THE BIFURCATION MEASURE HAS MAXIMAL ENTROPY

by

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Abstract. — Let Λ be a complex manifold and let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of rational maps of degree $d \geq 2$ of $\mathbb{P}^1$. We define a natural notion of entropy of bifurcation, mimicking the classical definition of entropy, by the parametric growth rate of critical orbits. We also define a notion a measure-theoretic bifurcation entropy for which we prove a variational principle: the measure of bifurcation is a measure of maximal entropy. We rely crucially on a generalization of Yomdin's bound of the volume of the image of a dynamical ball.

Applying our results to complex dynamics in several variables, we notably define and compute the entropy of the trace measure of the Green currents of a holomorphic endomorphism of $\mathbb{P}^k$.

Keywords. Families of rational maps, bifurcation currents and measure, entropy of bifurcation, entropy of rational maps

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Contents

1. Introduction

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of degree $d \geq 2$. The ergodic study of f is well understood:

– Gromov [**Gro2**] showed that the topological entropy of f is $\leq k \log d$.
– The *Green current* T_f of f is an invariant positive closed current of bidegree $(1,1)$ and mass 1 whose support is the *Julia set*, the set where the dynamics is chaotic. Following Fornæss and Sibony [**FS**], one can define, for $j \leq k$, the *j-th Green current* as its self-intersection T_f^j. It is a positive closed (j,j)-current whose support is the *j-th intermediate Julia set*: the set of points where the dynamics is chaotic in at least j directions in a independent way. In particular, the *Green measure* μ_f of f is the maximal self-intersection $\mu_f = T_f^k$ of T_f and is supported on the locus of maximal

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chaos. Fornæss and Sibony showed that μ_f is mixing ([**FS**]) and has maximal entropy $k \log d$.
– Briend and Duval then showed that μ_f is hyperbolic (the Lyapunov exponents are positive) and μ_f equidistributes the repelling cycles [**BrD1**]. Furthermore, μ_f is the unique measure of maximal entropy [**BrD2**].

More generally, for a dominant meromorphic map of a compact Kähler manifold, one want to construct a measure of maximal entropy, to show that it is hyperbolic and to prove the equidistribution of saddle cycles towards that measure (e.g. for complex Hénon maps, this is done in [**BS1, BS2, BLS**]).

On the other hand, let now Λ be a complex Kähler manifold and let $\hat{f} : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1$ be a holomorphic family of rational maps of degree $d \geq 2$: $\hat{f}$ is holomorphic and $\hat{f}(\lambda, z) = (\lambda, f_\lambda(z))$ where f_λ is a rational map of degree d. Though the object of study is the notion of J-stability, this situation shares many similarity with the iteration of a holomorphic map of $\mathbb{P}^k$.

Indeed, DeMarco [**De**] introduced a *current of bifurcation* T_{bif} on Λ, it is a positive closed current of bidegree $(1,1)$ whose support is exactly the unstability locus (the closure of the set where the Julia set does not move continuously) and it is defined as dd^cL where L is the Lyapunov function. Bassanelli and Berteloot [**BB1**] then defined its self-intersections T_{bif}^j for $j \leq \dim(\Lambda)$. Parallel to the j-th Green current, the support of T_{bif}^j is the set where there are unstabilities in at least j directions in a independent way [**Du3**]. The maximal intersection $\mu_{\text{bif}} := T_{\text{bif}}^{\dim(\Lambda)}$ is known as the *bifurcation measure*. Parallel to the equidistribution of repelling cycles, several authors have proved various equidistribution properties of specific dynamical parameters towards μ_{bif}: parameters having a maximal numbers of periodic cycles of given multipliers letting the periods go to ∞, strictly post-critically finite parameters letting the preperiods/periods go to ∞ (e.g. [**Le, FRL, BB2, FG, GV2, GV2, GOV**]).

Is it possible to continue the analogy and show that μ_{bif} is a measure of maximal entropy? This is the main goal of this paper. Of course, this requires to define a notion of entropy in this situation.

To do that, we assume that the family is critical marked: the $2d-2$ critical points can be followed holomorphically (this is always possible up to taking a finite branched cover of Λ). In other words, there exist holomorphic maps $c_1, \ldots, c_{2d-2} : \Lambda \to \mathbb{P}^1$ with $f'_\lambda(c_j(\lambda)) = 0$ and the critical set of f_λ is the collection, with multiplicity, $(c_1(\lambda), \ldots, c_{2d-2}(\lambda))$.

For $n \in \mathbb{N}$, we consider the *n-bifurcation distance* on Λ defined by

$$d_n(\lambda, \lambda') := \max_{1 \leq j \leq 2d-2} \max_{0 \leq q \leq n-1} d\left(f_\lambda^q\left(c_j(\lambda)\right), f_{\lambda'}^q\left(c_j(\lambda')\right)\right),$$

where $d(x,y)$ denotes the Fubini-Study distance on $\mathbb{P}^1$. We say that a set $E \subset \Lambda$ is (d_n, ε)-separated if :

$$\min_{\lambda, \lambda' \in E,\ \lambda \neq \lambda'} d_n(\lambda, \lambda') \geq \varepsilon.$$

We now define the bifurcation entropy on a subset X of Λ. We are mainly interested in the case where X is compact but it is convenient to give a more general definition.

Definition 1.1. — *Let $X \subset \Lambda$. We define $h_{\mathrm{bif}}(\hat{f}, X)$, the* bifurcation entropy *of the family $\hat{f}$ in X, as the quantity*

$$h_{\mathrm{bif}}(\hat{f}, X) := \lim_{\varepsilon \to 0} \limsup_{n} \frac{1}{n} \log \max \{\mathrm{card}(E),\ E \subset X \text{ is } (d_n, \varepsilon) - \text{separated}\}.$$

We let $h_{\mathrm{bif}}(\hat{f}) := \sup\{h_{\mathrm{bif}}(\hat{f}, K), K \text{ compact},\ K \subset \Lambda\}$ be the bifurcation entropy *of the family $\hat{f}$.*

A priori, $h_{\mathrm{bif}}(\hat{f}, K) \in [0, +\infty]$, but refining an argument of [**Gro2**] on the growth rate of the volume of the graph, we first show the following bound of the bifurcation entropy outside the support of T^l_{bif}.

Theorem A. — *Pick $1 \leq l \leq \dim(\Lambda)$ and K a compact subset of Λ. If $K \cap \mathrm{supp}(T^l_{\mathrm{bif}}) = \varnothing$, then*

$$h_{\mathrm{bif}}(\hat{f}, K) \leq (l - 1) \cdot \log d.$$

We now want to define a measure-theoretic entropy of bifurcation. The classical definition uses partitions and at some point relies on the invariance of the measure so having a variational principle seems very difficult with that respect. We thus proceed as in [**K**] and give a definition of (bifurcation) measure-theoretic entropy based on the concept of (d_n, ε)-separated set.

Pick a positive Radon measure ν on Λ (for example a probability measure). Let $\mathcal{B}(\Lambda)$ denote the Borel σ-algebra.

Definition 1.2. — *Let $X \in \mathcal{B}(\Lambda)$ with $\nu(X) > 0$. For $0 < \kappa < \nu(X)$, we then let:*

$$h_{\nu,\mathrm{bif}}(\hat{f}, X, \kappa) := \inf\{h_{\mathrm{bif}}(\hat{f}, X'),\ X' \in \mathcal{B}(\Lambda),\ X' \subset X,\ \nu(X') > \nu(X) - \kappa\}.$$

We defined the metric bifurcation entropy *of ν in X, denoted by $h_{\nu,\mathrm{bif}}(\hat{f}, X)$, as*

$$h_{\nu,\mathrm{bif}}(\hat{f}, X) := \sup_{\kappa > 0} h_{\nu,\mathrm{bif}}(\hat{f}, X, \kappa).$$

We define the metric bifurcation entropy *of ν for the family $\hat{f}$ as*

$$h_{\nu,\mathrm{bif}}(\hat{f}) := \sup\{h_{\nu,\mathrm{bif}}(\hat{f}, K), K \text{ compact},\ K \subset \Lambda\}.$$

Observe that $h_{\nu,\mathrm{bif}}(\hat{f}, K \cup K') = \max(h_{\nu,\mathrm{bif}}(\hat{f}, K), h_{\nu,\mathrm{bif}}(\hat{f}, K'))$ and that $h_{\nu,\mathrm{bif}}(\hat{f}) \leq h_{\mathrm{bif}}(\hat{f})$ for any ν, though there is no natural notion of ergodicity for ν.

Denote by μ_{bif} the bifurcation measure of the family $\hat{f}$ (see Section 3.4 for a precise definition). We prove the following

Theorem B. — *For any compact set K such that $\mu_{\mathrm{bif}}(K) > 0$ then*

$$h_{\mu_{\mathrm{bif}},\mathrm{bif}}(\hat{f}, K) = \dim(\Lambda) \log d.$$

In particular, if $\mu_{\mathrm{bif}} \neq 0$, one has

$$h_{\mu_{\mathrm{bif}},\mathrm{bif}}(\hat{f}) = h_{\mathrm{bif}}(\hat{f}) = \dim(\Lambda) \log d.$$

Notice that the hypothesis $\mu_{\mathrm{bif}} \neq 0$ is satisfied if and only if there exists a parameter in Λ which admits $\dim(\Lambda)$ critical points that are, in a non persistent way, strictly preperiodic to a repelling cycle ([**BE, Ga, Du2**]). It is in particular satisfied in any smooth orbifold parametrization of the moduli space of rational maps of degree d with marked critical points.

In particular, the theorem asserts that μ_{bif} has maximal entropy in a very strong sense: it only sees sets of maximal entropy and by Theorem A, any compact set outside its support does not carry maximal entropy. This gives a very precise interpretation of the bifurcation measure. A natural question is to know whether any measure satisfying those properties is equivalent to μ_{bif}.

From the two above theorems, we deduce that μ_{bif} satisfies a parametric Brin-Katok formula (see Theorem 3.8). We show similarly that the trace measure of T^l_{bif} is a measure of maximal entropy in $\mathrm{supp}(T^l_{\mathrm{bif}})\backslash\mathrm{supp}(T^{l+1}_{\mathrm{bif}})$ (see Theorem 3.7).

To prove Theorem B, we use Yomdin's bound of the volume of the image of a dynamical ball [**Y**]. The use of such ideas to compute the entropy of measures in complex dynamics in several variables has been introduced in [**BS2**]; the first and third authors generalized this idea to give a very general criterion under which we can produce a measure of maximal entropy for a meromorphic map of a compact Kähler manifold ([**DTV**], a great difficulty arises from the need to control precisely the derivatives near the indeterminacy set). Nevertheless, in both articles, one does not work with the measure directly (but with a Cesàro mean of approximations) and one uses the Misiurewicz's proof of the variational principle to conclude.

Here, the idea is to apply Yomdin's estimate on the parametric balls (with respect to d_n) directly for μ_{bif}. We need a precise control on the convergence towards the bifurcation current. Our proof leads us to deal with terms of the form

$$\int_{B_{d_n}(x,\varepsilon)\cap M} \bigwedge_{j=1}^{k} (F^{i_j})^*(\Omega)$$

where all i_j are $\leq n-1$, F is a holomorphic map on some manifold X and M is a complex submanifold of X of codimension k endowed with a metric Ω. If all the i_j were either 0 or $n-1$, this would be the classical Yomdin's bound. The idea of the proof is to work in the product space X^k and to replace the manifold M with $M^k \cap \Delta$ where Δ is the diagonal of X^k which still has bounded geometry (Proposition 2.2).

Going further in the analogy between the dynamics of an endomorphism of $\mathbb{P}^k$ and bifurcation in a holomorphic family of rational maps, it is natural to try and define *parametric Lyapunov exponents* by $\chi_j(\lambda) = \lim_n n^{-1} \log |(f^n_\lambda)'(c_j(\lambda))|$ and show that $\chi_j(\lambda) = L(f_\lambda)$ for μ_{bif}-almost every parameter λ (at least in the case of the moduli space of rational maps), where $L(f_\lambda)$ is the Lyapunov exponent of f_λ with respect to its unique measure of maximal entropy $\log d$. This has been done successfully in [**GS2**] in the very particular case of the unicritical family ($f_\lambda(z) = z^d + \lambda$). The proof relies on subtle properties of external rays and Makarov theorem. Generalizing such result is a challenging question that goes beyond the scope of this article.

In a second part of the article, we use the previous techniques (especially our variation of Yomdin's estimates) in the case of ergodic theory in several complex variables.

First, we give an alternate proof of the computation of the entropy of the Green measure μ of a Hénon maps ([**BS2**]). We apply for that our estimate on the dynamical ball $B_n(x,\varepsilon)$ directly for the measure μ. This allows us to get rid of Misiurewicz' proof of the variational principle (we explain as an application how we can retrieve Brin-Katok formula for Hénon maps).

Finally, we define a notion of entropy for the trace measure of the Green currents T_f^l of a holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ of degree d. Applying the above results to the (constant) holomorphic family $\hat{f} : \mathbb{P}^k \times \mathbb{P}^k \to \mathbb{P}^k \times \mathbb{P}^k$ defined by $\hat{f}(\lambda, z) = (\lambda, f(z))$, we have that this entropy is always bounded from above by $\geq l \log d$ and that it is equal to $l \log d$ on compact sets of $\mathrm{supp}(T_f^l)\backslash\mathrm{supp}(T_f^{l+1}) \neq \varnothing$ having positive trace measure. Nevertheless, we give examples where it is equal to α for infinitely many distinct $\alpha \in [l \log d, k \log d]$. This makes the study of the entropy of the trace measure of Green currents richer than the entropy of Green measures (since the latter is always $k \log d$). Finally, note that the idea to develop some ergodic theory for the trace measure of the Green currents was already present in [**Du1**] where Dujardin defined, for those measures, a notion of Fatou direction, similar to that of Lyapunov characteristic direction.

In Section 2, we shall give a general cut-off lemma for dynamical balls and prove our generalization of Yomdin's bound. Then, in Section 3, we recall the construction of the bifurcation currents and properties we need. We then prove Theorems A and B in the setting of families of holomorphic endomorphism of $\mathbb{P}^q$ with marked points and explain how to get back to the above setting. Finally, Section 4 is devoted to our results in complex dynamics in several variables.

Acknowledgments. — We thank the referee for a very careful reading and multiple remarks that allow us to improve the exposition. In particular, we follow his idea to consider a single holomorphic map as a constant holomorphic family; this unifies the treatment of the entropy for the trace measure of the Green currents of an holomorphic endomorphism of $\mathbb{P}^k$ and the entropy of bifurcation of a holomorphic family (section 4.2).

We also thank Neil Dobbs for Claim 4.4.

2. Preliminaries

2.1. A dynamical cut-off lemma

Let X be a Kähler manifold endowed with a Kähler form Ω and let $f : X \to X$ be a holomorphic map. Let d be the distance associated to Ω. For $n \geq 0$, we have on X the Bowen distance:

$$d_n(x, y) := \max_{i \in \{0,\dots,n-1\}} d(f^i(x), f^i(y)).$$

We denote by $B_{d_n}(x, \varepsilon)$ the ball centered at x and radius ε for d_n.

Let $Y \Subset X$ be a relatively compact set such that $f(Y) \subset Y$ (if X is compact, one can simply take $Y = X$). Assume also that there exists $\varepsilon_0 > 0$ such that $Y_{2\varepsilon_0}$, the $2\varepsilon_0$-neighborhood of Y, is still relatively compact in X and $f(Y_{2\varepsilon_0}) \subset Y_{2\varepsilon_0}$.

Lemma 2.1. — *We take the above notations. There exists a constant C such that for every $x \in Y$, every $0 < \varepsilon < \varepsilon_0$ and every $n \in \mathbb{N}$, there exists a smooth function θ_n satisfying:*

- $\theta_n \equiv 1$ *in* $B_{d_n}(x, \varepsilon)$ *and* $\mathrm{supp}(\theta_n) \subset B_{d_n}(x, 2\varepsilon)$.
- $C \frac{n^2}{\varepsilon^2} \sum_{i=0}^{n-1} (f^i)^*(\Omega) \pm dd^c \theta_n \geq 0$.

Proof. — Using a finite cover, one can construct for every $x \in Y$ a smooth cut-off function θ_x such that $\theta_x = 1$ in $B(x, \varepsilon)$ and $\mathrm{supp}(\theta_x) \subset B(x, 2\varepsilon)$ (for the distance d). Let $C > 0$ be such that for every $x \in Y$, $\varepsilon < \varepsilon_0$:

$$\frac{C}{\varepsilon^2} \cdot \Omega \pm dd^c\theta_x \geq 0 \text{ and } d\theta_x \wedge d^c\theta_x \leq \frac{C}{\varepsilon^2} \cdot \Omega.$$

Fix $x \in Y$. We then define $\theta_n := \prod_{i=0}^{n-1} \theta_{f^i(x)} \circ f^i$. By construction, $\theta_n \equiv 1$ in $B_{d_n}(x, \varepsilon)$ and $\mathrm{supp}(\theta_n) \subset B_{d_n}(x, 2\varepsilon)$. We compute:

$$dd^c\theta_n = \sum_{i=0}^{n-1} \left(\prod_{j \neq i} \theta_{f^j(x)} \circ f^j \right) dd^c\theta_{f^i(x)} \circ f^i$$
$$+ \sum_{\ell \neq \ell'} \left(\prod_{j \neq \ell,\ j \neq \ell'} \theta_{f^j(x)} \circ f^j \right) d\theta_{f^\ell(x)} \circ f^\ell \wedge d^c\theta_{f^{\ell'}(x)} \circ f^{\ell'}.$$

Using that $\pm(d\psi \wedge d^c\varphi + d\varphi \wedge d^c\psi) \leq d\psi \wedge d^c\psi + d\varphi \wedge d^c\varphi$ and the properties of θ_x gives

$$0 \leq \pm dd^c\theta_n + \frac{C}{\varepsilon^2} \sum_{i=0}^{n-1} (f^i)^*(\Omega) + \frac{2C}{\varepsilon^2} \sum_{\ell=0}^{n-1} \sum_{\ell'=0,\ \ell' \neq \ell}^{n-1} (f^\ell)^*(\Omega) + (f^{\ell'})^*(\Omega).$$

The result follows, up to changing the constant C. □

2.2. A Yomdin's Lemma

We keep the notations of the above subsection ($f(Y) \subset Y \subset Y_{2\varepsilon_0} \Subset X$). We will need the following variation of Yomdin's bound on the growth of the size of the image of a dynamical ball which uses the Algebraic Lemma (first stated in [**Y**], see [**Bu**] for a complete proof). We recall the definition of C^r-size.

Let l be an integer between 1 and $2\dim_{\mathbb{C}}(X)$. Put some coordinates charts on X. If Y is a subset of X (for example a submanifold of real dimension l), we call *C^r-size* (with $r \in \mathbb{N}^*$) of Y, the lower bound of the numbers $t \geq 0$ for which there exists a C^r-map of the unit l-cube into X, $h : [0,1]^l \mapsto \mathbb{C}^k$, with $Y \subset h([0,1]^l)$ and $\|D_r h\| \leq t$. Here $D_r h$ is the vector of the partial derivatives of h of order $1, \ldots, r$ in the above charts. The norm refers to the supremum over $x \in [0,1]^l$:

$$\|D_r h\| = \sup_x \|D_r h(x)\|.$$

The key point is that C^r-size increases with r and the C^1-size bounds the (real) l-dimensional volume of Y:

$$C^1 - \text{size of } Y \geq (\text{l-dimensional volume } (Y))^{1/l}.$$

The following is our main result.

Proposition 2.2. — *Consider a family of smooth manifolds of X of dimension k with* uniformly bounded geometry*: for each r, each manifold M can be covered by a uniform number of pieces of C^r-size equal to* 1*. Then, for every $\gamma > 0$, there exists $\varepsilon > 0$ such that for any M in the family, there exists an integer n_0 such that for any $n \geq n_0$, any $0 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n-1$ and any $x \in Y$, then:*

$$\int_{B_{d_n}(x,\varepsilon) \cap M} \bigwedge_{j=1}^{k} (f^{i_j})^*(\Omega) \leq e^{\gamma n}.$$

Proof. — We first briefly recall the strategy of the proof of the bound:

$$\mathrm{Vol}_\Omega(f^{n-1}(B_{d_n}(x,\varepsilon)\cap M)) \leq e^{\gamma n}.$$

We follow Gromov's exposition [**Gro1**]. Fix some M in the family and a regularity $r \gg 1$. Up to reducing M, assume that its C^r-size is 1 (by hypothesis, there exist n_0 large enough, independent of M, such that we can take $\leq e^{\gamma n_0}$ pieces of C^r-size 1 covering M). We work in some charts given by a finite atlas. Then, for any unit cubes $\square_1, \dots, \square_i$, let $M_i := f^i\left(M \cap f^{-1}(\square_1) \cap \dots \cap f^{-i}(\square_i)\right)$, then the Algebraic Lemma implies (see [**Gro1**][(*) p 233]):

$$\mathrm{Vol}_\Omega(M_i) \leq (C\|D_r f\|^{\frac{2k}{r}} + 1)^i \tag{1}$$

where C depends on r, the real dimensions of M $(= 2k)$ and X but not on f nor M. As it is crucial for our purpose, note that the volume of M_i is counted with multiplicity: indeed, Gromov's proof uses graphs so the multiplicity is automatically taken into account in the Algebraic Lemma.

Take some $j \geq 1$ and some dynamical ball $B_{d_n}(x, 1/j)$. We take $1 \ll m \ll n-1$. Let us assume to simplify that $n-1 = mi$ for some $i \in \mathbb{N}$. Consider the $1/j$-cubes $\tilde{\square}_l$ centered at $f^{ml}(x)$ for each $1 \leq l \leq i$. Then

$$f^{n-1}(B_{d_n}(x,1/j)\cap M) \subset (f^m)^i(M \cap (f^m)^{-1}(\tilde{\square}_1) \cap \dots \cap (f^m)^{-i}(\tilde{\square}_i)).$$

Rescaling to 1-cubes, we deduce from (1) that:

$$\mathrm{Vol}_\Omega(f^{n-1}(B_{d_n}(x,1/j)\cap M)) \leq (C\|D_r f_j^m\|^{\frac{2k}{r}} + 1)^i,$$

where f_j is the rescaled map in each unit cube $\square_l := j.\tilde{\square}_l$ so $f_j(t) := jf(j^{-1}t)$ (working in some charts). We choose j large enough so that $\|D_r f_j^m\| \leq 2\|Df^m\|$ (rescaling reduces the norm of the derivatives of order > 1 and has no effect on order 1). In particular:

$$\mathrm{Vol}_\Omega(f^{n-1}(B_{d_n}(x,1/j)\cap M)) \leq (C'\|Df^m\|^{\frac{2k}{r}} + 1)^i,$$

where C' is another constant that depends only on r, the dimensions of M and X (we assume that $\|Df\| \geq 1$, if not then the result is already obvious). In particular, using $\|Df^m\| \leq \|Df\|^m$, $n-1 = mi$, we recognize:

$$\mathrm{Vol}_\Omega(f^{n-1}(B_{d_n}(x,1/j)\cap M)) \leq \left((C')^{\frac{1}{m}}\right)^{n-1}.(\|Df\|^{\frac{2k}{r}})^n.$$

In fact, r was chosen so that $\|Df\|^{\frac{2k}{r}} \leq e^{\gamma/2}$ and we now choose m large enough so that $(C')^{\frac{1}{m}} \leq e^{\gamma/2}$ which proves the result (if $n-1 \neq im$ we simply prove the bound for $n' = im$ and we have a extra $\|Df\|^m$ that appears).

We now prove the Proposition. We take $\varepsilon \leq \varepsilon_0$, in what follows, the norms are taken on $Y_{2\varepsilon_0}$ (we are only interested in points whose orbits stay in $B_{d_n}(x,\varepsilon) \subset Y_{2\varepsilon_0}$). Observe that if all i_j are equal to $n-1$, then the proposition means that (taking into account the multiplicity):

$$\int_{B_{d_n}(x,\varepsilon)\cap M} (f^{n-1})^*(\Omega^k) = \mathrm{Vol}_\Omega(f^{n-1}(B_{d_n}(x,\varepsilon)\cap M)) \leq e^{\gamma n}$$

which is the classical bound. Similarly if $i_j = 0$ for $j \leq j_0$ and $i_j = n-1$ for $j > j_0$, then we can bound, near x, $\bigwedge_{j \leq j_0} \Omega$ by a finite sum of currents of integration on lamination by linear subspaces of codimension j_0:

$$\bigwedge_{j\leq j_0}\Omega \leq C\sum_\alpha\int_\alpha [L_\alpha(u)]d\lambda_\alpha(u)$$

where C is a constant that depends (locally uniformly) in x, the α are the subspaces of dimension j_0 given by the coordinates (we work in some chart), $L_\alpha(u)$ is the subspace of codimension j_0 directed by the remaining coordinates that intersects α at u and λ_α is the Lebesgue measure on α. In particular, it is sufficient to bound the term

$$\int_{B_{d_n}(x,\varepsilon)\cap M}(f^{n-1})^*(\Omega^{k-j_0})\wedge[L(u_1)\cap\cdots\cap L(u_{j_0})]$$

uniformly in $u_1,\dots u_{j_0}$. Since it can be rewritten as (taking into account the multiplicity):

$$\mathrm{Vol}_\Omega\left(f^{n-1}(B_{d_n}(x,\varepsilon)\cap M\cap L(u_1)\cap\cdots\cap L(u_{j_0}))\right)$$

and $M\cap L(u_1)\cap\cdots\cap L(u_{j_0}))$ has uniformly bounded geometry, the wanted inequality is again the classical Yomdin's result.

Let $\gamma>0$. Fix $\iota\ll 1$, independent of n, and let $m=E(\iota n)$ (E being the integer part). For each i_j, write $i_j=l_jm+r_j$ with $0\leq r_j<m$. Then, as $f^*(\Omega)\leq\|Df\|^2\Omega$ (up to changing the norm $\|Df\|$ by a constant), we have:

$$\int_{B_{d_n}(x,\varepsilon)\cap M}\bigwedge_{j=1}^k(f^{i_j})^*(\Omega)\leq\|Df\|^{2km}\int_{B_{d_n}(x,\varepsilon)\cap M}\bigwedge_{j=1}^k(f^{l_jm})^*(\Omega).$$

In particular, taking ι small enough gives $\|Df\|^{2km}\leq(\|Df\|^{2k\iota})^n\leq e^{\gamma n}$.

Assume that $l_j=0$ for $j\leq j_0$ Proceeding as above, we can replace Ω by a finite sum of currents of integration on lamination by linear subspaces of codimension j_0. So we are reduced to the case of terms of the form $\int_{B_{d_n}(x,\varepsilon)\cap M\cap L}\bigwedge_{j_0<j\leq k}(f^{i_j})^*(\Omega)$ where L is a linear subspace: in other words, we are reduced to prove the wanted estimate with M replaced by $M\cap L$.

Let us thus assume that $l_j\neq 0$ for every j. Write $\underline{l}:=(l_1,\dots,l_k)$. Let $f_{\underline{l}}:=(f^{l_1},\dots,f^{l_k}):X^k\to X^k$ and Δ be the diagonal in X^k:

$$\Delta:=\{(x,\dots,x)\in X^k\}.$$

Let δ_k be the distance on X^k defined by $\delta_k((x_i)_{i\leq k},(y_i)_{i\leq k}):=\max_i d(x_i,y_i)$ and let $d_{k,p,f_{\underline{l}}}$ be the Bowen distance in X^k for the $(p-1)^{\text{th}}$-iterate associated to $f_{\underline{l}}$. We let $B_{d_{k,p,f_{\underline{l}}}}((x_i)_{i\leq k},\varepsilon)$ be the associated Bowen ball. Finally, let $\Omega_k:=\sum_j\pi_j^*\Omega$ where π_j is the projection from X^k to the j-th factor and $\tilde{M}:=M^k\cap\Delta$. With these notations, we have:

$$\begin{aligned}\int_{B_{d_n}(x,\varepsilon)\cap M}\bigwedge_{j=1}^k(f^{l_jm})^*(\Omega)&\leq\int_{(\pi_1^{-1}(B_{d_n}(x,\varepsilon)))\cap\tilde{M}}\bigwedge_{j=1}^k\pi_j^*((f^{l_jm})^*(\Omega))\\&\leq\int_{B_{d_{k,m,f_{\underline{l}}}}((x,x,\dots,x),\varepsilon)\cap\tilde{M}}(f_{\underline{l}}^m)^*(\Omega_k^k).\end{aligned}$$

By the above proof of the bound

$$\mathrm{Vol}_\Omega(f^{n-1}(B_{d_n}(x,\varepsilon)\cap M))\leq e^{\gamma n},$$

where we replace M by $\tilde{M}$, f by $f_{\underline{l}}$ and n by m, we infer that:

$$\int_{B_{d_{k,m,f_{\underline{l}}}}((x,x,\ldots,x),1/j)\cap\tilde{M}} (f_{\underline{l}}^m)^*(\Omega_k^k) \leq (C''\|Df_{\underline{l}}\|^{\frac{2\tilde{k}}{r}})^m$$
$$\leq (C'')^{\iota n}\|Df\|^{\frac{n}{r}}$$

where C'' is a constant that depends on r, the dimension $2k$ of $\tilde{M}$ and the dimension of X^k. Observe that it is crucial that, since ι is fixed, there are only finitely many $\underline{l}$ (roughly $\leq \iota^{-k}$) so we can find a j such that for every $\underline{l}$ $\|D_r f_{\underline{l},j}\| \leq 2\|Df_{\underline{l}}\|$. We conclude as above by taking r large enough and adding the constraint $(C'')^{\iota} \leq e^{\gamma}$. □

3. Bifurcation entropy

3.1. Background in bifurcation theory

3.1.1. Defining the bifurcation currents and (locally uniform) estimates

For this section, we follow the presentation of [**DF, Du2**]. Though everything is stated there in the case $q = 1$ and for marked *critical* points, the exact same arguments apply in our settings.

Let Λ be a complex manifold and let $\hat{f} : \Lambda \times \mathbb{P}^q \to \Lambda \times \mathbb{P}^q$ be a holomorphic family of endomorphisms of $\mathbb{P}^q$ of algebraic degree $d \geq 2$: $\hat{f}$ is holomorphic and $\hat{f}(\lambda, z) = (\lambda, f_\lambda(z))$ where f_λ is an endomorphism of $\mathbb{P}^q$ of algebraic degree d.

Let $\omega_{\mathbb{P}^q}$ be the standard Fubini-Study form on $\mathbb{P}^q$ and $\pi_\Lambda : \Lambda \times \mathbb{P}^q \to \Lambda$ and $\pi_{\mathbb{P}^q} : \Lambda \times \mathbb{P}^q \to \mathbb{P}^q$ be the canonical projections. Finally, let $\widehat{\omega} := (\pi_{\mathbb{P}^q})^*\omega_{\mathbb{P}^q}$. It is known that the sequence $d^{-n}(\widehat{f^n})^*\widehat{\omega}$ converges to a closed positive $(1,1)$-current $\widehat{T}$ on $\Lambda \times \mathbb{P}^q$ with continuous potential. More precisely, for any $n \geq 1$, we have $\widehat{T} = d^{-n}(\hat{f}^n)^*\hat{\omega} + d^{-n}dd^c\widehat{u}_n$, where $(\widehat{u}_n)_n$ is a locally uniformly bounded sequence of continuous functions. Moreover, for any $1 \leq j \leq q$, it satisfies

$$\widehat{f}^*\widehat{T}^j = d^j \cdot \widehat{T}^j$$

and $\widehat{T}^q|_{\{\lambda_0\}\times\mathbb{P}^1} = \mu_{\lambda_0}$ is the unique measure of maximal entropy $q\log d$ of f_{λ_0} for every $\lambda_0 \in \Lambda$.

Assume now that the family $\hat{f}$ is endowed with k marked points i.e. we are given holomorphic maps $a_1, \ldots, a_k : \Lambda \to \mathbb{P}^q$. Let Γ_{a_j} be the graph of the map a_j and set

$$\mathfrak{a} := (a_1, \ldots, a_k).$$

Definition 3.1. — *For $1 \leq i \leq k$, the* bifurcation current T_{a_i} *of the point a_i is the closed positive $(1,1)$-current on Λ defined by*

$$T_{a_i} := (\pi_\Lambda)_*\left(\widehat{T} \wedge [\Gamma_{a_j}]\right)$$

and we define the bifurcation current $T_{\mathfrak{a}}$ *of the k-tuple $\mathfrak{a}$ as*

$$T_{\mathfrak{a}} := T_{a_1} + \cdots + T_{a_k}.$$

For any $\ell \geq 0$, write

$$\mathfrak{a}_\ell(\lambda) := \left(f_\lambda^\ell(a_1(\lambda)), \ldots, f_\lambda^\ell(a_k(\lambda))\right),\ \lambda \in \Lambda.$$

Let now $K \Subset \Lambda$ be a compact subset of Λ and let Ω be some compact neighborhood of K, then the trace measure of $(a_\ell)^*(\omega_{\mathbb{P}^q})$ of K is bounded from above by Cd^ℓ, where C depends on Ω but not on ℓ.

Applying verbatim the proof of [**DF**, Theorem 3.2] (see also the first section of [**Du3**]), we have the following

Lemma 3.2. — *For any $1 \leq i \leq k$, the support of T_{a_i} is the set of parameters $\lambda_0 \in \Lambda$ such that the sequence $\{\lambda \mapsto f_\lambda^n(a_i(\lambda))\}$ is not a normal family at λ_0.*

Moreover, writing $a_{i,\ell}(\lambda) := f_\lambda^\ell(a_i(\lambda))$, there exists a locally uniformly bounded family $(u_{i,\ell})$ of continuous functions on Λ such that

$$(a_{i,\ell})^*(\omega_{\mathbb{P}^q}) = d^\ell T_{a_i} + dd^c u_{i,\ell}, \text{ on } \Lambda.$$

3.1.2. Higher bifurcation currents of marked points

Using the uniform convergence on compact subsets of Λ in Lemma 3.2, for every $j \geq 1$, we deduce

$$T_{a_i}^{q+1} = 0 \text{ on } \Lambda. \tag{2}$$

Let us still denote $\pi_\Lambda : \Lambda \times (\mathbb{P}^q)^k \to \Lambda$ the projection onto the first coordinate and for $1 \leq i \leq k$, let $\pi_i : \Lambda \times (\mathbb{P}^q)^k \to \Lambda \times \mathbb{P}^q$ be the projection onto Λ times the i-th factor of the product $(\mathbb{P}^q)^k$. Finally, we denote by $\Gamma_\mathfrak{a}$ the graph of $\mathfrak{a}$:

$$\Gamma_\mathfrak{a} := \{(\lambda, (z_j)),\ \forall j, z_j = a_j(\lambda)\} \subset \Lambda \times (\mathbb{P}^q)^k.$$

Following verbatim the proof of [**AGMV**, Lemma 2.6], for any i, we get

$$T_\mathfrak{a}^i = \sum_{\substack{j_1,\dots,j_k \geq 0 \\ j_1+\dots+j_k=i}} \bigwedge_{\ell=1}^k T_{a_\ell}^{j_\ell} = \sum_{\substack{j_1,\dots,j_k \geq 0 \\ j_1+\dots+j_k=i}} (\pi_\Lambda)_* \left(\bigwedge_{\ell=1}^k \pi_\ell^* \left(\widehat{T}^{j_\ell}\right) \wedge [\Gamma_\mathfrak{a}] \right).$$

3.2. Bifurcation Entropy for a k-tuple of marked points

We now assume that Λ is a complex Kähler manifold endowed with a Kähler form ω_Λ and that the family $\hat{f}$ comes with k marked points. Recall that we have set

$$\mathfrak{a} := (a_1, \dots, a_k).$$

In analogy with topological entropy, we define a notion of *bifurcation entropy* of the marked family $(\hat{f}, \mathfrak{a})$ in the following way. For $n \in \mathbb{N}$, we consider the *n-bifurcation distance* on Λ associated with $\mathfrak{a}$ defined by

$$d_{\mathfrak{a},n}(\lambda, \lambda') := \max_{1 \leq j \leq k} \max_{0 \leq i \leq n-1} d\left(f_\lambda^i\left(a_j(\lambda)\right), f_{\lambda'}^i\left(a_j(\lambda')\right)\right),$$

where $d(x, y)$ denotes the Fubini-Study distance on $\mathbb{P}^1$. We say that a set $E \subset \Lambda$ is $(d_{\mathfrak{a},n}, \varepsilon)$-separated if :

$$\min_{\lambda, \lambda' \in E,\ \lambda \neq \lambda'} d_{\mathfrak{a},n}(\lambda, \lambda') \geq \varepsilon.$$

Definition 3.3. — *Let $X \subset \Lambda$ be a Borel set, we define the* bifurcation entropy $h_\mathfrak{a}(\hat{f}, X)$ *of the marked family $(\hat{f}, \mathfrak{a})$ in X as the quantity:*

$$h_\mathfrak{a}(\hat{f}, X) := \lim_{\varepsilon \to 0} \limsup_n \frac{1}{n} \log \max\left\{\text{card}(E),\ E \subset X \text{ is } (d_{\mathfrak{a},n}, \varepsilon) - \text{separated}\right\}.$$

A priori, for K compact, $h_{\mathfrak{a}}(\hat{f},K)\in[0,+\infty]$, but using the fact that, by definition, the bifurcation entropy is bounded by the topological entropy on $K\times(\mathbb{P}^q)^k$ of $\hat{f}_k:\Lambda\times(\mathbb{P}^q)^k\to\Lambda\times(\mathbb{P}^q)^k$ defined by $\hat{f}_k(\lambda,z_1,\dots z_k)=(\lambda,f_\lambda(z_1),\dots,f_\lambda(z_k))$, we have $h_{\mathfrak{a}}(\hat{f},K)\le kq\log d$ (see [**Gro2**] or the proof below). Observe that the entropy of $\hat{f}_k$ is already $kq\log d$ on each (invariant set) $\{\lambda\}\times(\mathbb{P}^q)^k$ so it is not related to bifurcation phenomena. We show here that bifurcation entropy can be bounded from above outside the support of $T^i_{\mathfrak{a}}$. The proof follows an idea of the first author ([**DT1**], see also [**Di2**]):

Theorem 3.4. — *Pick* $1\le i\le\dim(\Lambda)$. *Assume that* $K\cap\operatorname{supp}(T^i_{\mathfrak{a}})=\varnothing$. *Then*

$$h_{\mathfrak{a}}(\hat{f},K)\le(i-1)\log d.$$

Proof. — Fix $n\ge1$ and $\varepsilon>0$. Let $p_j:(\mathbb{P}^q)^k\to\mathbb{P}^q$ be the projection onto the j-th coordinate. We define a Kähler metric on $(\mathbb{P}^q)^k$ by letting $\omega:=p_1^*(\omega_{\mathbb{P}^q})+\cdots+p_k^*(\omega_{\mathbb{P}^q})$, here $\omega_{\mathbb{P}^q}$ is the Fubini-Study metric on $\mathbb{P}^q$ of mass 1.

Let ω_Λ be a Kähler metric on Λ and $\omega_\ell:=P_\ell^*(\omega)$, where $P_\ell:\left((\mathbb{P}^q)^k\right)^n\to(\mathbb{P}^q)^k$ is the projection onto the ℓ-th factor of the form $(\mathbb{P}^q)^k$. If $\Pi_\Lambda:\Lambda\times(\mathbb{P}^q)^{kn}\to\Lambda$ is the canonical projection onto Λ, we still denote by ω_Λ the pull-back $\Pi_\Lambda^*(\omega_\Lambda)$. We endow $\Lambda\times(\mathbb{P}^q)^{kn}$ with the product Kähler metric

$$\Omega:=\omega_\Lambda+\sum_{\ell=1}^n\omega_\ell$$

and denote by $\tilde{d}$ the induced distance on $\Lambda\times(\mathbb{P}^q)^{kn}$.

As before, set $\mathfrak{a}_\ell(\lambda):=(f^\ell_\lambda(a_1(\lambda)),\dots,f^\ell_\lambda(a_k(\lambda)))$ for every $\lambda\in\Lambda$ and every $\ell\ge0$. Let $\Gamma_n\subset\Lambda\times(\mathbb{P}^q)^{kn}$ be the graph of the map $\mathfrak{A}^n:=(\mathfrak{a}_0,\dots,\mathfrak{a}_{n-1}):\Lambda\to(\mathbb{P}^q)^{kn}$, and pick a set $E\subset\Lambda$ which is $(d_{\mathfrak{a},n},\varepsilon)$-separated and let $N:=\mathrm{Card}(E)$. For $\lambda\in E$, let $\tilde{\lambda}:=\mathfrak{A}^n(\lambda)$. If $\tilde{E}:=\mathfrak{A}^n(E)$, we have $\tilde{d}(\tilde{\lambda}_1,\tilde{\lambda}_2)\ge\varepsilon$, for any distinct $\tilde{\lambda}_1,\tilde{\lambda}_1\in\tilde{E}$. In particular, if $K_\varepsilon:=\{\lambda\in\Lambda\,;d_\Lambda(\lambda,K)\le\varepsilon\}$, we have

$$\bigcup_{\tilde{\lambda}\in\tilde{E}}B_{\tilde{d}}\left(\tilde{\lambda},\frac{\varepsilon}{2}\right)\cap\Gamma_n\subset\Pi_\Lambda^{-1}(K_\varepsilon)\cap\Gamma_n,$$

and the union is disjoint. So

$$\mathrm{Vol}_\Omega\left(\Pi_\Lambda^{-1}(K_\varepsilon)\cap\Gamma_n\right)\ge\sum_{\tilde{\lambda}\in\tilde{E}}\mathrm{Vol}_\Omega\left(B_{\tilde{d}}\left(\tilde{\lambda},\frac{\varepsilon}{2}\right)\cap\Gamma_n\right).$$

Since Γ_n is an analytic subvariety of $\Lambda\times(\mathbb{P}^q)^{kn}$ of complex dimension $\dim(\Lambda)$ passing through the center of the balls $B_{\tilde{d}}(\tilde{\lambda},\varepsilon/2)$, Lelong's inequality implies

$$\mathrm{Vol}_\Omega\left(B_{\tilde{d}}\left(\tilde{\lambda},\frac{\varepsilon}{2}\right)\cap\Gamma_n\right)\ge c\varepsilon^{2\dim(\Lambda)}$$

where c is a constant that does not depend on n. Since $\mathrm{Card}(\tilde{E})=\mathrm{Card}(E)=N$, we get

$$\mathrm{Vol}_\Omega\left(\Pi_\Lambda^{-1}(K_\varepsilon)\cap\Gamma_n\right)\ge N\cdot c\varepsilon^{2\dim(\Lambda)}. \tag{3}$$

We now bound $\mathrm{Vol}_\Omega\left(\Pi_\Lambda^{-1}(K_\varepsilon)\cap\Gamma_n\right)$ from above. Let A_s denote the set of $\alpha:=(\alpha_1,\dots,\alpha_k)$ such that $0\le\alpha_j\le q$ and $\sum\alpha_j=s$. Let L_n be the set of k-tuples $\ell=(\ell_1,\dots,\ell_k)$ of distinct integers in $\{1,\dots,n\}$. Note that the cardinality of A_s is $\le(q+1)^s$ and the cardinality of L_n^s is $\le n^k$.

Write $d_\Lambda := \dim(\Lambda)$. Up to reducing $\varepsilon > 0$, we can assume $K_{(d_\Lambda+1)\varepsilon} \cap \mathrm{supp}(T_{a_1}^{\alpha_1} \wedge \cdots \wedge T_{a_k}^{\alpha_k}) = \varnothing$ for every $\alpha \in A_i$. Choose C^2 non-negative functions $\theta_1, \ldots, \theta_i$ on Λ such that $\theta_j \equiv 1$ on $K_{j\varepsilon}$ and $\mathrm{supp}(\theta_j) \subset K_{(j+1)\varepsilon}$ for every $1 \le j \le i$. We then have

$$\begin{aligned}
\mathrm{Vol}_\Omega\left(\Pi_\Lambda^{-1}(K_\varepsilon) \cap \Gamma_n\right) &\le \int_{\Lambda\times(\mathbb{P}^q)^{kn}} (\theta_1 \circ \Pi_\Lambda)\ \Omega^{d_\Lambda} \wedge [\Gamma_n] \\
&\le \int_\Lambda \theta_1 \cdot (\Pi_\Lambda)_* \left(\sum_{s=0}^{d_\Lambda} \binom{d_\Lambda}{s} (\omega_\Lambda)^{d_\Lambda - s} \wedge \sum_{\alpha\in A_s} \sum_{\ell \in L_n} \bigwedge_{j=1}^{s} \omega_{\ell_j}^{\alpha_j} \wedge [\Gamma_n]\right) \\
&\le \int_\Lambda \theta_1 \left(\sum_{s=0}^{d_\Lambda} \binom{d_\Lambda}{s} \omega_\Lambda^{d_\Lambda - s} \wedge \sum_{\alpha\in A_s} \sum_{\ell \in L_n} \bigwedge_{j=1}^{k} (\mathfrak{a}_{\ell_j})^*(\omega^{\alpha_j})\right).
\end{aligned}$$

Fix an integer $s \le d_\Lambda$, a k-tuple $\alpha \in A_s$ and a k-tuple $\ell \in L_n$. Recall that, by definition, we have $T_\mathfrak{a} = \sum_{j\le k} T_{a_j}$. As seen in Section 3.1, there exists a locally uniformly bounded family (u_ℓ) of continuous functions on Λ such that

$$(\mathfrak{a}_\ell)^*(\omega) = d^\ell T_\mathfrak{a} + dd^c u_\ell$$

for every $\ell \ge 0$ and that $(\mathfrak{a}_\ell)^*(\omega^j) = ((\mathfrak{a}_\ell)^*\omega)^j$ for every $1 \le j \le q$. Assume for simplicity that $\alpha_1 \ne 0$. Then, letting $S = \bigwedge_{j=2}^k (\mathfrak{a}_{\ell_j})^*(\omega^{\alpha_j})$, by Stokes and using $\theta_2 \equiv 1$ on $\mathrm{supp}(\theta_1)$:

$$\begin{aligned}
\int_\Lambda \theta_1 \cdot \omega_\Lambda^{d_\Lambda - s} \wedge \bigwedge_{j=1}^{k} (\mathfrak{a}_{\ell_j})^*(\omega^{\alpha_j}) &= \int_\Lambda \theta_1 \cdot \omega_\Lambda^{d_\Lambda - s} \wedge \left(d^{\ell_1} T_\mathfrak{a} + dd^c u_{\ell_1}\right)^{\alpha_1} \wedge S \\
&= \int_\Lambda \theta_1 \cdot \omega_\Lambda^{d_\Lambda - s} \wedge d^{\ell_1} T_\mathfrak{a} \wedge \left(d^{\ell_1} T_\mathfrak{a} + dd^c u_{\ell_1}\right)^{\alpha_1 - 1} \wedge S \\
&\quad + \int_\Lambda u_{\ell_1} dd^c(\theta_1) \wedge \omega_\Lambda^{d_\Lambda - s} \wedge \left(d^{\ell_1} T_\mathfrak{a} + dd^c u_{\ell_1}\right)^{\alpha_1 - 1} \wedge S \\
&\le d^n \int_\Lambda \theta_1 \cdot \omega_\Lambda^{d_\Lambda - s} \wedge T_\mathfrak{a} \wedge \left(d^{\ell_1} T_\mathfrak{a} + dd^c u_{\ell_1}\right)^{\alpha_1 - 1} \wedge S \\
&\quad + C \int_\Lambda \theta_2 \omega_\Lambda^{d_\Lambda - s + 1} \wedge \left(d^{\ell_1} T_\mathfrak{a} + dd^c u_{\ell_1}\right)^{\alpha_1 - 1} \wedge S,
\end{aligned}$$

where C is a constant that depends on the C^2-norm of θ_1 and the supremum of the L^∞-norm of the (u_ℓ) but not on n. Iterating the process, we get the bound:

$$\int_\Lambda \theta_1 \cdot \omega_\Lambda^{d_\Lambda - s} \wedge \bigwedge_{j=1}^{s} (\mathfrak{a}_{\ell_j})^*(\omega^{\alpha_j}) \le C \sum_{j \le s} d^{jn} \int_\Lambda \theta_s \cdot \omega_\Lambda^{d_\Lambda - j} \wedge T_\mathfrak{a}^j$$

where C is (another) constant that does not depend on n. The quantity $\int_\Lambda \theta_s \cdot \omega_\Lambda^{d_\Lambda - j} \wedge T_\mathfrak{a}^j$ is bounded by $\int_{K_{(d_\Lambda+1)\varepsilon}} \omega_\Lambda^{d_\Lambda - j} \wedge T_\mathfrak{a}^j$. By hypothesis, we have $T_\mathfrak{a}^j = 0$ on $K_{(d_\Lambda+1)\varepsilon}$ for every $j \ge i$. In particular, $\int_{K_{(d_\Lambda+1)\varepsilon}} \omega_\Lambda^{d_\Lambda - j} \wedge T_\mathfrak{a}^j = 0$ for $j \ge i$. It follows that:

$$\int_\Lambda \theta_1 \cdot \omega_\Lambda^{d_\Lambda - s} \wedge \bigwedge_{j=1}^{k} (\mathfrak{a}_{\ell_j})^*(\omega^{\alpha_j}) \le C \sum_{j<i} d^{jn} \int_{K_{(d_\Lambda+1)\varepsilon}} \omega_\Lambda^{d_\Lambda - j} \wedge T_\mathfrak{a}^j \le C' d^{(i-1)n}$$

where C' depends on (a neighborhood of) K and j but not on n. Summing over all $\alpha \in A_s$ and all $\ell \in L_n$ implies

$$\mathrm{Vol}_\Omega\left(\Pi_\Lambda^{-1}(K_\varepsilon)\cap\Gamma_n\right) \le C''\cdot n^{kq}\cdot d^{n(i-1)}$$

again for some constant $C''>0$ which depends on K but not on n. The inequality (3) gives

$$\frac{1}{n}\log N \le (i-1)\log d + kq\frac{\log n}{n} + \frac{1}{n}\log C'' - \frac{1}{n}\log(c\varepsilon^{2\dim(\Lambda)})$$

and the conclusion follows letting $n\to\infty$. □

3.3. Metric bifurcation entropy

3.3.1. Metric entropy of a probability measure

Pick a positive Radon measure ν on Λ. Let $X\subset\Lambda$ be a Borel set with $\nu(X)>0$. For $0<\kappa<\nu(X)$, we then let

$$h_\nu(\hat f,\mathfrak{a},X,\kappa) := \inf\left\{h_{\mathfrak{a}}(\hat f,X')\,;\ X'\subset X,\ \nu(X')>\nu(X)-\kappa\right\}.$$

Finally, we define:

$$h_\nu(\hat f,\mathfrak{a},X) := \sup_{\kappa\to 0} h_\nu(\hat f,\mathfrak{a},X,\kappa)$$

Observe that for any compact sets K_1 and K_2, it follows from our definition that

$$h_\nu(\hat f,\mathfrak{a},K_1\cup K_2) = \max(h_\nu(\hat f,\mathfrak{a},K_1),h_\nu(\hat f,\mathfrak{a},K_2)).$$

We define the *metric bifurcation entropy* of ν as

$$h_\nu(\hat f,\mathfrak{a}) := \sup_{K\text{ compact}} h_\nu(\hat f,\mathfrak{a},K).$$

It will be convenient, in what follows, to consider a small variation of the Bowen bifurcation distance defined as:

$$\tilde d_{\mathfrak{a},n}(\lambda,\lambda') := \max\left(d_{\mathfrak{a},n}(\lambda,\lambda'),d_\Lambda(\lambda,\lambda')\right).$$

Notice that if we define accordingly $\tilde h_{\mathfrak{a}}(\hat f,K)$, $\tilde h_\nu(\hat f,\mathfrak{a},K,\kappa)$, $\tilde h_\nu(\hat f,\mathfrak{a},K)$ and $\tilde h_\nu(\hat f,\mathfrak{a})$ using $\tilde d_{\mathfrak{a},n}$ instead of $d_{\mathfrak{a},n}$ then as $\tilde d_{\mathfrak{a},n}\ge d_{\mathfrak{a},n}$ we have that $\tilde h_{\mathfrak{a}}(\hat f,K)\ge h_{\mathfrak{a}}(\hat f,K)$, $\tilde h_\nu(\hat f,\mathfrak{a},K,\kappa)=h_\nu(\hat f,\mathfrak{a},K,\kappa)$ $\tilde h_\nu(\hat f,\mathfrak{a},K)\ge h_\nu(\hat f,\mathfrak{a},K)$, $\tilde h_\nu(\hat f,\mathfrak{a})\ge h_\nu(\hat f,\mathfrak{a})$. On the other hand, by compactness of $K\subset\Lambda$, a d_Λ-separated set in K has bounded cardinality so the above inequalities are in fact equalities. In particular, we will still denote the above entropy as h (and not $\tilde h$).

Fix $\varepsilon>0$. For any integer n and any $\alpha\gg\gamma>0$, we let

$$X_{n,\gamma}(\nu,\alpha) := \left\{\lambda\in\Lambda\,;\ \nu\left(B_{\tilde d_{\mathfrak{a},n}}(\lambda,\varepsilon)\right)\le e^{-n(\alpha-\gamma)}\right\}.$$

Using the bound of the ν-volume of the ball $B_{\tilde d_{\mathfrak{a},n}}(\lambda,\varepsilon)$, we prove the following using a covering argument.

Proposition 3.5. — *Fix $\alpha>0$ and a compact set $K\subset\Lambda$. Assume that for any $\kappa>0$ and any $\gamma>0$, there exists $n_0\ge 1$ such that for every $n\ge n_0$, $\nu(X_{n,\gamma}(\nu,\alpha)\cap K)\ge\nu(K)-\kappa>0$. Then, $h_\nu(\hat f,\mathfrak{a},K)\ge\alpha$.*

Proof. — Let $X \subset K$ such that $\nu(X) > 0$. Choose $\kappa > 0$ small enough and pick $n_0 \geq 1$ so that $\nu(X_{n,\gamma}(\nu,\alpha) \cap K) \geq \nu(K) - \nu(X)/2$ and let $X' := X_{n,\gamma}(\nu,\alpha) \cap K$. By construction, $\nu(X') > 0$.

Choose $\lambda_0 \in X'$ and, recursively choose $\lambda_{k+1} \in X' \setminus \bigcup_{j\leq k} B_{\tilde{d}_{\mathfrak{a},n}}(\lambda_j,\varepsilon)$, which is possible as long as $X' \setminus \bigcup_{j\leq k} B_{\tilde{d}_{\mathfrak{a},n}}(\lambda_j,\varepsilon) \neq \varnothing$. Let E be the collection of those λ_j and let $N \geq 1$ be the cardinality of E. Remark that E is $(\tilde{d}_{\mathfrak{a},n},\varepsilon)$-separated and for every $k \leq N$,

$$\nu\left(\bigcup_{j=0}^{k-1} B_{\tilde{d}_{\mathfrak{a},n}}(\lambda_j,\varepsilon)\right) \leq ke^{-n(\alpha-\gamma)} \leq \nu(X').$$

In particular, this construction is possible, as long as $k \leq \nu(X')e^{n(\alpha-\gamma)}$. Whence $N \geq \nu(X')e^{n(\alpha-\gamma)}$ and

$$\frac{1}{n}\log(N) \geq \frac{1}{n}\log\nu(X') + \alpha - \gamma,$$

and making $n \to \infty$, we find $h_{\mathfrak{a}}(\hat{f},X) \geq \alpha - \gamma$ for all $X \subset K$ with $\nu(X) > 0$, thus $h_\nu(\hat{f},\mathfrak{a},K) \geq \alpha - \gamma$. Making $\gamma \to 0$, we find $h_\nu(\hat{f},\mathfrak{a},K) \geq \alpha$. □

3.3.2. The entropy of the bifurcation measure of a k-tuple of marked points

We now come to the heart of the section. Let $d_\Lambda := \dim\Lambda$ and let $\mu_{\mathfrak{a}} := T_{\mathfrak{a}}^{d_\Lambda}$. The measure $\mu_{\mathfrak{a}}$ is the *bifurcation measure* of the k-tuple $\mathfrak{a} = (a_1,\ldots,a_k)$ in Λ.

Theorem 3.6. — *Let K be a compact set in Λ. For any $\gamma > 0$, there exists n_0 and $\varepsilon > 0$ such that for every $\lambda_0 \in K$ and all $n \geq n_0$:*

$$\mu_{\mathfrak{a}}\left(B_{\tilde{d}_{\mathfrak{a},n}}(\lambda_0,\varepsilon)\right) \leq e^{-nd_\Lambda\log d + n\gamma}.$$

Consequently, for any compact set K such that $\mu_{\mathfrak{a}}(K) > 0$ then

$$h_{\mu_{\mathfrak{a}}}(\hat{f},\mathfrak{a},K) = d_\Lambda \log d.$$

Thus $h_{\mu_{\mathfrak{a}}}(\hat{f},\mathfrak{a}) = d_\Lambda\log d$.

Proof. — Choose $\varepsilon > 0$ and $\lambda_0 \in K$. Pick an integer $n \geq 1$. We let $X := \Lambda \times (\mathbb{P}^q)^k$ and $\hat{f}_k : X \to X$ be the map defined by $\hat{f}_k(\lambda,(z_j)_{j\leq k}) = (\lambda, f_\lambda(z_1),\ldots,f_\lambda(z_k))$. We consider the distance on X defined by

$$d((\lambda,(z_j)),(\lambda',(z'_j))) := \max\left(d_\Lambda(\lambda,\lambda'), \max_j d_{\mathbb{P}^q}(z_j,z'_j)\right)$$

where d_Λ is the distance on Λ induced by ω_Λ and $d_{\mathbb{P}^q}$ the distance on $\mathbb{P}^q$ induced by $\omega_{\mathbb{P}^q}$. We let d_n denote the Bowen distance on X associated to d:

$$d_n((\lambda,(z_j)),(\lambda',(z'_j))) := \max_{0\leq i\leq n-1} d(\hat{f}_k^i(\lambda,(z_j)),\hat{f}_k^i(\lambda',(z'_j))),$$

and we denote by $B_{d_n}((\lambda,(z_j)),\varepsilon)$ the associated ball. With the notations of Section 3.1, recall that (up to a multiplicative constant)

$$\mu_{\mathfrak{a}} = \sum_{\substack{j_1,\ldots,j_k\geq 0\\ j_1+\cdots+j_k=d_\Lambda}} (\pi_\Lambda)_*\left(\bigwedge_{\ell=1}^k \pi_\ell^*\left(\widehat{T}^{j_\ell}\right)\wedge[\Gamma_{\mathfrak{a}}]\right).$$

It is enough to prove the wanted estimate for each term of the sum. So from now on, fix a k-tuple $J := (j_1, \dots, j_k)$ with $j_1 + \dots + j_k = d_\Lambda$ and let

$$\mu_{\mathfrak{a}}^J := (\pi_\Lambda)_* \left(\bigwedge_{\ell=1}^k \pi_\ell^* \left(\widehat{T}^{j_\ell}\right) \wedge [\Gamma_{\mathfrak{a}}] \right).$$

Then

$$\begin{aligned}\mu_{\mathfrak{a}}^J \left(B_{\tilde{d}_{\mathfrak{a},n}}(\lambda_0, \varepsilon)\right) &= \int_{(\pi_\Lambda)^{-1}(B_{\tilde{d}_{\mathfrak{a},n}}(\lambda_0,\varepsilon))} \bigwedge_{\ell=1}^k \pi_\ell^* \left(\widehat{T}^{j_\ell}\right) \wedge [\Gamma_{\mathfrak{a}}] \\ &= \int_{B_{d_n}((\lambda_0,\mathfrak{a}(\lambda_0)),\varepsilon)} \bigwedge_{\ell=1}^k \pi_\ell^* \left(\widehat{T}^{j_\ell}\right) \wedge [\Gamma_{\mathfrak{a}}].\end{aligned}$$

Moreover, since $\widehat{T} = d^{-n+1}(\hat{f}^{n-1})^*\hat{\omega} + d^{-n+1}dd^c\widehat{u}_n$, where $(\widehat{u}_n)$ is a locally uniformly bounded family of continuous functions, letting $u_{n,j} := \widehat{u}_n \circ \pi_j$, we get

$$\mu_{\mathfrak{a}}^J \left(B_{\tilde{d}_{\mathfrak{a},n}}(\lambda_0, \varepsilon)\right) \le d^{-(n-1)d_\Lambda} \int_{B_{d_n}((\lambda_0,\mathfrak{a}(\lambda_0)),\varepsilon)} \bigwedge_{\ell=1}^k (\hat{f}_k^{n-1})^*(\Omega + dd^c u_{n,\ell})^{j_\ell} \wedge [\Gamma_{\mathfrak{a}}],$$

where $\Omega := \omega_\Lambda + \sum_{\ell=1}^k \pi_\ell^*(\omega_{\mathbb{P}^q})$. Let θ_n be the cut-off function of Lemma 2.1 in $B_{d_n}((\lambda_0, \mathfrak{a}(\lambda_0)), \varepsilon)$. Let $S := \bigwedge_{\ell=2}^k (\hat{f}_k^{n-1})^*(\Omega + dd^c u_{n,\ell})^{j_\ell} \wedge [\Gamma_{\mathfrak{a}}]$. By Stokes formula and Lemma 2.1, we deduce:

$$\begin{aligned}\mu_{\mathfrak{a}}^J \left(B_{\tilde{d}_{\mathfrak{a},n}}(\lambda_0, \varepsilon)\right) &\le d^{-(n-1)d_\Lambda} \int \theta_n \bigwedge_{\ell=1}^k (\hat{f}_k^{n-1})^*(\Omega + dd^c u_{n,1})^{j_1} \wedge [\Gamma_{\mathfrak{a}}] \\ &\le d^{-(n-1)d_\Lambda} \int \theta_n (\hat{f}_k^{n-1})^*(\Omega) \wedge (\hat{f}_k^{n-1})^*(\Omega + dd^c u_{n,1})^{j_1-1} \wedge S + \\ &\quad d^{-(n-1)d_\Lambda} \int u_{n,1} dd^c\theta_n \wedge (\hat{f}_k^{n-1})^*(\Omega + dd^c u_{n,1})^{j_1-1} \wedge S \\ &\le \frac{Cn^2}{\varepsilon^2 d^{nd_\Lambda}} \int_{B_{d_n}((\lambda_0,\mathfrak{a}(\lambda_0)),2\varepsilon)} \left(\sum_{r=0}^{n-1} (\hat{f}_k^r)^*(\Omega)\right) \\ &\qquad \wedge (\hat{f}_k^{n-1})^*(\Omega + dd^c u_{n,1})^{j_1-1} \wedge S,\end{aligned}$$

where C is a constant that depends only on the supremum of the $(u_{n,j})$ on the 2ε-neighborhood of $\{\lambda_0\} \times (\mathbb{P}^q)^k$. We iterate the process with all the $j_1 - 1$ terms then for every $\ell \le k$. We deduce

$$\mu_{\mathfrak{a}}^J \left(B_{\tilde{d}_{\mathfrak{a},n}}(\lambda_0, \varepsilon)\right) \le C \frac{n^{2d_\Lambda}}{d^{nd_\Lambda}} \int_{B_{d_n}\left((\lambda_0,\mathfrak{a}(\lambda_0)),2^{d_\Lambda}\varepsilon\right)} \sum_{0 \le m_1,\dots,m_k \le n-1} \bigwedge_{\ell=1}^k (\hat{f}_k^{m_\ell})^*(\Omega^{j_l}) \wedge [\Gamma_{\mathfrak{a}}],$$

where C is a constant that depends only on ε and the supremum of the $(u_{n,j})$ on the $2^{d_\Lambda}\varepsilon$-neighborhood of $\{\lambda_0\} \times (\mathbb{P}^q)^k$. Using Proposition 2.2 (in a neighborhood of K) implies that there exists some constant C' such that:

$$\mu_{\mathfrak{a}}^J \left(B_{\tilde{d}_{\mathfrak{a},n}}(\lambda_0, \varepsilon)\right) \le C \cdot C' \cdot \frac{n^{2d_\Lambda}}{d^{nd_\Lambda}} e^{\gamma n},$$

where γ can be chose arbitrarily small by taking ε small enough. This gives the wanted inequality in the theorem. Then, Proposition 3.5 implies the inequality $h_\nu(\hat{f}, \mathfrak{a}, K) \geq d_\Lambda \log d$ so we have the equality by Theorem 3.4. □

Arguing similarly one proves that

Theorem 3.7. — *Let K be a compact set in Λ. Pick an integer $1 \leq j < d_\Lambda$. For any $\gamma > 0$, there exists n_0 and $\varepsilon > 0$ such that for every $\lambda_0 \in K$ and all $n \geq n_0$:*

$$T_{\mathfrak{a}}^j \wedge \Omega^{d_\Lambda - j}\left(B_{\tilde{d}_{\mathfrak{a},n}}(\lambda_0, \varepsilon)\right) \leq e^{-nj\log d + n\gamma}$$

Consequently, if $T_{\mathfrak{a}}^j \wedge \Omega^{d_\Lambda - j}(K) > 0$, we have

$$h_{T_{\mathfrak{a}}^j \wedge \Omega^{d_\Lambda - j}}(\hat{f}, \mathfrak{a}, K) \geq j \log d.$$

If furthermore $\mathrm{supp}(T_{\mathfrak{a}}^{j+1}) \cap K = \varnothing$*, then*

$$h_{\mathfrak{a}}(\hat{f}, K) = j \log d.$$

For the proof, we proceed as above though we can only replace j-terms $\widehat{T}$ by $d^{-n}(\hat{f}^n)^*\hat{\omega} + d^{-n}dd^c\widehat{u}_n$. We conclude by Proposition 3.5 and Theorem 3.4. We also have the following parametric Brin-Katok formula for the bifurcation measure, similar to the dynamical one ([**BK**], the ideas of our proof are similar).

Theorem 3.8. — *For $\mu_{\mathfrak{a}}$-a.e. λ, one has:*

$$\lim_{\varepsilon\to 0}\liminf_{n\to\infty} \frac{-1}{n}\log \mu_{\mathfrak{a}}\left(B_{\tilde{d}_{\mathfrak{a},n}}(\lambda, \varepsilon)\right) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty} \frac{-1}{n}\log \mu_{\mathfrak{a}}\left(B_{\tilde{d}_{\mathfrak{a},n}}(\lambda, \varepsilon)\right) = d_\Lambda \log d.$$

Proof. — Fix some compact set K with $\mu_a(K) > 0$. Observe first that Theorem 3.6 above states that

$$\lim_{\varepsilon\to 0}\liminf_{n\to\infty} \frac{-1}{n}\log \mu_{\mathfrak{a}}\left(B_{\tilde{d}_{\mathfrak{a},n}}(\lambda, \varepsilon)\right) \geq d_\Lambda \log d,$$

for any $\lambda \in K$ (not necessarily in the support of $\mu_{\mathfrak{a}}$). So all there is left to prove is that:

$$\limsup_{n\to\infty} \frac{-1}{n}\log \mu_{\mathfrak{a}}\left(B_{\tilde{d}_{\mathfrak{a},n}}(\lambda, \varepsilon)\right) \leq d_\Lambda \log d$$

for $\mu_{\mathfrak{a}}$-a.e. $\lambda \in K$. Take $\alpha > d_\Lambda \log d$ and let $\gamma \ll 1$ be such that $\alpha - \gamma > d_\Lambda \log d$. Since $h_{\mathfrak{a}}(\hat{f}, K) = d_\Lambda \log d$, we know that for ε small enough, there exists $n_0(\varepsilon)$ such that for every $n \geq n_0(\varepsilon)$, the cardinality of a (n, ε)-separated set in K is $\leq e^{n(d_\Lambda \log d + \gamma)}$. Consider the set:

$$X_n(\mu_{\mathfrak{a}}, \alpha) := \left\{\lambda \in K\,;\ \mu_{\mathfrak{a}}\left(B_{\tilde{d}_{\mathfrak{a},n}}(\lambda, \varepsilon)\right) \leq e^{-n\alpha}\right\}.$$

Take λ_0 in $X_n(\mu_{\mathfrak{a}}, \alpha)$, then take inductively $\lambda_l \in X_n(\mu_{\mathfrak{a}}, \alpha) \setminus \cup_{i\leq l-1} B_{\tilde{d}_{\mathfrak{a},n}}(\lambda_i, \varepsilon)$. This is possible as long as

$$\mu_{\mathfrak{a}}(X_n(\mu_{\mathfrak{a}}, \alpha)) > l e^{-n\alpha},$$

so, in particular, we can find a (n, ε)-separated set of cardinality $\mu_{\mathfrak{a}}(X_n(\mu_{\mathfrak{a}}, a))e^{n\alpha}$. By the bound of the entropy:

$$\mu_{\mathfrak{a}}(X_n(\mu_{\mathfrak{a}}, \alpha)) \leq e^{-n(\alpha - \gamma - d_\Lambda \log d)}.$$

By Borel-Cantelli, $\mu_{\mathfrak{a}}(\limsup X_n(\mu_{\mathfrak{a}}, \alpha)) = 0$. As α is arbitrary, the result follows. □

Remark. — Using the same argument, we also have a Brin-Katok formula for the measures $T_{\mathfrak{a}}^j \wedge \Omega^{d_\Lambda - j}$ on any compact sets K such that $T_{\mathfrak{a}}^j \wedge \Omega^{d_\Lambda - j}(K) > 0$ and $\mathrm{supp}(T_{\mathfrak{a}}^{j+1}) \cap K = \varnothing$.

3.4. Bifurcation entropy of a holomorphic family of rational maps

Pick a holomorphic family $\hat{f} : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1$ of degree d rational maps which is critically marked, i.e. such that there exists $c_1, \dots, c_{2d-2} : \Lambda \to \mathbb{P}^1$, holomorphic and such that for every λ, the points $c_1(\lambda), \dots, c_{2d-2}(\lambda)$ describe all critical points of f_λ counted with multiplicity.

Let $\mathfrak{c} := (c_1, \dots, c_{2d-2})$. A theorem of DeMarco [**De**] states that the support of the closed positive $(1,1)$-current $T_\mathfrak{c}$ coincides with the bifurcation locus in the classical sense of Mañé-Sad-Sullivan [**MSS, Ly**]. The bifurcation measure of the family $\hat{f}$ is the positive measure μ_{bif} on Λ defined as

$$\mu_{\text{bif}} := T_\mathfrak{c}^{\dim(\Lambda)}.$$

Note that, the formula (2) reads as $T_{c_i}^2 = 0$ for any $1 \leq i \leq 2d-2$, so that the i-th bifurcation current T_{bif}^i decomposes as

$$T_{\text{bif}}^i = \sum_{\substack{j_1, \dots, j_i \\ \text{distinct}}} T_{c_{j_1}} \wedge \cdots \wedge T_{c_{j_i}}.$$

In particular, Theorem A is just a consequence of Theorem 3.4 and Theorem B a consequence of Theorem 3.6.

Observe that we can easily generalize Theorems A and 3.6 to general families (i.e. non necessarily critically marked nor smooth). Indeed, pick a holomorphic family $\hat{f} : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1$ of degree d rational maps. Take a finite branched cover $\pi : \tilde{\Lambda} \to \Lambda$ above the space of critically marked rational maps where $\tilde{\Lambda}$ is smooth, then the family $\tilde{f}$ defined by

$$\tilde{f}(\lambda, z) = (\lambda, f_{\pi(\lambda)}(z)), \ (\lambda, z) \in \tilde{\Lambda} \times \mathbb{P}^1,$$

is critically marked. We then define the bifurcation entropy of $\hat{f}$ as the bifurcation entropy of the lift $\tilde{f}$.

Question. — In [**IM**], the authors showed that the bifurcation locus of the anti-quadratic family: $(\lambda, z) \mapsto \bar{z}^2 + \lambda$, the so-called Tricorn, contains undecorated real-analytic arcs at its boundary. In [**GV2**], we built a bifurcation measure which is supported by the closure of PCF parameters (so it does not see those arcs). It is easy to extend the notion of bifurcation entropy in that setting and it would be interesting to show that the bifurcations in the anti-quadratic family in the real-analytic arcs have no positive entropy and to show that the above bifurcation measure has maximal positive entropy.

3.5. Application to point-wise dimension of the bifurcation measure

Let f be a rational map in the moduli space $\mathcal{M}_d$ of rational maps of degree d (two rational maps are identified when conjugated by a Moebius map). We identify f with its class and let $\mathcal{C}(f)$ denote its critical set and $J(f)$ its Julia set. Assume f is not a flexible Lattès map; for simplicity we also assume that f has simple critical points and we let $F : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1$ be a holomorphic family of rational maps that parameterizes a neighborhood of $f = f_0$ in $\mathcal{M}_d$. Up to reducing Λ, we thus can follow holomorphically the critical points of f in Λ.

We assume that f is *Misurewicz*: $\mathcal{C}(f) \subset J(f)$ and $\forall c \in \mathcal{C}(f)$, $\omega(c) \cap \mathcal{C}(f) = \varnothing$ ($\omega(c)$ is the ω-limit set). We shall also assume that for all $n \in \mathbb{N}$ and all $c \neq c' \in \mathcal{C}_f$, $f^n(f(c)) \neq$

$f(c')$. In particular, the orbit of each critical points is captured by an hyperbolic set so $\forall c \in \mathcal{C}(f)$, we denote $\exp(\underline{\chi}_c) := \liminf |(f^n)'(f(c))|^{1/n} > 1$.

Following [**AGMV**, section 4], we see that the ball:

$$\Omega_n := \mathbb{B}\left(f, C \cdot \frac{1}{\max_c |(f^n)'(f(c))|}\right)$$

is sent into a ε-neighborhood of $(f^{n+1}(c_1), \dots, f^{n+1}(c_{2d-2}))$ by the map

$$\lambda \to (f_\lambda^{n+1}(c_1(\lambda)), \dots, f_\lambda^{n+1}(c_{2d-2}(\lambda))),$$

where C does not depend on n and $(c_k(\lambda))$ is the collection of marked critical points of f_λ. Let $\exp \overline{\chi}_c := \limsup |(f^n)'(f(c))|^{1/n} > 1$. It follows that:

$$\mathbb{B}\left(f, C \cdot \frac{1}{\max_c \exp(n\overline{\chi}_c)}\right) \subset B_{\tilde{d}_{\mathfrak{c},n}}(0, \varepsilon).$$

On one hand, by definition of $\overline{d}_{\mu_{\mathrm{bif}}}(f)$, we have that

$$(\max_c \exp(\overline{\chi}_c))^{-n\overline{d}_{\mu_{\mathrm{bif}}}(f)} \lesssim \mu_{\mathrm{bif}}\left(\mathbb{B}\left(f, C \cdot \frac{1}{\max_c \exp(n\overline{\chi}_c)}\right)\right).$$

On the other hand, from Theorem 3.6, we have

$$\mu_{\mathrm{bif}}(B_{\tilde{d}_{\mathfrak{c},n}}(0, \varepsilon)) \lesssim e^{-n(2d-2)\log d}.$$

So taking the bifurcation measure μ_{bif} of $\mathbb{B}\left(f, C \cdot \frac{1}{\max_c \exp(n\overline{\chi}_c)}\right)$ and $B_{\tilde{d}_{\mathfrak{c},n}}(0, \varepsilon)$ and comparing the growth-rate, we deduce

$$(\max_c \exp \overline{\chi}_c)^{\overline{d}_{\mu_{\mathrm{bif}}}(f)} \geq d^{2d-2} \tag{4}$$

which gives another proof of the second author's results [**Ga**, Corollary 7.4].

Finally, recall that the bifurcation measure of the family $f(\lambda, z) = (\lambda, z^2 + \lambda)$ for which 0 is the only marked critical point is $\mu_{\mathbf{M}}$, the harmonic measure of the Mandelbrot set. A result of Graczyk and Swiatek [**GS2**] states:

Theorem 3.9 ([GS2]). — *For $\mu_{\mathbf{M}}$-almost every λ, the map f_λ is Collet-Eckmann and*

$$\underline{\chi}_0 = \overline{\chi}_0 = \lim_n \frac{1}{n} \log \left|(f_\lambda^n)'(0)\right| = \log 2.$$

As their proof relies crucially on fine properties of external rays, and on the fact that the parameter space is $\mathbb{C}$, it would be interesting to give a different proof, using the notion of bifurcation entropy which might also work in higher degree.

4. Measure-theoretic entropy in several complex variables

4.1. Entropy of the Green measure of Hénon maps

The purpose of this section is to give an alternate proof of the following result of Bedford and Smillie [**BS2**, p-411, Theorem 4.4].

Theorem 4.1. — $h_\mu(f) = \log d$.

Recall that f is a Hénon map of $\mathbb{C}^2$, that is a polynomial automorphism of degree $d > 1$ of $\mathbb{C}^2$. The measure μ is defined as $\mu := T^+ \wedge T^-$ where $T^\pm$ is the Green current of $f^\pm$: $T^\pm := \lim_{n\to\infty} d^{-n}(f^{\pm n})^*(\omega)$ (ω is the Fubini-Study form on $\mathbb{P}^2$). As I^+, the indeterminacy point of f, is a super-attracting fixed point of f^{-1}, one can take an open set U^+ which is the complementary set of a neighborhood of I^+ such that $f(U^+) \subset U^+$. One constructs similarly U^- and we can choose them so that the support of μ is relatively compact in $U := U^+ \cap U^-$. We can write $d^{-n+1}(f^{\pm(n-1)})^*(\omega) = T^\pm + dd^c u_n^\pm$ with the estimate $\|u_n^\pm\|_{\infty,U^\pm} \leq Cd^{-n}$ (the constant C depends on $U^\pm$ but not on n). As in [**BS2**], observe that by Gromov's result and the variational principle, $h_\mu(f) \leq \log d$ so all there is to prove is the reverse inequality.

For that, in [**BS2**], the authors applied Yomdin's estimate on the dynamical ball $B_n(x,\varepsilon)$ for the measure $d^{-n}(f^n)^*(\omega) \wedge \omega$ then using Misiurewicz' proof of the variational principle, they obtain the wanted lower bound of the entropy.

As explained in the introduction, here, we apply Yomdin's estimate on the dynamical ball $B_n(x,\varepsilon)$ directly for the measure μ. This allows us to get rid of Misiurewicz' proof of the variational principle; in exchange, we need a precise control on the convergence towards the Green currents. For that, we lift the different objects on the product space $\mathbb{P}^2 \times \mathbb{P}^2$ where the map (f, f^{-1}) acts and we intersect with the diagonal (this construction already appeared in [**Di1**] to prove the exponential decay of correlations for Hénon maps).

Let F be the birational map $F := (f, f^{-1}) : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2 \times \mathbb{P}^2$, Π_i be the projection to the i-th coordinate and Δ be the diagonal in $\mathbb{P}^2 \times \mathbb{P}^2$. Write $\Omega = \Pi_1^*(\omega) + \Pi_2^*(\omega)$. Then $\mu = (\Pi_1)_*(\Pi_1^*(T^+) \wedge \Pi_2^*(T^-) \wedge [\Delta])$. If

$$B_n(x,\varepsilon) := \{x' \in \mathbb{P}^2,\ \forall |k| \leq n-1,\ d(f^k(x'), f^k(x)) < \varepsilon\}$$

is the two-sided dynamical ball (d is the metric in $\mathbb{P}^2$ induced by the Fubini-Study form), then $B_n(x,\varepsilon) = \Pi_1(B_n^2((x,x),\varepsilon) \cap \Delta)$ where $B_n^2((x,x),\varepsilon)$ denotes the Bowen ball on $\mathbb{P}^2 \times \mathbb{P}^2$ for the map F with respect to the distance $\tilde{d}$ on $\mathbb{P}^2 \times \mathbb{P}^2$ defined by $\tilde{d}((x,y),(x',y')) = \max(d(x,x'),(y,y'))$. In other words:

$$B_n^2((x,y),\varepsilon) := \{(x',y') \in \mathbb{P}^2,\ \forall k \leq n-1,\ \tilde{d}(F^k(x',y'), F^k(x,y)) < \varepsilon\}.$$

We want an upper bound of $\mu(B_n(x,\delta))$ for $x \in \mathrm{supp}(\mu)$. Observe that $(x,x) \in U^+ \times U^-$ and $F(U^+ \times U^-) \subset U^+ \times U^-$. Reducing ε, we can assume that $B_n^2((x,x), 4\varepsilon) \subset U^+ \times U^-$. Using the cut-off function θ_n of Lemma 2.1, the estimates on the convergence toward the

Green currents and Stokes formula, we have:

$$\begin{aligned}
\mu(B_n(x,\varepsilon)) &= \int_{B_n^2((x,x),\varepsilon)} \Pi_1^*(T^+)\wedge\Pi_2^*(T^-)\wedge[\Delta]\\
&= \int_{B_n^2((x,x),\varepsilon)} \Pi_1^*(d^{-n+1}(f^{n-1})^*(\omega)-dd^cu_n^+)\wedge\\
&\qquad\qquad \Pi_2^*(d^{-n+1}(f^{-n+1})^*(\omega)-dd^cu_n^-)\wedge[\Delta]\\
&\le \int \theta_n\Pi_1^*(d^{-n+1}(f^{n-1})^*(\omega)-dd^cu_n^+)\wedge\Pi_2^*(d^{-n+1}(f^{-n+1})^*(\omega)-dd^cu_n^-)\wedge[\Delta]\\
&\le \int \theta_n\Pi_1^*(d^{-n+1}(f^{n-1})^*(\omega))\wedge\Pi_2^*(d^{-n+1}(f^{-n+1})^*(\omega)-dd^cu_n^-)\wedge[\Delta]\\
&\quad+\int \Pi_1^*(u_n^+)dd^c\theta_n\wedge\Pi_2^*(d^{-n+1}(f^{-n+1})^*(\omega)-dd^cu_n^-)\wedge[\Delta]\\
&\le C\frac{n^2}{\varepsilon^2d^n}\int_{B_n^2((x,x),2\varepsilon)}\sum_{k\le n-1}(F^k)^*(\Omega)\wedge\Pi_2^*(d^{-n+1}(f^{-n+1})^*(\omega)-dd^cu_n^-)\wedge[\Delta],
\end{aligned}$$

where C is a constant that does not depend on n. We proceed similarly for the term in $dd^cu_n^-$ using a dynamical cut-off function for $B_n^2((x,x),2\varepsilon)$ and we get:

$$\mu(B_n(x,\delta)) \le C\frac{n^4}{\varepsilon^2d^{2n}}\int_{B_n^2((x,x),4\varepsilon)}\sum_{k,l\le n-1}(F^k)^*(\Omega)\wedge(F^l)^*(\Omega)\wedge[\Delta],$$

where C is again a constant that does not depend on n. We apply Proposition 2.2 to the above term for F and Δ, so we have the bound:

$$\mu(B_n(x,\varepsilon)) \le Ce^{\gamma n},$$

for γ arbitrarily small, reducing ε if necessary. Arguing as in Proposition 3.5 gives back Theorem 4.1.

Remark. — Proceeding as in the proof of Theorem 3.8 allows us to prove Brin-Katok formula for μ directly. In particular we do not need to use the ergodicity of μ. It would be interesting and a priori difficult to get a speed in the convergence in Brin-Katok formula (for generic x). This raises the question of proving a quantitative Algebraic Lemma for holomorphic maps.
The above proof also works in the case of holomorphic maps or the so-called regular birational maps ([**DS**]).

4.2. Entropy for the trace measure of the Green currents

Let $f:\mathbb{P}^k\to\mathbb{P}^k$ be a holomorphic map of algebraic degree $d\ge 2$. Its Green current T_f is a positive closed current of bidegree $(1,1)$ and mass 1 defined by:

$$T_f := \lim_{n\to\infty} d^{-n}(f^n)^*\omega$$

where ω is the Fubini-Study form on $\mathbb{P}^k$. We can write $T_f = d^{-n+1}(f^{n-1})^*\omega + dd^cu_n$ with $\|u_n\|_\infty = O(d^{-n})$. We can thus define the j-th Green current as the self-intersection of the current T_f for $l\le k$: $T_f^l := \bigwedge_1^l T_f$. Its trace measure μ_l is then the well-defined probability measure:

$$\mu_l := T_f^l\wedge\omega^{k-l}.$$

When $k = l$, μ_k is known to be the (unique) ergodic measure of maximal entropy $k \log d$ [**BrD1**] so we shall assume that $l < k$. Our aim is to show similar results for μ_l in $\mathrm{supp}(T_l^k)$. Since μ_l is not invariant, we define its entropy in term of the asymptotic cardinality of (n, ε)-separated sets using the usual Bowen distance d_n on $\mathbb{P}^k$: $d_n(z, z') := \max_{j \leq n-1} d(f^j(z), f^j(z'))$. In particular, observe that $d_n = d_{\mathfrak{a},n}$ where

$$\begin{cases} \hat{f} : \mathbb{P}^k \times \mathbb{P}^k & \to \mathbb{P}^k \times \mathbb{P}^k \\ \qquad (z_1, z_2) & \mapsto (z_1, f(z_2)) \end{cases} \text{ and } \begin{cases} \mathfrak{a} : \mathbb{P}^k & \to \mathbb{P}^k \\ \qquad z_1 & \mapsto z_1 \end{cases}.$$

Then, $T_f^l = T_{\mathfrak{a}}^l$ so that for any compact set K, $h_{\mathrm{top}}(f, K) = h_{\mathfrak{a}}(\hat{f}, K)$ and $h_{\mu_l}(f, K) = h_{T_{\mathfrak{a}}^l \wedge \omega^{k-j}}(\hat{f}, \mathfrak{a}, K)$. In particular, applying Theorem 3.7 in this setting gives:

Theorem 4.2. — *For any $\gamma > 0$, there exists an integer n_0 and $\varepsilon > 0$ such that for any $x \in \mathbb{P}^k$ and any $n \geq n_0$*

$$\mu_l(B_n(x, \varepsilon)) \leq e^{-nl \log d + n\gamma}.$$

In particular, for any compact set $K \subset \mathrm{supp}(T_f^l) \backslash \mathrm{supp}(T_f^{l+1})$ such that $\mu_l(K) > 0$, we have $h_{\mu_l}(f, K) = l \log d$.

Finally, we can also consider the map $f \to h_{\mu_l}(f)$.

Proposition 4.3. — *Let f be an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$. Then $h_{\mu_l}(f) \geq l \log d$. Furthermore, for infinitely many distinct $\alpha \in [l \log d, k \log d]$ (including $k \log d$), there exists an endomorphism f such that $h_{\mu_l}(f) = \alpha$.*

Proof. — We start with an easy example where $h_{\mu_l}(f) = k \log d$. Take for that f a Lattès map of $\mathbb{P}^k$. Then μ_l is absolutely continuous with respect to the Fubini-Study measure on $\mathbb{P}^k$ and with respect to μ_k (this is even a characterization of Lattes [**BeD**]). Now, any set of positive μ_l measure contains a set of positive μ_k measure so it gives an entropy $k \log d$.

For simplicity, let us restrict ourselves to the case where $l = 1$, $k = 2$. We recall for that a construction in [**Du1**, p. 603]. Take h a rational map of $\mathbb{P}^1$ of degree d which admits an ergodic measure ν absolutely continuous with respect to the Lebesgue measure of $\mathbb{P}^1$ of entropy $0 < \beta < \log d$ given by the below claim. Let μ_h denote the measure of maximal entropy $\log d$ of h.

Consider now the map $\hat{f} := (h, h)$ acting on $\mathbb{P}^1 \times \mathbb{P}^1$. Taking the quotient of $\mathbb{P}^1 \times \mathbb{P}^1$ by $(z, w) \equiv (w, z)$, we get a holomorphic map f of degree d on $\mathbb{P}^2$. Let π_i denote the projection to the i-th factor of $\mathbb{P}^1 \times \mathbb{P}^1$. The trace measure of the Green current of $\hat{f}$ is $\pi_1^*(\mu_h) \wedge \pi_2^*(\omega) + \pi_2^*(\mu_h) \wedge \pi_1^*(\omega)$ which is absolutely continuous with respect to $\pi_1^*(\mu_h) \wedge \pi_2^*(\nu) + \pi_2^*(\mu_h) \wedge \pi_1^*(\nu)$. In particular, its entropy is $\log d + \beta$. Descending to f, we deduce that the entropy of the trace measure of the Green current is $\log d + \beta$. □

Claim 4.4. — *For infinitely many distinct $\beta \in]0, \log d]$, there exists a rational map h of $\mathbb{P}^1$ of degree d, which admits an ergodic measure ν, absolutely continuous with respect to the Lebesgue measure of $\mathbb{P}^1$ and of entropy β.*

The claim can be shown by the following (fairly standard) construction, as communicated by Neil Dobbs, that we only sketch. To construct h, we start by perturbing a map with $2d - 3$ critical points that are strictly preperiodic to a repelling cycle and one parabolic fixed point: choose the perturbation h so that the dynamics of the first $2d - 3$ is not changed and make the parabolic point slightly repelling. We can do that by making the last critical point strictly preperiodic so that h admits an ergodic measure ν, absolutely continuous with respect to the Lebesgue measure. By persistence of the Fatou coordinates, we can find a small open set U in the (former) parabolic basin which is stable by h^N for some N, so that neither U nor N depend on the perturbation. But then $|(h^N)|'$ is arbitrarily small in U so that the ν-almost everywhere existing limit of $\frac{1}{nN}\log|(h^{nN})'|$ on U can be taken in turn arbitrarily small. By ergodicity, the Lyapunov exponent χ of ν can be taken arbitrarily small. Mañé formula then gives that $h_\nu(h) = 2\chi$ can be taken arbitrarily small.

We do not know if it is possible to get all $\beta \in]0, \log d]$.

Remark. — To study the entropy on $\mathrm{supp}(T_f)\backslash\mathrm{supp}(\mu_f)$ for an endomorphism of $\mathbb{P}^2$, the first author defined saddle measures, under general assumptions [**DT2**]. The advantage of that approach is that we deal with a nice ergodic, invariant measure but such measure is not always known to exist.

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