Lemma 7.4. — *For any $d \geq 2$, the central hyperbolic component of the set $\mathcal{M}(d, 0)$ contains $\mathbb{D}(0, 1/4)$.*

Proof. — Let x_n be the sequence defined by $x_0 = 0$ and $x_{n+1} = x_n^2 + 1/4$. The sequence (x_n) is strictly increasing and converges to $1/2$, which is the only fixed point of $z^2 + 1/4$.

Let now $d \geq 2$ and pick $t \in \mathbb{D}(0, 1/4)$. An easy induction shows

$$|P_t^n(0)| \leq P_{1/4}^n(0) \leq x_n \leq \frac{1}{2}$$

for all $n \geq 1$. This ends the proof. □

7.2. Unlikely intersection in the unicritical family

Recall from the introduction our strengthening of the original Baker and DeMarco's result which deals with families of unicritical polynomials of possibly different degrees.

Theorem F. — *Let $d, \delta \geq 2$. If a, b are polynomials of the same degree and such that* $\mathrm{Preper}(z^d + t, a(t)) \cap \mathrm{Preper}(z^\delta + t, b(t))$ *is infinite, then $d = \delta$ and $a(t)^d = b(t)^d$.*

Remark 7.5. — With similar techniques, it is possible to treat the case $d = \delta$ and $\deg(a) \neq \deg(b)$, but the general case remains elusive.

Proof. — Observe that since the parameter space is the affine line, the divisors at infinity of the two pairs are supported at a single point, and Propositions 7.1 and 7.2 imply $\mathsf{D}_{P,a} = \mathsf{D}_{P,b} = [\infty]$ (when $\deg(a) = 0$), and $= \kappa[\infty]$ (otherwise). Condition $(\triangle)$ is therefore satisfied and we may apply Theorem 5.27.

Observe that $n = m = 1$ and $d^N = \delta^M$. It follows that there exists a root of unity ζ, and an integer $L \geq 1$ such that

$$\zeta(z^\delta + t)^{\circ ML} = ((\zeta z)^d + t)^{\circ NL}$$

and

$$(z^d + t)^{\circ NL}(a(t)) = \zeta(z^\delta + t)^{\circ ML}(b(t))$$

Expanding the first equation, and using (45) yields

$$\zeta(z^{\delta^{ML}} + t\delta^{ML} z^{\delta^{ML} - \delta} + \text{l.o.t}) = (\zeta z)^{d^{NL}} + t d^{NL} (\zeta z)^{d^{NL} - d} + \text{l.o.t}$$

which implies $d = \delta$, and $N = M = 1$. Note that $\zeta(z^d + t)^{\circ L} = ((\zeta z)^d + t)^{\circ L}$ hence $\zeta \in \Sigma(z^d + t)^{\circ L} = \Sigma(z^d + t) = \mathbb{U}_d$, and $\zeta = 1$. We have thus proved that

$$(z^d + t)^{\circ L}(a(t)) = (z^d + t)^{\circ L}(b(t)) \ .$$

We now exploit the equality $g_a(t) = g_b(t)$ together with (47) and (48).

When $\kappa = 0$, we follow the arguments of Baker and DeMarco. The two analytic functions $\varphi_{d,t}(\alpha^d + t)$ and $\varphi_{d,t}(\beta^d + t)$ have the same modulus near ∞, and are both tangent to the identity. They are hence equal. By injectivity of $\varphi_{d,t}$ near infinity, we get $\alpha^d = \beta^d$.

When $\kappa \geq 2$, then $\varphi_{d,t}(a(t)) = a(t) + O(t^{-1})$ and $\varphi_{d,t}(b(t)) = b(t) + O(t^{-1})$. Since these functions have the same modulus near ∞, we get $b(t) = \xi a(t)$ with $|\xi| = 1$.

We claim that in fact $\xi \in \mathbb{U}_d$. Write $a(t) = t^j(\alpha_0 + \alpha_1 t + O(t^2))$ with $\alpha_0 \neq 0$, and $j \geq 0$. When $j \geq 1$, we get for all $n \geq 1$

$$P_t^n(a(t)) = t + Q_n(t) + d^{n-1}\alpha_0^d t^{(n-1)(d-1)+dj} + O(t^{n(d-1)+dj+1})$$

where Q_n is a polynomial with constant (integral) terms vanishing up to order at least 2 at 0. And

$$0 = P_t^L(a(t)) - P_t^L(\xi a(t)) = (\xi^d - 1)d^{n-1}\alpha_0^d t^{(n-1)(d-1)+dj} + O(t^{n(d-1)+dj+1})$$

which implies $\xi^d = 1$. Otherwise $j = 0$, and we have

$$P_t^n(a(t)) = \alpha_0^{d^n} + t(d^n \alpha_0^{d^n - 1}\alpha_1 + R_n(\alpha_0)) + O(t^2)$$

with $R_1(T) = 1$, and $R_{n+1}(T) = dT^{d^n(d-1)}R_n(T) + 1$. We have $P_t^L(a(t)) = P_t^L(\xi a(t))$ hence $\xi^{d^L} = 1$, and $R_L(\xi T) = R_L(T)$. This implies $R_{L-1}(\xi T) = \xi^{d^{L-1}} R_{L-1}(T)$ so that

$$d(\xi T)^{d^{L-1}(d-1)} R_{L-2}(\xi T) + 1 = \xi^{d^{L-1}}(dT^{d^{L-1}(d-1)} R_{L-2}(T) + 1),$$

hence $\xi^{d^{L-1}} = 1$. By induction we get $\xi^d = 1$ as required.

When $\kappa = 1$, we replace a and b by $P(a)$ and $P(b)$ respectively, and by the previous argument we obtain $P^2(a) = P^2(b)$ which implies $P(a) = P(b)$. Details are left to the reader. □

7.3. Archimedean rigidity

We expect that under suitable conditions two complex dynamical pairs parametrized by the same algebraic curve and having the same (complex) bifurcation measures are entangled, see (Q1) from the Introduction.

We explore here this problem in the unicritical family, and obtain

Theorem 7.6. — *Fix $d \geq 2$, and pick $a, b \in \mathbb{C}[t]$ of the same degree. Then $\mathcal{M}(d, a) = \mathcal{M}(d, b)$ iff $a^d = b^d$.*

Remark 7.7. — It would be interesting to characterize dynamical pairs $(z^d+t, a(t))$ and $(z^\delta + t, b(t))$ with $d \neq \delta$ having the same bifurcation locus. It is not clear however how to show that d and δ are multiplicatively dependent.

Proof. — Write $a = \alpha t^\kappa + o(t^\kappa)$, and $b = \beta t^\kappa + o(t^\kappa)$ with $\kappa \geq 0$, and $\alpha\beta \neq 0$.

By Propositions 7.1 and 7.2, the function $d \times g_{d,a}$ (respectively $\frac{1}{\kappa} g_{d,a}$) is the Green function of $\mathcal{M}(d,a)$ when $\deg(a) = 0$ (respectively $\deg(a) \geq 1$). Since by assumption $\mathcal{M}(d,a) = \mathcal{M}(d,b)$, the two functions $g_{d,a}$ and $g_{d,b}$ are proportional, and even equal since $\deg(a) = \deg(b)$ and the capacity of $\mathcal{M}(d,a)$ and $\mathcal{M}(d,b)$ are equal. We deduce from this that for all t large enough, one has

$$\varphi_{d,t}(a^d(t) + t) = \zeta\, \varphi_{d,t}(b^d(t) + t) \text{ for some } |\zeta| = 1 \ .$$

If a and b are constants, then looking at the expansion of $\varphi_{d,t}$ yields $\zeta = 1$, and $a^d = b^d$. When $\kappa \geq 1$, we get $a^d(t) + t = \zeta(b^d(t) + t)$, and $(a^d(t)+t)^d + t = \zeta^d((b^d(t) + t)^d + t)$. Looking at the order 1 terms, $a(t) = \alpha_0 + t\alpha_1 + O(t^2)$, $b(t) = \beta_0 + t\beta_1 + O(t^2)$ we get

$$\alpha_0^d + (1 + d\alpha_0^{d-1}\alpha_1)t = \zeta\beta_0^d + \zeta(1 + d\beta_0^{d-1}\beta_1)t$$

$$\alpha_0^{d^2} + \left(1 + d\alpha_0^{d(d-1)}(1 + d\alpha_0^{d-1}\alpha_1)\right) t = \zeta^d\beta_0^{d^2} + \zeta^d \left(1 + d\beta_0^{d(d-1)}(1 + d\beta_0^{d-1}\beta_1)\right) t \ .$$

This implies $\zeta^d = 1$ hence $P^2(a) = P^2(b)$. We may then repeat the proof of Theorem F starting from "When $\kappa \geq 2$", and we conclude that $P(a) = P(b)$. □

7.4. Connectedness of the bifurcation locus

We explore in this section the connectedness of the Mandelbrot-type set $\mathcal{M}(d,a)$ under suitable assumptions on the marked point.

Theorem 7.8. — *Assume $a \in \mathbb{C}[t]$ is either a constant or that its degree is a power of d and its leading coefficient lies in the closed unit disk. The following assertions are equivalent:*

1. *the set $\mathcal{M}(d,a)$ is connected,*
2. *$a(t) = \zeta P_t^n(0)$ for some $n \geq 0$ and some $\zeta \in \mathbb{U}_d$.*

Proof. — When (2) is satisfied, the connectedness of $\mathcal{M}(d,a)$ is a famous theorem of Douady-Hubbard-Sibony, see e.g. [**39**, Chapter VIII, Theorem 1.2 page 124].

Suppose (2) is not satisfied, and observe that our assumptions imply the estimate $\mathrm{cap}(\mathcal{M}(d,a)) \geq 1$. We claim that $\mathcal{M}(d,a) \setminus \mathcal{M}(d,0)$ is non empty.

Suppose by contradiction that $\mathcal{M}(d,a) \subset \mathcal{M}(d,0)$. This implies that the series of inequalities

$$0 \geq -\log\operatorname{cap}(\mathcal{M}(d,a)) = \mathcal{E}\left(\mu_{\mathcal{M}(d,a)}\right) \geq \mathcal{E}\left(\mu_{\mathcal{M}(d,0)}\right) = -\log\operatorname{cap}(\mathcal{M}(d,0)) = 0,$$

so that $\mathcal{E}\left(\mu_{\mathcal{M}(d,a)}\right) = \mathcal{E}\left(\mu_{\mathcal{M}(d,0)}\right)$ which implies $\mu_{\mathcal{M}(d,a)} = \mu_{\mathcal{M}(d,0)}$. Since the support of these measures are $\mathcal{M}(d,a)$ and $\mathcal{M}(d,0)$ respectively, we obtain $\mathcal{M}(d,a) = \mathcal{M}(d,0)$. Since $\deg(a) = d^n$ for some integer n, we may apply the previous theorem to $P^n(0)$ and a, and we get $a = P^n(0)$ or $P(a) = P^n(0)$ which contradicts our standing assumption.

We conclude using the next lemma (note that the set $\{a' = 0\}$ is finite). □

Lemma 7.9. — *The set $\mathcal{M}(d,a)$ is totally disconnected in a punctured neighborhood of any point $t_0 \in \mathcal{M}(d,a) \setminus \mathcal{M}(d,0)$ such that $a'(t_0) \neq 0$.*

Proof. — Recall that the family $P_t(z) = z^d + t$ is stable in a neighborhood of t_0 and that the Julia set of P_{t_0} is a Cantor set. Thus there exists an open disk U centered at t_0 and a holomorphic motion $h\colon U \times J(P_{t_0}) \to \mathbb{C}$ conjugating the dynamics. By Theorem 2.33 (see [**125**]), we may reduce U and extend the holomorphic motion to the full complex plane $h\colon U \times \mathbb{C} \to \mathbb{C}$ so that $P_t(h(t,z)) = h(t, P_{t_0}(z))$ remains valid.

By Proposition 4.14, we have $\partial\mathcal{M}(d,a) \cap U = \{t \in U,\ a(t) \in J(P_t)\}$. Since $a'(t_0) \neq 0$, we may reduce U and find a polydisk $W = U \times V \subset \mathbb{C}^2$ containing $\Gamma := \{(t, a(t)),\ t \in U\}$ such that the intersection of Γ with $\{(t, h(t, a(t_0))),\ t \in U\}$ is transversal and reduced to $(t_0, a(t_0))$. By Rouché's theorem, it follows that any curve $\{(t, h(t,z)),\ t \in U\}$ intersects transversally $\{(t, a(t)),\ t \in U\}$ at a single point $(t, H(z)) \in U$, for any z close enough to $a(t_0)$. By continuity of the roots the map $z \mapsto H(z)$ is a homeomorphism, and $\partial\mathcal{M}(d,a) \cap \mathbb{D} = H^{-1}(J(P_{t_0}))$ is totally discontinuous as required. □

7.5. Some experiments

Theorem 7.8 leaves open the characterization of those marked points for which $\mathcal{M}(d,a)$ is connected in most cases, for instance when $\deg(a) = 1$. We propose to investigate the situation when the marked point is given by $a(t) = \lambda^{-1}t$ for some $\lambda \in \mathbb{C}^*$ In particular, we are interested in the description of the set $\mathbb{M}$ of complex numbers $\lambda \in \mathbb{C}^*$ such that $M_\lambda = \{t \in \mathbb{C},\ \lambda^{-1}t \in K(z^d + t)\}$ is connected. This set may be viewed as some kind of "higher" Mandelbrot set. The reason of choosing λ^{-1} instead of λ is motivated by the fact that $\mathbb{M}$ is then bounded.

We sum up in the next theorem what we know about this set.

Theorem 7.10. —

1. *A point* $\lambda \in \mathbb{C}^*$ *belongs to* $\mathbb{M}$ *iff* $M_\lambda \subset \mathcal{M}(d,0)$.
2. $\mathbb{M} \cap \{|\lambda| \geq 1\} = \mathbb{U}_d$.
3. $\mathbb{M} \supset \{0 < |\lambda| \leq 1/8\}$ *and* ∂M_λ *is a quasi-circle for all* $0 < |\lambda| \leq 1/8$.
4. *The set* $\mathbb{M} \cup \{0\}$ *is closed and perfect.*

Proof. — If M_λ is not included in $\mathcal{M}(d,0)$, then Lemma 7.9 implies that $\lambda \notin \mathbb{M}$. Suppose now that $M_\lambda \subset \mathcal{M}(d,0)$ so that $g_{M_\lambda} \geq g_{\mathcal{M}(d,0)}$. It follows from Propositions 7.1 and 7.2 that the Green functions of M_λ and $\mathcal{M}(d,0)$ are respectively $g_{P_t}(\lambda^{-1}t)$ and $dg_{P_t}(0)$ so that

$$g_{P_t}(\lambda^{-1}t) \geq dg_{P_t}(0) \ .$$

We may thus evaluate the Böttcher coordinate at $\lambda^{-1}t$ for any $t \notin M_\lambda$, and $t \mapsto \varphi_{d,t}(\lambda^{-1}t)$ defines a conformal map $\mathbb{C} \setminus M_\lambda \to \mathbb{C} \setminus \bar{\mathbb{D}}(0,1)$ which is tangent to $\lambda^{-1}t$ at infinity, and tends to 1 in modulus when $t \to \partial M_\lambda$. It thus defines a conformal isomorphism, hence M_λ is connected.

Point (2) follows from Theorem 7.8.

For (3), we start by observing that if $|z| \geq \max\{2, |t|\}$, then $|P_t(z)| = |z^d + t| \geq |z|^d - |t| > 2|z| - |t| \geq |z|$ so that by induction the sequence $|P_t^n(z)|$ is strictly increasing to infinity. If $|\lambda| \geq 8$ and $|t| \geq 1/4$, then $|\lambda t| \geq \max\{2, |t|\}$, so that $M_\lambda \subset \mathbb{D}(0, 1/4)$. But $\mathbb{D}(0,1/4)$ is included in the central hyperbolic component $\heartsuit$ of $\mathcal{M}(d,0)$ consisting of those parameters t for which 0 converges to an attracting fixed point of P_t. In particular, $M_\lambda \subset \mathcal{M}(d,0)$ in this case, whence $\lambda \in \mathbb{M}$ by the first point. It remains to prove that, when $0 < |\lambda| \leq 1/8$, the set ∂M_λ is a quasi-circle. Indeed, the component $\heartsuit$ is a J-stable family, whence there exists a holomorphic motion $h : \heartsuit \times \mathbb{S}^1 \to \mathbb{C}$ of the Julia set. According to Proposition 4.14,

$$\partial M_\lambda = \{t \in \heartsuit \ : \ \lambda^{-1}t \in h_t(\mathbb{S}^1)\} \Subset \heartsuit,$$

i.e. t satisfies $t = \lambda h_t(e^{i\theta})$ for some $\theta \in \mathbb{R}$. Recall that h_t is quasi-conformal with quasi-conformality constant $K_t := (1 + |\rho(t)|)/(1 - |\rho(t)|)$, where $\rho(t) \in \mathbb{D}$ is the multiplier of the attracting fixed point of P_t. In particular, $\partial M_\lambda = \tilde{h}(\mathbb{S}^1)$, where $\tilde{h}$ is the quasi-conformal extension of $t \mapsto h_t^{-1}(\lambda^{-1}t)$.

Let us now argue why $\mathbb{M}$ is closed in $\mathbb{C}^*$. Suppose that M_{λ_0} is not connected. Write $\mu_\lambda = \Delta g_{P_t}(\lambda^{-1}t)$. By what precedes, we can find a small disk U whose closure does not intersect $\mathcal{M}(d,0)$ and such that $\mu_{\lambda_0}(U) > 0$. Since the Green function $g_{P_t}(z)$ is continuous in both variables, it follows that $(\lambda, t) \mapsto g_{P_t}(\lambda^{-1}t)$ is also continuous hence the measures $\lambda \mapsto \mu_\lambda$ also varies continuously. For any

λ close enough to λ_0, we get $\mu_\lambda(U) > 0$, hence M_λ is not included in $\mathcal{M}(d,0)$, which implies the complement of $\mathbb{M}$ to be open.

Define

$$G_{\mathbb{M}}(\lambda) := \int_{\mathbb{C}} g_{\mathcal{M}(d,0)}\, d\mu_\lambda \ .$$

Since $g_{P_t}(\lambda^{-1}t) = \log|t| - \log|\lambda| + o(1)$, an integration by parts yields

$$G_{\mathbb{M}}(\lambda) = \int_{\mathbb{C}} \left(g_{P_t}(\lambda^{-1}t) + \log|\lambda|\right)\, \Delta(g_{\mathcal{M}(d,0)}) = \log|\lambda| + \int_{\mathbb{C}} g_{P_t}(\lambda^{-1}t)\, \Delta(g_{\mathcal{M}(d,0)}) \ .$$

Since the support of $\Delta(g_{\mathcal{M}(d,0)})$ is compact, and $g_{P_t}(\lambda^{-1}t)$ is continuous in both variables and subharmonic, $G_{\mathbb{M}}$ is continuous and subharmonic on $\mathbb{C}$. When $\lambda \neq 0$, observe that $G_{\mathbb{M}}(\lambda) = 0$ iff $g_{\mathcal{M}(d,0)} \equiv 0$ on M_λ and this is equivalent to $M_\lambda \subset \mathcal{M}(d,0)$. In other words, $\{G_{\mathbb{M}}(\lambda) = 0\} = \mathbb{M}$. It follows from the maximum principle applied to $G_{\mathbb{M}}$ that $\mathbb{M}$ has no isolated point. □

Remark 7.11. — Observe that $g_{P_t}(\lambda^{-1}t) \to d \times g_{\mathcal{M}(d,0)}$ uniformly on compact subsets of $\mathbb{C}^*$ as $\lambda \to \infty$, hence

$$\lim_{\lambda\to\infty} G_{\mathbb{M}}(\lambda) - \log|\lambda| = \int_{\mathbb{C}} g_{\mathcal{M}(d,0)}\, \Delta(g_{\mathcal{M}(d,0)}) = 0 \ .$$

Since the capacity of $\mathbb{M}$ is < 1, the function $G_{\mathbb{M}}$ is however not the Green function of $\mathbb{M}$, and is not harmonic on $\mathbb{C} \setminus \mathbb{M}$. Note that the boundary of $\mathbb{M}$ is included in $\mathrm{supp}(\Delta G_{\mathbb{M}})$.

Let us ask a couple of natural questions related to the geometry of $\mathbb{M}$.

Question. — Is the set $\mathbb{M}$ a union of a closed topological disk and finitely many isolated points on the unit circle? What is its logarithmic capacity?

Let us include three pictures (all courtesy of Arnaud Chéritat) illustrating this chapter.

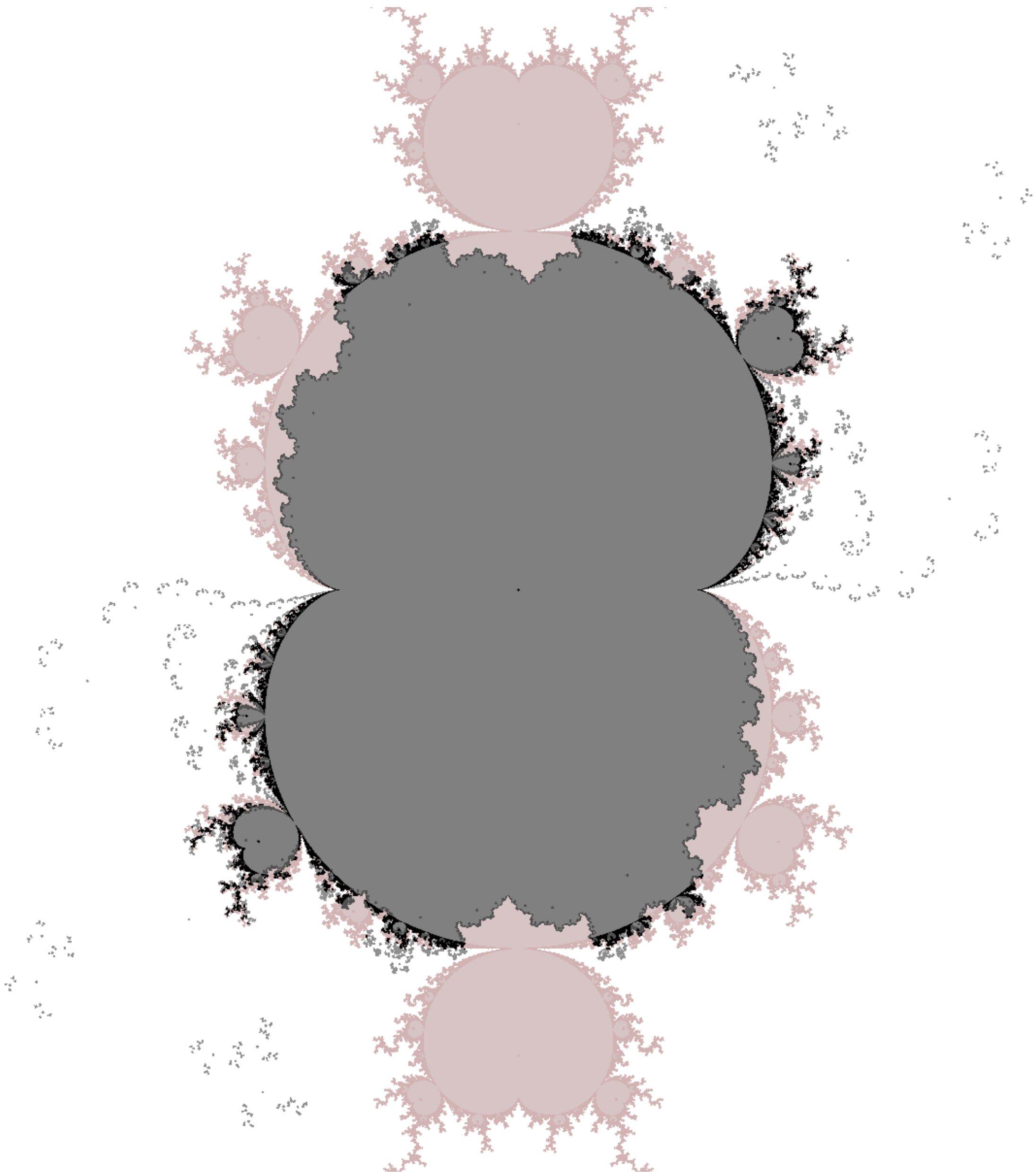


FIGURE 1. In light gray the Multibrot set $\mathcal{M}(3,0)$, and in black the set $\mathcal{M}(3,\lambda^{-1}t)$ with $\lambda = i$. Note that the intersection of $\mathcal{M}(3,\lambda^{-1}t)$ with the central hyperbolic component is a quasicircle. The point i does not belong to $\mathbb{M}$: we see some "dust" popping out of one of the cusp of the Multibrot set. These points correspond to parameters for which $\lambda^{-1}t$ belongs to a totally disconnected Julia set see Lemma 7.9. Finally observe the similarity of between $\mathcal{M}(3,\lambda^{-1}t)$ and the Multibrot set near its intersection with $\partial\mathcal{M}(3,0)$.

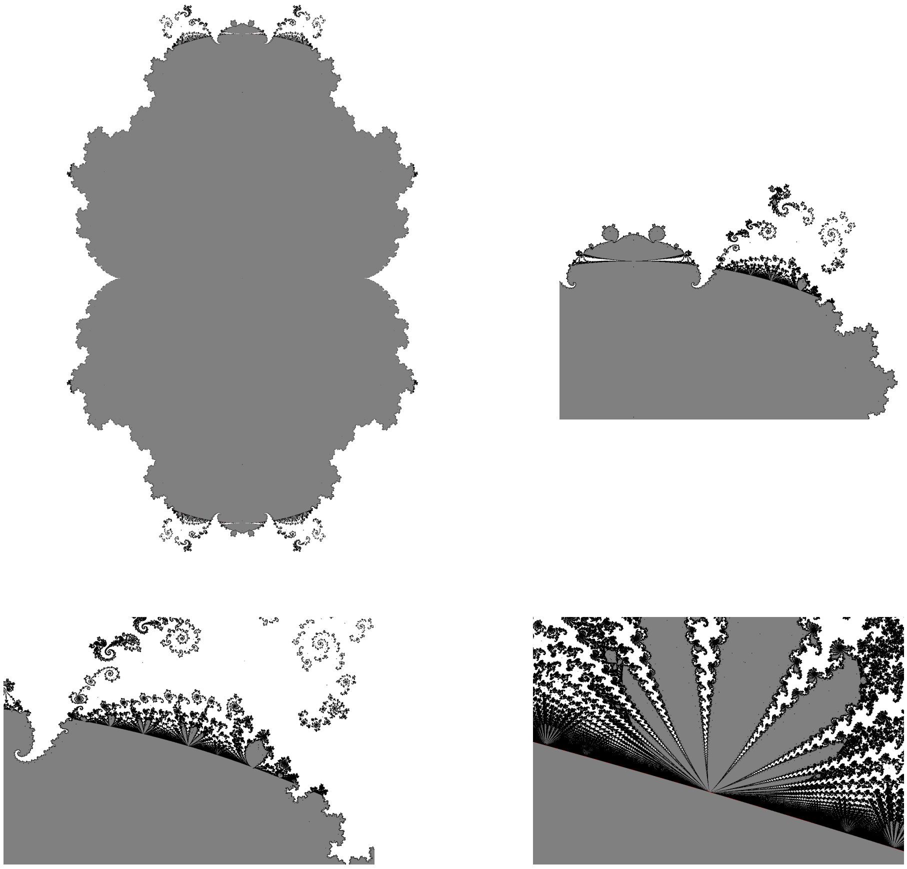


FIGURE 3. The figure on the upper left depicts $\mathcal{M}(3, \lambda^{-1}t)$ with $\lambda \approx 0,7$. Note that this parameter is very close to the boundary of $\mathbb{M}$ which explains why $\mathcal{M}(3, \lambda^{-1}t)$ is close to be connected in that case. The three other pictures shows successive zooms of a region centered at a window in the upper part of $\mathcal{M}(3, \lambda^{-1}t)$. The lotus (or finger) shape is reminiscent of pictures of transcendental Julia sets. N. Fagella obtained similar pictures in 1999, see the cover of "International Journal of Bifurcation and Chaos" Vol. 5, n.3. Recent works by M. Shishikura and D. Marti-Pete propose a theoretical interpretation of the appearance of these fingers using parabolic bifurcation schemes.

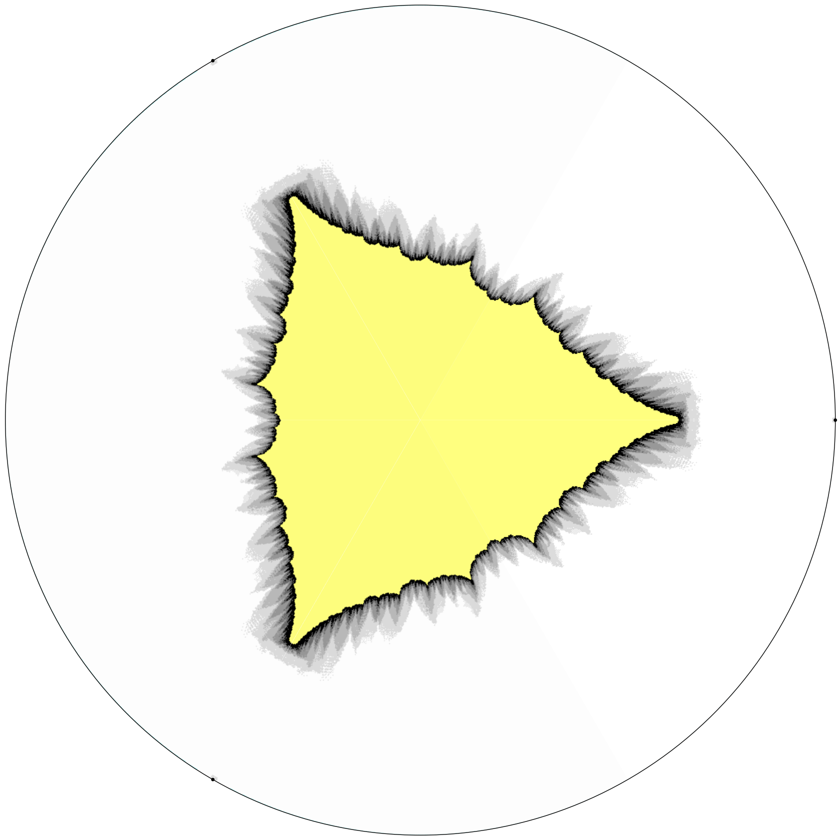


FIGURE 4. An approximation of $\mathbb{M}$. The drawned circle is the unit circle, and the three marked points on this circle are the third roots of unity which are the only isolated points of $\mathbb{M}$. The picture is obtained by computing for each parameter λ the set $\mathcal{M}(3, \lambda^{-1}t)$, and by checking "by hand" its connectedness. The difference in the shades of grey correspond to various approximations of $\mathbb{M}$. The computation is not very efficient and not very precise The inner part of the picture (in light grey) shows all parameters λ such that $\mathcal{M}(3, \lambda^{-1}t)$ is included in the central hyperbolic component. This is a subset of $\mathbb{M}$ and we believe the two sets are actually equal.

CHAPTER 8

SPECIAL CURVES IN THE PARAMETER SPACE OF POLYNOMIALS

We collect all informations from the previous chapters, and prove in §8.1 Baker-DeMarco's conjecture characterizing curves in the moduli space of complex polynomials containing an infinite set of PCF parameters (Theorem G from the Introduction).

In the subsequent sections, we investigate a combinatorial classification of special curves inspired by the classification of PCF polynomials in terms of Hubbard trees. More precisely, we seek a one-to-one correspondence between special curves and decorated graphs, but due to the presence of symmetries, this task turns out to be delicate to achieve. We thus limited ourselves to present a partial correspondence encoding a large class of special curves by a combinatorial gadget that we call critically marked dynamical graphs.

We define and study this notion in §8.2. To any polynomial P is attached a natural marked dynamical graph $\Gamma(P)$ that encodes its critical relations and its symmetries. In §8.3, we show how to associate a marked dynamical graph to an irreducible curve, and define the category of special graphs.

In Section §8.4, we give quite general sufficient conditions on a special graph Γ in order to ensure the existence of a special curve C whose dynamical graph is isomorphic to Γ (Theorem 8.15). Its proof is strongly inspired by previous works by DeMarco and McMullen. Although the overall strategy that we follow is similar to the proof of [**57**, Theorem 1.2], our approach is somewhat more involved since we need to control when orbits of critical points merge whereas DeMarco and McMullen only quantified when the Green functions at critical points agree.

In §8.5, we exhibit a correspondence between special curves and the class of marked dynamical graphs that we have defined (Theorem 8.30).

The applicability of this theorem relies on our ability to realize PCF polynomials whose critical points have prescribed period and preperiod. In §8.6, we state and prove a series of results on the realizability of some combinatorics by PCF polynomials. Our results are complete for unicritical polynomials, and fairly optimal in any degree for strictly PCF polynomials (Theorem 8.35).

We conclude this chapter by discussing the classification of special curves in low degrees (§8.7), and by a series of questions and open problems on the geometry of special curves (§8.8).

8.1. Special curves in the moduli space of polynomials

In this section we give a proof of Theorem G. Let us first recall its statement.

Theorem G. — *Pick any non-isotrivial complex family P of polynomials of degree $d \geq 2$ with marked critical points, parameterized by an algebraic curve C, and containing infinitely many PCF parameters.*

If the family is primitive, then possibly after a base change, there exists a subset A of the set of critical points of P such that for any pair $c_i, c_j \in \mathsf{A}$, there exists a symmetry $\sigma \in \Sigma(P)$ and integers $n, m \geq 0$ such that

$$P^n(c_i) = \sigma \cdot P^m(c_j) \ ; \tag{2}$$

and for any $c_i \notin \mathsf{A}$ there exist integers $n_i > m_i \geq 0$ such that

$$P^{n_i}(c_i) = P^{m_i}(c_i) \ . \tag{3}$$

Proof. — Since all critical points are marked and the family is not isotrivial, we can make a base change, and suppose that C is an algebraic curve in $\mathbb{A}^{d-1}$ so that P can be written under the form

$$P_{c,a}(z) := \frac{1}{d}z^d + \sum_{j=2}^{d-1}(-1)^{d-j}\sigma_{d-j}(c)\frac{z^j}{j} + a^d,$$

where $\sigma_j(c)$ is the monic symmetric polynomial in $(c_1, \ldots, c_{d-2})$ of degree j, so that $c_0 = 0, c_1, \cdots, c_{d-2}$ are the critical points of P (and a^d is a critical value).

Recall that PCF polynomials of the form $P_{c,a}$ have algebraic coefficients by Corollary 2.25. Since C contains a Zariski-dense subset of $\bar{\mathbb{Q}}$-points, it can be defined by equations with coefficients lying in a number field (see [**77**, §4.1 page 384] for a proof).

By Theorem 4.30, for any passive critical points c_i there exist integers $n_i > m_i$ such that $P^{n_i}(c_i) = P^{m_i}(c_i)$. Let A be the set of active critical points. If it is empty then $P_{c,a}$ is PCF for all $(c, a) \in C$ by Theorem 2 and the statement is clear. Otherwise, since family is defined over a number field, we may apply Theorem D to each pair of active critical points (P, c_i) and (P, c_j) and the result follows. □

One can complement the previous result by the following theorem. Recall the definition of the critical Green function G_v on the parameter space, and the critical heights h_{bif}, $\widetilde{h_{\mathrm{bif}}}$ on p.66.

Theorem 8.1. — *Pick any non-isotrivial family P of polynomials of degree $d \geq 2$ with marked critical points which is parameterized by an algebraic curve C containing infinitely many PCF parameters and defined over a number field $\mathbb{K}$. Let c be any active critical point.*

Then $\mathrm{Preper}(P, c) = \mathrm{PCF}(P)$*, and there exists $\alpha \in \mathbb{Q}_+^*$ such that for any place $v \in M_{\mathbb{K}}$ we have*

$$\alpha \cdot g_{P,v}(c) = G_v(P);\ \textit{and}\ \alpha \cdot \mu_{P,c,v} = \mu_{\mathrm{bif},v}.$$

In particular, the height functions $h_{P,c}$, h_{bif} and $\widetilde{h_{\mathrm{bif}}}$ are proportional on $C(\bar{\mathbb{K}})$.

Proof. — All passive critical points are persistently preperiodic so that $\mathrm{PCF}(P)$ is the intersection of the loci $\mathrm{Preper}(P, c)$ for all active critical points c. But the previous theorem implies that these loci are all equal, hence $\mathrm{Preper}(P, c) = \mathrm{PCF}(P)$.

Let A be the set of active critical points as above. Observe that for any place $v \in M_{\mathbb{K}}$, we have

$$G_v(P) = \max\{g_{P,v}(c_i),\ P'(c_i) = 0\} = \max\{g_{P,v}(c_i),\ c_i \in \mathsf{A}\}\ .$$

Apply Theorem 5.22 to all dynamical pairs (P, c) with $c \in \mathsf{A}$. Item (6) implies that $g_{P,c,v}$ are all proportional which leads to $\alpha \cdot g_{P,v}(c) = G_v(P)$, for some $\alpha \in \mathbb{Q}_+^*$ independent of v.

Taking the Laplacian on both sides yields $\alpha' \cdot \mu_{P,c_i,v} = \mu_{\mathrm{bif},v}$ for some constant $\alpha' \in \mathbb{Q}_+^*$. Similarly, we have

$$\alpha \cdot h_P(c) = \widetilde{h_{\mathrm{bif}}}$$

for any $c \in \mathsf{A}$, hence $h_{P,c}$ and h_{bif} are proportional, as claimed. □

8.2. Marked dynamical graphs

We define and study the notion of marked dynamical graphs, which encode the dynamical relations between critical orbits of a polynomial.

8.2.1. Definition

A vector field ξ on a graph Γ is an orientation of each edge of Γ such that for each vertex v there exists a unique outgoing edge at v. A flow on a graph is a (continuous) graph map $\pi\colon \Gamma \to \Gamma$ such that there exists a vector field ξ for which the unique outgoing edge at a vertex v is the edge $[v, \pi(v)]$.

Suppose Γ is a connected graph endowed with a vector field. If it is not a tree, then it is the union of a single loop with finitely many trees attached to it, and the induced flow π maps a point in any of the trees into the loop after finitely many iterations, whereas the restriction of π to the loop is a periodic rotation. In case Γ is a tree, there exists a height function $H\colon V(\Gamma) \to \mathbb{Z}$ such that $H(\pi(z)) = \pi(H(z)) + 1$ which is uniquely defined once the value at one point is fixed.

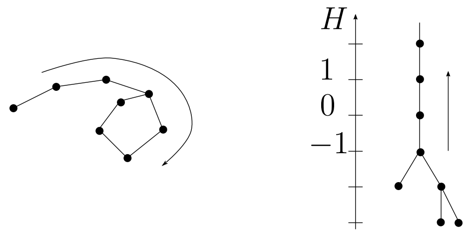


Figure 1. Flows on graphs

A finite or infinite graph Γ with vertex set $V(\Gamma)$ is a *dynamical graph of degree $\leq d$ marked by a finite set $\mathcal{F}$* when it is endowed with the following data:

(G1) a map $\mu\colon \mathcal{F} \to V(\Gamma)$;
(G2) a flow $\pi\colon \Gamma \to \Gamma$;
(G3) a function $d_\pi\colon V(\Gamma) \to \mathbb{N}^*$ such that $\sum_{\pi(w)=v} d_\pi(w) \leq d$ for all $v \in V(\Gamma)$;
(G4) an action of $\mathbb{U}_k$ on $V(\Gamma)$ such that:
- the action is free on the complement of at most one point;
- $\pi(g \cdot v) = \rho(g) \cdot \pi(v)$ for some morphism $\rho\colon \mathbb{U}_k \to \mathbb{U}_k$;
- $d_\pi(g \cdot v) = d_\pi(v)$;

(G5) an action of $\mathbb{U}_k$ on $\mathcal{F}$ such that $\mu(g \cdot i) = g \cdot \mu(i)$.

We also impose the following minimality condition:

(G6) Γ is the orbit by the flow of the set $\{g \cdot \mu(i),\, g \in \mathbb{U}_k,\, i \in \mathcal{F}\}$.

The group $\mathbb{U}_k$ is called the symmetry group of the marked dynamical graph. One can always replace $\mathcal{F}$ by its image in $V(\Gamma)$ and suppose that μ is injective. For our purposes it is however convenient not to do so.

Remark 8.2. — When $k = 1$, we say that Γ has no symmetry. In this case, Γ is completely determined by a finite graph, e.g. by the union of its finite components and the convex hull of the points $\mu(i)$ lying in infinite tree components.

Remark 8.3. — When $k \geq 2$, observe that the action of $\mathbb{U}_k$ does not extend in general to a continuous action on the edges of Γ (except if $\rho = \mathrm{id}$). Such a graph is depicted in the down left corner of Figure 2.

Marked dynamical graphs encode the relations between iterates of a finite set of points.

Example 8.4. — Let $\Phi\colon U \to U$ be any self-map on a set U, and let $\mathcal{F} \subset U$ be any finite collection of points. We define a marked dynamical graph $G(\mathcal{F}, \Phi)$ as follows. Vertices of $G(\mathcal{F}, \Phi)$ are points in U lying in the forward orbit of at least one point in $\mathcal{F}$; an edge joins two vertices v and v' iff $v = \Phi(v')$ or $v' = \Phi(v)$; the flow is defined by $\pi_{\mathcal{F}}(z) = \Phi(z)$; the marking $\mu_{\mathcal{F}}\colon \mathcal{F} \to G(\mathcal{F}, \Phi)$ is given by the natural inclusion; and $d_\pi(v) = \#\Phi^{-1}(v)$.

8.2.2. Critically marked dynamical graphs

We say that Γ is a *critically marked dynamical graph* of degree d when $\mathcal{F} = \{0, \cdots, d-2\}$, $d_\pi(v) = 1 + \operatorname{Card}\mu^{-1}(v)$, and the action of $\mathbb{U}_k$ on $V(\Gamma)$ satisfies the extra condition (G4'):

1. either it is free on $\mu(\mathcal{F})$ and $\pi(g \cdot v) = g \cdot \pi(v)$ for all $v \in V(\Gamma)$;
2. or there exists a vertex $v_* \in \mu(\mathcal{F})$ which is fixed by both $\mathbb{U}_k$ and π, and $\pi(g \cdot v) = g^{d_\pi(v_*)} \cdot \pi(v)$ for all $v \in V(\Gamma)$;
3. or there exists a vertex $v_* \in \mu(\mathcal{F})$ which is fixed by $\mathbb{U}_k$, but not by π, $k = d_\pi(v_*)$, and $\pi(g \cdot v) = \pi(v)$ for all $v \in V(\Gamma)$.

Lemma 8.5. — *For any critically marked dynamical graph of degree d having a symmetry group of order k, we have* $d - 1 = \sum_v (d_\pi(v) - 1)$ *and* $k \leq d$.

Proof. — The first equality follows from $\operatorname{Card}(\mathcal{F}) = d - 1$. For the second inequality, suppose first that $\mathbb{U}_k$ acts freely on $\mu(\mathcal{F})$. Then by (G5), we have $d_\pi(g \cdot v) = d_\pi(v)$ hence k divides $d - 1$. When $\mathbb{U}_k$ fixes a point $v_* \in \mu(\mathcal{F})$, then $d = d_\pi(v_*) + \sum_{v \neq v_*} (d_\pi(v) - 1)$ hence k divides $d - d_\pi(v_*)$, or $k = 1$. □

Remark 8.6. — Given any critically marked dynamical graph Γ and any subgroup G of its symmetry group $\mathbb{U}_k$ such that $G \cdot \mu(\mathcal{F}) = \mathbb{U}_k \cdot \mu(\mathcal{F})$, we can build a new marked dynamical graph by replacing $\mathbb{U}_k$ by G.

This remark leads to the following notion.

Definition 8.7. — *A critically marked dynamical graph* Γ *of degree d is said to be asymmetric if it cannot be embedded into a critically marked dynamical graph* Γ' *of degree d having a non-trivial group of symmetry.*

There are simple criteria detecting whether a graph is asymmetric or not. We refer to §8.4.1 for results in that direction.

Let us discuss graphs with $\mathbb{U}_2$-symmetries. Let Γ be any critically marked dynamical graph having this group of symmetries. Two possibilities arise.

Either $\rho(-1) = -1$ so that the action commutes with π, and $\mathbb{U}_2$ acts continuously on the graph Γ. In that case the action of $\mathbb{U}_2$ on $\mathcal{F}$ is free by (G4'). It may or may not have a fixed point in $V(\Gamma) \setminus \mathcal{F}$.

Or $\rho(-1) = +1$, and we have $\pi(-v) = \pi(v)$. In this case, (G4') implies there exists a vertex $v_* \in \mu(\mathcal{F})$ which is fixed by $\mathbb{U}_2$ but not by π. Observe that $\pi(-\pi^n(v)) = \pi^{n+1}(v)$ for all vertices.

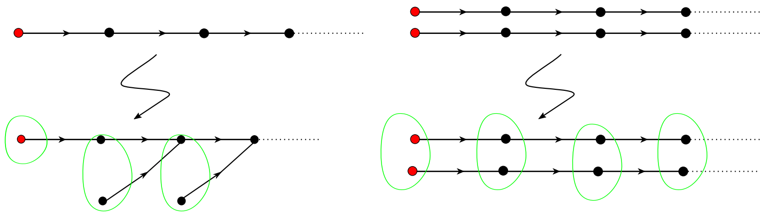


FIGURE 2. Two examples of non-asymmetric graphs

8.2.3. The critical graph of a polynomial

To any polynomial P of degree $d \geq 2$ with marked critical points $c_0, \dots c_{d-2}$, we can attach a (possibly disconnected) critically marked dynamical graph $\Gamma(P)$ which encodes the critical relations as follows. Vertices of Γ are given by $\sigma \cdot P^n(c_i)$, with $i \in \{0, \cdots, d-2\}$, $\sigma \in \Sigma(P)$ and $n \geq 0$, and we draw an oriented edge between any point z and its image $P(z)$ so that Γ carries a canonical flow $\pi \colon \Gamma \to \Gamma$ sending z to $P(z)$. The marking is given by the map $\mu \colon \{0, \cdots, d-2\} \to \Gamma$ which sends i to c_i. It satisfies $\pi \circ \mu = \mu \circ P$. Observe that for any vertex v of Γ, the degree $d(v)$ equals the local degree of P at v, so that (G3) holds since the number of preimages of a point by P counted with multiplicity equals d.

Finally recall that $\Sigma(P)$ is canonically isomorphic to $\mathbb{U}_k$ for some $1 \leq k \leq d$. We let the action of $\mathbb{U}_k$ on the set of vertices of Γ be the one induced by $\Sigma(P)$ on the orbits of the critical points.

Since $\mathbb{U}_k$ acts by rotation on the complex plane, its action is free off the origin which is fixed so that the action of $\mathbb{U}_k$ on Γ satisfies (G4). If P is a monic and centered polynomial and if we write its minimal decomposition $P(z) = z^m Q(z^k)$ with $Q(0) \neq 0$, then either $m = 1$ so that the fixed point of the action by $\mathbb{U}_k$ is not a critical point; or $m \geq 2$ and 0 is a critical point or order m; or $m = 0$ and 0 is a critical point which is not fixed by P. This implies (G4') and $\Gamma(P)$ is a critically marked dynamical graph in the sense above.

For any polynomial outside a countable union of algebraic subvarieties in the moduli space, the marked dynamical graph is a union of $d-1$ rays whose extremities are $\mu(i), i = 0, \cdots, d-2$. The marked dynamical graph of a PCF polynomial is a finite union of finite graphs (having a single loop with finitely many trees attached to it).

The marked dynamical graph of a polynomial encodes the dynamical relations between critical points in the following sense.

Lemma 8.8. — *Let P and Q be two polynomials with marked critical points of the same degree $d \geq 2$. Then the two critically marked dynamical graphs $\Gamma(P)$ and $\Gamma(Q)$ are equal iff the following two properties hold.*

1. *There exists $k \leq d$ such that $\mathbb{U}_k = \Sigma(P) = \Sigma(Q)$ and the two morphisms $\rho_P \colon \Sigma(P) \to \Sigma(P)$ and $\rho_Q \colon \Sigma(Q) \to \Sigma(Q)$ are identical;*
2. *for each pair of integers $i, j \in \{0, \cdots, d-2\}$ and for each $\sigma \in \mathbb{U}_k$, the two sets $\{(n,m),\, P^n(c_i) = \sigma \cdot P^m(c_j)\}$ and $\{(n,m),\, Q^n(c_i) = \sigma \cdot Q^m(c_j)\}$ are equal.*

Proof. — Suppose that P and Q have identical critically marked dynamical graphs. Then we have $\Sigma(P) = \Sigma(Q)$. Pick any vertex v of the graph $\Gamma(P)$ whose image by π is not fixed by $\mathbb{U}_k$. For any $g \in \mathbb{U}_k$, we have $\pi(g \cdot v) = \rho_P(g) \cdot \pi(v) = \rho_Q(g) \cdot \pi(v)$. Since $\mathbb{U}_k$ acts freely on the orbit of $\pi(v)$, we conclude that $\rho_P = \rho_Q$ hence (1) is satisfied. Choose $i, j \in \{0, \cdots, d-2\}$ and $\sigma \in \mathbb{U}_k$. Since

$$\{(n,m),\, P^n(c_i) = \sigma \cdot P^m(c_j)\} = \{(n,m),\, \pi^n(\mu(i)) = \sigma \cdot \pi^m(\mu(j))\} \tag{49}$$

condition (2) holds.

Conversely suppose that P and Q satisfy (1) and (2). We first show that one can recover the marked dynamical graph of P from $\rho = \rho_P$ and the sets $\{(n,m),\, P^n(c_i) = \sigma \cdot P^m(c_j)\}$ where i, j range over all pairs in $\{0, \cdots, d-2\}$ and σ over all elements of $\mathbb{U}_k = \Sigma(P)$. We first build the infinite graph $\hat{\Gamma}$ whose vertices are triple (i, n, σ) with $i \in \{0, \cdots, d-2\}$, $n \geq 0$ and $\sigma \in \mathbb{U}_k$; and edges join (i, n, σ) to $(i, n+1, \rho(\sigma))$. On this graph the map $\hat{\pi}(i, n, \sigma) = (i, n+1, \rho(\sigma))$ is a flow, and we have a marking $\hat{\mu}(i) = (i, 0, 1)$. We also have a natural action by $\mathbb{U}_k$ given by composition on the last factor $\sigma' \cdot (i, n, \sigma) = (i, n, \sigma'\sigma)$.

We observe now that the map sending (i, n, σ) to $\sigma \cdot P^n(c_i)$ identifies $\Gamma(P)$ as the quotient of $\hat{\Gamma}$ by the relation identifying (i_1, n_1, σ_1) and (i_2, n_2, σ_2) iff

$(n_1, n_2) \in \{(n,m),\ P^n(c_i) = \sigma_1^{-1}\sigma_2 \cdot P^m(c_j)\}$. Moreover the flow on $\Gamma(P)$ is induced by $\hat{\pi}$ and similarly the marking is induced by $\hat{\mu}$.

This shows that the two graphs $\Gamma(P)$ and $\Gamma(Q)$ are isomorphic, that their markings, their flows, and their corresponding actions of $\mathbb{U}_k$ coincide. □

8.2.4. The critical graph of an irreducible subvariety in the moduli space of polynomials

We observe that one can attach to any irreducible subvariety of the moduli space of critically marked polynomials a marked dynamical graph.

Proposition 8.9. — *Let V be any irreducible subvariety in the moduli space of critically marked polynomials. Then there exists a marked dynamical graph $\Gamma(V)$ such that $\Gamma(P) = \Gamma(V)$ for all $P \in V$ outside a countable union of subvarieties.*

Remark 8.10. — It is in general not possible to get equality $\Gamma(P) = \Gamma(V)$ on a Zariski dense open subset. When V is a special curve, then the set of PCF polynomials in V is infinite countable (hence Zariski dense) and for each of these polynomials the marked dynamical graph is finite, although $\Gamma(V)$ is not.

However we expect that $\Gamma(P) = \Gamma(V)$ for a (euclidean) dense subset of $P \in V$, see Question (Q7) at the end of this section. This fact would follow from Conjecture 4.

Proof. — By Proposition 3.13, there exists a open Zariski dense subset $V^\circ \subset V$ such that $\Sigma(P) = \mathbb{U}_k$ for all $P \in V^\circ$. Reducing V° if necessary, we may also assume that the morphism $\rho_P \colon \Sigma(P) \to \Sigma(P)$ is induced by the same morphism $\rho \colon \mathbb{U}_k \to \mathbb{U}_k$ for all $P \in V^\circ$.

For any $i, j \in \{0, \cdots, d-2\}$, and $\sigma \in \mathbb{U}_k$, observe that the set $Z(n,m,i,j,\sigma) = \{t \in V^\circ,\ P_t^n(c_i) = \sigma \cdot P_t^m(c_j)\}$ is Zariski closed. The union $\mathcal{Z}$ of all sets $Z(n,m,i,j,\sigma)$ which have empty interior is thus a countable union of strict subvarieties of V.

All polynomials P_t with $t \in V^\circ \setminus \mathcal{Z}$ have the same group of symmetries, and the same sets $\{(n,m),\ P^n(c_i) = \sigma \cdot P^m(c_j)\}$ so that the marked dynamical graph $\Gamma(P)$ is constant on this set by the previous lemma. □

8.3. Dynamical graphs attached to special curves

We aim at characterizing marked dynamical graphs attached to special curves. Before doing so let us begin with the following observation.

Lemma 8.11. — *Let v be any vertex of a marked dynamical graph Γ and pick any symmetry g of Γ. Then v has a finite π-orbit iff $g \cdot v$ does.*

Proof. — This result follows from the fact that the symmetry group of Γ is a finite group and $\pi^n(g \cdot v) = \rho^n(g) \cdot \pi^n(v)$ for any integer n. □

A geometric consequence of the previous lemma is the following. For any symmetry g, a vertex v belongs to an infinite connected component of Γ iff $g \cdot v$ does.

Definition 8.12. — *A critically marked dynamical graph is said to be* special *if it contains exactly one infinite connected component up to symmetry. In other words, for any two infinite connected components T and T' of the dynamical graph, there exists a symmetry σ such that $\sigma(T) \cap T' \neq \varnothing$.*

Lemma 8.13. — *Let Γ be any special critically marked dynamical graph. Then there exists a partition $\{0, \cdots, d-2\} = \mathsf{A} \sqcup \mathsf{P}$ such that*

1. *the π-orbit of a point $\mu(i)$, $i \in \{0, \cdots, d-2\}$ is finite if and only if $i \in \mathsf{P}$;*
2. *for any $i, j \in \mathsf{A}$, there exist $n, m \geq 0$, and a symmetry σ such that $\pi^n(\mu(j)) = \sigma \cdot \pi^m(\mu(i))$.*

Proof. — Let P (resp. A) be the set of indices i such that $\mu(i)$ belongs to finite (resp. infinite) component. Any point in $\mu(\mathsf{P})$ has finite orbit. Since π is a flow, any point in an infinite component has infinite orbit proving (1).

Let $i, j \in \mathsf{A}$. Denote by T and T' the components of Γ containing $\mu(i)$ and $\mu(j)$ respectively. Suppose $T \neq T'$. Since $\mathbb{U}_k$ is cyclic, there exists $k \geq 0$ such that $\pi(\sigma(v)) = \sigma^k \cdot \pi(v)$ for all v.

As Γ is special, we may find a symmetry σ (possibly the identity), and $v \in T$ such that $v' := \sigma(v) \in T'$. Write $\pi^n(v) = \pi^m(\mu(i))$ for some $n, m \geq 0$, and $\pi^{n'}(v') = \pi^{m'}(\mu(j))$ for some $n', m' \geq 0$. We may suppose $n \geq n'$ and $m \geq m'$. Then

$$\pi^{m'}(\mu(j)) = \pi^{n'}(\sigma(v)) = \sigma^{k^{n'}} \cdot \pi^{n'}(v) = \sigma^{k^{n'}} \cdot \pi^{n'-n+m}(\mu(i))$$

as required. □

Theorem 8.14. — *Let C be any irreducible curve in the moduli space of critically marked polynomials. If the family of polynomials induced by C is primitive, then the following are equivalent:*

- *the curve C is special (i.e. contains infinitely many PCF polynomials);*
- *the critically marked dynamical graph $\Gamma(C)$ is special.*

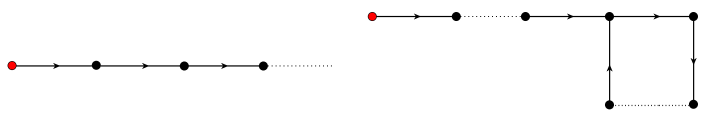
Proof. — Suppose first that the curve C is special. Observe that at least one critical point c is active on C so that for all $t \in C$ outside a countable set, the orbit of c under P_t is infinite. Pick any $t \in C$ such that $\Gamma(P_t) = \Gamma(C)$ and $\{P_t^n(c)\}_{n\geq 0}$ is infinite. By assumption the family $\{P_t\}_{t\in C}$ is primitive, and Theorem G implies that $\Gamma(P)$ has at most one infinite component (up to symmetry). Since c is not preperiodic, it has exactly one infinite component, and $\Gamma(C)$ is special.

Conversely, when the marked dynamical graph $\Gamma(C)$ is special, we let A be the set of indices $i \in \{0, \cdots, d-2\}$ such that $\mu(i)$ falls into the infinite component of $\Gamma(C)$. Observe that any critical point c_i for which $i \notin \mathsf{A}$ is stably preperiodic on C. Pick any $i \in \mathsf{A}$. Since C is a curve inside the moduli space of critically marked polynomials, its image in the moduli space of polynomials remains a curve, hence the family is not isotrivial. Theorem 4.30 implies the critical point c_i to be active.

It follows that the set of parameters $t \in C$ such that P_t is PCF coincides with the set $\{t \in C, c_i \text{ is preperiodic}\}$. The latter set is infinite countable by Montel's theorem (see, e.g., [**68**, Lemma 2.3]) which shows that C is special. □

The next two figures describe all special critically marked dynamical graphs of degree 2 and 3.

With no symmetry.



With $\mathbb{U}_2$-symmetry group

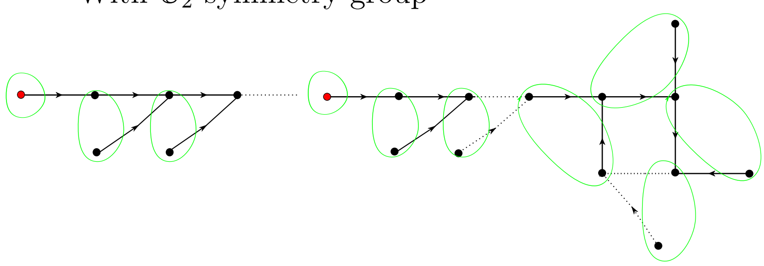


FIGURE 3. Special critically marked dynamical graphs of degree 2: the critical point is marked in red, and $\mathbb{U}_2$-orbits are circled in green.

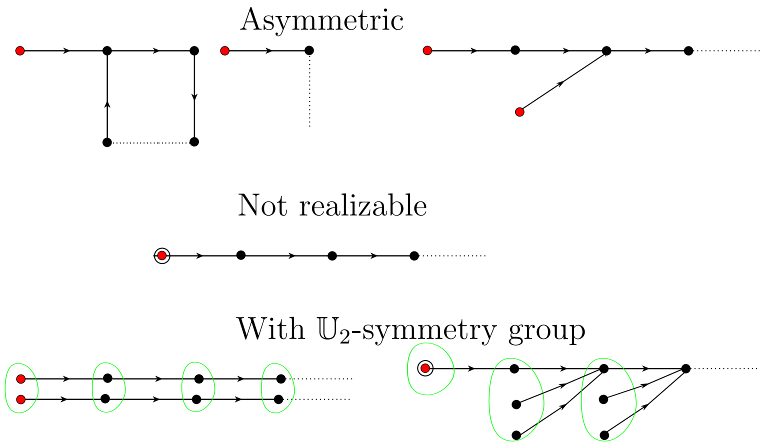


FIGURE 4. Special critically marked dynamical graphs of degree 3. Circled red dots are multiple critical critical points

8.4. Realization theorem

This section is devoted to the proof of the following result which forms the bulk of the proof of our correspondence theorem to be stated in the next section.

Observe that if Γ is critically marked dynamical graph, then the union of all its finite components is a finite critically marked dynamical graph Γ_{fin} (of smaller degree).

We shall say that a finite critically marked dynamical graph Γ_0 without symmetry is realizable by a PCF polynomial if there exists a PCF polynomial P such that Γ_0 is isomorphic to the critically marked dynamical graph obtained by forgetting the action of $\Sigma(P)$ on $\Gamma(P)$. In particular the graph Γ_0 is homeomorphic to $\Gamma(P)$ and the flows are conjugated.

Theorem 8.15. — *Let Γ be any special asymmetric critically marked dynamical graph such that two distinct marked points have different images.*

Then we can find a primitive polynomial P with disconnected Julia set such that $\Gamma(P) = \Gamma$.

A key ingredient in the proof is the realizability of the finite dynamically marked subgraph Γ_{fin} by a PCF polynomial, a theorem due to Floyd-Kim-Koch-Parry-Saenz [**84**]. We shall give a weaker version of this result in §8.6 using combinatorial arguments.

During the whole construction, we fix a constant $\rho > 1$, and write $M_d(z) = z^d$. A brief explanation of our strategy is given at the end of §8.4.2.

8.4.1. Asymmetric special graphs

Before embarking on the proof of the theorem above, we note the ubiquity of asymmetric graphs among special ones.

Proposition 8.16. — *Let Γ be any special dynamical graph without any symmetry. Suppose that Γ is not asymmetric. Then*

1. *either there exists a vertex $v_* \in \mu(\mathcal{F})$ which is fixed by π, an integer $k \geq 2$ which divides a power of $d_\pi(v_*)$, and a free action of $\mathbb{U}_k$ on $\mu(\mathcal{F}) \setminus \{v_*\}$ which preserves d_π;*
2. *or there exists an integer $k \geq 2$, a vertex $v_* \in \mu(\mathcal{F})$ with infinite π-orbit and $d_\pi(v_*) = k$, and a free action of $\mathbb{U}_k$ on $\mu(\mathcal{F}) \setminus \{v_*\}$ which preserves d_π, and such that $\pi(g \cdot v) = \pi(v)$ for all $v \neq v_*$.*

Remark 8.17. — Suppose Γ is a special dynamical graph without any symmetry. Then 2. implies Γ to be non-asymmetric. It is however not true that 1. implies Γ to be non-asymmetric.

Proof. — Since Γ is special and has no symmetry, it has a unique infinite component Γ_{esc} which is a tree, see the discussion in §8.2. Suppose that Γ is not asymmetric, embed $\Gamma \subset \Gamma'$ where Γ' is a critically marked dynamical graph with symmetry group $\mathbb{U}_k$ with $k \geq 2$. By (G4'), we have three possibilities.

If $\mathbb{U}_k$ acts freely on $\mu(\mathcal{F})$ then $\pi(g \cdot v) = g \cdot \pi(v)$, and $g \cdot \Gamma_{\text{esc}} \cap \Gamma_{\text{esc}} = \varnothing$ if $g \neq 1$. But points in $\mu(\mathcal{F})$ having an infinite π-orbit are permuted by $\mathbb{U}_k$ hence all belong to Γ_{esc} which is impossible.

The second option is that $\mathbb{U}_k$ fixes a vertex v_* which is also fixed by π, and that $\pi(g \cdot v) = g^{d_\pi(v_*)} \cdot \pi(v)$. Pick any vertex $v \in \mu(\mathcal{F}) \cap \Gamma_0$, and g a generator of $\mathbb{U}_k$. Since Γ_0 is connected, we have $\pi^n(g \cdot v) = \pi^n(v)$ for some n, hence k divides $d_\pi(v_*)^n$. We thus fall into Case 1.

The last option is when $\mathbb{U}_k$ fixes a vertex v_* which has infinite π-orbit. Then $\pi(g \cdot v) = \pi(v)$ and the fact that d_π is $\mathbb{U}_k$-invariant implies the result. □

8.4.2. Truncated marked dynamical graphs

We thus fix once and for all a special marked dynamical graph Γ satisfying the assumption of Theorem 8.15. As in Lemma 8.13, we define A to be the set of $i \in \{0, \cdots, d-2\}$ such that the π-orbit of $\mu(i)$ is infinite; and let P be its complement.

We let $H\colon V(\Gamma) \to \mathbb{Z} \cup \{-\infty\}$ be the unique (height) function such that $H(\pi(v)) = H(v) + 1$, which we normalize by the condition $\max\{H(\mu(i)), i = 0, \cdots, d-2\} = 0$. By convention we set $H|_{\Gamma_{\text{fin}}} = -\infty$.

Write $\Gamma_{\rm esc} = \Gamma \backslash \Gamma_{\rm fin}$. We first build from $\Gamma_{\rm esc}$ a sequence of marked dynamical graphs as follows. For any $n \in \mathbb{Z}$, let $\Gamma_n = \Gamma_{\rm esc} \cap \{H \geq 1-n\}$. Observe that Γ_n is connected since for any two vertices v, v' of $\Gamma_{\rm esc}$, one may find integers m, m' such that $\pi^m(v) = \pi^{m'}(v')$. It is also naturally a marked dynamical graph of Γ as follows. We have a canonical (injective) marking $\mu_n \colon \partial\Gamma_n \to \Gamma_n$, and the flow π preserves Γ_n, so that the data (Γ_n, μ_n, π) determines a marked dynamical graph as defined in §8.2.

Note that $\partial\Gamma_n$ contains $\{H = 1-n\}$ but might be strictly larger. Note also that $d_\pi(v) = 1$ for all vertices of Γ_n, $n \leq 0$, whereas $d_\pi(v) \geq 2$ for at least one vertex in Γ_1. Finally, we have $\Gamma_n = \Gamma_{\rm esc}$ for all n sufficiently large ($n \geq 1 - \min_{\Gamma_{\rm esc}} H$).

Recall the construction of marked dynamical graphs from Example 8.4, and $M_d(z) = z^d$.

Lemma 8.18. — *There exists a finite set $\mathcal{F} \subset \{|z| = \rho\}$ such that $G(\mathcal{F}, M_d) = \Gamma_0$.*

The domain of definition of M_d can be chosen to be $\{|z| > \rho^{1/d}\}$.

Proof. — Observe first that by (G6) any point in $\{H = 1\}$ lies in the orbit of at least one point $\mu(i)$ so that the cardinality n of $\{H = 1\}$ is at most $d-1$.

We build a map $\theta \colon V(\Gamma_0) \to \mathbb{C}$ whose image will be the vertices of $G(\mathcal{F}, M_d)$. Let v_1 be the branched point of Γ_0 of maximal height equal to $H(v_*) = H_*$. This point is uniquely determined and we let $\theta(v_*)$ be any complex number of modulus ρ^{H_*}.

Since π is a flow, the number of preimages by π of v_* is equal to the number of branches of Γ at v_1 hence is $\leq (d-1)$. We may thus find an injective map $\theta \colon \pi^{-1}(v_*) \to \{|z| = \rho^{H_*-1}\}$ such that $\theta(\pi(v)) = M_d(\theta(v))$. Applying the same argument to each point of height H_*-2, H_*-3, etc. we construct by induction an injective map $\theta \colon V(\Gamma_0) \to \mathbb{C}$ such that $\theta(\pi(v)) = M_d(\theta(v))$. We conclude by setting $\mathcal{F} = \theta(\{H = 1\})$. □

Remark 8.19. — From the previous lemma, we get a canonical injective map $\mu_0 \colon \Gamma_0 \to \{|z| \geq \rho\}$ such that $\mu_0(\pi(v)) = M_d(\mu_0(v))$. Observe that $\mathcal{F}$ is in bijection with $\{H = 1\}$.

We can now explain our strategy for the proof of Theorem 8.15. We shall construct by induction on n, an increasing sequence of Riemann surfaces S_n and finite map $\Phi_n \colon S_n \to S_n$ such that $\Phi_n|_{S_{n-1}} = \Phi_{n-1}$ in such a way that the dynamical graph associated to the critical points of Φ_n equals Γ_n. The construction of the sequence S_n is given in §8.4.4. At step n, we shall need to pick a polynomial whose critical points satisfy some constraints forced by the geometry of the graph $\{n \leq H \leq n-1\}$. We discuss in the next section the construction of such a polynomial.

When $\Gamma_{\mathrm{fin}} = \emptyset$, then the union of S_n is a planar domain and it follows from an argument of McMullen that Φ_n extends through the complement of $\cup S_n$ in $\mathbb{C}$ thus defining the required polynomial. When $\Gamma_{\mathrm{fin}} \neq \emptyset$, the construction is more involved as we have to patch a suitable disk containing the filled-in Julia of a PCF polynomial realizing Γ_{fin} with S_n for some large n. Once this is done, the argument proceeds as in the former case.

Remark 8.20. — Our assumption on having marked points with distinct images is only used to get condition (D2) in Theorem 8.21 below, and it is likely it is in fact superfluous.

8.4.3. Polynomials with a fixed portrait

We now discuss the construction of polynomials with prescribed ramification locus. We shall need a more precise result than [**57**, Proposition 7.3]. Our treatment is slightly different from op. cit. and more topological in nature.

To avoid a statement with too many assumptions, we first describe our setup. Let γ be any simple closed curve in the complex plane. We fix a finite set of points $\mathcal{G} \subset \gamma$ and for each $p \in \mathcal{G}$ a non-empty finite set of positive integers $\mathcal{D}(p) = \{n_{i,p}\}$. We also fix a (possibly empty) subset $\mathcal{D}_0(p) \subset \mathcal{D}(p)$.

Theorem 8.21. — *Suppose $\gamma, \mathcal{G}, \mathcal{D}(p)$ and $\mathcal{D}_0(p)$ are given as above, and pick any two positive integers d' and N such that*

(D1) $(d'-1) = (N-1) + \sum_{\mathcal{G}} \sum_{i \in \mathcal{D}(p)} (n_{i,p} - 1)$;
(D2) *for all* $p \in \mathcal{G}$, $d' \geq \sum_{i \in \mathcal{D}(p)} n_{i,p}$;
(D3) *for all* $p \in \mathcal{G}$, $N \geq \sum_{p \in \mathcal{G}} \operatorname{Card} \mathcal{D}_0(p)$.

Then for any point z lying in the bounded connected component of $\mathbb{C} \setminus \gamma$, there exist a polynomial Q of degree d', and a simple closed curve $\gamma' \subset Q^{-1}(\gamma)$, such that:

(R1) *the bounded component of $\mathbb{C} \setminus \gamma'$ contains a unique point $z' \in Q^{-1}(z)$ and* $\deg_{z'}(Q) = N$;
(R2) *for each* $p \notin Q^{-1}(\mathcal{G}) \cup \{z\}$, $\deg_p(Q) = 1$;
(R3) *for each $p \in \mathcal{G}$, there exists a finite set $\mathcal{Q}(p) \subset Q^{-1}(p)$ such that the function $\delta_p(q) := \deg_q(Q)$ defines a bijective map $\delta_p \colon \mathcal{Q}(p) \to \mathcal{D}(p)$;*
(R4) *for each $p \in \mathcal{G}$, the set $\delta_p(\mathcal{Q}(p) \cap \gamma')$ contains $\mathcal{D}_0(p)$.*

In plain words, it is always possible to find a polynomial with a prescribed branched portrait (determined by $\mathcal{D}$) , with critical values $\mathcal{G}$ in γ and branched locus in a fixed closed disk (the bounded component of $\mathbb{C} \setminus \gamma'$).

We include below two examples. The first one (Figure 5) shows that in general $\delta_p(\mathcal{Q}(p) \cap \gamma')$ might exceed $\mathcal{D}_0(p)$. The second one (Figure 6) proves that the polynomial Q satisfying the conditions above may not be unique (up to a composition by an affine transformation).

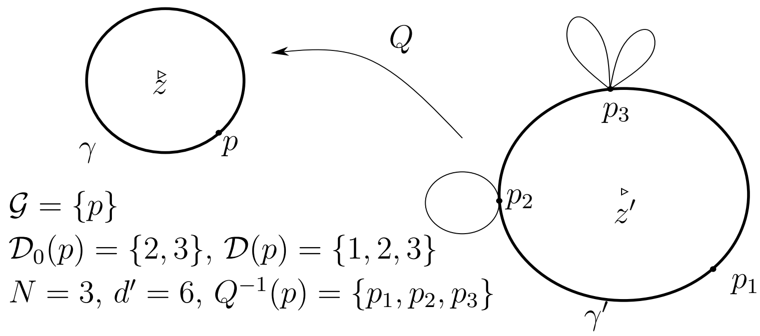


FIGURE 5. Example where $\mathcal{Q}(p) \subset \gamma'$ but $\mathcal{D}_0(p) \subsetneq \mathcal{D}(p)$

Proof. — Recall that any topological branched cover over a complex domain admits a canonical structure of Riemann surface for which the cover is holomorphic. In order to prove the theorem, it is thus only necessary to produce a topological branched cover of the Riemann sphere having the prescribed branch portrait, see e.g. [**63**, §6.1.10].

To simplify the discussion, we shall use the following convenient terminology. A disk is a domain of the complex plane homeomorphic to the unit disk whose boundary is a simple closed curve. Given a disk D and a point $p \in \partial D$, we say that we attach $k \geq 1$ disks to D at p, when we choose k disjoint disks $D_1, \ldots, D_k$ such that $\overline{D_i} \cap \overline{D_j} = \overline{D_i} \cap \overline{D} = \{p\}$ for all $i \neq j$. Observe that the pull-back by $z \mapsto (1 - z)^{k+1}$ of the unit disk, gives a (topological) disk containing 0 in its interior with k disks attached to it at 1. When the boundary of D is a general closed simple curve, one can use Jordan-Schoenflies theorem to reduce the situation to the unit disk and attach disks to D.

Let D be the bounded connected component of $\mathbb{C} \setminus \gamma$ which is a disk by Jordan's theorem. Fix any other disk D_0, and take any branched cover $h\colon \overline{D_0} \to \bar{D}$ of degree N which is totally ramified only at one point z' which is mapped to z. Observe that each point $p \in \mathcal{G}$ has exactly N preimages on $\gamma' := \partial D_0$.

Suppose first $\mathcal{G}$ is a singleton. By (D1) and (D2), we have $d' \leq N + d' - \mathrm{Card}(\mathcal{D}(p))$ so that $N \geq \mathrm{Card}(\mathcal{D}(p))$. We may thus select $\mathcal{Q}(p) \subset h^{-1}(p) \subset \gamma'$,

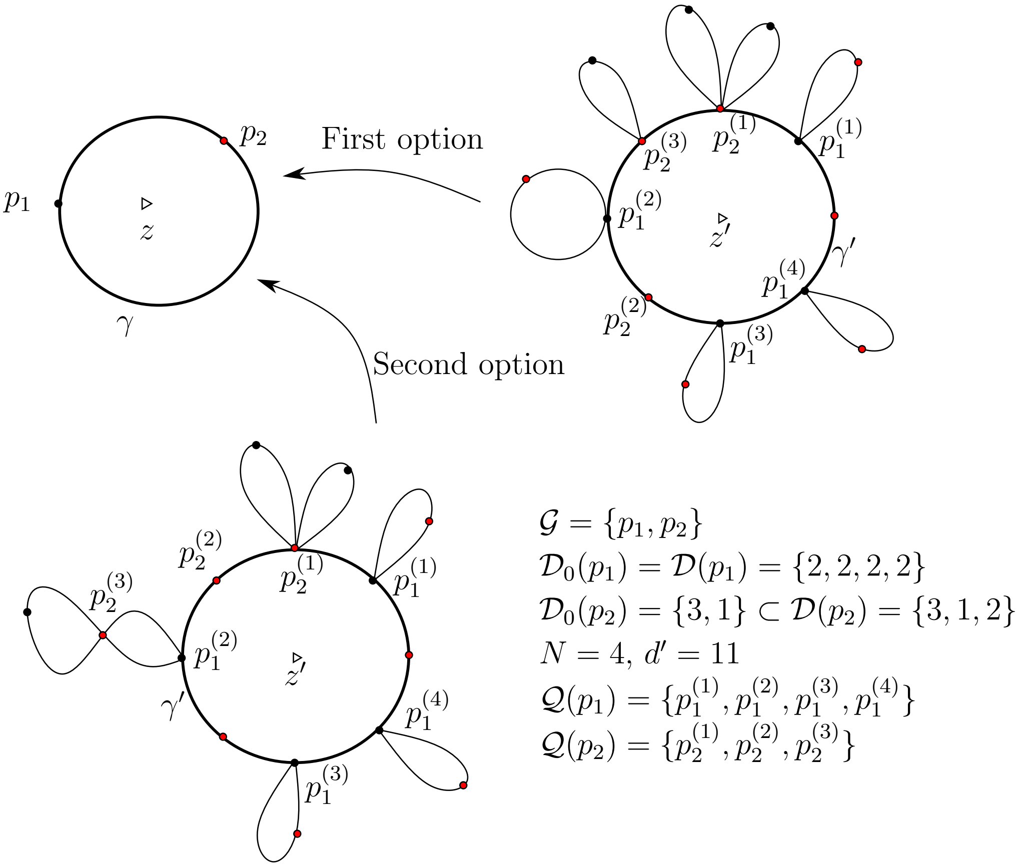


FIGURE 6. Non-uniqueness of the realization of a critical portrait

together with a bijection $\delta_p \colon \mathcal{Q}(p) \to \mathcal{D}(p)$. To each point $q \in \mathcal{Q}(p)$, we attach $\delta_p(q) - 1$ disks, and extend h to be a homeomorphism from the closure of each of the attached disks onto $\bar{D}$, and mapping q to p. At this point, we have a union of open disks U whose boundary Γ is a union of simple closed curves whose two-by-two intersections are either empty or reduced to a point in $\mathcal{Q}(p)$. The number of connected components of U equals $1 + \sum(n_{i,p} - 1) = d' - N + 1$. The map $h \colon \Gamma \to \gamma$ is a branched cover of degree $(d' - N) + N = d'$. We may thus extend h to a branched cover of the Riemann sphere of degree d' which leaves ∞ totally invariant, and is unramified over $\mathbb{C} \setminus \overline{U}$. This concludes the proof in this case.

The case $\mathrm{Card}(\mathcal{G}) \geq 2$ is harder to treat since it may happen that $\mathrm{Card}(\mathcal{D}(p)) \geq N$ for some p. Observe though that $\mathrm{Card}(\mathcal{D}_0(p)) \leq N$ for all p by (D3). For each $p \in \mathcal{G}$, we may thus select $\mathcal{Q}_0(p) \subset h^{-1}(p) \subset \gamma'$ and a bijective map $\delta_p \colon \mathcal{Q}_0(p) \to \mathcal{D}_0(p)$. We attach $\delta_p(q) - 1$ disks $D_i(q)$ to each point $q \in \mathcal{Q}_0(p)$, and extend h as a homeomorphism from each $\overline{D_i(q)}$ onto $\bar{D}$ as above. Denote by $U_0 = \cup_{i,q} D_i(q)$ the union of these disks.

The proof now proceeds as follows. We attach one step at a time $(n_{i,p} - 1)$ disks at a well-chosen point in $h^{-1}(p)$, and for some $n_{i,p}$, and extend h as a homeomorphism from each of these disks onto $\bar{D}$. At each step $k \geq 0$, we build a domain U_k which is a union of open disks, whose boundary is a union of circles whose two-by-two intersection is either a point or empty, and we get a finite branched cover $h \colon \overline{U}_k \to \bar{D}$.

We also have a sequence of sets $\mathcal{Q}_k(p) \subset h^{-1}(p)$, and injective maps $\delta_p^{(k)} \colon \mathcal{Q}_k(p) \to \mathcal{D}(p)$ such that $\delta_p^{(0)} = \delta_p$ is the map defined on $\mathcal{Q}_0(p)$ above. We shall say that $n_{i,p} \in \mathcal{D}(p)$ has been *allocated* at step k, when it belongs to the image of $\delta^{(k)}(p)$. The goal is to reach a situation where $\delta_p^{(k)}$ is surjective for all p (i.e. all elements of $\mathcal{D}(p)$ have been allocated for all p).

Introduce the sets

$$\mathcal{D}_1(p) = \{i \notin \mathcal{D}_0(p), n_{i,p} = 1\} \text{ and } \mathcal{D}_+(p) = \{i \notin \mathcal{D}_0(p), n_{i,p} \geq 2\} \ .$$

We claim that there exists a procedure such that after finitely many steps, all elements of $\mathcal{D}_+(p)$ have been allocated for all p. Grant this claim, and let k be the number of steps needed to allocate all elements of $\mathcal{D}_+(p)$. Note that in this case, the number of points in $h^{-1}(p)$ which are not in $\mathcal{Q}_k(p)$ is equal to

$$\begin{aligned}
\mu &:= N - \mathrm{Card}(\mathcal{D}_0(p)) - \mathrm{Card}(\mathcal{D}_+(p)) + \sum_{q \neq p} \sum_{\mathcal{D}(q)} (n_{i,q} - 1) \\
&\overset{(D1)}{=} d' - \sum_{\mathcal{D}(p)} (n_{i,p} - 1) - \mathrm{Card}(\mathcal{D}_0(p)) - \mathrm{Card}(\mathcal{D}_+(p)) \\
&= \left(d' - \sum_{\mathcal{D}(p)} n_{i,p} \right) + \mathrm{Card}(\mathcal{D}_1(p)) \overset{(D2)}{\geq} \mathrm{Card}(\mathcal{D}_1(p)) \ .
\end{aligned}$$

Since the number of points in $\mathcal{D}(p)$ which remains to be allocated is equal to $\mathrm{Card}(\mathcal{D}_1(p))$, we can extend the function δ_p to $\mathcal{D}(p)$ injectively as required.

To prove our claim, we need to allocate elements in $\mathcal{D}_+(p)$. For each p and at each step $k \geq 0$ of the construction, we let $\Delta_k(p)$ be the number of elements of $\mathcal{D}_+(p)$ which have not been allocated, and let $F_k(p)$ be the number of preimages of $h^{-1}(p)$ that are free in the sense that they do not belong to $\mathcal{Q}_k(p)$.

At step 0, we have $\Delta_0(p) = \mathrm{Card}(\mathcal{D}(p)) - \mathrm{Card}(\mathcal{D}_0(p))$, and

$$\begin{aligned} F_0(p) &= \sum_{p' \neq p} \sum_{i \in \mathcal{D}_0(p')} (n_{i,p'} - 1) + N - \mathrm{Card}(\mathcal{Q}_0(p)) \\ &= \sum_{p' \neq p} \sum_{i \in \mathcal{D}_0(p')} (n_{i,p'} - 1) + N - \mathrm{Card}(\mathcal{D}_0(p)). \end{aligned}$$

Order the points in $\mathcal{G}$ such that

$$\mathrm{Card}(\mathcal{D}_+(p_1)) \geq \cdots \geq \mathrm{Card}(\mathcal{D}_+(p_s)) > 0 = \mathrm{Card}(\mathcal{D}_+(p_j))$$

for all $j \geq s+1$, and let us first suppose that $\mathrm{Card}(\mathcal{D}_0(p)) < N$ for all p. At step 1, we may thus allocate one (randomly chosen) element $n_{i_1,p_1}, \ldots, n_{i_s,p_s}$ of each set $\mathcal{D}_+(p_1), \ldots, \mathcal{D}_+(p_s)$. Observe that

$$\begin{cases} \Delta_1(p_i) & = \Delta_0(p_i) - 1, \text{ and} \\ F_1(p_i) & = F_0(p_i) - 1 + \sum_{q \neq p_i} (n_{i,q} - 1) \geq F_0(p_i). \end{cases}$$

for all $1 \leq i \leq s$ since $\mathrm{Card}(\mathcal{G}) \geq 2$. If $\Delta_1(p_i) = 0$ for all i, then we are done. Otherwise, $\Delta_1(p_1), \ldots, \Delta_1(p_{s_1}) \geq 1$ and $\Delta_1(p_{s_1+1}), \ldots, \Delta_1(p_s) = 0$. At step 2, we allocate one element of each set $\mathcal{D}_+(p_1), \ldots, \mathcal{D}_+(p_{s_1})$. We may continue in this way until $\Delta_k(p_i) = 0$ for all $i \geq 2$ and $\Delta_k(p_1) > 0$. At this step, we get

$$\begin{aligned} F_k(p_1) &= N - \mathrm{Card}(\mathcal{D}_0(p_1)) - \mathrm{Card}(\mathcal{D}_+(p_2)) + \sum_{p \neq p_1} \sum_{i \in \mathcal{D}(p)} (n_{i,p} - 1) \\ &= d' - \mathrm{Card}(\mathcal{D}_0(p_1)) - \mathrm{Card}(\mathcal{D}_+(p_2)) - \sum_{i \in \mathcal{D}(p_1)} (n_{i,p_1} - 1) \\ &= \left(d' - \sum_{i \in \mathcal{D}(p_1)} n_{i,p_1} \right) + \mathrm{Card}(\mathcal{D}(p_1)) - \mathrm{Card}(\mathcal{D}_0(p_1)) - \mathrm{Card}(\mathcal{D}_+(p_2)) \\ &\geq \mathrm{Card}(\mathcal{D}(p_1)) - \mathrm{Card}(\mathcal{D}_0(p_1)) - \mathrm{Card}(\mathcal{D}_+(p_2)) = \Delta_k(p_1)\ , \end{aligned}$$

so that we may allocate all remaining points in $\mathcal{D}_+(p_1)$. This proves the claim when $\mathrm{Card}(\mathcal{D}_0(p)) < N$ for all p.

Suppose finally that $\mathrm{Card}(\mathcal{D}_0(p_*)) = N$. Then (D3) implies that $\mathcal{D}_0(p) = \varnothing$ for all $p \neq p_*$. Write $\varepsilon := \sum_{q \neq p_*} \sum_i (n_{i,q} - 1)$, so that

$$\begin{aligned} \mathrm{Card}(\mathcal{D}_+(p_*)) + \mathrm{Card}(\mathcal{D}_1(p_*)) &= \sum_i n_{i,p_*} - \sum_i (n_{i,p_*} - 1) - \mathrm{Card}(\mathcal{D}_0(p_*)) \\ &\overset{(D1)}{=} \sum_i n_{i,p_*} - d' + \varepsilon \overset{(D2)}{\leq} \varepsilon\ . \end{aligned}$$

One then removes p_* from the set $\{p_1, \dots, p_s\}$, and apply the sequence of steps above. The number of free points in $h^{-1}(p_*)$ is equal to ε, and one can thus allocate all elements of $\mathcal{D}_+(p_*) \cup \mathcal{D}_1(p_*)$. This concludes the proof. □

8.4.4. Construction of a suitable sequence of Riemann surfaces

This part is the key to the proof of Theorem 8.15. It is strongly inspired by the approach of [**57**] with an important extra difficult coming from gluing a connected filled-in Julia set containing all preperiodic critical points.

Recall that Γ is a fixed marked dynamical graph of degree d. We let $\Delta := 1 + \sum(d_\pi(\mu(i)) - 1)$ where the sum is taken over all indices i such that $\mu(i)$ lies in a bounded component of Γ (i.e. in Γ_{fin}).

To alleviate notations we identify a graph and its set of vertices.

Choose $\mathcal{F} \subset \gamma_0 := \{|z| = \rho\}$ as in Lemma 8.18 so that $G(\mathcal{F}, M_d) = \Gamma_0$. By definition $\mathcal{F} = \partial\Gamma_0$ and we have an injective map $\mu_0 \colon \Gamma_0 \to \{|z| = \rho\}$.

We shall build by induction a sequence of objects $S_n, \Phi_n, G_n, \mu_n, \gamma_n$ indexed by $n \in \mathbb{N}$ which satisfy the following conditions:

(C0) $S_0 = \{|z| > \rho^{1/d}\}$, $G_0 = \log^+|z|$, $\Phi_0 = M_d$, and $\mu_0 \colon \Gamma_0 \to \{|z| \geq \rho\}$ is the map above such that $G(\mathcal{F}, \Phi_0) = \Gamma_0$;

(C1) $S_0 \subset S_{n-1} \subset S_n$ is an increasing sequence of connected open Riemann surfaces such that ∂S_{n-1} is a real analytic curve in S_n, and $S_n \setminus \overline{S_{n-1}}$ is a finite union of conformal annuli of finite modulus;

(C2) $G_n \colon S_n \to \mathbb{R}_+$ is a harmonic function such that $G_n|_{S_{n-1}} = G_{n-1}$, and $G_n(z_k) \to \frac{1}{d^{n+1}} \log\rho$ for any diverging sequence $z_k \in S_n \setminus S_{n-1}$ (i.e. eventually leaving any compact subset of $S_n \setminus S_{n-1}$);

(C3) $\Phi_n \colon S_n \to S_n$ is a finite proper holomorphic map such that $\Phi_n(S_n) = S_{n-1}$, $\Phi_n|_{S_{n-1}} = \Phi_{n-1}$, and $G_n \circ \Phi_n = d\, G_n$;

(C4) $\mu_n \colon \Gamma_n \to S_n$ is an injective map such that $\mu_n|_{\Gamma_{n-1}} = \mu_{n-1}$, and $\mu_n(\pi(v)) = \Phi_n(\mu_n(v))$;

(C5) $d_\pi(v) = \deg_{\mu_n(v)}(\Phi_n)$ for all $v \in \Gamma_n$, and $\deg_z(\Phi_n) = 1$ if $z \notin \mu_n(\Gamma_n)$;

(C6) γ_n is a simple closed curve included in $\{G_n = \frac{1}{d^n}\log\rho\}$, which contains $\mu_n(\partial\Gamma_n \setminus \partial\Gamma_{n-1})$ and bounds a connected component $A_{\star,n}$ of $S_n \setminus \overline{S_{n-1}}$;

(C7) the restriction of Φ_n to any connected component $\neq A_{\star,n}$ of $S_n \setminus \overline{S_{n-1}}$ induces a conformal isomorphism onto its image;

(C8) the restriction of Φ_n to $A_{\star,n}$ induces a covering map onto its image of degree

$$d_\star(n) := \Delta + \sum_{H(v) \leq -n} (d_\pi(v) - 1) \ ,$$

and γ_n is included in the boundary of $\overline{\Phi_n(A_{\star,n})}$ in S_n.

We impose (C0). Since $G(\mathcal{F}, \Phi_0) = \Gamma_0$ we can extend μ_0 in a unique way such that (C4) is satisfied. Observe that (C2) and (C3) are satisfied since $G_0(z) \to \frac{1}{d} \log \rho$ when $|z| \to \rho^{1/d}$, and $G_0 \circ M_d = \log |z^d| = dG_0$. Similarly (C5) – (C7) hold, and (C8) holds with $d_\star(0) = d$ by Hurwitz formula since on the one hand

$$\Delta = 1 + \sum (d_\pi(\mu(i)) - 1)$$

where the sum ranges over all $\mu(i)$ lying in bounded components of Γ, and on the other hand $H(\mu(i)) \leq 0$ for all i in the unbounded component of Γ.

Assume now we have constructed $S_n, \Phi_n, G_n, \mu_n, \gamma_n$ for some $n \geq 0$. Before we explain our construction of S_{n+1}, we begin with some simple observations.

Adding ∞ to the surface S_0 yields a Riemann surface which we denote by $\hat{S}_0$. By (C1), the boundary of S_0 in S_1 is a curve, hence the natural inclusion $\imath \colon S_0 \to S_1$ extends to an injective holomorphic map $\hat{S}_0 \to \hat{S}_1 := S_1 \cup \{\infty\}$. By induction, we get an increasing sequence of Riemann surfaces $\hat{S}_n = S_n \cup \{\infty\}$, so that $\hat{S}_0 \subset \hat{S}_{n-1} \subset \hat{S}_n$.

The function G_n extends continuously (as a $\mathbb{R}_+ \cup \infty$-valued function) to $\hat{S}_n$ by setting $G_n(\infty) = +\infty$. It is superharmonic and $\Delta G_n = -\delta_\infty$.

Lemma 8.22. — *In S_n, we have*

$$S_{n-1} = \left\{ G_n > \frac{1}{d^n} \log \rho \right\}, \text{ and } \partial S_{n-1} = \left\{ G_n = \frac{1}{d^n} \log \rho \right\}.$$

More precisely, for any connected component A of $S_n \setminus \overline{S_{n-1}}$ there exists a conformal isomorphism $\psi \colon \{\rho_{n,A} < |z| < \rho^{1/d^n}\} \to A$ for some $\rho_{n,A} > 0$ such that $G_n(\psi(z))$ is proportional to $\log |z|$.

Proof. — By (C2), we know that $G_n \to \frac{1}{d^n} \log \rho$ when approaching the boundary of S_{n-1}, hence $\partial S_{n-1} \subset \{G_n = \frac{1}{d^n} \log \rho\}$. By the minimum principle applied to $\hat{G}_n$, we get the inclusion $S_{n-1} \subset \{G_n > \frac{1}{d^n} \log \rho\}$.

Now look at $S_n \setminus \overline{S_{n-1}}$. By (C1), it is a finite collection of annuli. Choose one of them say A: it has at least one boundary component included in ∂S_{n-1} over which one has $G_n = \frac{1}{d^n} \log \rho$. Since S_n is connected and $G_n(z) \to \infty$ as $z \to \infty$, $G_n|_A$ is harmonic and not constant, so that the other boundary component of A cannot be included in ∂S_{n-1}. It follows from (C2) that $G_n \to \frac{1}{d^{n+1}} \log \rho$ when approaching this boundary.

Pick any conformal isomorphism $\psi \colon \{\rho_{n,A} < |z| < \rho^{1/d^n}\} \to A$ with $\rho_{n,A} < \rho^{1/d^n}$ sending $\{z = \rho^{1/d^n}\}$ to the boundary of A included in ∂S_{n-1}. Then $G_n \circ \psi$ is harmonic, equal to $\frac{1}{d^n} \log \rho$ on ρ^{1/d^n}, and tends to $\frac{1}{d^{n+1}} \log \rho$ when

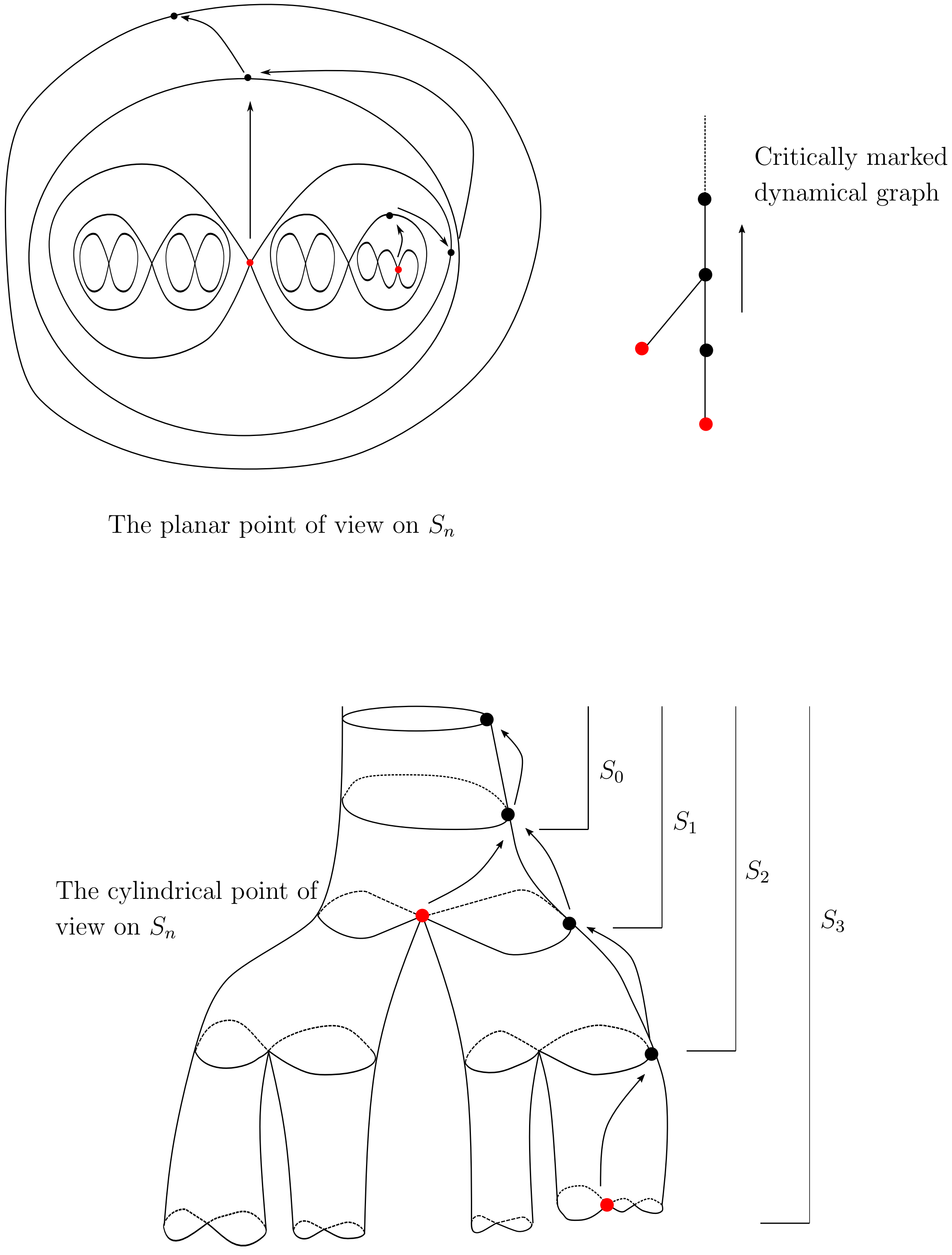


FIGURE 7. Constructing the surfaces S_n

one approaches the circle $\{|z| = \rho_{n,A}\}$. By circular symmetry, it follows that $G_n \circ \psi(z) = \lambda \log |z|$ for some $\lambda \neq 0$ and $\lambda \log \rho_{n,A} = \frac{1}{d^{n+1}} \log \rho$.

This implies the inclusions $\{G_n > \frac{1}{d^n} \log \rho\} \subset S_{n-1}$, $\{G_n = \frac{1}{d^{n-1}} \log \rho\} \subset \partial S_{n-1}$ and the second part of the lemma. □

We now need to analyze the ramification locus of Φ_n.

Lemma 8.23. — *The image of μ_n is included in $\cup_{l \leq n}\{G_n = \frac{1}{d^l} \log \rho\}$.*

Proof. — Observe that $\mu_0(\Gamma_0)$ is the union of the orbits of points in $\mathcal{F}$ under M_d, hence by construction it is included in $\cup_{j \geq 0}\{|z| = \rho^j\} = \cup_{l \leq 0}\{G_n = \frac{1}{d^l} \log \rho\}$.

Now the complement of Γ_0 in Γ_1 is precisely the set $\partial\Gamma_1 \setminus \partial\Gamma_0$, and by (C6) this set is mapped by μ_1 inside $\{G_1 = \frac{1}{d} \log \rho\}$. This implies the lemma in the case $n = 1$, and a simple induction on n allows one to conclude. □

We shall construct S_{n+1} by patching an open Riemann surface along each connected component of $S_n \setminus \overline{S_{n-1}}$. More precisely, for each component A, we shall find a planar domain V_A containing an annulus B_A and build a conformal isomorphism $\Phi_A \colon B_A \to A$. The surface S_{n+1} will be obtained as the disjoint union of S_n and all domains of the form V_A, each patched to S_n along the annulus B_A using Φ_A. We refer the reader to the figures 8 and 9 for a schematic view on the patching procedure.

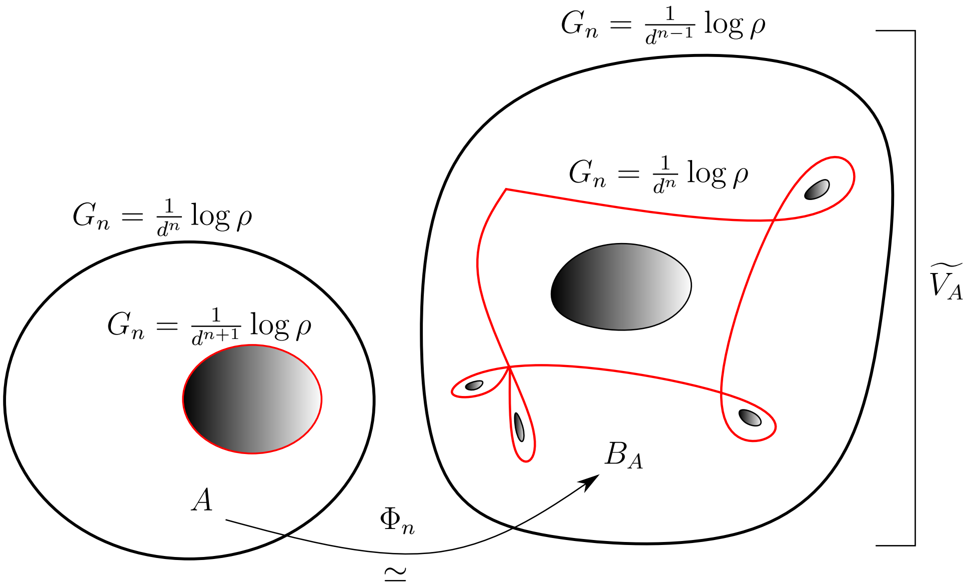


FIGURE 8. The patching procedure when $A \neq A_\star$

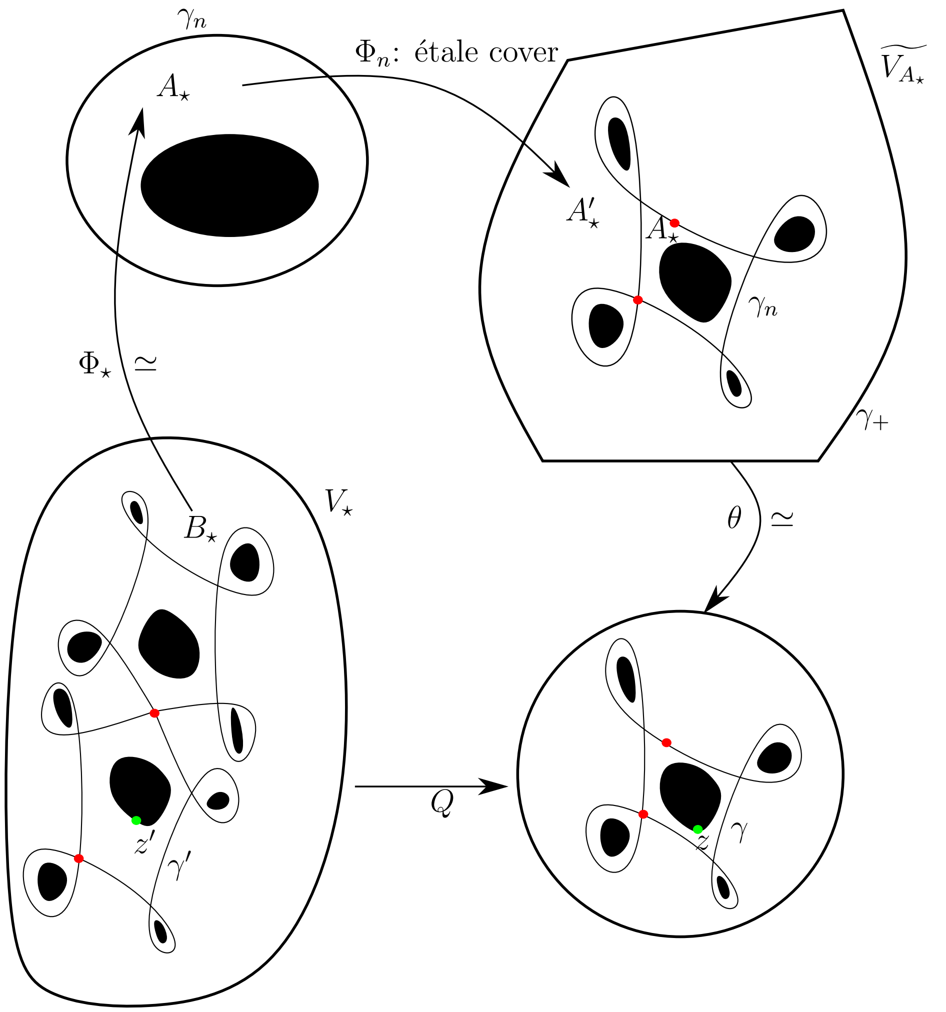


FIGURE 9. The patching procedure when $A = A_\star$

Construction of V_A, B_A and Φ_A. — Recall from (C6), that γ_n is a simple closed curve bounding a connected component $A_\star(= A_{\star,n})$ of $S_n \setminus \overline{S_{n-1}}$. We begin with the following general lemma.

Lemma 8.24. — *Let A be any connected component of $S_n \setminus \overline{S_{n-1}}$, and let $\widetilde{V_A}$ be the connected component of $\{G_n < \frac{1}{d^{n-1}} \log \rho\}$ in S_n which contains $\Phi_n(A)$. Then the following holds:*

1. *the boundary of $\widetilde{V_A}$ in S_n is a real-analytic simple closed curve γ'_A;*
2. *$\widetilde{V_A} \cap \{G_n < \frac{1}{d^n} \log \rho\}$ is a finite union of conformal annuli of finite modulus;*

3. *there exists a univalent holomorphic map* $\theta\colon \widetilde{V_A} \to \mathbb{D}$ *which extends continuously to* $\widetilde{V_A} \cup \{\gamma'_A\} \to \overline{\mathbb{D}}$ *whose image is the complement of finitely many connected full compact subsets of* $\mathbb{D}$.
4. $\widetilde{V_A}$ *is a planar domain.*

Moreover, when $A = A_\star$*, then* $\gamma'_{A_\star} = \Phi_n(\gamma_n)$ *and* $\widetilde{V_{A_\star}}$ *contains* $A_\star$.

For any connected component $A \neq A_\star$ of $S_n \setminus \overline{S_{n-1}}$, set $V_A := \widetilde{V_A}$. By (C3) and (C7), Φ_n induces a conformal isomorphism of A onto an annulus B_A which is a component of $\{\frac{1}{d^n}\log\rho < G_n < \frac{1}{d^{n-1}}\log\rho\}$, and we define $\Phi_A := \Phi_n\colon A \to B_A$.

Let us consider now the component $A_\star$ which is bounded by γ_n. By (C8), Φ_n induces an unramified cover of degree $d_\star(n) := \Delta + \sum_{H(v)\leq -n}(d_\pi(v) - 1)$ from $A_\star$ onto an annulus $A'_\star := \Phi_n(A_\star)$ bounded by two real analytic connected curves, $\Phi_n(\gamma_n) \subset \{G_n = \frac{1}{d^{n-1}}\log\rho\}$, and $\gamma_{A_\star} \subset \{G_n = \frac{1}{d^n}\log\rho\}$ which contains γ_n by the previous lemma.

Define $\mathcal{G}$ as the image under θ of $\mu_n\{H = 1-n\} = \mu_n(\partial\Gamma_n \setminus \partial\Gamma_{n-1})$. Since θ is injective, any point $p \in \mathcal{G}$ has a unique preimage $p = \theta(v)$, and we may set

$$\mathcal{D}(p) := \{d_\pi(v'),\ (v,v') \in \mathcal{B}(v)\} \tag{50}$$

$$\mathcal{D}_0(p) := \{d_\pi(v'),\ (v,v') \in \mathcal{B}(v) \text{ and } v' \notin \partial\Gamma_{n+2}\} \tag{51}$$

where $\mathcal{B}(v)$ is the set of edges (v,v') of Γ with $H(v') < H(v)$. In other words, $\mathcal{D}(p)$ encodes all multiplicities of those vertices w mapped by π to v; and $\mathcal{D}_0(p)$ consists of multiplicities of those vertices $w \in \pi^{-1}(v)$ for which $\pi(w') = w$ for at least one $w' \in \Gamma_{n+1}$. Finally set

$$\gamma := \theta(\gamma_n), \text{ and } N := \Delta + \sum_{H(v)\leq -n-1}(d_\pi(v) - 1).$$

Lemma 8.25. — *The three conditions (D1) – (D3) hold for the collection of objects* $\mathcal{G}$, $\mathcal{D}(p)$, $\mathcal{D}_0(p)$, γ, *and* $d' := d_\star(n)$.

Pick any point $z \in \partial\theta(A_\star) \cap \partial\theta(\widetilde{V_{A_\star}})$, and apply Theorem 8.21 to $\gamma, \mathcal{G}$, and the two collections of integers $\mathcal{D}_0$ and $\mathcal{D}$ defined above. We obtain a polynomial Q of degree d', a real analytic curve $\gamma' \subset Q^{-1}(\gamma)$ and a point z' in the bounded component of $\mathbb{C} \setminus \gamma'$ such that (R1) – (R3) hold.

By construction, all critical values of Q are included in $\mathcal{G} \cup \{z\} \subset \gamma$ hence in $\mathbb{D}$. It follows that $Q\colon Q^{-1}(\mathbb{D}) \to \mathbb{D}$ is a ramified cover of degree $d' = \deg(Q)$, hence $Q^{-1}(\mathbb{D})$ is connected. We also get that $A'_\star$ contains no critical values of $\tilde{Q} := \theta^{-1} \circ Q$, hence $B_\star = \tilde{Q}^{-1}(A'_\star)$ is an annulus, and $\tilde{Q}\colon B_\star \to A'_\star$ is an unramified covering map of degree d'.

By (C8), Φ_n is a covering map of degree d' from $A_\star$ onto $A'_\star$, so that we may find a conformal isomorphism $\Phi_\star\colon B_\star \to A_\star$ such that $\Phi_n \circ \Phi_\star = \tilde{Q}$.

We let $V_\star := \tilde{Q}^{-1}(\widetilde{V_{A_\star}})$, and we patch $V_\star$ to S_n using the conformal isomorphism $\Phi_\star$.

Proof of Lemma 8.24. — By (C1), the boundary γ_A of A in S_n is real analytic. It is a simple closed curve since S_{n-1} is connected. By (C7) and (C8), Φ_n induces a finite cover of A onto its image so that $A' := \Phi_n(A)$ is a conformal annulus which is a component of $S_{n-1}\setminus\overline{S_{n-2}}$. The boundary of A' in S_n has two connected components, one of which is the image under Φ_n of the boundary of A in S_n. It follows that $\partial A'$ is the disjoint union of $\gamma'_A = \Phi_n(\gamma_A)$ which is a simple closed curve proving 1., and a closed subset γ of S_n. By (C2) and (C3), $G_n|_\gamma = \frac{1}{d^n}\log\rho$, hence γ is a compact real-analytic curve since G_n is harmonic.

We claim that γ may be written as the union of finitely simple closed curves $\gamma = \ell_1 \cup \cdots \cup \ell_k$.

Grant this claim. Recall that $\widetilde{V_A}$ is the connected component of $\{G_n < \frac{1}{d^{n-1}}\log\rho\}$ containing A'_∞.

Since $\ell_i \subset \{G_n = \frac{1}{d^n}\log\rho\}$, this curve is included in ∂S_{n-1} hence by (C1) it bounds an annulus of finite modulus in S_n. The union of these annuli is equal to $V_A \cap \{G_n < \frac{1}{d^n}\log\rho\}$ proving 2.

We may thus attach to each ℓ_i a closed conformal disk $\bar{\mathbb{D}}_i$, and the union of A' and these disks contains $\widetilde{V_A}$ and is simply connected (any path is homotopic to one in A', and the path generating the fundamental group of A' is homotopic to the union of the ℓ_i's). This shows the existence of a univalent holomorphic map $\theta\colon \widetilde{V_A} \to \mathbb{D}$ satisfying condition 3. (the complement of the image of V_A in $\bar{\mathbb{D}}_i$ is the decreasing intersection of closed disks hence is connected and full).

When $A = A_\star$, the boundary of $A_\star$ in S_n is γ_n by (C8), hence $A_\star$ is included in $\widetilde{V_{A_\star}}$.

To prove the claim, observe that the singular locus of γ is included in the intersection $\mathcal{S}$ of γ with the critical locus of G_n: the latter is a finite (possibly empty) set. The gradient flow of G_n induces a continuous map from S^1 to γ which is a local diffeomorphism onto $\gamma \setminus \mathcal{S}$ and displays γ as the quotient of the circle by equivalence relation of the following form: there exists a finite set $F \subset S^1$ such that all equivalence class of points $\notin F$ are trivial. An induction on the cardinality of F shows that any quotient of the circle is a union of circles as claimed. □

Remark 8.26. — Although we shall not use it, note that in our situation the equivalence relation is induced by a family of finite subsets of S^1 that are unlinked (see §8.6.2 below for a definition). This implies γ to be a tree of simple closed curves, see Figure 10.

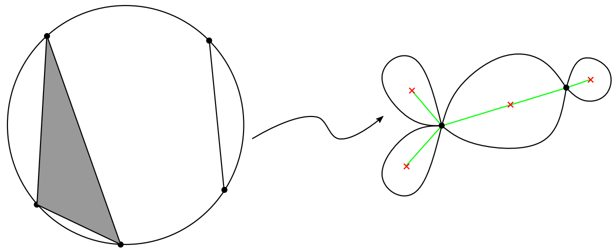


FIGURE 10. A tree of circles

Remark 8.27. — In fact $\widetilde{V_A}$ is the complement in a conformal disk of finitely many compact domains with real-analytic boundary. Indeed, this is clear if $n = 1$, and by induction on n, $\widetilde{V_A}$ is isomorphic to the pull-back by a polynomial of a region with real-analytic boundary.

Proof of Lemma 8.25. — Recall that the collection of integers $\{n_{i,p}\}$ with $i \in \mathcal{D}(p)$ is in bijection with the set $\{d_\pi(v')\}$ with $\pi(v') = v$ and $p = \theta(v)$. Condition (D1) is an easy check:

$$d' - 1 = \sum_{H \le -n} (d_\pi(v) - 1) = \left(1 + \sum_{H \le -n-1} (d_\pi(v) - 1)\right) - 1 + \sum_{H=-n} (d_\pi(v) - 1)$$
$$= N - 1 + \sum_{\mathcal{G}} \sum_{\mathcal{D}(p)} (n_{i,p} - 1) \ .$$

For the proof of (D2) we use our standing assumption (R2). Pick any point $p \in \mathcal{G}$, say $p = \theta(v)$ with $H(v) = -n + 1$. By (R2), $\mathcal{B}(v)$ contains at most one vertex v_0 such that $d_\pi(v_0) \ge 2$. By the minimality condition (G6), for any other vertex $v' \in \mathcal{B}(v)$ there exists a vertex $w = w(v')$ such that $d_\pi(w) \ge 2$ and $\pi^s(w) = v'$ for some $s \ge 1$. We infer

$$\sum n_{i,p} = \sum_{\mathcal{B}(v)} d_\pi(v') = d_\pi(v_0) - 1 + \mathrm{Card}(\mathcal{B}(v))$$
$$\le d_\pi(v_0) - 1 + \sum_{H \le -n-1} (d_\pi(w) - 1) \le d'$$

which implies (D2).

Observe finally that $\bigcup_{\mathcal{G}} \mathcal{D}_0(p)$ is in bijection with the set of edges of Γ_{n+1} that connect a vertex in $\{H = -n - 1\}$ to a vertex in $\{H = -n\}$. By the minimality condition (G6), for each of these edge e there exists a vertex $v \in \Gamma$

and an integer $s \geq 1$ such that $e = (\pi^s(v), \pi^{s+1}(v))$ and $d_\pi(v) \geq 2$. Since we have $H(v) = -s + H(\pi^{s+1}(v)) = -s - n$, we obtain

$$\sum_{\mathcal{G}} \mathrm{Card}\,(\mathcal{D}_0(p)) = \mathrm{Card}\left(\bigcup_{\mathcal{G}} \mathcal{D}_0(p)\right) \leq \sum_{H \leq -n-1} (d_\pi(v) - 1) \leq N$$

which implies (D3). □

Construction of Φ_{n+1}, G_{n+1}, and μ_{n+1}. — Now that we defined S_{n+1} as the union of S_n and annuli V_A attached to any component A of $\{\frac{1}{d^n} < G_n < \frac{1}{d^{n-1}} \log \rho\}$. To avoid confusion, denote by $I_A \colon V_A \to \widetilde{V_A}$ the canonical isomorphism between the open subset V_A of S_{n+1} and $\widetilde{V_A} \subset S_n$. We define the map:

$$\Phi_{n+1} = \begin{cases} \Phi_n \text{ on } S_n, \\ I_A \text{ on } V_A \text{ for } A \neq A_\star, \\ \tilde{Q} \text{ on } V_\star. \end{cases}$$

Observe that by construction Φ_{n+1} is a well-defined holomorphic map such that $\Phi_{n+1}(S_{n+1}) \subset S_n$. Similarly we set

$$G_{n+1} = \begin{cases} G_n \text{ on } S_n, \\ \frac{1}{d} G_n \text{ on } V_A \text{ for } A \neq A_\star, \\ \frac{1}{d} G_n \circ \tilde{Q} \text{ on } V_\star. \end{cases}$$

When $A \neq A_\star$, it follows from (C3) that G_{n+1} is well-defined and harmonic on V_A. On $V_\star$, we observe that $\frac{1}{d} G_n \circ \tilde{Q}$ is a harmonic function on $B_\star$ whose restriction to the outer (resp. inner) boundary of $V_\star$, i.e. on $\tilde{Q}^{-1}(\gamma_+)$ (resp. on $\tilde{Q}^{-1}(\gamma_-)$) is constant equal to $\frac{1}{d^n} \log \rho$ (resp. to $\frac{1}{d^{n+1}} \log \rho$) so that $\frac{1}{d} G_n \circ \tilde{Q} = G_n \circ \Phi_\star$ on $B_\star$.

We conclude that G_{n+1} is a well-defined harmonic function on S_{n+1} satisfying $G_{n+1} \circ \Phi_{n+1} = d G_{n+1}$.

Let us now define the map μ_{n+1}. We set $\mu_{n+1} = \mu_n$ on Γ_n. Observe that

$$T := \Gamma_{n+1} \setminus \Gamma_n = \partial \Gamma_{n+1} \setminus \partial \Gamma_n = \{H = -n\}\ .$$

Recall that we defined $\mathcal{G} := \theta(\mu_n(\{H = 1 - n\}))$. We attach to a vertex $w \in T$ a point $p(w) \in \mathcal{G}$ by setting $p(w) = (\theta \circ \mu_n)(\pi(w))$.

From the definition of $\mathcal{D}(p)$ and $\mathcal{D}_0(p)$, see (50) and (51), and since μ_n is injective on $\{H = 1 - n\}$, and θ is a conformal isomorphism, we also get a canonical bijection

$$\alpha\colon T \to \bigcup_{p\in\mathcal{G}} \mathcal{D}(p)$$

such that: for all $w \in T$ we have $\alpha(w) \in \mathcal{D}(p(w))$; and $\alpha(w) \in \mathcal{D}_0(p(w))$ iff $w \notin \partial\Gamma_{n+2}$ (i.e. $w = \pi(w')$ for some $w' \in \Gamma_{n+2}$).

We now observe that the polynomial Q obtained by applying Theorem 8.21 comes with a family of bijections $\delta_p\colon \mathcal{Q}(p) \to \mathcal{D}(p)$ where $\mathcal{Q}(p)$ is a subset of $Q^{-1}(p)$ (see condition (R3)), and that $\delta_p^{-1}(\mathcal{D}_0(p))$ is included in γ'. For any $w \in T$, we may thus set $\mu_{n+1}(w) = \delta_{p(w)}^{-1}(\alpha(w))$. The latter point naturally belongs to $V_\star$ hence to S_{n+1} since the latter is obtained by patching $V_\star$ to S_n using the map $\Phi_\star$.

Verification that all conditions (C1) – (C8) are satisfied. — By construction S_{n+1} is the union $S_n \cup \bigcup_A V_A$ where A ranges over all connected components of $S_n \setminus \overline{S_{n-1}}$. We further have a conformal isomorphism $\Phi_A\colon B_A \to A$ from an annulus $B_A \subset V_A$, and V_A is patched with S_n using this biholomorphism. To check (C1) we need to prove that $V_A \setminus B_A$ is a finite union of conformal annuli of finite modulus, and that the boundary of B_A inside V_A is real analytic.

When $A \neq A_\star$, we have

$$V_A \setminus B_A = \widetilde{V_A} \setminus \Phi_n(A) = \widetilde{V_A} \cap \left\{ G_n < \frac{1}{d^n} \log \rho \right\}$$

which is a union of annuli by Lemma 8.24 (2). Note also that $\partial B_A = \{G_n = \frac{1}{d^n}\log\rho\}$, hence ∂B_A is real-analytic since G_n is harmonic.

Otherwise $A = A_\star$, and

$$V_\star \setminus B_\star = \tilde{Q}^{-1}(\widetilde{V_{A_\star}} \setminus A'_\star) \ ,$$

with $A'_\star = \Phi_n(A_\star)$. The latter equality implies that $\partial A'_\star = \{G_n = \frac{1}{d^n}\log\rho\}$ as before so that $\partial A'_\star$ is real-analytic. By Lemma 8.23, the image of μ_n is included in $\bigcup_{l\leq n}\{G_n = \frac{1}{d^l}\log\rho\}$, hence $\mu_n(\Gamma_n) \cap \widetilde{V_{A_\star}} \subset \{G_n = \frac{1}{d^n}\log\rho\}$. Now by Lemma 8.24 (2), $\widetilde{V_{A_\star}} \setminus A'_\star$ is a finite union of annuli, and since $\tilde{Q}$ is ramified only over the image of μ_n, it follows that $\tilde{Q}$ induces a finite cover from $V_\star \setminus B_\star$ onto $\widetilde{V_{A_\star}} \setminus A'_\star$. This proves (C1).

The function G_{n+1} is a composition of a harmonic function and a holomorphic map, hence is harmonic. By construction, we also have $G_{n+1}|_{S_n} = G_n$. If z_k is a diverging sequence in $S_{n+1} \setminus S_n$, then after extraction it belongs to some open set V_A for some A. If $A \neq A_\star$, then the sequence z_k can be identified with a sequence $\widetilde{z_k}$ diverging in $\widetilde{V_A} \setminus B_A$, so that $G_{n+1}(z_k) = \frac{1}{d}G_n(\widetilde{z_k}) \to \frac{1}{d^{n+2}}\log\rho$.

When $z_k \in A_\star$, then $\tilde{Q}(z_k)$ diverges in $\widetilde{V_{A_\star}} \setminus A'_\star$, hence again $G_{n+1}(z_k) = \frac{1}{d} G_n(\tilde{Q}(z_k)) \to \frac{1}{d^{n+2}} \log \rho$ proving (C2).

We have $\Phi_{n+1}|_{S_n} = \Phi_n$, $\Phi_{n+1}(V_A) \subset S_n$ for all A, and $G_{n+1} \circ \Phi_{n+1} = dG_{n+1}$ by the very definitions of Φ_{n+1} and G_{n+1}. The properness is a consequence of the properness of Φ_n and of (C7) and (C8) to be proved below.

Let W be any connected component of $S_n \setminus \overline{S_{n-1}}$, and consider the component B of $S_n \setminus \overline{S_{n-2}}$ containing it. By induction we know that there exists a component W' of $S_n \setminus \overline{S_{n-1}}$ whose image by Φ_n is included in B. Let V' be the component of $S_{n+1} \setminus \overline{S_{n-1}}$ containing W'. Its image by Φ_{n+1} is a connected component of $S_n \setminus \overline{S_{n-2}}$ by the properness of Φ_{n+1}, which contains B hence it is W. This proves Φ_{n+1} is surjective.

Observe that $\mu_{n+1} = \mu_n$ on Γ_n by definition, and that $\mu_{n+1}(w) = \delta^{-1}_{p(w)}(\alpha(w))$ for any $w \in \{H = -n\} \subset \Gamma_{n+1}$. From the previous section, we also infer

$$\begin{aligned}\Phi_{n+1}(\mu_{n+1}(w)) &= \theta^{-1}\left(Q\left(\delta^{-1}_{p(w)}(\alpha(w))\right)\right)\\ &= \theta^{-1}(p(w)) = \mu_n(\pi(w)) = \mu_{n+1}(\pi(w))\end{aligned}$$

which completes the proof of (C4).

Pick any $v \in \Gamma_{n+1}$. If $v \in \Gamma_n$, then $d_\pi(v) = \deg_{\mu_n(v)}(\Phi_n) = \deg_{\mu_{n+1}(v)}(\Phi_{n+1})$. Otherwise $v \in T$, so that $\alpha(v) = d_\pi(v) \in \mathcal{D}(p(v))$, and

$$\deg_{\mu_{n+1}(v)}(\Phi_{n+1}) = \deg_{\mu_{n+1}(v)}(\tilde{Q}) = \deg_{\mu_{n+1}(v)}(Q) \overset{(R3)}{=} \delta_{p(v)}(\mu_{n+1}(v)) = d_\pi(v)$$

proving (C5).

We now define the curve γ_{n+1}. Recall that Theorem 8.21 yields a polynomial Q and a real-analytic curve γ' bounding a disk D' such that $Q(\gamma') = \gamma$. In particular, $Q(D')$ is the disk D bounded by the simple closed curve γ. Note further that the restriction of Q to D' has a single ramification point z' of degree $N = \Delta + \sum_{H(v) \le -n-1}(d_\pi(v) - 1)$, and that $Q(z')$ is a point z which was fixed on $\partial\theta(A_\star) \cap \partial\theta(\widetilde{V_{A_\star}})$. We let γ_{n+1} be the image of $\gamma' \subset V_\star$ in S_{n+1}.

Before proving (C6) – (C8), we discuss first the structure of $S_{n+1} \setminus \overline{S_n}$. This set is equal to $\{G_{n+1} < \frac{1}{d^{n+1}} \log \rho\}$, hence is included in the union $\cup V_A$ where A ranges over all connected components of $S_n \setminus \overline{S_{n-1}}$. By definition of G_{n+1} and by Lemma 8.24 (2), it follows that $S_{n+1} \setminus \overline{S_n}$ is a finite union of annuli, each of finite moduli.

Observe that $G_{n+1}|_{\gamma_{n+1}} = \frac{1}{d^{n+1}} \log \rho$, hence γ_{n+1} bounds a unique component of $S_{n+1} \setminus \overline{S_n}$ that we define to be $A_{\star,n+1}$. But γ' also contains $\bigcup_{\mathcal{G}} \delta_p^{-1}(\mathcal{D}_0(p))$, and the latter set is precisely $\mu_{n+1}(\partial\Gamma_{n+1} \setminus \partial\Gamma_n)$ proving (C6).

Pick any component A' of $S_{n+1} \setminus \overline{S_n}$ different from $A_{\star,n+1}$. Then either A' is contained in a component of $S_{n+1} \setminus \overline{S_{n-1}}$ that does not contain $A_{\star,n}$ and Φ_{n+1} is an isomorphism on A by definition. Or we may view A' inside $V_\star$ where is it a component of $V_\star \setminus \overline{B_\star}$ (see Figure 9). Critical points of Q outside $\{z'\}$ are mapped to γ, hence are included in the boundary of $B_\star$ (inside $V_\star$). It follows that Q has no critical point in the smallest conformal disk in $Q^{-1}(\mathbb{D})$ which contains A'. This proves that $Q|_{A'}$ is a conformal isomorphism onto its image concluding (C7).

Finally, the smallest conformal disk in $Q^{-1}(\mathbb{D})$ containing $A_{\star,n+1}$ contains a single critical point z' for Q of multiplicity $d_\star(n+1) = N$, hence $\Phi_{n+1}|_{A_{\star,n+1}}$ induces a covering map onto its image of degree $d_\star(n+1)$. When viewed in $V_\star$, then the set $\overline{\Phi_{n+1}(A_{\star,n+1})}$ equals $B_\star$ whose boundary contains γ' so that γ_{n+1} is included in the boundary of $\overline{\Phi_{n+1}(A_{\star,n+1})}$ concluding the proof of (C8).

Addendum. — Let us include here the following important remark.

Lemma 8.28. — *Each surface S_n is a planar domain.*

Proof. — Observe that $\hat{S}_n \setminus \overline{\hat{S}_{n-1}}$ is a finite union of annuli of finite modulus so that we may attach to $\hat{S}_n$ finitely many disks to obtain a compact Riemann surface $\overline{S}_n$. We prove by induction on n that $\overline{S}_n$ is conformally equivalent to the Riemann sphere.

Pick any connected component A of $S_n \setminus \overline{S_{n-1}}$, and let B be the connected component of $\{G_n < \frac{1}{d^{n-1}} \log \rho\}$ containing it. Since Φ_n is surjective onto S_{n-1}, one can find a component $\tilde{A}$ of $S_n \setminus \overline{S_{n-1}}$ whose image by Φ_n is included in B, and Lemma 8.24 proves that B is conformally equivalent to the complement of finitely many connected sets $K_1, \cdots, K_r$ in the unit disk. We know moreover that the function G_n is harmonic on B, tends to $\frac{1}{d^{n-1}} \log \rho$ when $z \to \partial B$, (resp. to $\frac{1}{d^{n+1}} \log \rho$ when $z \to K_i$), that $B_+ = \{z \in B, G_n > \frac{1}{d^n} \log \rho\}$ is an annulus, and that $\{z \in B, \frac{1}{d^{n+1}} \log \rho < G_n < \frac{1}{d^n} \log \rho\}$ is a union of annuli $A_1, \cdots, A_r$.

Note that $\overline{S_{n-1}}$ is obtained by attaching a disk to B_+, whereas $\overline{S_n}$ is obtained by attaching r disks to each of the annuli A_i, $i = 1, \ldots, r$. In both cases, we obtain a disk with boundary ∂B. We may thus patch together diffeomorphisms of the disk being the identity on its boundary to obtain a smooth diffeomorphism between $\overline{S_{n-1}}$ and $\overline{S_n}$. By induction the latter is thus compact and simply connected. □

8.4.5. End of the proof of Theorem 8.15

Let us first suppose that $\mathsf{P} = \varnothing$. In this case, one can use geometric arguments based on estimating moduli of annuli. Apply the construction of §8.4.4 with $\Delta = 0$: we obtain an open Riemann surface $\hat{S} = \cup_{n\geq 0}\hat{S_n}$ with a marked point $\infty \in \hat{S}$, and a holomorphic map $\Phi\colon \hat{S} \to \hat{S}$ leaving ∞ totally invariant, and such that $\Phi|_{S_n} = \Phi_n$ for all n. By Lemma 8.28, $S_n = \hat{S}_n \setminus \{\infty\}$ is a planar domain for each n, hence we may find a sequence of univalent maps $\kappa_n\colon \hat{S}_n \to \mathbb{C} \cup \{\infty\}$ such that $\kappa_n(\infty) = \infty$. We normalize them so that in the chart $S_0 = \{|z| > \rho^{1/d}\}$ we have the expansion $\kappa_n(z) = z + O(1)$.

By Koebe's 1/4-theorem, ψ_n forms a normal family, so that one can extract a subsequence and get a univalent map $\psi\colon S \to \mathbb{C}$. We thus get a domain $U = \psi(S) \subset \mathbb{C}$ and a holomorphic map $\Phi\colon U \to U$.

Observe that for n large enough (larger than $1 - \min_\Gamma H$), then $d_\star(n) = 1$, and $S_n \setminus \overline{S_{n-1}}$ is a finite union of annuli on which the restriction of Φ_n is a biholomorphism by (C7) and (C8). We thus get a system of nested annuli of modulus bounded from below by a positive constant, and it follows from [**124**, §2.8] (or [**149**, §8D]) that $K := \mathbb{C} \setminus U$ has absolute area zero in the sense of Ahlfors, so that Φ extends through K (see e.g. [**167**, §4]).

By (C0) we get a polynomial map Φ of degree d, whose marked dynamical graph is equal to Γ by (C4) as required.

To handle the general case $\mathsf{P} \neq \varnothing$, we take a slightly different approach using quasi-conformal deformation arguments.

The graph Γ is the union of an infinite connected marked dynamical graph $\Gamma_{\rm esc}$ and a finite marked dynamical graph $\Gamma_{\rm fin}$. By [,] (see the discussion in §8.6 below) the latter is realizable by a PCF polynomial so that we may find a PCF polynomial P_0 whose marked dynamical graph (with the action of the symmetry group removed) is isomorphic to $\Gamma_{\rm fin}$.

Apply the construction of §8.4.4 with $\Delta = \deg(P_0)$. Fix any integer n_0 larger than the depth of the infinite part of the marked dynamical graph, i.e. $n_0 \geq \min_{\Gamma_{\rm esc}} H$. Build a Riemann surface S by patching a conformal disc $\mathbb{D}_A = \{|z| < 1\}$ to each annuli component A of $S_{n_0}\setminus\overline{S_{n_0-1}}$. By Lemma 8.28, S is a conformal plane which contains $S_{n_0} \supset S_{n_0-1}$ as open subsets, and $S_{n_0} \setminus \overline{S_{n_0-1}}$ is a finite union of annuli of finite modulus.

Recall that for any of these components A, the map $\Phi_{n_0}|_A$ induces an unramified cover onto its image, whose degree is equal to 1 when $A \neq A_\star$ and to Δ when $A = A_\star$.

We may thus find a smooth map $\Phi\colon S \to S$ such that:

– Φ is an orientation preserving finite branched cover;

– $\Phi = \Phi_{n_0}$ on a neighborhood of S_{n_0-1};
– for each $A \neq A_\star$, Φ is conformal and univalent on a conformal disk $\hat{\mathbb{D}}_A \Subset \mathbb{D}_A$ such that $\overline{\mathbb{D}_A \setminus \hat{\mathbb{D}}_A}$ is a compact subset of A;
– there exists a conformal disk $\hat{\mathbb{D}}_\star \Subset \mathbb{D}_\star$ such that $\overline{\mathbb{D}_\star \setminus \hat{\mathbb{D}}_\star}$ is a compact subset of A, and $\Phi|_{\hat{\mathbb{D}}_\star}$ is conformally conjugated to P_0 in a neighborhood of its Julia set;
– critical points of Φ in $S \setminus S_{n_0}$ are contained in $\hat{\mathbb{D}}_\star$.

Observe that the set $K(\Phi)$ of points $x \in S$ such that $\Phi^n(x) \in \cup_A \mathbb{D}_A$ for all n is compact and that Φ is conformal in a neighborhood of $K(\Phi)$.

We now modify the complex structure on S as follows. Let σ_0 be the standard complex structure, and set $\sigma_n := \Phi^{n*}\sigma_0$ with corresponding Beltrami form μ_n. Note that $\mu_0 \equiv 0$. Moreover, as Φ is a finite branched cover, we have

$$\mu_1(z) = \frac{\partial_{\bar{z}}\Phi(z)}{\partial_z \Phi(z)} \frac{d\bar{z}}{dz}, \quad \text{for a.e } z \in S,$$

and μ_1 is non-zero exactly where Φ is not conformal, i.e. on a finite union of conformal annuli and $\|\mu_1\|_{L^\infty} < 1$. Now, outside a neighborhood of $K(\Phi)$, all the maps Φ^n being conformal, we have $\mu_n = \mu_0$. Finally, in any disk $\mathbb{D}_A$, we may write

$$\Phi^* \left(\mu_n \frac{d\bar{z}}{dz} \right) = \mu_n \circ \Phi \, \frac{\overline{\Phi_z}}{\Phi_z} \frac{d\bar{z}}{dz}$$

so that μ_n converges in a neighborhood of $K(\Phi)$ to an L^∞ Beltrami form μ_∞ of sup-norm < 1 since Φ is orientation preserving, and equal to 0 on $K(\Phi)$.

We may thus apply the Ahlfors-Bers theorem which yields a quasi-conformal homeomorphism $\psi \colon S \to S$ such that $\psi^*\mu_\infty$ corresponds to the standard complex structure. The map $P := \psi^{-1} \circ \Phi \circ \psi$ is a conformal map of the complex plane which is conjugated near ∞ to Φ hence is a polynomial of degree d.

Since critical points and their multiplicities are topological invariant, it follows that the marked critical graph of P is isomorphic to Γ which completes the proof.

Remark 8.29. — By construction P admits a renormalization which is hybrid equivalent to P_0.

8.5. Special curves and special critically marked dynamical graphs

Theorem 8.30. — *Let Γ be any special asymmetric critically marked dynamical graph such that two distinct marked points have different images.*

Then the Zariski closure in $\mathrm{MPoly}^d_{\mathrm{crit}}$ *of the set of primitive polynomials* P *with disconnected Julia set and such that* $\Gamma(P) = \Gamma$ *is a (possibly reducible) special curve* $C(\Gamma)$.

Moreover for any irreducible component C_i *of* $C(\Gamma)$, *we have* $\Gamma(C_i) = \Gamma$.

In other words, there is a natural one-to-one correspondence between large classes of critically marked dynamical graphs and of special curves.

8.5.1. Wringing deformations and marked dynamical graphs

Let us review the construction of wringing deformations by Branner and Hubbard. We refer to [**30**, Chapter II] for details on the construction which relies on quasiconformal deformation techniques.

Let P be any monic and centered complex polynomial of degree $d \geq 2$. For any $\tau := \rho + i\theta$ with $\rho > 0$ (i.e. $i\tau \in \mathbb{H}$), we define the analytic diffeomorphism $\ell_\tau : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \overline{\mathbb{D}}$ by setting

$$\ell_\tau(e^{r+i\psi}) := \exp(r\rho + i(\theta r + \psi)),$$

so that

$$\ell_\tau^{-1}(e^{r+i\psi}) = \exp(r\rho^{-1} + i(-\theta\rho^{-1}r + \psi)).$$

Denote by σ_0 the standard complex structure on $\mathbb{C}$. We let σ_τ be the unique measurable almost complex structure on the complex plane satisfying the following conditions:

- $\sigma_\tau = \sigma_0$ on $K(P)$;
- $\sigma_\tau = \varphi_P^* \ell_\tau^*(\sigma_0)$ on $\{g_P > G(P)\}$;
- σ_τ is invariant under P.

It follows from [**30**, Proposition 6.1] that there exists a unique quasiconformal map $h_\tau : \mathbb{P}^1 \to \mathbb{P}^1$ solving the Beltrami equation

$$\frac{\partial h_\tau}{\partial \bar{z}} = \mu_\tau \frac{\partial h_\tau}{\partial z}$$

and such that

- $h_\tau(\infty) = \infty$;
- $P_\tau := h_\tau \circ P \circ h_\tau^{-1}$ is a monic and centered polynomial of degree d;
- the map $\ell_\tau \circ \varphi_P \circ h_\tau^{-1}$ is conformal in a neighborhood of infinity and satisfies $\ell_\tau \circ \varphi_P \circ h_\tau^{-1}(z) = z + O(1)$.

We infer from [**30**, Proposition 7.2] that for each z the map $\tau \mapsto h_\tau(z)$ is analytic in τ, and that the family of polynomials P_τ is also analytic. When $K(P)$ is connected, then $P_\tau = P$ for all τ, see [**30**, Proposition 8.3]. When K_P

is disconnected, then the map $\tau \mapsto P_\tau$ is not constant in the space of monic centered polynomials of degree d, see [**30**, Proposition 8.4].

Let us now mark the critical points of P so that $\mathrm{Crit}(P) = \{c_0, \dots, c_{d-2}\}$. Since being critical is a purely topological property, the critical points of P_τ are given by $c_i(\lambda) := h_\tau(c_i)$ so that P_τ is also critically marked. We may thus talk about the critically marked dynamical graph $\Gamma(P_\tau)$ for all $\tau \in -i\mathbb{H}$.

Recall the definition of the symmetry group $\Sigma(P)$ from §3.1 and its associated morphism $\rho_P \colon \Sigma(P) \to \Sigma(P)$ such that $P \circ g = \rho_P(g) \circ P$.

Proposition 8.31. — *For any critically marked monic and centered polynomial P, and for any $\tau \in -i\mathbb{H}$, we have $\Sigma(P) = \Sigma(P_\tau)$, $\rho_P = \rho_{P_\tau}$, and $\Gamma(P) = \Gamma(P_\tau)$.*

Proof. — Since P and P_τ are conjugated by a homeomorphism of the plane, condition (2) from Lemma 8.8 is satisfied so that it is sufficient to prove that $\Sigma(P) = \Sigma(P_\tau)$ and $\rho_P = \rho_{P_\tau}$.

For any $g \in \Sigma(P)$ and any $\tau \in -i\mathbb{H}$, we set $g_\tau = h_\tau \circ g \circ h_\tau^{-1}$. We claim that g_τ is holomorphic. This shows that $g \mapsto g_\tau$ defines an injective morphism $\Sigma(P) \to \Sigma(P_\tau)$. Reversing the argument we get that the morphism is bijective, and it is clear that $P_\tau \circ g_\tau = g_{\rho(\tau)} \circ P_\tau$.

It thus remains to prove the claim. Let N be the minimal integer so that $\Sigma(P) \supset \rho(\Sigma(P)) \supset \cdots \supset \rho^N(\Sigma(P)) = \rho^{N+1}(\Sigma(P))$.

Pick any $g \in \rho^N(\Sigma(P))$. Observe that ρ induces a group isomorphism of $\rho^N(\Sigma(P))$. Replacing P by a suitable iterate we may thus suppose that $P \circ g = g \circ P$, and write $g(z) = \zeta z$ for some $\zeta^d = \zeta$.

Observe that ℓ_τ and M_d commute so that $\varphi_\tau := \ell_\tau \circ \varphi_P \circ h_\tau^{-1}$ satisfies $\varphi_\tau \circ P_\tau = (\varphi_\tau)^d$, $\varphi_\tau(z) = z + O(1)$ hence is the Böttcher coordinate of P_τ. In fact, since P_τ is also monic and centered then $\varphi_\tau(z) = z + O(z^{-1})$, and $h_\tau(z) = \ell_\tau(\varphi_P(z)) + O(z^{-1}) = \ell_\tau(z) + O(z^{-1})$.

Let us look at $g'_\tau = \varphi_\tau \circ g_\tau \circ \varphi_\tau^{-1}$. Then the equation $P_\tau g_\tau = g_\tau P_\tau$ translates as

$$(\varphi_\tau g_\tau \varphi_\tau^{-1})^d(z) = \varphi_\tau g_\tau P_\tau \varphi_\tau^{-1} = \varphi_\tau g_\tau \varphi_\tau^{-1}(z^d)$$

so that

$$(g'_\tau(z))^d = g'_\tau(z^d) = h_\tau(\zeta h_\tau^{-1}(z^d)) = \ell_\tau(\zeta \ell_\tau^{-1}(z^d)) = \zeta z^d = (\zeta z)^d \ .$$

which implies g'_τ and g_τ to be holomorphic near infinity. Using $P_\tau g_\tau = g_\tau P_\tau$ again, one obtains that g_τ is holomorphic outside the filled-in Julia set. Since by construction the Beltrami coefficient of h_τ is also 0 on $K(P_\tau)$, it follows that g_τ is holomorphic everywhere so that the claim is proved for any $g \in \rho^N(\Sigma(P))$.

One now prove by decreasing induction on n that g_τ is holomorphic for all $g \in \rho^n(\Sigma(P))$. Suppose this is proved for some $n \leq N$ and pick $g \in \rho^{n-1}(\Sigma(P))$. Then we have $P_\tau \circ g_\tau = g_{\rho(\tau)} \circ P_\tau$, and by the inductive hypothesis $g_{\rho(\tau)}$ is holomorphic. Outside finitely many points, one may locally find an inverse branch of P_τ and write $g_\tau = P_\tau^{-1} \circ g_{\rho(\tau)} \circ P_\tau$ so that g_τ is holomorphic. Since it is continuous, it is holomorphic everywhere which concludes the proof. □

Observe that the previous proof uses quasi-conformal conjugacies in families in a fixed wringing plaque. We conjecture the following more general result.

Conjecture 4. — *Let P and Q be degree $d \geq 2$ polynomials. Assume there exists a quasi-conformal homeomorphism $h : \mathbb{P}^1 \to \mathbb{P}^1$ which conjugates P to Q. Then we have the equalities $\Sigma(P) = \Sigma(Q)$, $\rho_P = \rho_Q$ and $\Gamma(P) = \Gamma(Q)$.*

8.5.2. Proof of Theorem 8.30

We shall work in the affine space $\mathcal{M} := \mathbb{C}^{d-1}$ with critically marked polynomials of the form $P_{c,a}(z) = \frac{1}{d}z^d + \sum_{j=2}^{d-1}(-1)^{d-j}\sigma_{d-j}(c)\frac{z^j}{j} + a^d$. There is a natural finite ramified cover from this space onto $\mathrm{MPoly}^d_{\mathrm{crit}}$.

Pick any special asymmetric critically marked dynamical graph Γ. Let us denote by $\mathcal{Z}$ the set of parameters $(c, a) \in \mathcal{M}$ such that $P_{c,a}$ has degree d, $\Gamma(P_{c,a}) = \Gamma$ and $J_{P_{c,a}}$ is disconnected, and let Z be the Zariski closure of $\mathcal{Z}$ in $\mathcal{M}$.

By Theorem 8.15, $\mathcal{Z}$ is non-empty, therefore Z is a non-empty algebraic variety.

We start with the following observation.

Lemma 8.32. — *The algebraic subvariety Z is of pure dimension* 1, *and $\Gamma(W)$ is special for any irreducible component W of Z.*

Proof. — Pick any irreducible component W of Z. Let A (resp. P) be the set of active (resp. passive) critical points of the family induced by W. Since W is a component of Z, one can find an infinite set of polynomials $P_k \in W$ such that $\Gamma(P_k) = \Gamma$. Note that a priori $\Gamma(P_k)$ may be different from $\Gamma(W)$.

Since P_k have all the same critically marked dynamical graph, the set $\mathsf{A}_\star$ of critical points of P_k with infinite orbit does not depend on k, and $\mathsf{A}_\star \subset \mathsf{A}$. Moreover for each $i \notin \mathsf{A}_\star$ one can find integers $m > n \geq 0$ such that $P_k^n(c_i) = P_k^m(c_i)$ for all k, hence $\mathsf{A} = \mathsf{A}_\star$. Now since Γ is special, $\mathsf{A}_\star$ is non-empty and for each $i, j \in \mathsf{A}_\star$, there exist integers n and m such that $P^n(c_i) = P^m(c_j)$. This implies $\Gamma(W)$ to have at most one infinite connected component so that this graph is special. Note that these arguments prove in particular that W cannot be of dimension 0 (one can also use Proposition 8.31 to see this).

Fix any critical point, say c_0 such that $P^n(c_0) = P^m(c_j)$ implies $m \geq n$. It follows that the two functions $G := G(P_{c,a})$ and $g_0(c,a) := g_{P_{c,a}}(c_0)$ coincide on W. Suppose by contradiction that $\dim(W) = l \geq 2$. Since G is continuous, psh and its growth at infinity is $G(c,a) = \log^+ \max\{|c|, |a|\} + O(1)$ and $G(c,a) - \log^+ \max\{|c|, |a|\}$ extends continuously at infinity in $\bar{\mathcal{M}} = \mathbb{P}^{d-1}$, it induces a continuous semi-positive metrization on $\mathcal{O}(1)_{\bar{W}}$ where $\bar{W}$ is the closure of W in $\mathbb{P}^{d-1}_{\mathbb{C}}$. It follows that the mass of $(dd^c G|_W)^l$ is positive equal to $\deg(W)$, so that the positive closed $(2,2)$-current $(dd^c G|_W)^2$ is non-zero. On the other hand, we have

$$(dd^c G|_W)^2 = \lim_{\epsilon \to 0} dd^c g_0 \wedge dd^c \max\{g_0, \epsilon\} = 0$$

since the current $dd^c g_0$ is supported on $\{g_0 = 0\}$ by [**68**, Proposition 6.9], which gives the required contradiction. This proves $\dim(W) = 1$ which concludes the proof. □

Note that by Theorem 8.14 the curve W is special. It remains to prove that $\Gamma(W) = \Gamma$. Pick any polynomial $P \in W$ for which $\Gamma(P) = \Gamma$, and which is a smooth point of $Z \supset W$. Note that any polynomial $P_{c,a}(z) = \frac{1}{d} z^d - \sigma_1(c) \frac{z^{d-1}}{d-1} + o(z^{d-1})$ is conjugated by $z \mapsto \delta z + \frac{\sigma_1(c)}{d-1}$ for some fixed $\delta^{d-1} = d$ to a monic and centered polynomial, so that we have a finite ramified cover from $\mathcal{M}$ onto the space of monic and centered polynomials. It follows from Proposition 8.31 that the lift of the wringing plaque to $\mathcal{M}$ yields a holomorphic disk $\mathbb{D}_P$ which contains P and such that $\Gamma(Q) = \Gamma$ for all $Q \in \mathbb{D}_P$. We get $\mathbb{D}_P \subset Z$, and since P is a regular point of Z, we conclude that $\mathbb{D}_P \subset W$ which implies $\Gamma(W) = \Gamma$ as required.

8.6. Realizability of PCF maps

The applicability of Theorem 8.30 for a graph relies on our ability to realize a given finite critically marked dynamical graph as the marked graph of a PCF polynomial (up to symmetry). The following definite result was recently obtained by Floyd, Kim, Koch, Parry and Saenz.

Theorem 8.33 **([84]).** — *Any finite critically marked dynamical graph of degree $d \geq 2$ is realizable by a PCF polynomial.*

Recall that a finite critically marked dynamical graph Γ without symmetry is realizable by a PCF polynomial if there exists a PCF polynomial P such that Γ is equal to $\Gamma(P)$ with the action of $\Sigma(P)$ removed. The strategy of Floyd et al is to first realize the graph by a finite topological branched cover of the affine plane, and then to show that this topological cover can be made holomorphic using Thurston's characterization of rational maps.

In this section, we present two combinatorial approaches to this delicate problem that lead to weaker results. First in the case of a single marked point, the realizability amounts to the existence of a solution to a polynomial equation which have been studied in details by Buff [**33**]. As we shall see in §8.6.1 below, a simple count then gives:

Proposition 8.34. — *Any finite critically marked dynamical graph of degree $d \geq 2$ with a single marked point is realizable by a PCF polynomial.*

The situation is much harder in the presence of several marked points.

Building on the realization theorem of Bielefeld, Fisher, Hubbard [**25**] of strictly PCF polynomials with prescribed combinatorics, we prove the realizability of a large class of PCF polynomials that we describe now. Note that the results in [**25**] were was further expanded by Kiwi [**112**] and has later lead Poirier to a complete combinatorial classification of all PCF polynomials [**140**].

Let us introduce the following numerical invariant attached to a critically marked dynamical graph. Enumerate the marked vertices of Γ such that if v_j lies in the orbit of v_i then $j \geq i$. For each i, let n_i be the minimum integer so that either $\pi^{n_i}(v_i)$ is periodic, or belongs to the orbit of v_j for some $j < i$. We set $\delta(\Gamma)$ to be the minimum of those positive numbers of the form $\max\{d^{-n_1}, \sum_{i\geq 2} d^{-n_i}\}$ over all orderings of marked vertices satisfying the condition above.

We shall prove:

Proposition 8.35. — *Suppose that Γ is a finite critically marked dynamical graph such that no two distinct cycles have the same period. If $\delta(\Gamma) \leq \frac{1}{2d}$, then Γ is realizable by a PCF polynomial.*

The condition on $\delta(\Gamma)$ requires all marked vertices to be strictly preperiodic (so that the polynomial realizing Γ is strictly PCF). Note that the condition is satisfied when we have $n_i \geq 3$ for all integers defined above since $\sum_{i\geq 2} d^{-n_i} \leq \frac{(d-2)}{d^3} < \frac{1}{2d}$.

The proof of this result is based on the realization theorem of Bielefeld, Fisher, Hubbard [**25**] of strictly PCF polynomials with prescribed combinatorics. This theorem was further expanded by Kiwi [**112**] and has later lead Poirier to a complete combinatorial classification of all PCF polynomials [**140**].

8.6.1. Proof of Proposition 8.34

Write $P_c(z) = cz^d + 1$. We need to show that for any $n \geq 0$ and $k \neq 1$ there exists one $c \in \mathbb{C}$ such that 0 is mapped by P_c^k to a periodic point of exact period n and $P_c^{k-1}(0)$ is not periodic.

Let us first treat the case $k = 0$. By [**33**, Lemma 3] the following holds. For any integer n, the polynomial $Q_n(c) := P_c^n(0)$ has degree $\frac{d^{n-1}-1}{d-1}$ and its roots are simple. Observe that $Q_n^{-1}(0)$ is the set of $c \in \mathbb{C}$ such that the critical point 0 is periodic for P_c of period divisible by n. By Möbius inversion formula, it follows that the cardinality $\delta(n)$ of the set of $c \in \mathbb{C}$ such that the critical point 0 is periodic for P_c of exact period n is equal to

$$\delta(n) := \sum_{m \leq n} \mu\left(\frac{n}{m}\right) \frac{d^{m-1} - 1}{d - 1} \ .$$

Since $\sum_{m \leq n} \mu\left(\frac{n}{m}\right) = 0$ for all $n \geq 2$, we infer $\delta(n) > 0$ from the next lemma [(1)].

Lemma 8.36. — *For any $\rho \geq 2$ and for any $n \geq 1$, we have*

$$\sum_{m|n} \mu\left(\frac{n}{m}\right) \rho^m > 0 \ .$$

Proof. — We may assume $n \geq 2$, so that

$$\sum_{m|n} \mu\left(\frac{n}{m}\right) \rho^m \geq \rho^n - \sum_{1 \leq m \leq n/2} \rho^m \geq \rho^n - \left(\frac{\rho^{1+n/2} - 1}{\rho - 1}\right)$$
$$> \frac{\rho^n(\rho - 1) - \rho^{1+n/2}}{\rho - 1} \geq 0 \ ,$$

since $n \geq 1 + n/2$. □

Suppose now that k is positive. Observe that $k \geq 2$ since P_c is unicritical. Let $Q_{k,n}$ be the polynomial with simple roots whose zero locus is the set of $c \in \mathbb{C}$ such that 0 is mapped by P_c^k to a periodic point of exact period n and $P_c^{k-1}(0)$ is not periodic. We need to show that $\delta_{k,n} := \deg(Q_{k,n}) > 0$.

By [**33**, Lemma 10], we have

$$\frac{Q_{k+n-1}^d(c) - Q_{k-1}^d(c)}{Q_{k+n-1}(c) - Q_{k-1}(c)} = Q_{\gcd\{k-1,n\}}^{d-1} \prod_{m|n} Q_{k,m}(c)$$

---

(1) Our proof actually yields $\delta(n) \geq cd^n$ for some positive constant $c > 0$, see [**76**].

so that

$$\delta_{k,n} = \sum_{m|n} \mu\left(\frac{n}{m}\right) \left(d^{m+k-2} - d^{-1+\gcd\{k-1,m\}}\right) .$$

Write $l = k-1$. By the lemma below, we have

$$d \times \delta_{k,n} = \sum_{m|n} \mu\left(\frac{n}{m}\right) \left(d^{m+l} - d^{\gcd\{l,m\}}\right) \geq d^l \left(d^n - \left(\frac{d^{1+n/2}-1}{d-1}\right)\right) - d^l > 0 .$$

This concludes the proof.

Lemma 8.37. — *For any $\rho \geq 2$, for any integer $l \geq 1$, and for any $n \geq 1$, we have*

$$0 \leq \sum_{m|n} \mu\left(\frac{n}{m}\right) \rho^{\gcd\{l,m\}} \leq \rho^l .$$

Proof. — Set $\Delta(n) = \sum_{m|n} \mu\left(\frac{n}{m}\right) \rho^{\gcd\{l,m\}}$ so that $\rho^{\gcd\{l,n\}} = \sum_{m|n} \Delta(m)$. It is sufficient to show that $\Delta(n) \geq 0$ for all n. Observe that for each divisor n of l we have $\Delta(n) = \sum_{m|n} \mu\left(\frac{n}{m}\right) \rho^m$ so that $\Delta(n) \geq 0$ follows from the previous lemma. We claim that by induction $\Delta(n) = 0$ if n is not a divisor of l. Indeed write $\gcd\{n,l\} = L$. Then

$$\rho^L = \sum_{l'|L} \left(\Delta(l') + \left(\sum_{\substack{l'<m,\, m|n \\ \gcd\{m,l\}=l'}} \Delta(m) \right) \right) = \sum_{l'|L} \Delta(l') + \Delta(n)$$

hence $\Delta(n) = 0$ as claimed. □

8.6.2. Combinatorics of strictly PCF polynomials

We review briefly the classification of PCF polynomial in terms of critical portraits. A PCF polynomial is said to be strict when none of its critical point is periodic.

Critical portrait. — The notion of a critical portrait was introduced in [**25**] which encodes the combinatorics of a given polynomial. Recall that two finite disjoint subsets F, F' of $\mathbb{R}/\mathbb{Z}$ are said to be unlinked when F lies in a single connected component of $(\mathbb{R}/\mathbb{Z}) \setminus F'$. Two subsets are unlinked iff there exist disjoint open segments I and I' such that $F \subset I$ and $F' \subset I'$.

Definition 8.38. — *A critical portrait is a collection $(\Theta_1, \cdots, \Theta_N)$ of disjoint finite sets in $\mathbb{R}/\mathbb{Z}$ satisfying the following three conditions:*

(CP1) *for any fixed* i, $\Theta_i = \{\theta_{i,1}, \cdots, \theta_{i,d(i)}\}$ *with* $d(i) \geq 2$ *and* $d\theta_{i,j} = d\theta_{i,1}$ *for all* j;
(CP2) $\sum_i (\mathrm{Card}(\Theta_i) - 1) = d - 1$;
(CP3) *for any* $i \neq j$, *the sets* Θ_i *and* Θ_j *are unlinked.*

Landing map. — Let P be any polynomial of degree $d \geq 2$ with connected Julia set so that all its critical points have a bounded orbit. Recall from Proposition 2.22 that the Böttcher coordinate φ_P defines a univalent map from $\{g_P > 0\} = \mathbb{C} \setminus K_P$ onto $\{|z| > 1\}$. The external ray R_θ of angle $\theta \in \mathbb{R}/\mathbb{Z}$ is the image under φ_P^{-1} of the ray $\{te^{i\pi\theta}\}_{t>1}$. Equivalently external rays are gradient lines of the Green function. One says that the external ray R_θ lands at a point $z \in K_P$ if z is the unique boundary point of the ray.

When K_P is locally connected, it follows from Caratheodory theorem that all external rays land at a point, so that we get a continuous surjective map $\mathsf{e}\colon \mathbb{R}/\mathbb{Z} \to J_P$ which semi-conjugates M_d to P.

Critical portrait of PCF maps. — When P is a strictly PCF polynomial with critical points $c_1, \ldots, c_N$, its Julia set is connected and locally connected (see [**127**, Theorem 17.5]). For each i, one may thus choose a ray R_{η_i} landing at $P(c_i)$, and set $\Theta_i = M_d^{-1}(\eta_i)$. One can prove that the collection of sets $\{\Theta_i\}_{1\leq i\leq N}$ defines a critical portrait, see [**25**, Proposition 2.10]. Note that the critical portrait is only well-defined once a choice of external rays landing at critical values is made.

To state the next result we recall the definition the Θ-equivalence relation given in [**25**]:

- $x, y \in \mathbb{R}/\mathbb{Z}$ are Θ-unlinked if $\{x, y\}$ and Θ_i are unlinked for all i;
- x and y are Θ-unlinkable if one can find a pair $\{x', y'\}$ arbitrarily close to $\{x, y\}$ which is Θ-unlinked;
- x and y are Θ-equivalent if for all $n \geq 0$ the points $d^n x$ and $d^n y$ are Θ-unlinkable.

We shall write $x \sim_\Theta y$ whenever x and y are Θ-equivalent.

Theorem 8.39. — *Suppose* $\Theta = \{\Theta_1, \cdots, \Theta_N\}$ *is a critical portrait such that any* $\theta \in \bigcup_i \Theta_i$ *is strictly preperiodic for* M_d. *Then the landing map* $\mathsf{e}\colon \mathbb{R}/\mathbb{Z} \to J_P$ *induces a conjugacy* $(\mathbb{R}/\mathbb{Z}/\sim_\Theta, M_d) \to (J_P, P)$.

Realization of a critical portrait. — The next result follows from [**25**, Theorem II], see also [**112**, Corollary 5.3].

Theorem 8.40. — *Suppose $\Theta = \{\Theta_1, \cdots, \Theta_N\}$ is a critical portrait such that any $\theta \in \bigcup_i \Theta_i$ is strictly preperiodic for M_d.*

Then there exist a strictly PCF polynomial P_Θ and a choice of external rays landing at its critical values for which the critical portrait is equal to Θ.

Moreover, for each i all external rays R_θ with $\theta \in \Theta_i$ land at the same critical point, and c_i which has multiplicity $\mathrm{Card}(\Theta_i)$.

We now explain how to understand the critically marked dynamical graph of P_Θ from its critical portrait. Pick an angle $\theta_i \in \Theta_i$ for each i, and consider the dynamical graph $\Gamma_\Theta := G(\{\theta_1, \ldots, \theta_N\}, M_d)$ as follows (see Example 8.4). Vertices are $\{M_d^n(\theta_i)\}_{i,n\geq 0}$ and an edge joins two vertices v and v' when $M_d(v) = v'$ or $M_d(v') = v$. The map M_d induces a natural flow on Γ. By (CP2), we may partition $\{0, \cdots, d-2\}$ into N subsets $I_1, \ldots, I_N$ such that each set I_j has the same cardinality as Θ_j. We define the marking $\mu\colon \{0, \cdots, d-2\} \to \Gamma$ sending each integer $i \in I_j$ to θ_j. We obtain in this way a critically marked dynamical graph (with no symmetry) that we denote by Γ_Θ.

Since the landing map is a semi-conjugacy between M_d and P_Θ, Theorem 8.40 implies e to induce a canonical map $\mathsf{e}\colon \Gamma_\Theta \to \Gamma(P_\Theta)$ which is surjective by construction and semi-conjugates the flows.

Proposition 8.41. — *If Γ_Θ and $\Gamma(P_\Theta)$ have the same number of cycles of any given period, then e induces an isomorphism between Γ_Θ and the critically dynamical graph without symmetry obtained from $\Gamma(P_\Theta)$ by forgetting the action of $\Sigma(P_\Theta)$.*

Proof. — We show that $\mathsf{e}\colon \Gamma_\Theta \to \Gamma(P_\Theta)$ induces a conjugacy. For each n denote by $\Gamma_{\Theta,n}, \Gamma(P_\Theta)_n$ the dynamical subgraphs of Γ_Θ and $\Gamma(P_\Theta)$ respectively of those points at (graph) distance $\leq n$ from the periodic cycles.

Since both graphs are finite, for n sufficiently large we have $\Gamma_{\Theta,n} = \Gamma_\Theta$ and $\Gamma(P_\Theta)_n = \Gamma(P_\Theta)$. Observe that e maps $\Gamma_{\Theta,n}$ onto $\Gamma(P_\Theta)_n$. Also $\Gamma_{\Theta,0}$ and $\Gamma(P_\Theta)_0$ are unions of periodic cycles, and by assumption $\mathsf{e}\colon \Gamma_{\Theta,0} \to \Gamma(P_\Theta)_0$ is a conjugacy.

We assume by induction that $\mathsf{e}\colon \Gamma_{\Theta,n} \to \Gamma(P_\Theta)_n$ is a conjugacy. Pick $\theta, \theta' \in \Gamma_{\Theta,n+1}$ such that $\mathsf{e}(\theta) = \mathsf{e}(\theta')$. By the inductive assumption $M_d(\theta) = M_d(\theta')$, so that the two angles differ by a multiple of $\frac{1}{d}$. We conclude using the next lemma. □

Lemma 8.42. — *Let $\theta \neq \theta' \in \mathbb{R}/\mathbb{Z}$ be two angles such that $\theta - \theta' \in \frac{\mathbb{Z}}{d}$, and $\theta \sim_\Theta \theta'$. Then $\{\theta, \theta'\} \subset \Theta_i$ for some i.*

Proof. — Consider the equivalence relation on $(\mathbb{R}/\mathbb{Z}) \setminus \bigcup_i \Theta_i$ defined by $x \equiv_\Theta y$ iff $\{x, y\}$ is Θ-unlinked. Equivalent classes for $\equiv_\Theta$ are in bijection with connected components of the complement in the unit disk of the union of the convex hulls of Θ_i (see Figure below).

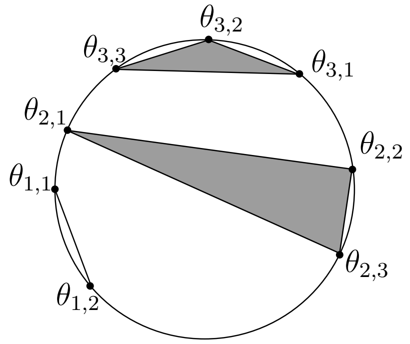


FIGURE 11. A critical portrait

In particular, there is exactly $(d-1)$ equivalent classes, and each of them is a finite union of open segments $\cup_{i=1}^k I_i$ that we may order cyclically clockwise. We observe now that there is a unique translation T_2 by a multiple of $\frac{1}{d}$ such that $T_2(I_2)$ has one common boundary point with I_1. Proceeding inductively we find translations T_j such that $T_j(I_i)$ has one common boundary point with $T_{j-1}(I_{j-1})$, so that the union $\overline{I_1} \cup \overline{T_j(I_j)}$ is a segment of length $\frac{1}{d}$.

If θ and θ' differ by a multiple of $\frac{1}{d}$ and none of them belong to $\bigcup_i \Theta_i$, then θ and θ' cannot belong to the same $\equiv_\Theta$-equivalence class so that in particular we have $\theta \not\sim_\Theta \theta'$. In fact if $\theta \in \Theta_i$ and $\theta \sim_\Theta \theta'$, then $\theta' \in \Theta_i$ too. □

8.6.3. Proof of Proposition 8.35

Let Γ be a critically marked dynamical finite graph as in the statement of the theorem. We may suppose $d \geq 3$ by Proposition 8.34. Recall the definition of a marked dynamical graph attached to an arbitrary dynamical system from §8.2.2.

Step 1. Choosing adapted periodic orbits for M_d.

It is convenient to work in d-ary base for points in $\mathbb{R}/\mathbb{Z}$. If $p = \sum_{k\geq 1} \frac{\epsilon_k}{d^k}$ with $\epsilon_k \in \{0, \ldots, d-1\}$ then we have $M_d(p) = \sum_{k\geq 1} \frac{\epsilon_{k+1}}{d^k}$.

Let $n_1, \ldots, n_r$ be the period of each periodic cycle of Γ. By assumption, we have $n_i \neq n_j$ for $i \neq j$. For each $i = 1, \ldots, r$, define

$$p_i := \left(\sum_{k=1}^{n_i} \frac{\epsilon_k^{(i)}}{d^k} \right) \cdot \frac{d}{d-1} ,$$

where $\epsilon_k^{(i)} = 2$ if $(k-1)|n_i$ and $\epsilon_k^{(i)} = 0$ otherwise.

Observe that $\frac{d}{d-1} = \sum_{k\geq 0} \frac{1}{d^k}$ hence p_i is periodic of period divisible by n_i. By construction the first digit of $M_d^k(p_i)$ when $k|n_i$ and $k < n_i$ is 2, whereas the first digit of p_i is 0, hence the exact period of p_i is n_i.

Lemma 8.43. — *For each i and for each each divisor m of n_i such that $m < n_i$, we have $d_{\mathbb{R}/\mathbb{Z}}(M_d^m(p_i), p_i) \geq \frac{1}{d}$.*

Proof. — Recall that for any two points $z, w \in [0,1]$, we have $d_{\mathbb{R}/\mathbb{Z}}(z,w) = \min\{|z-w|, 1-|z-w|\}$. To simplify notation we drop the index i, and estimate the distance between p and $M_d^m(p)$ for m dividing n. We have

$$\begin{aligned}\tau := d^m p - p \mod 1 &= \left(\sum_{k=1}^{n} \frac{\epsilon_{k+m} - \epsilon_k}{d^k}\right) \cdot \frac{d}{d-1} \\ &= \left(\frac{2}{d} + \sum_{k=2}^{n} \frac{\epsilon_{k+m} - \epsilon_k}{d^k}\right) \cdot \frac{d}{d-1}\ .\end{aligned}$$

We thus get the lower bound

$$\tau \geq \left(\frac{2}{d} - \frac{1}{d}\right) \cdot \frac{d}{d-1} = \frac{1}{d-1}\ .$$

To get an upper bound, we let r be the smallest integer ≥ 2 not dividing n. Then

$$\tau \leq \frac{2}{d} - \frac{2}{d^r} + \frac{1}{d^r} \leq \frac{2}{d}$$

hence

$$d_{\mathbb{R}/\mathbb{Z}}(M_d^m(p), p) = \min\{\tau, 1-\tau\} \geq \min\left\{\frac{1}{d-1}, 1 - \frac{2}{d}\right\}$$

which concludes the proof. □

Step 2. Find a critical portrait Θ such that $\Gamma_\Theta = \Gamma$.

We enumerate the marked points $\mu(\mathcal{F}) = \{v_1, \ldots, v_r\}$ in such a way that $\pi^n(v_i) = v_j$ for some $n \geq 0$ implies $i \geq j$.

Choose the first angle θ_1 with the same combinatorics as v_1. Let $n_1 \geq 2$ be the minimum integer such that $\pi^{n_1}(p_1)$ is periodic. Then there are $(d-1)$ possibilities for $\pi^{n_1-1}(p_1)$ at distance either $1/d$ or $2/d$ from one to the other. We thus have $(d-1)d^{n_1-1}$ possibilities for θ_1 at distance either $\frac{1}{d^{n_1}}$ or $\frac{2}{d^{n_1}}$. We may thus choose the angle such that $0 < \theta_1 < \frac{2}{d^{n_1}}$, and set $\Theta_1 = \{\theta_1, \ldots, \theta_1 + \frac{d_1-1}{d}\}$.

In the same way, we choose the angle θ_2 realizing the combinatorics of p_2, and such that $\theta_1 + \frac{d_1-1}{d} < \theta_2 < \theta_1 + \frac{d_1-1}{d} + \frac{2}{d^{n_2}}$, and set $\Theta_2 = \{\theta_2, \ldots, \theta_2 + \frac{d_2-1}{d}\}$.

Observe that

$$\theta_2 + \frac{d_2 - 1}{d} < \theta_1 + \frac{d_1 - 1}{d} + \frac{2}{d^{n_2}} + \frac{d_2 - 1}{d} \leq \theta_1 + \frac{d - 1}{d} + \frac{1}{d} \leq \theta_1 + 1$$

so that Θ_1 and Θ_2 are unlinked. We may thus define inductively a critical portrait $\Theta_1, \ldots, \Theta_N$ such that $\Theta_j = \{\theta_j, \ldots, \theta_j + \frac{d_j-1}{d}\}$, $\theta_{j-1} + \frac{d_j-1}{d} < \theta_j < \theta_{j-1} + \frac{d_j-1}{d} + \frac{2}{d^{n_j}}$ hence

$$\theta_N + \frac{d_N - 1}{d} < \theta_1 + \sum_{j=1}^{N} \frac{d_j - 1}{d} + \sum_{j=2}^{N} \frac{2}{d^{n_j}} \overset{\delta(\Gamma) \leq \frac{1}{2d}}{\leq} \theta_1 + \frac{d-1}{d} + \frac{1}{d} \leq \theta_1 + 1 \ .$$

By construction, we have $\Gamma_\Theta = \Gamma$.

Step 3. Proof that $\Gamma_\Theta = \Gamma(P_\Theta)$.

Recall that P_Θ is a PCF polynomial obtained by applying Theorem 8.40. We claim that Γ_Θ and $\Gamma(P_\Theta)$ have the same number of periodic cycles of any given period.

Then by Proposition 8.42, Γ_Θ is isomorphic to $\Gamma(P_\Theta)$ with the action of the symmetry group removed, and the proof is complete. It thus remains to prove our claim.

Pick any periodic point as in Step 1 of the proof. We drop the index i for simplicity. Then p has period n and its d-ary expansion is of the form $p = \frac{2}{d^2} + \sum_{k\geq 3} \frac{\epsilon_k}{d^k}$. Note that for any m dividing n, we have $M_d^m(p) = \frac{2}{d} + \sum_{k\geq 3} \frac{\epsilon_{k+m}}{d^k}$ mod 1. It follows that p belongs to the segment $(2/d^2, 3/d^2)$. Since $\theta_1 \in (0, \frac{2}{d^{n_1}})$ and $n_1 \geq 2$, we conclude that $p \in (\theta_1, \theta_1 + \frac{1}{d})$. Since $d_{\mathbb{R}/\mathbb{Z}}(M_d^m(p), p) \geq \frac{1}{d}$ by Lemma 8.43, $M_d^m(p)$ does not belong to the segment $(\theta_1, \theta_1 + \frac{1}{d})$ hence $\{M_d^m(p), p\}$ is not Θ-unlinked. In particular, $\mathsf{e}(M_d^m(p)) \neq \mathsf{e}(p)$ which implies the point $\mathsf{e}(p)$ to have exact period n.

This concludes the proof.

8.7. Special curves in low degrees

We discuss special curves of degree $d \leq 5$. Observe that (up to a finite branched cover) the only non-isotrivial one-parameter family of degree 2 polynomials is a curve: the family $P_t(z) = z^2 + t$, and it forms a special family.

Recall that we denoted by $\Sigma(d, k, \mu)$ the set of monic centered polynomials of degree $d \geq 2$ which can be written as $z^\mu Q(z^k)$ with $k \geq 2$ maximal and $Q(0) \neq 0$.

Classification in degree 3. — The special curves of cubic polynomials can be classified in the following way:

- either one critical point is preperiodic;
- or the two distinct critical points belong to the same grand orbit. Remark that the unicritical family $\Sigma(3,3,0)$ is a particular example where this happens.
- or the curve is the Zariski (or Euclidean) closure of $\Sigma(3,2,1)$: for a general polynomial P in the curve, $\Sigma(P) = \mathrm{Aut}(P) = \mathbb{U}_2$. This curve can be parametrized by $P_t(z) = z(z^2+t)$, $t \in \mathbb{A}^1$.

One has the following combinatorial classification in degree 3: if the curve is not $\Sigma(3,2,1)$, then the dynamical graph is asymmetrical and we have a correspondence special curves/dynamical graphs as stated in the Introduction (Theorem 4). We refer to Figure 4 for the description of special marked dynamical graphs of degree 3.

Classification in degree 4. — The special curves of degree 4 polynomials can be classified in the following way:

- either the curve is non-primitive: it can be parametrized as $P(z) = z^4 + az^2 + c$, with $4c = a^2 - 2a\zeta$ and $\zeta^3 = -1$.
- or two critical points are preperiodic,
- or one critical point is periodic, and the other two lie in the same grand orbit,
- or the three distinct critical points belong to the same grand orbit. Remark that the unicritical family $\Sigma(4,4,0)$ is a particular example where this happens,
- or the curve is the Zariski (or Euclidean) closure of a curve in $\Sigma(4,2,0)$ such that $\Sigma(P) = \mathrm{Aut}(P) = \mathbb{U}_2$, two critical points are permuted by a symmetry and
  - either the third critical point is preperiodic,
  - or the third critical point shares the same grand orbit as one of the swapped critical points.
- or the curve is the Zariski closure of $\Sigma(4,3,1)$: for a general polynomial P in the curve, the three critical points are permuted by $\mathbb{U}_3$. This curve can be parametrized by $P_t(z) = z(z^3+t)$, $t \in \mathbb{A}^1$,
- or the curve is the Zariski closure of $\Sigma(4,2,2)$: for a general polynomial P in the curve, one critical point is fixed and the other two are swapped by the $\mathbb{U}_2$.This curve can be parametrized by $P_t(z) = z^2(z^2+t)$, $t \in \mathbb{A}^1$.

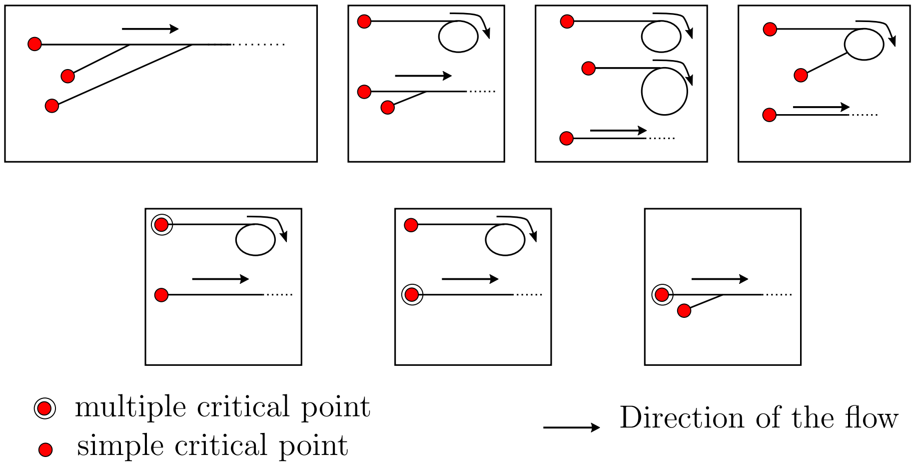


FIGURE 12. Asymmetrical special critically marked dynamical graphs of degree 4

Classification in degree 5. — Since 5 is prime, all special curves are primitive. The special curves of degree 5 polynomials can be classified in the following way:

- three critical points are preperiodic,
- two critical points are preperiodic and the other two lie in the same grand orbit,
- one critical point is preperiodic and the other three lie in the same grand orbit,
- all four critical points lie in the same grand orbit. A particular example is the closure of $\Sigma(5,5,0)$ which is the unicritical family and can be parametrized by $P_t(z) = z^5 + t$, $t \in \mathbb{A}^1$,
- the curve is the closure of $\Sigma(5,2,3)$: for a general P in the curve, $\Sigma(P) = \mathrm{Aut}(P) = \mathbb{U}_2$, two critical point are fixed and equal and the other two critical points are permuted by a symmetry,
- the curve is the closure of $\Sigma(5,3,2)$: for a general P in the curve, $\Sigma(P) = \mathbb{U}_3$ and $\mathrm{Aut}(P) = \mathbb{U}_1$, one critical point is fixed and the other three critical points are permuted by a symmetry,
- the curve is the closure of $\Sigma(5,4,1)$: for a general P in the curve, $\Sigma(P) = \mathrm{Aut}(P) = \mathbb{U}_4$ and all four critical points are permuted by a symmetry,
- the curve is the closure of a curve contained in $\Sigma(5,2,1)$ such that, for a general P in the curve, $\Sigma(P) = \mathrm{Aut}(P) = \mathbb{U}_2$, critical points are permuted by a symmetry by pairs and one critical point in each pair share the same grand orbit.

8.8. Open questions on the geometry of special curves

We know very little about the geometry of special curves.

(SC1) Are special curves smooth in the space of monic and centered polynomials with marked critical points? Beware that this space is a finite cover of the moduli space of critically marked polynomials, and is isomorphic to $\mathbb{A}^{d-1}$ (it is the quotient of $\{P_{c,a}\}$ by the action of $\mathbb{U}_d$ defined by $\zeta \cdot (c, a) = (c, \zeta a)$).

In degree 3, Milnor [**129**, §5] proved that the curve $\mathcal{S}_p$ for which the marked critical point has exact period p is smooth.

(SC2) What is the Euler characteristic and the genus of special curves? Again in degree 3, for curves for which one critical point is periodic, Bonifant, Kiwi and Milnor [**28**] proved that the Euler characteristic of the curve $\mathcal{S}_p$ satisfies

$$\chi(\mathcal{S}_p) = \deg(\mathcal{S}_p) \cdot (2 - p) + N_p$$

where N_p is the number of branches at infinity of $\mathcal{S}_p$. Dujardin [**66**] subsequently showed that $3^{-p}\chi(\mathcal{S}_p) \to -\infty$, as $p \to +\infty$. DeMarco-Schiff [**59**] gave an algorithm to compute N_p for all $p \geq 1$, and implemented it for $p \leq 26$.

(SC3) Milnor [**129**] also conjectured that in degree 3, the curves $\mathcal{S}_p$ are connected (or equivalently irreducible since they are smooth). Pick any integer $d \geq 3$ and any special marked dynamical graph Γ. Is the curve $C(\Gamma)$ connected (or irreducible)?

(SC4) Estimate (or better compute) the degree of $C(\Gamma)$ in terms of the geometry of their special marked dynamical graph Γ? Note that, in degree 3, for any integer p, the graph $\Gamma(\mathcal{S}_p)$ consists of a single loop of length p, together with an infinite half-line, so that their degrees satisfy $\deg(\mathcal{S}_p) \sim \alpha 3^p$ for some constant $\alpha > 0$.

(SC5) How are special curves distributed in the moduli space of critically marked degree d polynomials (or equivalently in $\{P_{c,a}\}$)?

For each $i = 0, \cdots, d-2$, define the bifurcation current of the i-th critical point by $T_{\mathrm{bif},i} = dd^c g_i(c, a)$. For each multi-index $I = (i_0, \cdots, i_{d-2})$ define

$$T^I_{\mathrm{bif}} = T^{i_0}_{\mathrm{bif},0} \wedge \cdots \wedge T^{i_{d-2}}_{\mathrm{bif},d-2} \ .$$

Recall that $T^I_{\mathrm{bif}} = 0$ iff $i_j \geq 2$ for some j or $|I| \geq d$. Recall also that an algebraic subvariety is said to be special if it contains a Zariski dense subset of PCF polynomials.

Now suppose that Γ_k is a sequence of special marked dynamical graphs such that the special curve $C_k = C(\Gamma_k)$ is well-defined (by Theorem 8.30). Is any weak limit of the sequence of closed positive $(d-2, d-2)$-currents

$$\frac{1}{\deg(C_k)}[C_k]$$

equal to some bifurcation current $T^I_{\mathrm{bif}} \wedge [Z]$ for some multi-index I and some special algebraic subvariety Z such that $|I| + \mathrm{codim}(Z) = d - 2$?

This is proved in degree 3 by Dujardin-Favre [**68**] for curves for which one critical point is preperiodic.

(SC6) Can we remove the assumption on marked points to have distinct images from Theorems 8.15 and 8.30? In other words, are all asymmetric special marked dynamical graph realizable? Can one extend the correspondence to graphs with no-trivial symmetries?

(SC7) Let Γ be any special marked dynamical graph. Is it true that the euclidean closure of the set of polynomials having Γ as marked dynamical graph is a special curve?

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BIBLIOGRAPHY

[1] Lars V. Ahlfors. *Complex analysis.* McGraw-Hill Book Co., New York, third edition, 1978. An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics.

[2] Pascal Autissier. Points entiers sur les surfaces arithmétiques. *J. Reine Angew. Math.*, 531:201–235, 2001.

[3] Irvine N. Baker and Alexandre Erëmenko. A problem on Julia sets. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 12(2):229–236, 1987.

[4] Matthew Baker. A finiteness theorem for canonical heights attached to rational maps over function fields. *J. Reine Angew. Math.*, 626:205–233, 2009.

[5] Matthew Baker and Laura DeMarco. Preperiodic points and unlikely intersections. *Duke Math. J.*, 159(1):1–29, 2011.

[6] Matthew Baker and Laura DeMarco. Special curves and postcritically finite polynomials. *Forum Math. Pi*, 1:e3, 35, 2013.

[7] Matthew Baker, Sam Payne, and Joseph Rabinoff. On the structure of non-Archimedean analytic curves. In *Tropical and non-Archimedean geometry*, volume 605 of *Contemp. Math.*, pages 93–121. Amer. Math. Soc., Providence, RI, 2013.

[8] Matthew Baker, Sam Payne, and Joseph Rabinoff. Nonarchimedean geometry, tropicalization, and metrics on curves. *Algebr. Geom.*, 3(1):63–105, 2016.

[9] Matthew Baker and Robert Rumely. Equidistribution of small points, rational dynamics, and potential theory. *Ann. Inst. Fourier (Grenoble)*, 56(3):625–688, 2006.

[10] Matthew Baker and Robert Rumely. *Potential theory and dynamics on the Berkovich projective line*, volume 159 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[11] Giovanni Bassanelli and François Berteloot. Bifurcation currents in holomorphic dynamics on $\mathbb{P}^k$. *J. Reine Angew. Math.*, 608:201–235, 2007.

[12] Eric Bedford, Mikhail Y. Lyubich, and John Smillie. Polynomial diffeomorphisms of $\mathbf{C}^2$. IV. The measure of maximal entropy and laminar currents. *Invent. Math.*, 112(1):77–125, 1993.

[13] Jason P. Bell, Yohsuke Matsuzawa, and Matthew Satriano. On dynamical cancellation. *arXiv e-prints*, 2021.

[14] Robert Benedetto, Patrick Ingram, Rafe Jones, Michelle Manes, Joseph H. Silverman, and Thomas J. Tucker. Current trends and open problems in arithmetic dynamics. *Bull. Amer. Math. Soc. (N.S.)*, 56(4):611–685, 2019.

[15] Robert L. Benedetto. Heights and preperiodic points of polynomials over function fields. *Int. Math. Res. Not.*, (62):3855–3866, 2005.

[16] Robert L. Benedetto. Wandering domains and nontrivial reduction in non-Archimedean dynamics. *Illinois J. Math.*, 49(1):167–193, 2005.

[17] Robert L. Benedetto. Wandering domains in non-Archimedean polynomial dynamics. *Bull. London Math. Soc.*, 38(6):937–950, 2006.

[18] Robert L. Benedetto. *Dynamics in One Non-Archimedean Variable*, volume 198 of *Graduate Studies in Mathematics*. McGraw-Hill Book Co., New York, 2019.

[19] Anna Miriam Benini. A survey on MLC, Rigidity and related topics. *arXiv e-prints*, 2017.

[20] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.

[21] Vladimir G. Berkovich. Étale cohomology for non-Archimedean analytic spaces. *Inst. Hautes Études Sci. Publ. Math.*, (78):5–161 (1994), 1993.

[22] Ricardo Berlanga. A mapping theorem for topological sigma-compact manifolds. *Compositio Mathematica*, 63(2):209–216, 1987.

[23] François Berteloot. Bifurcation currents in holomorphic families of rational maps. In *Pluripotential Theory*, volume 2075 of *Lecture Notes in Math.*, pages 1–93. Springer-Verlag, Berlin, 2013.

[24] François Berteloot and Volker Mayer. *Rudiments de dynamique holomorphe*, volume 7 of *Cours Spécialisés*. Société Mathématique de France, Paris, 2001.

[25] Ben Bielefeld, Yuval Fisher, and John H. Hubbard. The classification of critically preperiodic polynomials as dynamical systems. *J. Amer. Math. Soc.*, 5(4):721–762, 1992.

[26] Yuri Bilu. Limit distribution of small points on algebraic tori. *Duke Math. J.*, 89(3):465–476, 1997.

[27] E. Bombieri, D. Masser, and U. Zannier. On unlikely intersections of complex varieties with tori. *Acta Arith.*, 133(4):309–323, 2008.

[28] Araceli Bonifant, Jan Kiwi, and John Milnor. Cubic polynomial maps with periodic critical orbit. II. Escape regions. *Conform. Geom. Dyn.*, 14:68–112, 2010.

[29] Siegfried Bosch and Werner Lütkebohmert. Stable reduction and uniformization of abelian varieties. I. *Math. Ann.*, 270(3):349–379, 1985.

[30] Bodil Branner and John H. Hubbard. The iteration of cubic polynomials. I. The global topology of parameter space. *Acta Math.*, 160(3-4):143–206, 1988.

[31] Bodil Branner and John H. Hubbard. The iteration of cubic polynomials. II. Patterns and parapatterns. *Acta Math.*, 169(3-4):229–325, 1992.

[32] Hans Brolin. Invariant sets under iteration of rational functions. *Ark. Mat.*, 6:103–144 (1965), 1965.

[33] Xavier Buff. On postcritically finite unicritical polynomials. *New York J. Math.*, 24:1111–1122, 2018.

[34] Xavier Buff and Adam L. Epstein. From local to global analytic conjugacies. *Ergodic Theory Dynam. Systems*, 27(4):1073–1094, 2007.

[35] Xavier Buff and Adam L. Epstein. Bifurcation measure and postcritically finite rational maps. In *Complex dynamics : families and friends / edited by Dierk Schleicher*, pages 491–512. A K Peters, Ltd., Wellesley, Massachussets, 2009.

[36] Gregory S. Call and Joseph H. Silverman. Canonical heights on varieties with morphisms. *Compositio Math.*, 89(2):163–205, 1993.

[37] César Camacho. On the local structure of conformal mappings and holomorphic vector fields in $\mathbf{C}^2$. In *Journées Singulières de Dijon (Univ. Dijon, Dijon, 1978)*, volume 59 of *Astérisque*, pages 3, 83–94. Soc. Math. France, Paris, 1978.

[38] David G. Cantor. On an extension of the definition of transfinite diameter and some applications. *J. Reine Angew. Math.*, 316:160–207, 1980.

[39] Lennart Carleson and Theodore W. Gamelin. *Complex dynamics*. Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.

[40] Guy Casale. Enveloppe galoisienne d'une application rationnelle de $\mathbb{P}^1$. *Publ. Mat.*, 50(1):191–202, 2006.

[41] Antoine Chambert-Loir. Points de petite hauteur sur les variétés semi-abéliennes. *Ann. Sci. École Norm. Sup. (4)*, 33(6):789–821, 2000.

[42] Antoine Chambert-Loir. Théorèmes d'algébricité en géométrie diophantienne (d'après J.-B. Bost, Y. André, D. & G. Chudnovsky). Number 282, pages Exp. No. 886, viii, 175–209. 2002. Séminaire Bourbaki, Vol. 2000/2001.

[43] Antoine Chambert-Loir. Mesures et équidistribution sur les espaces de Berkovich. *J. Reine Angew. Math.*, 595:215–235, 2006.

[44] Antoine Chambert-Loir. Heights and measures on analytic spaces. A survey of recent results, and some remarks. In *Motivic integration and its interactions with model theory and non-Archimedean geometry. Volume II*, volume 384 of *London Math. Soc. Lecture Note Ser.*, pages 1–50. Cambridge Univ. Press, Cambridge, 2011.

[45] Antoine Chambert-Loir and Amaury Thuillier. Mesures de Mahler et équidistribution logarithmique. *Ann. Inst. Fourier (Grenoble)*, 59(3):977–1014, 2009.

[46] Ivan Chio and Roland Roeder. Chromatic zeros on hierarchical lattices and equidistribution on parameter space. *arXiv e-prints*, 2019.

[47] Brian Conrad. Several approaches to non-Archimedean geometry. In *p-adic geometry*, volume 45 of *Univ. Lecture Ser.*, pages 9–63. Amer. Math. Soc., Providence, RI, 2008.

[48] Jean-Pierre Demailly. Complex analytic and differential geometry, 2011. Book in open access.

[49] Laura DeMarco. Dynamics of rational maps: a current on the bifurcation locus. *Math. Res. Lett.*, 8(1-2):57–66, 2001.

[50] Laura DeMarco. Iteration at the boundary of the space of rational maps. *Duke Math. J.*, 130(1):169–197, 2005.

[51] Laura DeMarco. Bifurcations, intersections, and heights. *Algebra Number Theory*, 10(5):1031–1056, 2016.

[52] Laura DeMarco. Critical orbits and arithmetic equidistribution. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited lectures*, pages 1867–1886. World Sci. Publ., Hackensack, NJ, 2018.

[53] Laura DeMarco. Dynamical moduli spaces and elliptic curves. *Ann. Fac. Sci. Toulouse Math. (6)*, 27(2):389–420, 2018.

[54] Laura DeMarco and Dragos Ghioca. Rationality of dynamical canonical height. *Ergodic Theory and Dynamical Systems*, 39(9):2507–2540, 2019.

[55] Laura DeMarco, Dragos Ghioca, Holly Krieger, Khoa Dang Nguyen, Thomas Tucker, and Hexi Ye. Bounded height in families of dynamical systems. *Int. Math. Res. Not. IMRN*, (8):2453–2482, 2019.

[56] Laura DeMarco, Holly Krieger, and Hexi Ye. Common preperiodic points for quadratic polynomials. *arXiv e-prints*, 2019.

[57] Laura DeMarco and Curtis T. McMullen. Trees and the dynamics of polynomials. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(3):337–382, 2008.

[58] Laura DeMarco and Mavraki Niki Myrto. Variation of Canonical Height on $\mathbb{P}^1$ for Fatou points. *arXiv e-prints*, 2021.

[59] Laura DeMarco and Aaron Schiff. The geometry of the critically periodic curves in the space of cubic polynomials. *Exp. Math.*, 22(1):99–111, 2013.

[60] Laura DeMarco, Xiaoguang Wang, and Hexi Ye. Bifurcation measures and quadratic rational maps. *Proceedings of the London Mathematical Society*, 111(1):149–180, 2015.

[61] Adrien Douady and John H. Hubbard. *Étude dynamique des polynômes complexes. Partie I*, volume 84 of *Publications Mathématiques d'Orsay [Mathematical Publications of Orsay]*. Université de Paris-Sud, Département de Mathématiques, Orsay, 1984.

[62] Adrien Douady and John H. Hubbard. *Étude dynamique des polynômes complexes. Partie II*, volume 85 of *Publications Mathématiques d'Orsay [Mathematical Publications of Orsay]*. Université de Paris-Sud, Département de Mathématiques, Orsay, 1985. With the collaboration of P. Lavaurs, Tan Lei and P. Sentenac.

[63] Régine Douady and Adrien Douady. *Algèbre et théories galoisiennes. 2.* CEDIC, Paris, 1979. Théories galoisiennes. [Galois theories].

[64] Antoine Ducros. La structure des courbes analytiques, 2014. Preprint.

[65] Romain Dujardin. Structure properties of laminar currents on $\mathbb{P}^2$. *J. Geom. Anal.*, 15(1):25–47, 2005.

[66] Romain Dujardin. Cubic polynomials: a measurable view on parameter space. In *Complex dynamics : families and friends / edited by Dierk Schleicher*, pages 451–490. A K Peters, Ltd., Wellesley, Massachussets, 2009.

[67] Romain Dujardin. The supports of higher bifurcation currents. *Ann. Fac. Sci. Toulouse Math. (6)*, 22(3):445–464, 2013.

[68] Romain Dujardin and Charles Favre. Distribution of rational maps with a preperiodic critical point. *Amer. J. Math.*, 130(4):979–1032, 2008.

[69] Romain Dujardin and Charles Favre. The dynamical manin-mumford problem for plane polynomial automorphisms. *J. Eur. Math. Soc. (JEMS)*, 19(11):3421–3465, 2017.

[70] Romain Dujardin, Charles Favre, and Thomas Gauthier. When do two rational functions have locally biholomorphic julia sets? *Preprint*, 2021.

[71] Dominik Eberlein, Sabyasachi Mukherjee, and Dierk Schleicher. Rational parameter rays of the multibrot sets. In *Dynamical systems, number theory and applications*, pages 49–84. World Sci. Publ., Hackensack, NJ, 2016.

[72] Howard T. Engstrom. Polynomial substitutions. *Amer. J. Math.*, 63:249–255, 1941.

[73] Alexandre È. Erëmenko and Sebastian van Strien. Rational maps with real multipliers. *Trans. Amer. Math. Soc.*, 363(12):6453–6463, 2011.

[74] Pierre Fatou. Sur les équations fonctionnelles. *Bull. Soc. Math. France*, 48:208–314, 1920.

[75] Charles Favre. Degeneration of endomorphisms of the complex projective space in the hybrid space. *Journal of the Institute of Mathematics of Jussieu*, pages 1–43, 2018.

[76] Charles Favre and Thomas Gauthier. Distribution of postcritically finite polynomials. *Israel Journal of Mathematics*, 209(1):235–292, 2015.

[77] Charles Favre and Thomas Gauthier. Classification of special curves in the space of cubic polynomials. *Int. Math. Res. Not. IMRN*, (2):362–411, 2018.

[78] Charles Favre and Thomas Gauthier. Continuity of the Green function in meromorphic families of polynomials. *Algebra Number Theory*, 12(6):1471–1487, 2018.

[79] Charles Favre and Mattias Jonsson. *The valuative tree*, volume 1853 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2004.

[80] Charles Favre, Jan Kiwi, and Eugenio Trucco. A non-Archimedean Montel's theorem. *Compos. Math.*, 148(3):966–990, 2012.

[81] Charles Favre and Juan Rivera-Letelier. Equidistribution quantitative des points de petite hauteur sur la droite projective. *Math. Ann.*, 335(2):311–361, 2006.

[82] Charles Favre and Juan Rivera-Letelier. Théorie ergodique des fractions rationnelles sur un corps ultramétrique. *Proc. Lond. Math. Soc. (3)*, 100(1):116–154, 2010.

[83] Paul Fili. A metric of mutual energy and unlikely intersections for dynamical systems. *arXiv e-prints*, 2017.

[84] William Floyd, Daniel Kim, Sarah Koch, Walter Parry, and Edgar Saenz. Realizing polynomial portraits. *arXiv e-prints*, 2021.

[85] Hang Fu. Uniform unlikely intersections for unicritical polynomials. *arXiv e-prints*, 2020.

[86] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[87] Thomas Gauthier. Strong bifurcation loci of full Hausdorff dimension. *Ann. Sci. Éc. Norm. Supér. (4)*, 45(6):947–984, 2012.

[88] Thomas Gauthier. Dynamical pairs with an absolutely continuous bifurcation measure. *to appear in Ann. Fac. Sci. Toulouse Math.*, 2018.

[89] Thomas Gauthier. Good height functions on quasi-projective varieties: equidistribution and applications in dynamics. *arXiv e-print*, 2021.

[90] Thomas Gauthier and Gabriel Vigny. Distribution of postcritically finite polynomials II: Speed of convergence. *J. Mod. Dyn.*, 11:57–98, 2017.

[91] Thomas Gauthier and Gabriel Vigny. Distribution of postcritically finite polynomials III: Combinatorial continuity. *Fund. Math.*, 244(1):17–48, 2019.

[92] Thomas Gauthier and Gabriel Vigny. The geometric dynamical Northcott and Bogomolov properties. *arXiv e-print*, 2019.

[93] Dragos Ghioca. Unlikely intersections in arithmetic dynamics. *CMS Notes*, 49(4):12–13, 2017.

[94] Dragos Ghioca, Liang-Chung Hsia, and Khoa Dang Nguyen. Simultaneously preperiodic points for families of polynomials in normal form. *Proc. Amer. Math. Soc.*, 146(2):733–741, 2018.

[95] Dragos Ghioca, Liang-Chung Hsia, and Thomas J. Tucker. Preperiodic points for families of polynomials. *Algebra & Number Theory*, 146(3):701–732, 2013.

[96] Dragos Ghioca, Liang-Chung Hsia, and Thomas J. Tucker. Unlikely intersection for two-parameter families of polynomials. *Int. Math. Res. Not. IMRN*, (24):7589–7618, 2016.

[97] Dragos Ghioca, Holly Krieger, and Khoa Nguyen. A case of the dynamical André-Oort conjecture. *Int. Math. Res. Not. IMRN*, (3):738–758, 2016.

[98] Dragos Ghioca, Holly Krieger, Khoa Dang Nguyen, and Hexi Ye. The dynamical André-Oort conjecture: unicritical polynomials. *Duke Math. J.*, 166(1):1–25, 2017.

[99] Dragos Ghioca and Khoa Dang Nguyen. Dynamics of split polynomial maps: uniform bounds for periods and applications. *Int. Math. Res. Not. IMRN*, (1):213–231, 2017.

[100] Dragos Ghioca, Khoa Dang Nguyen, and Hexi Ye. The dynamical Manin-Mumford conjecture and the dynamical Bogomolov conjecture for endomorphisms of $(\mathbb{P}^1)^n$. *Compos. Math.*, 154(7):1441–1472, 2018.

[101] Dragos Ghioca, Khoa Dang Nguyen, and Hexi Ye. The dynamical Manin-Mumford conjecture and the dynamical Bogomolov conjecture for split rational maps. *J. Eur. Math. Soc. (JEMS)*, 21(5):1571–1594, 2019.

[102] Dragos Ghioca, Thomas J. Tucker, and Shouwu Zhang. Towards a dynamical Manin-Mumford conjecture. *Int. Math. Res. Not. IMRN*, (22):5109–5122, 2011.

[103] Dragos Ghioca and Hexi Ye. A dynamical variant of the André-Oort conjecture. *Int. Math. Res. Not. IMRN*, (8):2447–2480, 2018.

[104] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[105] Marc Hindry and Joseph H. Silverman. *Diophantine geometry*, volume 201 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. An introduction.

[106] Lars Hörmander. *An introduction to complex analysis in several variables*, volume 7 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[107] Lars Hörmander. *Notions of convexity*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2007. Reprint of the 1994 edition.

[108] John H. Hubbard. *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1*. Matrix Editions, Ithaca, NY, 2006. Teichmüller theory, With contributions by Adrien Douady, William Dunbar, Roland Roeder, Sylvain Bonnot, David Brown, Allen Hatcher, Chris Hruska and Sudeb Mitra, With forewords by William Thurston and Clifford Earle.

[109] Patrick Ingram. A finiteness result for post-critically finite polynomials. *Int. Math. Res. Not. IMRN*, (3):524–543, 2012.

[110] Patrick Ingram. Variation of the canonical height for a family of polynomials. *J. Reine Angew. Math.*, 685:73–97, 2013.

[111] Mattias Jonsson. Dynamics of Berkovich spaces in low dimensions. In *Berkovich spaces and applications*, volume 2119 of *Lecture Notes in Math.*, pages 205–366. Springer, Cham, 2015.

[112] Jan Kiwi. Combinatorial continuity in complex polynomial dynamics. *Proc. London Math. Soc. (3)*, 91(1):215–248, 2005.

[113] Lars Kühne. Points of small height on semiabelian varieties. *to appear in J. Eur. Math. Soc.*, 2019.

[114] Lars Kühne. Equidistribution in families of abelian varieties and uniformity. *arXiv e-print*, 2021.

[115] Serge Lang. Division points on curves. *Ann. Mat. Pura Appl. (4)*, 70:229–234, 1965.

[116] G. Levin and F. Przytycki. When do two rational functions have the same Julia set? *Proc. Amer. Math. Soc.*, 125(7):2179–2190, 1997.

[117] Guenadi M. Levin. Symmetries on Julia sets. *Mat. Zametki*, 48(5):72–79, 159, 1990.

[118] Alon Levy. The space of morphisms on projective space. *Acta Arith.*, 146(1):13–31, 2011.

[119] Luna Lomonaco and Sabyasachi Mukherjee. A rigidity result for some parabolic germs. *Indiana Univ. Math. J.*, 67(5):2089–2101, 2018.

[120] Mikhail Y. Lyubich. Some typical properties of the dynamics of rational mappings. *Uspekhi Mat. Nauk*, 38(5(233)):197–198, 1983.

[121] Mikhail Y. Lyubich. Investigation of the stability of the dynamics of rational functions. *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.*, 42:72–91, 1984. Translated in Selecta Math. Soviet. **9** (1990), no. 1, 69–90.

[122] Ricardo Mañé, Paulo Sad, and Dennis Sullivan. On the dynamics of rational maps. *Ann. Sci. École Norm. Sup. (4)*, 16(2):193–217, 1983.

[123] David Masser and Umberto Zannier. Torsion anomalous points and families of elliptic curves. *Amer. J. Math.*, 132(6):1677–1691, 2010.

[124] Curtis T. McMullen. *Complex dynamics and renormalization*, volume 135 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1994.

[125] Curtis T. McMullen and Dennis P. Sullivan. Quasiconformal homeomorphisms and dynamics. III. The Teichmüller space of a holomorphic dynamical system. *Adv. Math.*, 135(2):351–395, 1998.

[126] Alice Medvedev and Thomas Scanlon. Invariant varieties for polynomial dynamical systems. *Ann. of Math. (2)*, 179(1):81–177, 2014.

[127] John Milnor. *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, third edition, 2006.

[128] John Milnor. On Lattès maps. In *Dynamics on the Riemann sphere*, pages 9–43. Eur. Math. Soc., Zürich, 2006.

[129] John Milnor. Cubic polynomial maps with periodic critical orbit. I. In *Complex dynamics*, pages 333–411. A K Peters, Wellesley, MA, 2009.

[130] Arman Mimar. On the preperiodic points of an endomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ which lie on a curve. *Trans. Amer. Math. Soc.*, 365(1):161–193, 2013.

[131] Atsushi Moriwaki. Adelic divisors on arithmetic varieties. *Mem. Amer. Math. Soc.*, 242(1144):v+122, 2016.

[132] Yûsuke Okuyama. Repelling periodic points and logarithmic equidistribution in non-archimedean dynamics. *Acta Arith.*, 152(3):267–277, 2012.

[133] F. Pakovich. On rational functions sharing the measure of maximal entropy. *Arnold Math. J.*, 6(3-4):387–396, 2020.

[134] Fedor Pakovich. On polynomials sharing preimages of compact sets, and related questions. *Geom. Funct. Anal.*, 18(1):163–183, 2008.

[135] Fedor Pakovich. On semiconjugate rational functions. *Geom. Funct. Anal.*, 26(4):1217–1243, 2016.

[136] Fedor Pakovich. Polynomial semiconjugacies, decompositions of iterations, and invariant curves. *Ann. Sc. Norm. Super. Pisa*, XVII(5):1417–1446, 2017.

[137] Fedor Pakovich. Invariant curves for endomorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$. *arXiv e-prints*, 2019.

[138] Fedor Pakovich. On iterates of rational functions with maximal number of critical values. *arXiv e-prints*, 2021.

[139] Fabien Pazuki. Polarized morphisms between abelian varieties. *Int. J. Number Theory*, 9(2):405–411, 2013.

[140] Alfredo Poirier. Critical portraits for postcritically finite polynomials. *Fund. Math.*, 203(2):107–163, 2009.

[141] Feliks Przytycki and Juan Rivera-Letelier. Statistical properties of topological Collet-Eckmann maps. *Ann. Sci. École Norm. Sup. (4)*, 40(1):135–178, 2007.

[142] Feliks Przytycki and Juan Rivera-Letelier. Nice inducing schemes and the thermodynamics of rational maps. *Comm. Math. Phys.*, 301(3):661–707, 2011.

[143] Thomas Ransford. *Potential theory in the complex plane*, volume 28 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1995.

[144] Joseph F. Ritt. Prime and composite polynomials. *Trans. Am. Math. Soc.*, 23:51–66, 1922.

[145] Juan Rivera-Letelier. Dynamique des fonctions rationnelles sur des corps locaux. *Astérisque*, 287:xv, 147–230, 2003. Geometric methods in dynamics. II.

[146] Julius Ross and Richard Thomas. Weighted projective embeddings, stability of orbifolds, and constant scalar curvature Kähler metrics. *J. Differential Geom.*, 88(1):109–159, 2011.

[147] Matteo Ruggiero. Contracting rigid germs in higher dimensions. *Ann. Inst. Fourier (Grenoble)*, 63(5):1913–1950, 2013.

[148] Matteo Ruggiero. Classification of one-dimensional superattracting germs in positive characteristic. *Ergodic Theory Dynam. Systems*, 35(7):2242–2268, 2015.

[149] Leo Sario and Mitsuru Nakai. *Classification theory of Riemann surfaces*. Die Grundlehren der mathematischen Wissenschaften, Band 164. Springer-Verlag, New York-Berlin, 1970.

[150] A. Schinzel. *Polynomials with special regard to reducibility*, volume 77 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2000. With an appendix by Umberto Zannier.

[151] Walter Schmidt and Norbert Steinmetz. The polynomials associated with a Julia set. *Bull. London Math. Soc.*, 27(3):239–241, 1995.

[152] Mitsuhiro Shishikura. The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets. *Ann. of Math. (2)*, 147(2):225–267, 1998.

[153] Nessim Sibony. Dynamique des applications rationnelles de $\mathbf{P}^k$. In *Dynamique et géométrie complexes (Lyon, 1997)*, volume 8 of *Panor. Synthèses*, pages ix–x, xi–xii, 97–185. Soc. Math. France, Paris, 1999.

[154] Joseph H. Silverman. The space of rational maps on $\mathbf{P}^1$. *Duke Math. J.*, 94(1):41–77, 1998.

[155] Joseph H. Silverman. *The arithmetic of dynamical systems*, volume 241 of *Graduate Texts in Mathematics*. Springer, New York, 2007.

[156] Joseph H. Silverman. *Moduli spaces and arithmetic dynamics*, volume 30 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2012.

[157] The Stacks Project Authors. *Stacks Project.* https://stacks.math.columbia.edu, 2019.

[158] Lucien Szpiro, Emmanuel Ullmo, and Shou-Wu Zhang. Équirépartition des petits points. *Invent. Math.*, 127(2):337–347, 1997.

[159] Lei Tan. Similarity between the Mandelbrot set and Julia sets. *Comm. Math. Phys.*, 134(3):587–617, 1990.

[160] Amaury Thuillier. Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. applications à la théorie d'arakelov, 2005. Thèse de l'Université de Rennes 1, viii + 184 p.

[161] Eugenio Trucco. Wandering Fatou components and algebraic Julia sets. *Bull. Soc. Math. France*, 142(3):411–464, 2014.

[162] Masatsugu Tsuji. *Potential theory in modern function theory*. Chelsea Publishing Co., New York, 1975. Reprinting of the 1959 original.

[163] Ryuji Tsushima. Rational maps to varieties of hyperbolic type. *Proc. Japan Acad. Ser. A Math. Sci.*, 55(3):95–100, 1979.

[164] Alexander P. Veselov. What is an integrable mapping? In *What is integrability?*, Springer Ser. Nonlinear Dynam., pages 251–272. Springer, Berlin, 1991.

[165] Junyi Xie. Intersections of valuation rings in $k[x, y]$. *Proc. Lond. Math. Soc. (3)*, 111(1):240–274, 2015.

[166] Hexi Ye. Rational functions with identical measure of maximal entropy. *Adv. Math.*, 268:373–395, 2015.

[167] Malik Younsi. On removable sets for holomorphic functions. *EMS Surv. Math. Sci.*, 2(2):219–254, 2015.

[168] Xinyi Yuan. Big line bundles over arithmetic varieties. *Invent. Math.*, 173(3):603–649, 2008.

[169] Xinyi Yuan and Shou-Wu Zhang. Adelic line bundles over quasi-projective varieties. *arXiv e-print*, 2021.

[170] Umberto Zannier. Ritt's second theorem in arbitrary characteristic. *J. Reine Angew. Math.*, 445:175–203, 1993.

[171] Umberto Zannier. *Some problems of unlikely intersections in arithmetic and geometry*, volume 181 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2012. With appendixes by David Masser.

[172] Anna Zdunik. Parabolic orbifolds and the dimension of the maximal measure for rational maps. *Invent. Math.*, 99(3):627–649, 1990.

[173] Shou-Wu Zhang. Positive line bundles on arithmetic varieties. *J. Amer. Math. Soc.*, 8:187–221, 1995.

[174] Shou-Wu Zhang. Small points and adelic metrics. *J. Algebraic Geom.*, 4(2):281–300, 1995.

[175] Michael E. Zieve and Peter Müller. On Ritt's polynomial decomposition theorems. *arXiv e-prints*, 2008.