Strong bifurcations have maximal dimension

Thomas Gauthier

Institut de Mathématiques de Toulouse

September 27, 2010

Theorem (Shishikura, 1991)

Denote by M the Mandelbrot set. Then $\dim_H(\partial M) = 2$.

Let $d \geq 2$ be an integer and denote by Rat_d the space of all the degree d rational maps.

Theorem (Tan Lei, 1998)

In Rat_d the bifurcation locus is of maximal Hausdorff dimension, i.e. $2(2d+1)$.

On the other hand, Shishikura proved using quasiconformal surgery that a rational map $f \in \mathrm{Rat}_d$ has at most $2d-2$ distinct neutral cycles. Set

$$\mathcal{Z}_d := \{f \in \mathrm{Rat}_d \ / \ f \text{ has } 2d-2 \text{ distinct neutral cycles}\}.$$

The set $\mathcal{Z}_d$ is included in the bifurcation locus of Rat_d.

Theorem (Gauthier, 2010)

$$\dim_H(\overline{\mathcal{Z}_d}) = 2(2d+1).$$

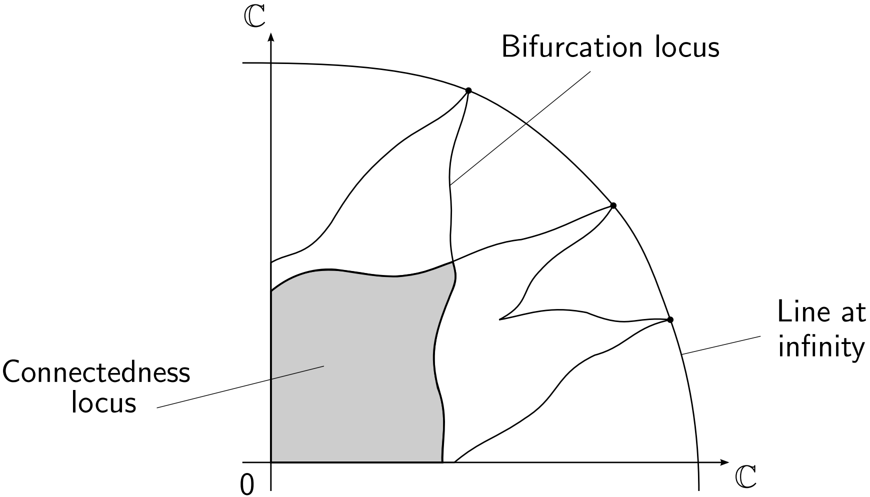

Figure: Behavior of the bifurcation locus at infinity in degree 3 polynomials (Branner-Hubbard 1992)

Misiurewicz maps

A rational map $f \in \mathrm{Rat}_d$ is said *Misiurewicz* if :

- $C(f) \cap \mathcal{J}_f \neq \emptyset$,
- f has no parabolic cycle, and
- $\omega(c) \cap C(f) = \emptyset$ for every $c \in C(f) \cap \mathcal{J}_f$.

Example

Every strictly postcritically finite rational map (denoted by SPFC) is Misiurewicz. (recall that f is SPCF if every critical point of f is preperiodic but not periodic).

A theorem of Mañé gives us the following proposition :

Proposition

Let $f \in \mathrm{Rat}_d$ be Misiurewicz. Then :

1. *f has no neutral cycle,*
2. *every periodic Fatou component of f is an attracting basin,*
3. *there exists an integer $k_0 \geq 1$ such that*
$$P^{k_0}(f) := \overline{\{f^n(c) \ / \ n \geq k_0, \ c \in C(f) \cap \mathcal{J}_f\}}$$
is a f-hyperbolic set,
4. *either $\mathcal{J}_f = \mathbb{P}^1$ or $\mathrm{Leb}(\mathcal{J}_f) = 0$.*

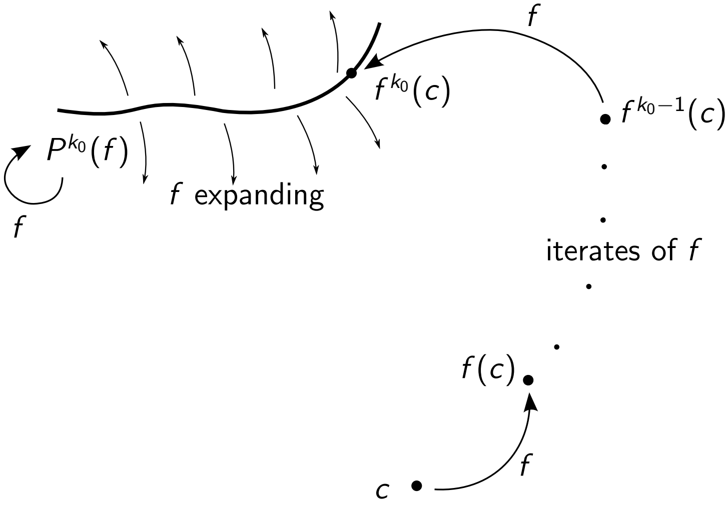

Figure: An expanding set capturing the critical point c.

We say that $X \subset \mathbb{P}^1$ is *homogeneous* if for every $x \in X$ and every neighborhood V of x in $\mathbb{P}^1$, we have

$$\dim_H(X \cap V) = \dim_H(X).$$

The *hyperbolic dimension* of $f \in \mathrm{Rat}_d$ is defined by

$$\dim_{\mathrm{hyp}}(f) := \sup\{\dim_H(X) \ / \ X \text{ is homogeneous } f\text{-hyperbolic}\}.$$

Using the thermodynamical formalism we get :

Theorem (Przytycki 1996, Urbanski 1994, McMullen 1997)

If $f \in \mathrm{Rat}_d$ is Misiurewicz, then $\dim_{\mathrm{hyp}}(f) = \dim_H(\mathcal{J}_f)$.

Let $1 \leq k \leq 2d-2$ be an integer. We say that $f \in \mathrm{Rat}_d$ is *k-Misiurewicz* if f is Misiurewicz and $C(f) \cap \mathcal{J}_f$ contains k critical points of f counted with multiplicities.

We define the set $\mathfrak{M}_k$ by :

$$\mathfrak{M}_k := \{f \in \mathrm{Rat}_d \ / \ f \text{ is } k\text{-Misiurewicz and } f \text{ is not a flexible Lattès map}\}$$

Local minoration of the dimension of $\mathfrak{M}_k$

Let $f \in \mathrm{Rat}_d$. A critical point $c \in C(f)$ is *marked* at f if there exists a neigborhood $\mathbb{B}(0,r) \subset \mathrm{Rat}_d$ of $f_0 := f$ and a holomorphic map $c : \mathbb{B}(0,r) \longrightarrow \mathbb{P}^1$ such that :

- $f_\lambda'(c(\lambda)) = 0$ for all $\lambda \in \mathbb{B}(0,r)$,
- $c(0) = c$.

Let $\mathbb{B}(0,r) \subset \mathrm{Rat}_d$ be centered at $f_0 := f$ and let $X \subset \mathbb{P}^1$ be f_0-hyperbolic. Then there exists a holomorphic motion

$$h : \mathbb{B}(0,r) \times X \longrightarrow \mathbb{P}^1$$

which conjugates the dynamics on X, i.e.

$$h_\lambda \circ f_0 = f_\lambda \circ h_\lambda \text{ on } X.$$

Let $f \in \mathfrak{M}_k$ and $\{c_1, \dots, c_k\} = C(f) \cap \mathcal{J}_f$. Suppose that the critical points $c_1, \dots, c_k$ are marked at f. Let $\mathbb{B}(0, r) \subset \mathrm{Rat}_d$ be a small enough neighborhood of $f_0 := f$ and let h_λ be the holomorphic motion of $P^{k_0}(f)$. Set

$$\chi_i(\lambda) := f_\lambda^{k_0}(c_i(\lambda)) - h_\lambda(f_0^{k_0}(c_i(0))) \text{ for } \lambda \in \mathbb{B}(0, r).$$

The *activity map* of f is given by :

$$\begin{array}{rcl} \chi : \mathbb{B}(0, r) & \longmapsto & \mathbb{C}^k \\ \lambda & \longmapsto & (\chi_1(\lambda), \dots, \chi_k(\lambda)). \end{array}$$

Theorem (Weak transversality)

The analytic set $\chi^{-1}\{0\}$ has codimension k in $\mathbb{B}(0, r)$.

Generalizing Shishikura's and Tan Lei's methods we can prove the following result :

Theorem (Gauthier, 2010)

For every $f \in \mathfrak{M}_k$ *and every neighborhood* $V_0 \subset \mathrm{Rat}_d$ *of* f *:*

$$\dim_H(\mathfrak{M}_k \cap V_0) \geq 2(2d+1-k) + k.\dim_{\mathrm{hyp}}(f).$$

In particular, if $k = 2d-2$ *we get* $\dim_H(\mathfrak{M}_{2d-2}) = 2(2d+1)$.

The bifurcation currents

The Lyapounov function $L : \mathrm{Rat}_d \longrightarrow \mathbb{R}$ of Rat_d is defined by

$$L(f) := \int_{\mathbb{P}^1} \log |f'| d\mu_f, \ \ f \in \mathrm{Rat}_d,$$

where μ_f is the maximal entropy measure of f.

Proposition

The function L is p.s.h and continuous on Rat_d.

The *bifurcation current* is defined by $T_{\mathrm{bif}} := dd^c L$. The main result about T_{bif} is hte following :

Theorem (DeMarco, 2000)

The support of T_{bif} coincides with the bifurcation locus of the family Rat_d.

Bassanelli and Berteloot studied the auto-intersections of the current T_{bif}. The k-th auto-intersection of T_{bif} is defined by induction by putting :

$$T_{\text{bif}}^k := dd^c(LT_{\text{bif}}^{k-1}),\ 2 \leq k \leq 2d-2.$$

Theorem (Bassanelli-Berteloot, 2007)

For $1 \leq k \leq 2d-2$ the support of the current T_{bif}^k is not empty and :

$$\text{supp}(T_{\text{bif}}^k) \subset \overline{\{f \in \text{Rat}_d \ / \ f \text{ has } k \text{ distinct neutral cycles}\}}.$$

Theorem (Buff-Epstein, 2009)

If f is SPCF and f is not a flexible Lattès map, then $f \in \mathrm{supp}(T_{\mathrm{bif}}^{2d-2})$.

We generalized this result to Misiurewicz maps proving the following theorem :

Theorem (Gauthier, 2010)

Pour $1 \leq k \leq 2d-2$, $\mathfrak{M}_k \subset \mathrm{supp}(T_{\mathrm{bif}}^k)$.

There were two main difficulties in generalizing Buff-Epstein result :

- Their approach does not work for Misiurewicz maps which are not SPCF.
- Instead of linearizing f at repelling cycles we need to linearize f along infinite orbits.

Conclusion

For $k = 2d - 2$ we obtain :

- $\mathfrak{M}_{2d-2} \subset \overline{\{f \in \mathrm{Rat}_d \ / \ f \text{ has } 2d-2 \text{ neutral cycles}\}}$
- $\dim_H(\mathfrak{M}_{2d-2}) = 2(2d+1)$.

Thank you for your attention!