# COMPLEX DYNAMICS OF BIRATIONAL MAPS OF $\mathbb{P}^k$ DEFINED OVER A NUMBER FIELD

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Abstract. Jonsson and Reschke [JR] showed that birational selfmaps on projective surface defined over a number field satisfy the energy condition of Bedford and Diller [BD] so their ergodic properties are very well understood. Under suitable hypotheses on the indeterminacy loci, we extend that result to birational maps $\mathbb{P}^k \dashrightarrow \mathbb{P}^k$, $k \geq 2$, defined over a number field, showing that they satisfy a similar energy condition introduced by De Thélin and the second author [DTV]. As a consequence, we can construct for such maps their Green measure and deduce several important ergodic consequences.

Under a mild additional hypothesis, we show that generic sequences of Galois invariant subset of periodic points equidistribute toward the Green measure.

## 1. Introduction

Let $f : X \dashrightarrow X$ be a birational map defined on a complex projective surface. Assume that the action of $f$ on the cohomology $H^{1,1}(X)$ has spectral radius $\lambda_1(f) > 1$, which is necessary to have positive entropy. In [BD], Bedford and Diller introduced an *energy condition* which can be restated as

$$\sum_{n\in\mathbb{N}} \frac{1}{\lambda_1(f)^n} \log d(f^{-n}(I_f), I_{f^{-1}}) > -\infty \text{ and } \sum_{n\in\mathbb{N}} \frac{1}{\lambda_1(f)^n} \log d(f^n(I_{f^{-1}}), I_f)) > -\infty,$$

where $I_f$ (resp. $I_{f^{-1}}$) is the indeterminacy locus of $f$ (resp. of $f^{-1}$). Under that condition, the authors managed to construct a natural mixing hyperbolic measure for $f$ which does not charge curves, and combining with results of Dujardin [Du], we have that this measure is of maximal entropy $\log \lambda_1(f)$ and describes the equidistribution of saddle periodic points. In other words, the energy condition is a natural condition to extend the ergodic properties of Hénon maps to birational maps on a complex projective surface (e.g. [BS, BLS]).

This condition is not always satisfied for birational maps of $\mathbb{P}^2$ ([DG, B]). Still, it is natural to produce examples of birational maps satisfying the energy condition. Our source of inspiration in this article is the work of Jonsson and Reschke [JR] which says the following: assume $f : X \dashrightarrow X$ is a birational maps defined on a complex projective surface $X$ where $X$ and $f$ are defined over a number field, assume that $\lambda_1(f) > 1$, then, up to birational change of coordinates

$$\begin{array}{ccc} X' & \overset{f'}{\dashrightarrow} & X' \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ X & \underset{f}{\dashrightarrow} & X \end{array}$$

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the map $f'$ satisfies the energy condition, also, as the measure $\mu_{f'}$ does not charge curve, it descends to a measure $\mu_f$ with the same ergodic properties. Note that the choice of the birational model is to ensure that the map $f'$ is *algebraically stable* (see below), and when $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ already is algebraically stable, then there is no need to change the model. Roughly speaking, combining [JR] and [BD] tells us that we have a very good understanding of birational maps defined over a number field.

In the present note, we focus on higher dimension and we consider the dynamics of birational maps $f : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ defined over a number field. Just like birational maps of $\mathbb{P}^2(\mathbb{C})$ generalize Hénon maps, the maps we consider are natural generalizations of Hénon-Sibony maps, which are the polynomials automorphisms of $\mathbb{C}^k$ for which $I_{f^{-1}} \cap I_f = \varnothing$ and whose dynamics is very well understood, see [Sib, Chapitre 2]. So, we make the assumption that, as for Hénon-Sibony maps, there is an integer $1 \leq s \leq k-1$ such that

$$(\dagger) \qquad \dim I_f = k - s - 1 \quad \text{and} \quad \dim I_{f^{-1}} = s - 1,$$

where $I_f$ (resp. $I_{f^{-1}}$) is the indeterminacy locus of $f$ (resp. of $f^{-1}$). We say that $f$ satisfies the *improved algebraic stability condition* if we have

$$(\star) \qquad \bigcup_{n\geq 0} f^n(I_{f^{-1}}) \cap I_f = \bigcup_{n\geq 0} f^{-n}(I_f) \cap I_{f^{-1}} = \varnothing.$$

Indeed, algebraic stability means that no hypersurface is sent to the indeterminacy set under iteration [Sib] and this is implied by $(\star)$.

Let $d$ (resp. $\delta$) be the algebraic degree of $f$ (resp. $f^{-1}$). Under the hypothesis $(\dagger)$ and $(\star)$, one can show that $d^s = \delta^{k-s}$ and that the algebraic degree of $f^n$ (resp. $f^{-n}$) is $d^n$ (resp. $\delta^n$), see [DTV]. De Thélin and the second author introduced for such maps a finite energy condition which generalizes Bedford-Diller's condition [DTV, Definition 3.1.9], which can be stated as

$$(1) \qquad \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{1}{d^{sn}} \int_{f^n(I_{f^{-1}})} \phi \cdot f^*(\omega^{s-1}) > -\infty \text{ and} \\ \displaystyle\sum_{n=0}^{\infty} \frac{1}{\delta^{n(k-s)}} \int_{f^{-n}(I_f)} \psi \cdot (f^{-1})^*(\omega^{k-s-1}) > -\infty. \end{cases}$$

where $\omega$ is the Fubini-Study form on $\mathbb{P}^k(\mathbb{C})$ and where $\phi$ (resp. $\psi$) is a quasi-potential of $d^{-1}f^*(\omega)$, i.e. $d^{-1}f^*(\omega) = \omega + dd^c\phi$ (resp. a quasi-potential of $\delta^{-1}(f^{-1})^*(\omega)$, i.e. $\delta^{-1}(f^{-1})^*(\omega) = \omega + dd^c\psi$). Under that condition, we can construct a mixing and hyperbolic measure of maximal entropy $s \log d$, see [DTV]. The main result of the paper is the following.

**Theorem 1.1.** *Any birational map $f : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ which satisfies $(\star)$ and $(\dagger)$ and which is defined over a number field satisfies the finite energy condition* (1).

*In particular, its Green measure $\mu_f$ is well-defined, mixing, hyperbolic, and has maximal entropy $s \log d$.*

As in [JR], the idea is to interpret (1) as the local contribution of a canonical height of the subvariety $I_f$ at an archimedean place of the field of definition of $f$. The difficulty is that $I_f$ is not a finite set of points anymore so we have to use the notion of height of subvarieties following Zhang [Z] and use Chambert-Loir's interpretation [CL] of the arithmetic intersection to relate local contributions of the height with (1).

In [JR], the complex projective surface is not necessarily $\mathbb{P}^2$ and algebraic stability is not assumed. In our case, we work on $\mathbb{P}^k$ and assume algebraic stability for two reasons. First, (1) was only defined for maps on $\mathbb{P}^k(\mathbb{C})$ where the super-potentials theory of Dinh and Sibony [DS3] is efficient, so the last part of Theorem 1.1 will only work on $\mathbb{P}^k(\mathbb{C})$. Second, it is not clear that one can even find a suitable model where a birational map is algebraically stable in dimension $> 2$ (Diller and Favre made it possible in dimension 2 [DF] but Kim and Bedford [BK] shows it is not true anymore for $k = 3$, see also the recent work [BDJK] of Bell, Diller, Jonsson, and Krieger). Nevertheless, we show in § 4.3 how to produce many examples which are not Hénon-Sibony maps.

Let us now move to the problem of the equidistribution of periodic points toward $\mu_f$. For Hénon-Sibony maps, the equidistribution is due to Dinh and Sibony [DS4] using complex methods and to Lee [L1] in the arithmetic setting. We want to extend Lee's result to maps satisfying ($\star$), (†), and (1). For that, we rely on De Thélin and Nguyen Van Sang [DTNVS] who proved that for such maps, isolated periodic points are Zariski dense in $\mathbb{P}^k$. We then apply an arithmetic equidistribution theorem of the first author [Ga, Theorem 6] to deduce the following from Theorem 1.1.

**Theorem 1.2.** *Let $f : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ be a birational map defined over a number field and satisfying ($\star$) and (†). Assume in addition there is a constant $C \geq 0$ such that*

$$\frac{1}{d} h \circ f + \frac{1}{\delta} h \circ f^{-1} \geq \left(1 + \frac{1}{d\delta}\right) h - C \tag{2}$$

*on $(\mathbb{P}^k \setminus (I_f \cup I_{f^{-1}}))(\bar{\mathbb{Q}})$, where $h$ is the naive logarithmic height. Then, there exist generic sequence $(F_i)_i$ of finite Galois invariant subsets of $\mathbb{P}^k(\bar{\mathbb{Q}})$ of periodic points of $f$ and, for such sequence, we have*

$$\frac{1}{\# F_i} \sum_{x \in F_i} \delta_x \to \mu_f,$$

*in the weak sense of probability measures on $\mathbb{P}^k(\mathbb{C})$.*

We construct the canonical height under ($\star$) and (†) (Proposition 3.2.1), but we need the additional assumption (2) to derive the equidistribution theorem 1.2 (even though we expect this hypothesis is not necessary). The condition (2) is not true in general but by Kawaguchi [K2, Corollary C] (or Lee [L2, Theorem 1.2]), it is true for Hénon-Sibony maps where, roughly speaking, it is possible to construct a nice birational model of $\mathbb{P}^k$ for both $f$ and $f^{-1}$. Thus condition (2) holds for maps of the form $A \circ f$ where $A$ is an automorphism and $f$ is a Hénon-Sibony map both defined over a number field, see §4.3 for examples.

It is worth noticing that apart from the case of Hénon-Sibony maps (see [L1] and [DS4]), this is the first equidistribution result for birational maps in dimension at least 3.

## 2. Preliminaries

2.1. **Metrized line bundles and mutual energy.** Let $(K, |\cdot|)$ be an algebraically closed field of characteristic zero which is complete with respect to a non-trivial absolute value. Let $X$ be a smooth projective variety of dimension $q \geq 2$ and let $L$ be an ample line bundle on $X$, both defined over $K$. In what follows, we denote by $X^{\mathrm{an}}$ the Berkovich analytification of $X$.

We let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semi-positive continuous metrics on $L$ and denote by $c_1(L,\|\cdot\|_i)$ the curvature form associated with $\|\cdot\|_i$ (see [Gu]). The continuous function

$$\phi := \log \frac{\|\cdot\|_1}{\|\cdot\|_2} : X^{\mathrm{an}} \to \mathbb{R}$$

defines a continuous metric on $\mathscr{O}_X$, which is a model metric if both $\|\cdot\|_1$ and $\|\cdot\|_2$ are model metrics (see [Z]). The *mutual energy* of $\|\cdot\|_1$ and $\|\cdot\|_2$ on $X$ is defined as

$$\mathrm{E}_X(L,\|\cdot\|_1,\|\cdot\|_2) := \frac{1}{(q+1)}\sum_{j=0}^{q}\int_{X^{\mathrm{an}}} \phi\cdot c_1(L,\|\cdot\|_1)^j \wedge c_1(L,\|\cdot\|_2)^{q-j}. \tag{3}$$

2.2. **Adelic metrics and arithmetic intersection.** Let $X$ be a projective variety of dimension $k$, and let $L_0,\ldots,L_k$ be $\mathbb{Q}$-line bundles on $X$, all defined over a number field $\mathbb{K}$. Assume $L_i$ is equipped with an adelic continuous metric $\{\|\cdot\|_{v,i}\}_{v\in M_{\mathbb{K}}}$ and denote $\bar{L}_i := (L_i,\{\|\cdot\|_v\}_{v\in M_{\mathbb{K}}})$. Assume $\bar{L}_i$ is semi-positive for $1\leq i\leq k$. Fix a place $v\in M_{\mathbb{K}}$. Denote by $X_v^{\mathrm{an}}$ the Berkovich analytification of $X$ at the place $v$. We also let $c_1(\bar{L}_i)_v$ be the curvature form of the metric $\|\cdot\|_{v,i}$ on $L_v^{\mathrm{an}}$.

Note that the hypothesis that $\bar{L}_i$ is adelic means in particular that for all but finitely many $v\in M_{\mathbb{K}}$, the metric $\|\cdot\|_{v,i}$ is a model metric on $L_{i,v}^{\mathrm{an}}$.

In the sequel, for a given place $v\in M_{\mathbb{K}}$, denote by $\mathbb{C}_v$ an algebraically closed and complete extension of $(\mathbb{K},|\cdot|_v)$.

For any closed subvariety $Y$ of dimension $q$ of $X$, the arithmetic intersection number $\left(\bar{L}_0\cdots\bar{L}_q|Y\right)$ is symmetric and multilinear with respect to the $L_i$'s. As observed by Chambert-Loir [CL], we can define $\left(\bar{L}_0\cdots\bar{L}_q|Y\right)$ inductively by

$$\left(\bar{L}_0\cdots\bar{L}_q|Y\right) = \left(\bar{L}_1\cdots\bar{L}_q|\mathrm{div}(s)\cap Y\right) + \sum_{v\in M_{\mathbb{K}}} n_v \int_{Y_v^{\mathrm{an}}} \log\|s\|_v^{-1} \bigwedge_{j=1}^{q} c_1(\bar{L}_i)_v,$$

for any global section $s\in H^0(X,L_0)$ such that the intersection $\mathrm{div}(s)\cap Y$ is proper. In particular, if $L_0$ is the trivial bundle and $\|\cdot\|_{v,0}$ is the trivial metric at all places but a finite set $S$ of places of $\mathbb{K}$, this gives

$$\left(\bar{L}_0\cdots\bar{L}_q|Y\right) = \sum_{v\in S} n_v \int_{Y_v^{\mathrm{an}}} \log\|s\|_{v,0}^{-1} \bigwedge_{j=1}^{q} c_1(\bar{L}_i)_v. \tag{4}$$

When $\bar{L}$ is an ample $\mathbb{Q}$-line bundle endowed with a semi-positive continuous adelic metric, following Zhang [Z], we can define $h_{\bar{L}}(Y)$ as

$$h_{\bar{L}}(Y) := \frac{\left(\bar{L}^{q+1}|Y\right)}{(q+1)[\mathbb{K}:\mathbb{Q}]\deg_L(Y)},$$

where $\deg_L(Y) = (L_{|Y})^q$ is the volume of the line bundle $L$ restricted to $Y$.

2.3. **Arithmetic intersection and mutual energies.**

**Lemma 2.3.1.** *Let $X$ be a smooth projective variety endowed with an ample line bundle $L$, both defined over a number field $\mathbb{K}$. Let $\{\|\cdot\|_{v,1}\}_{v\in M_{\mathbb{K}}}$ and $\{\|\cdot\|_{v,2}\}_{v\in M_{\mathbb{K}}}$ be two adelic semi-positive metrics on $L$ and denote by $\bar{L}_i := (L,\{\|\cdot\|_{v,i}\}_{v\in M_{\mathbb{K}}})$. For a given place*

*$v \in M_{\mathbb{K}}$, we denote by $\mathrm{E}_{X,v}(L, \|\cdot\|_{v,1}, \|\cdot\|_{v,2})$ the mutual energy (3) of the metrics $\|\cdot\|_{v,1}$ and $\|\cdot\|_{v,2}$ on $L_v^{\mathrm{an}}$. Then*

$$\frac{1}{q+1}\left((\bar{L}_1^{q+1}|X) - (\bar{L}_2^{q+1}|X)\right) = \sum_{v \in M_{\mathbb{K}}} n_v \cdot \mathrm{E}_{X,v}(L, \|\cdot\|_{v,1}, \|\cdot\|_{v,2}).$$

*Proof.* Since the arithmetic intersection product is multilinear, we have

$$(\bar{L}_1^{q+1}|X) - (\bar{L}_2^{q+1}|X) = \sum_{j=0}^{q}\left((\bar{L}_1 - \bar{L}_2)\cdot \bar{L}_1^j \cdot \bar{L}_2^{q-j}|Y\right).$$

Note that by assumption, $\bar{L}_1 - \bar{L}_2$ is the trivial bundle endowed with an adelic continuous metric which is trivial for all $v \notin S$ ($S \subset M_{\mathbb{K}}$ is a finite set such that for any $v \notin S$, we have $\|\cdot\|_{v,2} = \|\cdot\|_{v,1}$ as metrics on $L_v^{\mathrm{an}}$). Using (4) with the constant section $s \equiv 1$ of the trivial bundle, we find

$$\left((\bar{L}_1 - \bar{L}_2)\cdot \bar{L}_1^j \cdot \bar{L}_2^{q-j}|Y\right) = \sum_{v \in S} n_v \int_{X_v^{\mathrm{an}}} \phi_v \cdot c_1(\bar{L}_1)_v^j \wedge c_1(\bar{L}_2)_v^{q-j},$$

where $\phi_v = \log(\|\cdot\|_{1,v}/\|\cdot\|_{2,v})$. We thus get

$$(\bar{L}_1^{q+1}|X) - (\bar{L}_2^{q+1}|X) = (q+1)\sum_{v \in S} n_v \cdot \mathrm{E}_{X,v}(L, \|\cdot\|_{v,1}, \|\cdot\|_{v,2}).$$

Together with the fact that $\mathrm{E}_{X,v}(L, \|\cdot\|_{v,1}, \|\cdot\|_{v,2}) = 0$ for all $v \notin S$, this gives the lemma. □

2.4. **Dynamical degrees and algebraic stability.** Let $K$ be an algebraically closed field of characteristic zero and let $f : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ be a dominant rational map defined over $K$. Recall that the *$j$-dynamical degree* of $f$ can be computed as

$$\lambda_j(f) := \lim_{n\to\infty}\left((f^n)^*(L^j)\cdot L^{k-j}\right)^{1/n},$$

for any ample line bundle $L$ on $\mathbb{P}^k_K$ (see [RS, DS2] for the complex case and [T, Da] for arbitrary characteristic).

By [DTV, § 3.1], we have

**Proposition 2.4.1.** *Assume $f : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ is a birational map which satisfies the improved algebraic stability condition ($\star$) and the dimension hypothesis (†). Let $d$ be the algebraic degree of $f$ and $\delta$ be the algebraic degree of $f^{-1}$. Then*

$$\forall 1 \le j \le s,\ \lambda_j(f) = \lambda_1(f)^j = d^j \text{ and } \forall s \le \ell \le k,\ \lambda_\ell(f) = \lambda_1(f^{-1})^{k-\ell} = \delta^{k-\ell}.$$

*In particular, $d^s = \delta^{k-s}$.*

2.5. **The finite energy condition over a metrized field.** Let $f : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ be a dominant rational map defined over an algebraically closed field $K$ of characteristic zero. Let $d =$ be the algebraic degree of $f$. Let $1 \le s \le k-1$ be such that $\dim I_f = k - s - 1$ where $I_f$ is the indeterminacy locus of $f$. Assume that for all $q \le s$,

$$(\lambda_q(f))^n = \left((f^n)^*(\mathscr{O}_{\mathbb{P}^k}(1)^q)\cdot \mathscr{O}_{\mathbb{P}^k}(1)^{k-q}\right) = d^{nq}.$$

**Definition 2.5.1.** *Let $X \subsetneq \mathbb{P}^k$ be a closed subvariety. We say $X$ is $f$-good if*

$$\bigcup_{n\ge 0} f^n(X) \cap I_f = \varnothing.$$

Observe that if $X$ is $f$-good, then necessarily, $\dim X + \dim I_f \leq k-1$ so $\dim X \leq s$. Assume in addition that $(K, |\cdot|)$ is complete. Let $F : \mathbb{A}^{k+1} \to \mathbb{A}^{k+1}$ be a polynomial lift of $f$ defined over $K$, i.e. $F = (F_0, \ldots, F_k)$ with $F_i \in K[X_0, \ldots, X_k]$ homogeneous of degree $d$ with $\pi \circ F = f \circ \pi$ and $I_f = \pi(F^{-1}\{0\})$, where $\pi : \mathbb{A}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ is the canonical projection. We define

$$\varphi_f(x) := \frac{1}{d} \log \|F(p)\| - \log \|p\|,$$

for all $x \in (\mathbb{P}^k \setminus I_f)(K)$ and $p \in \mathbb{A}^{k+1}(K) \setminus \{0\}$ with $\pi(p) = x$. If we equip $\mathscr{O}_{\mathbb{P}^k}(1)$ with a model metric $\|\cdot\|_0$, we define a (singular) metric on $\mathscr{O}_{\mathbb{P}^k}(1)$ by letting $\|\cdot\|_f := \|\cdot\|_0 e^{-\varphi_f}$. The singularities of the metric $\|\cdot\|_f$ are contained in $I_f$ and, for any closed subvariety $X \subsetneq \mathbb{P}^k$ with $X \cap I_f = \varnothing$, the line bundle $(\frac{1}{d} f^* \mathscr{O}_{\mathbb{P}^k}(1) - \mathscr{O}_{\mathbb{P}^k}(1))|_X$ is nothing but the trivial bundle $\mathscr{O}_X$ on $X$ and $\|\cdot\|_f$ is a model metric on $L := \mathscr{O}_{\mathbb{P}^k}(1)|_X$.

**Definition 2.5.2.** *Let $X \subsetneq \mathbb{P}^k$ be an $f$-good closed subvariety of dimension $q$ and let $L_n := \mathscr{O}_{\mathbb{P}^k}(1)|_{f^n(X)}$ and $\|\cdot\|_{f,n}$ and $\|\cdot\|_n$ be the respective restrictions of $\|\cdot\|_f$ and $\|\cdot\|$ to $L_n$. We say $(f, X)$ satisfies the* finite energy condition (E) *if*

$$\sum_{n=0}^{\infty} \frac{1}{d^{n(q+1)}} \mathrm{E}_{f^n(X)}(L_n, \|\cdot\|_{f,n}, \|\cdot\|_n) > -\infty. \tag{E}$$

Remark that, when $K = \mathbb{C}$, the curvature form $c_1(\mathscr{O}_{\mathbb{P}^k}(1), \|\cdot\|)$ is the Fubini-Study form $\omega$. In particular, $c_1(L_n, \|\cdot\|_{f,n})$ is the restriction of $d^{-1} f^* \omega$ to $f^n(X)$. So that (E) rewrite as

$$\sum_{n=0}^{\infty} \frac{1}{d^{(q+1)n}} \sum_{j=0}^{q} \int_{f^n(X)} \varphi_f \cdot \omega^j \wedge f^*(\omega^{q-j}) > -\infty.$$

When $X = I_{f^{-1}}$, this condition implies the first part of (1) for a birational maps of $\mathbb{P}^k$ satisfying (†) and ($\star$).

## 3. Finite energy condition for maps defined over a number field

3.1. **A general result for dominant rational maps.** Let now $f : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ be a dominant rational map of degree $d$ defined over a number field $\mathbb{K}$. Let $1 \leq s \leq k-1$ be such that $\dim I_f = k - s - 1$ where $I_f$ is the indeterminacy locus of $f$. Assume that for all $q \leq s$,

$$(\lambda_q(f))^n = \left((f^n)^*(\mathscr{O}_{\mathbb{P}^k}(1)^q) \cdot \mathscr{O}_{\mathbb{P}^k}(1)^{k-q}\right) = d^{nq}.$$

For any place $v \in M_{\mathbb{K}}$, we let $\|\cdot\|_{f,v}$ and $\varphi_{f,v}$ be defined as in § 2.5. The metric $\|\cdot\|_{f,v}$ induces a singular metric on $\mathscr{O}_{\mathbb{P}^k}(1)_v^{\mathrm{an}}$ with singular locus exactly $I_{f,v}^{\mathrm{an}}$ and the function $\varphi_{f,v}$ extends as a continuous function on $\mathbb{P}_v^{k,\mathrm{an}} \setminus I_{f,v}^{\mathrm{an}}$.

The main result of this section is the following version of [JR, Theorem 5.1] to our case, in which we use the notations of Definition 2.5.2 and add a $v$ to precise the dependence on the choice of the place $v \in M_{\mathbb{K}}$. Note that it only relies on the product formula over a number field as well as the non-negativity of the naive height.

**Theorem 3.1.1.** *Let $X \subsetneq \mathbb{P}^k$ be an $f$-good subvariety defined over $\mathbb{K}$. There is a constant $C \geq 0$ such that for any place $v \in M_{\mathbb{K}}$, we have*

$$\sum_{n=0}^{\infty} \frac{1}{d^{n(q+1)}} \mathrm{E}_{f^n(X)}(L_n, \|\cdot\|_{f,n,v}, \|\cdot\|_{n,v}) \geq -\frac{[\mathbb{K} : \mathbf{k}] \deg(X)}{n_v} \left(h_{\mathrm{nv}}(X) + C\right).$$

*In particular, the pair $(X,f)$ satisfies the finite energy condition* (E) *over $\mathbb{C}_v$.*

*Proof.* As $X$ is $f$-good, we must have $q:=\dim X\leq s$. In particular, the degree of $(f^n)(X)$, counted with multiplicity, can be computed by

$$(f^n(X)\cdot \mathscr{O}_{\mathbb{P}^k}(1)^q)=(X\cdot (f^n)^*(\mathscr{O}_{\mathbb{P}^k}(1)^q))=d^{nq}\deg(X). \tag{5}$$

Fix an integer $n\geq 0$ and recall that $L_n:=\mathscr{O}_{\mathbb{P}^k}(1)|_{f^n(X)}$. Let $\bar{L}_n=(L_n,\{\|\cdot\|_{n,v}\}_{v\in M_{\mathbb{K}}})$ be the semi-positive adelic ample line bundle induced by $\bar{\mathscr{O}}_{\mathbb{P}^k}(1)$ on $f^n(X)$.

Note that, since $X$ is $f$-good, the collection $\{\|\cdot\|_{f,n,v}\}_{v\in M_{\mathbb{K}}}$ of singular metrics on $\mathscr{O}_{\mathbb{P}^k}(1)$ induces a model metric on $L_n$. We denote by $\bar{L}_{f,n}$ the induced adelically metrized line bundle. By construction, we have $f^*\bar{L}_{n+1}=d\bar{L}_{f,n}$ and

$$(\bar{L}_{f,n}^{q+1}|f^n(X))=d^{-(q+1)}((f^*\bar{L}_{n+1})^{q+1}|f^n(X))=d^{-(q+1)}(\bar{L}_{n+1}^{q+1}|f^{n+1}(X)).$$

We thus find

$$\begin{aligned}
I_N&:=\sum_{n=0}^N\frac{1}{(q+1)d^{n(q+1)}}\left((\bar{L}_{f,n}^{q+1}|f^n(X))-(\bar{L}_n^{q+1}|f^n(X))\right)\\
&=\sum_{n=0}^N\frac{1}{(q+1)d^{n(q+1)}}\left(\frac{1}{d^{q+1}}(\bar{L}_{n+1}^{q+1}|f^{n+1}(X))-(\bar{L}_n^{q+1}|f^n(X))\right)\\
&=\frac{1}{(q+1)d^{(q+1)(N+1)}}(\bar{L}_{N+1}^{q+1}|f^{N+1}(X))-\frac{1}{q+1}(\bar{L}_0^{q+1}|X)\\
&\geq -[\mathbb{K}:\mathbb{Q}]\deg(X)h_{\bar{L}_0}(X),
\end{aligned}$$

since $(\bar{L}_{N+1}^{q+1}|f^{N+1}(X))=(q+1)[\mathbb{K}:\mathbb{Q}]\deg(f^{N+1}(X))h_{\bar{L}_{N+1}}(X)\geq 0$. According to Lemma 2.3.1, for any $n\geq 0$,

$$\frac{1}{q+1}\left((\bar{L}_{f,n}^{q+1}|f^n(X))-(\bar{L}_n^{q+1}|f^n(X))\right)=\sum_{v\in M_{\mathbb{K}}}n_v\cdot \mathrm{E}_{f^n(X),v}(L_n,\|\cdot\|_{f,n,v},\|\cdot\|_{n,v}).$$

For any $N\geq 0$, this leads to

$$\sum_{n=0}^N\sum_{v\in M_{\mathbb{K}}}\frac{n_v}{d^{n(q+1)}}\mathrm{E}_{f^n(X),v}(L_n,\|\cdot\|_{f,n,v},\|\cdot\|_{n,v})\geq -[\mathbb{K}:\mathbb{Q}]\deg(X)h_{\bar{L}_0}(X). \tag{6}$$

We now recall that, by the (strong) triangular inequality, for any $v\in M_{\mathbb{K}}$, there is $C_v\geq 0$ such that

$$\varphi_{f,v}\leq C_v\quad\text{on}\quad (\mathbb{P}^k\setminus I_f)(\mathbb{C}_v), \tag{7}$$

with $C_v=0$ for all but finitely many $v\in M_{\mathbb{K}}$, see, e.g., [Sil, JR]. As a consequence, for all $n\geq 0$ and all $v\in M_{\mathbb{K}}$

$$\mathrm{E}_{f^n(X),v}(L_n,\|\cdot\|_{f,n,v},\|\cdot\|_{n,v})\leq C_v\cdot \deg_{L_n}(f^n(X))=C_v\cdot d^{nq}\deg(X),$$

where we used (5). Set $S:=\{v\in M_{\mathbb{K}}:C_v\neq 0\}\subset M_{\mathbb{K}}$. For all $v\notin S$, we get $\mathrm{E}_{f^n(X);v}(L,\|\cdot\|_{f,n,v},\|\cdot\|_{n,v})\leq 0$. Finally, we pick a place $v_0\in M_{\mathbb{K}}$ and $n\geq 0$. By (6) and

(7), we have

$$\sum_{n=0}^{N} \frac{n_{v_0} d^{-n(q+1)}}{[\mathbb{K}:\mathbb{Q}]\deg(X)} \mathrm{E}_{f^n(X),v_0}(L_n, \|\cdot\|_{f,n,v_0}, \|\cdot\|_{n,v_0}) \geq -h_{\bar{L}_0}(X) - \sum_{v\in M_{\mathbb{K}}\setminus\{v_0\}} \sum_{n=0}^{N} \frac{n_v}{d^n} C_v$$
$$\geq -h_{\bar{L}_0}(X) - \frac{dC}{d-1},$$

where $C := \sum_{v\in M_{\mathbb{K}}} n_v C_v = \sum_{v\in S} n_v C_v$, since $C_v \geq 0$ for all $v \in M_{\mathbb{K}}$ and $C_v = 0$ for all $v \notin S$. Since $\bar{L}_0 = \mathscr{O}_{\mathbb{P}^k}(1)|_X$ and since this metrization induces the naive height, this concludes the proof. □

3.2. **Finite energy and canonical heights for birational maps.** In this section, we let $f : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ be a birational map defined over a number field $\mathbb{K}$ satisfying the improved algebraic stability assumption ($\star$) and (†).

*Proof of Theorem 1.1.* Fix a complex place $v \in M_{\mathbb{K}}$. As $f$ satisfies ($\star$), $(I_{f^{-1}}, f)$ is a good pair. In particular, we can apply Theorem 3.1.1 over $\mathbb{C}_v$, which means

$$\sum_{n=0}^{\infty} \frac{1}{d^{ns}} \mathrm{E}_{f^n(I_{f^{-1}})}(L_n, \|\cdot\|_{f,n}, \|\cdot\|_n) > -\infty,$$

with

$$\mathrm{E}_{f^n(I_{f^{-1}})}(L_n, \|\cdot\|_{f,n}, \|\cdot\|_n) := \frac{1}{s}\sum_{j=0}^{s-1} \int_{f^n(I_{f^{-1}})} \varphi \cdot c_1(L_n, \|\cdot\|_{f,n})^j \wedge c_1(L_n, \|\cdot\|_n)^{s-1-j},$$

where $L_n = \mathscr{O}_{\mathbb{P}^k}(1)|_{f^n(I_{f^{-1}})}$, $\|\cdot\|_n$ is the naive metric $\|\cdot\|_0$ on $L_n$, and $\|\cdot\|_{f,n}$ is the metric $\|\cdot\|_0 e^{-\varphi_f}$ on $L_n$ with

$$\varphi_f(x) = \frac{1}{d}\log\|F(p)\| - \log\|p\|$$

for some lift $F$ of $f$. The finiteness of the sum implies (and is in fact equivalent) to the finiteness of the sum for $j = s-1$, see [DTV, Proof of Theorem 3.2.1], and $\varphi_f$ is indeed a quasi-potential of $d^{-1}f^*(\omega)$. Using that $c_1(L_n, \|\cdot\|_{f,n})$ is the restriction of $d^{-1}f^*(\omega)$ to $f^n(I_{f^{-1}})$ implies the first part of (1). Finally, working with $(I_f, f^{-1})$ implies $f$ satisfies (1).

The fact that its Green measure $\mu_f$ is well-defined, mixing, hyperbolic, and has maximal entropy $s\log d$ is an immediate consequence of [DTV, Theorems 4 & 5]. □

The set of points with a well defined grand orbit is

$$\mathbb{P}^k_f := \mathbb{P}^k \setminus \left(\bigcup_{n\geq 0} f^n(I_{f^{-1}}) \cup f^{-n}(I_f)\right).$$

We prove here the following.

**Proposition 3.2.1.** *Assume $f$ satisfies assumption ($\star$) and (†). There exist canonical height functions $\widehat{h}^+_f, \widehat{h}^-_f : \mathbb{P}^k_f(\bar{\mathbb{Q}}) \to \mathbb{R}_+$ such that*

$$\widehat{h}^+_f(f(x)) = d\widehat{h}^+_f(x) \text{ and } \widehat{h}^-_f(f^{-1}(x)) = \delta\widehat{h}^-_f(x) \text{ for all } x \in \mathbb{P}^k_f(\bar{\mathbb{Q}}).$$

*In particular, if $x \in \mathbb{P}^k(\bar{\mathbb{Q}})$ is periodic, then $\widehat{h}^+_f(x) = \widehat{h}^-_f(x) = 0$.*

*Moreover, if we assume there exists a constant $C \in \mathbb{R}$ such that*

$$\frac{1}{d}h(f(x)) + \frac{1}{\delta}h(f^{-1}(x)) \geq \left(1 + \frac{1}{d\delta}\right) h(x) + C, \quad x \in \mathbb{P}^k(\bar{\mathbb{Q}}) \setminus (I_f(\bar{\mathbb{Q}}) \cup I_{f^{-1}}(\bar{\mathbb{Q}})), \tag{8}$$

*then there is a sequence of positive numbers $(\epsilon_n)_n$ such that $\epsilon_n \to 0$ as $n \to \infty$ and*

$$\frac{1}{d^n} h \circ f^n(x) + \frac{1}{\delta^n} h \circ f^{-n}(x) \leq \hat{h}^+_f(x) + \hat{h}^-_f(x) + \epsilon_n, \quad x \in \mathbb{P}^k_f(\bar{\mathbb{Q}}). \tag{9}$$

*Proof.* As before, we use that $\max\{\varphi_{f,v}, \varphi_{f^{-1},v}\} \leq C_v$ on $\mathbb{P}^k_f(\bar{\mathbb{Q}})$ where $C_v = 0$ for all but finitely many $v \in M_{\mathbb{K}}$. We deduce that

$$\max\{\varphi_{f,v} \circ f^n, \varphi_{f^{-1},v} \circ f^{-n}\} \leq C_v \quad \text{on } \mathbb{P}^k_f(\bar{\mathbb{Q}}).$$

In particular, if $C_1$ is the constant $C_1 := \sum_{v \in M_{\mathbb{K}}} n_v \cdot C_v \in \mathbb{R}_+$, summing over all places and over all Galois conjugates of a point $x \in \mathbb{P}^k_f(\bar{\mathbb{Q}})$, we find

$$\begin{cases} \dfrac{1}{d^{n+1}} h_{\text{nv}}(f^{n+1}(x)) - \dfrac{1}{d^n} h_{\text{nv}}(f^n(x)) \leq \dfrac{1}{d^n} C_1, \qquad \text{and} \\ \dfrac{1}{\delta^{n+1}} h_{\text{nv}}(f^{-n-1}(x)) - \dfrac{1}{\delta^n} h_{\text{nv}}(f^{-n}(x)) \leq \dfrac{1}{\delta^n} C_1, \end{cases} \tag{10}$$

where $C_1$ is independent of $x \in \mathbb{P}^k_f(\bar{\mathbb{Q}})$ and of $n \geq 0$. As in Kawaguchi [K1], we deduce, following the arguments of [CS], that the limits

$$\widehat{h}^+_f := \limsup_{n \to \infty} \frac{1}{d^n} h_{\text{nv}} \circ f^n \quad \text{and} \quad \widehat{h}^-_f := \limsup_{n \to \infty} \frac{1}{\delta^n} h_{\text{nv}} \circ f^{-n}$$

are well-defined functions $\widehat{h}^\pm_f : \mathbb{P}^k_f(\bar{\mathbb{Q}}) \to \mathbb{R}_+$ and satisfy $\widehat{h}^+_f \circ f = d\widehat{h}^+_f$ and $\widehat{h}^-_f \circ f^{-1} = \delta \widehat{h}^-_f$.

We now assume (8) holds. We again follow ideas of [K1]. Let $D := d\delta$ and $h' := h + \kappa$ where $\kappa = \frac{-CD}{D+1-d-\delta}$ is a constant chosen so that (8) rephrases as

$$\frac{1}{d} h'(f(x)) + \frac{1}{\delta} h'(f^{-1}(x)) \geq \left(1 + \frac{1}{D}\right) h'(x). \tag{11}$$

For $n \in \mathbb{N}^*$ and $x \in \mathbb{P}^k_f$, let us denote by $h'_n(x)$ the quantity

$$h'_n(x) := \frac{1}{d^n} h'(f^n(x)) + \frac{1}{\delta^n} h'(f^{-n}(x)).$$

We write $h'_0 = h'$ for $n = 0$. We let $c_n := (D^n + 1)/D^n$ for $n \geq 1$ and $c_0 = 1$. We shall prove by induction on $n$ that

$$h'_n \geq \frac{c_n}{c_{n-1}} h'_{n-1}$$

whenever all these quantities are well defined. The step $n = 1$ is (11). Assume now the inequality holds for some $n$ and compose (11) with $f^n$ and $f^{-n}$:

$$\frac{1}{d^{n+1}} h'(f^{n+1}(x)) + \frac{1}{\delta d^n} h'(f^{n-1}(x)) \geq c_1 \frac{1}{d^n} h'(f^n(x))$$
$$\frac{1}{d\delta^n} h'(f^{-(n-1)}(x)) + \frac{1}{\delta^{n+1}} h'(f^{-(n+1)}(x)) \geq c_1 \frac{1}{\delta^n} h'(f^{-n}(x)).$$

Summing we recognize

$$h'_{n+1} + \frac{1}{D} h'_{n-1} \geq c_1 h'_n.$$

Using the induction hypothesis gives $h'_{n+1}+\frac{c_{n-1}}{Dc_n}h'_n\geq c_1h'_n$ so $h'_{n+1}\geq\left(c_1-\frac{c_{n-1}}{Dc_n}\right)h'_n=\frac{c_{n+1}}{c_n}h'_n$ by a straightforward computation. So $h'_n\geq\frac{c_n}{c_{n-1}}h'_{n-1}$ holds for all $n$. Multiplying that inequality for $m\geq n+1$ and passing to the limit give

$$\limsup h'_m\geq c_nh'_n.$$

Recall that $\hat{h}^+_f=\limsup d^{-n}h\circ f^n$ and $\hat{h}^-_f=\limsup\delta^{-n}h\circ f^{-n}$ so, replacing $h'_n$ by $d^{-n}h\circ f^n+\delta^{-n}h\circ f^{-n}+\kappa(d^{-n}+\delta^{-n})$ implies (9) for points in $\mathbb{P}^k_f$. □

## 4. Distribution of generic periodic points of birational maps

4.1. **Arithmetic equidistribution for quasi-heights.** In this section, we let $X$ be a projective variety of dimension $k$ defined over a number field $\mathbb{K}$ and we fix a place $v\in M_{\mathbb{K}}$. For any $n\geq 0$, we let $\psi_n:X_n\to X$ be a birational morphism and we let $L_n$ be a big and nef $\mathbb{Q}$-line bundle on $X_n$ endowed with a semi-positive adelic continuous metrization $\bar{L}_n$. We assume that

(1) the sequence $\mathrm{vol}(L_n)$ converges to constant $\mathrm{V}>0$ and the sequence of probability measures $(\mathrm{vol}(L_n)^{-1}(\psi_n)_*c_1(\bar{L}_n)^k_v)_n$ converges weakly to a probability measure $\mu_v$ on $X^{\mathrm{an}}_v$,

(2) For any ample line bundle $M_0$ on $X$ and any adelic semi-positive continuous metrization $\bar{M}_0$ on $M_0$, there is a constant $C\geq 0$ such that

$$\left(\psi^*_n(\bar{M}_0)\right)^j\cdot\left(\bar{L}_n\right)^{k+1-j}\leq C,$$

for any $2\leq j\leq k+1$ and any $n\geq 0$.

**Definition 4.1.1.** *The data* $(X,\mu_v,X_n,\bar{L}_n)$ *is a* quasi-height *on* $X$ *at the place* $v$.

A sequence $(F_i)_i$ of Galois-invariant finite subsets of $X(\bar{\mathbb{Q}})$ is *quasi-small* if $\psi_n^{-1}\{F_i\}$ is a finite subset of $X_n(\bar{\mathbb{Q}})$ for any $n\geq 0$ and any $i$ and if the sequence

$$\varepsilon_n(\{F_i\}_i):=\limsup_i h_{\bar{L}_n}(\psi_n^{-1}(F_i))-h_{\bar{L}_n}(X_n)$$

satisfies $\limsup_{n\to\infty}\varepsilon_n(\{F_i\})\leq 0$.

The following is proved in [Ga] (see also [YZ]):

**Theorem 4.1.2** (Equidistribution of quasi-small points)**.** *Let* $X$ *be a projective variety defined over a number field* $\mathbb{K}$, *let* $v\in M_{\mathbb{K}}$ *and let* $(X,\mu_v,X_n,\bar{L}_n)$ *be a quasi-height on* $X$ *at the place* $v$. *For any quasi-small sequence* $(F_m)_m$ *of Galois-invariant finite subsets of* $X(\bar{\mathbb{Q}})$ *such that for any hypersurface* $H\subset V$ *defined over* $\mathbb{K}$, *we have*

$$\#(F_n\cap H)=o(\#F_n),\quad \text{as } n\to+\infty,$$

*the probability measure* $\mu_{F_m,v}$ *on* $X^{\mathrm{an}}_v$ *which is equidistributed on* $F_m$ *converges to* $\mu_v$ *in the weak sense of measures, i.e. for any continuous function with compact support* $\varphi\in\mathscr{C}^0(X^{\mathrm{an}}_v)$, *we have*

$$\lim_{m\to\infty}\frac{1}{\#F_m}\sum_{y\in F_m}\varphi(y)=\int_{X^{\mathrm{an}}_v}\varphi\,\mu_v.$$

4.2. **Dynamical quasi-heights for birational maps.** We now prove Theorem 1.2, applying Theorem 4.1.2 above. Let $f:\mathbb{P}^k \dashrightarrow \mathbb{P}^k$ be a birational selfmap of $\mathbb{P}^k$ defined over a number field $\mathbb{K}$ and satisfying $(\star)$ and (2). Let $d:=\deg(f)$ and $\delta:=\deg(f^{-1})$. Recall that $d^s=\delta^{k-s}$. We now choose an embedding $\mathbb{K}\hookrightarrow\mathbb{C}$, and let $f:\mathbb{P}^k(\mathbb{C})\dashrightarrow\mathbb{P}^k(\mathbb{C})$ be the induced complex birational selfmap. By Theorem 1.1, $f$ satisfies the hypothesis of [DTV, Theorems 4 & 5]. More precisely, the Green currents $T_f^j$ and $T^\ell_{f^{-1}}$ are well-defined for $1\leq j\leq s$ and for $1\leq \ell\leq k-s$ and satisfy

$$\lim_{n\to\infty}\frac{1}{d^{nj}}(f^n)^*\omega^j=T_f^j \quad\text{and}\quad \lim_{n\to\infty}\frac{1}{\delta^{n\ell}}(f^{-n})^*\omega^\ell=T_f^\ell.$$

By [DTV, Theorem 3.2.8], the above convergence is in the Hartogs' sense (which means that the super-potentials are almost decreasing to the super-potentials of the limits [DS3].) Moreover, the measure $\mu_f:=T_f^s\wedge T_{f^{-1}}^{k-s}$ is mixing (with an exponential speed by [V]) hence ergodic, and of maximal entropy $s\log d>0$. Since the currents $T_f^s$ and $T_{f^{-1}}^{k-s}$ are wedgeable by [DTV, Theorem 3.4.1], continuity of the wedge product under Hartogs convergence for wedgeable currents (see again [DS3, Proposition 4.2.6]) implies

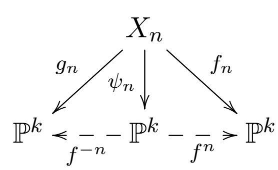
$$\mu_f=\lim_{n\to\infty}\frac{1}{d^{ns}}(f^n)^*(\omega^s)\wedge\frac{1}{\delta^{n(k-s)}}(f^{-n})^*(\omega^{k-s}). \tag{12}$$

The measure $\mu_f$ satisfies $\int\log d(x,I_f)d\mu_f>-\infty$ and is hyperbolic

$$\chi_1\geq\cdots\geq\chi_s>0>\chi_{s+1}\geq\cdots\geq\chi_k,$$

where $\chi_i$ is the $i$-th Lyapunov exponent of $\mu_f$.

We define $X_n$ as a finite sequence of blowups $\psi_n:X_n\to\mathbb{P}^k$ of $\mathbb{P}^k$ such that the maps $f^n\circ\psi_n$ and $f^{-n}\circ\psi_n$ extend as morphisms $f_n,g_n:X_n\to\mathbb{P}^k$. This amounts to the fact that the following diagram commutes:




We let $\bar{L}_0$ be the classical adelic metrization on $\mathscr{O}_{\mathbb{P}^k}(1)$, so that in particular $h_{\bar{L}_0}\geq 0$ on $\mathbb{P}^k(\bar{\mathbb{Q}})$ and, if $\omega$ be the Fubini Study form on $\mathbb{P}^k$, then $\omega$ is the curvature form of $\bar{L}_0$ over $\mathbb{C}$. Let

$$\bar{L}_n:=\frac{1}{d^n}f_n^*\bar{L}_0+\frac{1}{\delta^n}g_n^*\bar{L}_0.$$

In what follows, we denote by $h$ the naive logarithmic height on $\mathbb{P}^k$. We prove here the following.

**Proposition 4.2.1.** *Let $f$ be a birational selfmap of $\mathbb{P}^k$ satisfying $(\star)$ and $(\dagger)$. Assume in addition there is a constant $C\geq 0$ such that*

$$\frac{1}{d}h\circ f+\frac{1}{\delta}h\circ f^{-1}\geq\left(1+\frac{1}{d\delta}\right)h-C$$

*on $(\mathbb{P}^k\setminus(I_f\cup I_{f^{-1}}))(\bar{\mathbb{Q}})$. With the above notations, $(\mathbb{P}^k,\mu_f,X_n,\bar{L}_n)$ is a quasi-height at the complex place and any sequence $(F_i)$ of Galois-invariant finite sets $F_i\subset\mathbb{P}^k(\bar{\mathbb{Q}})$ of periodic points of $f$ is quasi-small.*

*Proof.* We first check condition (1). Note that, since $f_n$ and $g_n$ are generically finite dominant morphisms, and since $\mathscr{O}_{\mathbb{P}^k}(1)$ is ample, the $\mathbb{Q}$-line bundle $L_n$ is big and nef. In particular, $\mathrm{vol}(L_n)=(L_n^k)$. Then we can compute

$$\begin{aligned}\mathrm{vol}(L_n) &= \int_{X_n(\mathbb{C})}\left(\frac{1}{d^n}f_n^*\omega+\frac{1}{\delta^n}g_n^*\omega\right)^k\\ &= \sum_{j=0}^{k}\binom{k}{j}\frac{1}{(d^j\delta^{k-j})^n}\int_{X_n(\mathbb{C})}f_n^*(\omega^j)\wedge g_n^*(\omega^{k-j}).\end{aligned}$$

Fix $0\leq j\leq k$. As $\omega$ is a smooth form and as $f_n$ and $g_n$ are morphisms, for any closed subvariety $Y\subsetneq X_n$, the measure $f_n^*(\omega^j)\wedge g_n^*(\omega^{k-j})$ does not give mass to $Y(\mathbb{C})$. Let $Y_n:=\psi_n^{-1}\left(\bigcup_{0\leq\ell\leq n}f^\ell(I_{f^{-1}})\cup f^{-\ell}(I_f)\right)$, so that $\psi_n$ is an isomorphism from $X_n\setminus Y_n$ to its image $Z_n:=\bigcup_{0\leq\ell\leq n}f^\ell(I_{f^{-1}})\cup f^{-\ell}(I_f)$. We then have

$$\begin{aligned}\int_{X_n(\mathbb{C})}f_n^*(\omega^j)\wedge g_n^*(\omega^{k-j}) &= \int_{X_n(\mathbb{C})\setminus Y_n(\mathbb{C})}f_n^*(\omega^j)\wedge g_n^*(\omega^{k-j})\\ &= \int_{X_n(\mathbb{C})\setminus Y_n(\mathbb{C})}\psi_n^*\left((f^n)^*(\omega^j)\wedge(f^{-n})^*(\omega^{k-j})\right)\\ &= \int_{\mathbb{P}^k(\mathbb{C})\setminus Z_n(\mathbb{C})}(f^n)^*(\omega^j)\wedge(f^{-n})^*(\omega^{k-j}).\end{aligned}$$

We now use Bézout Theorem for currents to find

$$\begin{aligned}&\int_{\mathbb{P}^k(\mathbb{C})\setminus Z_n(\mathbb{C})}(f^n)^*(\omega^j)\wedge(f^{-n})^*(\omega^{k-j})\leq\\ &\qquad\left(\int_{\mathbb{P}^k(\mathbb{C})}(f^n)^*(\omega^j)\wedge\omega^{k-j}\right)\times\left(\int_{\mathbb{P}^k(\mathbb{C})}(f^{-n})^*(\omega^{k-j})\wedge\omega^j\right),\end{aligned}$$

with equality when $j=s$. By assumption, this gives

$$\int_{X_n(\mathbb{C})}f_n^*(\omega^j)\wedge g_n^*(\omega^{k-j})\leq\lambda_j(f)^n\times\lambda_{k-j}(f^{-1})^n,$$

and we have proved that $(d^{-j}\delta^{j-k})^nf_n^*(\omega^j)\wedge g_n^*(\omega^{k-j})$ has mass at most

$$\left(\frac{\lambda_j(f)}{d^j}\frac{\lambda_{k-j}(f^{-1})}{\delta^{k-j}}\right)^n=\begin{cases}O\left(\delta^{-n}\right) & \text{if } j<s,\\ 1 & \text{if } j=s,\\ O\left(d^{-n}\right) & \text{if } j>s.\end{cases}$$

In particular, the volume of $L_n$ satisfies

$$\mathrm{vol}(L_n)\leq\sum_{j=0}^{k}\binom{k}{j}\left(\frac{\lambda_j(f)}{d^j}\frac{\lambda_{k-j}(f^{-1})}{\delta^{k-j}}\right)^n=\binom{k}{s}+o(1).$$

Moreover, for $j=s$, the measure $(d^{-s}\delta^{s-k})^nf_n^*(\omega^s)\wedge g_n^*(\omega^{k-s})$ is a probability measure, whence $\mathrm{vol}(L_n)\geq\binom{k}{s}$ for any $n$, so that $\lim_{n\to\infty}\mathrm{vol}(L_n)=\binom{k}{s}=:\mathrm{V}>0$.

We now show that $\mathrm{vol}(L_n)^{-1}(\psi_n)_* c_1(\bar{L}_n)^k$ converges to the measure $\mu_f = T_f^s \wedge T_{f^{-1}}^{k-s}$. As above, we have

$$\begin{aligned}(\psi_n)_* c_1(\bar{L}_n)^k &= \sum_{j=0}^{k} \binom{k}{j} \frac{1}{d^{nj}\delta^{n(k-j)}} (\psi_n)_* \left(f_n^*(\omega^j) \wedge g_n^*(\omega^{k-j})\right) \\ &= \frac{\binom{k}{s}}{d^{ns}\delta^{n(k-s)}} (\psi_n)_* \left(f_n^*(\omega^s) \wedge g_n^*(\omega^{k-s})\right) + \nu_n,\end{aligned}$$

where the mass of $\nu_n$ is $O(\min\{d,\delta\}^{-n})$, whence tends to 0 as $n \to \infty$. Also

$$\frac{1}{d^{ns}\delta^{n(k-s)}} (\psi_n)_* \left(f_n^*(\omega^s) \wedge g_n^*(\omega^{k-s})\right) = \frac{1}{d^{ns}\delta^{n(k-s)}} (f^n)^*(\omega^s) \wedge (f^{-n})^*(\omega^{k-s}),$$

which converges towards $\mu_f$ as $n \to \infty$ by (12). Since $\mathrm{vol}(L_n) \to \binom{k}{s}$, this gives

$$\lim_{n\to\infty} \mathrm{vol}(L_n)^{-1}(\psi_n)_* c_1(\bar{L}_n)^k = \mu_f,$$

as expected.

We now check condition (2). The metrized line bundle $\bar{M}_n := \bar{L}_n - 2\psi_n^*\bar{L}_0$ is integrable (in the sense of Zhang) with underlying trivial bundle. Fix a place $v \in M_{\mathbb{K}}$. Let $E_n$ be the exceptional divisor of $\psi_n$. Then the function

$$u_{n,v} := \varphi_{f,v} \circ \psi_n + \varphi_{f^{-1},v} \circ \psi_n : (X_n \setminus E_n)(\mathbb{C}_v) \to \mathbb{R}$$

extends continuously to $X_{n,v}^{\mathrm{an}} \setminus E_{n,v}^{\mathrm{an}}$, where $\varphi_{f^{\pm 1},v}$ are the functions introduced in § 2.5. Applying (7) to $f$ and $f^{-1}$, we see that there exists a constant $C_v \geq 0$ such that

$$u_{n,v} \leq C_v \quad \text{on} \quad X_{n,v}^{\mathrm{an}},$$

and there exists a finite set $S \subset M_{\mathbb{K}}$ such that $C_v = 0$ for all $v \notin S$.

We now pick an integer $N \geq 1$ and let $\bar{M}$ be the line bundle $M := \mathscr{O}_{\mathbb{P}^k}(N)$ endowed with an adelic semi-positive continuous metrization. Pick any integer $2 \leq j \leq k$. Then

$$\left(\psi_n^*(\bar{M})\right)^j \cdot \left(\bar{L}_n\right)^{k+1-j} = \left(\psi_n^*(\bar{M})\right)^j \cdot \left(\bar{L}_n\right)^{k-j} \cdot \left(\bar{M}_n\right) + 2\left(\psi_n^*(\bar{M})\right)^j \cdot \left(\bar{L}_n\right)^{k-j} \cdot \left(\psi_n^*\bar{L}_0\right).$$

As $M_n$ is the trivial bundle on $X_n$, we have

$$\begin{aligned}\left(\psi_n^*(\bar{M})\right)^j \cdot \left(\bar{L}_n\right)^{k-j} \cdot \left(\bar{M}_n\right) &= \sum_{v\in M_{\mathbb{K}}} n_v \int_{X_{n,v}^{\mathrm{an}}} u_{n,v} \cdot c_1(\bar{L}_n)_v^{k-j} \wedge c_1((\psi_n)^*\bar{M})_v^j \\ &\leq C_1 \cdot ((\psi_n)^* M^j \cdot L_n^{k-j}) \leq C_1 \max\{N,2\}^k,\end{aligned}$$

where $C_1 := \sum_{v\in M_{\mathbb{K}}} n_v \cdot C_v \in \mathbb{R}_+$. In particular, the constant $C_2 := C_1 \max\{N,2\}^k$ depends only on $c_1(M)$ and

$$\left(\psi_n^*(\bar{M})\right)^j \cdot \left(\bar{L}_n\right)^{k+1-j} \leq C_2 + 2\left(\psi_n^*(\bar{M})\right)^j \cdot \left(\bar{L}_n\right)^{k-j} \cdot \left(\psi_n^*\bar{L}_0\right).$$

In particular, iterating the process and using the projection formula, we deduce

$$\begin{aligned}\left(\psi_n^*(\bar{M})\right)^j \cdot \left(\bar{L}_n\right)^{k+1-j} &\leq \left(\sum_{\ell=0}^{k+1-j} 2^\ell C_2\right) + 2^{k+1-j}\left(\psi_n^*(\bar{M})\right)^j \cdot \left(\psi_n^*\bar{L}_0\right)^{k+1-j} \\ &\leq \left(\sum_{\ell=0}^{k+1-j} 2^\ell C_2\right) C_2 + 2^{k+1-j}\left(\bar{M}\right)^j \cdot \left(\bar{L}_0\right)^{k+1-j}\end{aligned}$$

The conclusion follows taking $C := \left(\sum_{\ell=0}^{k+1} 2^\ell C_2\right) C_2 + \max_j 2^{k+1-j}\left(\bar{M}\right)^j \cdot \left(\bar{L}_0\right)^{k+1-j}$.

To conclude the proof, we check that periodic points are quasi-small. By construction, we have

$$h_{\bar{L}_n} = \frac{1}{d^n} h_{\bar{L}_0} \circ f_n + \frac{1}{\delta^n} h_{\bar{L}_0} \circ g_n = \frac{1}{d^n} h \circ f_n + \frac{1}{\delta^n} h \circ g_n + O(\min\{d,\delta\}^{-n}) \quad \text{on } X_n(\bar{\mathbb{Q}}),$$

by our choice of $\bar{L}_0$. In particular, we have $h_{\bar{L}_n} \geq 0$ on $X_n(\bar{\mathbb{Q}})$ and $h_{\bar{L}_n}(X_n) \geq 0$ by, e.g., [Ga, Lemma 7]. As above, we denote by $\mathbb{P}^k_f(\bar{\mathbb{Q}})$ the set of points with well-defined orbits:

$$\mathbb{P}^k_f(\bar{\mathbb{Q}}) := \mathbb{P}^k(\bar{\mathbb{Q}}) \setminus \left( \bigcup_{n \geq 0} f^{-n}(I_f) \cup f^n(I_{f^{-1}}) \right).$$

Pick any $n \geq 1$. By construction of $X_n$ and $\bar{L}_n$, for $x \in \mathbb{P}^k_f(\bar{\mathbb{Q}})$, $x$ avoids the exceptional set of $\psi_n$ for any $n$ and

$$\begin{aligned} h_{\bar{L}_n}(\psi_n^{-1}(x)) &= \frac{1}{d^n} h_{\bar{L}_0} \circ f_n(\psi_n^{-1}(x)) + \frac{1}{\delta^n} h_{\bar{L}_0} \circ g_n(\psi_n^{-1}(x)) \\ &= \frac{1}{d^n} h(f^n(x)) + \frac{1}{\delta^n} h(f^{-n}(x)) + O(\min\{d,\delta\}^{-n}). \end{aligned}$$

Together with Proposition 3.2.1, this implies

$$h_{\bar{L}_n}(\psi_n^{-1}(x)) \leq \widehat{h}^+_f(x) + \widehat{h}^-_f(x) + \epsilon_n + O(\min\{d,\delta\}^{-n}), \quad x \in \mathbb{P}^k_f(\bar{\mathbb{Q}}),$$

and $h_{\bar{L}_n}(\psi_n^{-1}(x)) \leq \epsilon_n + O(\min\{d,\delta\}^{-n})$ for all periodic points $x \in \mathbb{P}^k(\bar{\mathbb{Q}})$ (since $\widehat{h}^+_f(x) = \widehat{h}^-_f(x) = 0$ for $x$ periodic). As $\min\{d,\delta\} \geq 2$ and $\epsilon_n \to 0$, this concludes the proof of the Proposition. □

4.3. **Proof of Theorem 1.2 and examples.** Let us explain quickly how to deduce from [DTNVS] that the set of isolated periodic points of $f$ is Zariski dense in $\mathbb{P}^k$. By [DTV, Theorem 3.4.13], we know that the measure $\mu_f$ does not charge (pluripolar hence) strict algebraic sets. We are thus in the settings of [DTNVS, Théorème 5]: isolated hyperbolic periodic points accumulated to a set of measure arbitrarily close to 1 (recurring orbits are of full measure on the natural extension for the lift of $\mu_f$ since it is ergodic) and are thus Zariski dense.

In particular, there exist generic sequences $\{F_i\}_i$ of Galois invariant finite subsets of $\mathbb{P}^k(\bar{\mathbb{Q}})$ of periodic points of $f$ (we do not claim that all points in $\{F_i\}_i$ are hyperbolic). We apply Proposition 4.2.1 to end the proof of Theorem 1.2.

**Example.** It is easy to produce examples of birational maps of $\mathbb{P}^k$ defined over a number field $\mathbb{K}$, satisfying ($\star$), (†) and (2). To do so, start with a regular automorphism $f$ of $\mathbb{C}^k$ defined of $\mathbb{K}$, see [Sib], which obviously satisfies ($\star$), (†) and also (2) by [K2, Corollary C] or [L2, Theorem 1.2]. Then, for $A \in \mathrm{PSL}(k+1,\mathbb{K})$, $A \circ f$ still satisfies (†) and (2). For $A$ sufficiently close to the identity (at the complex place), $A \circ f$ will still satisfy the improved algebraic stability because it is a *regular birational map* of $\mathbb{P}^k(\mathbb{C})$: its indeterminacy sets are contained in two disjoint fixed open sets of $\mathbb{P}^k(\mathbb{C})$, see [DS1] for a detailed study of such maps. By [BD, DTV], we know that outside a pluripolar set of maps $A \in \mathrm{PSL}(k+1,\mathbb{C})$, then $A \circ f$ satisfies ($\star$) and the energy condition, nevertheless, countable sets are pluripolar so it could be that the $A \in \mathrm{PSL}(k+1,\mathbb{K})$ for which $A \circ f$ satisfies the improved algebraic stability are exactly those for which $A \circ f$ is a regular birational map.

Let us a give a slight modification of the above construction to show it can produce many birational maps that satisfy the improved algebraic stability. Let us stick to the case of dimension 2 and degree 2 for simplicity. Pick $a, b$ in $\mathbb{K}$, a prime number $p$ and a $p$-adic absolute value $|.|_p$ on $\mathbb{K}$ with $|a|_p = |b|_p = 1$ and consider the Hénon map

$$f([x:y:t]) = ([x^2+yt+at^2 : bxt : t^2]) \text{ and } f^{-1}([x:y:t]) = ([yt/b : xt - y^2/b^2 - at^2 : t^2])$$

so $I_f = [0:1:0]$ and $I_{f^{-1}} = [1:0:0]$. Take $A \in \mathrm{PSL}(3, \mathbb{K})$ such that

$$A^{-1}([x,y,t]) = [a_1x+b_1y+c_1t : a_2x+b_2y+c_2t : a_3x+b_3y+c_3t].$$

with $|b_2|_p$ strictly larger than the $p$-adic absolute values of all the others numbers $a_i$, $b_i$, $c_i$. Then we claim that the map $f \circ A$ is algebraically stable.

For that, we claim by induction on $n$ that, writing $(f \circ A)^{-n}(A^{-1}(I_f)) = [x_n : y_n : t_n]$, then $|y_n|_p > \max |x_n|_p, |t_n|_p$ which implies the algebraic stability as $\{(f\circ A)^{-n} I_{f\circ A}\} \cap \{I_{(f\circ A)^{-1}}\} = \varnothing$. Indeed, observe that $I_{f\circ A} = A^{-1}(I_f) = [b_1 : b_2 : b_3]$ so the case $n = 0$ is clear. Now, assume $|y_n|_p > \max |x_n|_p, |t_n|_p$ for some $n$, then

$$[x_{n+1} : y_{n+1} : t_{n+1}] = A^{-1}[y_nt_n/b : x_nt_n - y_n^2/b^2 - at_n^2 : t_n^2]$$

and by the strong triangular inequality $|x_nt_n - y_n^2/b^2 - at_n^2|_p = |y_n|_p^2$ so applying $A^{-1}$ concludes the induction as $b_2$ dominates all the other coefficients.

To go further, we show that we can impose the condition that the backward orbit of $I_f$ is Zariski dense. Let us sketch the construction: assume $|b_2|_p \gg |b_1|_p \gg |b_3|_p \gg \max |a_i|_p, |c_j|_p$ so that $|y_0|_p > 2|x_0|_p > 4|t_0|_p \neq 0$. An immediate induction then shows that

$$|y_n|_p > 2^{n+1}|x_n|_p > 4^{n+1}|t_n|_p \neq 0.$$

Take any hypersurface over $\mathbb{Q}$. It is given by the equation $P(x, y, t) = 0$ for some homogeneous polynomial of degree $p$:

$$P(x,y,t) = \sum_{i_1+i_2+i_3=p} a_{i_1,i_2,i_3} x^{i_1} y^{i_2} t^{i_3}$$

where the $a_{i_1,i_2,i_3}$ are in $\bar{\mathbb{Q}}$. Then, it is easy to see that for $n$ large enough, one cannot have $[x_n : y_n : t_n] \overset{\forall n}{\in} (P = 0)$.

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