HÖLDER ESTIMATES AND UNIFORMITY IN ARITHMETIC DYNAMICS

by

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Abstract. — In this note we study common preperiodic points of rational maps of the Riemann Sphere. We show that given any degrees $d_1, d_2 \geq 2$, outside a Zariski closed subset of the space of pairs of rational maps (f, g) of degree d_1 and d_2 respectively, the maps f and g share at most a uniformly bounded number of common preperiodic points. This generalizes a result of DeMarco and Mavraki to maps of possibly different degrees. Our main contribution is the use of Hölder properties of the Green function of a rational map to obtain height estimates.

1. Introduction

In a recent work, DeMarco and Mavraki [**DM**] showed that for any $d \geq 2$, there is a constant $C \geq 1$, depending only on d, such that a general pair of degree d rational maps share at most C preperiodic points. This question was first addressed by DeMarco, Krieger and Ye [**DKY1, DKY2**] in the case of the Lattès family and the quadratic polynomial family, with a slightly different proof, even though they relied on the same tool: the Arakelov-Zhang pairing (see also [**MS**] and [**P**] for works on this question). We want here to give a proof of a generalization to the case when the rational maps can have different degrees - even multiplicatively independent degrees - which follows more or less the approach oy DeMarco, Krieger and Ye using a new ingredient: the Hölder regularity of the canonical metric invariant by an endomorphism of the projective space.

For any integers $d \geq 2$, we denote by Rat_d the space of all endomorphisms of degree d of $\mathbb{P}^1$, i.e. the space of all degree d rational maps on $\mathbb{P}^1$. This is a smooth affine variety defined over $\mathbb{Q}$ which can be identified with $\mathbb{P}^{2d+1} \setminus \{\mathrm{Res} = 0\}$, where Res is the homogeneous MacCaulay resultant (see, e.g., [**BB**]). For a given complex rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$, we denote

$$\mathrm{Preper}(f) := \{z \in \mathbb{P}^1(\mathbb{C}) \, : \, \text{there are } n > m, \text{ s.t. } f^n(z) = f^m(z)\}.$$

The main result of the present paper is the following.

Theorem A. — *For any $d_1, d_2 \geq 2$, there is a dense Zariski open subset $V \subseteq \mathrm{Rat}_{d_1} \times \mathrm{Rat}_{d_2}$ and an integer $N \geq 1$ such that for any $(f, g) \in V(\mathbb{C})$,*

$$\# \, \mathrm{Preper}(f) \cap \mathrm{Preper}(g) \leq N.$$

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As written above, to show this result, we follow the approach of [**DKY1, DKY2**]: the key ingredient is the *Arakelov-Zhang pairing* of two rational maps $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ defined over a number field $\mathbb{K}$: this pairing can be defined as

$$\langle f, g\rangle := -\frac{1}{2[\mathbb{K}:\mathbb{Q}]}\left(\bar{L}_f - \bar{L}_g\right)^2,$$

where $\bar{L}_f$ (resp. $\bar{L}_g$) is the adelic semi-positive continuous line bundle associated with f. By the arithmetic Hodge Index Theorem of Yuan and Zhang [**YZ1**], we have $\langle f, g\rangle \geq 0$ and $\langle f, g\rangle = 0$ if and only if $\bar{L}_f = \bar{L}_g$ (see Section 2.4 for more derails).

The proof of this result goes in two main steps: the first one relies on Yuan and Zhang's theory of adelic line bundles on quasi-projective varieties [**YZ2**] and the non-vanishing of an appropriate measure induced by the pairing on the quotient space $(\mathrm{Rat}_{d_1} \times \mathrm{Rat}_{d_2})/\mathrm{PGL}(2)$. This non-vanishing property has been established for the case $d_1 = d_2$ by DeMarco and Mavraki [**DM**] and the case when the d_1 and d_2 are multiplicatively dependent follows easily. The case when d_1 and d_2 are multiplicatively independent actually follows easily from properties of the local archimedean contribution of the Arakelov-Zhang pairing (see Section 2.4 for more details).

The second step is the main new ingredient we give here, and is directly inspired from an energy inequality relating the pairing of two maps f and g and the height of a finite set of points for the height function $\hat{h}_f + \hat{h}_g$, when f and g are defined over a number field. Here, the canonical height function, as a function on the $\bar{\mathbb{Q}}$-points, is defined as

$$\hat{h}_f = \lim_{n\to\infty} \frac{1}{d^n} h_{\mathrm{nv}} \circ f^n \quad \text{on } \mathbb{P}^1(\bar{\mathbb{Q}}).$$

To any endomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over a field K, we can associate a polynomial homogeneous lift $F : \mathbb{A}^2_K \to \mathbb{A}^2_K$ and two such lifts F, F' necessarily satisfy $F = \alpha F'$ with $\alpha \in \bar{K}^\times$. If $(a_0, \ldots, a_N)$ is the collection of coefficients of a lift F of the map f, one can identify f with $[a_0 : \cdots : a_{2d+1}] \in \mathbb{P}^{2d+1} \setminus \{\mathrm{Res} = 0\}$. If f is defined over a number field, let

$$h_{\mathrm{Rat}_d}(f) := \sum_{v\in M_{\mathbb{K}}} \frac{N_v}{[\mathbb{K}:\mathbb{Q}]} \log \|F\|_v,$$

where F is any lift of f defined over a number field $\mathbb{K}$ and $\|F\|_v = \max\{|a_0|_v, \ldots, |a_{2d+1}|_v\}$. By the product formula, the quantity $h_{\mathrm{Rat}_d}(f)$ is well-defined.

Inspired by [**DKY1**, Theorem 1.8] and [**DKY2**, Theorem 1.9], we prove the following.

Theorem B. — *For any $d_1, d_2 \geq 2$, there is a constant $A \geq 1$ depending only on d_1 and d_2 such that for any rational maps $(f, g) \in \mathrm{Rat}_{d_1} \times \mathrm{Rat}_{d_2}(\bar{\mathbb{Q}})$, for any $0 < \delta < 1$, and any finite Galois invariant subset $E \subset \mathbb{A}^1(\bar{\mathbb{Q}})$,*

$$\langle f, g\rangle \leq A\left(\delta - \frac{\log(\delta)}{\#E}\right)\left(h_{\mathrm{Rat}_{d_1}}(f) + h_{\mathrm{Rat}_{d_2}}(g) + 1\right) + \frac{1}{\#E}\sum_{x\in E}(\hat{h}_f(x) + \hat{h}_g(x)).$$

The proof of this result relies on a precise analysis of the adelically Hölder regularity properties of the invariant metric for a given rational map, as well as its dependence on the parameter. We then use the approximation of measures distributed on finite Galois-invariant finite sets introduced by Favre and Rivera-Letelier [**FRL**] as in [**DKY1, DKY2**] to conclude by choosing an appropriate adelic radius of approximation, see Section 4.2. Shang and Yap proved a variant of Theorem B form polynomials as Theorem 3.18 in [**AY**].

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2. Preliminaries

2.1. Basics over a metrized field

Let $(K, |\cdot|)$ be a complete and algebraically closed field of characteristic 0. Let $k \geq 1$ be an integer and $\pi : \mathbb{A}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ be the canonical projection. For $X = (X_1, \dots, X_{k+1}) \in K^{k+1}$, we also let

$$\|X\| := \max_j |X_j|.$$

We also define a distance on $\mathbb{P}^k$ by letting

$$\mathrm{dist}(x, y) := \frac{\max_{1 \leq i,j \leq k+1} |X_i Y_j - X_j Y_i|}{\|X\| \cdot \|Y\|},$$

for any $X, Y \in K^{k+1} \setminus \{0\}$ with $\pi(X) = x$ and $\pi(Y) = y$. For this distance, $\mathrm{diam}(\mathbb{P}^k) = 1$.

When X is a projective variety over K, and L is a line bundle over X, we denote by X^{an} (resp. L^{an}) the Berkovich analytification of X (resp. of L). A *continuous metric* on L^{an} is a the data for each local analytic section σ defined over an open subset $U \subset X^{\mathrm{an}}$ of a continuous function $\|\sigma\|_U : U \to \mathbb{R}_+$ such that $\|\sigma\|_U$ vanishes only at zeroes of σ, the restriction of $\|\sigma\|_U$ to an open subset V is $\|\sigma\|_V$ and for each analytic function f on U, $\|f\sigma\|_U = |f| \times \|\sigma\|_U$. When $(K, |\cdot|)$ is non-archimedean, we say that a metric is a *model* metric if there is an $\mathscr{O}_K$-model $\mathscr{X}$ of X – i.e. an integral $\mathscr{O}_K$-scheme with generic fiber $\mathscr{X}_\eta$ isomorphic to X and a line bundle $\mathscr{L}$ on $\mathscr{X}$ which restriction to the generic fiber isomorphic to L – such that the metric is on L is induced by this model $(\mathscr{X}, \mathscr{L})$ (see, e.g., [**FG**, Chapter 1] for more details).

2.2. Adelic metrizations over a number field

For the material of this section, we refer to [**CL**] and [**Z**]. When $\mathbb{K}$ is a number field, let $M_{\mathbb{K}}$ be its set of places, i.e. the set of equivalence places of non-trivial norms on $\mathbb{K}$. For any $v \in M_{\mathbb{K}}$, denote by $|\cdot|_v$ the corresponding norm. We let $\mathbb{K}_v$ be the completion of $(\mathbb{K}, |\cdot|_v)$, by $\bar{\mathbb{K}}_v$ the algebraic closure of $\mathbb{K}_v$, and finally by $\mathbb{C}_v$ the completion of $\bar{\mathbb{K}}_v$. We then denote by $\|\cdot\|_v$ the induced norm on the affine space $\mathbb{A}^k$ for any $k \geq 1$ and by dist_v the induced distance on $\mathbb{P}^k_{\mathbb{C}_v}$.

Finally, we assume the norms $|\cdot|_v$ are normalized so that the product formula holds:

$$\prod_{v \in M_{\mathbb{K}}} |x|_v^{N_v} = 1, \quad x \in \mathbb{K}^{\times},$$

with $N_v = [\mathbb{K}_v : \mathbb{Q}_p]$, where p is the residual characteristic of $(\mathbb{K}, |\cdot|_v)$.

We will use the following definitions in the whole text:

- An *adelic constant* (over $\mathbb{K}$) is a function $C : v \in M_{\mathbb{K}} \longmapsto C_v \in \mathbb{R}_+^*$ such that $C_v = 1$ for all but finitely many $v \in M_{\mathbb{K}}$,
- An adelic constant is *small* if $0 < C_v \leq 1$, for all $v \in M_{\mathbb{K}}$.

We will denote by $C = \{C_v\}_{v\in M_{\mathbb{K}}}$ an adelic constant over $\mathbb{K}$.

Let X be a projective variety defined over a number field $\mathbb{K}$. Fix a place $v \in M_{\mathbb{K}}$. Denote by X_v^{an} the Berkovich analytification of X at the place v. Let L be a line bundle on X, also defined over $\mathbb{K}$. An *adelic metric* on L is a collection of metrics $\{\|\cdot\|_v\}_{v\in M_{\mathbb{K}}}$ such that, for any $v \in M_{\mathbb{K}}$, $\|\cdot\|_v$ is a continuous metric on L_v^{an} and there exists an $\mathscr{O}_{\mathbb{K}}$-model $(\mathscr{X},\mathscr{L})$ of (X,L) such that, for all but finitely many $v \in M_{\mathbb{K}}$, $\|\cdot\|_v$ is a model metric on L_v^{an} induced by $(\mathscr{X},\mathscr{L})$. Denote $\bar{L} := (L, \{\|\cdot\|_v\}_{v\in M_{\mathbb{K}}})$. We say $\bar{L}$ is *semi-positive* if for any $v \in M_{\mathbb{K}}$, in any local chart $U \subset X_v^{\mathrm{an}}$, the metric $\|\cdot\|_v$ can be written under the form $|\cdot|_v e^{-u_v}$, where u_v is plurisubharmonic. We say $\bar{L}$ is *integrable* if it can be written as a difference of semi-positive adelic line bundles. We also let $c_1(\bar{L})_v$ be the curvature form of the metric $\|\cdot\|_v$ on X_v^{an}.

2.3. Arithmetic intersection and heights

Let $L_0,\dots,L_k$ be $\mathbb{Q}$-line bundle on X. Assume L_i is equipped with an adelic continuous metric $\{\|\cdot\|_{v,i}\}_{v\in M_{\mathbb{K}}}$ and we denote $\bar{L}_i := (L_i, \{\|\cdot\|_v\}_{v\in M_{\mathbb{K}}})$. Assume $\bar{L}_i$ is semi-positive for $1 \le i \le k$ and $\bar{L}_0$ is integrable.

We will use in the sequel that the arithmetic intersection number $\left(\bar{L}_0\cdots\bar{L}_k\right)$ is symmetric and multilinear with respect to the L_i and that

$$\left(\bar{L}_0\right)\cdots\left(\bar{L}_k\right) = \left(\bar{L}_1|_{\mathrm{div}(s)}\right)\cdots\left(\bar{L}_k|_{\mathrm{div}(s)}\right) + \sum_{v\in M_{\mathbb{K}}} N_v \int_{X_v^{\mathrm{an}}} \log\|s\|_v^{-1} \bigwedge_{j=1}^{k} c_1(\bar{L}_i)_v,$$

for any global section $s \in H^0(X, L_0)$ (whenever such a section exists) and wuere $N_v = [\mathbb{K}_v : \mathbb{Q}_p]$. In particular, if L_0 is the trivial bundle and $\|\cdot\|_{v,0}$ is the trivial metric at all places but v_0, this gives

$$\left(\bar{L}_0\right)\cdots\left(\bar{L}_k\right) = N_{v_0} \int_{X_{v_0}^{\mathrm{an}}} \log\|1\|_{v_0,0}^{-1} \bigwedge_{j=1}^{k} c_1(\bar{L}_i)_{v_0}.$$

When $\bar{L}$ is a big and nef $\mathbb{Q}$-line bundle endowed with a semi-positive continuous adelic metric, following Zhang [**Z**], we define $h_{\bar{L}}(X)$ as

$$h_{\bar{L}}(X) := \frac{\left(\bar{L}\right)^{k+1}}{(k+1)[\mathbb{K}:\mathbb{Q}]\mathrm{vol}(L)},$$

where $\mathrm{vol}(L) = (L)^k$ is the volume of the line bundle L (also denoted by $\deg_X(L)$ sometimes). We also define the height of a closed point $x \in X(\bar{\mathbb{Q}})$ as

$$h_{\bar{L}}(x) = \frac{\left(\bar{L}|x\right)}{[\mathbb{K}:\mathbb{Q}]} = \frac{1}{[\mathbb{K}:\mathbb{Q}]\#\mathsf{O}(x)} \sum_{v\in M_{\mathbb{K}}} \sum_{\sigma:\mathbb{K}(x)\hookrightarrow\mathbb{C}_v} \log\|s(\sigma(x))\|_v^{-1},$$

where $\mathsf{O}(x)$ is the Galois orbit of x, for any section $s \in H^0(X, L)$ which does not vanish at x. Finally, for any Galois-invariant finite set $F \subset X(\bar{\mathbb{K}})$, we define $h_{\bar{L}}(F)$ as

$$h_{\bar{L}}(F) := \frac{1}{\#F}\sum_{y\in F} h_{\bar{L}}(y).$$

We now assume X is a curve. The Zhang's inequalities [**Z**] can be then written as follows:

Lemma 2.1. — *If X is a smooth projective curve defined over a number field, if L is ample, if $\bar{L}$ is an adelic semi-positive metrized line bundle and if we let $e(\bar{L}) = \sup_Z \inf_{x\in(X\setminus Z)(\bar{\mathbb{K}})} h_{\bar{L}}(x)$, where the supremum is taken over all Zariski closed proper subsets Z of X defined over $\mathbb{K}$, then*

$$\frac{1}{2}\left(e(\bar{L}) + k \inf_{y\in X(\bar{\mathbb{K}})} h_{\bar{L}}(y)\right) \leq h_{\bar{L}}(X) \leq e(\bar{L}).$$

In particular, if $h_{\bar{L}}(x) \geq 0$ for all $x \in X(\bar{\mathbb{K}})$, then $h_{\bar{L}}(X) \geq 0$.

We also will use the following version of the arithmetic Hodge Index Theorem of Yuan and Zhang [**YZ1**].

Theorem 2.2 (Arithmetic Hodge Index). — *Let X be a smooth projective curve defined over $\mathbb{K}$. Let $\bar{L}$ be an integrable adelic line bundle over X with $L = \mathcal{O}_X$. Then*

$$(\bar{L})^2 \leq 0.$$

When $f : \mathbb{P}^k \to \mathbb{P}^k$ is a degree $d \geq 2$ endomorphism defined over a number field $\mathbb{K}$, one can define a canonical metrization on $\mathcal{O}_{\mathbb{P}^k}(1)$ as follows: let $\{\|\cdot\|_{0,v}\}_{v\in M_{\mathbb{K}}}$ be the naive metrization, i.e. the metrization inducing the naive height. For any $v \in M_{\mathbb{K}}$, if we let $\|\cdot\|_{n,v} := ((f^n)^*\|\cdot\|_{0,v})^{1/d^n}$, then the sequence

$$g_n := \log\left(\frac{\|\cdot\|_{n,v}}{\|\cdot\|_{n-1,v}}\right)$$

is a uniform Cauchy sequence, whence $\|\cdot\|_{n,v}$ onverges uniformly to a metric $\|\cdot\|_{f,v}$ on $\mathcal{O}_{\mathbb{P}^k}(1)$ which satisfies $d^{-1}f^*\|\cdot\|_{f,v} = \|\cdot\|_{f,v}$. Moreover, if $g_f := \sum_n g_n$, then $\|\cdot\|_{f,v} = e^{-g_f}\|\cdot\|_{0,v}$. The induced metrized line bundle $\bar{L}_f$ is adelic and semi-positive and satisfies $\hat{h}_f = h_{\bar{L}_f}$.

2.4. The mutual energy pairing

2.4.1. The global mutual energy

Pick integers $d_1, d_2 \geq 2$, pick rational maps $f_1 \in \mathrm{Rat}_{d_1}(\bar{\mathbb{Q}})$ and $f_2 \in \mathrm{Rat}_{d_2}(\bar{\mathbb{Q}})$ and let $\mathbb{K}$ be a number field such that f_1 and f_2 are both defined over $\mathbb{K}$.

Let $\bar{L} := \frac{1}{2}(\bar{L}_{f_1} + \bar{L}_{f_2})$, where $\bar{L}_{f_i}$ is the canonical metric of f_i on $\mathcal{O}_{\mathbb{P}^1}(1)$.

Lemma 2.3. — *For any $x \in \mathbb{P}^1(\bar{\mathbb{Q}})$, we have $h_{\bar{L}}(x) = \frac{1}{2}(\hat{h}_{f_1}(x) + \hat{h}_{f_2}(x))$. Moreover,*

$$h_{\bar{L}}(\mathbb{P}^1) = \frac{1}{2}\langle f_1, f_2\rangle.$$

Proof. — The first equality comes from directly from the definition of $\bar{L}$. Indeed, if $x \in \mathbb{P}^1(\bar{\mathbb{Q}})$, then $h_{\bar{L}}(x) = h_{\frac{1}{2}(\bar{L}_{f_1}+\bar{L}_{f_2})}(x) = \frac{1}{2}h_{\bar{L}_{f_1}}(x) + \frac{1}{2}h_{\bar{L}_{f_2}}(x) = \frac{1}{2}(\hat{h}_{f_1}(x) + \hat{h}_{f_2}(x))$. For the second equality, we can compute

$$h_{\bar{L}}(\mathbb{P}^1) = \frac{(\bar{L})^2}{2[\mathbb{K}:\mathbb{Q}]} = \frac{(\bar{L}_{f_1} + \bar{L}_{f_2})^2}{8[\mathbb{K}:\mathbb{Q}]} = \frac{1}{8[\mathbb{K}:\mathbb{Q}]}\left((\bar{L}_{f_1})^2 + (\bar{L}_{f_2})^2 + 2\left(\bar{L}_{f_1}\cdot\bar{L}_{f_2}\right)\right).$$

We thus have proved that $h_{\bar{L}}(\mathbb{P}^1) = \frac{1}{4}\left(h_{\bar{L}_{f_1}}(\mathbb{P}^1) + h_{\bar{L}_{f_2}}(\mathbb{P}^1)\right) + \frac{(\bar{L}_{f_1}\cdot\bar{L}_{f_2})}{4[\mathbb{K}:\mathbb{Q}]}$. Similarly, by definition of Zhang's pairing, we have

$$\langle f_1, f_2\rangle = -\frac{(\bar{L}_{f_1} - \bar{L}_{f_2})^2}{2[\mathbb{K}:\mathbb{Q}]} = \frac{\left(\bar{L}_{f_1}\cdot\bar{L}_{f_2}\right)}{[\mathbb{K}:\mathbb{Q}]} - \frac{1}{2}\left(h_{\bar{L}_{f_1}}(\mathbb{P}^1) + h_{\bar{L}_{f_2}}(\mathbb{P}^1)\right).$$

By Zhang's inequalities (see, e.g., Lemma 2.1), we have $\hat{h}_{f_1}(\mathbb{P}^1) = \hat{h}_{f_2}(\mathbb{P}^1) = 0$. This concludes the proof. □

Remark. — Usually, the mutual energy is not described as above, but rather in terms of local contributions, see e.g. [**PST**, Formula (2)]. We will use properties of the archimedean contributions in the proof of Theorem A.

2.4.2. The complex mutual energy

The mutual energy of two signed measures ρ, ν of mass 0 with continuous potentials on $\mathbb{P}^1(\mathbb{C})$ is defined in [**FRL**] by

$$(\rho, \nu) := -\int_{\mathbb{C}^2\setminus\Delta} \log|z-w| \mathrm{d}\rho(z)\otimes \mathrm{d}\nu(w).$$

It is known that $(\rho,\rho)\geq 0$ and $(\rho,\rho)=0$ if and only if $\rho = 0$ (see [**FRL**, Propositions 2.6]). By the classification of rational maps sharing their maximal entropy measure [**LP**], we get

Lemma 2.4. — *Pick two integers $d_1, d_2 \geq 2$. Then*

1. *If d_1 and d_2 are multiplicatively independent, then $(\mu_{f_1}-\mu_{f_2}, \mu_{f_1}-\mu_{f_2}) > 0$, unless there exists $\varphi\in \mathrm{PLG}(2,\mathbb{C})$ such that both $\varphi\circ f_1\circ\varphi^{-1}$ and $\varphi\circ f_2\circ\varphi^{-1}$ are monomials, Chebychev polynomials of Lattès maps.*
2. *if d_1 and d_2 are multiplicatively dependent, the set of pairs $(f_1, f_2)\in \mathrm{Rat}_{d_1}\times\mathrm{Rat}_{d_2}(\mathbb{C})$ such that $(\mu_{f_1}-\mu_{f_2}, \mu_{f_1}-\mu_{f_2}) = 0$ is contained in a pluripolar set.*

Proof. — First remark that if $(\mu_{f_1}-\mu_{f_2}, \mu_{f_1}-\mu_{f_2}) = 0$, then $\mu_{f_1} = \mu_{f_2}$ by [**FRL**, Proposition 2.6]. The lemma then follows Theorem A from [**LP**]: it states that

(*a*) either there is $\varphi\in\mathrm{PGL}(2,\mathbb{C})$ such that $\varphi\circ f_1\circ\varphi^{-1}$ and $\varphi\circ f_2\circ\varphi^{-1}$ are monomials, Chebychev polynomials of Lattès maps,

(*b*) or there are iterates F_1 and F_2 of f_1 and f_2 respectively and integers $N, M\geq 1$ such that

$$(F_1^{-1}\circ F_1)\circ F_1^N = (F_2^{-1}\circ F_2)\circ F_2^M$$

where $(F_i^{-1}\circ F_i)$ is defined as a correspondence. In particular, $d_1 = \deg(f_1)$ and $d_2 = \deg(f_2)$ are multiplicatively dependent in this case.

When d_1 and d_2 are multiplicatively independent, we thus are in the case (*a*) above. This concludes the proof of the first point.

To prove the second point, we can remark that for any given $p, q, N, M\geq 1$, the set of pairs $(f_1, f_2)\in\mathrm{Rat}_{d_1}\times\mathrm{Rat}_{d_2}(\mathbb{C})$ that satisfy

$$(f_1^{-p}\circ f_1^p)\circ f_1^N = (f_2^{-q}\circ f_2^q)\circ f_2^M$$

is contained in a strict closed subvariety $Z_{p,q}^{N,M}$ of $\mathrm{Rat}_{d_1}\times\mathrm{Rat}_{d_2}(\mathbb{C})$. Whence we can write

$$\{(f_1,f_2)\in\mathrm{Rat}_{d_1}\times\mathrm{Rat}_{d_2}(\mathbb{C})\ :\ (\mu_{f_1}-\mu_{f_2}, \mu_{f_1}-\mu_{f_2}) = 0\} \subset \bigcup_{p,q,N,M\geq 1} Z_{p,q}^{N,M}\cup E,$$

where E is the union of the orbits of pairs of monomial maps, pairs of Chebychev polynomials and pairs of Lattès maps, under the action by simultaneous conjugacy. In particular E is a proper closed subvariety of $\mathrm{Rat}_{d_1} \times \mathrm{Rat}_{d_2}(\mathbb{C})$ and $\bigcup_{p,q,N,M\geq 1} Z_{p,q}^{N,M} \cup E$ is pluripolar. □

3. Hölder estimates for the dynamical Green functions

In this section, we let $(K, |\cdot|)$ be a complete and algebraically closed field of characteristic 0 and $k \geq 1$ be an integer. Our main aim here is to show that, when varying in an algebraic family, the Green function of an endomorphism of $\mathbb{P}^k_K$ has explicitly controlled Hölder constant and exponent with the parameter.

Pick a degree d endomorphism $f : \mathbb{P}^k_K \to \mathbb{P}^k_K$ defined over K and let $F : \mathbb{A}^{k+1}_K \to \mathbb{A}^{k+1}_K$ be a homogeneous polynomial lift of f. For any $z \in \mathbb{P}^k(K)$ and any $x \in K^{k+1} \setminus \{0\}$ with $\pi(x) = z$, we let

$$u_F(z) := \frac{1}{d} \log \|F(x)\| - \log \|x\|.$$

Note that it depends on a choice of lift F we have $u_{\alpha F} = u_F + \frac{1}{d} \log |\alpha|$ for all $\alpha \in K^*$. We also define the *Green function* of F as

$$g_F := \sum_{j\geq 0} d^{-j} u_F \circ f^j.$$

It also depends on a choice of lift F: we have $g_{\alpha F} = g_F + \frac{1}{d-1} \log |\alpha|$ for any $\alpha \in K^*$.

As the map F is a polynomial map, it can be identified with a point in $\mathbb{A}^{N+1}_K$ where $N + 1 = (k + 1)\frac{(d+k)!}{k!d!}$ via its coefficients. We let

$$\|F\| := \max\{|a| \ : \ a \text{ coefficient of } F\}.$$

This section is devoted to the proof of the following.

Theorem 3.1. — *The Green function g_F is Hölder continuous. More precisely, there are constants $C_1, C_2 \geq 1$ such that $C_1 = C_2 = 1$ if K is non-archimedean and C_1 and C_2 depend only on d and k if K is archimedean and a constant $C_3 \geq 1$ depending only on k and d such that*

$$|g_F(z) - g_F(w)| \leq C_3 \left(C_1 + \log^+ \|F\| + \log^+ \frac{\|F\|^{(k+1)d^k}}{|\mathrm{Res}(F)|} \right) \cdot \mathrm{dist}(z, w)^\alpha,$$

for all $z, w \in \mathbb{P}^k(K)$, where, if $(K, |\cdot|)$ is archimedean

$$\alpha := \log d / \left(C_2 + \log \max \left\{ 2d, \frac{\|F\|^{(k+1)d^k}}{|\mathrm{Res}(F)|} \right\} \right),$$

and if $(K, |\cdot|)$ is non-archimedean, α satisfies

$$\alpha = \begin{cases} 1 & \text{if } \|F\|^{(k+1)d^k} = |\mathrm{Res}(F)| \text{ and,} \\ \log d / 2 \log \left(\dfrac{\|F\|^{(k+1)d^k}}{|\mathrm{Res}(F)|} \right) & \text{otherwise.} \end{cases}$$

3.1. Estimates for building blocks of the Green function

From now on, unless specified, K is an algebraically closed field of characteristic zero which is complete with respect to a non-trivial norm $|\cdot|$. The following is a consequence of basic properties of McCaulay resultant.

Lemma 3.2. — *There is a constant $C_1(K,d,k)\geq 1$ such that $C_1(K,d,k)=1$ if K is non-archimedean and $C_1(K,d,k)$ depends only on d and k when K is archimedean, and such that for any F as above,*

$$\sup_{z\in\mathbb{P}^k(K)}|u_F(z)|\leq\log\max\{\|F\|,1\}+\log C_1(K,d,k).$$

Proof. — An application of the MacCaulay resultant gives a constant $C(K)\geq 1$ which is 1 whenever K is non-archimedean and depending only on d and k when K is archimedean, such that

$$\frac{|\mathrm{Res}(F)|}{C(K)\|F\|^{(k+1)d^k-1}}\|x\|^d\leq\|F(x)\|\leq C(K)\|F\|\cdot\|x\|^d,\quad x\in K^{k+1}\setminus\{0\}. \tag{1}$$

This rewrites, for $z\in\mathbb{P}^k(K)$ and $x\in K^{k+1}$ with $\pi(x)=z$, as

$$\begin{aligned}|u_F(z)|&=\left|\frac{1}{d}\log\frac{\|F(x)\|}{\|x\|^d}\right|\\&\leq\max\left\{0,\log\|F\|+\log C(K),\log\frac{|\mathrm{Res}(F)|}{\|F\|^{(k+1)d^k-1}}+\log C(K)\right\}.\end{aligned}$$

As the resultant map $\mathrm{Res}:\mathbb{A}^{N+1}\to\mathbb{A}^1$ is homogeneous with degree $(k+1)d^k$ and is defined over $\mathbb{Z}$, we have

$$\log\frac{|\mathrm{Res}(F)|}{\|F\|^{(k+1)d^k-1}}\leq\log\|F\|+C'(K),$$

where $C'(K)\geq 0$ and $C'(K)=0$ if K is non-archimedean and depending only on d and k when K is archimedean. This concludes the proof. □

Following the strategy of [**KS**, Theorem 13], we control the lipschitz constant of u_F.

Lemma 3.3. — *There is a constant $C_2(K,d,k)\geq 1$ such that $C_2(K,d,k)=1$ if K is non-archimedean and $C_2(K,d,k)$ depends only on d and k when K is archimedean, and such that for any F as above,*

$$|u_F(z)-u_F(w)|\leq\frac{C_2(K,d,k)^2\|F\|^{(k+1)d^k}}{d|\mathrm{Res}(F)|}\cdot\log\left(\frac{C_2(K,d,k)^2\|F\|^{(k+1)d^k}}{|\mathrm{Res}(F)|}\right)\cdot\mathrm{dist}(z,w).$$

Moreover, if $(K,|\cdot|)$ is non-archimedean and $\mathrm{dist}(x,y)\leq|\mathrm{Res}(F)|/\|F\|^{(k+1)d^k}$, then $u_F(x)=u_F(y)$.

Proof. — Let $x,y\in\mathbb{P}^k(K)$ and $X,Y\in K^{k+1}$ with $\pi(X)=x$ and $\pi(Y)=y$. First, if

$$\mathrm{dist}(x,y)\geq R:=\frac{|\mathrm{Res}(F)|}{C(K)^2\|F\|^{(k+1)d^k}},$$

the formula (1) gives

$$|u_F(x) - u_F(y)| = \left|\frac{1}{d}\log\frac{\|F(X)\|\|Y\|^d}{\|F(Y)\|\|X\|^d}\right| \le \frac{1}{d}\log\left(\frac{1}{R}\right)$$
$$\le \frac{1}{d}\frac{1}{R}\log\left(\frac{1}{R}\right)\cdot \mathrm{dist}(x,y).$$

We now assume $\mathrm{dist}(x,y) < R \le 1$. Pick $X, Y \in K^{k+1}$ such that $\pi(X) = x$ and $\pi(Y) = y$ with $\|X\| = \|Y\| = 1$. If K is non-archimedean, following the proof of [, Theorem 13], but we reproduce the argument here for the sake of completeness. By the strong triangle inequality, we can assume $|x_k| = |y_k| = 1$. Write

$$F(X+h) = F(X) + \sum_{j=0}^{k} h_i B_i(X,h),$$

where $B_i \in K[X,h]$ and its coefficients are linear combinations of coefficients of F. Since $|y_k| = 1$ and F is homogeneous, we can compute

$$\|F(X)\| = \|F(x_kY + y_kX - x_kY)\|$$
$$= \|F(x_kY) + \sum_{j=0}^{k}(y_kx_j - x_ky_j)B_j(x_yY, y_kX - x_kY)\|.$$

By our assumption, we have $R \le \frac{\|F(X)\|}{\|F(Y)\|} \le R^{-1}$ and

$$\left\|\sum_{j=0}^{k}(y_kx_j - x_ky_j)B_j(x_yY, y_kX - x_kY)\right\| < R\|F(X)\| \le \|F(x_kY)\| = \|F(Y)\|.$$

This implies $\|F(X)\| = \|F(Y)\|$ and

$$u_F(x) - u_F(y) = \frac{1}{d}\log\frac{\|F(X)\|}{\|F(Y)\|} = 0,$$

and the proof is complete in this case.

If K is archimedean, we can assume there is j such that $y_j = 1$ and there is $C \ge 1$ depending only on d and k such that

$$|x_j|^d\frac{\|F(Y)\|}{\|F(X)\|} = \frac{\|F(x_jY)\|}{\|F(X)\|} = \frac{\|F(X) + D_XF(x_jY - y_jX)\|}{\|F(X)\|}$$
$$\le 1 + \frac{\|D_XF(x_jY - y_jX)\|}{\|F(X)\|} \le 1 + C\|F\|\frac{\max_\ell |x_jy_\ell - y_jx_\ell|}{\|F(X)\|}$$
$$\le 1 + C\frac{1}{R}\mathrm{dist}(x,y).$$

Up to increasing C and up to replacing $C(K)^2$ by $C \cdot C(K)^2$ and R by R/C, we have $\log(R) \le -1$, whence, taking the supremum over j gives

$$\frac{\|F(Y)\|}{\|F(X)\|} \le 1 + \frac{1}{R}\log\left(\frac{1}{R}\right)\mathrm{dist}(x,y).$$

We now use that $log(1+t) \le t$ for all $t \ge 0$ and get

$$u_F(x) - u_F(y) \le \frac{1}{d}\frac{1}{R}\log\left(\frac{1}{R}\right)\mathrm{dist}(x,y).$$

We conclude reversing the roles of x and y. □

Finally, we control the lipschitz constant of F, relying on McCaulay resultant.

Lemma 3.4. — *There is a constant $C_3(K,d,k) \geq 1$ such that $C_3(K,d,k) = 1$ if K is non-archimedean and $C_3(K,d,k)$ depends only on d and k when K is archimedean, and such that for any F as above, and any $x, y \in \mathbb{P}^k(K)$,*

$$\mathrm{dist}(f(x), f(y)) \leq C_3(K,d,k) \left(\frac{\|F\|^{(k+1)d^k}}{|\mathrm{Res}(F)|} \right)^2 \cdot \mathrm{dist}(x,y).$$

Proof. — Pick $x, y \in \mathbb{P}^k(K)$. For $X, Y \in K^{k+1} \setminus \{0\}$, if $x = \pi(X)$ and $y = \pi(Y)$, we have

$$\begin{aligned}\frac{\mathrm{dist}(f(x), f(y))}{\mathrm{dist}(x,y)} &= \frac{\|X\| \cdot \|Y\| \cdot \max_{1\leq i,j\leq k+1} |F_i(X)F_j(Y) - F_i(Y)F_j(X)|}{\|F(X)\| \cdot \|F(Y)\| \cdot \max_{1\leq i,j \leq k+1} |X_iY_j - X_jY_i|} \\ &\leq C(K) \cdot \frac{\|F\|^{2(k+1)d^k-2}}{|\mathrm{Res}(F)|^2} \cdot \frac{\max_{1\leq i,j\leq k+1} |F_i(X)F_j(Y) - F_i(Y)F_j(X)|}{(\|X\| \cdot \|Y\|)^{d-1} \cdot \max_{1\leq i,j\leq k+1} |X_iY_j - X_jY_i|}.\end{aligned}$$

We now use that for any i, j, there is ℓ, m such that $X_mY_\ell - X_\ell Y_m$ divides $F_i(X)F_j(Y) - F_i(Y)F_j(X)$ as a polynomial in the variables $X_1, \ldots, X_{k+1}, Y_1, \ldots, Y_{k+1}$. Since F_j and F_i are homogeneous of degree d, for any i, j we have

$$|F_i(X)F_j(Y) - F_i(Y)F_j(X)| \leq C'(K)\|F\|^2 \cdot (\|X\| \cdot \|Y\|)^{d-1} \cdot \max_{m,\ell} |X_mY_\ell - X_\ell Y_m|,$$

where $C'(K) \geq 1$ depends only on d and k if K is archimedean and $C'(K) = 1$ if K is non-archimedean. This gives the wanted inequality. □

3.2. Hölder regularity of the Green function: proof of Theorem 3.1

Recall that we defined the Green function of the lift F as $g_F := \sum_{j\geq 0} d^{-j} u_F \circ f^j$. The proof consists in first giving a Hölder estimate involving the next three constants depending on F, and then estimating those constants. Define

$$\begin{cases} C_1(F) &:= \sup_{z\in\mathbb{P}^k(K)} |u_F(z)|, \\ C_2(F) &:= \inf\{L \geq 0 \ : \ \forall z, w \in \mathbb{P}^k(K), \ |u_F(z) - u_f(w)| \leq L \cdot \mathrm{dist}(z,w)\}, \\ C_3(F) &:= \inf\{L \geq 0 \ : \ \forall z, w \in \mathbb{P}^k(K), \ \mathrm{dist}(f(z), f(w)) \leq L \cdot \mathrm{dist}(z,w)\}. \end{cases}$$

We follow a classical argument from complex dynamics (see, e.g. [**DS**]): pick $z, w \in \mathbb{P}^k(K)$ and fix an integer $N \geq 1$. First assume $(K, |\cdot|)$ is archimedean. By definition of g_F,

$$\begin{aligned}|g_F(z) - g_F(w)| &\leq \sum_{n\geq 0} \frac{1}{d^n} |u_F(f^n(z)) - u_F(f^n(w))| \\ &\leq \sum_{n=0}^{N-1} \frac{1}{d^n} |u_F(f^n(z)) - u_F(f^n(w))| + 2C_1(F) \sum_{n\geq N} d^{-n} \\ &\leq C_2(F) \sum_{n=0}^{N-1} \frac{1}{d^n} \mathrm{dist}(f^n(z), f^n(w)) + 2C_1(F) \frac{d^{-N}}{d-1} \\ &\leq C_2(F) \sum_{n=0}^{N-1} \left(\frac{C_3(F)}{d} \right)^n \mathrm{dist}(z,w) + 2C_1(F) \frac{d^{-N}}{d-1}.\end{aligned}$$

We now use Lemmas 3.2, 3.3 and 3.4. We have

1. $C_1(F) \leq \log\max\{\|F\|, 1\} + \log C_4$,
2. $C_2(F) \leq C_4^2 \dfrac{\|F\|^{(k+1)d^k}}{d|\mathrm{Res}(F)|} \log\left(\dfrac{C_4^2\|F\|^{(k+1)d^k}}{|\mathrm{Res}(F)|}\right)$, and
3. $C_3(F) \leq C_4 \left(\dfrac{\|F\|^{(k+1)d^k}}{|\mathrm{Res}(F)|}\right)^2$,

where $C_4 = C_4(K,d,k) := \max\{C_1(K,d,k), C_2(K,d,k), C_3(K,d,k)\} \geq 1$ is a constant depending only on d and k when K is archimedean and $C_4 = 1$ when K is non-archimedean. We first treat the case where $(K, |\cdot|)$ is archimedean. We then can replace $C_3(F)$ by $\max\{2d, C_3(F)\} \geq d+2$ and the above gives

$$|g_F(z) - g_F(w)| \leq C_2(F)\left(\frac{\max\{2d, C_3(F)\}}{d}\right)^N \mathrm{dist}(z,w) + C_1(F)\cdot d^{-N}.$$

This in particular implies

$$C_2(F) \leq \frac{C_4^2}{d}\max\left\{2d, C_4\left(\frac{\|F\|^{(k+1)d^k}}{|\mathrm{Res}(F)|}\right)^2\right\} \times \log\left(C_4\frac{\|F\|^{(k+1)d^k}}{|\mathrm{Res}(F)|}\right),$$

so that the above rewrites as

$$\begin{aligned}|g_F(z) - g_F(w)| \leq\ & 2C_4^2 \log\max\left(C_4\frac{\|F\|^{(k+1)d^k}}{|\mathrm{Res}(F)|}\right)\\ &\times\left(\frac{C_4}{d}\cdot\max\left\{2d, \frac{\|F\|^{2(k+1)d^k}}{|\mathrm{Res}(F)|^2}\right\}\right)^{N+1}\\ &\times \mathrm{dist}(z,w) + \frac{\log^+\|F\| + \log C_4}{d^N}.\end{aligned}$$

Recall that $\mathrm{dist}(z,w) \leq 1$ by definition. We now choose $N \geq 0$ so that

$$N+1 \geq -\log\mathrm{dist}(z,w)/\log\left(C_4\max\left\{2d, \frac{\|F\|^{2(k+1)d^k}}{|\mathrm{Res}(F)|^2}\right\}\right) \geq N.$$

Then $d^{-(N+1)} \leq \mathrm{dist}(z,w)^{\log d/\log\left(C_4\max\left\{2d, \frac{\|F\|^{2(k+1)d^k}}{|\mathrm{Res}(F)|^2}\right\}\right)}$ and the above gives

$$|g_F(z)-g_F(w)| \leq C_4^2\left(2\log\left(\frac{\|F\|^{(k+1)d^k}}{|\mathrm{Res}(F)|}\right) + 2d\log C_4^2 + d\log^+\|F\|\right)\times\mathrm{dist}(z,w)^\alpha,$$

where $\alpha = \log d/\left(\log C_4 + \log\max\left\{2d, \frac{\|F\|^{2(k+1)d^k}}{|\mathrm{Res}(F)|^2}\right\}\right)$, ending the proof in this case.

We now assume $(K, |\cdot|)$ is non-archimedean. Then we have

1. $C_1(F) \leq \log^+\|F\|$,
2. $C_2(F) \leq \dfrac{\|F\|^{(k+1)d^k}}{d|\mathrm{Res}(F)|}\log\left(\dfrac{\|F\|^{(k+1)d^k}}{|\mathrm{Res}(F)|}\right)$, and

3. $C_3(F) \leq \left(\frac{\|F\|^{(k+1)d^k}}{|\mathrm{Res}(F)|}\right)^2$,

Let $R := |\mathrm{Res}(F)|/\|F\|^{(k+1)d^k} \leq 1$. By Lemma 3.3 and by definition of g_F, if $R = 1$, then $g_F \equiv 0$ and we can set $\alpha := 1$. We thus assume $R < 1$ and pick $x, y \in \mathbb{P}^k(K)$. If $\mathrm{dist}(f^N(x), f^N(y)) \leq R^2 \leq R \leq 1$ for any $N \geq 0$, we have $g_F(x) = g_F$ by construction and Lemma 3.3. In particular, this applies if $x = y$. Otherwise, there is N minimal such that $\mathrm{dist}(f^N(x), f^N(y)) > R$. As $C_3(F) \leq R^{-2}$ and N is minimal, this implies $R^2 \leq R^{-2N}\mathrm{dist}(x,y)$ and $R^2 > R^{-2(N-1)}\mathrm{dist}(x,y)$, i.e.

$$R^{2(N+1)} \leq \mathrm{dist}(x,y) \leq R^{2N}.$$

By Lemma 3.3, we deduce that

$$\begin{aligned}|g_F(z) - g_F(w)| &\leq \sum_{n\geq 0} \frac{1}{d^n}|u_F(f^n(z)) - u_F(f^n(w))| \\ &\leq \sum_{n\geq N} \frac{1}{d^n}|u_F(f^n(z)) - u_F(f^n(w))| \\ &\leq 2C_1(F)\sum_{n\geq N} d^{-n} \leq 2C_1(F)\frac{d^{-N}}{d-1}.\end{aligned}$$

By definition of N, one has $2N \leq \frac{\log \mathrm{dist}(x,y)}{\log R} \leq 2(N+1)$ and the above gives

$$|g_F(x) - g_F(y)| \leq 2C_1(F)\frac{d^{-\frac{\log \mathrm{dist}(x,y)}{2\log R}}}{d-1} = 2C_1(F)\frac{\mathrm{dist}(x,y)^{-\frac{\log d}{2\log R}}}{d-1}.$$

By definition of R, this concludes the proof.

4. Hölder estimates and an inequality for the mutual energy

4.1. A height estimate for Hölder adelic line bundle

As above, we use the notation $L = \mathcal{O}_{\mathbb{P}^1}(1)$. We say $\bar{L} = (L, \{\|\cdot\|_v\}_{v\in M_{\mathbb{K}}})$ is an *adelically Hölder line bundle* if $\bar{L}$ is semi-positive continuous line bundle in the sense of Zhang and if there exists

- an adelic constant $C := \{C_v\}_{v\in M_{\mathbb{K}}}$ with $C_v \geq e$ at archimedean places, and
- an adelic constant $\alpha := \{\alpha_v\}_{v\in M_{\mathbb{K}}}$ which is small at archimedean places,

such that for any $v \in M_{\mathbb{K}}$, if $g_v := \log\|\cdot\|_v/\|\cdot\|_{v,0}$,then any $z, w \in \mathbb{P}^1(\mathbb{C}_v)$, we have

$$|g_v(z) - g_v(w)| \leq \log(C_v)\mathrm{dist}_v(z,w)^{\alpha_v}.$$

Here dist_v denotes the chordal distance on $\mathbb{P}^1_{\mathbb{C}_v}$ defined in §2.1.

We prove here the following.

Theorem 4.1. — *Assume $\bar{L}$ is an adelically Hölder metrized line bundle with adelic constants $C = \{C_v\}_{v\in M_{\mathbb{K}}}$ and $\alpha = \{\alpha_v\}_{v\in M_{\mathbb{K}}}$. Let $\varepsilon = \{\varepsilon_v\}_{v\in M_{\mathbb{K}}}$ be a small adelic constant and let $E \subset \mathbb{A}^1(\bar{\mathbb{K}})$ be a $\mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$-invariant set. Then*

$$h_{\bar{L}}(\mathbb{P}^1) \leq \sum_{v\in M_{\mathbb{K}}} \frac{N_v}{[\mathbb{K}:\mathbb{Q}]}\left(2\log(C_v)\varepsilon_v^{\alpha_v} - \frac{\log(\varepsilon_v)}{2\#E}\right) + \frac{1}{\#E}\cdot\sum_{x\in E} h_{\bar{L}}(x).$$

Proof. — Let $s \in H^0(\mathbb{P}^1, L)$ be non-zero with $\mathrm{div}(s) = [\infty]$. We let $\bar{L}_0 := (L, \{\|\cdot\|_{v,0}\}_{v\in M_{\mathbb{K}}})$ be the standard (naive) adelic metrization on L.

For a given finite $\mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})-$invariant set $E \subset \mathbb{A}^1(\bar{\mathbb{K}})$ and a given small adelic constant $\varepsilon := \{\varepsilon_v\}_{v\in M_{\mathbb{K}}}$, define a metrization $\{\|\cdot\|_{\varepsilon_v,v}\}_{v\in M_{\mathbb{K}}}$ on L as follows: on $\mathbb{A}^1_{\mathbb{C}_v}$, let

$$g_{\varepsilon,v}(z) := \frac{1}{\#E}\sum_{x\in E}\log\max\{|z-x|_v, \varepsilon_v\}, \quad z \in \mathbb{A}^1(\mathbb{C}_v).$$

The function $g_{\varepsilon,v}$ satisfies $g_{\varepsilon,v}(z) = \log|z|_v$ for all $z \in \mathbb{A}^1(\mathbb{C}_v)$ with $|z|_v$ large enough. In particular, if for for any degree one polynomial function σ, we set

$$\|\sigma(z)\|_{\varepsilon,v} := |\sigma(z)|_v e^{-g_{\varepsilon,v}(z)}, \quad z \in \mathbb{A}^1(\mathbb{C}_v),$$

we define a metric $\|\cdot\|_{\varepsilon,v}$ on L which is continuous. Denote by $\bar{L}_\varepsilon$ the adelic line bundle $\bar{L}_\varepsilon := (L, \{\|\cdot\|_{\varepsilon,v}\}_{v\in M_{\mathbb{K}}})$. The adelic line bundle $\bar{L} - \bar{L}_\varepsilon$ is an integrable line bundle which underlying line bundle is the trivial bundle $\mathcal{O}_{\mathbb{P}^1}$. We thus can apply the Arithmetic Hodge Index Theorem of Yuan and Zhang (see Theorem 2.2): we have

$$(\bar{L})^2 - 2(\bar{L}\cdot\bar{L}_\varepsilon) + (\bar{L}_\varepsilon)^2 = (\bar{L}-\bar{L}_\varepsilon)^2 \le 0. \tag{2}$$

We can easily deduce the following from [**FRL**] and [**F**]:

Lemma 4.2. — *We have* $(\bar{L}_\varepsilon)^2 \ge \frac{1}{\#E}\sum_{v\in M_{\mathbb{K}}} N_v\log(\varepsilon_v)$.

Take Lemma 4.2 for granted. Combining (2) with Lemma 4.2, we find

$$(\bar{L})^2 \le 2(\bar{L}\cdot\bar{L}_\varepsilon) - (\bar{L}_\varepsilon)^2 \le 2(\bar{L}\cdot\bar{L}_\varepsilon) - \frac{1}{\#E}\sum_{v\in M_{\mathbb{K}}} N_v\log(\varepsilon_v). \tag{3}$$

To conclude the proof, we need to estimate $(\bar{L}\cdot\bar{L}_\varepsilon)$ from above. Up to replacing $\mathbb{K}$ by an extension $\mathbb{L}$ and to normalizing the computations by $1/[\mathbb{L}:\mathbb{K}]$, we can assume $E \subset \mathbb{A}^1(\mathbb{K})$. As above, let s be the section with $\mathrm{div}(s) = [\infty]$. Then

$$(\bar{L}\cdot\bar{L}_\varepsilon) = (\bar{L}_\varepsilon|\mathrm{div}(s)) + \sum_{v\in M_{\mathbb{K}}} N_v\int_{\mathbb{A}^{1,\mathrm{an}}_v}\log\|s\|_v^{-1}c_1(\bar{L}_\varepsilon)_v.$$

By construction, $(\bar{L}_\varepsilon|\mathrm{div}(s)) = (\bar{L}_0|\mathrm{div}(s)) = 0$ and, as s does not vanish on E, we have

$$\frac{[\mathbb{K}:\mathbb{Q}]}{\#E}\sum_{x\in E}h_{\bar{L}}(x) = \frac{1}{\#E}\sum_{x\in E}\sum_{v\in M_{\mathbb{K}}}N_v\log\|s(x)\|_v^{-1}.$$

By definition of $\bar{L}_\varepsilon$, for any $v \in M_{\mathbb{K}}$ one can write $c_1(\bar{L}_\varepsilon)_v = \frac{1}{\#E}\sum_{x\in E}\delta_{\zeta_{x,\varepsilon_v}}$ as measures on $\mathbb{A}^{1,\mathrm{an}}_v$. In particular, if we let ζ_{x,ε_v} be point of $\mathbb{A}^{1,\mathrm{an}}_v$ corresponding to the seminorm $\sup_{\bar{B}(x,\varepsilon_v)}|\cdot|$ on $\mathbb{C}_v[T]$, we find

$$\begin{aligned}\int_{\mathbb{A}^{1,\mathrm{an}}_v}\log\|s\|_v^{-1}c_1(\bar{L}_\varepsilon)_v &= \frac{1}{\#E}\sum_{x\in E}\int_{\mathbb{A}^{1,\mathrm{an}}_v}\log\|s\|_v^{-1}\delta_{\zeta_{x,\varepsilon_v}}\\ &= \frac{1}{\#E}\sum_{x\in E}\int_{\mathbb{A}^{1,\mathrm{an}}_v}\log\left(\frac{\|s(x)\|_v}{\|s\|_v}\right)\delta_{\zeta_{x,\varepsilon_v}} + \frac{1}{\#E}\sum_{x\in E}\log\|s(x)\|_v^{-1}.\end{aligned}$$

Denote as above $g_v = \log\|\cdot\|_v/\|\cdot\|_{v,0}$. The function g_v satisfies

$$|g_v(z) - g_v(w)| \le \log(C_v)\mathrm{dist}_v(z,w)^{\alpha_v} \le \log(C_v)\|z-w\|_v^{\alpha_v}.$$

for all $z, w \in \mathbb{A}^1(\mathbb{C}_v)$. The above gives

$$\int_{\mathbb{A}_v^{1,\mathrm{an}}} \log \|s\|_v^{-1} c_1(\bar{L}_\varepsilon)_v = \frac{1}{\#E} \sum_{x\in E} \int_{\mathbb{A}_v^{1,\mathrm{an}}} (g_v(x) - g_v)\delta_{\zeta_{x,\varepsilon_v}} + \frac{1}{\#E} \sum_{x\in E} \int_{\mathbb{A}_v^{1,\mathrm{an}}} \log\left(\frac{\|s(x)\|_{v,0}}{\|s\|_{v,0}}\right)\delta_{\zeta_{x,\varepsilon_v}} + \frac{1}{\#E}\sum_{x\in E} \log \|s(x)\|_v^{-1}.$$

By assumption, we deduce

$$\int_{\mathbb{A}_v^{1,\mathrm{an}}} \log \|s\|_v^{-1} c_1(\bar{L}_\varepsilon)_v \le \log(C_v)\varepsilon_v^{\alpha_v} + \frac{[\mathbb{K}:\mathbb{Q}]}{\#E}\sum_{x\in E}\log\|s(x)\|_v^{-1} + \frac{1}{\#E} \sum_{x\in E} \int_{\mathbb{A}_v^{1,\mathrm{an}}} \log\left(\frac{\|s(x)\|_{v,0}}{\|s\|_{v,0}}\right)\delta_{\zeta_{x,\varepsilon_v}}.$$

An easy computation gives the next lemma

Lemma 4.3. — $\displaystyle\int_{\mathbb{A}_v^{1,\mathrm{an}}} \log\left(\frac{\|s(x)\|_{v,0}}{\|s\|_{v,0}}\right)\delta_{\zeta_{x,\varepsilon_v}} \le \begin{cases} \varepsilon & \textit{if } v \textit{ is archimedean,} \\ 0 & \textit{if } v \textit{ is non-archimedean,}\end{cases}$

For v archimedean, since $C_v \ge e$ and $\alpha_v \le 1$, we have $\varepsilon_v \le \log(C_v)\varepsilon_v^{\alpha_v}$. Pick again $v \in M_{\mathbb{K}}$. This implies

$$\int_{\mathbb{A}_v^{1,\mathrm{an}}} \log \|s\|_v^{-1} c_1(\bar{L}_\varepsilon)_v \le 2\log(C_v)\varepsilon_v^{\alpha_v} + \frac{[\mathbb{K}:\mathbb{Q}]}{\#E}\sum_{x\in E}\log\|s(x)\|_v^{-1}. \tag{4}$$

Since $h_{\bar{L}}(\mathbb{P}^1) = \frac{(\bar{L})^2}{2[\mathbb{K}:\mathbb{Q}]}$, the conclusion follows from summing (4) over all places $v \in M_{\mathbb{K}}$ and using (3). □

Proof of Lemma 4.2. — By definition, we have

$$(\bar{L}_\varepsilon^2) = (\bar{L}_\varepsilon|\mathrm{div}(s)) + \sum_{v\in M_{\mathbb{K}}} N_v \int_{\mathbb{A}_v^{1,\mathrm{an}}} \log\|\cdot\|_{\varepsilon,v}^{-1} c_1(\bar{L}_\varepsilon)_v,$$

where s is a section of L with divisor $[\infty]$. By construction of $\bar{L}_\varepsilon$, we have $(\bar{L}_\varepsilon|C\cap H_\infty) = (\bar{L}_0|C\cap H_\infty) = 0$. Pick now $v \in M_{\mathbb{K}}$, then $c_1(\bar{L}_\varepsilon)_v = dd^c(g_{\varepsilon,v})$, so that

$$(\bar{L}_\varepsilon^2) = (\bar{L}_0|C\cap H_\infty) + \sum_{v\in M_{\mathbb{K}}} N_v \int_{C_v^{\mathrm{an}}} g_{\varepsilon,v} dd^c g_{\varepsilon,v}.$$

Fix $v \in M_{\mathbb{K}}$. Lemma 12 from [**F**] and lemma 4.11 from [**FRL**] rewrite as

$$\int_{C_v^{\mathrm{an}}} g_{\varepsilon,v} dd^c g_{\varepsilon,v} \ge \frac{1}{\#E}\log(\varepsilon_v) + \frac{1}{(\#E)^2}\sum_{x\neq y\in E}\log|x-y|_v$$

and the product formula implies

$$\sum_{v\in M_{\mathbb{K}}} N_v \log|x-y|_v = 0.$$

This concludes the proof. □

Proof of Lemma 4.3. — By the triangle inequality,

$$\int_{\mathbb{A}^{1,\mathrm{an}}_v} \log\left(\frac{\|s(x)\|_{v,0}}{\|s\|_{v,0}}\right)\delta_{\zeta_{x,\varepsilon_v}} \leq \begin{cases} \log^+(\|z-x\|_v+\|x\|_v)-\log^+\|x\|_v & \text{if } v \text{ archimedean,}\\ \log^+\max\{\|z-x\|_v,\|x\|_v\}-\log^+\|x\|_v & \text{otherwise,}\end{cases}$$

For z lying in the support of $dd^c_z \log\max\{\|z-x\|_v,\varepsilon_v\}$, this gives

$$\log\left(\frac{\|s(x)\|_{v,0}}{\|s(z)\|_{v,0}}\right) \leq \begin{cases} \log(1+\varepsilon_v)\leq\varepsilon_v & \text{if } v \text{ is archimedean,}\\ 0 & \text{if } v \text{ is non-archimedean,}\end{cases}$$

and the proof is complete. □

4.2. The mutual energy - height of points relation: proof of Theorem B

Pick integers $d_1, d_2 \geq 2$, pick rational maps $f_1 \in \mathrm{Rat}_{d_1}(\bar{\mathbb{Q}})$ and $f_2 \in \mathrm{Rat}_{d_2}(\bar{\mathbb{Q}})$ and let $\mathbb{K}$ be a number field such that f_1 and f_2 are both defined over $\mathbb{K}$.

Let $\bar{L} := \frac{1}{2}(\bar{L}_{f_1}+\bar{L}_{f_2})$, where $\bar{L}_{f_i}$ is the canonical metric of f_i on $\mathcal{O}_{\mathbb{P}^1}(1)$.

Fix $i=1,2$ and apply Theorem 3.1: Given a polynomial lift F_i of f_i, defined over a number field $\mathbb{K}$, we have $\mathrm{Res}(F_i)\in\mathbb{K}^\times$. Up to replacing $\mathbb{K}$ with an extension, we can assume there is $\alpha_i\in\mathbb{K}^\times$ such that $\mathrm{Res}(F_i)=\alpha_i^{2d_i}$. We thus may replace F_i by $\alpha_i^{-1}F_i$ to get $\mathrm{Res}(F_i)=1$. For a given $v\in M_{\mathbb{K}}$, if $X,Y\in\mathbb{A}^2(\bar{\mathbb{Q}})\setminus\{0\}$ with $\|X\|_v=\|Y\|_v=1$, by (1), we have

$$\frac{C_v(d_i)}{\|F_i\|_v^{2d_i}} \leq \frac{\|F_i(X)\|}{\|F_i(Y)\|} \leq C_v(d_i)\|F_i\|_v^{2d_i},$$

with $C_v(d_i)=1$ if v is non-archimedean and $C_v(d_i)\geq 1$ depending only on d if v is archimedean. In particular, we have $\|F_i\|_v\geq 1$ if v is non-archimedean and $\|F_i\|_v\geq 1/\tilde{C}(d_i)$ when v is non-archimedean, where $\tilde{C}(d_i)^{2d_i}=C(d_i)$.

For any $v\in M_{\mathbb{K}}$, we denote by $u_{F,v}$, $g_{F,v}$ and $\|F\|_v$ the objects defined in Section 3 using the v-adic norm.

We let $C_v(F_i)$ and $\alpha_v(F_i)$ be the constants of the Hölder property of $\bar{L}_{f_i}$ at place v. By Theorem 3.1, if v is archimedean we have

$$\begin{cases} C_v(F_i)=C_3\left(C_1+(2d_i+1)\log\|F_i\|_v\right), & \text{and}\\ \alpha_v(F_i)^{-1}=\frac{1}{\log d_i}\left(C_2+\max\{\log(2d_i),2d_i\log\|F_i\|_v\}\right), & \end{cases}$$

where $C_1,C_2,C_3\geq 1$ depend only on d and, when v is non-archimedean,

$$\begin{cases} C_v(F_i)=C_3(2d_i+1)\log\|F_i\|_v, & \text{and}\\ \alpha_v(F_i)^{-1}=\begin{cases}1 & \text{if } \|F_i\|_v=1,\\ 4d_i\log\|F_i\|_v/\log(d_i) & \text{otherwise.}\end{cases} & \end{cases}$$

For any $v\in M_{\mathbb{K}}$ with $C_v(F_i)=0$, we have $\alpha_v(F_i)=1$ and $g_{F_i,v}\equiv 0$ in this case.

By this discussion, the adelic line bundle $\bar{L}$ is adelically Hölder with constants $\{C_v(\bar{L})\}_{v\in M_{\mathbb{K}}}$ and $\{\alpha_v(\bar{L})\}_{v\in M_{\mathbb{K}}}$ satisfying

$$\begin{cases} 1\leq C_v(\bar{L})\leq C_3\left(C_1+(2d_1+1)\log\|F_1\|_v+(2d_2+1)\log\|F_2\|_v\right), & \text{and}\\ \alpha_v(\bar{L})^{-1}\leq\sum_{i=1}^2\frac{1}{\log d_i}\left(C_2+\max\{\log(2d_i),2d_i\log\|F_i\|_v\}\right), & \end{cases}$$

where $C_1,C_2,C_3\geq 1$ depend only on d when v is archimedean, and

$$\begin{cases} 1\leq C_v(\bar{L})\leq C_3((2d_1+1)\log\|F_1\|_v+(2d_2+1)\log\|F_2\|_v), & \text{and}\\ \alpha_v(\bar{L})^{-1}\leq\begin{cases}1 & \text{if } \|F_1\|_v=\|F_2\|_v=1,\\ \sum_{i=1}^2 4d_i\log\|F_i\|_v/\log(d_i) & \text{otherwise,}\end{cases} & \end{cases}$$

when v is non-archimedean.

Choose a finite Galois-invariant subset $E \subset \mathbb{A}^1(\bar{\mathbb{Q}})$ and a small adelic constant $\{\varepsilon_v\}_{v\in M_{\mathbb{K}}}$. Combined with Theorem 4.1 and Lemma 2.3, the above gives

$$(5) \quad \langle f_1, f_2\rangle \leq \frac{1}{\#E}\sum_{x\in E}(\hat{h}_{f_1}(x)+\hat{h}_{f_2}(x)) + \sum_{v\in M_{\mathbb{K}}}\frac{N_v}{[\mathbb{K}:\mathbb{Q}]}\left(4\log C_v(\bar{L})\varepsilon_v^{\alpha_v(\bar{L})} - \frac{\log(\varepsilon_v)}{\#E}\right).$$

We now choose the small adelic constant $\{\varepsilon_v\}_{v\in M_{\mathbb{K}}}$. Fix $0<\delta<1$ and $v\in M_{\mathbb{K}}$ and set

$$\varepsilon_v := \begin{cases} \delta^{1/\alpha_v(\bar{L})} & \text{if } \log C_v(\bar{L})>0,\\ 1 & \text{otherwise.}\end{cases}$$

To conclude the proof, we use (5) and the choice of $\{\varepsilon_v\}_v$ to get:

$$\begin{aligned}\mathscr{E} :=& \langle f_1, f_2\rangle - \frac{1}{\#E}\sum_{x\in E}(\hat{h}_{f_1}(x)+\hat{h}_{f_1}(x))\\ \leq& \sum_{v\in M_{\mathbb{K}}}\frac{N_v}{[\mathbb{K}:\mathbb{Q}]}\left(4C_v(\bar{L})\varepsilon_v^{\alpha_v(\bar{L})} - \frac{\log(\varepsilon_v)}{2\#E}\right)\\ \leq& \sum_{v\in M_{\mathbb{K}}}\frac{N_v}{[\mathbb{K}:\mathbb{Q}]}4C_v(\bar{L})\delta - \sum_{v\in M_{\mathbb{K}},\ C_v(\bar{L})\neq 0}\frac{N_v}{[\mathbb{L}:\mathbb{Q}]}\frac{\log(\delta)}{\alpha_v(\bar{L})\#E}.\end{aligned}$$

Let $M_{\mathbb{K}}^\infty$ denote the set of archimedean places of $\mathbb{K}$. The above implies

$$\begin{aligned}\sum_{v\in M_{\mathbb{K}}}\frac{N_v}{[\mathbb{K}:\mathbb{Q}]}4C_v(\bar{L}) \leq& A\sum_{v\in M_{\mathbb{K}}}\frac{N_v}{[\mathbb{L}:\mathbb{Q}]}(\log\|F_1\|_v + \log\|F_2\|_v)\\ &+ 4C_3\sum_{v\in M_{\mathbb{K}}^\infty}\frac{N_v}{[\mathbb{K}:\mathbb{Q}]}\log(C_1)\\ \leq& A\cdot\left(h_{\mathrm{Rat}_{d_1}}(f_1)+h_{\mathrm{Rat}_{d_2}}(f_2)\right)+B,\end{aligned}$$

where $A = 4C_3(d_1+d_2+1)$ and $B := 4C_3\log(C_1)$. Similarly, there are $A_1\geq A$ anb $B_1\geq B$ depending only on d_1 and d_2 such that

$$\begin{aligned}\sum_{\substack{v\in M_{\mathbb{K}}\\ C_v(\bar{L})\neq 0}}\frac{N_v}{[\mathbb{K}:\mathbb{Q}]}\frac{1}{\alpha_v(\bar{L})} \leq& \sum_{\substack{v\in M_{\mathbb{K}}\\ C_v(\bar{L})\neq 0}}\frac{N_v}{[\mathbb{K}:\mathbb{Q}]}A_1\cdot(\log\|F_1\|_v+\log\|F_2\|_v)\\ &+ B_1\sum_{v\in M_{\mathbb{K}}^\infty}\frac{N_v}{[\mathbb{K}:\mathbb{Q}]}\log(C_2)\\ \leq& A_1\cdot\left(h_{\mathrm{Rat}_{d_1}}(f_1)+h_{\mathrm{Rat}_{d_2}}(f_2)\right)+B',\end{aligned}$$

where $B' = B_1\log(C_2)$. This concludes the proof.

5. A current and a measure on parameter spaces

5.1. The complex pairing and a bifurcation current

For the whole section, we fix two integers $d_1, d_2\geq 2$ and we consider families of pairs of rational maps $(f_{1,t}, f_{2,t})$ of degrees d_1 and d_2 parametrized by a complex quasi-projective variety S. Such a family is given by a morphism

$$\mathcal{F} : S \times \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow S \times \mathbb{P}^1 \times \mathbb{P}^1$$

such that, for any $t \in S$, $(f_{1,t}, f_{2,t}) = \mathcal{F}(t, \cdot, \cdot)$ is a pair of rational maps of respective degrees d_1 and d_2. In what follows, we denote by

- $\pi : S \times \mathbb{P}^1 \times \mathbb{P}^1 \to S$ the canonical projection and
- $\pi_i : S \times \mathbb{P}^1 \times \mathbb{P}^1 \to S \times \mathbb{P}^1$ the projection onto S and the i-th factor $\mathbb{P}^1$.

For $i = 1, 2$, following [**GV**], we let $\widehat{T}_{\mathfrak{f}_i}$ is the fibered Green current of the family $\mathfrak{f}_i : S \times \mathbb{P}^1 \to S \times \mathbb{P}^1$ of degree d rational maps induced by $\mathcal{F}$, we let $\widehat{T}_i := \pi_i^*(\widehat{T}_{\mathfrak{f}_i})$. Let also $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the diagonal. As in [**DM**] we define:

Definition 5.1. — *The* pairwise-bifurcation current *of the family $\mathcal{F}$ is*

$$T_{\mathcal{F},\Delta} := \pi_* \left(\left(\widehat{T}_1 + \widehat{T}_2 \right)^{\wedge 2} \wedge [S(\mathbb{C}) \times \Delta] \right).$$

We also define a function $U : S(\mathbb{C}) \longrightarrow \mathbb{R}_+$ by

$$U(f_{1,t}, f_{2,t}) := (\mu_{f_{1,t}} - \mu_{f_{2,t}}, \mu_{f_{1,t}} - \mu_{f_{2,t}}), \quad t \in S(\mathbb{C}).$$

Lemma 5.2. — *The function U is plurisubharmonic and continuous and satisfies*

$$T_{\mathcal{F},\Delta} = 2\mathrm{dd}^c U.$$

Proof. — Let $\Omega \subset S(\mathbb{C})$ be a small simply connected open subset. This defines two analytic families of rational maps $f_{1,t}$ and $f_{2,t}$ of degree d_1 and d_2 respectively. Up to reducing Ω, we can define holomorphic families $F_{1,t}$ and $F_{2,t}$ of polynomial homogeneous lifts. By definition, on has $\mu_{f_{1,t}} - \mu_{f_{2,t}} = \mathrm{dd}^c(g_{F_{1,t}} - g_{F_{2,t}})$, so that [**FRL**, §2] gives

$$(\mu_{f_{1,t}} - \mu_{f_{2,t}}, \mu_{f_{1,t}} - \mu_{f_{2,t}}) = -\int_{\mathbb{P}^1(\mathbb{C})} (g_{F_{1,t}} - g_{F_{2,t}})\mathrm{dd}^c(g_{F_{1,t}} - g_{F_{2,t}}).$$

The continuity of $(z, t) \mapsto g_{F_{1,t}}(z)$ and $(z, t) \mapsto g_{F_{2,t}}(z)$ implies that U is continuous on Ω.

We now prove that U is psh and that $\mathrm{dd}^c U = \pi_*((\widehat{T}_1 + \widehat{T}_2)^{\wedge 2} \wedge [S(\mathbb{C}) \times \Delta])$. Let ω be the standard Fubini-Study form on $\mathbb{P}^1(\mathbb{C})$ and $p_1 : S \times \mathbb{P}^1 \to S$ and $p_2 : S \times \mathbb{P}^1 \to \mathbb{P}^1$ be the canonical projections. Recall that $\widehat{T}_i = \lim_n d_i^{-n}(f_i^n)^*(p_i^*\omega)$ and that the local ppotentials converge uniformly. As $(p_i^*\omega)^{\wedge 2}) = 0$ for $i = 1, 2$, this implies $\widehat{T}_1^{\wedge 2} = \widehat{T}_2^{\wedge 2} = 0$. Denote by $w(t, z) := g_{F_{1,t}}(z) - g_{F_{2,t}}(z)$ and define a $(1, 1)$-current T_Ω on $\Omega \times \mathbb{P}^1$ by

$$T_\Omega := -w(\widehat{T}_1 - \widehat{T}_2).$$

By definition, since $\widehat{T}_1^2 = \widehat{T}_2^2 = 0$, we have $\mathrm{dd}^c T_\Omega = \widehat{T}_1 \wedge \widehat{T}_2$. By [**BB**, Proposition 4.3],

$$(T_\Omega)_t = -(g_{F_{1,t}}(z) - g_{F_{2,t}}(z))\mathrm{dd}^c(g_{F_{1,t}} - g_{F_{2,t}}),$$

which in turn implies

$$(p_1)_*(T_\Omega) = -\int_{\mathbb{P}^1(\mathbb{C})} (g_{F_{t,1}} - g_{F_{2,t}})\mathrm{dd}^c(g_{F_{1,t}} - g_{F_{2,t}}) = (\mu_{f_{1,t}} - \mu_{f_{2,t}}, \mu_{f_{1,t}} - \mu_{f_{2,t}}).$$

We thus have $(p_1)_*(T_\Omega) = U$ and for a test form ψ on Ω,

$$\langle \mathrm{dd}^c U, \psi \rangle = \int_\Omega U \mathrm{dd}^c \psi = \int_\Omega (p_1)_*(T_\Omega)\mathrm{dd}^c \psi = \int_\Omega \mathrm{dd}^c (p_1)_*(T_\Omega) \wedge \psi,$$

and if $T = (\widehat{T}_1+\widehat{T}_2)^{\wedge 2}\wedge[S(\mathbb{C})\times\Delta]$, we have

$$\begin{aligned}\langle \pi_*T,\psi\rangle = \langle T,\pi^*\psi\rangle &= 2\int_\Omega \widehat{T}_1\wedge\widehat{T}_2\wedge p_1^*\psi\\ &= 2\int_\Omega (p_1)_*(\mathrm{dd}^cT_\Omega)\wedge\psi = 2\int_\Omega \mathrm{dd}^c(p_1)_*(T_\Omega)\wedge\psi = 2\langle \mathrm{dd}^cU,\psi\rangle.\end{aligned}$$

Since T is a positive current, dd^cU is positive. This concludes the proof. □

Assume now that $S=\mathrm{Rat}_{d_1}\times\mathrm{Rat}_{d_2}$. The next lemma is important in what follows.

Lemma 5.3. — *For any* $(f,g)\in\mathrm{Rat}_{d_1}\times\mathrm{Rat}_{d_2}(\mathbb{C})$ *and any* $\varphi\in\mathrm{PGL}(2,\mathbb{C})$,

$$U(f,g)=U(\varphi^{-1}\circ f\circ\varphi,\varphi^{-1}\circ g\circ\varphi).$$

Proof. — Pick $(f,g)\in\mathrm{Rat}_{d_1}\times\mathrm{Rat}_{d_2}(\mathbb{C})$ and $\varphi\in\mathrm{PGL}(2,\mathbb{C})$. We use the notations $f^\varphi:=\varphi^{-1}\circ f\circ\varphi$ and $g^\varphi:=\varphi^{-1}\circ g\circ\varphi$ in this proof. By construction, we have

$$\mu_{f^\varphi}=\lim_{n\to\infty}d^{-n}((f^\varphi)^n)^*\omega=\lim_{n\to\infty}d^{-n}\varphi^*(f^n)^*((\varphi^{-1})^*\omega)=\varphi^*\mu_f.$$

In particular, for any choice of lifts F of f and G of g respectively, we have $\mu_{f^\varphi}=\varphi^*(\omega)+\mathrm{dd}^c(g_F\circ\varphi)$ and $\mu_{g^\varphi}=\varphi^*(\omega)+\mathrm{dd}^c(g_G\circ\varphi)$, whence $\mu_{f^\varphi}-\mu_{g^\varphi}=\mathrm{dd}^c(g_F\circ\varphi-g_G\circ\varphi)$. This in turn implies

$$\begin{aligned}(\mu_{f^\varphi}-\mu_{g^\varphi},\mu_{f^\varphi}-\mu_{g^\varphi}) &= -\int_{\mathbb{P}^1(\mathbb{C})}\varphi^*\left((g_F-g_G)\mathrm{dd}^c(g_F-g_G)\right)\\ &= -\int_{\mathbb{P}^1(\mathbb{C})}(g_F-g_G)\mathrm{dd}^c(g_F-g_G)\\ &= (\mu_f-\mu_g,\mu_f-\mu_g),\end{aligned}$$

by the change of variable formula. □

5.2. The measure on the quotient space

Pick two integers $d_1,d_2\geq 2$. Define $X_{d_1,d_2}:=(\mathrm{Rat}_{d_1}\times\mathrm{Rat}_{d_2})/\mathrm{PGL}(2)$, where the action of PGL(2) is by simultaneous conjugacy. The variety X_{d_1,d_2} is qusiprojective, irreducible of dimension $2(d_1+d_2)-1$ and defined over $\mathbb{Q}$, see [**DM**] for more details on the construction of such a space. As the function U is PGL(2)-invariant, it descends to a function

$$u: X_{d_1,d_2}(\mathbb{C})\longrightarrow\mathbb{R}_+$$

which is psh and continuous on $X_{d_1,d_2}(\mathbb{C})$.

Definition 5.4. — *The* bifurcation measure *of the family* X_{d_1,d_2} *is*

$$\mu_{d_1,d_2}:=(\mathrm{dd}^cu)^{\wedge(2(d_1+d_2)-1)}.$$

Denote by $\Pi:\mathrm{Rat}_{d-1}\times\mathrm{Rat}_{d_2}\to X_{d_1,d_2}$ the canonical projection. Let also

$$\mathcal{F}:\mathrm{Rat}_{d_1}\times\mathrm{Rat}_{d_2}\times\mathbb{P}^1\times\mathbb{P}^1\to\mathrm{Rat}_{d_1}\times\mathrm{Rat}_{d_2}\times\mathbb{P}^1\times\mathbb{P}^1$$

be the universal family of pairs of rational maps.

Lemma 5.5. — *For any* $d_1,d_2\geq 2$, *the measure* μ_{d_1,d_2} *is non-zero and satisfies*

$$\Pi^*(\mu_{d_1,d_2})=\left(\frac{1}{2}T_{\mathcal{F},\Delta}\right)^{\wedge(2(d_1+d_2)-1)}.$$

Proof. — We first compute $\Pi^*(\mu_{d_1,d_2})$: for a test $(3,3)$-formψ on $V := \mathrm{Rat}_{d_1} \times \mathrm{Rat}_{d_2}(\mathbb{C})$,

$$\begin{aligned}\langle \Pi^*(\mu_{d_1,d_2}), \psi \rangle &= \int_V (\mathrm{dd}^c u \circ \Pi)^{\wedge(2(d_1+d_2)-1)} \wedge \psi = \int_V (\mathrm{dd}^c U)^{\wedge(2(d_1+d_2)-1)} \wedge \psi \\ &= \int_V \left(\frac{1}{2} T_{\mathcal{F},\Delta}\right)^{\wedge(2(d_1+d_2)-1)} \wedge \psi,\end{aligned}$$

where we used Lemma 5.3 and Lemma 5.2 successively.

We now show μ_{d_1,d_2} is non-zero. First, assume d_1 and d_2 are multiplicatively independent. Let $t_0 \in X_{d_1,d_2}(\mathbb{C})$ be the equivalence class of the (T_{d_1}, T_{d_2}), where T_{d_i} is the Chebychev degree d_i polynomial. Note that the point t_0 is a smooth point if X_{d_1,d_2}. Indeed, as X_{d_1,d_2} is the geometric quotient $(\mathrm{Rat}_{d_1} \times \mathrm{Rat}_{d_2})//\mathrm{PGL}(2)$, the singularities of X_{d_1,d_2} arise from points in $\mathrm{Rat}_{d_1} \times \mathrm{Rat}_{d_2}$ with non-trivial stabilizer, i.e. paire $(f,g) \in \mathrm{Rat}_{d_1} \times \mathrm{Rat}_{d_2}$ with an automorphism $\varphi \in \mathrm{PGL}(2) \setminus \{\mathrm{id}\}$ with $\varphi \circ f = f \circ \varphi$ and $\varphi \circ g = g \circ \varphi$. Now, the group of automorphisms $\psi \in \mathrm{PGL}(2)$ with $\psi \circ T_{d_i} = T_{d_i} \circ \psi$ is reduced to the identity (see e.g. [**FG**, § 3.1]), and we proved that X_{d_1,d_2} is smooth at t_0. By the first point of Lemma 2.4, the function u has an isolated minimum at t_0. As u is psh continuous and $\mu_{d_1,d_2} = (2\mathrm{dd}^c u)^{\wedge(2(d_1+d_2)-1)}$, by Lemma 5.2, $t_0 \in \mathrm{supp}(\mu_{d_1,d_2})$.

We now assume there are $n, m \geq 1$ such that $d_1^n = d_2^m$. If $n = m = 1$, this is proved in [**DM**]. We thus assume $nm > 1$ and let $D := d_1^n = d_2^m$. The map

$$\iota : \{(f_1, f_2)\} \in X_{d_1,d_2} \longmapsto \{(f_1^n, f_2^m)\} \in X_{D,D}$$

is proper and finite by [**D**]. Whence μ_{d_1,d_2} is proportional to $\iota^*(\tilde{T})$, where

$$\tilde{T} := \pi_* \left(\left(\widehat{T}_{1,D} + \widehat{T}_{2,D} \right)^{\wedge 2} \wedge [X_{D,D} \times \Delta] \right)^{\wedge(2(d_1+d_2)-1)}$$

and it vanishes if and only if $\tilde{T}$ does. By [**DM**, Theorem 7.1] and the criterion established in [**DM**] and [**GTV**], we have $\tilde{T} \neq 0$. Indeed, DeMarco and Mavraki provide a family of pairs of rational maps parametrized by a quasiprojective variety S_D such that the canonical projection $p_D : S_D \to X_{D,D}$ is finite-to-one onto a dense Zariski open subset $V \subset X_{D,D}$. Moreover, they show that the pairwise-bifurcation measure ν of this family is non-zero. As ν has continuous potential, its support is not contained in a proper subvariety of S_D and $p_D^*(\tilde{T}) = \nu$, whence $\tilde{T}$ is non-zero. The proof is complete. □

6. Common preperiodic points for rational maps

6.1. Height inequalities and uniformity

In this paragraph, we let X and S be quasi-projective varieties defined over a number field $\mathbb{K}$. We assume $\pi : X \to S$ is a family of curves defined over $\mathbb{K}$, i.e. π is a projective and flat morphism with relative dimension 1 whose fibers are geometrically connected. Pick an embedding $\iota : X \hookrightarrow \mathbb{P}^N \times S$ so that $\iota|_{X_t}$ is an embedding of X_t into $\mathbb{P}^N$. Let D be the divisor $D := \iota^{-1}(H_\infty)$ and assume that $L = \mathcal{O}(D)$ Let $D_t := D \cap X_t$ for $t \in S$. The line bundle L is relatively ample and we endow L with an semi-positive continuous metrization $\{\|\cdot\|_v\}_{v \in M_\mathbb{K}}$ and denote $\bar{L} = (L, \{\|\cdot\|_v\}_{v \in M_\mathbb{K}})$. We assume that

(*a*) for any $t \in S(\bar{\mathbb{K}})$, the metrized line bundle $\bar{L}_t := \bar{L}|_{X_t}$ is adelic,
(*b*) $h_{\bar{L}} \geq 0$ on $X(\bar{\mathbb{K}})$.

The following is inspired by DeMarco, Krieger and Ye [**DKY1, DKY2**].

Lemma 6.1. — *Let B be a projective model of S and M be an ample line bundle on B. Assume there are constants $C > 0$ and $C' \geq 0$ such that*

$$h_{\bar{L}_t}(X_t) \geq C h_{B,M}(t) - C', \quad \text{for all } t \in S(\bar{\mathbb{K}}) \tag{6}$$

and that there are constant $C_1 \geq 1$ and $C_2 \geq 0$ such that for any finite $\mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$*-invariant set $E \subset X_t(\bar{\mathbb{K}}) \setminus \mathrm{supp}(D_t)$, and any $0 < \delta < 1$, then*

$$h_{\bar{L}_t}(X_t) \leq C_1 \left(\delta - \frac{\log(\delta)}{\#E}\right)(h_{B,M}(t) + C_2) + \frac{1}{\#E}\sum_{x\in E} h_{\bar{L}_t}(x). \tag{7}$$

Then the following properties are equivalent:

1. *there exists $\varepsilon > 0$ such that for all $t \in S(\bar{\mathbb{K}})$, we have $h_{\bar{L}_t}(X_t) \geq \varepsilon$,*
2. *there exist $\epsilon > 0$ and an integer $N \geq 1$ such that, for all $t \in S(\bar{\mathbb{K}})$,*

$$\#\{x \in X_t(\bar{\mathbb{K}}) \ : \ h_{\bar{L}_t}(x) \leq \epsilon\} \leq N.$$

Proof. — The implication 2. implies 1. follows from Zhang's inequalities: they give

$$h_{\bar{L}_t}(X_t) \geq \frac{1}{2}\left(e_1(\bar{L}|_{X_t}) + \inf_{x\in X_t(\bar{\mathbb{K}})} h_{\bar{L}}(x)\right),$$

if $e_1(\bar{L}|_{X_t})$ is the essential infimum of $h_{\bar{L}}$ on $X_t(\bar{\mathbb{K}})$. If we assume there exists $\epsilon > 0$ and $N \geq 1$ such that $\#\{x \in X_t(\bar{\mathbb{K}}) \ : \ h_{\bar{L}_t}(x) \leq \varepsilon\} \leq N$ for all $t \in S(\bar{\mathbb{K}})$, since $h_{\bar{L}} \geq 0$, this gives $h_{\bar{L}_t}(X_t) \geq \varepsilon := \epsilon/2$, for all $t \in S(\bar{\mathbb{K}})$.

To prove the converse implication, we may use the inequality (6). We assume there is a finite $\mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$-invariant set $E \subset X_t(\bar{\mathbb{K}}) \setminus \mathrm{supp}(D_t)$ which is non-empty and such that $h_{\bar{L}}(x) \leq \varepsilon/2$ for all $x \in E$. First, we assume $h_{B,M}(t)$ is large. More precisely, we assume $h_{B,M}(t) \geq 4(C' + CC_2 + \varepsilon/2)/C + C_2$. Combining (6) with (7) yields

$$C h_{B,M}(t) - C' \leq h_{\bar{L}_t}(X_t) \leq C_1 \left(\delta - \frac{\log(\delta)}{\#E}\right)(h_{B,M}(t) + C_2) + \frac{\varepsilon}{2},$$

for any $0 < \delta < 1$. This rewrites as

$$C - C_1\delta + C_1\frac{\log(\delta)}{\#E} \leq \frac{C' + CC_2 + \varepsilon/2}{h_{B,M}(t) - C_2}.$$

Without loss of generalities, we can assume $C \leq 1$. For $\delta = C/(2C_1) < 1$, this gives

$$\frac{C}{2} + C_1\frac{\log(C/(2C_1))}{\#E} \leq \frac{C' + CC_2 + \varepsilon/2}{h_{B,M}(t) - C_2}.$$

If we assume $h_{B,M}(t) \geq 4(C' + CC_2 + \varepsilon/2)/C + C_2$, we deduce

$$\frac{C}{4} \leq -C_1\frac{\log(C/(2C_1))}{\#E} = C_1\frac{\log(2C_1/C)}{\#E}.$$

We thus have proved $\#E \leq 4C_1/C \cdot \log(2C_1/C)$.

We now assume $h_{B,M}(t) \leq 4(C' + CC_2 + \varepsilon/2)/C + C_2$. Then (7) yields

$$\begin{aligned}
\varepsilon \leq h_{\bar{L}_t}(X_t) &\leq C_1\left(\delta - \frac{\log(\delta)}{\#E}\right)(h_{B,M}(t) + C_2) + \frac{\varepsilon}{2} \\
&\leq C_1\left(\delta - \frac{\log(\delta)}{\#E}\right)\left(\frac{4(C' + CC_2 + \varepsilon/2)}{C} + 2C_2\right) + \frac{\varepsilon}{2},
\end{aligned}$$

for any $0 < \delta < 1$. Letting $B := C_1(4(C' + CC_2 + \varepsilon/2)/C + 2C_2)$ and $\delta := \varepsilon/4B$, we deduce

$$\#E \leq -\frac{\log(\delta)}{\varepsilon/2B - \delta} = \frac{4B\log(4B/\varepsilon)}{\varepsilon}.$$

Taking $N \geq \max\{4B/\varepsilon \cdot \log(4B/\varepsilon), 4C_1/C \cdot \log(2C_1/C)\} + \deg_{L_t}(X_t)$ and letting $\epsilon := \varepsilon/2$, we conclude the proof. □

6.2. The fundamental height inequality and equidistribution

Pick any integers $d_1, d_2 \geq 2$. We say that a sequence $t_n = (f_n, g_n) \in X_{d_1,d_2}(\bar{\mathbb{Q}})$ is called

- *generic* if for any strict subvariety $Z \subset X_{d_1,d_2}$ that is defined over $\mathbb{Q}$, there is n_0 such that $Z \cap (\mathrm{Gal} \cdot t_n) = \varnothing$ for all $n \geq n_0$.
- *small* if $\langle f_n, g_n \rangle \to 0$, as $n \to \infty$.

Inspiring from the polarized case $d_1 = d_2$, one can derive the following from [**YZ2**].

Theorem 6.2. — *The following hold:*

1. *for any ample divisor N on a projective model of X_{d_1,d_2}, there are $C_1, C_2 > 0$ and Zariski open and dense subset $U \subseteq X_{d_1,d_2}$ such that*
$$\langle f_1, f_2 \rangle \geq C_1 \cdot h_N(t) - C_2, \quad (f_1, f_2) \in t \in U(\bar{\mathbb{Q}}).$$
2. *If there is a generic and small sequence $(t_n)_n \in X_{d_1,d_2}(\bar{\mathbb{Q}})$, then the sequence*
$$\mu_{t_n} = \frac{1}{\#\mathrm{Gal} \cdot t_n} \sum_{t \in \mathrm{Gal} \cdot \mathrm{t_n}} \delta_t$$
converges to $\mu_{d_1,d_2}/\mu_{d_1,d_2}(X_{d_1,d_2}(\mathbb{C}))$ in the weak sense of measures on $X_{d_1,d_2}(\mathbb{C})$.

Proof. — Pick $i = 1, 2$. By [**YZ2**, Theorem 6.1.1], the metrized line bundle $\bar{L}_{f_i}$ is a nef adelic line bundle on the quasi-projective variety $\mathbb{P}^1_{\mathrm{Rat}_{d_i}}$. Let $S := \mathrm{Rat}_{d_1} \times \mathrm{Rat}_{d_2}$ and denote by $\bar{L}_i$ the nef adelic line bundle induced by $\bar{L}_{f_i}$ on $\mathbb{P}^1_S$ by pullback and let $\bar{L} := \pi_1^*\bar{L}_1 + \pi_2^*\bar{L}_2$, where $\pi_i : S \times (\mathbb{P}^1)^2 \to S \times \mathbb{P}^1$ is as above. Then $\bar{L}$ is a nef adelic line bundle on $S \times (\mathbb{P}^1)^2$ and it induces a nef adelic line bundle on $X_{d_1,d_2} \times (\mathbb{P}^1)^2$. Denote by

$$\bar{M} := \langle \bar{L} | X_{d_1,d_2} \times \Delta \rangle^2.$$

By [**YZ2**, Theorem 4.1.3], $\bar{M}$ is a nef adelic line bundle on X_{d_1,d_2}. By Lemma 5.2, the equilibrium measure of the adelic line bundle is proportional to μ_{d_1,d_2} and is non-zero. According to [**YZ2**, Lemma 5.4.4], we deduce that $\deg_{\widetilde{M}}(X_{d_1,d_2}) > 0$, i.e. $\bar{M}$ is non-degenerate in the sense of Yuan and Zhang.

We thus can apply the second item of Theorem 5.3.5 from [**YZ2**] to get the existence, for any adelic line bundle $\bar{N}$ on X_{d_1,d_2}, of constants $C_1, C_2 > 0$ and a dense Zariski open subset $U \subseteq X_{d_1,d_2}$ such that

$$h_{\bar{M}}(t) \geq C_1 h_{\bar{N}}(t) - C_2, \quad t \in U(\bar{\mathbb{Q}}).$$

We also can apply Theorem 5.4.3 from [**YZ2**] to obtain that for any generic sequence t_n with $h_{\bar{M}}(t_n) \to 0$, then μ_{t_n} converges to $\mu_{d_1,d_2}/\mu_{d_1,d_2}(X_{d_1,d_2}(\mathbb{C}))$ in the weak sense of measures on $X_{d_1,d_2}(\mathbb{C})$.

Pick $\phi \in \mathrm{GL}_2(\bar{\mathbb{Q}})$ and let $(f, g) \in (\mathrm{Rat}_{d_1} \times \mathrm{Rat}_{d_2})(\bar{\mathbb{Q}})$. Proceeding as in the proof of Lemma 5.3, we find

$$\langle \phi^{-1} \circ f \circ \phi, \phi^{-1} \circ g \circ \phi \rangle = \langle f, g \rangle,$$

so that, by construction, for all $t = \{(f,g)\} \in X_{d_1,d_2}(\bar{\mathbb{Q}})$, define

$$h_{\bar{M}}(t) = h_{\bar{L}}(\Delta) = \frac{(\bar{L}_t^2|\Delta)}{2\deg_L(\Delta)} = \frac{((\bar{L}_f + \bar{L}_g)^2|\Delta)}{4} = h_{\bar{L}_f+\bar{L}_g}(\mathbb{P}^1) = \langle f, g\rangle.$$

This concludes the proof. □

6.3. Proof of Theorem A

We now conclude the proof of Theorem A. Assume there is a Zariski dense sequence $t_n \in X_{d_1,d_2}(\bar{\mathbb{Q}})$ which is small. Up to extracting a subsequence, we can assume (t_n) is actually generic. By the second point of Theorem 6.2, for any continuous compactly supported function $\varphi \in \mathscr{C}^0_c(X_{d_1,d_2}(\mathbb{C}),\mathbb{R})$, we have

$$\lim_{n\to\infty}\int_{X_{d_1,d_2}(\mathbb{C})} \varphi \, \mathrm{d}\mu_{t_n} = \int_{X_{d_1,d_2}(\mathbb{C})} \varphi \, \mathrm{d}\mu,$$

where $\mu = \mu_{d_1,d_2}/\mu_{d_1,d_2}(X_{d_1,d_2}(\mathbb{C}))$ and $\mu_{t_n} = \frac{1}{\#\mathrm{Gal}\cdot t_n}\sum_{t\in\mathrm{Gal}\cdot \mathrm{t_n}} \delta_t$. Take $t_0 \in \mathrm{supp}(\mu)$ and pick a cut-off function $\psi \in \mathscr{C}^0_c(X_{d_1,d_2}(\mathbb{C}),\mathbb{R}_+)$ such that $0 \le \psi \le 1$ and such that there is an euclidean neighborhood $\Omega \subset X_{d_1,d_2}(\mathbb{C})$ with $\psi \equiv 1$ on Ω. Recall that u is the potential of μ defined in Section 5.1 and that $u \ge 0$ and set $\varphi := u\cdot\psi$. Then $\varphi \in \mathscr{C}^0_c(X_{d_1,d_2}(\mathbb{C}),\mathbb{R}_+)$ and $\varphi = u$ on Ω. By Lemma 2.4, the set $E = \{t \in X_{d_1,d_2}(\mathbb{C});\ u(t) = 0\}$ is contained in a pluripolar set. Now, as we chose Ω to be a neighborhood of $t_0 \in \mathrm{supp}(\mu)$, we have

$$\mu(\Omega \setminus E) = \mu(\Omega) > 0,$$

since μ has contiunous potentials. Whence

$$\int_{X_{d_1,d_2}(\mathbb{C})} \varphi \, \mathrm{d}\mu \ge \int_{\Omega} u \, \mathrm{d}\mu = \int_{\Omega\cap\{u>0\}} u \, \mathrm{d}\mu > 0.$$

Now, as we chose Ω to be a neighborhood of $t_0 \in \mathrm{supp}(\mu)$, we have $\mu(\Omega) > 0$ and, as $\mu_{t_n} \to \mu$ and Ω is open, this implies we have $\mu_{t_n}(\Omega) \ge \mu(\Omega)/2 > 0$ for all n large enough. Also, by non-negativity of the local mutual energy pairing (see [**FRL**]), we have

$$\int_{X_{d_1,d_2}(\mathbb{C})} u \, \mathrm{d}\mu_{t_n} \le \frac{1}{\#\mathrm{Gal}\cdot t_n}\sum_{(f,g)\in\mathrm{Gal}\cdot t_n} \langle f,g\rangle = \langle f_n, g_n\rangle.$$

The hypothesis that the sequence t_n is small implies that

$$\lim_{n\to\infty}\int_{X_{d_1,d_2}(\mathbb{C})} u \, \mathrm{d}\mu_{t_n} = 0.$$

As $0 \le \varphi \le u$, this implies

$$\int_{X_{d_1,d_2}(\mathbb{C})} \varphi \, \mathrm{d}\mu = \lim_{n\to\infty}\int_{X_{d_1,d_2}(\mathbb{C})} \varphi \, \mathrm{d}\mu_{t_n} = 0.$$

This is a contradiction and we have proved the existence of $\varepsilon > 0$ and of a Zariski open an dense subset $U \subseteq X_{d_1,d_2}$ such that

$$\langle f,g\rangle \ge \varepsilon > 0, \quad (f,g) \in U(\bar{\mathbb{Q}}).$$

The conclusion follows from the combination of Theorem B (combined with Lemma 17 from [**I**]), Lemma 2.3, Lemma 6.1 and the first point of Theorem 6.2.

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