DISTRIBUTION OF POSTCRITICALLY FINITE POLYNOMIALS III: COMBINATORIAL CONTINUITY

by

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Abstract. — In the first part of the present paper, we continue our study of the distribution of postcritically finite parameters in the moduli space of polynomials: we show the equidistribution of PCF Misiurewicz parameters with prescribed combinatorics toward the bifurcation measure. Our results essentially rely on a combinatorial description of the escape locus and of the bifurcation measure developed by Kiwi and Dujardin-Favre.

In the second part of the paper, we construct a bifurcation measure for the connectedness locus of the quadratic anti-holomorphic family which is supported by a strict subset of the boundary of the Tricorn. We also establish an approximation property by PCF Misiurewicz parameters in the spirit of the previous one.

Contents

Introduction

In this article, we study equidistribution problems in parameter spaces of polynomials. In any holomorphic family of rational maps, DeMarco [**De**] introduced a current T_{bif} which is supported exactly on the bifurcation locus, giving a measurable point of view to study bifurcations. Bassanelli and Berteloot [**BB1**] considered the self-intersections of this current which enable to study higher bifurcations phenomena. In the moduli space $\mathcal{P}_d$ of degree d polynomials, the maximal self-intersection of the bifurcation current induces a *bifurcation measure*, μ_{bif}, which is the analogue of the harmonic measure of the Mandelbrot set when $d \geq 3$. It measures the sets of maximal bifurcation phenomena (see [**DF**]).

In particular, we want to understand the distribution of the *postcritically finite Misiurewicz parameters* (the parameters for which all the critical points are strictly preperiodic). Such parameters play a central role in complex dynamics. They allow computations of Hausdorff dimension of parametric fractal sets ([**Sh, G1**]). They also are special from the arithmetic point of view, since they are points of small height for a well-chosen Weil height ([**I, FG**]). They also play a central role in the geometry of the bifurcation locus: it is around such parameters that we can exhibit rich geometric phenomena (for example small copies of the Mandelbrot set [**Mc, G2**] or similarity between the Julia set and the bifurcation locus [**DH2, Ta**]).

In $\mathcal{P}_d$, Dujardin and Favre [**DF**] proved the density of Misiurewicz parameters in the support of the bifurcation measure (see [**BE**] for the case of rational maps, which relies on the results of [**BB1**]). Our goal here is to give a *measurable* version of this statement to better understand the distribution of these parameters. More precisely, we want to show that the probability measure equidistributed on such Misiurewicz parameters converges to the bifurcation measure (letting the length of critical orbits go to ∞). This can be seen as a parametric version of Birkhoff's ergodic Theorem and also as an arithmetic equidistribution statement. Several distinct approaches exist to get such a convergence of measures, making it a deep and rich subject.

Indeed, such convergence has already been achieved using pluripotential theoretic tools. Levin [**L**] showed the equidistribution of PCF (postcritically finite) parameters toward the bifurcation measure in the quadratic family using extremal properties of the Green function of the Mandelbrot set. That approach has been extended by Dujardin and Favre to prove the equidistribution of maps having a preperiodic marked critical point toward the bifurcation current of that given critical point [**DF**] (see also [**O**] for a simplified proof in the hyperbolic case). Still relying on pluripotential theory, the authors proved the equidistribution of hyberbolic PCF parameters with exponential speed of convergence in [**GV**]. Notice also the results of [**Du**] for the case of intersections of bifurcation currents of given marked critical points.

Another fruitful approach was the use of arithmetic methods, using the Theorem of equidistribution of points of small height. Indeed, PCF parameters are arithmetic. Favre and Rivera-Letelier [**FRL**] first used this approach for the quadratic family in order to prove the equidistribution of PCF parameters toward the bifurcation measure, with an exponential speed of convergence. This result was extended to higher degrees by Favre and the first author in [**FG**]. Notice that the equidistribution statement proved in [**FG**] requires technical assumptions on the preperiods and periods of the critical orbits of the PCF parameters considered in the approximation.

Here, we develop instead a combinatorial approach, based on the impression of external rays. For that, we impose conditions on the combinatorics of angles with given period and preperiod landing at critical points instead of giving conditions for the parameter itself (see Section 3.1). On the other hand, we make no technical assumption on the periods and preperiods of the critical orbits. The first part of this article is dedicated to the proof of Theorem A in that setting. For that, we develop further the arguments of Dujardin and Favre [**DF**], using Kiwi's results on the combinatorial space and the landing of external rays ([**K4, K3**]) and the results of Przytycki and Rohde [**PR**] on the rigidity of Topological Collet-Eckmann repellers.

Let Cb be the space of combinatorics of degree d polynomials (see Section 1 for a precise definition). Pick now any $(d-1)$-tuples $\underline{n} = (n_0, \ldots, n_{d-2})$ and $\underline{m} = (m_0, \ldots, m_{d-2})$ of non-negative integers with $m_i > n_i$. We let

$$\mathsf{C}(\underline{m}, \underline{n}) := \{(\Theta_0, \ldots, \Theta_{d-2}) \in \mathsf{Cb}\,;\ \forall i,\ \exists \theta \in \Theta_i,\ d^{m_i}\theta = d^{n_i}\theta\ \}.$$

When $m_i > n_i \geq 1$ for all i, we also let

$$\mathsf{C}^*(\underline{m}, \underline{n}) := \mathsf{C}(\underline{m}, \underline{n}) \setminus \mathsf{C}(\underline{m} - \underline{n}, \underline{0}).$$

A combinatorics $\Theta \in \mathsf{C}^*(\underline{m}, \underline{n})$ is called *Misiurewicz.* The space Cb is known to admit a natural probability measure μ_{Cb}. Dujardin and Favre have also built a *landing map* $e : \mathsf{Cb} \longrightarrow \mathcal{P}_d$ which satisfies $e_*(\mu_{\mathsf{Cb}}) = \mu_{\text{bif}}$ and which sends Misiurewicz combinatorics to Misiurewicz polynomials (see Sections 3.1 and 3 for more details).

Our first result can be stated as follows.

Theorem A. — *Let $(\underline{n}_k)_k$ and $(\underline{m}_k)_k$ be two sequences of $(d-1)$-tuples with $m_{k,j} > n_{k,j} \geq 1$ and $m_{k,j} \to \infty$ as $k \to \infty$ for all j. Let $X_k := e(\mathsf{C}^*(\underline{m}_k, \underline{n}_k))$ and let μ_k be the measure*

$$\mu_k := \frac{1}{\text{Card}(\mathsf{C}^*(\underline{m}_k, \underline{n}_k))} \sum_{\{P\} \in X_k} \mathcal{N}_{\mathsf{Cb}}(P) \cdot \delta_{\{P\}},$$

where $\mathcal{N}_{\mathsf{Cb}}(P) \geq 1$ is the (finite) number of distinct combinatorics of the polynomial P. Then μ_k converges to μ_{bif} as $k \to \infty$ in the weak sense of probablility measures on $\mathcal{P}_d$.

Notice that the support of μ_k is contained in the set of classes $\{P\} \in \mathcal{P}_d$ such that $P^{n_{k,j}}(c_j) = P^{m_{k,j}}(c_j)$ and c_j is not periodic. Remark also that the above result does not deal with an *equidistribution* property, since the considered measures take into account the *combinatorial multiplicity* $\mathcal{N}_{\mathsf{Cb}}(P)$ of Misiurewicz parameters. We give in Section 3.1 a description of the range of $\mathcal{N}_{\mathsf{Cb}}$.

Building on the same idea, we give a sufficient condition for polynomials with $(d-1)$ distinct parabolic cycles to equidistribute the bifurcation measure.

In the second part of the present work, we adapt the above combinatorial methods to the case of the parameters space of quadratic antiholomorphic polynomials, i.e. the family

$$f_c(z) := \bar{z}^2 + c\ ,\ c \in \mathbb{C}.$$

The connectedness locus, in this setting, is known as the Tricorn $\mathbf{M}_2^*$. As observed by Inou and Mukherjee in [**IM**], the harmonic measure of the Tricorn is not a good candidate to measure bifurcation phenomena: the existence of (real analytic) stable parabolic arcs is an obstruction for the density of PCF parameters in the boundary of tricorn. We develop further the theory of the landing map of external rays in this setting. Precisely, we prove the following.

Theorem B. — *Almost any external ray of the Tricorn $\mathbf{M}_2^*$ lands and, if $\ell : \mathbb{R}/\mathbb{Z} \longrightarrow \partial\mathbf{M}_2^*$ is the landing map, then ℓ is measurable and there exists a set $\mathsf{R} \subset \mathbb{R}/\mathbb{Z}$ of full Lebesgue measure such that $\ell|_{\mathsf{R}}$ is continuous.*

The use of external rays for the Tricorn has been initiated by Nakane in [**N**] to prove the connectedness of the Tricorn. A finer study of the topological and combinatorial properties was developed by several authors (e.g. [**HS**] where the authors showed that the Tricorn is not path connected).

To prove Theorem B, we imbed the quadratic antiholomorphic polynomials family in a complex family of degree 4 polynomials maps in order to use again Kiwi's results on the combinatorial space to prove the equidistribution of Misiurewicz parameters. We now define the *bifurcation measure* of the Tricorn $\mathbf{M}_2^*$ as

$$\mu_{\mathrm{bif}}^* := (\ell)_* \left(\lambda_{\mathbb{R}/\mathbb{Z}}\right) .$$

We believe that this measure should equidistribute other dynamical phenomena, as hyperbolic postcritically finite parameters for example. For $n > k > 0$, we consider the following set of *Misiurewicz parameters*:

$$\mathrm{Per}^*(n,k) := \{c \in \mathbb{C}\,;\, f_c^n(0) = f_c^k(0) \text{ and } f_c^{n-k}(0) \neq 0\} ,$$

similarly we consider the following set of *Misiurewicz combinatorics*:

$$\mathsf{C}^*(n,k) := \{\theta \in \mathbb{R}/\mathbb{Z}\,;\, (-2)^{n-1}\theta = (-2)^{k-1}\theta \text{ and } (-2)^{n-k}(\theta) \neq \theta\} .$$

Building on the above definition of the bifurcation measure, we can describe the distribution of the sets $(\ell)_*(\mathsf{C}^*(n,k))$ which is a subset of $\mathrm{Per}^*(n,k)$. This is the content of our next result.

Theorem C. — *For any $1 < k < n$, the set $\mathrm{Per}^*(n,k)$ is finite and $(\ell)_*(\mathsf{C}^*(n,k)) \subset$ Per$^*(2n,2k)$. Moreover, for any sequence $1 < k(n) < n$, the measure*

$$\mu_n^* := \frac{1}{\mathrm{Card}(\mathsf{C}^*(n,k(n)))} \sum_{c \in (\ell)_*(\mathsf{C}^*(n,k(n)))} \mathcal{N}_{\mathbb{R}/\mathbb{Z}}(c) \cdot \delta_c ,$$

where $\mathcal{N}_{\mathbb{R}/\mathbb{Z}}(c) := \mathrm{Card}\{\theta \in \mathsf{C}^*(n,k(n))\,;\ \ell(\theta) = c\} \geq 1$, converge to μ_{bif}^* in the weak sense of measures on $\mathbb{C}$.

In a certain sense, parameters of $(\ell)_*(\mathsf{C}^*(n,k(n)))$ are *truly* of pure period $n - k(n)$ and preperiod $k(n)$, since their combinatorics also have the same property.

Notice that the question of counting parameters such that $f_c^n(0) = f_c^k(0)$ is of *real* algebraic nature and is difficult. On this matter, notice the difficult work [**MNS**] where the authors notably count the number of hyperbolic components of the Tricorn.

In order to prove the above result, we relate the Misiurewicz character of f_c to the Misiurewicz character of the induced degree 4 polynomials for which we can apply known results about landing of external rays. We also relate the measure μ_{bif}^* to the bifurcation measure ν_{bif} of this family of degree 4 polynomials by the inclusion $\mathrm{supp}(\mu_{\mathrm{bif}}^*) \subset \mathrm{supp}(\nu_{\mathrm{bif}}) \cap \mathbb{R}^2$. This follows from the fact that Misiurewicz parameters belong to the support of ν_{bif} and are dense in it.

In a first section, we start with general preliminaries, notably on the combinatorial space and the landing of external rays. We then give the proof of Theorem A in the particular case of the quadratic family (for parabolic and Misiurewicz combinatorics). That proof is of folklore nature in this case but we believe it will help the global understanding of the reader. In Part I, we prove Theorem A and explore the basic landing properties of parabolic combinatorics à la Douady and Hubbard. In Part II, we treat the case of the Tricorn. We start by exploring the combinatorial space in this setting. Finally, we prove Theorems B and C.

Acknowledgement. — Both authors are partially supported by the ANR project Lambda ANR-13-BS01-0002.

1. General preliminaries

1.1. The moduli space and the visible shift locus

The *moduli space* $\mathcal{P}_d$ of degree d polynomials is the space of affine conjugacy classes of degree d polynomials with $d-1$ marked critical points. A point in $\mathcal{P}_d$ is represented by a d-tuple $(P, c_0, \ldots, c_{d-2})$ where P is a polynomial of degree d, and the c_i's are complex numbers such that $\{c_0, \ldots, c_{d-2}\}$ is the set of all critical points of P. For each i, $\mathrm{Card}\{j,\ c_j = c_i\}$ is the order of vanishing of P' at c_i. Two points $(P, c_0, \ldots, c_{d-2})$ and $(\tilde{P}, \tilde{c}_0, \ldots, \tilde{c}_{d-2})$ are identified when there exists an affine map ϕ such that $\tilde{P} = \phi \circ P \circ \phi^{-1}$, and $\tilde{c}_i = \phi(c_i)$ for all $0 \leq i \leq d-2$.

The set $\mathcal{P}_d$ is a quasiprojective variety of dimension $d-1$, and is isomorphic to the quotient of $\mathbb{C}^{d-1}$ by the finite group of $(d-1)$-th roots of unity acting linearly and diagonally on $\mathbb{C}^{d-1}$ (see [**Si**]). When $d \geq 3$, this space admits a unique singularity at the point $(z^d, 0, \ldots, 0)$.

Recall that for $(P, c_0, \ldots, c_{d-2}) \in \mathcal{P}_d$, the *Green function of* P is defined by

$$g_P(z) := \lim_{n \to +\infty} d^{-n} \log^+ |P^n(z)|,\ z \in \mathbb{C} .$$

It satisfies $\mathcal{K}_P = \{g_P = 0\}$ and it is a psh and continuous function of $(P, z) \in \mathcal{P}_d \times \mathbb{C}$. Let

$$G(P) := \max_{0 \leq j \leq d-2} g_P(c_j) .$$

The *connectedness locus* $\mathcal{C}_d := \{(P, c_0, \ldots, c_{d-2}) \in \mathcal{P}_d\,;\ \mathcal{J}_P \text{ is connected}\}$ is a compact set and satisfies $\mathcal{C}_d = \{(P, c_0, \ldots, c_{d-2}) \in \mathcal{P}_d\,;\ G(P) = 0\}$ (see [**BH**]).

We also call *Böttcher coordinate* of P at infinity the unique biholomorphic map

$$\phi_P : \mathbb{C} \setminus \{z \in \mathbb{C}\,;\ g_P(z) \leq G(P)\} \longrightarrow \mathbb{C} \setminus \overline{\mathbb{D}}(0, \exp(G(P)))$$

which is tangent to the identity (assuming that P is monic) at infinity and satisfies

1. $\phi_P \circ P = (\phi_P)^d$ on $\mathbb{C} \setminus \{z \in \mathbb{C}\,;\ g_P(z) \leq G(P)\}$,
2. $g_P(z) = \log |\phi_P(z)|$ for all $z \in \mathbb{C} \setminus \{z \in \mathbb{C}\,;\ g_P(z) \leq G(P)\}$.

For an angle $\theta \in [0, 1]$, consider the set:

$$\phi_P^{-1}(]\exp(G(P)), +\infty[e^{2i\pi\theta}) .$$

We define the *external ray* R_θ for P of angle θ as the maximal flow line of the gradient ∇g_P in $\{g_P > 0\} = \mathbb{C} \setminus \mathcal{K}_P$ that contains that set. If it meets a critical point c_i of P, we say that R_θ terminates at c_i.

Definition 1.1. — *We say that a* $(P, c_0, \ldots, c_{d-2}) \in \mathcal{P}_d$ *lies in the* shift locus $\mathcal{S}_d$ *if all critical points of* P *escape under iteration. We also say that a class* $(P, c_0, \ldots, c_{d-2}) \in \mathcal{S}_d$ *lies in the* visible shift locus $\mathcal{S}_d^{\mathrm{vis}}$ *if for all* $0 \leq i \leq d-1$, *there exists* $\deg_{c_i}(P)$ *external rays terminating at* c_i *and* $P(c_i)$ *belongs to an external ray.*

When $(P, c_0, \ldots, c_{d-2}) \in \mathcal{S}_d^{\mathrm{vis}}$, we denote by $\Theta(P)$ the *combinatorics* (or *critical portrait*) of P, i.e. the $(d-1)$-tuple $\Theta(P) := (\Theta_0, \ldots, \Theta_{d-2})$ of finite subsets of $\mathbb{R}/\mathbb{Z}$ for which Θ_i is exactly the collection of angles of rays landing at c_i.

1.2. The combinatorial space

We follow the definition given by Dujardin and Favre [**DF**]. Two finite and disjoint subsets $\Theta_1, \Theta_2 \subset \mathbb{R}/\mathbb{Z}$ are said to be *unlinked* if Θ_2 is included in a single connected component of $(\mathbb{R}/\mathbb{Z}) \setminus \Theta_1$. We let S be the set of pairs $\{\alpha, \alpha'\}$ contained in the circle $\mathbb{R}/\mathbb{Z}$, such that $d\alpha = d\alpha'$ and $\alpha \neq \alpha'$. First, we can define the simple combinatorial space.

Definition 1.2. — *We let* Cb_0 *be the set of* $(d-1)$*-tuples* $\Theta = (\Theta_0, \ldots, \Theta_{d-2}) \in \mathsf{S}^{d-1}$ *such that for all* $i \neq j$*, the two pairs* Θ_i *and* Θ_j *are disjoint and unlinked.*

It is known that Cb_0 has a natural structure of translation manifold. It is also known to carry a natural invariant probability measure that we will denote μ_{Cb_0} (see [**DF**, §7]). We now may define the full combinatorial space.

Definition 1.3. — *The set* Cb *is the collection of all* $(d-1)$*-tuples* $(\Theta_0, \ldots, \Theta_{d-2})$ *of finite sets in* $\mathbb{R}/\mathbb{Z}$ *satisfying the following four conditions:*

- *for any fixed* i*,* $\Theta_i = \{\theta_1, \ldots, \theta_{k(i)}\}$ *and* $d\theta_j = d\theta_1$ *for all* j*;*
- *for any* $i \neq j$*, either* $\Theta_i = \Theta_j$ *or* $\Theta_i \cap \Theta_j = \varnothing$*;*
- *if* N *is the total number of distinct* Θ_i*'s, then* $\mathrm{Card} \bigcup_i \Theta_i = d + N - 1$*;*
- *for any* $i \neq j$ *such that* $\Theta_i \cap \Theta_j = \varnothing$*, the sets* Θ_i *and* Θ_j *are unlinked.*

Remark. — *When* $(P, c_0, \ldots, c_{d-2}) \in \mathcal{S}_d^{\mathrm{vis}}$*, then* $\Theta(P) \in \mathsf{Cb}$*.*

We will use the following.

Proposition 1.4 (Kiwi, Dujardin-Favre). — *The set* Cb *is compact and path connected and contains* Cb_0 *as a dense open subset.*

Then, we define, as Dujardin and Favre, the combinatorial measure μ_{Cb} as the only probability measure on μ_{Cb} which coincides with μ_{Cb_0} on Cb_0 and does not charge $\mathsf{Cb} \setminus \mathsf{Cb}_0$.

Following Kiwi [**K4**], we will use the following definition.

Definition 1.5. — *Pick* $\Theta \in \mathsf{Cb}$*. We say that* P *lies in the* impression *of* Θ *if there exists a sequence* $P_n \in \mathcal{S}_d^{\mathrm{vis}}$ *converging to* P *such that the corresponding combinatorics* $\Theta(P_n)$ *converge to* Θ*.*

We denote by $\mathcal{I}_{\mathcal{C}_d}(\Theta)$ the impression of any $\Theta \in \mathsf{Cb}$. Kiwi proved the following result concerning basic properties of the impression of a combinatorics (see [**K4**]).

Proposition 1.6. — *For any* $\Theta \in \mathsf{Cb}$*, the impression* $\mathcal{I}_{\mathcal{C}_d}(\Theta)$ *is a non-empty connected compact subset of* $\partial\mathcal{C}_d \cap \partial\mathcal{S}_d$*.*

According to Theorem 5.12 of [**K3**], whenever $\mathcal{J}_P = \mathcal{K}_P$ is locally connected and P has no irrationally neutral cycle, the map $P : \mathcal{J}_P \longrightarrow \mathcal{J}_P$ is conjugate to the maps induced by $z \mapsto z^d$ on a quotient $\mathbb{S}^1/\sim_P$ of $\mathbb{S}^1$ by a dynamically defined equivalence relation. Moreover, Theorem 1 of [**K4**] guarantees that if $P, P' \in \mathcal{I}_{\mathcal{C}_d}(\Theta)$ have only repelling cycles and have locally connected Julia sets $\mathcal{J}_P = \mathcal{K}_P$ and $\mathcal{J}_{\tilde{P}} = \mathcal{K}_{\tilde{P}}$, the quotient spaces $\mathbb{S}^1/\sim_P$ and $\mathbb{S}^1/\sim_{P'}$ depend only on the combinatorics Θ, and in particular are homeomorphic. In fact, Theorem 5.13 of [**K3**] guarantees that they are actually conjugate on $\mathbb{C}$

All this summarizes as follows.

Theorem 1.7 (Kiwi). — *Let $\Theta \in \mathsf{Cb}_0$ and let $(P, c_0, \ldots, c_{d-2}), (\tilde{P}, \tilde{c}_0, \ldots, \tilde{c}_{d-2}) \in \mathcal{I}_{\mathcal{C}_d}(\Theta)$. Assume that $\mathcal{J}_P = \mathcal{K}_P$ and $\mathcal{J}_{\tilde{P}} = \mathcal{K}_{\tilde{P}}$ are locally connected and that P and $\tilde{P}$ have only repelling cycles. Then there exists an orientation preserving homeomorphism $h : \mathbb{C} \longrightarrow \mathbb{C}$ which conjugates P to $\tilde{P}$ on $\mathbb{C}$.*

In the sequel, we will use this result in the following way: when the impression $\mathcal{I}_{\mathcal{C}_d}(\Theta)$ of Θ contains a polynomial which is topologically rigid, with locally connected Julia set and having only repelling cycles, then $\mathcal{I}_{\mathcal{C}_d}(\Theta)$ is reduced to the singleton $\{P\}$. We will be particularly interested in the case where P is Topological Collet-Eckmann.

Recall that a polynomial P satisfies the *topological Collet-Eckmann* (or TCE) condition if for some $A \geq 1$ there exist constants $M > 1$ and $r > 0$ such that for every $x \in \mathcal{J}_P$ there is an increasing sequence (n_j) with $n_j \leq A \cdot j$ such that for every j,

$$\mathrm{Card}\left\{i\,;\ 0 \leq i < n_j, \mathrm{Comp}_{f^i(x)} f^{-(n_j - i)} \mathbb{D}(f^{n_j}(x), r) \cap C(P) \neq \varnothing\right\} \leq M,$$

where $\mathrm{Comp}_x(X)$ is the connected component of the set X containing x and $C(P)$ is the critical set of P. It is known that if P is TCE, then $\mathcal{J}_P$ is locally connected, $C(P) \subset \mathcal{J}_P = \mathcal{K}_P$ and P only has repelling cycles (see e.g. [**PRLS**, Main Theorem]).

1.3. Measure theoretic tools

A classical result states that if $f : X \longrightarrow Y$ is a map between metric spaces, ν is a probability measure on X and ν_n converges weakly to ν and if the set D_f of discontinuities of f satisfies $\nu(D_f) = 0$, then $f_*(\nu_n)$ converges weakly to $f_*(\nu)$. This is known as the *mapping theorem.* We prove the following slight generalization that we will use in a crucial way.

Theorem 1.8. — *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \longrightarrow Y$ be a measurable map. Let ν be a probability measure on X such that there exists a Borel subset $S \subset X$ with $\nu(S) = 1$ and such that $f|_S : S \longrightarrow Y$ is continuous. Pick any sequence ν_n of probability measures on X with $\nu_n(S) = 1$. Assume in addition that ν_n converges weakly to ν on X. Then $f_*(\nu_n)$ converges weakly to $f_*(\nu)$.*

We rely on the following classical fact (see e.g. [**B**, Theorem 2.1 p. 16]).

Fact. — *A sequence of probability measures μ_n on a metric space converges weakly to a probability measure μ if and only if $\limsup_{n\to\infty} \mu_n(F) \leq \mu(F)$ for any closed set F, or equivalently, if and only if $\liminf_{n\to\infty} \mu_n(U) \geq \mu(U)$ for any open set U.*

Proof of Theorem 1.8. — First, $f_*(\nu)$ is a probability measure on Y and its restriction to $W := f(S)$ is a probability measure since $\nu(S) = 1$. Let $g := f|_S : S \to W$ and $\tilde{\nu}_n := \nu_n|_S$ and $\tilde{\nu} := \nu|_S$. Notice that, by assumption, g is a continuous map between metric spaces and $f_*(\nu_n)|_W = g_*(\tilde{\nu}_n)$ and $f_*(\nu)|_W = g_*(\tilde{\nu})$. Moreover, $\tilde{\nu}_n$ and $\tilde{\nu}$ are probability measures on the metric space S and $\tilde{\nu}_n$ converges weakly to $\tilde{\nu}$ on S.

Pick now any closed subset $B \subset Y$ and let $B' := W \cap B$. We have that $\nu_n(f^{-1}(B)) = \tilde{\nu}_n(g^{-1}(B'))$ and $\nu(f^{-1}(B)) = \tilde{\nu}(g^{-1}(B'))$, since $\nu_n(X \setminus S) = \nu(X \setminus S) = 0$. Recall also that B' is a closed subset of W and $g^{-1}(B')$ is a closed subset of S. Hence, by the above

Fact,

$$\begin{aligned}\limsup_{n\to\infty} f_*(\nu_n)(B) &= \limsup_{n\to\infty} g_*(\tilde{\nu}_n)(B') = \limsup_{n\to\infty} \tilde{\nu}_n(g^{-1}(B'))\\ &\leq \tilde{\nu}\left(g^{-1}(B')\right) = \nu\left(f^{-1}(B)\right) = f_*(\nu)(B)\ .\end{aligned}$$

Again by the Fact, this ends the proof. □

We say that a sequence $(A_n)_{n\geq 0}$ of finite subsets of $\mathbb{R}/\mathbb{Z}$ is equidistributed if $\lim_{n\to\infty}\mathrm{Card}(A_n) = +\infty$ and if for any open interval $I \subset \mathbb{R}/\mathbb{Z}$ we have

$$\lim_{n\to+\infty} \frac{\mathrm{Card}(A_n \cap I)}{\mathrm{Card}(A_n)} = \lambda_{\mathbb{R}/\mathbb{Z}}(I),$$

where $\lambda_{\mathbb{R}/\mathbb{Z}}$ denotes the Lebesgue measure on the circle $\mathbb{R}/\mathbb{Z}$. We shall also use the following easy lemma.

Lemma 1.9. — *Pick $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ two sequences of finite sets of $\mathbb{R}/\mathbb{Z}$. Assume that (A_n) and (B_n) are equidistributed, that $B_n \subset A_n$ and that*

$$\liminf_{n\to\infty} \frac{\mathrm{Card}(A_n \setminus B_n)}{\mathrm{Card}(A_n)} > 0.$$

Then the sequence $(A_n \setminus B_n)$ is equidistributed. In other words, the probability measure μ_n equidistributed on $A_n \setminus B_n$ converges weakly towards $\lambda_{\mathbb{R}/\mathbb{Z}}$.

Proof. — From the above fact, it is sufficient to check that for any open interval $I \subset \mathbb{R}/\mathbb{Z}$, we have $\liminf_{n\to\infty}\mu_n(I) \geq \lambda_{\mathbb{R}/\mathbb{Z}}(I)$, since any open subset of $\mathbb{R}/\mathbb{Z}$ is a disjoint union of open intervals. Pick an open interval $I \subset \mathbb{R}/\mathbb{Z}$ and $\epsilon > 0$. As A_n and B_n are equidistributed, there exists $n_0 \geq 1$ such that for any $n \geq n_0$,

$$\left|\frac{\mathrm{Card}(A_n \cap I)}{\mathrm{Card}(A_n)} - \lambda_{\mathbb{R}/\mathbb{Z}}(I)\right| \leq \epsilon \text{ and } \left|\frac{\mathrm{Card}(B_n \cap I)}{\mathrm{Card}(B_n)} - \lambda_{\mathbb{R}/\mathbb{Z}}(I)\right| \leq \epsilon.$$

Let $\alpha := \liminf_n \mathrm{Card}(A_n \setminus B_n)/\mathrm{Card}(A_n)$. By assumption, we have $\alpha > 0$ and up to increasing n_0, for any $n \geq n_0$ we may assume $\mathrm{Card}(A_n \setminus B_n)/\mathrm{Card}(A_n) \geq \alpha/2 > 0$. Hence

$$\left|\frac{\mathrm{Card}((A_n \setminus B_n) \cap I)}{\mathrm{Card}(A_n \setminus B_n)} - \lambda_{\mathbb{R}/\mathbb{Z}}(I)\right| \leq \frac{4}{\alpha}\epsilon.$$

This concludes the proof. □

2. In the quadratic family

This section serves as a model to the sequel: we illustrate our strategy in the family

$$p_c(z) := z^2 + c\ ,\ (c, z) \in \mathbb{C}^2$$

which parametrizes the moduli space of quadratic polynomials.

In the present section, we prove a continuity property for the Riemann map of the complement of the Mandelbrot set which we combine with well known landing properties of rational rays to deduce theorems A for $d = 2$ (see e.g. [**DH1, DH2, Sc1**]). We also prove an equidistribution result for parabolic parameters.

2.1. Prime-End Impressions and Collet-Eckmann parameters

The content of this section is classical (see e.g. [**DH1, DH2**]). Recall that the bifurcation measure of the quadratic family $(p_c)_{c\in\mathbb{C}}$ is the harmonic measure $\mu_{\mathbf{M}}$ of the Mandelbrot set $\mathbf{M} := \{c \in \mathbb{C}\,;\ |p_c^n(0)| \leq 2\ , \forall n \geq 0\} = \{c \in \mathbb{C}\,;\ \mathcal{J}_c \text{ is connected}\}$. Moreover, the map

$$\Phi : \mathbb{C} \setminus \mathbf{M} \longrightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$$

defined by $\Phi(c) := \phi_{p_c}(c)$ is a biholomorphism which is tangent to the identity at ∞. The *external ray* of the Mandelbrot set of angle $\theta \in \mathbb{R}/\mathbb{Z}$ is the set

$$\mathcal{R}_{\mathbf{M}}(\theta) := \Phi^{-1}\left(\{Re^{2i\pi\theta}\,;\ R > 1\}\right).$$

The combinatorial space Cb is then $\mathsf{Cb} = \{\{\alpha, \alpha + \frac{1}{2}\}\,;\ \alpha \in \mathbb{R}/\mathbb{Z}\}$. The impression of the combinatorics $\Theta = \{\alpha, \alpha + \frac{1}{2}\}$ can be described as the impression at angle $\theta = 2\alpha$ under the map Φ^{-1}, i.e. as the set

$$\mathcal{I}_{\mathbf{M}}(\Theta) = \bigcap_{\rho>1,\ \epsilon>0} \overline{\Phi^{-1}\left(\{Re^{2i\pi\tau}\,;\ |\theta - \tau| < \epsilon,\ 1 < R < \rho\}\right)}.$$

We say θ is *Misiurewicz* if there exists $n > k \geq 1$ such that $2^n\theta = 2^k\theta$ and $2^{n-k}\theta \neq \theta$. Combining [**Sc1**, Lemma 4.1] with [**K4**, Theorem 5.3], we have the following.

Proposition 2.1. — *For any Misiurewicz angle θ, the prime-end impression of θ is reduced to a singleton. Moreover, this singleton consists in a Misiurewicz parameter c. In particular, the ray $\mathcal{R}_{\mathbf{M}}(\theta)$ lands at c.*

For any Misiurewicz parameter c, at least one ray lands at c and the angles of the rays that land at c are exactly the angles of the dynamical rays of p_c that land at its critical value c.

We also say that θ is parabolic if there exists $n \geq 1$ such that $2^n\theta = \theta$. The following is classical (see [**DH1, DH2**] and [**Sc2**, Corollary 5.3]).

Proposition 2.2. — *Pick any parameter c for which p_c admits a parabolic cycle. Either $c = 1/4$, in which case exactly one external ray of $\mathbf{M}$ lands at c, or exactly two external rays of $\mathbf{M}$ land at c. Furthermore, the corresponding impressions are reduced to singletons.*

We shall now give a short proof of the following toy-model for Theorem 3.6 (see Section 3.3 for a more detailed proof of this result).

Theorem 2.3. — *There exists a set $\mathsf{C} \subset \mathbb{R}/\mathbb{Z}$ of full Lebesgue measure such that*

1. *the map $\Phi^{-1} :]1, +\infty[\times \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{C} \setminus \mathbf{M}$ extends continuously to $\{0\} \times \mathsf{C}$,*
2. *the set C contains the Misiurewicz and parabolic angles.*

Proof. — From [**Sm**], we know that *Collet-Eckmann* angles have full Lebesgue measure and that their impression contains the limit of the corresponding ray which is a Collet-Eckmann parameter. Pick such a θ, let $\Theta := \{\theta, \theta + \frac{1}{2}\}$ and let c_0 be a Collet-Eckmann parameter contained in $\mathcal{I}_{\mathbf{M}}(\Theta)$. According to [**K3**, Theorem 1] and [**Sm**, §2], any parameter c in $\mathcal{I}_{\mathbf{M}}(\Theta)$ has locally connected Julia set $\mathcal{J}_c = \mathcal{K}_c$ and all its cycles are repelling. By Theorem 1.7, this implies that p_c and p_{c_0} are topologically conjugate on their Julia sets. By [**PR**, Corollary C], they are affine conjugate, hence $c = c_0$. Since $\mathcal{I}_{\mathbf{M}}(\Theta)$ is connected, it is reduced to a singleton hence Φ^{-1} extends continuously to $\{(0, \theta)\}$.

Item 2 follows directly from the two above propositions. □

2.2. Distribution of Misiurewicz and Parabolic parameters

We now aim at proving the following, using Theorem 2.3.

Theorem 2.4. — *For any integer, let μ_n be the measure equidistributed on the set X_n of roots of hyperbolic components of period $k|n$. Then μ_n converges to $\mu_{\mathbf{M}}$ in the weak sense of measures on $\mathbb{C}$ as $n \to \infty$.*

Proof. — Let $\ell : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbf{M}$ be the landing map of rays, i.e. the radial limit almost everywhere of the map Φ^{-1}. For any $n \geq 1$, let also

$$\mathsf{P}(n) := \{\theta \in \mathbb{R}/\mathbb{Z}\, ;\ 2^n\theta = \theta\}\ .$$

It is known that ℓ is a well-defined measurable map which satisfies $\mu_{\mathbf{M}} = \ell_*(\lambda_{\mathbb{R}/\mathbb{Z}})$ (see e.g. [**GŚ**]). By Theorem 2.3, it restricts as a continuous function on a set of full measure which contains the set $\mathsf{P}(n)$ for any n. It is clear that the sequence $\{\mathsf{P}(n)\}_n$ is equidistributed. Let ρ_n be the probability measure equidistributed on $\mathsf{P}(n)$. By Theorem 1.8, the above implies that

$$\ell_*(\rho_n) = \frac{1}{\mathrm{Card}(\mathsf{P}(n))} \sum_{\ell(\mathsf{P}(n))} \mathcal{N}_{\mathbf{M}}(c) \cdot \delta_c$$

converges weakly to $\ell_*(\lambda_{\mathbb{R}/\mathbb{Z}}) = \mu_{\mathbf{M}}$, where $\mathcal{N}_{\mathbf{M}}(c)$ is the number of external rays of $\mathbf{M}$ that land at c. Remark now that $\mathrm{Card}(\mathsf{P}(n)) = 2^n - 1$. Using Proposition 2.2, we deduce that $\mathrm{Card}(X_n) = 2^{n-1}$ and

$$\ell_*(\rho_n) - \mu_n = \frac{1}{2^n - 1}\mu_n - \frac{1}{2^{n-1}}\delta_{1/4}$$

converges weakly to 0. This concludes the proof. □

Remark that, for any $\lambda \in \mathbb{C}$, it is known that the set of parameters $c \in \mathbb{C}$ for which p_c admits a n-cycle of multiplier λ equidistribute towards $\mu_{\mathbf{M}}$ by [**BG2**].

For any integers $n > k > 1$, we let

$$\mathsf{C}(n,k) := \{\theta \in \mathbb{R}/\mathbb{Z}\, ;\ 2^{n-1}\theta = 2^{k-1}\theta \ \text{ and } 2^{n-k}\theta \neq \theta\}$$

and we let $d(n,k) := \mathrm{Card}(\mathsf{C}(n,k)) = 2^{n-1} - 2^{k-1} - 2^{n-k} + 1$. A similar proof gives the following.

Theorem 2.5. — *Pick any sequence $1 < k(n) < n$ and let $d_n := d(n,k(n))$. Let also ν_n*

$$\nu_n := \frac{1}{d_n} \sum_{\ell(\mathsf{C}(n,k(n)))} \mathcal{N}_{\mathbf{M}}(c) \cdot \delta_c\ ,$$

where $\mathcal{N}_{\mathbf{M}}(c)$ is the number of external rays of $\mathbf{M}$ that land at c. Then ν_n converges to $\mu_{\mathbf{M}}$ in the weak sense of measures on $\mathbb{C}$ as $n \to \infty$.

PART I
IN THE MODULI SPACE OF POLYNOMIALS

3. The bifurcation measure and combinatorics

3.1. Misiurewicz combinatorics

We define the map $M_d : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$ by letting

$$M_d(\theta) = d \cdot \theta \mod 1.$$

Recall that we say that a combinatorics $\Theta = (\Theta_0, \ldots, \Theta_{d-2}) \in \mathsf{Cb}$ is *Misiurewicz* if any $\alpha \in \bigcup_i \Theta_i$ is strictly preperiodic under the map $M_d : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. We denote by Cb_{mis} the set of all Misiurewicz combinatorics.

We will use the following (see [**K4**, Theorem 5.3]).

Theorem 3.1 (Kiwi). — *The impression of a Misiurewicz combinatorics is reduced to a singleton and corresponds to the only degree d critically marked Misiurewicz polynomial with the chosen combinatorics.*

As noticed by Dujardin and Favre [**DF**, Theorem 7.18], this induces a bijection between Cb_{mis} and the set of Misiurewicz parameters in the moduli space of *combinatorially* marked degree d polynomials (see also [**BFH**, Theorem III]).

We now want to describe how many Misiurewicz combinatorics can have the same impression in $\mathcal{P}_d$. To this aim, for any $0 \leq i \leq d-2$ and any $0 \leq n < m$, we let

$$\mathsf{C}_i(m,n) := \{\Theta \in \mathsf{Cb}\,;\ \Theta_i = \{\alpha_1, \ldots, \alpha_{k_i}\},\ \exists j,\ d^m\alpha_j = d^n\alpha_j\ \}\ .$$

Notice that the periods and preperiods do not depend on the combinatorics, i.e. for any Θ, Θ' with the same impression, the periods and preperiods of Θ_i and Θ'_i coincide. Relying on a result of Kiwi [**K2**], we can prove

Proposition 3.2. — *Pick any two $(d-1)$-tuples of positive integers $(n_0, \ldots, n_{d-2})$ and $(m_0, \ldots, m_{d-2})$ such that $m_i > n_i$. Let also $(P, c_0, \ldots, c_{d-2}) \in \mathcal{P}_d$ be such that $P^{n_i}(c_i) = P^{m_i}(c_i)$, $P^{m_i-n_i}(c_i) \neq c_i$ and $P^{n_i}(c_i)$ is exactly $(m_i - n_i)$-periodic. Set*

$$\mathcal{N}_{\mathsf{Cb}}(P) := \mathrm{Card}\left(\{\Theta \in \mathsf{Cb}_{\text{mis}}\,;\ \{(P, c_0, \ldots, c_{d-2})\} = \mathcal{I}_{\mathcal{C}_d}(\Theta)\}\right).$$

Then $\mathcal{N}_{\mathsf{Cb}}(P)$ is finite. More precisely, if $\Theta = (\Theta_0, \ldots, \Theta_{d-2})$ and q_i is the exact period of the cycle contained in the orbit $\{M_d^k(\Theta_i)\}_{k\geq 1}$, then $(m_i - n_i)|q_i$ and

$$\prod \deg_{P(c_i)}(P^{n_i-1}) \cdot \left(\frac{q_i}{m_i - n_i}\right) \leq \mathcal{N}_{\mathsf{Cb}}(P) \leq \prod \deg_{P(c_i)}(P^{n_i-1}) \cdot \max\left(\ell + 1, \ell\frac{q_i}{m_i - n_i}\right)$$

where the product ranges over the set of geometrically distinct critical points of P and ℓ is the number of geometrically distinct critical values of P.

The following lemma is a reformulation of [**K2**, Theorem 3.2].

Lemma 3.3. — *Let P be any degree $d \geq 3$ polynomial with connected Julia set. Assume that P has ℓ distinct critical values. Let z be a repelling or parabolic periodic point of P of exact period p. Then*

1. *the number N of cycles of rays that land on the orbit of z satisfies $1 \leq N \leq \ell + 1$,*

2. *if $N=\ell+1$, then the exact period of any of those cycles of rays is p.*

Proof of Proposition 3.2. — By Theorem 3.1, if α is periodic and lies in the orbit under iteration of M_d of Θ_i, then the point z at which it lands lies in the orbit under iteration of P of c_i. In particular, if R_α is the dynamical ray of angle α of P, then $P^{q_i}(R_\alpha)=R_\alpha$, hence $P^{q_i}(z)=z$, i.e. $(m_i-n_i)|q_i$.

Up to reordering, write now $c_1,\ldots,c_\ell$ the geometrically distinct critical points of P, $d_1,\ldots,d_\ell\geq 2$ the local degree of P at $c_1,\ldots,c_\ell$ respectively. As long as $P^r(c_i)$ is not a critical point, a ray landing at $P^{r+1}(c_i)$ has one and only one preimage under P which lands at $P^r(c_i)$. On the other hand, if $P^r(c_i)=c_j$ for some $j\neq i$, then any ray landing at $P^{r+1}(c_i)$ has exactly d_j preimages landing at $P^r(c_i)=c_j$. As a conclusion, the number N_i of rays landing at $P(c_i)$ is exactly $\deg_{P(c_i)}(P^{n_i-1})$ times the number of rays landing at $P^{n_i}(c_i)$, which satisfies

$$\deg_{P(c_i)}(P^{n_i-1})\cdot\left(\frac{q_i}{m_i-n_i}\right)\leq N_i\leq \deg_{P(c_i)}(P^{n_i-1})\cdot\max\left(\ell+1,\ell\frac{q_i}{m_i-n_i}\right)$$

by Lemma 3.3.

Finally, each ray landing at $P(c_i)$ has exactly d_i preimages. For any i, pick θ_i landing at $P(c_i)$ and let Θ_i be the set of angles whose rays land at c_i and $M_d(\alpha)=\theta_i$ for any $\alpha\in\Theta_i$. Then $\Theta:=(\Theta_0,\ldots,\Theta_{d-2})$ (with repetitions if critical points are multiple) is a combinatorics for P and we can associate to each collection $(\theta_0,\ldots,\theta_{d-2})$ of angles landing respectively at $P(c_i)$ one and only one combinatorics for P. The conclusion follows. □

3.2. The bifurcation measure and the Goldberg and landing maps

We recall here material from [**DF**, §6 & 7]. Recall that we defined the psh and continuous function $G:\mathcal{P}_d\longrightarrow\mathbb{R}_+$ by letting $G(P):=\max_{0\leq j\leq d-2}g_P(c_j)$ for any $(P,c_0,\ldots,c_{d-2})\in\mathcal{P}_d$. We can define the *bifurcation measure* μ_{bif} of the moduli space $\mathcal{P}_d$ as the Monge-Ampère measure associated to the function G, i.e.

$$\mu_{\mathrm{bif}}:=(dd^cG)^{d-1}\ .$$

This measure was introduced first by Dujardin and Favre [**DF**] and they proved that it is a probability measure whose support is the Shilov boundary of the connectedness locus $\partial_S\mathcal{C}_d$ (see [**DF**, §6]).

The Goldberg and landing maps after Dujardin and Favre. — For $r>0$, let $\mathcal{G}(r):=\{(P,c_0,\ldots,c_{d-2})\in\mathcal{P}_d\,;\ g_P(c_i)=r,\ \forall 0\leq i\leq d-2\}$. The set $\mathcal{G}(r)$ is contained in $\mathcal{S}_d^{\mathrm{vis}}$. Moreover, there exists a unique continuous map

$$\begin{aligned}\Phi_g:\mathsf{Cb}\times\mathbb{R}_*^+ &\longrightarrow \mathcal{P}_d\\ (\Theta,r)&\longmapsto (P(\Theta,r),c_0(\Theta,r),\ldots,c_{d-2}(\Theta,r))\end{aligned}$$

such that the following holds:

- $P(\Theta,r)\in\mathcal{S}_d^{\mathrm{vis}}$ and the $(d-1)$-tuple Θ of subsets is the combinatorics of $P(\Theta,r)$ and $g_{P(\Theta,r)}(c_i(\Theta,r))=r$ for each $0\leq i\leq d-2$,
- the map $\Phi_g(\cdot,r)$ is a homeomorphism from Cb onto $\mathcal{G}(r)$. Moreover, $\Phi_g(\cdot,r)$ restricts to a homeomorphism from Cb_0 onto the subset of $\mathcal{G}(r)$ of polynomials for which all critical points are simple.

The map Φ_g is the *Goldberg* map of the moduli space $\mathcal{P}_d$. The radial limit of the map $\Phi_g(\cdot, r)$ as $r \to 0$ exists μ_{Cb}-almost everywhere and defines a map $e : \mathsf{Cb} \longrightarrow \mathcal{P}_d$ (see [**DF**, Proposition 7.19]). By construction, its image is contained in $\partial\mathcal{C}_d \cap \partial\mathcal{S}_d^{\mathrm{vis}}$.

Definition 3.4. — *The map* $e : \mathsf{Cb} \longrightarrow \mathcal{P}_d$ *is called the* landing map.

The main result relating this landing map with the bifurcation measure is the following (see [**DF**, Theorem 8]).

Theorem 3.5 (Dujardin-Favre). — $e_*(\mu_{\mathsf{Cb}}) = \mu_{\mathrm{bif}}$.

3.3. Continuity of the landing map on a set of μ_{Cb}-full measure

The main goal of this section is to prove the following result.

Theorem 3.6. — *There exists a set* $\mathsf{Cb}_1 \subset \mathsf{Cb}_0$ *of full* μ_{Cb}*-measure such that the map* $e|_{\mathsf{Cb}_1}$ *is continuous. Moreover, the set* Cb_1 *contains the Misiurewicz combinatorics* $\mathsf{Cb}_{\mathrm{mis}}$.

In fact, we rely on the stronger statement below, which is essentially the combination of Theorem 1.7 with [**K4**, Theorem 1] and with the rigidity property established in [**PR**, Corollary C].

Theorem 3.7. — *Pick* $\Theta \in \mathsf{Cb}_0$ *such that there exists* $(P, c_0, \ldots, c_{d-2}) \in \mathcal{I}_{\mathcal{C}_d}(\Theta)$ *with* $\mathcal{J}_P = \mathcal{K}_P$ *and which satisfies the TCE condition. Then the impression* $\mathcal{I}_{\mathcal{C}_d}(\Theta)$ *is reduced to a singleton.*

Proof. — Pick $\Theta \in \mathsf{Cb}$ and $(P, c_0, \ldots, c_{d-2}), (\tilde{P}, \tilde{c}_0, \ldots, \tilde{c}_{d-2}) \in \mathcal{I}_{\mathcal{C}_d}(\Theta)$ such that $(P, c_0, \ldots, c_{d-2})$ satisfies the TCE condition. According to [**K4**, Theorem 1], the real lamination of P is equal to that of Θ and has aperiodic kneading since P has only repelling cycles. Again by [**K4**, Theorem 1], P and $\tilde{P}$ have the same real lamination and do not satisfy the *Strongly Recurrent Condition* (see e.g. [**Sm**, §2]). In particular, $\tilde{P}$ also has only repelling periodic points and its Julia set is locally connected. Moreover, $\tilde{P} \in \mathcal{C}_d \cap \partial\mathcal{S}_d$ and all its cycles are repelling, hence $\mathcal{K}_{\tilde{P}}$ has no interior, i.e. $\mathcal{J}_{\tilde{P}} = \mathcal{K}_{\tilde{P}}$.

We now apply Theorem 1.7: the polynomials P and $\tilde{P}$ are conjugate on their Julia sets by an orientation preserving homeomorphism. Finally, since P satisfies the TCE property and $C(P) \subset \mathcal{J}_P = \mathcal{K}_P$ (recall that $C(P)$ is the critical set of P), [**PR**, Corollary C] states that P and $\tilde{P}$ are affine conjugate and there exists σ in the symmetric group of $d-1$ elements such that $\tilde{c}_i = c_{\sigma(i)}$. Hence $\mathcal{I}_{\mathcal{C}_d}(\Theta)$ is contained in a finite subset of $\mathcal{P}_d$.

Since $\mathcal{I}_{\mathcal{C}_d}(\Theta)$ is connected, it is reduced to a singleton. □

We now are in position to prove Theorem 3.6.

Proof of Theorem 3.6. — Dujardin and Favre [**DF**, Theorem 10] prove that there exists a Borel set $\mathsf{Cb}_1^* \subset \mathsf{Cb}_0$ such that

- Cb_1^* has full μ_{Cb}-measure,
- for any $\Theta \in \mathsf{Cb}_1^*$ the impression $\mathcal{I}_{\mathcal{C}_d}(\Theta)$ contains a polynomial P satisfying the TCE condition.

Let us now set

$$\mathsf{Cb}_1 := \mathsf{Cb}_1^* \cup \mathsf{Cb}_{\mathrm{mis}}.$$

Pick $\Theta \in \mathsf{Cb}_1^* \cup \mathsf{Cb}_{\mathrm{mis}}$. According to Theorem 3.7, Theorem 3.1 of Kiwi, the impression $\mathcal{I}_{\mathcal{C}_d}(\Theta)$ is reduced to a singleton. By definition of the impression $\mathcal{I}_{\mathcal{C}_d}(\Theta)$, the map Φ_g extends continuously to $\mathsf{Cb}_1 \times \{0\}$. Recall that the landing map e is the radial limit almost everywhere of the map $\Phi_g(\cdot, r)$, as $r \to 0$. The landing map e thus coincides μ_{Cb}-almost everywhere with the extension of the Goldberg map Φ_g, which ends the proof. □

4. Distribution of Misiurewicz Combinatorics

Our goal here is to apply the combinatorial tools studied above to equidistribution problems concerning Misiurewicz parameters with prescribed combinatorics.

4.1. Preliminary properties

Recall that, for any $0 \leq i \leq d-2$ and any $0 \leq n < m$, we have denoted

$$\mathsf{C}_i(m,n) := \{\Theta \in \mathsf{Cb}\,;\ \Theta_i = \{\alpha_1, \ldots, \alpha_{k_i}\},\ \exists \alpha_j \in \Theta_i,\ d^m\alpha_j = d^n\alpha_j\}\ .$$

For any i, pick any sequences $0 < n_{k,i} < m_{k,i}$ such that $m_{k,i} \to \infty$ as $k \to \infty$ and let

$$\mathsf{C}^*_{k,i} := \mathsf{C}_i(m_{k,i}, n_{k,i}) \setminus \mathsf{C}_i(m_{k,i} - n_{k,i}, 0) \text{ and } \mathsf{C}^*_k := \bigcap_{i=0}^{d-2} \mathsf{C}^*_{k,i}.$$

Notice that the set C^*_k is finite and that $\mathrm{Card}(\mathsf{C}^*_k) \geq c \cdot d^{\sum_i m_{k,i}}$, where $c > 0$ is a constant depending only on d and not on the sequences $(m_{k,i})$ and $(n_{k,i})$ (see [**FG**, §5.3]). Finally, we let ν_k be the probability measure on Cb which is equidistributed on C^*_k.

Lemma 4.1. — *The sequence $\nu_k(\mathsf{Cb} \setminus \mathsf{Cb}_0)$ converges to 0 as $k \to +\infty$.*

Proof. — To do so, it is sufficient to prove that

$$\limsup_{k\to+\infty} \frac{\mathrm{Card}\,(\mathsf{C}^*_k \setminus \mathsf{Cb}_0)}{\mathrm{Card}\,\left(\mathsf{C}^*_k\right)} = 0\ .$$

An element in the set $\mathsf{C}^*_k \setminus \mathsf{Cb}_0$ coincides with the union over the j of the set $\bigcap_{i\neq j} \mathsf{C}^*_{k,i}$ intersected with $\bigcup_{i\neq j}\{\Theta \in \mathsf{Cb}\,;\ \Theta_j = \Theta_i\}$. The equation $d^m\alpha_1 = d^n\alpha_1$ has $d^m - d^n - 1 \leq d^m$ solutions, the conditions $d\alpha_1 = d\alpha_j$ implies that they are at most $c(d)d^m$ possible Θ_i in $C_i(m,n)$ ($c(d)$ is an (explicit) constant that depends only on d). As a consequence,

$$\mathrm{Card}\,(\mathsf{C}^*_k \setminus \mathsf{Cb}_0) \leq C \sum_j d^{-m_{k,j} + \sum m_{k,i}}$$

where C depends only on d. Hence

$$\frac{\mathrm{Card}\,(\mathsf{C}^*_k \setminus \mathsf{Cb}_0)}{\mathrm{Card}\,\left(\mathsf{C}^*_k\right)} \leq \sum_j \frac{C}{c} d^{-m_{k,j}} \longrightarrow 0\ ,$$

as $k \to +\infty$, which ends the proof. □

We now give a more precise description of the spaces Cb and S and of the measure μ_{Cb} we will need in our proof. We refer to [**DF**, §7.1] for more details.

Recall that S is the set of pairs $\{\alpha, \alpha'\}$ contained in the circle $\mathbb{R}/\mathbb{Z}$, such that $d\alpha = d\alpha'$ and $\alpha \neq \alpha'$. The set S is a translation manifold of dimension 1 which has $\lfloor d/2 \rfloor$ connected components, each of them being isomorphic to $\mathbb{R}/\mathbb{Z}$. We endow each of these components with a copy of the probability measure $\lambda_{\mathbb{R}/\mathbb{Z}}$ and let λ_{S} be the probability measure which is proportional to the obtained finite measure.

The set Cb_0 can be seen as an open subset of S^{d-1}. Notice that $\lambda_{\mathsf{S}}^{\otimes(d-1)}(\mathsf{Cb}_0) > 0$ and let μ_{Cb_0} be the measure

$$\mu_{\mathsf{Cb}_0} := \frac{1}{\lambda_{\mathsf{S}}^{\otimes(d-1)}(\mathsf{Cb}_0)} \mathbf{1}_{\mathsf{Cb}_0} \cdot \lambda_{\mathsf{S}}^{\otimes(d-1)}.$$

The set Cb has Cb_0 as an open and dense subset. The measure μ_{Cb} is then the trivial extension of μ_{Cb_0} to Cb.

4.2. An equidistribution result: Theorem A

For any $0 \leq i \leq d-2$ and any $0 \leq n < m$, we let

$$\mathsf{S}(m,n) := \{\{\alpha,\alpha'\} \in \mathsf{S}\,;\ d^m\alpha = d^n\alpha \text{ or } d^m\alpha' = d^n\alpha'\} \text{ and } \mathsf{S}^*(m,n) := \mathsf{S}(m,n)\backslash \mathsf{S}(m-n,0)\,.$$

For any i, pick any sequences $0 < n_{k,i} < m_{k,i}$ such that $m_{k,i} \to \infty$ as $k \to \infty$ and let

$$\mathsf{S}_k^* := \prod_{i=0}^{d-2} \mathsf{S}^*(m_{k,i}, n_{k,i}) \subset \mathsf{S}^{d-1}.$$

Finally, we let m_k be the probability measure equidistributed on the finite set S_k^* of S^{d-1}.

As in [**DF**, §7.1], for any collection of open intervals $I_0, \ldots, I_{d-2} \subset \mathbb{R}/\mathbb{Z}$, and any collection of integers $q_0, \ldots, q_{d-2} \in \{1, \ldots, \lfloor d/2 \rfloor\}$, we let

$$\mathsf{I}_i(q_i) := \left\{\{\alpha,\alpha'\} \in \mathsf{S}\,;\ \{\alpha,\alpha'\} \subset I_i \cup \left(I_i + \frac{q_i}{d}\right)\right\}$$

and we can define an open set of S^{d-1} by setting

$$U(I,q) := \left\{\Theta \in \mathsf{S}^{d-1}\,;\ \Theta_i \in \mathsf{I}_i(q_i)\right\},$$

where $I := (I_0, \ldots, I_{d-2})$ and $q = (q_0, \ldots, q_{d-2})$. The open sets $U(I,q)$, for all I and q, span the topology of S^{d-1}.

We rely on the following key intermediate result.

Lemma 4.2. — *The sequence (m_k) is equidistributed with respect to $\lambda_{\mathsf{S}}^{d-1}$ on S^{d-1}. More precisely, if $I = (I_0, \ldots, I_{d-2})$ is a $(d-1)$-tuple of intervals and $q = (q_0, \ldots, q_{d-2})$ is a $(d-1)$-tuple of integers with $1 \leq q_i \leq \lfloor d/2 \rfloor$, we have*

$$\lim_{k\to\infty} m_k(U(I,q)) = \lambda_{\mathsf{S}}^{\otimes(d-1)}(U(I,q)) = \prod_{i=0}^{d-2} \lambda_{\mathsf{S}}(I_i(q_i)).$$

Proof. — As the measure m_k is a product measure $m_k = m_{k,0} \otimes \cdots \otimes m_{k,d-1}$, where $m_{k,i}$ is the probability measure equidistributed on the set $S^*(m_{k,i}, n_{k,i})$, by Fubini Theorem, it is sufficient to prove that $m_{k,i}$ is equidistributed with respect to λ_{S} as $k \to +\infty$.

Let $d_k := \mathrm{Card}(\mathsf{S}^*(m_{k,i}, n_{k,i}))$. Since for any $m > n > 0$,

$$d^m - d^n \leq \mathrm{Card}(\mathsf{S}(m,n)) \leq \lfloor d/2 \rfloor \cdot (d^m - d^n)\ ,$$

we find

$$\begin{aligned} d_k &= \mathrm{Card}(\mathsf{S}(m_{k,i}, n_{k,i})) - \mathrm{Card}(\mathsf{S}(m_{k,i} - n_{k,i}, 0)) \\ &\geq d^{m_{k,i}} - d^{n_{k,i}} - \lfloor d/2 \rfloor \cdot (d^{m_{k,i}-n_{k,i}} - 1) \\ &\geq d^{m_{k,i}} - d^{n_{k,i}} - \frac{d/2+1}{d} \cdot (d^{m_{k,i}} - d^{n_{k,i}}) \\ &\geq \left(1 - \frac{d/2+1}{d}\right) \cdot (d^{m_{k,i}} - d^{n_{k,i}}). \end{aligned}$$

Notice that $1 - \frac{d/2+1}{d} > 0$. Now, the natural measure λ_{S} is the renormalization of $\lfloor d/2 \rfloor$ copies of $\lambda_{\mathbb{R}/\mathbb{Z}}$, hence we can directly apply Lemma 1.9. This gives the equidistribution of $m_{k,i}$ with respect to λ_{S}, as $k \to +\infty$ and the proof is complete. □

As a consequence, using classical measure theory, we easily get the following:

Corollary 4.3. — *The sequence (m_k) converges towards $\lambda_{\mathsf{S}}^{\otimes(d-1)}$ in the weak sense of probability measures on S^{d-1}.*

We now can end the proof of Theorem A.

Proof of Theorem A. — Write again $\lambda := \lambda_{\mathsf{S}}^{\otimes(d-1)}$. Recall that ν_k is the probability measure equidistributed on C_k^* and μ_k is the measure defined in Theorem A. By Theorem 3.1, one has $e_*(\nu_k) = \mu_k$ for any k. Notice also that $\mu_{\mathrm{bif}} = e_*(\mu_{\mathsf{Cb}})$, by Theorem 3.5. According to Theorem 3.6 and Theorem 1.8, it is sufficient to prove that (ν_k) converges weakly to μ_{Cb}.

First, remark that, since μ_{Cb} is the trivial extension of μ_{Cb_0} to Cb, Lemma 4.1 implies that it is actually sufficient to prove that ν_k converges weakly towards μ_{Cb_0}. Let K be any compact subset of Cb_0. Then

$$\nu_k(K) - \mu_{\mathsf{Cb}_0}(K) = \frac{m_k(K)}{m_k(\mathsf{Cb}_0)} - \frac{\lambda(K)}{\lambda(\mathsf{Cb}_0)}.$$

According to Corollary 4.3 and to the Fact of Section 1.3, for any $\epsilon > 0$, there exists $k_0 \geq 1$ such that for any $k \geq k_0$,

$$m_k(K) \leq \lambda(K) + \epsilon \text{ and } m_k(\mathsf{Cb}_0) \geq \lambda(\mathsf{Cb}_0) - \epsilon$$

since Cb_0 is open and K is compact in Cb_0, hence in Cb. In particular, for $k \geq k_0$, we find

$$\nu_k(K) - \mu_{\mathsf{Cb}_0}(K) \leq \epsilon \cdot \frac{\lambda(\mathsf{Cb}_0) + \lambda(K)}{\lambda(\mathsf{Cb}_0)(\lambda(\mathsf{Cb}_0) - \epsilon)}.$$

Taking the limsup as $k \to \infty$ and then making $\epsilon \to 0$ gives

$$\limsup_{k \to \infty} \nu_k(K) \leq \mu_{\mathsf{Cb}_0}(K).$$

This ends the proof, using again the Fact of Section 1.3. □

4.3. The case of Parabolic Combinatorics: a triviality criterion

Pick any $(d-1)$-tuple of integers $\underline{m}$. A combinatorics $\Theta \in \mathsf{C}^*(\underline{m}, 0)$ is called *parabolic*.

The same proof as that of Theorem A gives the following.

Theorem 4.4. — *Let $(\underline{n}_k)_k$ be any sequence of $(d-1)$-tuples with $n_{k,j}\to\infty$ as $k\to\infty$ for all j and $\gcd(n_{k,j},n_{k,i})=1$ for all k and $j\neq i$. Let $Y_k:=e(\mathsf{C}(\underline{n}_k,\underline{0}))$ and μ'_k be the measure*

$$\mu'_k:=\frac{1}{\mathrm{Card}(\mathsf{C}(\underline{n}_k,\underline{0}))}\sum_{\{P\}\in Y_k}\mathcal{M}_{\mathsf{Cb}}(P)\cdot\delta_{\{P\}},$$

where $\mathcal{M}_{\mathsf{Cb}}(P)$ is the (finite) number of combinatorics in $\mathsf{C}(\underline{n}_k,\underline{0})$ whose landing point is reduced to $\{P\}$. If for any k and any $\Theta\in\mathsf{C}(\underline{n}_k,\underline{0})$ the impression $\mathcal{I}_d(\Theta)$ is trivial, then μ'_k converges to μ_{bif} as $k\to\infty$ in the weak sense of probability measures on $\mathcal{P}_d$.

Unfortunately, apart from the case $d=2$, we don't know how to prove the triviality of impressions. However, we can prove that the parameters involved in the statement of Theorem 4.4 have $(d-1)$ distinct parabolic cycles. We call such parameters *totally parabolic.*

Let P be a degree d polynomial. We say that a parabolic periodic point z of P is *n-degenerate* if it has period $k|n$ and if n is minimal so that $(P^n)'(z)=1$. The following shows that for all parabolic combinatorics $\Theta\in\mathsf{C}(\underline{n}_k,\underline{0})$, the landing map is well defined at Θ and that such lading parameter is totally parabolic with $d-1$ parabolic cycles which are $n_0,\dots,n_{d-2}$-degenrate respectively.

Theorem 4.5. — *Pick $\underline{n}=(n_0,\dots,n_{d-2})$ with $\gcd(n_i,n_j)=1$ for $i\neq j$ and $n_i\geq 2$ for all i. Let $\Theta=(\Theta_0,\dots,\Theta_{d-2})\in\mathsf{Cb}_{\mathrm{par}}$ be a combinatorics such that for any j, there exists $\theta_j\in\Theta_j$ which is exactly n_j-periodic for M_d. There exists a unique critically marked polynomial $(P,c_0,\dots,c_{d-2})\in\mathcal{I}_{\mathcal{C}_d}(\Theta)$ having $d-1$ distinct parabolic periodic cycles which are respectively n_j-*degenerate *such that any sequence $(Q_n)_{n\geq1}\subset\mathcal{S}_d^{\mathrm{vis}}$ with $\Theta(Q_n)=\Theta$ converges to P.*

In particular, the landing map e is well-defined at Θ. Moreover, for all $0\leq j\leq d-2$, if $\Theta_j=\{\theta_j,\theta'_j\}$, the following holds

- *the ray $R_{\theta_j}(P)$ lands at a parabolic point z_j of period $k_j|n_j$ of P whose basin contains c_j and*
- *the ray $R_{\theta'_j}(P)$ lands at the preimage z'_j of $P(z_j)$ which satisfies $z_j\neq z'_j$ and which lies on the boundary of the bounded Fatou component of P that contains c_j.*

For our proof, we deeply rely on the seminal work [**DH1**] of Douady and Hubbard. Moreover, we follow closely the proof of [**DH1**, Exposé VIII Théorème 2].

Recall the following (see [**Si**, p. 225], [**Mi2**, Appendix D] or [**BB2**, Theorem 2.1]):

Theorem 4.6 (Milnor, Silverman). — *For any $n\geq1$, there exists a polynomial map $p_n:\mathcal{P}_d\times\mathbb{C}\to\mathbb{C}$ such that for any $(P,c_0,\dots,c_{d-2})\in\mathcal{P}_d$ and any $w\in\mathbb{C}$,*

1. *if $w\neq1$, then $p_n(P,w)=0$ if and only if P has a cycle of exact period n and multiplier w,*
2. *otherwise, $p_n(P,1)=0$ if and only if there exists $q\geq1$ such that P has a cycle of exact period n/q and multiplier η a primitive q-root of unity.*

We can define an algebraic hypersurface by letting

$$\mathrm{Per}_n(w):=\{(P,c_0,\dots,c_{d-2})\in\mathcal{P}_d\mid p_n(P,w)=0\}\ ,$$

for $n\geq1$ and $w\in\mathbb{C}$. By the Fatou-Shishikura inequality and using the compactness of the connectedness locus, we have the following:

Lemma 4.7. — *Pick* $n_0, \ldots, n_{d-2} \geq 2$ *and assume that* $\gcd(n_i, n_j) = 1$ *for all* $i \neq j$ *and* $n_i \geq 2$ *for all* i. *Pick any* $w_0, \ldots, w_{d-2} \in \overline{\mathbb{D}}$. *Then the algebraic variety* $\bigcap_i \mathrm{Per}_{n_i}(w_i)$ *is a finite set.*

Proof of Theorem 4.5. — Pick $(P, c_0, \ldots, c_{d-2}) \in \mathcal{I}_{\mathcal{C}_d}(\Theta)$. Recall that $\mathcal{I}_{\mathcal{C}_d}(\Theta) \subset \mathcal{C}_d$ so that the Böttcher coordinate of P at infinity is a biholomorphism $\phi_P : \mathbb{C} \setminus \mathcal{K}_P \longrightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$. According to [**DH1**, Exposé VIII, §2, Proposition 2], the dynamical external rays of P of respective angles $\theta_0, \ldots, \theta_{d-2}$ land at periodic points $z_0, \ldots, z_{d-2}$ of P. Moreover, the period of z_i divides n_i and either z_i is repelling, or $(P^{n_i})'(z_i) = 1$.

Notice that, since $\gcd(n_i, n_j) = 1$ for $i \neq j$, the points z_i and z_j can not lie in the same cycle. We now assume by contradiction that there exists $0 \leq i \leq d-2$ such that z_i is repelling. Moreover, by the implicit function theorem we can follow z_i holomorphically as a repelling n_i-periodic point $z_i(Q)$ of Q, in a neighborhood of P in $\mathcal{P}_d$. Notice also that, since $P \in \partial\mathcal{S}_d$, the only possible periodic Fatou components of P are Siegel disks and parabolic basins.

Pick now a sequence $Q_n \in \mathcal{S}_d^{\mathrm{vis}}$ such that $\Theta(Q_n) = \Theta$ for all n and $Q_n \to P$ as $n \to +\infty$. We may apply [**DH1**, Exposé VIII, §2, Proposition 3]: there exist a neighborhood W of $(P, c_0, \ldots, c_{d-2})$ in $\mathcal{P}_d$ and a continuous map

$$\psi : (Q, s) \in W \times \mathbb{R}_+ \longmapsto \psi(Q, s) \in \mathbb{C}$$

which depends holomorphically of $Q \in W$ and such that the following holds

- for any $s \geq 0$ and any $Q \in W$, $\psi(Q, s) = \phi_Q^{-1}\left(e^{s+2i\pi\theta_i}\right)$ and in particular $g_Q\left(\psi(Q, s)\right) = s$,
- for any $Q \in W$, the dynamical ray of Q of angle θ_i lands at $z_i(Q) = \psi(Q, 0)$.

According to [**K4**, Lemma 3.19], the visible shift locus is dense in the shift locus, and since $(P, c_0, \ldots, c_{d-2}) \in \mathcal{I}_{\mathcal{C}_d}(\Theta)$, $W \cap \mathcal{S}_d \neq \varnothing$ and $P \in \overline{W \cap \mathcal{S}_d^{\mathrm{vis}}}$; hence we may assume $Q_n \in W$ for n large enough. Let $s_n := g_{Q_n}(c_i(Q_n))$. Set now

$$H_s(Q) := \psi(Q, s) - c_i(Q), Q \in W.$$

By the above, $(Q, s) \in W \times \mathbb{R}_+ \mapsto H_s(Q) \in \mathbb{C}$ is continuous and we have $H_{s_n}(Q_n) = \psi(Q_n, s_n) - c_i(Q_n) = 0$ for all n large enough. As $n \to \infty$, we get $c_i(P) = z_i$, which is a contradiction since z_i is repelling. We thus have shown that P has $d-1$ distinct parabolic cycles which are $n_0, \ldots, n_{d-2}$-degenerate respectively, i.e. $(P, c_0, \ldots, c_{d-2}) \in \bigcap_i \mathrm{Per}_{n_i}(1)$.

On the other hand, by Theorem 3.7 of [**K4**], for all $n \geq 1$, the set $X_n := \{Q \in \mathcal{S}_d^{\mathrm{vis}}\,;\ \Theta(Q) = \Theta \text{ and } G(Q_n) \leq \frac{1}{n}\}$ is connected. In particular, the set

$$X(\Theta) := \bigcap_{n \geq 1} \overline{X}_n$$

is compact and connected. The above implies $X(\Theta)$ is contained in the finite set $\bigcap_i \mathrm{Per}_{n_i}(1)$, i.e. $X(\Theta)$ is reduced to a singleton.

Since the set $R_{\theta_i}(Q_n) \cup \{\infty\}$ is a compact connected subset of $\mathbb{P}^1$, the set $R := \limsup_{n\to\infty} R_{\theta_i}(Q_n) \cup \{\infty\}$ is a connected compact subset of $\mathbb{P}^1$. Moreover, $R \cap (\mathbb{C} \setminus \mathcal{K}_P) = R_{\theta_i}(P)$, $c_i(P) \in R$ and, reasonning as in [**K1**, Lemma 7.6], we find that any $z \in R \cap \mathcal{J}_P$ satisfies $P^{n_i}(z) = z$. In particular, $c_i(P) \in R \cap \mathring{\mathcal{K}}_P$, and the component U of $\mathring{\mathcal{K}}_P$ which contains $c_i(P)$ is periodic of period dividing n_i, hence $c_i(P)$ lies in the parabolic basin of z_i by the above.

Similarly, if $\Theta_i = \{\theta_i, \theta_i'\}$ the ray θ_i' lands at a preimage by P of $P(z_i)$. Let now $R' := \limsup_{n\to\infty} R_{\theta_i'}(Q_n) \cup \{\infty\}$. As above, we have $R' \cap (\mathbb{C} \setminus \mathcal{K}_P) = R_{\theta_i'}(P)$ and $c_i(P) \in R'$, hence $R_{\theta_i'}(P)$ lands at the boundary of the bounded Fatou component of P which contains $c_i(P)$. This ends the proof. □

Note also that we lack a precise control on the cardinality of combinatorics that land at a given parabolic polynomial to have a better result in the spirit of Theorem 2.4.

An easy estimate follows from Theorem 4.5 up to considering all the possible permutations of the given combinatorics (a priori, two permuted combinatorics may land at the same parameter). Given such a polynomial P with periodic parabolic cycles of exact periods k_i and combinatorial periods n_i, the number $\mathcal{M}_{\mathsf{Cb}}(P)$ is bounded above:

$$\mathcal{M}_{\mathsf{Cb}}(P) \leq ((d-1)!)^2 \cdot \prod_{i=0}^{d-2} \left(k_i \cdot \max\left\{ d, (d-1)\frac{n_i}{k_i} \right\} \right).$$

Following [**Sc1**] and [**K2**], one can expect that the only angles which actually belong to Θ_i are the *characteristic rays* of the parabolic cycle whose parabolic basin contains c_i, i.e. the rays separating the petals containing $P(c_i)$ from the other petals clustering at the same point of the considered parabolic cycle. This would give an exact formula for $\mathcal{M}_{\mathsf{Cb}}(P)$.

PART II
IN THE QUADRATIC ANTI-HOLOMORPHIC FAMILY

5. The anti-holomorphic quadratic family

5.1. Anti-holomorphic polynomials and the Tricorn

We now aim at studying the family of *quadratic anti-holomorphic* dynamical systems, i.e. the family

$$f_c(z) := \bar{z}^2 + c\ ,\ \ z \in \mathbb{C}\ ,$$

parametrized by $c \in \mathbb{C}$. It is classical to proceed by analogy with the holomorphic case, i.e. to define the filled Julia set of f_c and the Julia set of f_c by letting

$$\mathcal{K}_c := \{z \in \mathbb{C}\,;\ (f_c^n(z))_n \text{ is bounded}\} \ \text{ and } \ \mathcal{J}_c := \partial\mathcal{K}_c\ .$$

We also define the *Tricorn* as the set

$$\mathbf{M}_2^* := \{c \in \mathbb{C}\,;\ (f_c^n(0))_n \text{ is bounded}\}\ .$$

Again, as in the holomorphic case, for $n > k > 0$, we let

$$\mathrm{Per}(n,k) := \{c \in \mathbb{C}\,;\ f_c^n(0) = f_c^k(0)\} \ \text{ and } \ \mathrm{Per}^*(n,k) := \{c \in \mathrm{Per}(n,k)\,;\ f_c^{n-k}(0) \neq 0\}.$$

Definition 5.1. — *We say that a parameter* $c \in \bigcup_{n>k\geq 1} \mathrm{Per}^*(n,k)$ *is a* Misiurewicz *parameter.*

Notice that we chose this definition by analogy to the holomorphic case. Observe that we do not have to consider the case $\mathrm{Per}^*(n,1)$ since this set is empty. Indeed, since the map f_c has local degree 2 at 0, the point $f_c(0)$ cannot have a preimage distinct from 0. In particular, any parameters for which $f_c^n(0) = f_c(0)$ satisfies $f_c^{n-1}(0) = 0$.

We now want to address the following question.

Question. — *Is the set* $\mathrm{Per}(n,k(n))$ *(resp. the set* $\mathrm{Per}^*(n,k(n))$*) finite and can we describe its distribution as* $n\to\infty$*, for any sequence* $0\leq k(n)<n$*?*

The rest of the paper gives a partial answer to the above question for $k(n)>1$ (see the above remark for $k(n)=1$).

For convenience, define the family of quadratic anti-holomorphic polynomials

$$f_\lambda(z):=\bar{z}^2+(a+ib)^2\ ,\ z\in\mathbb{C}\ ,$$

for $\lambda=(a,b)\in\mathbb{C}^2$. A classical observation is that $f_\lambda\circ f_\lambda$ defines a family of holomorphic degree 4 polynomials P_λ. An easy computation shows that for $\lambda=(a,b)\in\mathbb{C}^2$,

$$P_\lambda(z)=z^4+2(a-ib)^2z^2+(a+ib)^2+(a-ib)^4\ ,\ z\in\mathbb{C}.$$

This family has (complex) dimension 2. The critical points of P_λ are exactly $c_0:=0$, $c_1:=ia+b$ and $c_2=-(ia+b)$. It is also easy to check that for any $\lambda=(a,b)\in\mathbb{C}^2$, we have $c_1=c_2$ if and only if $c_0=c_1$ if and only if $c_0=c_2$ if and only if $b=-ia$.

Lemma 5.2. — *The family* $(P_\lambda,0,ia+b,-(ia+b))_{\lambda\in\mathbb{C}^2}$ *projects, in the moduli space* $\mathcal{P}_4$*, to the surface* $\mathcal{X}:=\{(P,c_0,c_1,c_2)\in\mathcal{P}_4\,;\ P(c_1)=P(c_2)\}$*. Moreover, the projection* $\pi:\mathbb{C}^2\longrightarrow\mathcal{X}$ *is a degree* 6 *branched covering ramifying exactly at* $\lambda=(0,0)$*. Moreover, if* $\lambda=(a,b)\in\mathbb{R}^2$ *then* P_λ *is the only real representative of* $\{P_\lambda\}$ *in the family.*

Proof. — We first show that the surface $\mathcal{X}$ is irreducible. Let $(P,c_0,c_1,c_2)\in\mathcal{X}$, we can choose a representative $\tilde{P}$ of the form:

$$\tilde{P}(z)=z^4-2\alpha z^2+\gamma$$

so that the critical points are 0, c_1 and c_2 with $c_1^2=\alpha$, $c_1+c_2=0$. Conversely, such a marked polynomials is in $\mathcal{X}$. Consider the analytic set $\mathcal{X}'$ of $\mathbb{C}^4$ defined by the equations:

$$\mathcal{X}':=\{(\alpha,\gamma,c_1,c_2)\in\mathbb{C}^4,\ c_1^2=\alpha,\ c_1+c_2=0\}.$$

Then $\mathcal{X}'$ is irreducible since $c_1^2=\alpha$ is irreducible. Hence $\mathcal{X}$ is irreducible by considering the canonical map $\mathcal{X}'\to\mathcal{X}$.

Then, since π is proper, it is surjective, hence has finite degree. Solve the equations:

$$(\alpha z+\beta)\circ P_\lambda=P_{\lambda'}\circ(\alpha z+\beta)$$
$$(\alpha z+\beta)(0)=0$$
$$(\alpha z+\beta)(ia+b)=ia'+b'$$

where $\lambda=ia+b$ and $\lambda'=ia'+b'$. We get that $\beta=0$ and $\alpha^3=1$ and we can rewrite the previous equations as:

$$\alpha(a-ib)=a'-ib'$$
$$\alpha(a+ib)^2=(a'+ib')^2.$$

By Bézout theorem, we get six solutions. One can see they give six different λ' by looking at the parameter $\lambda=(1,-i)$.

One easily sees that if (a,b) in $\mathbb{R}^2$ then P_λ is the only real representative of $\{P_\lambda\}$ in the family. Moreover, it is clear that π ramifies exactly at $\lambda=(0,0)$ (else $c_1\neq 0$). □

We also will rely on the following which is essentially obvious.

Lemma 5.3. — *The map $\pi|_{\mathbb{R}^2} : \mathbb{R}^2 \longrightarrow \mathcal{P}_4$ is a real-analytic homeomorphism onto its image. Moreover, if $\lambda \in \mathbb{R}^2$, then $f_\lambda(c_1) = f_\lambda(c_2) = c_0$.*

Proof. — If $\lambda \in \mathbb{R}^2 \setminus \{0\}$, the polynomials P_λ and $P_{-\lambda}$ are affine conjugate, but the conjugacy exchanges c_1 with c_2 and the conclusion follows from Lemma 5.2. The fact that $f_\lambda(c_1) = f_\lambda(c_2) = c_0$ follows from a direct computation. □

We have the first following result on the finiteness of the sets $\mathrm{Per}(n,k)$ and $\mathrm{Per}^*(n,k)$.

Lemma 5.4. — *Pick $n > k > 0$. Then the sets $\mathrm{Per}(n,k)$ and $\mathrm{Per}^*(n,k)$ are finite.*

Proof. — We first prove that $\mathrm{Per}(n,k)$ is a finite set (hence $\mathrm{Per}^*(n,k) \subset \mathrm{Per}(n,k)$ is also finite). Pick $\lambda = (a,b) \in \mathbb{R}^2$ with $c = (a+ib)^2$. Assume $c \in \mathrm{Per}(n,k)$. According to Lemma 5.3,

$$P_\lambda^n(c_0) = f_c^{2n}(0) = f_c^{2k}(0) = P_\lambda^k(c_0)$$

and again by Lemma 5.3,

$$\begin{aligned} P_\lambda^{n+1}(c_1) &= f_c \circ f_c^{2n}(f_c(c_1)) = f_c \circ f_c^{2n}(0) \\ &= f_c \circ f_c^{2k}(0) = f_c \circ f_c^{2k}(f_c(c_1)) = P_\lambda^{k+1}(c_1). \end{aligned}$$

Since $P_\lambda(c_1) = P_\lambda(c_2)$, we get $P_\lambda^{n+1}(c_2) = P_\lambda^{k+1}(c_2)$. We write

$$\mathrm{Per}_j(m,l) := \{(P, c_0, c_1, c_1) \in \mathcal{P}_4\, ;\ P^m(c_j) = P^l(c_j)\}.$$

The set $\mathrm{Per}_j(m,l)$ is an algebraic subvariety of $\mathcal{P}_4$ and $\mathrm{Per}_0(n,k) \cap \mathrm{Per}_1(n+1,k+1) \cap \mathrm{Per}_2(n+1,k+1)$ is contained in the compact set $\mathcal{C}_4$, hence it is finite. By Lemma 5.2, the set of $\lambda \in \mathbb{C}^2$ with $\pi(\lambda) \in \mathrm{Per}_0(n,k) \cap \mathrm{Per}_1(n+1,k+1) \cap \mathrm{Per}_2(n+1,k+1)$ is thus finite. The finiteness of $\mathrm{Per}(n,k)$ follows directly. □

5.2. The combinatorial space

For the material of this section, we follow [**N**]. Let ψ_c be the Böttcher coordinate of the anti-holomorphic polynomial $f_c(z) = \bar{z}^2 + c$, with $c \in \mathbb{C}$, i.e. the holomorphic map conjugating f_c near ∞ to $\bar{z}^2$ near ∞ which is tangent to the identity. Let $\lambda = (a,b) \in \mathbb{R}^2$ be such that $c = (a+ib)^2$. The map ψ_c is known to be a biholomorphic map from $\{z \in \mathbb{C}\, ;\ g_{P_\lambda}(z) > g_{P_\lambda}(c_0)\}$ onto $\mathbb{C} \setminus \overline{D}(0, \exp(g_{P_\lambda}(c_0)))$. Moreover, we also have $\psi_c \circ P_\lambda = (\psi_c)^4$, i.e. $\psi_c = \phi_\lambda$ (recall that ϕ_λ is the Böttcher coordinate of P_λ).

For $c \in \mathbb{C}$, we let $\Psi^*(c) := \psi_c(c)$, when $c \in \mathbb{C} \setminus \mathbf{M}_2^*$. The map

$$\Psi^* : \mathbb{C} \setminus \mathbf{M}_2^* \longrightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$$

is known to be a real-analytic isomorphism (see [**N**]).

For $\theta \in \mathbb{R}/\mathbb{Z}$, the *external ray* of $\mathbf{M}_2^*$ of angle θ is the curve $\mathcal{R}^*(\theta)$ defined by

$$\mathcal{R}^*(\theta) := (\Psi^*)^{-1}\left(\{Re^{2i\pi\theta}\, ;\ 1 < R < +\infty\}\right).$$

We will need the following.

Lemma 5.5. — *Let $c = (a+ib)^2 \in \mathbb{C}$ with $\lambda := (a,b) \in \mathbb{R}^2$. Assume that $c \in \mathcal{R}^*(\theta)$ and $\Psi^*(c) = e^{2r+2i\pi\theta}$ with $\theta \in \mathbb{R}/\mathbb{Z}$ and $r > 0$. Then $\{P_\lambda\} \in \mathcal{S}_4^{\mathrm{vis}}$, P_λ has simple critical points, i.e. $\Theta(P_\lambda) \in \mathsf{Cb}_0$ and $r = g_{P_\lambda}(c_0) = 2g_{P_\lambda}(c_1) = 2g_{P_\lambda}(c_2)$. Moreover, if $\Theta(P_\lambda) = (\Theta_0, \Theta_1, \Theta_2)$, then $\Theta(P_{-\lambda}) = (\Theta_0, \Theta_2, \Theta_1)$ and*

$$-2\Theta_0 = 4\Theta_1 = 4\Theta_2 = \{\theta\}\ .$$

Proof. — As seen above, P_λ has simple critical points if and only if $c \neq 0$, which is the case here since $g_{P_\lambda}(c_0) > 0$. Notice that $P_\lambda(c_1) = P_\lambda(c_2) = f_\lambda(c_0) = c$ in our case, hence this point belongs to the ray of angle θ by assumption. Moreover, since $g_{P_\lambda} = \log|\phi_\lambda| = \log|\psi_c|$ on $\{g_{P_\lambda} > 0\}$, we have $G(P_\lambda) = g_{P_\lambda}(c_0) = 2g_{P_\lambda}(c_1) = 2g_{P_\lambda}(c_2)$. As a consequence $g_{P_\lambda}(P_\lambda(c_0)) = 4g_{P_\lambda}(c_0) > G(P_\lambda)$ which means that $P_\lambda(c_0) = f_\lambda(c)$ belongs to the ray of angle -2θ.

Finally, let α and $\alpha + \frac{1}{2}$ be the angles so that $-2\alpha = -2(\alpha + \frac{1}{2}) = \theta$. In particular, $4\alpha = 4(\alpha + \frac{1}{2}) = -2\theta$ and the two dynamical rays of angle α and $\alpha + \frac{1}{2}$ don't cross critical points of P_λ until they terminate at c_0 by [**K4**, Lemma 3.9]. Since $c_1, c_2 \notin \bigcup_{n\geq 1} P_\lambda^{-n}\{c_0\}$, using again [**K4**, Lemma 3.9], we have 2 distinct rays terminating at c_1 (resp. at c_2), hence $P_\lambda \in \mathcal{S}_4^{\mathrm{vis}}$.

The last assertion follows immediately, since taking $P_{-\lambda}$ instead of P_λ only exchanges the roles of c_1 and c_2. □

Mimicking the quadratic case, we define the impression at angle $\theta = -2\alpha$ under the map $(\Psi^*)^{-1}$ as the set

$$\bigcap_{\rho>1,\ \epsilon>0} \overline{(\Psi^*)^{-1}\left(\{Re^{2i\pi\tau}\,;\ |\theta-\tau|<\epsilon,\ 1<R<\rho\}\right)}.$$

Remark that, when $c = (a+ib)^2 \in \mathbb{C}$ and $\lambda := (a,b) \in \mathbb{R}^2$, then both P_λ and $P_{-\lambda}$ are the polynomial map f_c^2. As an immediate consequence of Lemma 5.5, we get the following crucial property.

Corollary 5.6. — *Let $c = (a+ib)^2 \in \mathbb{C}$ with $\lambda := (a,b) \in \mathbb{R}^2$, let $\theta \in \mathbb{R}/\mathbb{Z}$ and let $r > 0$. Assume that $c \in \mathcal{R}^*(\theta)$ and $\Psi^*(c) = e^{2r+2i\pi\theta}$. Then $\mathcal{I}_{\mathcal{C}_4}(\Theta(P_\lambda))$ and $\mathcal{I}_{\mathcal{C}_4}(\Theta(P_{-\lambda}))$ both contain a copy of the prime end impression of the angle θ under the map Ψ^*.*

6. A bifurcation measure for the Tricorn

We now want to define a good bifurcation measure for the Tricorn $\mathbf{M}_2^*$ and prove equidistribution properties of specific parameters towards this bifurcation measure.

6.1. Misiurewicz combinatorics

We let $\mathsf{R}_{\mathrm{mis}}$ be the set of angles $\theta \in \mathbb{R}/\mathbb{Z}$ such that there exists integers $n > k > 1$ for which θ satisfies $(-2)^{n-1}\theta = (-2)^{k-1}\theta$ and such that $(-2)^{n-k}\theta \neq \theta$. For $n > k \geq 1$, we also let $\mathsf{C}^*(n,k) := \{\theta \in \mathbb{R}/\mathbb{Z}\,;\ (-2)^{n-1}\theta = (-2)^{k-1}\theta\}$. We now want to relate Misiurewicz combinatorics with Misiurewicz parameters.

Lemma 6.1. — *Pick $\theta \in \mathsf{R}_{\mathrm{mis}}$ and let $n > k > 1$ be minimal such that $\theta \in \mathsf{C}^*(n,k)$.*

1. *There exists a Misiurewicz parameter $c \in \partial\mathbf{M}_2^*$ such that the prime end impression of θ under Ψ^* is reduced to $\{c\}$ and $c \in \mathrm{Per}^*(2n,2k)$,*
2. *Moreover, if $n-k$ is even, then $c \in \mathrm{Per}^*(n,k)$.*

Proof. — First, we prove 1. Let $\Theta_0 := \{\alpha, \alpha + \frac{1}{2}\}$ be such that $-2\alpha = \theta$. Let also $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ be such that $4\beta_i = \theta$ for $1 \leq i \leq 4$, and let Θ_1, Θ_2 be such that $\Theta_1 \cup \Theta_2 = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ and $\Theta_1 \cap \Theta_2 = \varnothing$ and $\Theta := (\Theta_0, \Theta_1, \Theta_2) \in \mathsf{Cb}_0$.

By assumption, $4^n\alpha = (-2)^{2n}\alpha = (-2)^{2k}\alpha = 4^k\alpha$. Moreover, if $4^{(n-k)}\alpha \in \Theta_0$, then $(-2)^{2(n-k)}\theta = \theta$, which is excluded since $\theta \in \mathsf{R}_{\mathrm{mis}}$. As a consequence, α is strictly preperiodic under the map M_4. Similarly, β_i is strictly M_4-preperiodic for all i, hence $\Theta \in \mathsf{Cb}_{\mathrm{mis}}$.

Moreover, according to Theorem 3.1, the impression $\mathcal{I}_{\mathcal{C}_4}(\Theta)$ is reduced to a singleton $\{P_\lambda\}$ where P_λ Misiurewicz and Θ_i is a set of angles landing at the critical point c_i of P_λ.

By Corollary 5.6, this implies that the prime end impression of θ is reduced to a singleton $\{c\}$. Writing $\lambda = (a, b)$, we thus have $c = (a + ib)^2$ and $f_c^{2n}(0) = P_\lambda^n(0) = P_\lambda^k(0) = f_c^{2k}(0)$, i.e. $c \in \mathrm{Per}(2n, 2k)$. As c is contained in the prime end impression of θ by Ψ^*, c lies on the boundary of $\mathbf{M}_2^*$. If we had $f_c^{2(n-k)}(0) = 0$, then c would be a center of a hyperbolic component of $\mathbf{M}_2^*$ which contradicts the fact that $c \in \partial\mathbf{M}_2^*$ (see e.g. [**HS**]).

To prove 2, if n is even, we may proceed exactly as above, replacing n and k respectively with $n/2$ and $k/2$. Otherwise, we replace n and k with $(n+1)/2$ and $(k+1)/2$ respectively. This ends the proof. □

Notice that this result can be understood as follows: Misiurewicz combinatorics have to cluster to Misiurewicz parameters which are countable, i.e. we naturally have a rigidity property for the impression of such combinatorics. On the other hand, the existence of (real analytic) arcs of stable parabolic parameters is an obstruction to the rigidity of those parameters ([**IM**]). Parabolic combinatorics, i.e. periodic ones under multiplication by -2, should then cluster on parabolic parameters (see Corollary 5.6) so we cannot expect rigidity of the impression.

6.2. Landing of rays and the bifurcation measure

We want first to prove the following, in the spirit of Theorem 3.6.

Theorem 6.2. — *There exists a set* $\mathsf{R} \subset \mathbb{R}/\mathbb{Z}$ *of full Lebesgue measure such that the map* $\Phi : (\theta, r) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}_+^* \longmapsto (\Psi^*)^{-1}\left(e^{2r+2i\pi\theta}\right) \in \mathbb{C} \setminus \mathbf{M}_2^*$ *extends continuously to* $\mathsf{R} \times \{0\}$. *Moreover, this set contains the set* $\mathsf{R}_{\mathrm{mis}}$ *and the extended map* Φ *induces a surjection between* $\mathsf{R}_{\mathrm{mis}} \times \{0\}$ *and the set of Misiurewicz parameters.*

We follow Smirnov [**Sm**] and Dujardin and Favre [**DF**].

Proof. — As for the proof of Theorem 3.6, we rely on Theorem 3.7. To an angle $\theta \in \mathbb{R}/\mathbb{Z}$, owing to Lemma 5.5, we can associate $\Theta \in \mathsf{Cb}_0$ and we let R_1 be the set of angles θ such that the associated Θ satisfies that the impression $\mathcal{I}_{\mathcal{C}_4}(\Theta)$ contains a polynomial P_λ satisfying the TCE condition. Let $\mathsf{R} := \mathsf{R}_1 \cup \mathsf{R}_{\mathrm{mis}}$. According to Corollary 5.6 and to Theorem 3.1, the map Φ extends as a continuous map to $\mathsf{R} \times \{0\}$ and the extended map Φ induces a surjection from $\mathsf{R}_{\mathrm{mis}} \times \{0\}$ to the set of Misiurewicz parameters. Indeed, any $\theta \in \mathsf{R}_{\mathrm{mis}}$ corresponds exactly to 2 distinct $\Theta, \Theta' \in \mathsf{Cb}_{\mathrm{mis}}$. According to Lemma 5.5 they correspond respectively to distinct $(a, b), (a', b') \in \mathbb{R}^2$ with $c = (a + ib)^2$. The conclusion follows from Theorem 3.1.

It remains to prove that R has full-Lebesgue measure. Notice that $\mathsf{R}_{\mathrm{mis}}$ is countable, hence satisfies $\dim_H(\mathsf{R}_{\mathrm{mis}}) = 0$. Hence it has Lebesgue measure 0 and the full measure property will be fulfilled by the set R_1. Let

$$\mathsf{Cb}^1 := \{\Theta = (\Theta_0, \Theta_1, \Theta_2) \in \mathsf{Cb}_0\,; -2\Theta_0 = 4\Theta_1 = 4\Theta_2\}.$$

Lemma 5.5 allows to define a map $\pi : \mathsf{Cb}^1 \longrightarrow \mathbb{R}/\mathbb{Z}$. It is clear that π is a degree 2 unbranched cover. In particular, $\dim_H(\mathbb{R}/\mathbb{Z} \setminus \mathsf{R}) = \dim_H(\mathsf{Cb}^1 \setminus \pi^{-1}\mathsf{R}_1)$. Assume (see the proof below) we have the following :

Claim. — $\dim_H(\mathbb{R}/\mathbb{Z} \setminus \mathsf{R}) \leq \log 3/\log 4 < 1$.

In particular, $\lambda_{\mathbb{R}/\mathbb{Z}}(\mathbb{R}/\mathbb{Z} \setminus \mathsf{R}) = 0$ which ends the proof. □

Proof of The Claim. — We follow the proof of [**DF**, Lemma 7.25]. Pick $\Theta \in \mathsf{Cb}^1$ and let $c = (a+ib)^2$ be such that $P_{a,b} \in \mathcal{I}_{\mathcal{C}_4}(\Theta)$. According to [**DF**, Lemma 7.24], there exists a partition of $\mathcal{J}_{a,b}$ into three sets J_0, J_1 and the impression of the external ray of angle θ of $P_{a,b}$ in $\mathcal{J}_{a,b}$, where $\{\theta\} = -2\Theta_0$. Let $\Sigma_2 := \{0,1\}^{\mathbb{N}}$ and $\kappa : (\mathbb{R}\setminus\mathbb{Q})/\mathbb{Z} \longrightarrow \Sigma_2$ be defined as follows: we say that $\kappa(\theta)_n = \epsilon \in \{0,1\}$ if $4^n\theta \in I_\epsilon$, where I_ϵ is the connected component of $\mathbb{R}/\mathbb{Z} \setminus \{\alpha, \alpha+\frac{1}{2}\}$, with $-2\alpha = \theta$, such that angles in I_ϵ land in J_ϵ.

Following Smirnov [**Sm**], we see that $\theta \in \mathbb{R}/\mathbb{Z}$ fails the TCE condition if and only if $\theta \in \kappa^{-1}(SR)$ i.e. is strongly recurrent. The precise definition of SR can be found in [**Sm**]. It is known that SR has Hausdorff dimension 0 and, following Dujardin and Favre, if C_n is any cylinder of depth n in Σ_2, then $\kappa^{-1}(C_n)$ consists of the intersection of $(\mathbb{R}\setminus\mathbb{Q})/\mathbb{Z}$ with at most $A \times n3^n$ intervals of length at most 2^{-n}, where A is a constant independent of n.

Indeed, if $4^n\theta$ turns once around $\mathbb{R}/\mathbb{Z}$ so that $4^n\theta = \theta$, then $\theta \in \mathbb{Q}$, which is excluded. We now proceed by induction on n: let I be an interval of $\mathbb{R}/\mathbb{Z}$ of length $\ell_I < 1/4$. Then, either $4\theta \notin I$ and $M_4^{-1}(I) \cap I_\epsilon$ consists in 2 intervals, or $4\theta \in I$ and $M_4^{-1}(I) \cap I_\epsilon$ consists in 3 intervals, which can occur only for one of the $N(n)$ intervals. As a consequence, $N(n+1) \leq 2\cdot(N(n)-1)+3 \leq 2N(n)+1$, whence $N(n) \leq A \times n3^n$. The estimate for the Hausdorff dimension then easily follows. □

Thanks to Theorem 6.2, we can define the landing map and the bifurcation measure for the Tricorn.

Definition 6.3. — *The* landing map of the Tricorn *is the measurable map* $\ell : \mathbb{R}/\mathbb{Z} \to \partial\mathbf{M}_2^*$ *defined by* $\ell(\theta) := \Phi(e^{2i\pi\theta}, 0)$ *for any* $\theta \in \mathsf{R}$. *We define the* bifurcation measure of the Tricorn *as the probability measure*

$$\mu^*_{\mathrm{bif}} := \ell_*\left(\lambda_{\mathbb{R}/\mathbb{Z}}\right).$$

As an immediate consequence of Theorem 6.2, we have proven Theorem B. Notice also that we have $\mathsf{R}_{\mathrm{mis}} \subset \mathsf{R}$.

6.3. Distribution of Misiurewicz combinatorics: Theorem C

Recall that, for $n > k > 1$, we denoted by $\mathrm{Per}^*(n,k)$ the set of parameters such that $f_c^n(0) = f_c^k(0)$ and $f_c^{n-k}(0) \neq 0$.

For any $n \geq 4$, pick $1 < k(n) < n$. Let $\mathsf{X}_n := \mathsf{C}^*(n, k(k)) \setminus \mathsf{C}^*(n-k(n)+1, 1)$, $d_n := \mathrm{Card}(\mathsf{X}_n) = 2^{n-1} - 2^{k(n)-1} - 2^{n-k(n)} + 1$ and

$$\nu_n := \frac{1}{d_n}\sum_{\theta\in\mathsf{X}_n}\delta_\theta.$$

Let also $X_n := \ell(\mathsf{X}_n) \subset \mathrm{Per}^*(2n, 2k(n))$, and let μ_n^* be the probability measure

$$\mu_n^* := \frac{1}{d_n}\sum_{c\in X_n}\mathcal{N}_{\mathbb{R}/\mathbb{Z}}(c)\cdot\delta_c\ ,$$

where $\mathcal{N}_{\mathbb{R}/\mathbb{Z}}(c) \geq 1$ is the number of angles $\theta \in \mathsf{R}_{\mathrm{mis}}$ for which $\ell(\theta) = c$, i.e. $\mu_n^* = \ell_*(\nu_n)$.

Our aim here is to prove the following.

Theorem 6.4. — *The sequence of measures* $(\nu_n)_{n\geq 4}$ *converges weakly to* $\lambda_{\mathbb{R}/\mathbb{Z}}$.

Proof. — As in the proof of the above lemma, if $n-k(n)$ is even, we have

$$\begin{aligned} d_n &= 2^{n-1} - 2^{k(n)-1} - 2^{n-k(n)} + 1 \\ &\geq 2^{n-1} - 2^{k(n)-1} - \frac{1}{2}\left(2^{n-1} - 2^{k(n)-1}\right) \\ &\geq \frac{1}{2}\mathrm{Card}(\mathsf{C}^*(n,k(n))) \ , \end{aligned}$$

and if $n-k(n)$ is odd, we also have

$$\begin{aligned} d_n &= 2^{n-1} + 2^{k(n)-1} - 2^{n-k(n)} - 1 \\ &\geq 2^{n-1} + 2^{k(n)-1} - \frac{1}{2}\left(2^{n-1} + 2^{k(n)-1}\right) \\ &\geq \frac{1}{2}\mathrm{Card}(\mathsf{C}^*(n,k(n))) \ . \end{aligned}$$

Moreover, it is easy to see that the sequence $(\mathsf{C}^*(n,k(n)))_{n\geq 2}$ is equidistributed for $\lambda_{\mathbb{R}/\mathbb{Z}}$ and that $(\mathsf{C}^*(n-k(n)+1,1))_{n\geq 2}$ is either equidistributed or $k(n)\geq n-K$ for some integer $K\geq 1$. By Lemma 1.9, the sequence $(\mathsf{X}_n)_n$ is equidistributed. In particular, the sequence of probability measures (ν_n) converges weakly towards $\lambda_{\mathbb{R}/\mathbb{Z}}$. □

We now can prove Theorem C.

Proof of Theorem C. — It follows directly from Theorem 6.2, Lemmas 5.4 and 6.1 combined with Theorem 6.4 and Theorem 1.8. □

Remark. — We expect the measure which is *equidistributed* on the set $\mathrm{Per}^*(n,k(n))$ converges towards to the bifurcation measure μ^*_{bif}, whenever $1<k(n)<n$.

7. The bifurcation measure of the Tricorn and real slices

Let $\mu_{\mathrm{bif},d}$ (resp. $\nu_{\mathrm{bif},d}$) denote the bifurcation measure on the moduli space $\mathcal{P}_d$ of degree d polynomials with marked critical points (resp. on the moduli space $\mathcal{M}_d$ of degree d rational maps with marked critical points). Let $\{P\}\in\mathcal{P}_d$ (resp. $\{R\}\in\mathcal{M}_d$), then $\{P^{\circ m}\}\in\mathcal{P}_{d^m}$ (resp. $\{R^{\circ m}\}\in\mathcal{M}_{d^m}$). Let $\mathrm{it}_m:\{P\}\mapsto\{P^{\circ m}\}$ (resp. $\mathrm{it}_m:\{R\}\mapsto\{R^{\circ m}\}$). We have the following interesting result that relates the image of the support of the bifurcation measure under the iteration map it_m with the support of the bifurcation measure. It is new to the authors' knowledge. More precisely:

Proposition 7.1. — *Let $m\geq 2$, then with the above notations, we have that:*

– *if $\{P\}\in\mathcal{P}_d$ satisfies $\{P\}\in\mathrm{supp}(\mu_{\mathrm{bif},d})$ then $\{P^{\circ m}\}\in\mathrm{supp}(\mu_{\mathrm{bif},d^m})$. In other words*

$$\mathrm{it}_m(\mathrm{supp}(\mu_{\mathrm{bif},d}))\subset\mathrm{supp}(\mu_{\mathrm{bif},d^m})\cap\mathrm{it}_m(\mathcal{P}_d);$$

– *if $\{R\}\in\mathcal{M}_d$ satisfies $\{R\}\in\mathrm{supp}(\nu_{\mathrm{bif},d})$ then $\{R^{\circ m}\}\in\mathrm{supp}(\nu_{\mathrm{bif},d^m})$. In other words*

$$\mathrm{it}_m(\mathrm{supp}(\nu_{\mathrm{bif},d}))\subset\mathrm{supp}(\nu_{\mathrm{bif},d^m})\cap\mathrm{it}_m(\mathcal{M}_d).$$

Proof. — We first prove the case where $\{P\}\in\mathcal{P}_d$. By density of Misiurewicz parameters in $\mathrm{supp}(\mu_{\mathrm{bif},d})$ (see [**DF**]), we can assume that $\{P\}$ is Misiurewicz, i.e. all critical points of P are preperiodic to repelling cycles. Then $P^{\circ m}$ is also Misiurewicz since its critical points are the preimages of the critical points of P by $P^{\circ k}$ for $k\leq m-1$. But then, it is known that such conjugacy classes $\{P^{\circ m}\}$ belong to $\mathrm{supp}(\mu_{\mathrm{bif},d^m})$ (see again [**DF**]). This ends the proof in $\mathcal{P}_d$.

The case of rational maps is similar (see [**BE**], [**G1**] and [**BG1**]). □

Question. — Does the reverse inclusion hold? Namely, if $\{P^{\circ m}\} \in \text{supp}(\mu_{\text{bif},d^m})$, is it true that $\{P\} \in \text{supp}(\mu_{\text{bif},d})$? We expect the answer to be positive.

We proceed similarly for μ^*_{bif}. Let $\pi : \mathbb{C} \to \mathcal{P}_4$ be the map defined by

$$\pi(c) := ([f^2_{c^2}], 0, c_1, c_2),\ c \in \mathbb{C},$$

where $(c_1, c_2) := (i\bar{c}, -i\bar{c})$. Let also μ be the bifurcation measure of the complex surface $\mathcal{X} := \{c_1 = -c_2\}$ of the moduli space $\mathcal{P}_4$ of degree 4 critically marked polynomials and let $\mathcal{S} \subset \mathcal{X}$ be the smooth real surface defined as the image of the map π. Then proceeding as in the proof of Proposition 7.1, we have:

Proposition 7.2. — *Let $c \in \mathbb{C}$ be such that $c \in \text{supp}(\mu^*_{\text{bif}})$, then $\pi(c) \in \text{supp}(\mu)$. In other words, $\pi(\text{supp}(\mu^*_{\text{bif}})) \subset \text{supp}(\mu) \cap \mathcal{S}$.*

The fact that Misiurewicz parameters belong to the support of μ follows from [**G1**].

Question. — Is μ^*_{bif} the *slice* $\langle \mu, \mathcal{S} \rangle$ in the sense of measures of μ along $\mathcal{S}$? One of our initial strategies (that we could not conduct) to define μ^*_{bif} was to define it as this slice. Notice that it is not clear that $\langle \mu, \mathcal{S} \rangle$ is even well-defined (see for instance [**Ti**] for the delicate question of the real slicing of the harmonic measure of the Mandelbrot set). Numerical evidences of such results, in the spirit of Milnor's explorations ([**Mi1**]), would be a first step.

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