Toric varieties

Nicolas Perrin

CMLS, CNRS, École Polytechnique
Email address: nicolas.perrin.cmls@polytechnique.edu

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Introduction

Let k be an algebraically closed field. The multiplicative group over k is the group $\mathbb{G}_m = (\mathsf{k}^\times, \times)$ of non-zero elements of k. An n-dimensional torus T is an algebraic group isomorphic to $(\mathbb{G}_m)^n$.

A toric variety is a variety X (an integral separated k-scheme of finite type over k) endowed with an algebraic action of a torus T such that T has a dense orbit in X. Actually, we will see that this is equivalent (up to replacing T by a quotient of T) to the requirement that T is isomorphic to a dense open subset of X.

Toric varieties are a quite special class of varieties and one could wonder why they should play an important role. One answer for this is the fact that their geometry can be translated into convex geometry, building a bridge between these two topics and enabling to replace complex arguments coming from algebraic geometry by explicit reasoning in convex geometry and, conversely, using the powerful tools of algebraic geometry to get some convex geometric results. There are other more involved application of toric varieties for example in birational geometry: toric varieties play a key role in the *Weak Factorisation Theorem*.

In these lectures, we will explain the dictionary between algebraic geometry of toric varieties and convex geometry, from the affine case to the general one with a special focus on classical geometric topics such as singularities, line bundles, birational geometry...

Convention, notation and prerequisites

We work over an algebraically closed field k. We assume $\mathrm{char}(\mathsf{k}) \geq 0$ and specify when the assumption $\mathrm{char}(\mathsf{k}) = 0$ is needed/

All the groups we shall consider will be linear algebraic groups (except in the first chapter) and mostly tori

We will also assume some familiarities with basics on algebraic geometry and use [**5**] as reference. A k-variety is a reduced and separated k-scheme of finite type over k (not necessarily irreducible). We will mainly consider k-varieties in these lectures.

There are many excellent references on toric varieties. A few of them are [**8**], [**4**] and [**2**]

We will try to use the following conventions in these notes: T will denote a torus, Γ or G will denote abtract and/or algebraic groups, S will denote a monoid (or semigroup) while M and N will denote abelian groups (and most of the time lattices).

CHAPTER 1

Affine pre-toric varieties

Let k be an algebraically closed field.

1. Toric varieties

We start with the main definitions of these lectures: tori and toric varieties.

DEFINITION 1.1.1. An algebraic group is a k-variety G with a structure of abstract group such that the multiplication map $\mu : G \times G \to G$ and the inverse $\iota : G \to G$ are morphisms.

EXAMPLE 1.1.2. Let $\mathbb{G}_m = \mathrm{Spec}(\mathsf{k}[T, T^{-1}] = \mathsf{k}^\times$ be the multiplicative group of the field. Then $\mathbb{G}_m$ is al algebraic group, the multiplication μ and inverse ι are defined on the level of functions by the maps

$$\mu^* : \mathsf{k}[T, T^{-1}] \to \mathsf{k}[T, T^{-1}] \otimes_{\mathsf{k}} \mathsf{k}[T, T^{-1}] \text{ and } \iota^* : \mathsf{k}[T, T^{-1}] \to \mathsf{k}[T, T^{-1}]$$

with $\mu^*(f) = f \otimes f$ and $\iota^*(T) = T^{-1}$.

DEFINITION 1.1.3. A torus is an algebraic variety T isomorphic to $(\mathbb{G}_m)^n$:

$$T \simeq (\mathsf{k}^\times)^n \simeq (\mathrm{Spec}(\mathsf{k}[X, X^{-1}]))^n \simeq \mathrm{Spec}(\mathsf{k}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]).$$

The torus T has a group structure defined by

$$(x_1, \cdots x_n) \cdot (y_1, \cdots, y_n) = (x_1 y_1, \cdots, x_n y_n) \text{ and } (x_1, \cdots, x_n)^{-1} = (x_1^{-1}, \cdots, x_n^{-1}).$$

Note that both maps are morphisms of algebraic varieties.

DEFINITION 1.1.4. Let T be a torus.

(1) A variety X is a T-variety if T acts on X and the action map $T \times X \to X$, $(t, x) \mapsto t.x$ is a morphism.
(2) A variety X is called pre-toric if there exists a torus T such that X is a T-variety and T has an open (and therefore dense) orbit in X.
(3) A variety X is called toric if X is pre-toric and normal.

Tori and toric varieties are very special cases of algebraic groups and of varieties endowed with an algebraic action of an algebraic group. We refer to Appendix A.1 for a few more details and to the classical books on the subject [**1**], [**6**] and [**9**] for much more on these topics.

REMARK 1.1.5. We shall see later on (see Proposition 1.3.14) that the dense orbit of T in a toric variety X is isomorphic to a torus so that we may replace the condition that T has a dense orbit in the previous definition by the fact that T is isomorphic to an open subset of X.

2. From lattices to tori...

We construct tori from lattices.

Definition 1.2.1. Let M be an abelian group.

(1) The group algebra $\mathsf{k}\lfloor M\rfloor$ of M over k is the set of function from M to k with finite support:

$$\mathsf{k}\lfloor M\rfloor = \{f : M \to \mathsf{k} \mid f(m) = 0 \text{ for all but finitely many } m \in M\}.$$

(2) The k-vector space structure is given by $(\lambda f + \mu g)(m) = \lambda f(m) + \mu g(m)$ while the (commutative) product is defined by

$$(fg)(m) = \sum_{\substack{(m',m'')\in A^2,\\ m'+m''=m}} f(m')g(m'').$$

(3) Define $\chi^m \in \mathsf{k}\lfloor M\rfloor$ by $\chi^m(n) = \delta_{m,n}$ for $n \in M$. We have $\chi^m\chi^n = \chi^{m+n}$.

Remark 1.2.2. Let M be a finitely generated group.

(1) We have the inclusion $\mathsf{k}\lfloor M\rfloor \subset \mathrm{Fun}(M, \mathsf{k})$.
(2) For M finite, $\mathsf{k}\lfloor M\rfloor = \mathsf{k}[M]$ as k-vector space but the product is different.
(3) This algebra structure is often called *convolution product*.
(4) The element $(\chi - m)_{m\in M}$ are linearly independant: if $\sum_m \lambda_m\chi^m = 0$ evaluate at $m' \in M$ to get $\lambda_{m'} = 0$.

Lemma 1.2.3. *The family $(\chi^m)_{m\in M}$ is a basis of $\mathsf{k}\lfloor M\rfloor$.*

Proof. The family $(\chi^m)_{m\in M}$ is clearly spanning and free by Remark 1.2.2.(4). □

Proposition 1.2.4. *$\mathsf{k}\lfloor M\rfloor$ is finitely generated iff M is finitely generated.*

Proof. Let $(m_i)_{i\in[1,r]}$ be a generating set of M. Up to replacing $(m_i)_{i\in[1,r]}$ by $(m_i)_{i\in[1,r]} \cup (-m_i)_{i\in[1,r]}$, any element of M is a linear combination with coefficients in $\mathbb{N}$ of this family. We claim that $(\chi^{m_i})_{i\in[1,r]}$ spans $\mathsf{k}\lfloor M\rfloor$ as k-algebra. Since $(\chi^m)_{m\in M}$ spans $\mathsf{k}\lfloor M\rfloor$ as vector space, it is enough to prove the result for χ^m. Write $m = \sum_{i=1}^r \lambda_i m_i$ with $\lambda_i \in \mathbb{N}$, then $\chi_m = \prod_{i=1}^r (\chi^{m_i})^{\lambda_i}$ proving the claim. Conversely let $(f_i)_{i\in[1,r]}$ be a generating set of $\mathsf{k}\lfloor M\rfloor$. Since $(\chi^m)_{m\in M}$ is spanning $\mathsf{k}\lfloor M\rfloor$ as k-vector space, up to adding elements in the family, we may assume that $f_i = \chi^{m_i}$ for some $m_i \in M$. Let $m \in M$, then $\chi^m = P(\chi^{m_1}, \cdots, \chi^{m_r})$ for some $P \in \mathsf{k}[X_1, \cdots, X_r]$. Expanding P and using Lemma 1.2.3, we get that there exist $\lambda_1, \cdots, \lambda_r \in \mathbb{N}$ such that $\chi^m = \chi^{\lambda_1 m_1+\cdots+\lambda_r m_r}$ and thus $am = \lambda_1 m_1 + \cdots + \lambda_r m_r$, proving the converse. □

Definition 1.2.5. For M finitely generated abelian M, set $T_M = \mathrm{Spec}(\mathsf{k}\lfloor M\rfloor)$.

Example 1.2.6. We compute T_M for some easy abelian groups.

(1) If $M = \mathbb{Z}$, then $T_M = \mathrm{Spec}(\mathsf{k}[T, T^{-1}]) = \mathbb{G}_m$.
(2) If $M = \mathbb{Z}/n\mathbb{Z}$, then $T_M = \mathrm{Spec}(\mathsf{k}[T]/(T^n - 1))$.
(3) If $M = \mathbb{Z}/p\mathbb{Z}$ with $p = \mathrm{char}(\mathsf{k})$, then $T_M = \mathrm{Spec}(\mathsf{k}[T]/(T^p - 1)) = \mathrm{Spec}(\mathsf{k}[T]/(T - 1)^p)$ is a non reduced point.

Remark 1.2.7. On T_M, we get a structure of algebraic group (or group scheme for non-reduced ones) given by the maps $\mu^* : \mathsf{k}\lfloor M\rfloor \to \mathsf{k}\lfloor M\rfloor\otimes_{\mathsf{k}}\mathsf{k}\lfloor M\rfloor$ and $\iota^*\mathsf{k}\lfloor M\rfloor \to \mathsf{k}\lfloor M\rfloor$ with $\mu^*(f) = f \otimes f$ and $\iota^*(\chi^m) = \chi^{-m}$.

LEMMA 1.2.8. *Let M and N be f.g. abelian groups, then $T_{M\times N} = T_M \times T_N$.*

Proof. For $f \in \mathsf{k}\lfloor M\rfloor$ and $g \in \mathsf{k}\lfloor N\rfloor$, define $[fg] \in \mathsf{k}\lfloor M \times N\rfloor$ by $[fg](m,n) = f(m)g(n)$. Define two maps $\Phi : \mathsf{k}\lfloor M\rfloor \otimes_{\mathsf{k}} \mathsf{k}\lfloor N\rfloor \to \mathsf{k}\lfloor M\times N\rfloor$ and $\Psi : \mathsf{k}\lfloor M\times N\rfloor \to \mathsf{k}\lfloor M\rfloor \otimes_{\mathsf{k}} \mathsf{k}\lfloor N\rfloor$ by $\Phi(f\otimes g) = [fg]$ and

$$\Psi(h) = \sum_{(m,n)\in M\times N} h(m,n)\chi^m \otimes \chi^n.$$

It is easy to check that Φ and Ψ are k-algebra morphisms inverse from each other, proving the result. □

PROPOSITION 1.2.9. *Let M be a finitely generated abelian group, then T_M is a torus if and only if M is a lattice. In this case* $\dim T_M = \mathrm{rk}(M)$.

Proof. Follows from Lemma 1.2.8, Example 1.2.6 and the structure of finitely generated abelian groups. The multiplication on T_M is given by the map $\mathsf{k}\lfloor M\rfloor \to \mathsf{k}\lfloor M\rfloor \otimes_k \mathsf{k}\lfloor M\rfloor, f \mapsto f\otimes f$ and the inverse map by $\mathsf{k}\lfloor M\rfloor \to \mathsf{k}\lfloor M\rfloor, \chi^m \mapsto \chi^{-m}$. □

3. ... and back from tori to lattices

We reverse the previous construction to a correspondence between tori and lattices. This correspondence extends to diagonalisable groups and finitely generated $\mathbb{Z}$-modules. We refer to Appendix A.2 for more details.

DEFINITION 1.3.1. Let Γ be an abstract group.

(1) A character of Γ is a group morphism $\Gamma \to \mathbb{G}_m$.
(2) The set of all characters of Γ is denoted by $\mathbb{X}(\Gamma)$.

REMARK 1.3.2. Let Γ be an abstract group.

(1) We have an inclusion $\mathbb{X}(\Gamma) \subset \mathrm{Fun}(\Gamma, \mathsf{k}) := \{f : \Gamma \to \mathsf{k}\}$ the vector space of k-valued functions on Γ.
(2) The set of all characters $\mathbb{X}(\Gamma)$ has a commutative group structure defined by $(\chi + \chi')(\gamma) = \chi(\gamma)\chi'(\gamma)$. In particular, we denote by 0 the trivial character: $0(\gamma) = 1$ for all $\gamma \in \Gamma$.

LEMMA 1.3.3. *Let Γ be a group, then $\mathbb{X}(\Gamma) \setminus \{0\}$ is linearly independent.*

Proof. Assume that $\mathbb{X}(\Gamma) \setminus \{0\}$ is not linearly independent and let r be minimal such that there exists $\chi_1, \cdots, \chi_r \in \mathbb{X}(\Gamma)\setminus\{0\}$ which is linearly dependent *i.e.* there exists $a, \cdots, a \in \mathbb{Z}$ with $\sum_{i=1}^r a_i\chi_i = 0$.

We clearly have $r > 1$ and by minimality $\chi_1 \neq \chi_r$ thus there exists $\gamma_0 \in \Gamma$ with $\chi_1(\gamma_0) \neq \chi_r(\gamma_0)$. Then for all $\gamma \in \Gamma$, we have

$$0 = \sum_{i=1}^r a_i\chi_i(\gamma_0\gamma) - \chi_r(\gamma_0)\sum_{i=1}^r a_i\chi_i(\gamma) = \sum_{i=1}^r a_i(\chi_i(\gamma_0) - \chi_r(\gamma_0))\chi_i(\gamma).$$

In particular, we get the relation $\sum_{i=1}^{r-1} a_i(\chi_i(\gamma_0) - \chi_r(\gamma_0))\chi_i = 0$ contradicting the minimality of the first relation. □

DEFINITION 1.3.4. Let G be an algebraic group.

(1) An algebraic character of G is a group morphism $G \to \mathbb{G}_m$.
(2) The set of all algebraic characters of G is denoted by $\mathfrak{X}^*(G)$.

REMARK 1.3.5. Let G be an algebraic group.

(1) We have an inclusion $\mathfrak{X}^*(G) \subset \mathsf{k}[G]$.
(2) $\mathfrak{X}^*(G)$ is a subgroup of $\mathbb{X}(G)$ (we have $\mathfrak{X}^*(G) = \mathbb{X}(G) \cap \mathsf{k}[T]$).
(3) Not all characters are algebraic. Set $\mathsf{k} = \mathbb{C}$, the map $\sigma : \mathbb{G}_m \to \mathbb{G}_m$, $z \mapsto \bar{z}$ lies in $\mathbb{X}(\mathbb{G}_m) \setminus \mathfrak{X}^*(\mathbb{G}_m)$.

LEMMA 1.3.6. *Let $f : \mathbb{G}_m \to \mathbb{G}_m$ be a morphism of algebraic groups. There exists a unique integer $m \in \mathbb{Z}$ such that $f(z) = z^m$.*

Proof. This follows from Theorem 1.3.13 below and the equality $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$. Here is an explicit proof. Write $\mathbb{G}_m = \mathrm{Spec}(\mathsf{k}[X, X^{-1}]$ and let $f^* : \mathsf{k}[X, X^{-1}] \to \mathsf{k}[X, X^{-1}]$ be the map associated to f on algebras of functions. Then f is determined by $f^*(X) = P(X, X^{-1}$ with P a polynomial in two variables. Since f is a group morphism, we have $P(X^m, X^{-m}) = P(X, X^{-1})^m$ for all $m \in \mathbb{Z}$. This easily implies that $P(X, X^{-1}) = X^k$ for some $k \in \mathbb{Z}$. □

DEFINITION 1.3.7. For $f : \mathbb{G}_m \to \mathbb{G}_m$ a morphism of group, the degree of f is the integer $\deg(f) \in \mathbb{Z}$ such that $f(z) = z^{\deg(f)}$ for all $z \in \mathbb{G}_m$.

COROLLARY 1.3.8. *We have $\mathfrak{X}^*(\mathbb{G}_m) = \mathbb{Z}$ given by $\chi \mapsto \deg(\chi)$.*

EXAMPLE 1.3.9. Let $D_n \subset \mathrm{GL}_n(\mathsf{k})$ be the subgroup of diagonal matrices. We write $\mathrm{diag}(d_1, \cdots, d_n) \in D_n$ the diagonal matrix with entries $d_1, \cdots, d_n \in \mathsf{k}^\times$. Using Lemma 1.3.6, it is easy to check that any $\chi \in \mathfrak{X}^*(D_n)$ is of the form $\chi(\mathrm{diag}(d_1, \cdots, d_n)) = \prod_{i=1}^n d_i^{n_i}$ for a unique $(n_i)_{i \in [1,n]} \in \mathbb{Z}^n$. In particular we get an isomorphism $\mathfrak{X}^*(D_n) \simeq \mathbb{Z}^n$.

COROLLARY 1.3.10. *For a torus T, the group $\mathfrak{X}^*(T)$ is a lattice.*

In particular this gives a correspondence between tori and lattices.

PROPOSITION 1.3.11. *Let T be a torus and M be a lattice.*

(1) *We have $T = T_{\mathfrak{X}^*(T)}$.*
(2) *We have $M = \mathfrak{X}^*(T_M)$.*

Proof. Exercise. □

This correspondence extends to diagonalisable groups.

DEFINITION 1.3.12. G is diagonalisable if it is a closed subgroup of D_n.

THEOREM 1.3.13. *The contravariant functor $G \mapsto \mathfrak{X}^*(G)$ is a fully faithful from the category of diagonalisable group to the category of finitely generated groups.*

As a consequence we get the following results on tori.

PROPOSITION 1.3.14. *Let $f : T \to T'$ be a group morphism between tori.*

(1) *The image of f is a torus and a closed subgroup of T'.*
(2) *The kernel of f is a closed subgroup of T and is a torus iff it is connected.*

We finish this section by some results on representations of tori. Indeed, tori (and diagonalisable groups) have very simple representations.

DEFINITION 1.3.15. Let T be a torus.

(1) A rational or algebraic representation of T (or a T-module) is a k-vector space V such that T acts on V by linear maps. Equivalently, such a representation is given by a morphism of algebraic groups $T \to \mathrm{GL}(V)$.

(2) For $\chi \in \mathfrak{X}^*(T)$ and V a T-module, set $V_\chi = \{v \in V \mid t.v = \chi(t)v, \forall t \in T\}$.
(3) A character χ is a weight of the T-module V if $V_\chi \neq 0$.

THEOREM 1.3.16. *For T a torus and V a T-module, we have $V = \oplus_{\chi \in \mathfrak{X}^*(G)} V_\chi$.*

4. The dual point of view: one parameter subgroups

DEFINITION 1.4.1. Let G be an algebraic group.

(1) A one parameter subgroup of G is a group morphism $\mathbb{G}_m \to G$ which is a morphism of algebraic varieties.
(2) The set of all one parameter subgroups of G is denoted by $\mathfrak{X}_*(G)$.
(3) $\mathfrak{X}_*(G)$ has a structure of abelian group defined by $(f+g)(z) = f(z)g(z)$ for $f, g \in \mathfrak{X}_*(G)$.

One parameter subgroups are dual to characters. Let T be a torus.

PROPOSITION 1.4.2. *We have $\mathfrak{X}_*(T) = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{X}^*(T), \mathbb{Z})$.*

Proof. Follows from Theorem 1.3.13 since $\mathbb{G}_m = T_{\mathbb{Z}}$ and $T = T_{\mathfrak{X}^*(T)}$. □

We have a pairing $\mathfrak{X}^*(T) \times \mathfrak{X}_*(T) \to \mathbb{Z}$ defined by $(\chi, \varphi) \mapsto \deg(\chi \circ \varphi)$.

PROPOSITION 1.4.3. *The pairing $\mathfrak{X}^*(T) \times \mathfrak{X}_*(T) \to \mathbb{Z}$ is non degenerate.*

Proof. Under the correspondence of Theorem 1.3.13, this pairing is the pairing $\mathfrak{X}^*(T) \times \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{X}^*(T), \mathbb{Z}) \to \mathbb{Z}$, $(\chi, f) \mapsto f(\chi)$ which is non degenerate. □

Working with a torus T is equivalent to working with its character lattice $\mathfrak{X}^*(T)$ or with its cocharacter lattice (or lattice of one parameter subgroups) $\mathfrak{X}_*(T)$. In particular we want to keep in mind the pair $(\mathfrak{X}^*(T), \mathfrak{X}_*(T))$ together with the pairing $\mathfrak{X}^*(T) \times \mathfrak{X}_*(T) \to \mathbb{Z}$ rather than only one of them. The cones and fans will live place in the vector space $V = \mathfrak{X}_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

5. Monoids and affine toric varieties

We extend the previous constructions to monoids.

DEFINITION 1.5.1. A monoid (or semigroup) is a set S with an associative binary operation such that there exists an identity element.

(1) S is commutative if the operation, denoted $+$, is commutative. The identity element is denoted 0.
Assume that S is a commutative monoid.
(2) For $\mathcal{S} \subset S$, the monoid generated by $\mathcal{S}$ is $\mathbb{N}\mathcal{S} = \{\sum_{s \in \mathcal{S}} a_s s \mid a_s \in \mathbb{N}\}$.
(3) S is finitely generated if there exists $\mathcal{S} \subset S$ finite such that $S = \mathbb{N}\mathcal{S}$.
(4) S is torsion free if $ks = 0$ implies $k = 0$ or $s = 0$ for $k \in \mathbb{N}$ and $s \in M$.
(5) S is affine if it is commutative, torsion free and finitely generated.

DEFINITION 1.5.2. Let S be a commutative monoid. Define the equivalence relation $\sim$ on S^2 by $(s,t) \sim (s',t') \Leftrightarrow s + t' + u = s' + t + u$ for some $u \in S$. Define the group completion $S_{\mathbb{Z}}$ of S as the set of equivalence classes: $S_{\mathbb{Z}} = S^2/\sim$.

LEMMA 1.5.3. *Let S be a commutative monoid.*

(1) *The group completion $S_{\mathbb{Z}}$ is a commutative group for the addition defined by $[s,t] + [s',t'] = [s+s', t+t']$.*
(2) *There is a morphism of monoid $\varphi_S : S \to S_{\mathbb{Z}}, s \mapsto [s, 0]$.*

(3) *The map* $\varphi_S : S \to S_{\mathbb{Z}}$ *is the solution of the following universal problem: for any commutative group* M *with a morphism of monoid* $f : S \to M$*, there exists a unique group morphism* $\bar{f} : S_{\mathbb{Z}} \to M$ *such that* $f = \bar{f} \circ \varphi_S$*.*
(4) *If* $\mathcal{S} \subset S$ *is generating (*i.e. $S = \mathbb{N}\mathcal{S}$*), then* $S_{\mathbb{Z}} = \mathbb{Z}\mathcal{S}$

Proof. Exercise. □

EXAMPLE 1.5.4. If $S = \mathbb{N}^n$, then $S_{\mathbb{Z}} = \mathbb{Z}^n$. We recover the map $\mathbb{N}^n \subset \mathbb{Z}^n$.

REMARK 1.5.5. Assume that S is an affine monoid.
(1) Then the equivalence relation $\sim$ can be simplified and defined by $(s, t) \sim (s', t') \Leftrightarrow s + t' = s' + t$.
(2) The group $S_{\mathbb{Z}}$ is a lattice and $S \subset S_{\mathbb{Z}}$ (prove this as an exercise).
(3) If $V = S_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ and we view S and $S_{\mathbb{Z}}$ as subsets of V, then

$$S_{\mathbb{Z}} = \{s_1 - s_2 \in V \mid s_1, s_2 \in S\}.$$

EXAMPLE 1.5.6. If M is a lattice and $\mathcal{S} \subset M$ a finite set, then
(1) $\mathbb{N}\mathcal{S} = \{\sum_{s\in\mathcal{S}} a_s s \mid a_s \in \mathbb{N}\}$ is a monoid
(2) $(\mathbb{N}\mathcal{S})_{\mathbb{Z}} = \mathbb{Z}\mathcal{S} = \{\sum_{s\in\mathcal{S}} a_s s \mid a_s \in \mathbb{Z}\} \subset M$ is the sublattice spanned by $\mathcal{S}$.

DEFINITION 1.5.7. Let S be a commutative monoid.
(1) The monoid algebra $\mathsf{k}\lfloor S\rfloor$ of S over k is the set of function from S to k with finite support:

$$\mathsf{k}\lfloor S\rfloor = \{f : S \to \mathsf{k} \mid f(s) = 0 \text{ for all but finitely many } s \in S\}.$$

(2) The k-vector space structure is given by $(\lambda f + \mu g)(s) = \lambda f(s) + \mu g(s)$ while the (commutative) product is defined by

$$(fg)(s) = \sum_{\substack{(t,u)\in M^2,\\ t+u=s}} f(t)g(u).$$

(3) Let $\chi^s \in \mathsf{k}\lfloor S\rfloor$ be defined by $\chi^s(t) = \delta_{s,t}$.

REMARK 1.5.8. If S is affine and $M = S_{\mathbb{Z}}$ is its group completion (recall that M is a lattice and that we have an inclusion $S \subset M = S_{\mathbb{Z}}$), then $\mathsf{k}\lfloor S\rfloor$ is the subalgebra of $\mathsf{k}\lfloor M\rfloor$ generated by the elements χ^s for $s \in S$.

DEFINITION 1.5.9. Let S be an affine monoid and $M = S_{\mathbb{Z}}$ its group completion. Recall that M is a lattice and that we have an inclusion $S \subset M = S_{\mathbb{Z}}$.
(1) Define $Y_S = \mathrm{Spec}(\mathsf{k}\lfloor S\rfloor)$.
(2) If M is a lattice, $\mathcal{S} \subset M$ a finite set and $S = \mathbb{N}\mathcal{S}$, we set $Y_{\mathcal{S}} := Y_S$.

REMARK 1.5.10. Let S be an affine monoid and let $\mathcal{S} \subset S$ be a set of generators of S. Then $S = \mathbb{N}\mathcal{S}$ inside the lattice $M = S_{\mathbb{Z}}$.

PROPOSITION 1.5.11. $\mathsf{k}\lfloor S\rfloor$ *is finitely generated iff* S *is finitely generated.*

Proof. Let $(s_i)_{i\in[1,r]}$ be a generating set of S. We claim that $(\chi^{s_i})_{i\in[1,r]}$ spans $\mathsf{k}\lfloor S\rfloor$ as k-algebra. Since $(\chi^s)_{s\in S}$ spans $\mathsf{k}\lfloor S\rfloor$ as vector space, it is enough to prove the result for χ^s. Write $s = \sum_{i=1}^r \lambda_i s_i$ with $\lambda_i \in \mathbb{N}$, then $\chi_s = \prod_{i=1}^r (\chi^{s_i})^{\lambda_i}$ proving the claim. Conversely let $(f_i)_{i\in[1,r]}$ be a generating set of $\mathsf{k}\lfloor S\rfloor$. Since $(\chi^s)_{s\in M}$ is spanning $\mathsf{k}\lfloor S\rfloor$ as k-vector space, we may assume that $f_i = \chi^{s_i}$ for some $s_i \in S$. Let $s \in S$, then $\chi^s = P(\chi^{s_1}, \cdots, \chi^{s_r})$ for some $P \in \mathsf{k}[X_1, \cdots, X_r]$.

Expanding P and using Lemma 1.2.3, we get that there exist $\lambda_1, \cdots, \lambda_r \in \mathbb{N}$ such that $\chi^s = \chi^{\lambda_1 s_1 + \cdots + \lambda_r s_r}$ and thus $s = \lambda_1 s_1 + \cdots + \lambda_r s_r$, proving the converse. □

EXAMPLE 1.5.12. For $S = \mathbb{N}^n$, then $M = S_{\mathbb{Z}} = \mathbb{Z}^n$. We have $\mathsf{k}\lfloor S \rfloor = \mathsf{k}[X_1, \cdots, X_n] \subset \mathsf{k}\lfloor M \rfloor = \mathsf{k}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$ and $T_M = (\mathbb{G}_m)^n \subset \mathbb{A}^n = Y_S$.

LEMMA 1.5.13. *Let S be an affine monoid and set $M = A_{\mathbb{Z}}$. Consider a generating set $\mathcal{S} = \{s_1, \cdots, s_r\} \subset S \subset M$ of the monoid S. For $i \in [1, r]$ define $\mathcal{S}_i := \mathcal{S} \cup \{-s_i\}$, $S_i = \mathbb{N}\mathcal{S}_i \subset M$ and set $f_i = \chi^{s_i} \in \mathsf{k}\lfloor S \rfloor$.*

(1) *Then $\mathsf{k}\lfloor S_i \rfloor = \mathsf{k}\lfloor S \rfloor [f_i^{-1}]$.*
(2) *The inclusion $\mathsf{k}\lfloor S \rfloor \subset \mathsf{k}\lfloor S_i \rfloor$ induces an open embedding $Y_{S_i} \subset Y_S$.*

Proof. The first statement follows from the fact that $(\chi^s)_{s \in M}$ and $(\chi^t)_{t \in S_i}$ are basis of $\mathsf{k}\lfloor S \rfloor$ and $\mathsf{k}\lfloor S_i \rfloor$ respectively and from the definition of the product which implies the equality $f_i \chi^{-s_i} = 1$. The second statement follows since $\mathsf{k}\lfloor S_i \rfloor$ is the localisation of $\mathsf{k}\lfloor S \rfloor$ at a non-zero divisor. □

COROLLARY 1.5.14. *Let S be an affine monoid and $M = S_{\mathbb{Z}}$. Set $n = \mathrm{rk}(M)$. Then $(\mathbb{G}_m)^n \simeq \mathrm{Spec}(\mathsf{k}\lfloor M \rfloor) \subset Y_S$ is an equivariant partial compactification of $(\mathbb{G}_m)^n$.*

Proof. We only need to prove that the map is equivariant. The action is given by the first line of the following diagram and the equivariance is equivalent to the commutativity of the diagram.

$$\begin{array}{ccc} \mathsf{k}\lfloor S \rfloor & \longrightarrow & \mathsf{k}\lfloor S \rfloor \otimes_{\mathsf{k}} \mathsf{k}\lfloor M \rfloor \\ \big\downarrow & & \big\downarrow \\ \mathsf{k}\lfloor M \rfloor & \longrightarrow & \mathsf{k}\lfloor M \rfloor \otimes_{\mathsf{k}} \mathsf{k}\lfloor M \rfloor, \end{array}$$

with horizontal maps $f \mapsto f \otimes f$. This proves the result. □

REMARK 1.5.15. We shall see later on that the action of $(\mathbb{G}_m)^n$ on the open subset extends to Y_S thus Y_S is an affine pre-toric variety. We shall also see that any affine pre-toric variety is of this form.

DEFINITION 1.5.16. Let S be an affine monoid and let $V = S_{\mathbb{Z}} \otimes \mathbb{R} \supset S_{\mathbb{Z}} \supset S$.

(1) The cone spanned by S in V is defined by
$$\sigma_S = \{\lambda s \in V \mid s \in M, \lambda \in \mathbb{R}_+\}.$$
(2) The saturation of S is the monoid $\widehat{S} = S_{\mathbb{Z}} \cap \sigma_S$.
(3) S is called saturated if $S = \widehat{S}$.

EXAMPLE 1.5.17. We construct all saturated affine monoids as follows.

(1) Let M be a lattice of rank n and $V = M_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\mathcal{S} \subset M$ be any finite set and define the cone σ generated by $\mathcal{S}$ by
$$\sigma = \left\{ \sum_{s \in \mathcal{S}} \lambda_s s \mid \lambda_s \in \mathbb{R}_+ \right\}.$$
Then by definition the monoid $S = \sigma \cap M$ is saturated and torsion free. We will see later that S is finitely generated.

(2) Let $S \subset S_{\mathbb{Z}} = \mathbb{Z}$ be the monoid generated by 2 and 3. Then $S = \mathbb{Z}_+ \setminus \{1\}$ while $\widehat{S} = \mathbb{Z}_+$. The monoid S is not saturated.

LEMMA 1.5.18. *Let S be an affine monoid, $S_{\mathbb{Q}} = S_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $S_{\mathbb{R}} = S_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$.*

(1) *Then $\sigma_S \cap S_{\mathbb{Q}} = \{\lambda s \mid s \in S, \lambda \in \mathbb{Q}_+\}$.*

(2) *We have $\widehat{S} = \{m \in S_{\mathbb{Z}} \mid \exists k \in \mathbb{N} \setminus \{0\} \text{ with } km \in S\}$.*

Proof. (1) Let $s_1, \cdots, s_n$ be generators of S and consider $f : \mathbb{Z}^n \to \mathbb{Z}^r$ defined by $f(x_1, \cdots, x_n) = \sum_{i=1}^n x_i s_i$. Extend f to a map $f : \mathbb{R}^n \to \mathbb{R}^r$. The condition $m \in \sigma_S \cap S_{\mathbb{Q}}$ is equivalent to $m \in f(\mathbb{Q}^n) \cap f(\mathbb{R}^n_+)$ while we want to prove that $m \in f(\mathbb{Q}^n_+)$. Let $K = \mathrm{Ker}(f : \mathbb{Q}^n \to \mathbb{Q}^r)$ and $K_{\mathbb{R}} = \mathrm{Ker}(f) = K \otimes_{\mathbb{Q}} \mathbb{R}$. Recall that there exists $x \in \mathbb{Q}^n$ such that $f(x) = m$. We thus have $f^{-1}(m) \cap \mathbb{Q}^n = x + K$ and $f^{-1}(m) = x + K_{\mathbb{R}}$. Note in particular that $f^{-1}(m) \cap \mathbb{Q}^n$ is dense in $f^{-1}(m)$. First assume that $f^{-1}(m) \cap \mathbb{R}^n_{>0} \neq \emptyset$. This is an open subset thus $f^{-1}(m) \cap \mathbb{R}^n_{>0} \cap \mathbb{Q}^n \neq \emptyset$ and the result follows. Otherwise, up to reordering, we may assume that $f^{-1}(m) \cap (\mathbb{R}^k_{>0} \times \{0\}^{n-k}) \neq \emptyset$. This is an open set in $f^{-1}(m) \cap (\mathbb{R}^k \times \{0\}^{n-k})$ but since $\mathbb{R}^k \times \{0\}^{n-k}$ is defined by equations with rational coefficients we easily check that $f^{-1}(m) \cap (\mathbb{Q}^k \times \{0\}^{n-k})$ is dense in $f^{-1}(m) \cap (\mathbb{R}^k \times \{0\}^{n-k})$ and the same argument gives the result.

(2) Assume that $m \in M_{\mathbb{Z}}$ with $km \in S$ for some $k \in \mathbb{N} \setminus \{0\}$. Then $m \in \sigma_S \cap S_{\mathbb{Z}} = \widehat{S}$. Conversely, assume that $m \in \widehat{S}$, then by (1), we have $m = \lambda s$ for $s \in S$ and $\lambda \in \mathbb{Q}_+$. The result follows easily. □

PROPOSITION 1.5.19. *Let S be an affine monoid.*

(1) *Y_S is irreducible.*

(2) *The normalisation of Y_S is $Y_{\widehat{S}}$.*

(3) *Y_S is normal if and only if S is saturated.*

Proof. Set $M = S_{\mathbb{Z}}$ and $n = \mathrm{rk}(M)$.

(1) Note that $\mathsf{k}\lfloor S \rfloor \subset \mathsf{k}\lfloor M \rfloor \simeq \mathsf{k}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$ and the later is a domain.

(3) follows from (2) since $\mathsf{k}[Y_S] = \mathsf{k}\lfloor S \rfloor$ and $\mathsf{k}[Y_{\widehat{S}}] = \mathsf{k}\lfloor \widehat{S} \rfloor$ have basis given by χ^s and χ^m with $s \in S$ and $m \in \widehat{S}$ respectively. We prove (1). Note that both Y_S and $Y_{\widehat{S}}$ contain $(\mathbb{G}_m)^n$ with $n = \mathrm{rk}(S)$ as a dense open subset thus Y_S and $Y_{\widehat{S}}$ are birational. Furthermore, we have $S \subset \widehat{S}$ thus $\mathsf{k}\lfloor S \rfloor \subset \mathsf{k}\lfloor \widehat{S} \rfloor \subset \mathrm{Frac}(\mathsf{k}\lfloor S \rfloor)$ (where $\mathrm{Frac}(R)$ is the fraction field of the domain R). We need to prove that $\mathsf{k}\lfloor \widehat{S} \rfloor$ is the integral closure of $\mathsf{k}\lfloor S \rfloor$.

Let $m \in \widehat{S}$, then $m \in \sigma_S$ thus there exists $k \in \mathbb{N} \setminus \{0\}$ such that $km \in S$ thus $(\chi^m)^k \in \mathsf{k}\lfloor S \rfloor$. In particular $\mathsf{k}\lfloor \widehat{S} \rfloor$ is integral over $\mathsf{k}\lfloor S \rfloor$. Conversely, let $f \in \mathrm{Frac}(\mathsf{k}\lfloor S \rfloor)$ be integral over $\mathsf{k}\lfloor M \rfloor$. In particular f is integral over $\mathsf{k}\lfloor S_{\mathbb{Z}} \rfloor = \mathsf{k}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$ and thus $f \in \mathsf{k}\lfloor S_{\mathbb{Z}} \rfloor$ since $(\mathbb{G}_m)^n$ is smooth therefore normal. Write $f = \sum_{m \in M} \lambda_m \chi^m$ with $\lambda_m \in \mathsf{k}$. We have a polynomial equation $f^n + x_1 f^{n-1} + \cdots + x_n = 0$ with $x_i \in \mathsf{k}\lfloor S \rfloor$. We may also write $x_i = \sum_{s \in S} \mu_{i,s} \chi^s$. Expanding f and all the x_i and choosing an order on M we get the implication $(\lambda_m \neq 0) \Rightarrow (\exists k \in \mathbb{N}, km \in S) \Leftrightarrow (m \in \widehat{S})$. In particular, we get $f \in \mathsf{k}\lfloor \widehat{S} \rfloor$. □

REMARK 1.5.20. In the next chapter we will give another more geometric proof of the fact that $\mathsf{k}\lfloor S \rfloor$ is normal if S is saturated.

CHAPTER 2

Convex geometric picture

1. Convex polyhedral cones

We have seen that any affine monoid S is generated by a finite set $\mathcal{S} \subset N = S_{\mathbb{Z}}$ thus the datum of a monoid S and a finite generating set $\mathcal{S} \subset S$ is equivalent to the datum $(\mathcal{S} \subset N)$ where N is a lattice and $\mathcal{S}$ is finite (in which case $S = \mathbb{N}\mathcal{S}$). We want to formalise this point of view keeping the pairing between N and $M := N^\vee = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ in mind (the fact that cones live in N and not M comes from the toric picture where the cone lives in the cocharacter lattice $N = \mathfrak{X}_*(T)$ dual to the character lattice $M = \mathfrak{X}^*(T)$).

Let N be a lattice and $M = N^\vee$. Set $V = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $W = M \otimes_{\mathbb{Z}} \mathbb{R}$. We have a natural identification $W = V^\vee = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. We denote by $\langle\ ,\ \rangle : W \times V \to \mathbb{R}$, the natural pairing defined by $\langle w, v\rangle = w(v)$.

Definition 2.1.1. Let V be a $\mathbb{R}$-vector space.

(1) A convex polyhedral cone σ in V is a set
$$\sigma = \left\{ \sum_{i=1}^n r_i v_i \in V \mid \forall i, r_i \in \mathbb{R}_+ \right\}$$
generated by finitely many elements $v_1, \cdots, v_n \in V$.

(2) The half lines $\mathbb{R}_+ v_i$ are called the rays of σ.

(3) The elements $(v_1, \cdots, v_n)$ are called generators of σ. We may replace v_i by any non-zero element in the ray $\mathbb{R}_+ v_i$.

(4) The dimension of σ is the dimension of its $\mathbb{R}$-span: $\langle \sigma \rangle = \sigma + (-\sigma)$.

(5) The dual $\sigma^\vee$ of any set $\sigma \subset V$ is defined by
$$\sigma^\vee = \{w \in W \mid \langle w, v\rangle \geq 0 \text{ for all } v \in \sigma\}.$$

Remark 2.1.2. Let $\sigma \subset V$ be a convex polyhedral cone.

(1) Then σ is convex: if $v, v' \in \sigma$ and $t \in [0,1] \subset \mathbb{R}_+$, then $tv + (1-t)v' \in \sigma$.

(2) Then σ is a cone: if $v \in \sigma$, then $tv \in \sigma$ for any $t \in \mathbb{R}_+$.

(3) Then $\sigma^\vee$ is also a convex cone (exercise).

Lemma 2.1.3. *Let σ be a convex polyhedral cone.*

(1) *If $v_0 \in V \setminus \sigma$, then there exists $w_0 \in \sigma^\vee$ with $\langle w_0, v_0\rangle < 0$.*

(2) *We have $(\sigma^\vee)^\vee = \sigma$.*

Proof. Note that the inclusion $\sigma \subset (\sigma^\vee)^\vee$ holds by definition of the dual cone. We first prove that (1) and (2) are equivalent. Assume that (1) holds, and let $v \notin \sigma$, then there exists $w \in \sigma^\vee$ such that $\langle w, v\rangle < 0$ and $v \notin (\sigma^\vee)^\vee$ proving (2). If (2) holds and $v_0 \notin \sigma$, then $v_0 \notin (\sigma^\vee)^\vee$ and there exists $w_0 \in \sigma^\vee$ with $\langle w_0, v_0\rangle$.

We now prove (1). Let $(-,-)$ be a scalar product on V and let $v_0 \notin \sigma$. Define $d_{v_0} : \sigma \to \mathbb{R}_+$ by $d_{v_0}(c) = (c - v_0, c - v_0)$. One easily checks that d_{v_0} has a positive minimum. Let $c_0 \in \sigma$ such that $d_{v_0}(c_0) = \min_\sigma d_{v_0}$. Note that the minimality implies that $f(t) = d_{v_0}(tc_0)$ with $t > 0$ has a minimum for $t = 1$. This implies that $(c_0, c_0 - v_0) = 0$. Let $w \in W$ be defined by $w(v) = (v, c_0 - v_0) = (v - c_0, c_0 - v_0)$. We have $\langle w, v_0\rangle = w(v_0) = -(v_0 - c_0, v_0 - c_0) < 0$ since $v_0 \notin \sigma$ and $c_0 \in \sigma$. We claim that $w \in \sigma^\vee$. Assume that there exists $c \in \sigma$ such that $\langle w, c\rangle < 0$ *i.e.* $(c - c_0, c_0 - v_0) < 0$. Then $c_t = tc + (1-t)c_0 \in \sigma$ for all $t \in [0, 1]$. We compute

$$d_{v_0}(c_t) = (c_t - v_0, c_t - v_0) = t^2(c - c_0, c - c_0) + 2t(c - c_0, c_0 - v_0) + d_{v_0}(c_0).$$

Since $(c - c_0, c_0 - v_0) < 0$, we get $d_{v_0}(c_t) < d_{v_0}(c_0)$ for small t contradicting the minimality of $d_{v_0}(c_0)$. □

Definition 2.1.4. Let σ be a convex polyhedral cone and $w \in \sigma^\vee$.
(1) A face τ of σ is $\tau = \sigma \cap w^\perp = \{v \in \sigma \mid \langle w, v\rangle = 0\}$. We write $\tau \preceq \sigma$.
(2) Note that σ is a face of itself.
(3) The faces of σ different from σ are called proper faces. We write $\tau \prec \sigma$.

Remark 2.1.5. If $\tau \subset \sigma$ is a face and $v, x \in \sigma$ are such that $v + x \in \tau$, then both v and x are in τ. Indeed, take w such that $\tau = \sigma \cap w^\perp$, then $\langle w, v\rangle + \langle w, x\rangle = 0$ while $\langle w, v\rangle \geq 0$ and $\langle w, x\rangle \geq 0$, thus both are 0.

Lemma 2.1.6. *Let σ be a convex polyhedral cone.*
(1) *Any face of σ is a convex polyhedral cone.*
(2) *Any linear subspace contained in σ is contained in all faces of σ.*
(3) *Any intersection of faces of σ is a face of σ.*
(4) *Any face of a face is a face.*

Proof. (1) Let $v_1, \cdots; v_n$ be generators of σ. Let $I = \{i \in [1, r] \mid v_i \in \tau\}$. We claim that τ is the convex polyhedral cone generated by $(v_i)_{i\in I}$. We have $v_i \in \tau$ for $i \in I$. Let $w \in \sigma^\vee$ such that $\tau = \sigma \cap w^\perp$ and let $v \in \tau$. Write $v = \sum_{i=1}^n r_i v_i$ with $r_i \geq 0$. We have $0 = \langle w, v\rangle = \sum_{i=1}^n r_i\langle w, v_i\rangle = \sum_{i\notin I}^n r_i\langle w, v_i\rangle$. Since $r_i \geq 0$ and $\langle w, v_i\rangle > 0$ for $i \notin I$, we get $r_i = 0$ for all $i \notin I$ and v is in the convex polyhedral cone generated by $(v_i)_{i\in I}$.

(2) Let $w \in \sigma^\vee$, let V_0 is a linear subspace contained in σ and let $v_0 \in V_0 \neq 0$. Then since $v_0, -v_0 \in \sigma$, we have $0 \leq \langle w, v_0\rangle = -\langle w, -v_0\rangle \leq 0$ thus $\langle w, v_0\rangle = 0$ and v_0 lies in all faces of σ.

(3) Let $(w_i)_{i\in I}$ be a family of elements in $\sigma^\vee$ and let $\tau_i = \sigma \cap w_i^\perp$ be the corresponding faces. Let $w = \sum_{i\in I} w_i$. We claim that

$$\bigcap_{i\in I} \tau_i = \sigma \cap w^\perp.$$

If v lies in the left hand side, then $\langle w_i, v\rangle = 0$ for all $i \in I$ thus $\langle w, v\rangle = 0$. If $\langle w, v\rangle = 0$, then $\sum_{i\in I}\langle w_i, v\rangle = 0$ but since $w_i \in \sigma^\vee$, we have $\langle w_i, v\rangle \geq 0$ thus $\langle w_i, v\rangle = 0$ for all $i \in I$.

(4) Let $w \in \sigma^\vee$, $\tau = \sigma \cap w^\perp$, $x \in \tau^\vee$ and $\phi = \tau \cap x^\perp$. Let $v_1, \cdots, v_n$ be generators of σ. Let $I = \{i \in [1, n] \mid v_i \in \tau\}$ and $J = \{i \in I \mid v_i \in \phi\}$. We claim that, for $\lambda \in \mathbb{R}_+$ large enough, we have $\lambda w + x \in \sigma^\vee$ and $\phi = \sigma \cap (\lambda w + x)^\perp$.

If $i \in I$, then $\langle \lambda w + x, v_i\rangle = \langle x, v_i\rangle \geq 0$ since $x \in \tau^\vee$. Otherwise $\langle w, v_i\rangle > 0$ and there exists $\lambda_i \in \mathbb{R}_+$ such that $\lambda_i\langle w, v_i\rangle \geq -\langle x, v_i\rangle$. Let $\lambda > \max_i \lambda_i$. We have $\lambda w + x \in \sigma^\vee$.

If $v \in \phi$, then $\langle w, v\rangle = 0$ and $\langle x, v\rangle = 0$ thus $\langle \lambda w + x, v\rangle = 0$ for all $\lambda \in \mathbb{R}_+$ thus $\phi \subset \sigma \cap (\lambda w + x)^\perp$. Now $\sigma \cap (\lambda w + x)^\perp$ is the polyhedral cone generated by $S = \{v_i \mid \langle \lambda w + x, v_i\rangle = 0\}$. If $v_i \notin \tau$, then by definition of λ, we have $\langle \lambda w + x, v_i\rangle > 0$ and $v_i \notin S$. Thus $S \subset \phi$ and $\sigma \cap (\lambda w + x)^\perp \subset \phi$. □

DEFINITION 2.1.7. A facet of σ is a face of codimension 1.

LEMMA 2.1.8. *Let σ be a convex polyhedral cone.*

(1) *Any proper face is contained in a facet.*
(2) *Any proper face of codimension 2 is the intersection of exactly 2 facets.*
(3) *Any proper face is the intersection of the facets containing it.*
(4) *The relative topological boundary of σ is the union of its proper facets.*

Proof. (1) Let $\tau = \sigma \cap w^\perp$ be a proper face and let $V' \subsetneq V$ be the span of τ. We denote with [ ] the images by the projection $V \to V/V'$. In particular $[\sigma]$ is a convex polyhedral cone (generated by the images of the generators of σ) and $[\tau]$ is the trivial cone: $[\tau] = 0$. Let $\sigma' = [\sigma] \cap \varphi$ with $\varphi \in (V/V')^\vee$ be a facet of $[\sigma]$ and let $w' \in W = V^\vee$ be the linear map defined by the composition $V \to V/V' \xrightarrow{\varphi} \mathbb{R}$. Then $w' \in \sigma^\vee$ and $\sigma \cap (w')^\perp$ is a facet containing τ.

(2) Applying the above construction, we may assume that $\dim \sigma = 2$, $\tau = 0$ and σ contains no linear subspace. The result easly follows from the fact that such cones are generated by 2 elements.

(3) Let $\tau \subset \sigma$ be a proper face. Then $\tau \subset \gamma$ with γ a facet of σ. In particular by induction on $\operatorname{codim}_\sigma \tau$, the fact τ is the intersection of facets in γ by (1). By (2) any facet in γ is the intersection of γ with a facet of σ proving the result.

(4) Replacing V by the span of σ, we may replace relative interior and boundary by interior and boundary. Let $\tau = \sigma \cap w^\perp$ be a proper face (thus $w \neq 0$) and let $v \in \tau$. Then, in any ball centered at v, there exists elements v' with $\langle w, v'\rangle < 0$ thus τ is in the boundary of σ. If $(v_n)_n$ is a sequence of elements outside σ converging to $v \in \sigma$. Then there exists a sequence of elements $(w_n)_n$ in $\sigma^\vee$ (that we may assume of norm 1) such that $\langle w_n, v_n\rangle < 0$. Replacing $(v_n)_n$ and $(w_n)_n$ by subsequences, we may assume that w_n converges to $w \in \sigma^\vee$ of norm 1 and we get $0 \geq \langle w, v\rangle \leq 0$ thus $\langle w, v\rangle = 0$ and v is in a proper face. □

REMARK 2.1.9. By definition, the relative interior and the relative boundary of σ coincide with the interior and the boundary of σ if σ spans V.

DEFINITION 2.1.10. We denote by $\mathring{\sigma}$ the relative interior of σ.

COROLLARY 2.1.11. *We have $\mathring{\sigma} = \{v \in \sigma \mid \langle w, v\rangle > 0$ for all $w \in \sigma^\vee \setminus \sigma^\perp\}$.*

Proof. The relative interior is $\sigma \setminus \cup_{\tau \prec \sigma} \tau$. In particular, if $w \in \sigma^\vee \setminus \sigma^\perp$, then $\sigma \cap w^\perp \prec \sigma$ thus for $v \in \mathring{\sigma}$, we have $\langle w, v\rangle > 0$. Conversely, if $\langle w, v\rangle > 0$ for any such w, then $v \notin \tau$ for all $\tau \prec \sigma$ and the result follows. □

FACT 2.1.12. *Assume that σ spans V and that τ is a facet of σ. Then there is, up to scalars, a unique w with $\tau = \sigma \cap w^\perp$.*

Proof. We know that w vanishes on the span of τ which has codimension 1 in V. The result follows from this. □

DEFINITION 2.1.13. Assume that σ spans V and let $\tau \subset \sigma$ be a facet. Denote by w_τ any non-zero element in $\sigma^\vee$ such that $\tau = \sigma \cap w_\tau^\perp$.

REMARK 2.1.14. If V is endowed with a scalar product identifying V and $W = V^\vee$, we may choose w_τ to be of norm 1. In this case, the vector w_τ is the normal vector to the facet τ.

LEMMA 2.1.15. *Let σ be a convex polyhedral cone s.t. $\sigma \subsetneq V$ spans V. Then*

$$\sigma = \bigcap_{\tau \text{ facet of } \sigma} \{v \in V \mid \langle w_\tau, v\rangle \geq 0\}.$$

Proof. Since $w_\tau \in \sigma^\vee$, we have $\sigma \subset \{v \in V \mid \langle w_\tau, v\rangle \geq 0\}$ for any facet τ of σ. Conversely if v is contained in the rights hand side but not in σ. Let u be an element in the interior of σ. Then there exists v' on the segment from u to v which is on the boundary of σ. In particular v' lies on a facet τ' of σ and $\langle w_{\tau'}, v'\rangle = 0$. Since u is in the interior of σ, we have $\langle w_{\tau'}, u\rangle > 0$. This implies $\langle w_{\tau'}, v\rangle < 0$ contradicting the definition of v. □

As a consequence of the previous result, for σ a convex polyhedral cone we have

COROLLARY 2.1.16. *The cone $\sigma^\vee$ is convex polyhedral.*

Proof. Assume first that σ spans V. If $\sigma = V$, then $\sigma^\vee = 0$ is convex polyhedral, thus we may assume that $\sigma \neq V$. Let γ be the convex polyhedral cone generated by the vectors w_τ where τ runs over all facets of σ. Lemma 2.1.15 implies that $\sigma = \gamma^\vee$ thus $\sigma^\vee = (\gamma^\vee)^\vee = \gamma$ proving the result in this case. If σ spans $V' \subsetneq V$. Let γ be the dual cone of σ in $(V')^\vee$. By what we just proved, γ is a convex polyhedral cone and $\sigma^\vee$ is the inverse image of γ under the projection map $W = V^\vee \to (V')^\vee$ proving the result. □

COROLLARY 2.1.17. *For σ convex polyhedral, there exists $w_1, \cdots, w_n \in W$ with*

$$\sigma = \bigcap_{i=1}^n \{v \in V \mid \langle w_i, v\rangle \geq 0\} = \{v \in V \mid \langle w_1, v\rangle \geq 0, \cdots, \langle w_n, v\rangle \geq 0\}.$$

Proof. Write $\sigma^\vee$ as the cone generated by $w_1, \cdots, w_n \in W$. □

REMARK 2.1.18. To construct $w_1, \cdots, w_n$ as above, choose first a basis of $(\sigma + (-\sigma))^\perp$ and then lift the elements $w_\tau \in (\sigma + (-\sigma))^\vee$ to $V^\vee$ for $\tau \subset \sigma$ a facet.

LEMMA 2.1.19. *Let $w \in \sigma^\vee$ and $\tau = \sigma \cap w^\vee$. Then $\tau^\vee = \sigma^\vee + \mathbb{R}_+(-w)$.*

Proof. Since $\tau^\vee$ and $\sigma^\vee + \mathbb{R}_+(-w)$ are convex polyhedral cones, it is enough to prove that their duals are equal. We have $(\tau^\vee)^\vee = \tau$ while $(\sigma^\vee + \mathbb{R}_+(-w))^\vee = (\sigma^\vee)^\vee \cap w^\perp = \sigma \cap w^\perp = \tau$ proving the result. □

There is a nice order reversing correspondence between faces of σ an $\sigma^\vee$.

PROPOSITION 2.1.20. *Let $\tau, \gamma \subset \sigma$ be two faces.*

(1) *Then $\tau^* = \sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$ and any face of $\sigma^\vee$ is of this form.*
(2) *For $\gamma \subset \tau$, we have $\tau^* \subset \gamma^*$.*
(3) *$\tau \mapsto \tau^*$ defines an order reversing bijection between faces of σ and $\sigma^\vee$.*
(4) *We have $(\tau^*)^* = \tau$.*
(5) *We have $\dim \tau + \dim \tau^* = \dim V$.*

Proof. (1) A face of $\sigma^\vee$ is of the form $\sigma^\vee \cap v^\perp$ for $v \in (\sigma^\vee)^\vee = \sigma$. Let τ be the unique face of σ containing v in its relative interior. We claim that $\tau^* = \sigma^\vee \cap v^\perp$. Since $v \in \tau$, we get $\tau^* \subset \sigma^\vee \cap v^\perp$. Assume that $w \in (\sigma^\vee \cap v^\perp) \setminus \tau^*$. Then v is in the face $\tau \cap w^\perp$ of τ but the assumption implies that there exists $v' \in \tau$ such that $\langle w, v'\rangle > 0$ and v would be in a proper face of τ, a contradition.

(2) The inclusion $\gamma \subset \tau$ implies $\tau^\perp \subset \gamma^\perp$ and the result.

(3) and (4) By (1), the map $\tau \mapsto \tau^*$ is surjective. We have an easy inclusion $\tau \subset (\tau^*)^*$ and thus $\tau^* \supset ((\tau^*)^*)^*$ (using (2)) but also $\tau^* \subset ((\tau^*)^*)^*$ (applying $\tau \subset (\tau^*)^*$ to τ^*) thus $\tau^* = ((\tau^*)^*)^*$. In particular, the map $\tau^* \mapsto (\tau^*)^*$ is injective. The result follows as well as the equality $\tau = (\tau^*)^*$.

(5) By (3), the smallest face of σ is $\gamma = (\sigma^\vee)^* = \sigma \cap (\sigma^\vee)^\perp = (\sigma^\vee)^\perp$ thus $\dim \gamma + \dim \gamma^* = \dim(\sigma^\vee)^* + \dim \sigma^\vee = \dim(\sigma^\vee)^\perp + \dim \sigma^\vee = \dim V$. The result follows by induction since if $\gamma \subset \tau$ is an inclusion of faces with $\dim \tau = \dim \gamma + 1$, we must have $\dim \tau^* = \dim \gamma^* - 1$ (use the order reversing bijection and the fact that we have chains of codimension 1 faces). □

Note that the same proof as (1) in Proposition 2.1.20 gives the following result.

LEMMA 2.1.21. *Let τ be a face, then $\tau = \sigma \cap w^\perp$ for w in $(\mathring{\tau^*})$.*

Proof. Let w in the relative interior of τ^*, then the proof of (1) in Proposition 2.1.20 gives that $\tau = (\tau^*)^* = (\sigma^\vee)^\vee \cap w^\perp = \sigma \cap w^\perp$. □

PROPOSITION 2.1.22. *Let σ and σ' be convex polyhedral cones such that $\tau = \sigma \cap \sigma'$ is a face of both. There exists $w \in \sigma^\vee \cap (-\sigma')^\vee$ such that $\tau = \sigma \cap w^\perp = \sigma' \cap w^\perp$.*

REMARK 2.1.23. In particular σ and σ' are contained in the opposite half-spaces defined by the hyperplane $H_w = \{v \in V \mid \langle w, v\rangle = 0\}$ while $\tau \subset H_w$. The hyperplane H_w (and thus the face τ) separates σ and σ'.

Proof. Let $\gamma = \sigma - \sigma'$ be the cone generated by σ and $-\sigma'$ and let w in the relative interior of $\gamma^\vee$. We claim that $\tau = \sigma \cap w^\perp = \sigma' \cap w^\perp$.

Note that $\gamma \cap w^\perp$ is the minimal face of γ thus $\gamma \cap w^\perp = (\gamma) \cap (-\gamma) = (\sigma - \sigma') \cap (\sigma' - \sigma)$. Since $\tau = \sigma \cap \sigma'$ and because we have $\sigma \subset \sigma - \sigma'$ and $\sigma' \subset \sigma' - \sigma$, we get $\tau \subset \gamma \cap w^\perp \subset w^\perp$. In particular, we have $\tau \subset \sigma \cap w^\perp$ and $\tau \subset \sigma' \cap w^\perp$.

Conversely, let $v \in \sigma \cap w^\perp$, then $v \in \gamma \cap w^\perp = (\sigma - \sigma') \cap (\sigma' - \sigma)$. In particular $v \in \sigma' - \sigma$. Write $v = x' - x$ with $x' \in \sigma'$ and $x \in \sigma$. We have $x' = v + x \in \sigma' \cap \sigma = \tau$. Thus v and x are both elements of σ such that the sum is in a face τ, this implies that both v and x are in τ (see Remark 2.1.5), in particular $v \in \tau$. Thus $\sigma \cap w^\perp \subset \tau$. The same argument using $(-w)$ gives the inclusion $\sigma' \cap w^\perp \subset \tau$ □

Let σ be a convex polyhedral cone.

DEFINITION 2.1.24. σ is strongly convex if it contains no non-trivial subspace.

PROPOSITION 2.1.25. *The following are equivalent*

(1) *σ is strongly convex*
(2) $\sigma \cap (-\sigma) = 0$
(3) *There exists $w \in \sigma^\vee$ with $\sigma \cap w^\perp = 0$.*
(4) *$\sigma^\vee$ spans W.*

Proof. (1) $\Leftrightarrow$ (2) follows from the fact that $\sigma \cap (-\sigma)$ is the maximal linear subspace contained in σ. Any w in the relative interior of $\sigma^\vee$ gives $\sigma \cap w^\perp = \sigma \cap (-\sigma)$ which is the minimal face of σ proving (2) $\Leftrightarrow$ (3). This last assertion gives $(\sigma \cap (-\sigma))^* = \sigma^\vee$ thus $\dim \sigma^\vee = n - \dim(\sigma \cap (-\sigma))$ proving (2) $\Leftrightarrow$ (4). $\square$

Corollary 2.1.26. *If σ is strongly convex of dimension n, then so is $\sigma^\vee$.*

Corollary 2.1.27. *Assume that σ is strongly convex and let $v_1, \cdots, v_n$ be a minimal set of generators.*

(1) *Then the one-dimensional faces of σ are of the form $\mathbb{R}_+ v_i$ for $1 \leq i \leq n$.*
(2) *The minimal generators are unique up to scalars.*
(3) *Minimal families of generators have the same cardinality.*

Proof. (2) and (3) follow from (1). Let $\mathbb{R}_+ v$ be a one dimensional face. Write $v = \sum_i \lambda_i v_i$ with $\lambda_i \in \mathbb{R}_+$. Since $\lambda_1 v_1$ and $\sum_{i \geq 2} \lambda_i v_i$ are in σ, Remark 2.1.5 and an easy induction implies that there exists an i with $v_i \in \mathbb{R}_+ v$. By minimality, v_i is unique and non-zero thus $\mathbb{R}_+ v = \mathbb{R}_+ v_i$ for a unique i. We prove that $\mathbb{R}_+ v_i$ is indeed a face, for simplicity take $i = 1$. Consider the cone σ' generated by $v_2, \cdots, v_n$. By minimality $v_1 \notin \sigma'$ thus there exists $w \in (\sigma')^\vee$ such that $\langle w, v_1 \rangle < 0$. Let $w' \in \mathring{\sigma}$, then $\langle w', v_i \rangle > 0$ for all i. Set $w'' = w - \frac{\langle w, v_1 \rangle}{\langle w', v_1 \rangle} w'$. Then $\langle w'', v_1 \rangle = 0$ and $\langle w'', v_i \rangle > 0$ for all $i \geq 2$ thus $w'' \in \sigma^\vee$ and $\mathbb{R}_+ v_1 = \sigma \cap (w'')^\perp$ is a face of σ. $\square$

Definition 2.1.28. A simplex is a cone spaned by linearly independent vectors.

2. Rational convex polyhedral cones

We now add an ingredient to the picture. Assume that N is a lattice such that $V = N \otimes_Z \mathbb{R}$. Note that $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ is a lattice spanning $W = V^\vee$. Let $\sigma \subset V$ be a convex polyhedral cone.

Definition 2.2.1. σ is called rational if its generators can be chosen in N.

Remark 2.2.2. Note that since we have a canonical lattice M in W, we may also define rational convex polyhedral cones in W.

Lemma 2.2.3. *If σ is rational, so is $\sigma^\vee$.*

Proof. If σ spans $V' \subsetneq V$, we may add elements in $M \cap (V')^\perp$ spanning $(V')^\perp$ in the generating set of $\sigma^\vee$ so we may assume that σ spans V. In that case we only need to check that we can choose w_τ in M which is clear. $\square$

We define monoids that will play a prominent role.

Definition 2.2.4. For $\sigma \subset V$ a rational polyhedral cone, set $S_\sigma = \sigma^\vee \cap M$.

Proposition 2.2.5. *If σ is rational, then S_σ is a finitely generated monoid.*

Remark 2.2.6. Note furthermore that $S_\sigma = \sigma^\vee \cap M$ is saturated. In particular any affine saturated monoid is obtained this way.

Proof. Let $w_1, \cdots, w_n \in \sigma^\vee \cap M$ be generators of $\sigma^\vee$ and consider the set $A = \{\sum_{i=1}^n a_i w_i \mid a_i \in [0,1]\}$. Then A is compact and since M is discrete, the intersection $\mathcal{S} = A \cap M$ is finite. Note that $w_i \in \mathcal{S}$ for all i. We claim that $\mathbb{N}\mathcal{S} = S_\sigma$. Since $A \subset \sigma^\vee$, we have $\mathcal{S} \subset S_\sigma$ thus $\mathbb{N}\mathcal{S} \subset S_\sigma$. Conversely, let $w \in \sigma^\vee \cap M$ and write $w = \sum_{i=1}^n \lambda_i w_i$ with $\lambda_i \in \mathbb{R}_+$. Write $\lambda_i = k_i + a_i$ with $a_i \in \mathbb{N}$ and $a_i \in [0,1]$. Then $w = w' + w''$ with $w' = \sum_{i=1}^n k_i w_i \in \mathbb{N}\mathcal{S}$ and $w'' = \sum_{i=1}^n a_i w_i \in A$. But $w'' = w - w' \in M$ thus $w'' \in \mathcal{S}$ and the result follows. $\square$

COROLLARY 2.2.7. *The variety* $Y_{S_\sigma} = \mathrm{Spec}(\mathsf{k}\lfloor S_\sigma \rfloor)$ *is a normal variety.*

DEFINITION 2.2.8. We set $U_\sigma = Y_{S_\sigma}$.

REMARK 2.2.9. As in Remark 1.5.15, we shall see later on that U_σ is an affine toric variety and that any toric variety is of this form.

PROPOSITION 2.2.10. *Let* σ *be a rational convex polyhedral cone and* $w \in S_\sigma$.

(1) *Then* $\tau = \sigma \cap w^\perp$ *is a rational convex polyhedral cone.*
(2) *All faces of* σ *are of this form.*
(3) $S_\tau = S_\sigma + \mathbb{Z}_+(-w)$.

Proof. (1) Recall that if $v_1, \cdots, v_n$ are elements in N generating σ, then τ is generated by the subfamily $\{v_i \mid 1 \leq i \leq n$ and $\langle w, v_i\rangle = 0$ thus τ is rational.

(2) Lemma 2.1.21 implies that if w lies in the interior of τ^*, then $\sigma \cap w^\perp = \tau$ thus we only need to find $w \in \tau^* \cap M$ which is possible since τ^* is rational (take a positive linear combination of its generators).

(3) We have $S_\tau = \tau^\vee \cap M$ and, by Lemma 2.1.19, we have $\tau^\vee = \sigma^\vee + \mathbb{R}_+(-w)$ thus $S_\sigma + \mathbb{Z}_+(-w) \subset S_\tau$. Conversely, since $S_\tau \subset \sigma^\vee + \mathbb{R}_+(-w)$, for any $w' \in S_\tau$, there exists $k \in \mathbb{N}$ with $w' + ku \in \sigma^\vee$ thus $S_\tau \subset S_\sigma + \mathbb{Z}_+(-w)$. □

PROPOSITION 2.2.11. *If* σ *and* σ' *are rational convex polyhedral and* $\tau = \sigma \cap \sigma'$ *is a common face, then* $S_\tau = S_\sigma + S_{\sigma'}$.

Proof. Since $\tau \subset \sigma$, we get $\sigma^\vee \subset \tau^\vee$ and the inclusion $S_\sigma \subset S_\tau$. The same argument gives $S_{\sigma'} \subset S_\tau$ thus $S_\sigma + S_{\sigma'} \subset S_\tau$. Recall from the proof of Proposition 2.1.22 that picking w in the relative interior of $\gamma^\vee$ with $\gamma = \sigma - \sigma'$, we have $w \in \sigma^\vee \cap (-\sigma')^\vee$ and $\tau = \sigma \cap w^\perp = \sigma' \cap w^\perp$. Note furthermore that since γ is rational, we may take $w \in M$. Now Proposition 2.2.10 gives $S_\tau = S_\sigma + \mathbb{Z}_+(-w)$. Since $-w \in S_{\sigma'}$, we get the result. □

DEFINITION 2.2.12. Let σ be rational and $m \in S_\sigma$. Call m irreducible if $m = m' + m''$ with $m', m'' \in S_\sigma$ implies $m' = m$ or $m' = m''$.

PROPOSITION 2.2.13. *Let* σ *be rational, strongly convex of maximal dimension. Set* $\mathcal{H} = \{m \in S_\sigma \mid m$ *is irreducible*$\}$. *Then* $\mathcal{H}$ *has the following properties*

(1) $\mathcal{H}$ *is finite and generates* S_σ.
(2) $\mathcal{H}$ *contain the generator of* $S_\tau = \tau^\vee \cap M$ *for any facet* $\tau \prec \sigma$.
(3) $\mathcal{H}$ *is the minimal generating set of* S_σ.

Proof. We first prove that $\mathcal{H}$ generates S_σ. Let $v \in \mathring{\sigma}$, then for any $m \in S_\sigma \setminus \{0\}$, we have $\langle m, v\rangle \in \mathbb{N} \setminus \{0\}$ thus by induction over $\langle m, v\rangle$ on easily check that m is a sum of elements in $\mathcal{H}$.

Let $m_1, \cdots, m_n$ be a generating set of S_σ and $m \in \mathcal{H}$. Then $m = \sum_i \lambda_i m_i$ with $\lambda_i \in \mathbb{N}$. Since m is irreducible, an easy induction implies that $m = m_i$ for some i. In particular, $\mathcal{H} \subset \{m_1, \cdots, m_n\}$ is finite and is a minimal generating set of S_σ, this finishes the proof of (1) and (3).

Let $\tau^\vee$ be a one dimensional face of σ and $m \in S_\tau$ be the minimal element. Since $\mathcal{H}$ generates S_σ, there exist $m' \in \mathcal{H}$ and $m'' \in S_\sigma$ such that $m = m' + m''$. Since $\tau^\vee$ is a face, we get that $m', m'' \in S_\tau$ and in particular m' is a multiple of m. Since m' is irreducible, we must have $m' = m$, proving (2). □

EXAMPLE 2.2.14. Let $N = \mathbb{Z}^2$ and σ be the cone generated by $(1,0)$ and $(3,2)$. Identify M to $\mathbb{Z}^2 = N$ using the standard scalar product. Then $\mathcal{H} = \{(0,1),(1,-1),(2,-3)\}$. In particular $\mathcal{H}$ contains more elements than the generator of $S_\tau = \tau^\vee \cap M$ for any facet $\tau \prec \sigma$.

We now define the analog of strongly convex for monoids.

DEFINITION 2.2.15. A monoid S is pointed of $S \cap (-S) = 0$.

3. Affine toric varieties via cones

We have seen that for σ a rational convex polyhedral cone, we will see that the scheme $U_\sigma = \mathrm{Spec}(\mathsf{k}\lfloor S_\sigma \rfloor)$ is an affine toric variety. We want to study the relations between the properties of σ and U_σ.

PROPOSITION 2.3.1. *Let σ be a rational polyhedral cone and $n = \mathrm{rk}(M)$.*

1. *T_M is a torus of dimension n.*
2. *T_M acts on U_σ.*
3. *U_σ is a toric variety.*
4. $\dim U_\sigma = \dim \sigma^\vee$.

Proof. (1) Since M is a lattice, this follows from Proposition 1.2.9.

(2) Assume first that $M = (S_\sigma)_\mathbb{Z}$. By Lemma 1.5.13, the inclusion $\mathsf{k}\lfloor S_\sigma \rfloor \to \mathsf{k}\lfloor M \rfloor$ gives an open embedding $(\mathbb{G}_m)^n \simeq T_M \subset U_\sigma$. The map $\mathsf{k}\lfloor S_\sigma \rfloor \to \mathsf{k}\lfloor M \rfloor \otimes_\mathsf{k} \mathsf{k}\lfloor S_\sigma \rfloor, \chi^s \mapsto \chi^s \otimes \chi^s$ defines an action of T_M on U_σ. This map extends to $\mathsf{k}\lfloor M \rfloor \to \mathsf{k}\lfloor M \rfloor \otimes_\mathsf{k} \mathsf{k}\lfloor M \rfloor, \chi^m \mapsto \chi^m \otimes \chi^m$ which is the map defining the multiplication on T_M, proving the result.

In general, let $M' = (S_\sigma)_\mathbb{Z} \subset M$, then U_σ is a toric variety for the torus $T_{M'}$. Note that T_M acts on U_σ via the map $T_M \to T_{M'}$ induced by the inclusion $\mathsf{k}\lfloor M' \rfloor \subset \mathsf{k}\lfloor M \rfloor$. Note in particular that $\dim U_\sigma = \dim T_{M'} = \mathrm{rk} M' = \dim \sigma^\vee$ proving (4).

(3) Follows from Proposition 1.5.19 ☐

PROPOSITION 2.3.2. *For $\tau \preceq \sigma$, we have an open embedding $U_\tau \subset U_\sigma$.*

Proof. Write $\tau = \sigma \cap w^\perp$ with $w \in S_\sigma$. Proposition 2.2.10 gives the equality $S_\tau = S_\sigma + \mathbb{Z}_+(-w)$. The result follows from Lemma 1.5.13. ☐

PROPOSITION 2.3.3. *Assume that σ is generated by a part of a basis of N.*

1. *Then $U_\sigma \simeq (\mathbb{G}_a)^{\dim \sigma} \times (\mathbb{G}_m)^{n - \dim \sigma}$.*
2. *In particular U_σ is smooth.*

Proof. Let $e_1, \cdots, e_n$ be a basis of N such that $e_1, \cdots, e_k$ generate σ. We have $\dim \sigma = k$. Let $m_1, \cdots, m_n$ be the dual basis in M. We have $S_\sigma = \sum_{i=1}^k \mathbb{N} m_i + \sum_{i=k+1}^n \mathbb{Z} m_i$, thus $\mathsf{k}\lfloor S_\sigma \rfloor = \mathsf{k}[X_1, \cdots, X_k, X_{k+1}^{\pm 1}, \cdots, X_n^{\pm 1}]$ proving the result. ☐

EXAMPLE 2.3.4. Recall Example 2.2.14: $N = \mathbb{Z}^2$ and σ is the cone generated by $(1,0)$ and $(3,2)$. Identifying M to $\mathbb{Z}^2$ using the standard scalar product, we have $S_\sigma = \mathbb{N}(0,1) + \mathbb{N}(1,-1) + \mathbb{N}(2,-3)$. We have a surjective map $f : \mathsf{k}[X,Y,Z] \to \mathsf{k}\lfloor S_\sigma \rfloor$ defined by $f(X) = \chi^{(0,1)}$, $f(Y) = \chi^{(1,-1)}$ and $f(Z) = \chi^{(2,-3)}$. Since $(0,1) + (3,-2) = 2(1,-1)$, we have $f(XZ - Y^2) = 0$. An easy computation proves that $\mathrm{Ker}(f) = (XZ - Y^2)$ thus U_σ is a quadratic cone of dimension 2. The jacobian criterion proves that U_σ is not smooth (but normal).

EXAMPLE 2.3.5. Let $N = \mathbb{Z}^3 = M$ with identification given by the standard scalar product. Let σ be the cone generated by $n_1 = (1,1,0)$, $n_2 = (1,0,1)$, $n_3 = (1,-1,0)$ and $n_4 = (1,0,-1)$. An easy computation gives that $\mathcal{H}$ has four element: $(1,1,1)$, $(1,1,-1)$, $(1,-1,1)$ and $(1,-1,-1)$. We thus have a surjective map $f : \mathsf{k}[X,Y,Z,T] \to \mathsf{k}\lfloor S_\sigma \rfloor$ defined by $f(X) = \chi^{(1,1,1)}$, $f(Y) = \chi^{(1,1,-1)}$, $f(Z) = \chi^{(1,-1,1)}$ and $f(T) = \chi^{(1,-1,-1)}$. On can check that $\mathrm{Ker}(f) = (XT - YZ)$ thus U_σ is a quadratic cone of dimension 3 and again singular.

DEFINITION 2.3.6. σ is smooth if it is generated by a part of a basis of N.

REMARK 2.3.7. We shall see later on that U_σ is smooth iff σ is smooth.

PROPOSITION 2.3.8. *Any affine toric variety is of the form U_σ.*

Proof. Let $X = \mathrm{Spec}(A)$ be an affine toric variety. Let $T = \mathrm{Spec}(\mathsf{k}\lfloor M \rfloor)$ be the torus having a dense orbit in X. Up to taking a quotient of T (which is again a torus), we may assume that T is isomorphic to its open dense orbit in X. Let $M = \mathfrak{X}^*(T)$ be the character group of T, then $T = \mathrm{Spec}(\mathsf{k}\lfloor M \rfloor)$. The action of T on X induces an action of T on A via $t.f(x) = f(t^{-1}.x)$ for $t \in T$, $f \in A$ and $x \in X$. By Theorem 1.3.16, the T-module A is diagonalisable *i.e.* is a direct sum of eigenspaces: $A = \oplus_{m \in M} A_m$ with $A_m = \{f \in A \mid t.f = m(t)f, \forall t \in T\}$. Let $x_0 \in X$ be in the dense T-orbit and let $f, g \in A_m \setminus \{0\}$. Then $f(t.x) = t^{-1}.f)(x) = m(t^{-1})f(x)$ thus $f(x) \neq 0$. The same argument implies $g(x) \neq 0$. Let $\lambda = f(x)/g(x) \in \mathsf{k}^\times$, then $g(t.x) = m(t^{-1})g(x) = \lambda m(t^{-1})f(x) = \lambda f(t.x)$. Since $T.x$ is dense in X and X is separated, we get that $g = \lambda f$. In particular $\dim_{\mathsf{k}}(A_m) \leq 1$. Let $S = \{m \in M \mid A_m \neq 0\}$ and for $m \in S$, let $\chi^m \in A_m$ such that $\chi^m(x) = 1$. Note that S is an affine monoid contained in M. The restriction of χ^m to $T \simeq T.x$ is a character of T and $A = \oplus_{m \in S} \mathsf{k}\chi^m = \mathsf{k}\lfloor S \rfloor$. Since X is normal, S is saturated and therefore of the form S_σ for some convex polyhedral cone $\sigma \subset N = M^\vee$. □

4. Local properties of affine toric varieties

We study the local properties of affine toric varieties. Since any toric variety will be covered by affine toric varieties, this will be anough to characterise toric varieties having these properties. We start with closed points in U_σ.

PROPOSITION 2.4.1. *The closed points in U_σ are in bijection with the morphisms of monoids $S_\sigma \to (\mathsf{k}, \times)$.*

Proof. A closed point in U_σ is a k-algebra morphism $\mathsf{k}\lfloor S_\sigma \rfloor \to \mathsf{k}$. The restriction to the basis $(\chi^s)_{s \in \S_\sigma}$ induces a morphism of monoids $S_\sigma \to (\mathsf{k}, \times)$. Conversely given a morphism of monoids $\varphi : S_\sigma \to (\mathsf{k}, \times)$ define $f : \mathsf{k}\lfloor S_\sigma \rfloor \to \mathsf{k}$ as the linear map with $f(\chi^s) = \varphi(s)$. The fact that φ is a morphism of monoids implies that f is a k-algebra morphism. □

EXAMPLE 2.4.2. Let $\sigma = 0$ thus $S_\sigma = M$ is the character lattice of T_M. Choose a basis $(e_1, \cdots, e_n)$ of M, then the closed points of U_σ are in bijection with $(\mathbb{G}_m)^n$ given by the morphisms of monoids $\varphi : M \to \mathsf{k}$ defined by $\varphi(e_i) = a_i \in \mathsf{k}^\times$ and $\varphi(-e_i) = a_i^{-1}$. The corresponding k-algebra morphisms $g : \mathsf{k}\lfloor M \rfloor \to \mathsf{k}$ are given by $g(\chi^{e_i}) = a_i$. In particular the unit of T_M is the closed point p_0 corresponding to $\varphi(e_i) = 1$ for all i.

LEMMA 2.4.3. *Let $p \in U_\sigma$ be a closed point associated to the morphism of monoids $\gamma : S_\sigma \to (\mathsf{k}, \times)$ and let $t \in T$. Then the point $t.p$ is represented by the morphism of monoids $S_\sigma \to (\mathsf{k}, \times), v \mapsto \chi^v(t)\gamma(t)$.*

Proof. We first claim that $t.\gamma : S_\sigma \to (\mathsf{k}, \times), v \mapsto \chi^v(t)\gamma(t)$ is indeed a morphism of monoids. We have $(t.\gamma)(v + v') = \chi^{v+v'}(t)\gamma(v + v') = (\chi^v\chi^{v'})(t)\gamma(v)\gamma(v') = (t.\gamma)(v)(t.\gamma)(v')$ proving the claim. Now recall that the action is given by the map $\mathsf{k}\lfloor S_\sigma\rfloor \to \mathsf{k}\lfloor M\rfloor \otimes \mathsf{k}\lfloor S_\sigma\rfloor, \chi^v \mapsto \chi^v \otimes \chi^v$ and the points $t \in T$ and $p \in U_\sigma$ by the maps $\mathsf{k}\lfloor M\rfloor \to \mathsf{k}, \chi^v \to \chi^v(t)$ and $\mathsf{k}\lfloor S_\sigma\rfloor \to \mathsf{k}, \chi^v \mapsto \gamma(v)$. The result follows from this. □

Note that T_M has no T_M-fixed point for $\dim T_M > 0$.

PROPOSITION 2.4.4. *Let σ be a rational convex polyhedral cone.*

(1) *U_σ has a closed T_M-fixed point if and only if S_σ is pointed.*
(2) *U_σ has a closed T_M-fixed point if and only if $\dim\sigma = \mathrm{rk}(M)$.*
(3) *In that case, the T_M-fixed point is given by the morphism of monoids $\varphi : S_\sigma \to \mathsf{k}$ defined by $\varphi(0) = 1$ and $\varphi(s) = 0$ for all $s \in S_\sigma \setminus \{0\}$ and the corresponding maximal ideal is $\oplus_{s\in S_\sigma\setminus\{0\}}\mathsf{k}\chi^s$.*

Proof. Let $f : \mathsf{k}\lfloor S_\sigma\rfloor \to \mathsf{k}$ be a k-algebra morphism corresponding to a closed point $u \in U_\sigma$ and $g : \mathsf{k}\lfloor M\rfloor \to \mathsf{k}$ corresponding to a closed point $t \in T_M$. The closed point $t.u \in U_\sigma$ is given by $\mathsf{k}\lfloor S_\sigma\rfloor \to \mathsf{k}, \chi^s \mapsto f(\chi^s)g(\chi^s)$. The point u is T_M-fixed if and only if $f(\chi^s)g(\chi^s) = f(\chi^s)$ for all $s \in S_\sigma$ and all g. For s non-trivial, we may choose g with $g(\chi^s) \neq 1$ (see Example 2.4.2), we get $f(\chi^s) = 0$ for $s \neq 0$ while $f(\chi^0) = f(1) = 1$.

To conclude, we claim that the morphism of monoids $\varphi : S_\sigma \to \mathsf{k}$ defined by $\varphi(0) = 1$ and $\varphi(s) = 0$ for all $s \in S_\sigma \setminus \{0\}$ exists if and only if S_σ is pointed. Indeed, if φ exists and $m, -m \in S_\sigma$, then $\varphi(m)\varphi(-m) = 1$ and we must have $m = 0$. Conversely, if S_σ is pointed, consider the map $\varphi : S_\sigma \to \mathsf{k}$ defined by $\varphi(0) = 1$ and $\varphi(s) = 0$ for all $s \in S_\sigma \setminus \{0\}$. We check that it is a morphism of monoids. Let $s, s' \in S_\sigma$, we need to prove that $\varphi(s + s') = \varphi(s)\varphi(s')$. If $0 \in \{s, s'\}$, an easy computation proves the claim. If both are non-zero, then $s + s' \neq 0$ since S_σ is pointed and the result follows.

Finally note that S_σ is pointed if and only if $\dim\sigma = \mathrm{rk}(M)$ *i.e.* σ is of maximal dimension. □

REMARK 2.4.5. If σ is not strongly convex, then the action of the torus T_M factors through the action of its quotient $T_\sigma = \mathrm{Spec}(\mathsf{k}\lfloor\langle\sigma^\vee\rangle \cap M\rfloor)$ (where $\langle\sigma^\vee\rangle$ is the span of $\sigma^\vee$ in $M_\mathbb{R}$) and U_σ is a T_σ-toric variety. In particular, we may assume that σ is strongly convex.

More precisely, we have the following statement.

LEMMA 2.4.6. *T_M is isomorphic to an open orbit in U_σ iff σ is strongly convex.*

Proof. The open T_M-orbit is given by $\mathrm{Spec}(\mathsf{k}\lfloor(S_\sigma)_\mathbb{Z}\rfloor)$ where $(S_\sigma)_\mathbb{Z}$ is the lattice generated by S_σ. The statement is equivalent to $(S_\sigma)_\mathbb{Z} = M$ iff σ is strongly convex. If $(S_\sigma)_\mathbb{Z} = M$, then $\mathrm{Rank}(S_\sigma)_\mathbb{Z} = \mathrm{rk}(M) = n$ thus $\sigma^\vee$ is of maximal dimension and σ is strongly convex. Conversely, if σ is strongly convex, then $\sigma^\vee$ is of maximal dimension and by definition $S_\sigma = \sigma^\vee \cap M = \{m \in M \mid \langle m, v\rangle \geq 0 \text{ for } v \in \sigma\}$ thus $(S_\sigma)_\mathbb{Z}$ is a sublattice of M of maximal rank. We only need to check that

$M/(S_\sigma)_{\mathbb{Z}}$ is torsion free. If $m \in M$ is such that there is $k > 1$ with $km \in (S_\sigma)_{\mathbb{Z}}$, then $km = m_1 - m_2$ for $m_1, m_2 \in S_\sigma = \sigma^\vee \cap M$. We have $m_3 = m + m_2 = \frac{1}{k}m_1 + \frac{k-1}{k}m_2 \in \sigma^\vee \cap M = S_\sigma$ thus $m = m_3 - m_2 \in (S_\sigma)_{\mathbb{Z}}$ proving the claim. □

We now want to characterise smooth affine toric varieties. Recall that Proposition 1.5.19 implies that U_σ is always normal.

Remark 2.4.7. We give an alternative proof of the fact that $\mathrm{Spec}(\mathsf{k}\lfloor S \rfloor)$ is normal if S is saturated (see Proposition 1.5.19 and Remark 1.5.20) using our results on cones. Assume that S is saturated and let $\sigma^\vee$ be the cone generated by σ, then $S = S_\sigma$ for $\sigma = (\sigma^\vee)^\vee$. Let $\rho_1, \cdots, \rho_r$ be the rays of σ. Since σ is generated by its rays, we have $\sigma^\vee = \cap_{i=1}^r \rho_i^\vee$ and therefore $S_\sigma = \cap_{i=1}^r S_{\rho_i}$. We thus have

$$\mathsf{k}\lfloor S_\sigma \rfloor = \bigcap_{i=1}^r \mathsf{k}\lfloor S_{\rho_i} \rfloor$$

and a classical results of commutative algebra implies that if $\mathsf{k}\lfloor S_{\rho_i} \rfloor$ is normal for all i, then so is $\mathsf{k}\lfloor S_\sigma \rfloor$. For simplicity assume $i = 1$ and let e_1 be a generator of ρ_1. Then (prove this as an exercise), we can complete e_1 so that $e_1, \cdots, e_n$ is a basis of N such. We get that $\mathsf{k}\lfloor S_{\rho_1} \rfloor = \mathsf{k}[x_1, x_2^{\pm 1}, \cdots, x_n^{\pm 1}] \supset \mathsf{k}[x_1, \cdots, x_n]$. Taking the spectra, U_{ρ_1} is an open subset of $\mathbb{A}^n$ which is smooth, proving the result.

We start with a description of the tangent space of U_σ at its closed fixed point. Recall that U_σ has a T_M-fixed point if and only if σ is of maximal dimension. We denote this fixed point by $p_\sigma \in U_\sigma$. Recall the definition and properties of $\mathcal{H} \subset S_\sigma$ the set of irreducible elements from Definition 2.2.12 and Proposition 2.2.13.

Lemma 2.4.8. *Let σ be strongly convex with $\dim \sigma = n$, then $|\mathcal{H}| = \dim T_{p_\sigma} U_\sigma$.*

Proof. Let $\mathfrak{m} \subset \mathsf{k}\lfloor S_\sigma \rfloor$ be the maximal ideal of p_σ. We compute $\mathfrak{m}/\mathfrak{m}^2$ which is the dual space to $T_{p_\sigma} U_\sigma$. We have

$$\mathfrak{m} = \bigoplus_{s \in S_\sigma \setminus \{0\}} \mathsf{k}\chi^s = \bigoplus_{s \text{ irreducible}} \mathsf{k}\chi^s \oplus \bigoplus_{s \text{ reducible}} \mathsf{k}\chi^s = \bigoplus_{s \in \mathcal{H}} \mathsf{k}\chi^s \oplus \mathfrak{m}^2.$$

In particular $\dim T_{p_\sigma} U_\sigma = \dim \mathfrak{m}/\mathfrak{m}^2 = |\mathcal{H}|$. □

Recall Definition 2.3.6 of smooth cones and let σ be strongly convex.

Theorem 2.4.9. *The toric variety U_σ is smooth iff σ is smooth.*

Proof. Proposition 2.3.3 proves the implication (σ smooth $\Rightarrow U_\sigma$ smooth).

Assume that U_σ is smooth. We start with the special case where σ has maximal dimension: $\dim \sigma = n = \dim U_\sigma$. In that case U_σ contains a unique closed T_M-fixed point: p_σ. The point p_σ is smooth in U_σ thus $n = \dim U_\sigma = \dim T_{p_\sigma} U_\sigma = |\mathcal{H}|$ (the last equality follows from Lemma 2.4.8). By Proposition 2.2.13, $\mathcal{H}$ generates S_σ and contains the rays of $\sigma^\vee$ which is also of dimension n (since σ is strictly convex). Thus $\mathcal{H}$ is made exactly of the generators of the rays of $\sigma^\vee$ which are of number exactly $n = \dim \sigma^\vee$. Furthermore, Lemma 2.4.6 implies that S_σ generates M thus $\mathcal{H}$ generates M thus $\mathcal{H}$ is a basis of M.

Assume that U_σ has no fixed point or equivalently that σ is not of maximal dimension. Let $r = \dim \sigma$. We will reduce to the previous case. Recall the definition of $N = M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ and let $N' = (\sigma \cap N)_{\mathbb{Z}}$. This is the lattice generated by the saturated cone $\sigma \cap N$, it has rank r and is saturated. Since N' is saturated, the quotient N/N' is torsion free thus there exists a sublattice $N'' \subset N$ of rank

$n-r$ such that $N = N' \oplus N''$ (the exact sequence $0 \to N' \to N \to N/N' \to 0$ splits since N/N' is free). Let $M' = \mathrm{Hom}_{\mathbb{Z}}(N', \mathbb{Z})$. The cone σ defines an affine variety U_σ which has an open dense orbit of T_M given by $\mathrm{Spec}(\mathsf{k}\lfloor (S_\sigma)_{\mathbb{Z}} \rfloor)$. Define $M' = M \cap (N'')^\perp$ and $M'' = M \cap (N')^\perp$. We have $M = M' \oplus M''$.

Note that $\sigma \subset N'$ thus $M'' \subset S_\sigma = \sigma^\vee \cap M$. Define $S'_\sigma = \sigma^\vee \cap M'$. We have $S'_\sigma \oplus M'' \subset S_\sigma$. We claim that there is an equality. Indeed if $m \in S_\sigma$, then $m = m' + m''$ with $m' \in M'$ and $m'' \in M''$ and the condition $m \in S_\sigma$ implies $m' \in S'_\sigma$ since m'' is orthogonal to σ. Define $U'_\sigma = \mathrm{Spec}(\mathsf{k}\lfloor S'_\sigma \rfloor)$. We have an isomorphism $U_\sigma = \mathrm{Spec}(\mathsf{k}\lfloor S_\sigma \rfloor) = \mathrm{Spec}(\mathsf{k}\lfloor S'_\sigma \oplus M'' \rfloor) \simeq \mathrm{Spec}(\mathsf{k}\lfloor S'_\sigma \rfloor) \times_{\mathsf{k}} \mathrm{Spec}(\mathsf{k}\lfloor M'' \rfloor) \simeq U'_\sigma \times_{\mathsf{k}} (\mathbb{G}_m)^{n-r}$. In particular U_σ is smooth if and only if U'_σ is smooth. This last toric variety for $T_{M'}$ has a $T_{M'}$-fixed point thus σ is smooth in N' and is generated by a basis of N'. Choosing a basis of N'' we complete this in a basis of N proving that σ is smooth. □

5. Equivariant morphisms between affine toric varieties

Let T and T' be algebraic groups, let X be a T-variety and Y be a T'-variety.

Definition 2.5.1. A morphism $f : X \to Y$ is equivariant if there exists a morphism of algebraic groups $g : T \to T'$ with $f(t.x) = g(t).f(x)$ for $t \in T$, $x \in X$.

We also call a map as above (T, T')-equivariant and T-equivariant if $T = T'$.

Remark 2.5.2. The previous definition can be expressed in terms of compatibility between the action morphisms $T \times X \to X$ and $T' \times Y \to Y$.

Definition 2.5.3. Let N_1 and N_2 be two lattices and let $\sigma_i \subset (N_i)_{\mathbb{R}}$, for $i = 1$ or 2 be two rational strictly convex polyhedral cones. A morphism of convex rational polyhedral cones $\phi : \sigma_1 \to \sigma_2$ is a morphism of abelian groups $\phi : N_1 \to N_2$ such that $\phi_{\mathbb{R}}(\sigma_1) \subset \sigma_2$ with $\phi_{\mathbb{R}} = \phi \otimes_{\mathbb{Z}} \mathbb{R} : (N_1)_{\mathbb{R}} \to (N_2)_{\mathbb{R}}$.

Remark 2.5.4. If $\phi : N_1 \to N_2$ is a morphism of abelian groups, we have a dual map $\phi^\vee : M_2 \to M_1$ where $M_i = \mathrm{Hom}_{\mathbb{Z}}(N_i, \mathbb{Z})$ which induces a morphism of algebraic tori $\Phi : T_{M_1} \to T_{M_2}$ defined by $\mathsf{k}\lfloor M_2 \rfloor \to \mathsf{k}\lfloor M_1 \rfloor, \chi^m \mapsto \chi^{\phi^\vee(m)}$.

Proposition 2.5.5. *Let N_1 and N_2 be two lattices and let $\sigma_i \subset (N_i)_{\mathbb{R}}$, for $i = 1$ or 2 be two rational strictly convex polyhedral cones.*

1. *A morphism $\phi : \sigma_1 \to \sigma_2$ of convex rational polyhedral cones induces a (T_{M_1}, T_{M_2})-equivariant morphism $f : U_{\sigma_1} \to U_{\sigma_2}$ with $f(T_{M_1}) \subset T_{M_2}$.*
2. *Any (T_{M_1}, T_{M_2})-equivariant morphism $f : U_{\sigma_1} \to U_{\sigma_2}$ with $f(T_{M_1}) \subset T_{M_2}$ comes from a morphism of convex rational polyhedral cones $\phi : \sigma_1 \to \sigma_2$.*

Proof. (1) We have a map $\phi : N_1 \to N_2$ such that $\phi_{\mathbb{R}}(\sigma_1) \subset \sigma_2$. This induces a morphism of tori $\Phi : T_{M_1} \to T_{M_2}$ (see Remark 2.5.4). Furthermore, we have a map $\phi^\vee : M_2^\vee \to M_1^\vee$ which satisfies $\phi^\vee(S_{\sigma_2}) \subset S_{\sigma_1}$. Indeed, if $m \in S_{\sigma_2}$, then $m \in M_2$ and $\phi^\vee(m) \in M_1$. Furthermore, for $v \in \sigma_1$, we have $\langle \phi^\vee(m), v \rangle = \langle m, \phi(v) \rangle \geq 0$ since $\phi(v) \in \phi(\sigma_1) \subset \sigma_2$. We thus have a morphism of monoids $\phi^\vee : S_{\sigma_2} \to S_{\sigma_1}$ which induces a k-algebra morphism $f^* : \mathsf{k}\lfloor S_{\sigma_2} \rfloor \to \mathsf{k}\lfloor S_{\sigma_1} \rfloor$ with $f^*(\chi^m) = \chi^{\phi^\vee(m)}$ and a morphism of schemes $f : U_{\sigma_1} \to U_{\sigma_2}$. We need to check that this map is equivariant. On the algebra level, this is equivalent to the commutativity of the following diagram

$$\begin{array}{ccc} \mathsf{k}\lfloor S_{\sigma_2}\rfloor & \longrightarrow & \mathsf{k}\lfloor S_{\sigma_2}\rfloor \otimes_{\mathsf{k}} \mathsf{k}\lfloor M_2\rfloor \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*\otimes f^*} \\ \mathsf{k}\lfloor S_{\sigma_1}\rfloor & \longrightarrow & \mathsf{k}\lfloor S_{\sigma_1}\rfloor \otimes_{\mathsf{k}} \mathsf{k}\lfloor M_1\rfloor, \end{array}$$

where the horizontal maps are given by the action and of the form $\chi^s \mapsto \chi^s \otimes \chi^s$. The commutativity is then an easy check. Now let p_0 be the closed point associated to the unit in T_{M_1} corresponding to the morphism of monoids $\gamma_1 : S_{\sigma_1} \mapsto \mathsf{k}$, $\gamma_1(v_1) = 1$ for all v_1 (see Example 2.4.2). Its image the closed point corresponding to the morphism of monoids $\gamma_2 : S_{\sigma_2} \mapsto \mathsf{k}$, $\gamma_2(v_2) = 1$ for all v_2 and therefore the unit of T_{M_2} proving that $f(T_{M_1}) \subset T_{M_2}$.

(2) Conversely $f : U_{\sigma_1} \to U_{\sigma_2}$ induces a commutative diagram

$$\begin{array}{ccc} \mathsf{k}\lfloor S_{\sigma_2}\rfloor & \longrightarrow & \mathsf{k}\lfloor S_{\sigma_2}\rfloor \otimes_{\mathsf{k}} \mathsf{k}\lfloor M_2\rfloor \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*\otimes f^*} \\ \mathsf{k}\lfloor S_{\sigma_1}\rfloor & \xrightarrow{\varphi} & \mathsf{k}\lfloor S_{\sigma_1}\rfloor \otimes_{\mathsf{k}} \mathsf{k}\lfloor M_1\rfloor, \end{array}$$

with $\varphi(\chi^r) = \chi^r \otimes \chi^r$ for $r \in S_{\sigma_1}$. For each $s \in S_{\sigma_2}$, write $f^*(\chi^s) = \sum_r \lambda_{s,r}\chi^r$ where r runs over S_{σ_1}. The equivariance implies $\varphi(f^*(\chi^s)) = f^*(\chi^s) \otimes f^*(\chi^s)$. We get the equality

$$\sum_r \lambda_{s,r}\chi^r \otimes \chi^r = \sum_{r,r'} \lambda_{s,r}\lambda_{s,r'}\chi^r \otimes \chi^{r'}$$

and by independance of characters we have $\lambda_{s,r} = \lambda_{r,s}^2$ and $\lambda_{s,r}\lambda_{s,r'} = 0$ for all s, r, r' with $r' \neq r$. This implies $f^*(\chi^s) = \chi^r$ for a unique $r \in S_{\sigma_1}$. Set $\phi^\vee(s) = r$. Since f^* is a morphism of algebras, $\phi^\vee$ is $\mathbb{Z}$-linear and thus defines a morphism of lattice $\phi^\vee : M_2 \to M_1$ (because $\sigma_2^\vee$ spans M_2). We have $\phi^\vee(\sigma_2^\vee) \subset \sigma_1^\vee$. Setting $\phi = (\phi^\vee)^\vee$, then $\phi : N_1 \to N_2$ is a morphism of lattices mapping σ_1 to σ_2. □

Remark 2.5.6. Note that the above constructions are inverse from each other.

Remark 2.5.7. On the level of k-algebra morphisms $f : U_{\sigma_1} \to U_{\sigma_2}$ is equivariant if and only if f^* maps characters to characters.

An example of such a morphism of cones is given by a face inclusion $\tau \subset \sigma$. The following was proved in Proposition 2.3.2.

Proposition 2.5.8. *Let $\tau \prec \sigma$ be a face.*

(1) *The map $U_\tau \to U_\sigma$ is an open embedding.*
(2) *More precisely, if $\tau = \sigma \cap w^\perp$ with $w \in S_\sigma$, then $\mathsf{k}\lfloor S_\tau\rfloor = \mathsf{k}\lfloor S_\sigma\rfloor[\chi^{-w}]$ is the localisation of $\mathsf{k}\lfloor S_\sigma\rfloor$ at χ^w.*

CHAPTER 3

Toric varieties and fans

In this chapter we consider general toric varieties.

1. Fans

Fans are the combinatorial data allowing to glue affine toric varieties together.

DEFINITION 3.1.1. Let N be a lattice. A fan is a finite set Δ of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ such that

(1) Each face τ of $\sigma \in \Delta$ is in Δ.
(2) The intersection of two cones in Δ is a face of each.

EXAMPLE 3.1.2. If σ is a cone, then $\Delta = \{\tau \mid \tau \preceq \sigma\}$ is a fan.

Let Δ be a fan and $\sigma \in \Delta$. For $\tau \prec \sigma$, let $\varphi_{\tau,\sigma} : U_\tau \to U_\sigma$ be the open embedding obtained in Proposition 2.3.2.

LEMMA 3.1.3. *Let $\sigma, \tau, \gamma \in \Delta$ with $\gamma \subset \tau \subset \sigma$. Then $\varphi_{\gamma,\sigma} = \varphi_{\tau,\sigma} \circ \varphi_{\gamma,\tau}$.*

Proof. This follows from the inclusions $S_\sigma \subset S_\tau \subset S_\gamma$. □

The previous result ensures that we can glue the affine toric varieties $(U_\sigma)_{\sigma\in\Delta}$ together along the open subsets corresponding to the intersection of two cones.

DEFINITION 3.1.4. $X(\Delta)$ is the scheme obtained by the above glueing.

LEMMA 3.1.5. *Let $\sigma, \tau \in \Delta$ be cones that intersect along a common face $\gamma = \sigma \cap \tau$. The map $U_\gamma \to U_\sigma \times U_\tau$ is a closed embedding.*

Proof. This is equivalent to the fact that the map $\mathsf{k}\lfloor S_\sigma \rfloor \otimes \mathsf{k}\lfloor S_\tau \rfloor \to \mathsf{k}\lfloor S_\gamma \rfloor$ is surjective. Recall from Proposition 2.2.11 that we have $S_\gamma = S_\sigma + S_\tau$ proving the surjectivity. □

COROLLARY 3.1.6. *The variety $X(\Delta)$ is separated.*

Proof. Lemma 3.1.5 implies that the map $X(\Delta) \to X(\Delta) \times X(\Delta)$ is closed. □

THEOREM 3.1.7. *The variety $X(\Sigma)$ is a toric variety.*

Proof. We prove that $T_N = \mathrm{Spec}(\mathsf{k}\lfloor N \rfloor)$ acts on $X(\Delta)$ and is isomorphic to the unique dense orbit. First note that $T_N = U_0$ where 0 is the trivial cone (since all cones in Δ are strongly convex 0 is a face of all these cones. In particular T_N is an open subset in $X(\Delta)$. Furthermore T_N acts on all U_σ with dense orbit T_N and the action is compatible by the glueing construction, thus T_N acts on $X(\Delta)$ with dense orbit T_N proving the result. □

EXAMPLE 3.1.8. Let σ be a cone and $\Delta = \{\tau \mid \tau \preceq \sigma\}$. Then $X(\Delta) = U_\sigma$.

Example 3.1.9. Let $N = \mathbb{Z}$ and $\Delta = \{\mathbb{R}_+, \mathbb{R}_-, \{0\}\} = \{\sigma_+, \sigma_-, \sigma_0\}$, then $U_{\sigma_+} = \mathbb{A}^1_{\mathsf{k}}$, $U_{\sigma_+} = \mathbb{A}^1_{\mathsf{k}}$ and $U_{\sigma_0} = \mathbb{G}_m$. To get $X(\Delta)$, we glue $\mathbb{A}^1_{\mathsf{k}}$ with itself along $\mathbb{G}_m$. Thus $X(\Delta) = \mathbb{P}^1_{\mathsf{k}}$.

Remark 3.1.10. To define a fan Δ it is enough to give its maximal cones.

Example 3.1.11. Let $N = \mathbb{Z}^2$ with canonical basis (e_1, e_2). Let $M = N^\vee$ with dual basis s_1, s_2. Set $x = \chi^{s_1}$ and $y = \chi^{s_2}$.

(1) Let $\sigma_1 = \mathbb{R}_+ e_1$ and $\sigma_2 = \mathbb{R}_+(-e_1)$. Let Δ be the fan containing all the faces of σ_1 and σ_2. Then $U_{\sigma_1} = \mathbb{A}^1_{\mathsf{k}} \times \mathbb{G}_m = \operatorname{Spec} \mathsf{k}[x, y, y^{-1}]$ and $U_{\sigma_2} = \mathbb{A}^1_{\mathsf{k}} \times \mathbb{G}_m = \operatorname{Spec} \mathsf{k}[x^{-1}, y, y^{-1}]$ while $U_{\sigma_1 \cap \sigma_2} = \mathbb{G}_m \times \mathbb{G}_m = \operatorname{Spec} \mathsf{k}[x, x^{-1}, y, y^{-1}]$. We get $X(\Delta) = \mathbb{P}^1_{\mathsf{k}} \times \mathbb{G}_m$.
(2) Let $\sigma_1 = \mathbb{R}_+ e_1 + \mathbb{R}_+ e_2$ and $\sigma_2 = \mathbb{R}_+(-e_1) + \mathbb{R}_+ e_2$. Let Δ be the fan containing all the faces of σ_1 and σ_2. Then $U_{\sigma_1} = \mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}} = \operatorname{Spec} \mathsf{k}[x, y]$ and $U_{\sigma_2} = \mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}} = \operatorname{Spec} \mathsf{k}[x^{-1}, y]$ while $U_{\sigma_1 \cap \sigma_2} = \mathbb{G}_m \times \mathbb{A}^1_{\mathsf{k}} = \operatorname{Spec} \mathsf{k}[x, x^{-1}, y]$. We get $X(\Delta) = \mathbb{P}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}}$. The action of the torus $\mathbb{G}_m^2$ is given by $(t, t').([x_0, x_1], y) = ([tx_0 : x_1], t'y)$.
(3) Let $\sigma_1 = \mathbb{R}_+ e_1 + \mathbb{R}_+(e_2 - e_1)$ and $\sigma_2 = \mathbb{R}_+(e_2 - e_1) + \mathbb{R}_+(-e_1)$. Let Δ be the fan containing all the faces of σ_1 and σ_2. Then $U_{\sigma_1} = \mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}} = \operatorname{Spec} \mathsf{k}[xy, y]$ and $U_{\sigma_2} = \mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}} = \operatorname{Spec} \mathsf{k}[(xy)^{-1}, y]$ while $U_{\sigma_1 \cap \sigma_2} = \mathbb{G}_m \times \mathbb{A}^1_{\mathsf{k}} = \operatorname{Spec} \mathsf{k}[xy, (xy)^{-1}, y]$. We get $X(\Delta) = \mathbb{P}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}}$. The action of the torus $\mathbb{G}_m^2$ is given by $(t, t').([x_0, x_1], y) = ([tt'x_0 : x_1], t'y)$.
(4) Let $\sigma_1 = \mathbb{R}_+ e_1 + \mathbb{R}_+ e_2$, $\sigma_2 = \mathbb{R}_+ e_1 + \mathbb{R}_+(-e_2)$, $\sigma_3 = \mathbb{R}_+(-e_1) + \mathbb{R}_+ e_2$ and $\sigma_4 = \mathbb{R}_+(-e_1) + \mathbb{R}_+(-e_2)$. Let Δ be the fan containing all the cones faces of σ_i for $i \in [1, 4]$. Then $X(\Delta) = \mathbb{P}^1_{\mathsf{k}} \times \mathbb{P}^1_{\mathsf{k}}$.

Exercise 3.1.12. Let $N = \mathbb{Z}^2$ with canonical basis (e_1, e_2). Let $M = N^\vee$ with dual basis s_1, s_2. Set $x = \chi^{s_1}$ and $y = \chi^{s_2}$. Let $\sigma_1 = \mathbb{R}_+ e_1 + \mathbb{R}_+ e_2$, $\sigma_2 = \mathbb{R}_+ e_1 + \mathbb{R}_+(-e_1 - e_2)$ and $\sigma_3 = \mathbb{R}_+ e_2 + \mathbb{R}_+(-e_1 - e_2)$. Let Δ be the fan containing all the faces of σ_1, σ_2 and σ_3. Prove that $X(\Delta) \simeq \mathbb{P}^2_{\mathsf{k}}$.

Exercise 3.1.13. Let $N = \mathbb{Z}^2$ with canonical basis (e_1, e_2). Let $M = N^\vee$ with dual basis s_1, s_2. Set $x = \chi^{s_1}$ and $y = \chi^{s_2}$. Let $\sigma_1 = \mathbb{R}_+ e_1 + \mathbb{R}_+ e_2$, $\sigma_2 = \mathbb{R}_+(-e_1) + \mathbb{R}_+(-e_2)$. Let Δ be the fan containing all the faces of σ_1 and σ_2. Describe $X(\Delta)$.

Example 3.1.14. More generally, let N and N' be lattices and Δ and Δ' be fans in N and N'. Then $\Delta \times \Delta' = \{\sigma \times \sigma' \mid \sigma \in \Delta, \sigma' \in \Delta'\}$. Then we have $X(\Delta \times \Delta') \simeq X(\Delta) \times X(\Delta')$.

Exercise 3.1.15. Consider $X = V(yx_0 - xx_1) \subset \mathbb{A}^2_{\mathsf{k}} \times \mathbb{P}^1_{\mathsf{k}}$ with coordinates (x, y) on $\mathbb{A}^2_{\mathsf{k}}$ and $[x_0 : x_1]$ on $\mathbb{P}^1_{\mathsf{k}}$.Let $T = (\mathbb{G}_m)^2$ act on X via $(t, t').(x, y, [x_0 : x_1]) = (tx, t'y, [tx_0 : t'x_1])$. For $i \in \{0, 1\}$, let $U_i = \{(x, y, [x_0 : x_1]) \in X \mid x_i \neq 0\}$.

(1) Compute the coordinate rings of U_0, U_1 and $U_0 \cap U_1$.
(2) Let $N = \mathbb{Z}^2$ with canonical basis (e_1, e_2). Let $M = N^\vee$ with dual basis s_1, s_2. Set $x = \chi^{s_1}$ and $y = \chi^{s_2}$. Let $\sigma_1 = \mathbb{R}_+ e_1 + \mathbb{R}_+(e_1 + e_2)$ and $\sigma_2 = \mathbb{R}_+(e_1 + e_2) + \mathbb{R}_+ e_2$. Let Δ be the fan containing all the faces of σ_1 and σ_2. Compute the coordinate rings of U_{σ_1}, U_{σ_2} and $U_{\sigma_1 \cap \sigma_2}$ and prove that $X(\Delta) \simeq X$.

Conversely, we will see that the following holds (see Corollary 3.3.2):

Theorem 3.1.16. *Any toric variety is of the form $X(\Delta)$ for a fan Δ.*

2. Orbits

Let N be a lattice and $M = N^\vee$ be its dual and set $V = N_\mathbb{R}$, $W = M_\mathbb{R}$. Let $T = \operatorname{Spec} \mathsf{k}\lfloor M \rfloor$ be the torus whose character lattice in M. For $n \in N$, we have a one-parameter subgroup $\lambda_n : \mathbb{G}_m \to T$. For Δ a fan in N consider the toric variety $X(\Delta)$ and let $x \in T$. Our goal is to understand $\lim_{t\to 0} \lambda_v(t).x$ for $v \in N$.

Fix $\sigma \in \Delta$. Recall the description of closed points in U_σ from Proposition 2.4.1: we have a bijective correspondence between

(1) Closed points in U_σ.
(2) Morphisms of monoid $S_\sigma \to (\mathsf{k}, \times)$.

Definition 3.2.1. For any convex polyhedral cone $\sigma \subset N$, define the closed point p_σ by the following morphism of monoids:

$$S_\sigma \to (\mathsf{k}, \times),\ m \mapsto \begin{cases} 1 & \text{if } m \in \sigma^\perp \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.2.2. This is a morphism of monoids since $\sigma^\vee \cap \sigma^\perp$ is a face of $\sigma^\vee$.

Remark 3.2.3. Let $\tau \prec \sigma$ be a face of σ. Then p_τ is a closed point in U_σ.

Remark 3.2.4. If Δ is a fan, any $\sigma \in \Delta$ is strictly convex, thus 0 is a face of σ and p_0 is a closed point in any U_σ for $\sigma \in \Delta$. Note that $p_0 \in U_0$ the affine open subset corresponding to the trivial cone and that $U_0 = \operatorname{Spec}(\mathsf{k}\lfloor M \rfloor) = T$ is the open dense T-orbit.

Lemma 3.2.5. *Let $\sigma \in \Delta$ and $v \in N$.*

(1) *Then $\lim_{t\to 0} \lambda_v(t).p_0$ exists in U_σ if and only if $v \in \sigma$.*
(2) *Furthermore if $v \in \mathring{\sigma}$, then $\lim_{t\to 0} \lambda_v(t).p_0 = p_\sigma$.*

Proof. (1) Note that we have a map $\phi : \mathbb{G}_m \to U_\sigma, t \mapsto \phi(t) = \lambda_v(t).p_0$. The limit exists if this map extends to a map $\psi : \mathbb{A}^1_\mathsf{k} \to U_\sigma$ and by definition $\lim_{t\to 0} \lambda_v(t).p_0 = \psi(0)$. The map ϕ is given by the composition $\mathbb{G}_m \xrightarrow{\lambda_v} T \to U_\sigma, t \mapsto \lambda_v(t) \mapsto \lambda_v(t).p_0$ and thus comes from the k-algebra morphism $\phi^* : \mathsf{k}\lfloor S_\sigma \rfloor \to \mathsf{k}\lfloor M \rfloor \to \mathsf{k}[T, T^{-1}], \chi^m \to \chi^m \to T^{\langle m,v\rangle}$. In particular this map extends to $\mathbb{A}^1_\mathsf{k}$ if and only if $\langle m, v\rangle \geq 0$ for all $m \in S_\sigma$ *i.e.* for all $m \in \sigma^\vee$ which is equivalent to $v \in \sigma$.

(2) The limit point is given by the morphism of monoids $S_\sigma \to (\mathsf{k}, \times), m \mapsto 0^{\langle m,v\rangle}$ which defines p_σ for $v \in \mathring{\sigma}$. ☐

We now describe all T-orbits in $X(\Delta)$.

Definition 3.2.6. For $\sigma \in \Delta$, set $O(\sigma) = T.p_\sigma$.

Before describing the T-orbits in $X(\Delta)$ we will need few definitions and lemmas.

Definition 3.2.7. Let $\sigma \in \Delta$.

(1) Let N_σ be the sublattice spanned $\sigma \cap N$ and set $N(\sigma) = N/N_\sigma$.
(2) For $w \in N$, let $\bar{w} \in N(\sigma)$ be its class in the quotient.
(3) Set $M(\sigma) = \sigma^\perp \cap M \subset \sigma^\vee \cap M = S_\sigma$.

Remark 3.2.8. Note that the linear span of $M(\sigma)$ is a face of $\sigma^\vee$.

Proposition 3.2.9. *Let $\sigma \in \Delta$.*

(1) *The closed points of $O(\sigma)$ correspond to morphisms of monoids of the form*

$$\gamma : S_\sigma \to (\mathsf{k}, \times), \gamma(m) \neq 0 \Leftrightarrow m \in M(\sigma).$$

(2) *$O(\sigma)$ is isomorphic to $T_{M(\sigma)}$ the torus with character group $M(\sigma)$.*

Proof. (1) Recall from Lemma 2.4.3 that if p is the closed point associated to the morphism of monoids $\gamma : S_\sigma \to \mathsf{k}$ and if $t \in T$, the point $t.p$ corresponds to the morphism of monoids $t.\gamma(m) = \chi^m(t)\gamma(m)$. This implies that any closed point of $O(\sigma)$ corresponds to a morphism of monoids of the correct form. Conversely, let $\gamma : S_\sigma \to \mathsf{k}$ be a morphism of monoids such that $\gamma(m) \neq 0$ iff $m \in M(\sigma)$. Consider the closed point $t \in T_{M(\sigma)}$ defined by $\varphi : \mathsf{k}\lfloor M(\sigma) \rfloor \to \mathsf{k}, \chi^m \mapsto \gamma(m)$. Then $t.p_\sigma$ corresponds to the morphism of monoids γ therefore $t.p_\sigma = p$. The injective map $M(\sigma) \subset M$ induces a surjection of tori $T \to T_{M(\sigma)}$ proving the result.

(2) The previous argument shows that $O(\sigma)$ is indeed a $T_{M(\sigma)}$-orbit. We only need to prove that the action is free *i.e.* that the stabiliser of p_σ in $T_{M(\sigma)}$ is trivial. Let $t \in T_{M(\sigma)}$ act trivially on p_σ. Considering the action on the corresponding morphisms of monoids, we get that $\chi^m(t) = 1$ for all $m \in M(\sigma)$ which implies the claim. $\square$

The following result computes the the group of cocharacters of $T_{M(\sigma)}$.

LEMMA 3.2.10. $\langle\!\langle\ ,\ \rangle\!\rangle : M(\sigma) \times N(\sigma) \to \mathbb{Z}$, $\langle\!\langle m, \bar{v} \rangle\!\rangle = \langle m, v \rangle$ *is a perfect pairing.*

Proof. The pairing is well defined. Let $m \in M(\sigma)$ such that $\langle\!\langle m, \bar{v} \rangle\!\rangle = 0$ for all $\bar{v} \in N(\sigma)$, then $\langle m, v \rangle = 0$ for all $v \in N$ and $m = 0$ since $\langle\ ,\ \rangle$ is perfect. Let $v \in N$ such that $\langle\!\langle m, \bar{v} \rangle\!\rangle = 0$ for all $m \in M(\sigma)$, then $v \in M(\sigma)^\perp \subset (\sigma^\perp)^\perp = \sigma$ thus $v \in N_\sigma$ and $\bar{v} = 0$. $\square$

LEMMA 3.2.11. *Let $\gamma : S_\sigma \to (\mathsf{k}, \times)$ be a morphism of monoids defining a closed point $p \in U_\sigma$, then $\{m \in S_\sigma \mid \gamma(m) \neq 0\} = S_\sigma \cap \tau^\perp$ for some face τ of σ and $p \in U_\tau$.*

Proof. Let $m_1, \cdots, m_r$ be generators of the monoid S_σ. Set $I = \{i \in [1, r] \mid \gamma(m_i) \neq 0\}$ and let F be the face of $\sigma^\vee$ generated by $(m_i)_{i \in I}$. Let $m \in S_\sigma$ thus $m = \sum_{i=1}^r a_i m_i$ and $\gamma(m) = \prod_{i \in I} \gamma(m_i)^{a_i}$. In particular $m \in \{w \in S_\sigma \mid \gamma(w) \neq 0\}$ if and only if $a_i = 0$ for $i \notin I$ thus if and only if $m \in F \cap M$. Now any face F of $\sigma^\vee$ is of the form $F = \sigma^\vee \cap \tau^\perp$ for a face τ of σ, proving the first claim. Define $\gamma' : S_\tau \to \mathsf{k}$ by

$$\gamma'(m) = \begin{cases} \gamma(m) & \text{if } m \in S_\sigma \\ 0 & \text{otherwise.} \end{cases}$$

Then since $\sigma^\vee$ is a face of $\tau^\vee$, this defines a morphism of monoids and thus a closed point in U_τ which maps to p via the inclusion $U_\tau \subset U_\sigma$ proving the last claim. $\square$

THEOREM 3.2.12. *Let Δ be a fan and $X = X(\Delta)$.*

(1) *Any T-orbit in X is of the form $O(\sigma)$ for some $\sigma \in \Delta$.*
(2) *We have $\dim O(\sigma) = \dim X - \dim \sigma$.*
(3) *For $\sigma \in \Delta$, we have the disjoint union*

$$U_\sigma = \bigcup_{\tau \prec \sigma} O(\tau).$$

Proof. (1) Let O be a T-orbit. Since X is covered by the T-stable open subsets $(U_\sigma)_{\sigma \in \Delta}$, there exists at least one $\sigma \in \Delta$ with $O \subset U_\sigma$. Furthermore, since $U_\sigma \cap U_\tau = U_{\sigma \cap \tau}$, we may choose σ minimal for this property. We claim that $O = O(\sigma)$ which will prove (1).

We now prove the claim. Let $p \in O$ and let $\gamma : S_\sigma \to (\mathsf{k}, \times)$ be the corresponding morphism of monoids. Consider $\{m \in S_\sigma \mid \gamma(m) \neq 0\}$. By Lemma 3.2.11 this set is of the form $S_\sigma \cap \tau^\perp = \sigma^\vee \cap \tau^\perp \cap M$ for some face τ of σ and $p \in U_\tau$. By minimality, we have $\tau = \sigma$. Thus $\{m \in S_\sigma \mid \gamma(m) \neq 0\} = M(\sigma)$ and Proposition 3.2.9 gives that $p \in O(\sigma)$.

(2) Proposition 3.2.9 implies that $\dim O(\sigma) = \dim T_{M(\sigma)} = \mathrm{rk}(M(\sigma))$ and by definition $\mathrm{rk}(M(\sigma)) = \dim X - \dim \sigma$.

(3) Since U_σ is T-stable it is the union of T-orbits. Furthermore, for $\tau \prec \sigma$, we have $O(\tau) \subset U_\tau \subset U_\sigma$. Conversely if $p \in U_\sigma$ corresponding to a morphism of monoids $\gamma : S_\sigma \to \mathsf{k}$, we have seen in the proof of (1) that $\{m \in S_\sigma \mid \gamma(m) \neq 0\} = S_\sigma \cap \tau^\perp = \sigma^\vee \cap \tau^\perp \cap M = M(\tau)$ for some face τ of σ and $p \in O(\tau)$ proving the converse inclusion. □

Lemma 3.2.13. *Let $\tau \in \Delta$. Set $N_{\tau,\mathbb{R}} = N_\tau \otimes_\mathbb{Z} \mathbb{R}$ and $N(\tau)_\mathbb{R} = N(\tau) \otimes_\mathbb{Z} \mathbb{R}$.*

(1) *Then $\overline{\tau} := \tau + N_{\tau,\mathbb{R}}/N_{\tau,\mathbb{R}} \subset N(\tau)_\mathbb{R}$ is a cone.*
(2) *The set $\{\overline{\sigma} \mid \tau \preceq \sigma \text{ in } \Delta\}$ is a fan in $N(\sigma)$.*

Proof. Exercise. □

Definition 3.2.14. Let Δ be a fan and $\tau \in \Delta$.

(1) The star of τ in Δ, denoted $\mathrm{Star}(\tau)$ is the fan $\mathrm{Star}(\tau) = \{\overline{\sigma} \mid \tau \preceq \sigma \text{ in } \Delta\}$.
(2) Set $V(\tau) = X(\mathrm{Star}(\tau))$.

Remark 3.2.15. Let $\tau \in \Delta$.

(1) Recall from Lemma 3.2.10 that $M(\tau)$ and $N(\tau)$ are dual lattices therefore $V(\tau)$ is a $T_{M(\tau)}$-toric variety.
(2) The inclusion $M(\tau) \subset M$ induces a surjective group morphism $T \to T_{M(\tau)}$ and therefore an action of T on $V(\tau)$.

For $\overline{\sigma} \in \mathrm{Star}(\tau)$, set $U_\sigma(\tau) = \mathrm{Spec}(\mathsf{k}\lfloor\overline{\sigma}^\vee \cap M(\tau)\rfloor) = \mathrm{Spec}(\mathsf{k}\lfloor\sigma^\vee \cap \tau^\perp \cap M\rfloor)$ the corresponding $T_{M(\tau)}$-stable affine open subset. Note that we have isomorphisms $U_\tau(\tau) \simeq T_{M(\tau)} \simeq O(\tau)$. In particular the open dense T-orbit of $V(\tau)$ can be embedded in $X(\Delta)$. The following is a generalisation of this remark.

Proposition 3.2.16. *Let $\tau \in \Delta$.*

(1) *There is a T-equivariant closed embedding $V(\tau) \to X(\Delta)$.*
(2) *The image of the open dense T-orbit $U_\tau(\tau)$ in $V(\tau)$ is $O(\tau)$.*
(3) *Identifying $V(\tau)$ with its image, we have $V(\tau) = \overline{O(\tau)}$ and*

$$V(\tau) = \bigcup_{\sigma \succeq \tau} O(\sigma).$$

Proof. (1) For $\sigma \succeq \tau$, we define a closed embedding $\varphi : U_\sigma(\tau) \to U_\sigma$. For this it is enough to define a surjective map $\varphi^* : \mathsf{k}\lfloor\sigma^\vee \cap M\rfloor \to \mathsf{k}\lfloor\sigma^\vee \cap \tau^\perp \cap M\rfloor$. Set

$$\varphi(\chi^m) = \begin{cases} \chi^m & \text{if } m \in \sigma^\vee \cap \tau^\perp \cap M \\ 0 & \text{otherwise.} \end{cases}$$

Since $\sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$ this map is a k-algebra morphism, it is obviously surjective and T-equivariant. To get a global map it is enough to check that these maps are compatible on the smaller T-stable affine open subsets. If $\sigma' \succeq \sigma$, we have a commutative diagram of inclusions

$$\begin{array}{ccc} (\sigma')^\vee \cap \tau^\perp \cap M & \longrightarrow & \sigma^\vee \cap \tau^\perp \cap M \\ \downarrow & & \downarrow \\ (\sigma')^\vee \cap M & \longrightarrow & \sigma^\vee \cap M \end{array}$$

which induces a commutative diagram

$$\begin{array}{ccc} U_\sigma(\tau) & \longrightarrow & U_{\sigma'}(\tau) \\ \downarrow & & \downarrow \\ U_\sigma & \longrightarrow & U_{\sigma'}. \end{array}$$

and proves that these maps are indeed compatible.

(2) Let $p_0(\tau)$ in $V(\tau)$ be the closed point corresponding to the morphism of monoids sending all elements in $\sigma^\vee \cap \tau^\perp \cap M$ to 1. We have $p_0(\tau) \in U_\tau(\tau)$. Its image in U_τ is the closed point p_τ proving the claim.

(3) The first assertion follows from (1) and (2). Let $\sigma \succeq \tau$. Define the closed point $p_\sigma(\tau) \in U_\sigma(\tau)$ by the morphism of monoids $\gamma : \sigma^\vee \cap \tau^\perp \cap M \to \mathsf{k}$ mapping elements in $\sigma^\perp$ to 1 and other elements to 0. Its image in $X(\Delta)$ is p_σ. This together with Theorem 3.2.12.(1) proves the last formula. □

Corollary 3.2.17. *We have $\tau \prec \sigma$ if and only if $O(\sigma) \subset \overline{O(\tau)}$.*

Corollary 3.2.18. *Orbits closures in $X(\Delta)$ are (normal) toric varieties.*

Corollary 3.2.19. *Closed orbits are the orbits $O(\sigma)$ for maximal cones $\sigma \in \Delta$.*

Corollary 3.2.20. *If $X(\Delta)$ is smooth then all orbit closures are smooth.*

Proof. Note that $X(\Delta)$ is smooth if U_σ is smooth for all $\sigma \in \Delta$ *i.e.* if and only if all the cones $\sigma \in \Delta$ are generated by a subset of a basis of N. This implies that the same is true for the fan $\mathrm{Star}(\tau)$ for any $\tau \in \Delta$ proving the claim. □

3. General toric varieties and morphisms

We now prove that any toric variety is of the form $X(\Delta)$ for a fan Δ. We will need the following general result on T-varieties (see [**10**]).

Theorem 3.3.1. *Any X normal T-variety, has an affine T-stable covering.*

Corollary 3.3.2. *Any toric variety is of the form $X(\Delta)$ for a fan Δ.*

Proof. Let T be a torus and X be a toric T-variety with dense orbit isomorphic to T. Let $M = \mathfrak{X}^*(T)$ and N its dual. By Theorem 3.3.1, X has an affine covering $(U_i)_{i\in I}$ with U_i an affine toric variety. By Proposition 2.3.8, for each $i \in I$, there exists a convex rational polyhedral cone $\sigma_i \subset N$ such that $U_i = U_{\sigma_i}$. Since the dense orbit is isomorphic to T (and not a quotent of T), σ_i is strictly convex. Since X is separated $U_i \cap U_j$ is closed in $U_i \times U_j$ and thus affine. It is of the form $U_{\sigma_{i,j}}$ for some strictly convex rational polyhedral cone $\sigma_{i,j} \subset N$. Lemma 3.2.5.(1) implies that $\sigma_{i,j} = \sigma_i \cap \sigma_j$. Let Δ be the family of cones consisting of all cones $(\sigma_i)_{i\in I}$, their faces and intersections of faces. By the above $X = X(\Delta)$. □

DEFINITION 3.3.3. Let N and N' be two lattices and $\Delta \subset N_{\mathbb{R}}$ and $\Delta' \subset N'_{\mathbb{R}}$ be two fans. A morphism of fans $\Delta \to \Delta'$ is a morphism of lattices $\varphi : N' \to N$ such that for each $\sigma \in \Delta$, there exists $\sigma' \in \Delta'$ such that $\varphi(\sigma) \subset \sigma'$.

PROPOSITION 3.3.4. *Let Δ and Δ' be two fans associated to lattices N and N'.*

(1) *Any morphism of fans $\phi : \Delta \to \Delta'$ induces a $(T_{N^\vee}, T_{(N')^\vee})$-equivariant morphism of toric varieties $\varphi : X(\Delta) \to X(\Delta')$ with $\varphi(T_{N^\vee}) \subset T_{(N')^\vee}$.*
(2) *Any $(T_{N^\vee}, T_{(N')^\vee})$-equivariant morphism of toric varieties $\varphi : X(\Delta) \to X(\Delta')$ with $\varphi(T_{N^\vee}) \subset T_{(N')^\vee}$ induces a morphism of fans $\phi : \Delta \to \Delta'$.*

Proof. (1) Let $\phi : N \to N'$ be the associated morphism of lattices. For any $\sigma \in \Delta$, the inclusion $\phi(\sigma) \subset \sigma'$ induces (see Proposition 2.5.5) a $(T_{N^\vee}, T_{(N')^\vee})$-equivariant morphism of toric varieties $U_\sigma \to U_{\sigma'} \subset X(\Delta')$ mapping $T_{N^\vee}$ to $T_{(N')^\vee}$. By construction (see again Proposition 2.5.5) for $\tau \prec \sigma$, the map $U_\tau \to U_{\sigma'} \subset X(\Delta')$ is the restriction of the previous map therefore all these maps glue together to define a $(T_{N^\vee}, T_{(N')^\vee})$-equivariant morphism of toric varieties $\varphi : X(\Delta) \to X(\Delta')$.

(2) First note that φ induces a morphism of tori $T_{N^\vee} \to T_{(N')^\vee}$ which in turn induces a morphism of lattices $\phi : N \to N'$. We check that ϕ map elements in Δ to elements in Δ'. Let $\sigma \in \Delta$. By equivariance, the orbit $O(\sigma)$ is mapped into an orbit $O(\sigma') \subset X(\Delta')$ for some $\sigma' \in \Delta'$: $\varphi(O(\sigma)) \subset O(\sigma')$. Now if $\tau \preceq \sigma$ is a face of σ, its orbit $\overline{O(\tau)}$ is mapped into an orbit $\overline{O(\tau')} \subset X(\Delta')$. We have $\varphi(O(\sigma)) \subset \varphi(\overline{O(\tau)}) \subset \overline{O(\tau')}$ thus $O(\sigma') \subset \overline{O(\tau')}$ and $\tau' \preceq \sigma'$. This implies that φ maps U_σ into U'_σ and by Proposition 2.5.5, we get $\phi(\sigma) \subset \sigma'$ proving the claim. □

We now describe a nice example. Let N be a lattice and $N' \subset N$ be a sublattice of finite index. If Δ is a fan for N', then Δ is also a fan for N and we get a map $\phi : N' \to N$ inducing a morphism of fans $\phi : \Delta \to \Delta$. Denote by $X_N(\Delta)$ and $X_{N'}(\Delta)$ the corresponding toric varieties. We have a morphism of toric varieties $\varphi : X_{N'}(\Delta) \to X_N(\Delta)$.

PROPOSITION 3.3.5. *Set $G = N/N'$ and assume that* $\mathrm{char}(\mathsf{k}) \not| \; |G|$. *Then the map $\varphi : X_{N'}(\Delta) \to X_N(\Delta)$ is the quotient of $X_{N'}(\Delta)$ under the action of G.*

Proof. Set $M = N^\vee$ and $M' = (N')^\vee$. We have an exact sequence $0 \to M \to M' \to G \to 0$ which induces an exact sequence of k-algebras $0 \to \mathsf{k}\lfloor M \rfloor \to \mathsf{k}\lfloor M' \rfloor \to \mathsf{k}\lfloor G \rfloor \to 0$. Passing to spectra, we get an exact sequence of groups $1 \to \mathrm{Spec}(\mathsf{k}\lfloor G \rfloor) \to T_{M'} \to T_M \to 1$. Since $\mathrm{char}(\mathsf{k}) \not| \; |G|$, one easily checks that $G \simeq \mathrm{Spec}(\mathsf{k}\lfloor G \rfloor)$. In particular this proves the result on the open orbit. Furthermore, this implies that G acts on $X_{N'}(\Delta)$ and that the map is G-invariant (the action on the target is trivial). To prove the result we only need to check that the results holds on affine covers so we may replace $X_{N'}(\Delta)$ by $U_{\sigma,N'}$ and $X_N(\Delta)$ by $U_{\sigma,N}$ for some strictly convex rational polyhedral cone σ associated to N'. Let $S_{\sigma,N} = \sigma^\vee \cap M$ and $S_{\sigma,N'} = \sigma^\vee \cap M'$. We now prove that $\mathrm{Spec}(\mathsf{k}\lfloor S_{\sigma,N} \rfloor) = \mathrm{Spec}(\mathsf{k}\lfloor S_{\sigma,N'} \rfloor)^G$ which proves the result by Proposition 3.3.6. The action of G on $\mathsf{k}\lfloor S_{\sigma,N'} \rfloor$ is given by a map $\mathsf{k}\lfloor S_{\sigma,N'} \rfloor \to \mathsf{k}\lfloor S_{\sigma,N'} \rfloor \otimes \mathsf{k}[G]$. Using the isomorphism $\mathsf{k}[G] \simeq \mathsf{k}\lfloor N/N' \rfloor \simeq \mathsf{k}\lfloor M'/M \rfloor$ we get a map $\mathsf{k}\lfloor S_{\sigma,N'} \rfloor \to \mathsf{k}\lfloor S_{\sigma,N'} \rfloor \otimes \mathsf{k}\lfloor M'/M \rfloor$ given by $\chi^m \to \chi^m \otimes \chi^{[m]}$. If $g \in G$ is a closed point associated to a morphism of monoids $\gamma_g : M'/M \to \mathsf{k}$, then $g.\chi^m = \gamma_g([m])\chi^m$ and invariant elements are of the form χ^m for $[m] = 0$ *i.e.* of the form χ^m for $m \in M$ proving the claim. □

PROPOSITION 3.3.6. *Let X be an affine variety and G a finite group.*

(1) *Then* $\mathsf{k}[X]^G$ *is a finitely generated* k*-algebra.*
(2) *The map* $X \to Y := \mathrm{Spec}(\mathsf{k}[X]^G)$ *is constant on* G*-orbits.*
(3) *There is a bijection* $Y \to X/G$*.*

Proof. Exercise. $\square$

EXERCISE 3.3.7. Let Δ be a fan. Prove that the following are equivalent.

(1) $X(\Delta)$ has a torus factors
(2) There is a non-constant morphism $X(\Delta) \to \mathbb{G}_m$.
(3) The one dimensional cones in Δ do not span N.

4. Completeness and properness

DEFINITION 3.4.1. Let N be a lattice and Δ a fan in $N_{\mathbb{R}}$.

(1) The support of the fan is $|\Delta| = \bigcup_{\sigma\in\Delta} \sigma$.
(2) The fan is called complete if $|\Delta| = N_{\mathbb{R}}$.

PROPOSITION 3.4.2. *The variety* $X(\Delta)$ *is proper if and only if* Δ *is complete.*

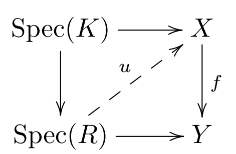
Proof. We prove the easy direction. Assume that Δ is not complete. Then there exists $v \in N$ with $v \notin |\Delta|$. By Lemma 3.2.5, $\lim_{t\to 0} \lambda_v(t).p_0$ does not exist in any U_σ thus it does not exist in $X(\Delta)$ proving that $X(\Delta)$ is not proper. The converse will follow from a relative version. $\square$

Recall that an equivariant map of toric varieties $\varphi : X(\Delta) \to X(\Delta')$ mapping the dense orbit to the dense orbit is given by a morphism of fans $\phi : \Delta \to \Delta'$.

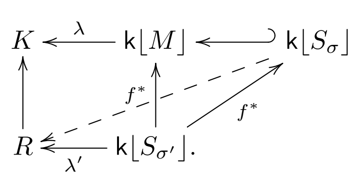
THEOREM 3.4.3. *The map* φ *is proper if and only if* $\phi^{-1}(|\Delta'|) = |\Delta|$.

REMARK 3.4.4. Note that we always have $|\Delta| \subset \phi^{-1}(|\Delta'|)$.

Proof. We will use the valuative criterion of properness: a morphism $f : X \to Y$ is proper if and only if for any discrete valuation ring R with quotient field K and any commutative diagram



can be filled with the dashed arrow u so that the diagram remains commutative. Replacing $X(\Delta)$ with the closure of the orbit containing the image of $\mathrm{Spec}(K)$, we may assume that $\mathrm{Spec}(K)$ is mapped to the open orbit T_N in $X(\Delta)$. Let $\sigma' \in \Delta'$ be such that $\mathrm{Spec}(R)$ maps to $U_{\sigma'}$. The map $\mathrm{Spec}(K) \to T_N \subset X(\Delta)$ is given by $\lambda : \mathsf{k}\lfloor M \rfloor \to K, \chi^m \mapsto \gamma(m)$ where $\gamma : M \to (K, \times)$ is a morphism of monoids (actually of groups). We thus have a commutative diagram



Let ν be the valuation of K, define the element $n_f \in N$ via $\langle m, n_f\rangle = \nu(\lambda(\chi^m))$ for all $m \in M$. We have $\phi(n_f) \in ((\sigma')^\vee)^\vee = \sigma'$. Indeed, if $m' \in S_{\sigma'}$, we have

$\langle m', \phi(n_f)\rangle = \langle \phi^\vee(m'), n_f\rangle = \nu(\lambda(\chi^{\phi^\vee(m')})) = \nu(\lambda(f^*(\chi^{m'}))) = \nu(\lambda'(\chi^{m'})) \geq 0$ since λ' maps $\mathsf{k}\lfloor S_{\sigma'}\rfloor$ to R. By assumption, there exists a cone $\sigma \in \Delta$ with $n_f \in \sigma$ thus for any $m \in S_\sigma$, we have $\nu(\lambda(\chi^m) = n_f(m) \geq 0$ thus λ maps $\mathsf{k}\lfloor S_\sigma\rfloor$ to R. We thus get the above diagram. This proves that $\mathrm{Spec}(R) \to X(\Delta')$ lifts to $X(\Delta)$.

Conversely, if $v' \in |\Delta'|$ is not in the image of any element in $|\Delta|$, consider the cocharacter $\lambda_{v'}$ of T' and for $v \in N$ with $\phi(v) = v'$ consider λ_v the cocharacter of T. If $p_0 \in X(\Delta)$ and $p_0' \in X(\Delta')$ are the unit elements in T and T', then $\lim_{t\to 0} \lambda_{v'}(t).p_0'$ has a limit in $X(\Delta')$ but $\lambda_v(t).p_0$ has no limit in $X(\Delta)$ proving that the map is not proper. □

5. Polytopes and fans

We describe a combinatorial gadget that produces fans.

Definition 3.5.1. Let V be a real vector space.

(1) A convex polytope $P \subset V$ is the convex hull of finitely many points.
(2) A face of P is the set of point $\{v \in P \mid \langle w, v\rangle = r\}$ where $w \in W = V^\vee$ and $r \in \mathbb{R}$ is such that $\langle w, v\rangle \geq r$ for all $v \in P$.
(3) If P' is a face of P, we write $P' \preceq P$.
(4) The dimension of a polytope is the dimension of its span.
(5) A face of codimension 1 is called a facet.
(6) A face of dimension 0 is called a vertex.
(7) A face of dimension 1 is called an edge.

Remark 3.5.2. P and $\emptyset$ are faces of P.

Remark 3.5.3. P is the convex hull of its vertices.

We construct a cone out of a polytope. Let $E = \mathbb{R} \times V$ and embbed P in E via $\varphi : P \to E, v \mapsto (1, v)$. Define the cone $\sigma_P \subset E$ as the cone generated by $\varphi(P)$. If $w \in W = V^\vee$ and $r \in \mathbb{R}$, define $f_{r,w} \in E^\vee$ via $f_{r,w}(\lambda, v) = r\lambda + \langle w, v\rangle$.

Remark 3.5.4. If P is the convex hull of a finite set S, then σ_P is the cone generated by $\varphi(S) = \{(s, 1) \in E \mid s \in S\}$.

Proposition 3.5.5. *Let P be a polytope. There is a bijection $\{P' \mid P' \preceq P\} \to \{\tau \mid \tau \preceq \sigma_P\}$ given by $P' \mapsto \sigma_{P'}$.*

Proof. If P' is defined by the condition $\langle w, v\rangle = r$, then $f_{-r,w} \in \sigma_P^\vee$ and $\sigma_{P'} = \sigma_P \cap f_{-r,w}^\perp$ is a face of σ_P. Conversely, if $f \in \sigma_P^\vee$ and $\tau = \sigma_P \cap f^\perp$ is a face, then set $w = f|_{0\times V}$, $r = -f(1, 0)$ and $P' = \{v \in P \mid \langle w, v\rangle = r\}$. Note that if $v \in P$, then $(1, v) \in \sigma_P$ and $0 \leq f(1, v) = f(1, 0) + f(0, v) = -r + \langle w, v\rangle$ thus $\langle w, v\rangle \geq r$. In particular $P' \preceq P$ and the previous computation shows that $\sigma_{P'} = \tau$. These maps are inverse from each other proving the claim. □

Appying results in Section 1, we get.

Corollary 3.5.6. *Let P be a convex polytope.*

(1) *Any face of P is a convex polytope.*
(2) *Any intersection of faces of P is a face of P.*
(3) *Any face of a face is a face.*
(4) *Any proper face is contained in a facet.*
(5) *Any proper face of codimension 2 is the intersection of exactly 2 facets.*

(6) *Any proper face is the intersection of the facets containing it.*
(7) *The relative topological boundary of P is the union of its proper facets.*

As for cones, there is a duality theory for polytopes. For simplicity we assume that P contains 0 in its interior. Set $W = V^\vee$.

DEFINITION 3.5.7. Let $P \subset V$ be a polytope. The polar P° of P is the set

$$P^\circ = \{w \in W \mid \langle w, v\rangle \geq -1, \text{ for all } v \in P\}.$$

REMARK 3.5.8. Mapping P° in $\mathbb{R} \times W$ via $w \mapsto f_{1,w}$, we may define $\sigma_{P^\circ} \subset E^\vee$ as the cone generated by the image of P°. It is easy to check that $\sigma_{P^\circ} = \sigma_P^\vee$.

EXAMPLE 3.5.9. We recover the duality between regular solid polyhedra:

P	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
P°	Tetrahedron	Octahedron	Cube	Icosahedron	Dodecahedron

PROPOSITION 3.5.10. *Let P be a polytope.*
(1) *P° is a convex polytope (since 0 is an interior point of P).*
(2) *We have $P^{\circ\circ} = P$*
(3) *If $F \preceq P$, then $F^* := \{w \in P^\circ \mid \langle w, v\rangle = -1, \forall v \in F\}$ is a face of P°.*
(4) *The correspondence $F \mapsto F^*$ is an involutive order reversing bijection between faces of P and faces of P°.*
(5) *We have* $\dim F + \dim F^* = \dim V - 1$.

Proof. The only non trivial statement is (1), we leave it as an exercise. For (2)-(5), apply Lemma 2.1.3, Corollary 2.1.20 and Proposition 2.1.20 to the cone construction of Proposition 3.5.5. □

Let M be a lattice such that $V = M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. Set $N = M^\vee \subset W = V^\vee$.

DEFINITION 3.5.11. A convex polytope $P \subset V$ is rational if P is the convex hull of points in N (equivalently the vertices of P lie in N).

PROPOSITION 3.5.12. *The faces of a rational polytope are rational polytopes.*

Proof. Faces are convex hulls of subset of vertices in P. □

We construct fans from a polytope P. We write $\mathring{P}$ for the interior of P.

DEFINITION 3.5.13. Let $P \subset V$ be a rational polytope and assume that $0 \in \mathring{P}$.
(1) For $F \preceq P$, let $\sigma(F)$ be the cone generated by F.
(2) Let $\Delta(P) = \{\sigma(F) \mid F \prec P\}$.

PROPOSITION 3.5.14. *$\Delta(P)$ is a complete fan in V.*

Proof. Exercise. □

We now construct a fan from a rational polytope $P \subset W = M \otimes_Z \mathbb{R}$.

DEFINITION 3.5.15. Let $P \subset W$ be a rational polytope of maximal dimension.
(1) For $Q \preceq P$, set $\sigma_Q = \{v \in V \mid \langle w', v\rangle \geq \langle w, v\rangle \text{ for all } w \in Q, w' \in P\}$.
(2) Define $\Delta_P = (\sigma_Q)_{Q \preceq P}$.

Let P be a rational polytope of maximal dimension in W.

PROPOSITION 3.5.16. *Let $Q \preceq P$.*

(1) *σ_Q is a rational convex polyhedral cone in V.*
(2) *$\sigma_Q^\vee$ is the cone generated by the vectors $w' - w$ with $w \in Q$, $w' \in P$.*
(3) *Δ_P is a fan in V.*
(4) *If $0 \in \mathring{P}$, then Δ_P is the fan of cones over the faces of P°: $\Delta_P = \Delta(P^\circ)$.*

Proof. (1) and (2) Since elements in Q and P are linear combination of the vertices of Q and P respectively, the elements of the form $w' - w$ with $w \in Q$, $w' \in P$ are linear combination with non-negative coefficients of finitely many elements in M.

(3) Let $w_1, \cdots, w_r$ be the vertices of P. Up to reordering, we may assume that the vertices of Q are $v_1, \cdots, v_s$. Then $\sigma_Q^\vee$ is generated by $(\pm w_i)_{i \in [1,s]}$ and $(w_j - w_i)_{i \leq s < j}$. A facet of σ_Q is obtained by taking $\sigma_Q \cap (w_j - w_i)^\perp$ for some pair $i \leq s < j$. We get that the dual cone is generated by the generators of $\sigma_Q^\vee$ and $w_j - w_i$ thus $\sigma_Q \cap (w_j - w_i)^\perp = \sigma_F$ with F generated by Q and $w_j - w_i$. Faces of σ_Q are therefore in Δ_P. If Q and F are faces of P generated by $(w_i)_{i \in I}$ and $(w_i)_{i \in J}$ for some subsets I and J of $[1, r]$, then $\sigma_Q \cap \sigma_F$ has dual cone generated by $(w_i)_{i \in I \cup J}$ and if R is the face generated by these vectors, we have $\sigma_Q \cap \sigma_F = \sigma_R$.

(4) Exercise. □

CHAPTER 4

Divisors and line bundles

1. Reminders on divisors

Recall the following definitions. We refer to [5, Chapter II.6] for more details.

Definition 4.1.1. Let X be a normal irreducible variety.

(1) A prime divisor is a reduced irreducible closed subset of codimension 1.
(2) A Weil divisor is a finite formal sum $\sum_i n_i D_i$ with D_i a prime divisor.
(3) Denote by $\mathrm{Weil}(X)$ the group of all Weil divisors in X.
(4) $D = \sum_i n_i D_i \in \mathrm{Weil}(X)$ is effective if $n_i \geq 0$ for all i. We write $D \geq 0$.
(5) A Cartier divisor D is the data of a covering $(U_i)_i$ of X by affine open subset and non-zero rational functions $f_i \in \mathsf{k}(X)$ such that f_i/f_j is a nowhere vanishing regular function on $U_{i,j} = U_i \cap U_j$.
(6) Denote by $\mathrm{Ca}(X)$ the group of all Cartier divisors in X.
(7) $D = (U_i, f_i) \in \mathrm{Ca}(X)$ is effective if $f_i \in \mathsf{k}[U_i]$ for all i. We write $D \geq 0$.
(8) If D is a Cartier divisor, the ideal sheaf $\mathcal{O}_X(-D)$ of D is the subsheaf of the constant sheaf of rational functions generated by f_i on U_i:

$$\mathcal{O}_X(-D)(U_i) = f_i \mathcal{O}_X(U_i) \subset \mathsf{k}(X).$$

(9) The sheaf $\mathcal{O}_X(-D)$ is invertible with inverse $\mathcal{O}_X(D)$ given by

$$\mathcal{O}_X(D)(U_i) = \frac{1}{f_i} \mathcal{O}_X(U_i) \subset \mathsf{k}(X).$$

(10) The line bundle $\mathcal{O}_X(D)$ has a canonical section $\mathcal{O}_X \to \mathcal{O}_X(D)$ defined on each U_i by $\mathcal{O}_X(U_i) \to f_i \mathcal{O}_X(U_i), f \mapsto f/f_i$.
(11) If $D = (U_i, f_i) \in \mathrm{Ca}(X)$ is effective then $\mathcal{O}_X(-D) \subset \mathcal{O}_X$ is an ideal subsheaf of $\mathcal{O}_X$. We call D the closed subset defined by this ideal sheaf.
(12) Since X is normal, for D a Weil divisor, the local ring $\mathcal{O}_{X,D}$ is a discrete valuation ring. In particular, we may define $\mathrm{ord}_D(f)$ for any $f \in \mathsf{k}(X)$.
(13) If $D = (U_i, f_i)_i$ is a Cartier divisor, define

$$[D] = \sum_{D'} \mathrm{ord}_{D'}(f_i) D'.$$

Since f_i/f_j is nowhere vanishing and regular, the value of $\mathrm{ord}_{D'}(f_i)$ does not depend on i. Then $[D]$ is a Weil divisor on X.
(14) $D \in \mathrm{Ca}(X)$ is effective if and only if $[D] \in \mathrm{Weil}(X)$ is effective.
(15) If $f \in \mathsf{k}(X)$, define

$$\mathrm{div}(f) = \sum_D \mathrm{ord}_D(f) D.$$

A divisor of this form is called principal.
(16) Denote by $\mathrm{Princ}(X)$ the group of principal divisors.

Remark 4.1.2. The map $\mathrm{Weil}(X) \to \mathrm{Ca}(X)$, $D = (U_i, f_i)_i \mapsto [D]$ is injective. The group structure on $\mathrm{Ca}(X)$ is multiplicative on functions. This inclusion is a group morphism. We will view $\mathrm{Ca}(X)$ as a subgroup of $\mathrm{Weil}(X)$ and use an additive notation.

2. T-divisors

From now $X = X(\Delta)$ is a T-toric variety.

Definition 4.2.1. Let $X = X(\Delta)$ be a T-toric variety.

(1) A T-prime divisor is prime divisor stable by T.
(2) A T-Weil divisor is a Weil divisor $\sum_i n_i D_i$ where D_i is a T-prime divisor.
(3) Denote by $\mathrm{Weil}_T(X)$ the group of T-Weil divisors.
(4) A T-Cartier divisor is a Cartier divisor D such that $[D] \in \mathrm{Weil}_T(X)$.
(5) Denote by $\mathrm{Ca}_T(X)$ the group of T-Cartier divisors.
(6) We write $\Delta(k)$ for the k-dimensional cones in Δ.

Remark 4.2.2. T-prime divisors are the closures of T-orbits of codimension 1 and thus of the form $V(\tau)$ for $\tau \in \Delta(1)$. Set $D_\tau = V(\tau)$ for $\tau \in \Delta(1)$.

Lemma 4.2.3. *Assume that $X = U_\sigma$ with $\dim \sigma = \dim X$ and let $D \in \mathrm{Ca}_T(X)$. Then there exists a unique $m \in M$ such that $[D] = \mathrm{div}(\chi^m)$.*

Proof. If $[D] = 0$, then $[D] = \mathrm{div}(1) = \mathrm{div}(\chi^0)$. Furthermore, if $[D] = \mathrm{div}(\chi^m)$ for some $m \in M$, then $\chi^m \in \mathsf{k}[U_\sigma]$ is invertible *i.e.* both m and $-m$ are in S_σ which means that $m = 0$, since σ is of maximal dimension.

We now assume $D > 0$. In that case $\mathcal{O}_X(-D)$ is a proper ideal sheaf of $\mathcal{O}_X$ and $I = H^0(X, \mathcal{O}_X(-D)) = \{a \in \mathcal{O}_X(X) \mid a = 0 \text{ or } \mathrm{div}(a) \geq [D]\}$. We thus have an ideal $I \subsetneq \mathsf{k}[U_\sigma] = \mathcal{O}_X(X)$ and since D is T-stable, so is I. In particular

$$I = \bigoplus_{\chi^m \in I} \mathsf{k}\chi^m \subset \bigoplus_{m \neq 0} \mathsf{k}\chi^m = \mathfrak{M},$$

where $\mathfrak{M} \subset \mathsf{k}[U_\sigma]$ is the maximal ideal of the point $p_\sigma \in U_\sigma$. We want to prove that I is generated by some χ^m for some $m \in M$. Since D is Cartier, there is an affine open subset $U \ni p_\sigma$ and $f \in \mathsf{k}[U]$ such that $H^0(U, \mathcal{O}_X(-D)) = f\mathsf{k}[U]$. Let $\mathsf{k}[U_\sigma] \to \mathsf{k}[U] \to \mathsf{k}[U_\sigma]_{(\mathfrak{M})}$ be the localisation map and for $J \subset \mathsf{k}[U_\sigma]$, set $J_{(\mathfrak{M})} = J\mathsf{k}[U_\sigma]_{(\mathfrak{M})}$. We have $I_{(\mathfrak{M})} = (f)\mathsf{k}[U_\sigma]_{(\mathfrak{M})}$ thus $I/I\mathfrak{M} = I_{(\mathfrak{M})}/(I\mathfrak{M})_{(\mathfrak{M})}$ is generated by one element (the class of f). By the graded version of Nakayama's Lemma, we get that $I = (\chi^m)$ for some $m \in M$ thus $[D] = \mathrm{div}(\chi^m)$.

Now assume that D is general and write $D = D_+ - D_-$ with $D_+, D_- \geq 0$ and apply the previous result to get $m_+, m_- \in M$ with $\mathrm{div}(\chi^{m_+}) = [D_+]$, $\mathrm{div}(\chi^{m_-}) = [D_-]$ and thus $[D] = \mathrm{div}(\chi^{m_+ - m_-})$.

The fact that m is unique works as in the case $D = 0$. □

Definition 4.2.4. Let $\sigma \in \Delta$.

(1) Define $N_\sigma = \langle \sigma \rangle \cap N$ and $N(\sigma) = N/N_\sigma$.
(2) Define $M(\sigma) = \sigma^\perp \cap M$ and $M_\sigma = M/M(\sigma)$.
(3) The duality $M = N^\vee$ induces dualities $M_\sigma = N_\sigma^\vee$ and $M(\sigma) = N(\sigma)^\vee$.

Lemma 4.2.5. *Assume that $X = U_\sigma$ is affine and let $D \in \mathrm{Ca}_T(X)$.*

(1) $\mathrm{div}(\chi^m) = 0$ *for $m \in M(\sigma)$, so that* $\mathrm{div}(\chi^m)$ *for $m \in M_\sigma$ is well defined.*
(2) *Then there exists a unique $m \in M_\sigma$ such that $[D] = \mathrm{div}(\chi^m)$.*

Proof. (1) Let $m \in M(\sigma)$, then $\pm m \in S_\sigma$ so that $\chi^m \in \mathsf{k}[U_\sigma]^\times$ *i.e.* $\mathrm{div}(\chi^m) = 0$.

(2) Let σ' be the cone σ seen as a subset of N_σ and $S_{\sigma'} \subset M_\sigma$ the corresponding monoid. Set $U_{\sigma'} = \mathrm{Spec}(\mathsf{k}\lfloor S_{\sigma'} \rfloor)$. We have $S_\sigma = M(\sigma) \times S_{\sigma'}$ thus $U_\sigma = T_{M(\sigma)} \times U_{\sigma'}$. Now the prime T-divisors in U_σ are of the form $T_{M(\sigma)} \times D'$ with D' a prime T_{M_σ}-divisor in $U_{\sigma'}$ and we may apply Lemma 4.2.3 to $U_{\sigma'}$. □

COROLLARY 4.2.6. *We have equality* $\mathrm{Ca}_T(X) = \varprojlim M/M(\sigma)$ *or more explicitely*

$$\mathrm{Ca}_T(X) = \mathrm{Ker}\left(\bigoplus_{\substack{\sigma \in \Delta \\ \sigma \text{ maximal}}} M/M(\sigma) \to \bigoplus_{\substack{\sigma, \tau \in \Delta, \sigma \neq \tau \\ \sigma, \tau \text{ maximal}}} M/(M(\sigma) \cap M(\tau)) \right).$$

Let $\tau \in \Delta(1)$, since $X = X(\Delta)$ is normal, the local ring $\mathcal{O}_{X,D_\tau}$ is a DVR and we have a valuation $\nu_\tau : \mathsf{k}(X) \to \mathbb{Z}$ defined by $\nu_\tau(f) = \mathrm{ord}_{D_\tau}(f)$.

LEMMA 4.2.7. *Let* $X = X(\Delta)$ *and* $\tau \in \Delta(1)$. *Let* $v_\tau \in \tau \cap N$ *be the minimal generator of the ray* τ.

(1) *Then for any* $m \in M$, *we have* $\nu_\tau(\chi^m) = \langle m, v_\tau \rangle$.
(2) *In particular,* $\mathrm{div}(\chi^m) = \sum_{\tau \in \Delta(1)} \langle m, v_\tau \rangle D_\tau$, *for* $m \in M$.

Proof. (2) Follows from (1). Let's prove (1). Since v_τ is the minimal generator of the ray τ, we may find a basis $(e_1, \cdots, e_n)$ of the lattice N such that $e_1 = v_\tau$. If $(m_1, \cdots, m_n)$ is the dual basis in M, we have $S_\tau = \mathbb{Z}_{\geq 0} m_1 + \sum_{i=2}^n \mathbb{Z} m_i$ and $\mathsf{k}\lfloor S_\tau \rfloor = \mathsf{k}[X_1, X_2^{\pm 1} \cdots, X_n^{\pm 1}]$. We get $U_\tau = \mathrm{Spec}(\mathsf{k}\lfloor S_\tau \rfloor) = \mathsf{k} \times (\mathsf{k}^\times)^{n-1}$ and U_τ contains a unique T-prime divisor D_τ given by the equation $\chi^{m_1} = X_1 = 0$. We thus have $\mathcal{O}_{X,D_\tau} = \mathsf{k}[X_1, \cdots, X_n]_{(X_1)}$. If $m = \sum_i a_i m_i$ with $a_i \in \mathbb{Z}$ is a character, then $\chi^m = \prod_i (X_i)^{a_i}$ and $\nu_\tau(\chi^m) = m_1 = \langle m, e_1 \rangle = \langle m, v_\tau \rangle$. □

3. Line bundles

Recall the definition of Chow groups and Picard groups.

DEFINITION 4.3.1. Let X be a toric variety of dimension n.

(1) Define the Chow group of divisors $\mathrm{CH}_{n-1}(X) = \mathrm{Weil}(X)/\mathrm{Princ}(X)$.
(2) Define the Picard group $\mathrm{Pic}(X) = \mathrm{Ca}(X)/\mathrm{Princ}(X)$.

REMARK 4.3.2. Since $\mathrm{Ca}(X) \subset \mathrm{Weil}(X)$, we have $\mathrm{Pic}(X) \subset \mathrm{CH}_{n-1}(X)$.

THEOREM 4.3.3. *Let* $X = X(\Delta)$ *be a toric variety.*

(1) *We have an exact sequence* $M \to \mathrm{Weil}_T(X) \to \mathrm{CH}_{n-1}(X) \to 0$.
(2) *We have an exact sequence* $M \to \mathrm{Ca}_T(X) \to \mathrm{Pic}(X) \to 0$.
(3) *The above exact sequences are exact on the left iff* $(v_\tau)_{\tau \in \Delta(1)}$ *spans* V.

Proof. (1) We will first need the following result: if $U = \mathrm{Spec}(A)$ with A an UFD, then $\mathrm{CH}_{n-1}(U) = 0$ (see [5, Proposition II.6.2]). Consider $Z = \cup_{\tau \in \Delta(1)} D_\tau$ and $U = X \setminus Z$, then $U = T = \mathrm{Spec}(A)$ with $A = \mathsf{k}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$ which is an UFD. In particular $\mathrm{CH}_{n-1}(U) = 0$. Now use the exact sequence $\mathrm{CH}_{n-1}(Z) = \oplus_{\tau \in \Delta(1)} \mathbb{Z}[D_\tau] \to \mathrm{CH}_{n-1}(X) \to \mathrm{CH}_{n-1}(U) \to 0$ to get that $\mathrm{CH}_{n-1}(X)$ is generated by T-stable divisors. Now By Lemma 4.2.5, if $f \in \mathsf{k}(X)$ is such that $\mathrm{div}(f) \in \mathrm{Ca}_T(X)$ is a T-stable divisor, we have $f = \lambda \chi^m$ for some $m \in M$ and $\lambda \in \mathsf{k}^\times$. We thus have a surjective map $M \to \mathrm{Princ}(X)$, $m \mapsto \mathrm{div}(\chi^m)$. This proves (1). For (2), restrict the previous exact sequence to Cartier divisors.

(3) If m is such that $\mathrm{div}(\chi^m) = 0$, we have by Lemma 4.2.7, that $\langle m, v_\tau \rangle = 0$ for all $\tau \in \Delta(1)$. The equivalence follows from this. □

COROLLARY 4.3.4. *If* $\Delta(n) \neq \emptyset$, *then* $\mathrm{Pic}(X)$ *is torsion free.*

Proof. Assume that $D \in \mathrm{Ca}_T(X)$ is such that $kD \in \mathrm{Princ}(X)$ *i.e.* $kD = \mathrm{div}(\chi^m)$ for some $m \in M$. Write

$$D = \sum_{\tau \in \Delta(1)} n_\tau D_\tau.$$

Let $\sigma \in \Delta$ of maximal dimension n, then by Lemma 4.2.3, there exists a unique $m' \in M$ such that $D|_{U_\sigma} = \mathrm{div}(\chi^{m'})$. In particular, for $\tau \in \Delta(1)$ such that $\tau \prec \sigma$, we have $n_\tau = \langle m', v_\tau \rangle$ and $kn_\tau = \langle m, v_\tau \rangle$. Since σ is of dimension n, we get $m = km'$ thus $D = \mathrm{div}(\chi^{m'})$. □

COROLLARY 4.3.5. $\mathrm{Pic}(X) \simeq \mathbb{Z}^{|\Delta(1)|-n}$ *for* X *is smooth with* $\Delta(n) \neq \emptyset$.

Proof. Since X is smooth, we have $\mathrm{Pic}(X) = \mathrm{CH}_{n-1}(X)$. Furthermore $\mathrm{Pic}(X)$ and $\mathrm{CH}_{n-1}(X)$ are torsion free so we only need to compute the rank. The rsult now follows from the exact sequences in Theorem 4.3.3. □

EXAMPLE 4.3.6. Let $n = 2$ and $N = \mathbb{Z}^2$.

(1) Consider the cone σ generated by $2e_1 - e_2$ and $v_2 = e_2$ and Δ the fan whose elemenets are the faces of σ. Set $X = X(\Delta)$. Then $\Delta(1) = \{\tau, \gamma\}$ with $v_\tau = 2e_1 - e_2$ and $v_\gamma = e_2$. The image of M in $\mathbb{Z}D_\tau \oplus \mathbb{Z}D_\gamma$ is given by the image of the matrix

$$A = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$$

thus $\mathrm{CH}_1(X) = \mathrm{Coker}(A) = \mathbb{Z}/2\mathbb{Z}$ while $\mathrm{Pic}(X) = 0$.

(2) Keep the previous notation and let $\Delta' = \{0, \tau, \gamma\}$. Then $X' = X(\Delta') = X \setminus V(\sigma)$ (note that $V(\sigma)$ is a closed point) is smooth and $\mathrm{Pic}(X') = \mathrm{CH}_1(X') = \mathrm{CH}_1(X) = \mathbb{Z}/2\mathbb{Z}$. In particular the assumption in the previous corollary is necessary.

(3) Let $v_1 = 2e_1 - e_2$, $v_2 = -e_1 + 2e_2$, $v_3 = -e_1 - e_2$ and Δ be the complete fan with $\Delta(1) = \{v_1, v_2, v_3\}$ and $X = X(\Delta)$. For $i \in [1, 3]$, let $\tau_i = \mathbb{R}_{\geq 0} v_i$, $D_i = D_{\tau_i}$ and σ_i be the cone generated by $(v_j)_{j \neq i}$.

We have $\mathrm{CH}_1(X) = \mathrm{Coker}(A : \mathbb{Z}^2 \to \mathbb{Z}^3)$ with

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \\ -1 & -1 \end{pmatrix}$$

thus $\mathrm{CH}_1(X) = (\mathbb{Z}D_1 \oplus \mathbb{Z}D_2)/\mathbb{Z}(3(D_1 - D_2)) \simeq \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. In particular $\mathrm{CH}_1(X)$ has torsion even if Δ contains cones of maximal dimension.

On the other hand, for $D = a_1 D_1 + a_2 D_2 + a_3 D_3$, we have $D \in \mathrm{Ca}_T(X)$ if and only if there exists $m_1, m_2, m_3 \in \mathbb{Z}^2$ such that

(a) $\langle m_1, v_j \rangle = a_j$ for $j \neq 1$,
(b) $\langle m_2, v_j \rangle = a_j$ for $j \neq 2$,
(c) $\langle m_3, v_j \rangle = a_j$ for $j \neq 3$.

In particular, writing $m_i = (x_i, y_i)$, we see that $a_1 = 2x_2 - y_2$ and $a_3 = -x_2 - y_2$ thus $a_1 - a_3 = 3x_2$ so that $a_1 \equiv a_3 \pmod 3$. The same arguments show that a necessary condition for D to be Cartier is that $a_1 \equiv a_2 \equiv a_3 \pmod 3$. Conversely if this is true, set $m_1 = (2(a_2 - a_3)/3 - a_2, (a_2 - a_3)/3)$, $m_2 = ((a_1 - a_3)/3, 2(a_1 - a_3)/3 - a_1)$ and $m_3 = ((2a_1 + a_2)/3, 2(2a_1 + a_2)/3 - a_1)$. These elements satisfy the desired conditions.

In particular, we get $\mathrm{Pic}(X) = \mathbb{Z}(3D_1) \simeq \mathbb{Z}$.

Proposition 4.3.7. *Let X be a toric variety.The following are equivalent*

(1) $\mathrm{Weil}(X) = \mathrm{Ca}(X)$.
(2) $\mathrm{CH}_{n-1}(X) = \mathrm{Pic}(X)$.
(3) X *is smooth.*

Proof. The first two are clearly equivalent. If X is smooth then every Weil divisor is Cartier. Assume the converse and let $\sigma \in \Delta$. Since $\mathrm{CH}_{n-1}(X) \to \mathrm{CH}_{n-1}(U_\sigma)$ is surjective, we have that every Weil divisor in U_σ is Cartier and since every Cartier divisor in U_σ is principal by Theorem 4.3.3 and Lemma 4.2.5, we get that the map $M \to \oplus_{\tau\in\sigma(1)}\mathbb{Z}D_\tau$ is surjective and therefore the family $(v_\tau)_{\tau\in\sigma(1)}$ is part of a basis *i.e.* U_σ is smooth. □

Definition 4.3.8. Let Δ be a fan and σ be a cone.

(1) σ is simplicial if σ is generated by part of a basis of V.
(2) Δ is simplicial if σ is simplicial for all $\sigma \in \Delta$.

With the same proof as the previous proposition we can prove the following.

Proposition 4.3.9. *Let X be a toric variety.The following are equivalent*

(1) $\mathrm{Weil}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Ca}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.
(2) $\mathrm{CH}_{n-1}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.
(3) Δ *is simplicial.*

Definition 4.3.10. A variety such that $\mathrm{CH}_{n-1}(X) = \mathrm{Pic}(X)$ is called locally factorial. A variety with $\mathrm{CH}_{n-1}(X)\otimes_{\mathbb{Z}}\mathbb{Q} = \mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ is called locally $\mathbb{Q}$-factorial.

4. Piecewise linear functions

By the previous discussion, if $D \in \mathrm{Ca}_T(X)$, then for each cone $\sigma \in \Delta$, the divisor $D|_{U_\sigma}$ is principal and there exists a unique $m_\sigma \in M_\sigma$ such that $(D + \mathrm{div}(\chi^{m_\sigma})|_{U_\sigma} = 0$. If $D \in \mathrm{Weil}_T(X)$, then D is Cartier if and only if $D|_{U_\sigma}$ is Cartier for all $\sigma \in \Delta$. Furthermore it is enough to check this on maximal cones

In other words, if we write

$$D = \sum_{\tau\in\Delta(1)} n_\tau D_\tau,$$

then D is Cartier if and only if for each maximal cone $\sigma \in \Delta$, there exists $m_\sigma \in M_\sigma$ such that $\langle m_\sigma, v_\tau\rangle = -n_\tau$ for each $\tau \in \sigma(1)$ with the compatibility conditions $[m_\sigma] = [m_{\sigma'}] \in M/(M(\sigma) + M(\sigma')) = M/M(\sigma\cap\sigma')$ for maximal cones $\sigma, \sigma' \in \Delta$.

Definition 4.4.1. Let $D \in \mathrm{Ca}_T(X)$ and $m_\sigma \in M_\sigma$ with $(D + \mathrm{div}(\chi^{m_\sigma})|_{U_\sigma} = 0$ for $\sigma \in \Delta$ maximal. Define a piecewise linear function $\psi_D : |\Delta| \to \mathbb{R}$ by

$$\psi_D(v) = \langle m_\sigma, v\rangle \text{ for } v \in \sigma.$$

The compatibility conditions imply that ψ_D is well defined and continuous: for $\sigma, \sigma' \in \Delta$ maximal cones such that $v \in \sigma \cap \sigma'$, we have $\langle m_\sigma, v\rangle = \langle m_{\sigma'}, v\rangle$.

REMARK 4.4.2. The function ψ_D is characterised by the following formula:

$$D = \sum_{\tau\in\Delta(1)} (-\psi_D(v_\tau))D_\tau.$$

In particular, we have the following formulas for $D, E \in \mathrm{Ca}_T(X)$ and $k \in \mathbb{Z}$:

$$\psi_{D+E} = \psi_D + \psi_E \text{ and } \psi_{kD} = k\psi_D.$$

For principal divisors, we have $\psi_{\mathrm{div}(\chi^m)}(v) = -\langle m, v\rangle$ for all $v \in V$. In particular D and E are linearly equivalent (*i.e.* $D - E \in \mathrm{Princ}(X)$) iff $\psi_D - \psi_E$ is linear.

DEFINITION 4.4.3. Let Δ be a fan and let $\langle|\Delta|\rangle$ be the linear span of $|\Delta|$.

(1) A piecewise linear function on Δ is a function $\varphi : |\Delta| \to \mathbb{R}$ such that the restriction of φ on each cone of Δ is linear. We denote by $\mathrm{PL}(\Delta)$ the space of piecewise linear function on Δ. Such a function is called integral if $\varphi(N \cap |\Delta|) \subset \mathbb{Z}$. We denote by $\mathrm{PL}(\Delta, N)$ the space of integral piecewise linear functions.
(2) A linear function on Δ is a function $\varphi : |\Delta| \to \mathbb{R}$ such that φ is linear on $\langle|\Delta|\rangle$. We denote by $\mathrm{L}(\Delta)$ the space of linear function on Δ. Such a function is called integral if $\varphi(N \cap |\Delta|) \subset \mathbb{Z}$. We denote by $\mathrm{L}(\Delta, N)$ the space of integral linear functions.

The previous discussion implies the following.

PROPOSITION 4.4.4. *We have an isomorphism* $\mathrm{Pic}(X) \simeq \mathrm{PL}(\Delta, N)/\mathrm{L}(\Delta, N)$.

5. Sections of line bundles

DEFINITION 4.5.1. Let $D = \sum_{\tau\in\Delta(1)} n_\tau D_\tau \in \mathrm{Ca}_T(X)$. Define

$$P_D = \{w \in W \mid \langle w, v_\tau\rangle \geq -n_\tau, \forall\tau \in \Delta(1)\} = \{w \in W \mid w \geq \psi_D \text{ on } |\Delta|\}.$$

REMARK 4.5.2. The set P_D is convex: if $w, w' \in P_D$ and $t \in [0, 1]$, then $\langle tw + (1-t)w', v\rangle = t\langle w, v\rangle + (1-t)\langle w', v\rangle \geq t\psi_D(v) + (1-t)\psi_D(v) = \psi_D(v)$. However, in general P_D is not bounded and not rational.

PROPOSITION 4.5.3. *We have* $H^0(X, \mathcal{O}_X(D)) = \bigoplus_{m\in P_D\cap M} \mathsf{k}\chi^m$.

Proof. First consider the case of U_σ for $\sigma \in \Delta$. Then $H^0(U_\sigma, \mathcal{O}_X(D)) = \{f \in \mathsf{k}(X) \mid (\mathrm{div}(f) + D)|_{U_\sigma} \geq 0\}$. Since T-acts on $H^0(U_\sigma, \mathcal{O}_X(D))$, any element in $H^0(U_\sigma, \mathcal{O}_X(D))$ is a linear combination of characters $\chi^m \in H^0(U_\sigma, \mathcal{O}_X(D))$. Furthermore for $m \in M$, we have $\chi^m \in H^0(U_\sigma, \mathcal{O}_X(D))$ if and only if $m \geq \psi_D$ on $|\sigma|$. Let $P_D(\sigma) = \{w \in W \mid \langle w, v_\tau\rangle \geq -n_\tau, \forall\tau \in \sigma(1)\}$, then $H^0(U_\sigma, \mathcal{O}_X(D)) = \bigoplus_{m\in P_D(\sigma)\cap M} \mathsf{k}\chi^m$. We get

$$H^0(X, \mathcal{O}_X(D)) = \bigcap_{\sigma\in\Delta} H^0(U_\sigma, \mathcal{O}_X(D)) = \bigcap_{\sigma\in\Delta}\left(\bigoplus_{m\in P_D(\sigma)\cap M} \mathsf{k}\chi^m\right)$$

proving the result. ∎

COROLLARY 4.5.4. *If $\Delta(1)$ spans V as a cone, then $H^0(X, \mathcal{O}_X(D))$ is finite dimensional. In particular if Δ is complete $H^0(X, \mathcal{O}_X(D))$ is finite dimensional.*

Proof. Assume that the one-dimensional cones in Δ span V and that P_D is unbounded. Then there exists a sequence $(w_k)_{k\geq 0}$ of elements in P_D such that $\lim_{k\to+\infty}\|w_k\| = +\infty$. Let $w_k = w_k/\|w_k\|$. Then, up to extracting a subsequence, we may assume that $(w_k)_{k\geq 0}$ converges to w with $\|w\| = 1$. We have $\langle w_k, v_\tau\rangle \geq \psi_D(v_\tau)$ for any $\tau \in \Delta(1)$, thus $\langle w_k/\|w_k\|, v_\tau\rangle \geq \psi_D(v_\tau)/\|w_k\|$ and taking the limit, we get $\langle w, v_\tau\rangle \geq 0$ for $\tau \in \Delta(1)$. Since $\Delta(1)$ spans V as a cone, this implies that $w = 0$, contradicting the condition $\|w\| = 1$. □

EXAMPLE 4.5.5. In general P_D is not a convex polytope. Indeed, let $N = \mathbb{Z}^2$ with canonical basis (e_1, e_2) and let $\Delta = \{\{0\}, \mathbb{R}_{\geq 0}e_1\}$.Then $X = X(\Delta) = \mathrm{Spec}(\mathsf{k}[X_1, X_2^{\pm 1}]) \simeq \mathbb{A}^1_{\mathsf{k}} \times \mathbb{G}_m$ Let $D = D_\tau$ with $\tau = \mathbb{R}_{\geq 0}e_1$. Then

$$P_D = \{(m_1, m_2) \in \mathbb{Z}^2 = M \mid m_1 \geq -1\}.$$

REMARK 4.5.6. Let $D \in \mathrm{Ca}_T(X)$ be a T-Cartier divisor and for $\sigma \in \Delta$, let $m_\sigma \in M_\sigma$ such that $(D + \mathrm{div}(\chi^{m_\sigma}))|_{U_\sigma} = 0$. Any $s \in H^0(U_\sigma, \mathcal{O}_X(D))$ induces a morphism of invertible sheaves

$$\mathcal{O}_X|_{U_\sigma} \to \mathcal{O}_X(D)|_{U_\sigma}.$$

Using the trivialisation $H^0(U_\sigma, \mathcal{O}_X(D)) \simeq \mathsf{k}[U_\sigma]$ given by $f \leftrightarrow f\chi^{-m_\sigma}$, we get that the above map of sheaves induces a map $\mathsf{k}[U_\sigma] \to H^0(U_\sigma, \mathcal{O}_X(D)) \simeq \mathsf{k}[U_\sigma]$ given by $g \mapsto gs\chi^{-m_\sigma}$. In particular if s is T-semiinvariant, *i.e.* $s = \chi^m$ for some $m \in M$ with $m + m_\sigma \in S_\sigma$, then the map is given by

$$g \mapsto g\chi^{m-m_\sigma}.$$

If $\tau \in \sigma(1)$, then s vanishes on D_τ is and only if $\langle m - m_\sigma, v_\tau\rangle > 0$. In particular $s = \chi^m$ does not vanish on U_σ if and only if $m = m_\sigma$ on σ.

6. Globally generated line bundles

In this section, we assume that Δ is a fan such that all maximal cones are of dimension n. Recall the following basic definition.

DEFINITION 4.6.1. A function $\psi : A \to \mathbb{R}$ is convex if

$$\psi(tv + (1-t)v') \geq t\psi(v) + (1-t)\psi(v')$$

for all $v, v' \in A$ and $t \in [0, 1]$ such that $v, v', tv + (1-t)v' \in A$.

Let $D \in \mathrm{Ca}_T(X)$ and recall the definition of $\psi_D : |\Delta| \to \mathbb{R}$ as $\psi_D(v) = \langle m_\sigma, v\rangle$ where $m_\sigma \in M_\sigma$ is the unique element such that $(D + \mathrm{div}(\chi^{m_\sigma}))|_{U_\sigma} = 0$.

LEMMA 4.6.2. *ψ_D is convex iff $\psi_D(v) \leq \langle m_\sigma, v\rangle$ for all $\sigma \in \Delta$ and $v \in |\Delta|$.*

Proof. Assume first that ψ_D is convex. Let $v \in \sigma$ and choose $v' \in \mathring{\sigma}'$ and $t > 0$ such that $tv + (1-t)v' \in \sigma'$. By convexity, we have $\psi_D(tv + (1-t)v') \geq t\psi_D(v) + (1-t)\psi_D(v')$. Evaluating these functions, we get $\langle m_{\sigma'}, tv + (1-t)tv'\rangle \geq t\psi_D(v) + (1-t)\langle m_{\sigma'}, v'\rangle$ proving the inequality $\langle m_{\sigma'}, v\rangle \geq \psi_D(v)$. Conversely, assume that for any $\sigma \in \Delta$ and any $v \in |\Delta|$, we have $\psi_D(v) \leq \langle m_\sigma, v\rangle$. Let $v, v' \in |\Delta|$ and $t \in [0, 1]$ such that $tv + (1-t)v' \in \Delta$. Then there exists $\sigma \in \Delta$ such that $tv + (1-t)v' \in \sigma$ and $\psi_D(tv + (1-t)v') = \langle m_\sigma, tv + (1-t)v'\rangle \geq t\psi_D(v) + (1-t)\psi_D(v')$. □

COROLLARY 4.6.3. *ψ_D is convex iff $m_\sigma \in P_D$ for all $\sigma \in \Delta$ maximal.*

Definition 4.6.4. The function $\psi_D : |\Delta| \to \mathbb{R}$ is strictly convex if for $\sigma, \sigma' \in \Delta(n)$ and $v' \in \mathring{\sigma}' \setminus \sigma$, we have $\langle m_\sigma, v' \rangle > \psi_D(v')$.

Assume that ψ_D is convex.

Corollary 4.6.5. *ψ_D is strictly convex iff $m_\sigma \neq m_{\sigma'}$ for $\sigma \neq \sigma'$ in $\Delta(n)$.*

Proof. Assume that ψ_D is strictly convex and let $\sigma \neq \sigma'$ in $\Delta(n)$. Let $v' \in \sigma' \setminus \sigma$, then $\langle m_\sigma, v' \rangle > \psi_D(v') = \langle m_{\sigma'}, v \rangle$ thus $m_\sigma \neq m_{\sigma'}$. Conversely, let $v' \in \mathring{\sigma}' \setminus \sigma$ with $\sigma, \sigma' \in \Delta(n)$. Since $m_\sigma \neq m_{\sigma'}$, there exists $v'' \in \sigma'$ such that $\langle m_{\sigma'}, v'' \rangle \neq \langle m_\sigma, v'' \rangle$. Since ψ_D is convex, we have $\langle m_\sigma, v'' \rangle \geq \psi_D(v'') = \langle m_{\sigma'}, v'' \rangle$ thus $\langle m_{\sigma'}, v'' \rangle > \langle m_\sigma, v'' \rangle$. Let $H = \mathrm{Ker}(m_\sigma - m_{\sigma'})$. Then σ' is contained in the halfspace on which $m_\sigma - m_{\sigma'}$ is non-negative, while σ is on the half space where it is non-positive. In particular, since $v' \in \mathring{\sigma}' \setminus \sigma$, we have $\langle m_\sigma - m_{\sigma'}, v' \rangle > 0$ proving the claim. □

Proposition 4.6.6. *Assume that all maximal cones in Δ are of dimension n and let $D \in \mathrm{Ca}_T(X)$. Then $\mathcal{O}_X(D)$ is globally generated if and only if ψ_D is convex.*

Proof. The line bundle $\mathcal{O}_X(D)$ is globally generated iff for any $x \in X$ there exists $s_x \in H^0(X, \mathcal{O}_X(D))$ such that $s_x(x) \neq 0$.

Since T acts on X and D is stable under this action, if there exists $x \in X$ and s_x such that $s_x(x) \neq 0$ as above, then the same is true on the T-orbit $T.x$. Indeed, we have $(t.s_x)(t.x) = s_x(t^{-1}.(t.x)) = s_x(x) \neq 0$. In particular, since the locus where this does not occur is closed and T-stable, we see that $\mathcal{O}_X(D)$ is globally generated if and only if for each maximal cone $\sigma \in \Delta$ (or equivalently each closed T-fixed point p_σ), there exists a global section $s_\sigma \in H^0(X, \mathcal{O}_X(D))$ with $s_\sigma(p_\sigma) \neq 0$.

Assume first that s_σ exists and write $s_\sigma = \sum_{m \in P_D \cap M} a_m \chi^m$ (with finitely many non-vanishing a_m). For all m with $a_m \neq 0$ and any $\tau \in \Delta(1)$, we have $\langle m, v_\tau \rangle \geq -n_\tau$. The condition $s_\sigma(p_\sigma) \neq 0$ implies that there is at least one m with $a_m \neq 0$ such that $\langle m, v_\tau \rangle = -n_\tau$ for all $\tau \in \sigma(1)$. This implies that $m = m_\sigma$ thus $m_\sigma \in P_D \cap M$ and $\langle m_\sigma, v_\tau \rangle \geq \psi_D(v_\tau)$ for any $\tau \in \Delta(1)$, ψ_D is convex. Conversely, if ψ_D is convex, let $s = \chi^{m_\sigma}$, then $m_\sigma \in P_D \cap M$ and $s \in H^0(X, \mathcal{O}_X(D))$ is such that $s(p_\sigma) \neq 0$ proving that $\mathcal{O}_X(D)$ is globally generated. □

Exercise 4.6.7. Prove that $P_{D+E} = P_D + P_E$ for D and E globally generated.

7. Ample and very ample line bundles

Recall that $\Delta(n)$ is the set of cones of dimension n in Δ. Assume that X is complete *i.e.* $|\Delta| = V$. In particular $\Delta(n)$ is the set of maximal cones in Δ. For any $D \in \mathrm{Ca}_T(X)$, the set $P_D \cap M$ is finite so that $H^0(X, \mathcal{O}_X(D))$ is finite dimensional. Assume that $\mathcal{O}_X(D)$ is globaly generated, then we have a map

$$\varphi_D : X \to \mathbb{P}(H^0(X, \mathcal{O}_X(D))^\vee), x \mapsto [s \mapsto s(x)].$$

Definition 4.7.1. Let $S_{D,\sigma}$ be the monoid spanned by $\{m - m_\sigma \mid m \in P_D \cap M\}$.

Remark 4.7.2. For any $m \in P_D \cap M$ and $\tau \in \sigma(1)$, we have $\langle m, v_\tau \rangle \geq \psi_D(v_\tau) = \langle m_\sigma, v_\tau \rangle$ thus $m - m_\sigma \in \sigma^\vee \cap M$ and $S_{D,\sigma} \subset S_\sigma$.

Proposition 4.7.3. *φ_D is a closed embedding if and only ψ_D is strictly convex and $S_{D,\sigma} = S_\sigma$ for all $\sigma \in \Delta(n)$.*

Proof. Assume that ψ_D is strictly convex and for any maximal cone $\sigma \in \Delta$, the monoid S_σ is generated by $\{m - m_\sigma \mid m \in P_D \cap M\}$. For σ such a maximal cone, let m_σ be such that $(D + \mathrm{div}(\chi^{m_\sigma}))|_{U_\sigma}$. Let $U \subset \mathbb{P}(H^0(X, \mathcal{O}_X(D))^\vee)$ be defined by $\{[\phi] \mid \phi(\chi^{m_\sigma}) \neq 0\}$ (recall that $m_\sigma \in P_D$ since D is globally generated). We have $\varphi_D^{-1}(U) = \{x \in X \mid \chi^{m_\sigma}(x) \neq 0\}$ and, since ψ_D is strictly convex, we get $\varphi_D^{-1}(U) = U_\sigma$. Furthermore $\mathsf{k}[U]$ is the polynomial algebra over $S_{D,\sigma} = \{m - m_\sigma \mid m \in P_D \cap M \setminus \{0\}\}$ and the second assumption implies that the map $\mathsf{k}[U] \to \mathsf{k}\lfloor S_\sigma \rfloor$ is surjective proving that φ_D is a closed embedding.

Assume conversely that φ_D is a closed embedding. Then $\varphi^{-1}(U)$ is an affine open subset which has to be T-stable and containing U_σ. By the classification of affine toric varieties, we have $\varphi_D^{-1}(U) = U_\sigma$ which proves that ψ_D is strictly convex. The second assertion follows readily from the fact that the map $\mathsf{k}[U] \to \mathsf{k}\lfloor S_\sigma \rfloor$ is surjective. □

Corollary 4.7.4. *D is ample iff ψ_D is strictly convex.*

Proof. If D is ample, then kD is very ample for some $k > 0$ large enough and φ_{kD} is a closed embedding. By the previous proposition, we get that $k\psi_D = \psi_{kD}$ is strictly convex thus ψ_D is strictly convex.

Conversely, assume that ψ_D is strictly convex. Let $m \in S_\sigma$. By strict convexity, we have $\langle m_\sigma, v_\tau \rangle > \psi_D(v_\tau)$ for any $\tau \in \Delta(1) \setminus \sigma(1)$ and $\langle m_\sigma, v_\tau \rangle = \psi_D(v_\tau)$ for $\tau \in \sigma(1)$. In particular for $k > 0$ large enough, we have $\langle m + km_\sigma, v_\tau \rangle \geq k\psi_D(v_\tau)$ for any $\tau \in \Delta(1)$ thus $m + km_\sigma \in P_{kD}$. Choose k such that all generators of S_σ satisfy the previous property, then $\{m' - km_\sigma \mid m' \in P_{kD} \cap M\}$ generate S_σ. □

Corollary 4.7.5. *If D is ample, then D is globally generated.*

We now get some more information on the set P_D. Let $D \in \mathrm{Ca}_T(X)$ and assume that D is globally generated.

Lemma 4.7.6. *We have $\psi_D(v) = \min_{w \in P_D} \langle w, v \rangle = \min_{\sigma \in \Delta(n)} \langle m_\sigma, v \rangle$.*

Proof. Since $m_\sigma \in P_D$ for $\sigma \in \Delta(n)$, for any $v \in |\Delta|$ and $m \in P_D$, we have $\langle m, v \rangle \geq \psi_D(v)$ thus $\min_{\sigma \in \Delta(n)} \langle m_\sigma, v \rangle \geq \min_{w \in P_D} \langle w, v \rangle \geq \psi_D(v)$. Furthermore $v \in |\Delta|$, thus $v \in \sigma$ for some $\sigma \in \Delta(n)$ and $\psi_D(v) = \langle m_\sigma, v \rangle$, proving the result. □

Proposition 4.7.7. *P_D is the convex hull of its vertices $(m_\sigma)_{\sigma \in \Delta(n)}$.*

Proof. We have already seen that since D is globally generated, we have $m_\sigma \geq \psi_D$ *i.e.* $m_\sigma \in P_D$ for all $\sigma \in \Delta(n)$. We first prove that if $\sigma \in \Delta(n)$, then m_σ is a vertex of P_D. Let $v \in \mathring{\sigma}$ *i.e.* $v = \sum_{\tau \in \sigma(1)} \lambda_\tau v_\tau$ with $\lambda_\tau > 0$. Set $a = \psi_D(v)$. Then for any $m \in P_D$, we have $\langle m, v \rangle \geq \psi_D(v) = a$ thus $H_{v,a} = \{w \in W \mid \langle w, v \rangle = a\}$ is a supporting hyperplane of P_D. Furthermore if $w \in P_D \cap H_{v,a}$, then $\sum_{\tau \in \sigma(1)} \lambda_\tau \langle w, v_\tau \rangle = \langle w, v \rangle = a = \psi_D(v) = \langle m_\sigma, v \rangle = \sum_{\tau \in \sigma(1)} \lambda_\tau \langle m_\sigma, v_\tau \rangle = \sum_{\tau \in \sigma(1)} \lambda_\tau \psi_D(v_\tau)$. We get $\sum_{\tau \in \sigma(1)} \lambda_\tau (\langle w, v_\tau \rangle - \psi_D(v_\tau)) = 0$. But since $w \in P_D$, we have $\langle w, v_\tau \rangle - \psi_D(v_\tau) \geq 0$ and since $\lambda_\tau > 0$, we get the equality $\langle w, v_\tau \rangle = \psi_D(v_\tau)$ for every $\tau \in \sigma(1)$. This implies that w and ψ_D agree on σ, thus w and m_σ agree on σ, and finally $w = m_\sigma$.

Assume now that $m \in P_D$ is a vertex and let $v \in V$ and $a \in \mathbb{R}$ such that $H_{v,a} = \{w \in W \mid \langle w, v \rangle = a\}$ is a supporting hyperplane of P_D such that $P_D \cap H_{v,a} = \{m\}$. We thus have $\langle w, v \rangle \geq a$ for any $w \in P_D$ with equality iff $w = m$. We compute

$\psi_D(v) = \min_{w\in P_D}\langle w, v\rangle = \langle m, v\rangle = a$. But for $\sigma \in \Delta(n)$ such that $v \in \sigma$, we have $a = \psi_D(v) = \langle m_\sigma, v\rangle$ and this implies $m_\sigma \in H_{v,a} \cap P_D$ thus $w = m_\sigma$. □

Corollary 4.7.8. *P_D is a rational polytope.*

Recall Definition 3.5.15 of the fan Δ_P associated to a rational polytope P.

Proposition 4.7.9. *If D is ample, then P_D is of dimension n and $\Delta = \Delta_{P_D}$.*

Proof. If P_D is not of dimension n, then there exists $v \in V\setminus\{0\}$ and $a \in \mathbb{R}$ such that $P_D \subset H_{v,a} = \{w \in W | \langle w, v\rangle = a\}$. Let $\sigma \in \Delta(n)$, then $m_\sigma \in P_D$ and $\langle m_\sigma, v\rangle = a$. But there exists $\sigma_0 \in \Delta(n)$ such that $v \in \sigma_0$ and we get $\psi_D(v) = \langle m_{\sigma_0}, v\rangle = a$. By strict convexity of ψ_D, this implies that $v \in \sigma$ for all $\sigma \in \Delta(n)$ and since Δ is complete, we get $v = 0$, a contradiction.

Let $\sigma \in \Delta(n)$, then m_σ is a vertex of P_D and we want to prove that the cone generated by $\{m - m_\sigma \mid m \in P_D\}$ is $\sigma^\vee$. If $m \in P_D$, and $\tau \in \sigma(1)$, we have $\langle m, v_\tau\rangle \geq \langle m_\sigma, v_\tau\rangle$, thus $m - m_\sigma \in \sigma^\vee$. Conversely, since Δ is complete, the cone $\sigma^\vee$ is generated by non-zero vectors $w_{\sigma,\sigma'} \in \sigma^\vee$ such that $w_{\sigma,\sigma'} \in (\sigma \cap \sigma')^\perp$ for $\sigma' \in \Delta(n)$ adjacent to σ. Now for such a σ', we have $m_{\sigma'} - m_\sigma \in \sigma^\vee \setminus \{0\}$ and $m_{\sigma'} - m_\sigma \in (\sigma \cap \sigma')^\perp$ so that we may choose $w_{\sigma,\sigma'} = m_{\sigma'} - m_\sigma$. We get that $\{m - m_\sigma \mid m \in P_D\}$ generates $\sigma^\vee$. □

Remark 4.7.10. We have $\sigma^\vee = \sigma_{S_{D,\sigma}}$ is the cone generated by $S_{D,\sigma}$.

Theorem 4.7.11. *If X is smooth, D is ample iff D is very ample.*

Proof. Assume that D is ample, then ψ_D is strictly convex. We only need to prove that $S_{D,\sigma} = S_\sigma$ for all $\sigma \in \Delta(n)$. By the previous remark, we have $\sigma_{S_{D,\sigma}} = \sigma^\vee = \sigma_{S_\sigma}$. Since X is smooth S_σ is generated by the minimal elements in M in the rays of the cone $\sigma^\vee$. These rays are generated by $m_{\sigma'} - m_\sigma$ for σ' adjacent to σ. Let $m = \lambda(m_{\sigma'} - m_\sigma)$ with $\lambda \in]0, 1]$ be the minimal element in $\mathbb{R}_{\geq 0}(m_{\sigma'} - m_\sigma) \cap M$ ($\lambda \leq 1$ since $m_{\sigma'} - m_\sigma \in M$). Then $\lambda(m_{\sigma'} - m_\sigma) + m_\sigma = \lambda m_{\sigma'} + (1 - \lambda)m_\sigma \in P_D$ thus $m \in S_{D,\sigma}$ proving the result. □

8. Proper versus projective

A toric variety $X = X(\Delta)$ is proper if and only if Δ is complete *i.e.* $|\Delta| = V$. A complete variety will be projectif if and only if it admits an ample line bundle. In this section, we start with an example of a proper but not projective toric variety and prove a toric version of Chow's Lemma. Recall that a morphism between irreducible varieties is birational if it is an isomorphism on non-empty open subsets.

Theorem 4.8.1 (Chow's Lemma). *Let Y be complete and irreducible. Then there exists a birational morphism $f : Y' \to Y$ with Y projective and irreducible.*

Let $N = \mathbb{Z}^3$ and $V = N_R = \mathbb{R}^3$. Let $(e_i)_{i\in[1,3]}$ be the canonical basis and consider the following elements in V:

$$v_1 = -e_1 \quad ; \quad v_2 = -e_2 \quad ; \quad v_3 = -e_3 \quad ; \quad v_4 = e_1 + e_2 + e_3$$
$$v_5 = v_3 + v_4 = e_1 + e_2 \quad ; \quad v_6 = v_1 + v_4 = e_2 + e_3 \quad ; \quad v_7 = v_2 + v_4 = e_1 + e_3.$$

For any $i, j, k \in [1, 7]$ distinct, let $\sigma_{i,j,k}$ be the cone generated by $\{v_i, v_j, v_k\}$. Consider the fan Δ whose set of maximal cones has the following 10 elements:

$$\Delta_{\max} = \{\sigma_{1,2,3}; (\sigma_{1,2,6}, \sigma_{2,6,7}, \sigma_{4,6,7}); (\sigma_{1,3,5}, \sigma_{1,5,6}, \sigma_{4,5,6}); (\sigma_{2,3,7}, \sigma_{3,5,7}, \sigma_{4,5,7})\}.$$

We gathered those which are in the same plane. Note that since v_1, v_2, v_3, v_4 span V as a cone, the fan Δ is complete thus $X = X(\Delta)$ is a complete variety.

PROPOSITION 4.8.2. *X is not projective.*

Proof. Assume that there exists $D \in \mathrm{Ca}_T(X)$ such that $\mathcal{O}_X(D)$ is ample. This means that ψ_D is strictly convex. We prove that there can be no such strictly convex piecewise linear function ψ on Δ.

Assume that ψ exists and for $\sigma \in \Delta_{\max}$, let $m_\sigma \in M$ be such that $\psi(v) = \langle m_\sigma, v\rangle$ for $v \in \sigma$. Consider the vector $v_1 + v_7 = e_3 = v_2 + v_6$. We have $e_3 \in \sigma_{1,2,6} \cap \sigma_{2,6,7}$, $v_7 \in \sigma_{2,6,7}$ but $v_1 \notin \sigma_{2,6,7}$. Thus $\psi(v_1 + v_7) = \langle m_{\sigma_{2,6,7}}, v_1 + v_7\rangle$ and $\psi(v_7) = \langle m_{\sigma_{2,6,7}}, v_7\rangle$, while $\langle m_{\sigma_{2,6,7}}, v_1\rangle > \psi(v_1)$ by strict convexity. This gives

$$\psi(v_1 + v_7) = \langle m_{\sigma_{2,6,7}}, v_1 + v_7\rangle = \langle m_{\sigma_{2,6,7}}, v_1\rangle + \langle m_{\sigma_{2,6,7}}, v_7\rangle > \psi(v_1) + \psi(v_7).$$

Since however $v_2, v_6 \in \sigma_{1,2,6}$, we have $\psi(v_1+v_7) = \psi(v_2+v_6) = \langle m_{\sigma_{1,2,6}}, v_2+v_6\rangle = \langle m_{\sigma_{1,2,6}}, v_2\rangle + \langle m_{\sigma_{1,2,6}}, v_6\rangle = \psi(v_2) + \psi(v_6)$. We finally get the inequality

$$\psi(v_2) + \psi(v_6) > \psi(v_1) + \psi(v_7).$$

The same argument gives the inequalities

(1) $\psi(v_2) + \psi(v_6) > \psi(v_1) + \psi(v_7)$
(2) $\psi(v_1) + \psi(v_5) > \psi(v_3) + \psi(v_6)$
(3) $\psi(v_3) + \psi(v_7) > \psi(v_2) + \psi(v_5)$.

Adding up these inequalities, we get a contradiction. □

We prove that up to subdivising the fan, we may obtain a projective variety.

DEFINITION 4.8.3. Let Δ be a fan. A fan Δ' is a refinement of Δ if

(1) $|\Delta'| = |\Delta|$ and
(2) for $\sigma' \in \Delta'$, there exists $\sigma \in \Delta$ such that $\sigma' \subset \sigma$.

Note from Proposition 3.3.4 that a refinement Δ' of Δ induces a proper birational (and thus surjective) T-equivariant morphism $\phi : X(\Delta') \to X(\Delta)$. Assume that Δ is complete.

THEOREM 4.8.4. *Δ has a refinement Δ' such that $X(\Delta')$ is projective.*

Proof. Let $\Delta(n-1)$ be the set of cones of codimension 1. For any $\gamma \in \Delta(n-1)$, there exists $m_\gamma \in M$ such that $\langle\gamma\rangle = \{v \in V \mid \langle m_\gamma, v\rangle = 0\}$, where $\langle\gamma\rangle$ is the span of γ. Consider the hyperplane arrangement $H = \bigcup_{\gamma\in\Delta(n-1)} H_\gamma$ and set $U = V \setminus H$. Each connected component of U is of the form $\mathring{\sigma}'$ for an n-dimensional rational convex polyhedral cone σ'. Define Δ' to be the fan whose maximal cones are the cones σ' obtained this way using all connected component of U. Note in particular, that for any $\gamma \in \Delta(n-1)$, the hyperplane H_γ is a union of elements in $\Delta'(n-1)$ and that any element in $\Delta'(n-1)$ is a maximal cone in H_γ for some $\gamma \in \Delta(n-1)$. Note that Δ' is a refinement of Δ. We thus have a proper surjective birational T-equivariant morphism $X' = X(\Delta') \to X(\Delta) = X$.

We prove that X' is projective. It is enough to prove that X' has an ample divisor and thus to produce a strictly convex integral piecewise linear function ψ on Δ'. Let $\Gamma \subset \Delta(n-1)$ be minimal such that $H = \bigcup_{\gamma\in\Gamma} H_\gamma$. Define

$$\psi(v) = -\sum_{\gamma\in\Gamma} |\langle m_\gamma, v\rangle|.$$

We claim that ψ is a strictly convex integral piecewise linear function. Since $m_\gamma \in M$, it is integral. Let σ' be the closure of a connected component of U. Then $\mathring{\sigma}' \cap H_\gamma = \emptyset$ for all $\gamma \in \Gamma$. In particular for any $\gamma \in \Gamma$), the function $|\langle m_\gamma, -\rangle|$ restricted to σ' is either equal to $\langle m_\gamma, -\rangle|_{\sigma'}$ or to $-\langle m_\gamma, -\rangle|_{\sigma'}$ and is therefore linear. This proves that ψ is piecewise linear. For $t \in [0,1]$ and $v, v' \in V$, we have $|\langle m_\gamma, tv + (1-t)v'\rangle| = |\langle m_\gamma, tv\rangle + \langle m_\gamma, (1-t)v'\rangle| \leq |\langle m_\gamma, tv\rangle| + |\langle m_\gamma, (1-t)v'\rangle| = t|\langle m_\gamma, v\rangle| + (1-t)|\langle m_\gamma, v'\rangle|$. This implies that $\psi(tv+(1-t)v') \geq t\psi(v) + (1-t)\psi(v')$ proving that ψ is convex.

We are left to proving that ψ is strictly convex. Let $\gamma'_0 \in \Delta'(n-1)$ and let $\sigma'_1; \sigma'_2 \in \Delta'(n)$ such that $\gamma' = \sigma'_1 \cap \sigma'_2$. There exists a unique $\gamma_0 \in \Gamma$ such that $\gamma'_0 \subset H_{\gamma_0}$ and σ'_1, σ'_2 are on both sides of H_{γ_0}. Assume that m_{γ_0} is non-negative on σ'_1 (and thus non-positive on σ'_2). We have

(1) $\psi(v) = -\langle m_{\gamma_0}, v\rangle - \sum_{\gamma\in\Gamma, \gamma\neq\gamma_0} |\langle m_\gamma, v\rangle|$ for $v \in \sigma'_1$ and
(2) $\psi(v) = \langle m_{\gamma_0}, v\rangle - \sum_{\gamma\in\Gamma, \gamma\neq\gamma_0} |\langle m_\gamma, v\rangle|$ for $v \in \sigma'_2$.

Note that σ'_1 and σ'_2 are on the same side of H_γ for any $\gamma \in \Gamma$ such that $\gamma \neq \gamma_0$. The map $v \mapsto \sum_{\gamma\in\Gamma,\gamma\neq\gamma_0} |\langle m_\gamma, v\rangle|$ is therefore linear on $\sigma'_1 \cup \sigma'_2$. In particular, ψ is represented by different linear functions on σ'_1 and σ'_2. The result follows from Lemma 4.8.5 below. □

LEMMA 4.8.5. *Let ψ be a convex piecewise linear function on a fan Δ such that for each maximal cone $\sigma \in \Delta(n)$ and $v \in \sigma$, we have $\psi(v) = \langle m_\sigma, v\rangle$. Assume that for any $\gamma \in \Delta(n-1)$ such that $\gamma = \sigma_1 \cap \sigma_2$ with $\sigma_1, \sigma_2 \in \Delta(n)$, we have $\sigma_1 \neq \sigma_2 \Rightarrow m_{\sigma_1} \neq m_{\sigma_2}$, then ψ is strictly convex.*

Proof. Let $\sigma, \sigma' \in \Delta(n)$ such that $\sigma \neq \sigma'$. Since ψ is convex, we have $\langle m_\sigma, v\rangle = \psi(v) \leq \langle m_{\sigma'}, v\rangle$ for $v \in \sigma$. Thus if $H^{\geq 0}_{\sigma,\sigma'} = \{v \in V \mid \langle m_\sigma - m_{\sigma'}, v\rangle \geq 0\}$ and $H^{\leq 0}_{\sigma,\sigma'} = \{v \in V \mid \langle m_\sigma - m_{\sigma'}, v\rangle \leq 0\}$, we have $\sigma \subset H^{\leq 0}_{\sigma,\sigma'}$ and $\sigma' \subset H^{\geq 0}_{\sigma,\sigma'}$.

In particular, for $\gamma \in \Delta(n-1)$ such that $\gamma = \sigma \cap \sigma'$ with $\sigma, \sigma' \in \Delta(n)$, we have $\psi(v) = \langle m_\sigma, v\rangle \leq \langle m_{\sigma'}, v\rangle$ for any $v \in \sigma$. If furthermore $v \in \mathring{\sigma}$, then $v \notin \mathrm{Ker}(m_\sigma - m_{\sigma'})$ thus $\psi(v) = \langle m_\sigma, v\rangle < \langle m_{\sigma'}, v\rangle$.

Let $v \in V$ and $\sigma \in \Delta(n)$ such that $v \notin \sigma$. We may choose $v' \in \mathring{\sigma}$ such that the segment $[v, v'] = \{tv + (1-t)v' \mid t \in [0,1]\}$ meets each $\gamma \in \Delta(n-1)$ transversaly. Let $\{\gamma_1, \cdots, \gamma_r\} = \{\gamma \in \Delta(n-1) \mid \gamma \cap [v, v'] \neq \emptyset\}$. For each $i \in [1, r]$, there exists a unique $t_i \in [0,1]$ such that $\gamma_i \cap [v, v'] = \{t_i v + (1-t_i)v'\}$. Up to reordering, we may assume that $t_1 < \cdots < t_r$. Let $\sigma_0, \cdots, \sigma_r \in \Delta(n-1)$ be such that $v \in \sigma_0$, $\sigma_r = \sigma$ and $\gamma_i = \sigma_{i-1} \cap \sigma_i$ for all $i \in [1, r]$. Since $\sigma_{i-1} \subset H^{\leq 0}_{\sigma_{i-1},\sigma_i}$ and $\sigma_i \subset H^{\geq 0}_{\sigma_{i-1},\sigma_i}$, we must have $v \in H^{\leq 0}_{\sigma_{i-1},\sigma_i}$. In particular, we get

$$\langle m_{\sigma_0}, v\rangle \leq \langle m_{\sigma_1}, v\rangle \leq \cdots \leq \langle m_{\sigma_r}, v\rangle.$$

Furthermore, there exists an index i such that $v \in \sigma_i$ and $v \notin \sigma_{i+1}$ (actually $i = 0$ if $v \notin \sigma_1$ and $i = 1$ if $v \in \sigma_0 \cap \sigma_1$) for which we have $\langle m_{\sigma_i}, v\rangle < \langle m_{\sigma_{i+1}}, v\rangle$, proving that ψ is strictly convex. □

CHAPTER 5

Birational maps and resolution of singularities

In this chapter we presend a view basic birational transformations and an explicit algorithmic resolution of toric singularities.

1. Blowing-up

As observed in the previous section, any refinement Δ' of a fan Δ induces a proper birational T-equivariant morphism $X(\Delta') \to X(\Delta)$. Furthermore, any proper birational T-equivariant morphism $X(\Delta') \to X(\Delta')$ is induced by a refinement. In this section, we describe the refinement associated to the blow-up of a smooth toric variety along the closure of a T-orbit and discuss resolutions of singularities.

Let Δ be a fan and for each $\tau \in \Delta(1)$, let $v_\tau \in \tau \cap N$ be the minimal generator.

Definition 5.1.1. Let $\sigma \in \Delta(n)$ and $\gamma \in \Delta$.

(1) If $\sigma \in \Delta$ is a smooth cone, let $v_1, \cdots, v_n \in \sigma \cap N$ such that $(v_1, \cdots, v_n)$ is a basis of N. Let $v_0 = v_1 + \cdots + v_n$. The start subdivision $\Delta^*(\sigma)$ of Δ along σ is defined by

$$\Delta^*(\sigma) = (\Delta \setminus \{\sigma\}) \cup \Delta'(\sigma)$$

where $\Delta'(\sigma)$ is the set of all cones generated by subsets of $\{v_0, \cdots, v_n\}$ not containing $\{v_1, \cdots, v_n\}$.

(2) If all cones containing γ are smooth, define $v_\gamma = \sum_{\tau \in \gamma(1)} v_\tau$ and for each $\sigma \in \Delta$ with $\gamma \preceq \sigma$, define $\Delta^*_\sigma(\tau)$ as the set of all cones generated by subsets of $\sigma(1) \cup \{v_\gamma\}$ not containing $\gamma(1)$. The star subdivision $\Delta^*(\gamma)$ of Δ relative to γ is the fan

$$\Delta^*(\gamma) = \{\sigma \in \Delta \mid \gamma \not\subset \sigma\} \cup \bigcup_{\tau \subset \sigma} \Delta^*_\sigma(\gamma).$$

Remark 5.1.2. For $\gamma = \sigma \in \Delta(n)$ smooth, $\Delta'(\sigma) = \Delta^*_\sigma(\gamma)$ and $\Delta^*(\gamma) = \Delta^*(\sigma)$.

Lemma 5.1.3. *Let $\sigma \in \Delta(n)$ be a smooth cone.*

(1) *Then $\Delta^*(\sigma)$ is a refinement of Δ.*

(2) *The map $X(\Delta^*(\sigma)) \to X(\Delta)$ is the blow-up of $X(\Delta)$ along the point p_σ.*

Proof. Since Δ and $\Delta^*(\sigma)$ agree outside of σ, we may assume that $\Delta = \{\gamma \preceq \sigma\}$. In that case $X(\Delta) = U_\sigma$ is affine.

(1) This is an easy exercise.

(2) Choosing a basis of σ which is a basis of N, we identify U_σ with $\mathbb{A}^n_{\mathsf{k}}$ and P_σ with the origin $0 \in \mathbb{A}^n_{\mathsf{k}}$. The blow-up X' of $0 \in \mathbb{A}^n_{\mathsf{k}}$ is described as follows:

$$X' = \{([x_1 : \cdots, x_n], (y_1, \cdots, y_n)) \in \mathbb{P}^{n-1}_{\mathsf{k}} \times \mathbb{A}^n_{\mathsf{k}} \mid (y_1, \cdots, y_n) \in \langle (x_0, \cdots, x_n) \rangle \}.$$

For $i \in [1,n]$, let $U_i = \{[x_1 : \cdots , x_n] \in \mathbb{P}^{n-1}_{\mathsf{k}} \mid x_i \neq 0\}$. Then we have a T-equivariant isomorphism

$$X'_i := X' \cap (U_i \times \mathbb{A}^n_{\mathsf{k}}) \simeq \operatorname{Spec}\left(\mathsf{k}\left[\frac{x_1}{x_i}, \cdots , \frac{x_n}{x_i}, y_i\right]\right).$$

These open subsets are glued together on $X'_i \cap X'_j$ via the identification

$$\mathsf{k}\left[\frac{x_1}{x_i}, \cdots , \frac{x_n}{x_i}, y_i, \left(\frac{x_j}{x_i}\right)^{-1}\right] \simeq \mathsf{k}\left[\frac{x_1}{x_j}, \cdots , \frac{x_n}{x_j}, y_j, \left(\frac{x_i}{x_j}\right)^{-1}\right],$$

$$\text{with } \frac{x_a}{x_i} = \frac{x_a}{x_j}\frac{x_j}{x_i} \text{ for all } a \in [1,n] \text{ and } y_i = y_j\frac{x_i}{x_j}.$$

Consider now the toric $X'' = X(\Delta^*(\sigma)$. It is covered by n affine open toric varieties X''_i given by the cone generated by $v_0, v_1, \cdots , v_{i-1}, v_{i+1}, \cdots , v_n$. If $(m_1, \cdots , m_n)$ is the basis of M dual to $(v_1, \cdots , v_n)$, then the monoids associated to the previous cone is generated by $m_1 - m_i, \cdots , m_{i-1} - m_i, m_i, m_{i+1} - m_i, \cdots m_n - m_i$. In particular, we have a T-equivariant isomorphism

$$X''_i = \operatorname{Spec}\left(\mathsf{k}\left[\frac{X_1}{X_i}, \cdots , \frac{X_n}{X_i}, X_i\right]\right)$$

where $X_a = \chi^{m_a}$ for all $a \in [1,n]$. These open subsets are glued together on $X''_i \cap X''_j$ via the identification

$$\mathsf{k}\left[\frac{X_1}{X_i}, \cdots , \frac{X_n}{X_i}, X_i\right] \simeq \mathsf{k}\left[\frac{X_1}{X_j}, \cdots , \frac{X_n}{X_j}, X_j\right],$$

$$\text{with } \frac{X_a}{X_i} = \frac{X_a}{X_j}\frac{X_j}{X_i} \text{ for all } a \in [1,n] \text{ and } X_i = X_j\frac{X_i}{X_j}.$$

This proves that both constructions agree and gives the result. □

PROPOSITION 5.1.4. *Let $\gamma \in \Delta$ such that all cones containing γ are smooth.*

(1) *Then $\Delta^*(\gamma)$ is a refinement of Δ.*
(2) *The map $X(\Delta^*(\gamma)) \to X(\Delta)$ is the blow-up of $X(\Delta)$ along $V(\tau)$.*

Proof. It is enough to prove this locally and since both maps restrict to the identity on $U_\sigma \cap V(\tau) = \emptyset$, we may assume that this intersection is non-empty. This is equivalent to $\tau \preceq \sigma$. Let $(e_1, \cdots , e_n)$ be a basis of N generating σ as a cone such that $(e_1, \cdots , e_r)$ generates τ as a cone. Then $U_\sigma \simeq \mathsf{k}^r \times \mathsf{k}^{n-r}$ and $V(\tau) \cap U_\sigma \simeq \{0\} \times \mathsf{k}^{n-r}$. By Lemma 5.1.3, the blow-up of $\{0\}$ in k^r is given by the star refinement and the product with the fan of the cone generated by $(e_{r+1}, \cdots , e_n)$ gives the star refinement of τ in σ. Glueing these affine descriptions together yields the result. □

2. Resolution of singularities

We have seen that birational proper maps are given by refinements of fans. It therefore becomes a combinatorial problem to construct a refinement of any fan to a smooth one, providing an explicit combinatorial resolution of singularities. We present this construction.

DEFINITION 5.2.1. Let X be an irreducible variety. A resolution of singularities of X is a proper birational morphism $f : X' \to X$ such that

(1) X' is smooth and irreducible and
(2) f induces an isomorphism $f^{-1}(X_{\rm sm}) \simeq X_{\rm sm}$,

where $X_{\rm sm}$ is the smooth locus of X.

2.1. The singular locus. We start with an explicit combinatorial description of the smooth and the singular loci of toric varieties. Write $X_{\rm sm}$ for the smooth locus of X and $X_{\rm sing}$ for its singular locus. Recall the definition of a smooth cone from Definition 2.3.6.

PROPOSITION 5.2.2. *We have*

$$X_{\rm sm} = \bigcup_{\sigma\in\Delta, \sigma \text{ smooth}} U_\sigma \text{ and } X_{\rm sing} = \bigcup_{\sigma\in\Delta, \sigma \text{ not smooth}} V(\sigma).$$

Proof. Recall from Theorem 2.4.9 that U_σ is smooth if and only if σ is smooth. In particular, we get the inclusion $U_\sigma \subset X_{\rm sm}$ for σ smooth. We thus need to prove that $V(\sigma) \subset X_{\rm sing}$ and since $X_{\rm sing}$ is closed, we only need to prove that $O(\sigma) \subset X_{\rm sing}$. We use the usual decomposition, let $N_\sigma = \langle\sigma\rangle \cap N$ and $M_\sigma = M/(\sigma^\perp \cap M)$. Since M_σ is torsion free, we may find $M' \subset M$ of so that $M = M_\sigma \oplus M'$ and we get $S_\sigma = S_{\sigma'} \oplus M'$ where $\sigma' \subset N_\sigma$ is the cone σ seen in the lattice N_σ. Then $U_\sigma = T_{M'} \times U_{\sigma'}$ where σ' is a cone of maximal dimesion in N_σ. In particular, since σ' is not smooth, $U_{\sigma'}$ is ingular thus $p_{\sigma'}$ is a singular point. Now $O(\sigma) = T_{M'} \times \{p_{\sigma'}$ is also singular proving te claim and the result. ☐

2.2. Toric surfaces. We start with the case of surfaces where the algorithm is a rewriting of Gauß algorithm for computing the gcd of two integers.

PROPOSITION 5.2.3. *Toric surfaces have resolutions of singularities.*

Proof. We construct an equivariant resolution. Note that we may assume that $X = U_\sigma$ is an affine open subset defined by a cone σ which we may assume to be generated by $v_1 = e_1$ and $v_2 = -ae_1 + be_2$ where (e_1, e_2) is a basis of N and $\gcd(a, b) = 1$ (otherwise replace v_2 by $v_2' = \frac{1}{\gcd(a,b)} v_2$). Without loss of generality, we may also assume that $b > 0$ (otherwise replace e_2 by $-e_2$). More generally replacing using the base change given by the matrix

$$A_x = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$$

we replace v_2 by $-(a - bx)e_1 + be_2$ so that we may assume that $0 \leq a < b$. Note that U_σ is smooth if and only if (v_1, v_2) is a basis of N which is equivalent to $b = 1$. Let $c, d \in \mathbb{Z}$ be such that $ac + bd = 1$.

We construct a refinement of σ into two cones, one of which is smooth and the other *improves* the situation. For that we consider $v_3 = e_2$. We get a refinement of σ by considering the cones generated by (v_1, v_3) and (v_2, v_3). Note that the first one is smooth. Using the matrix

$$A = \begin{pmatrix} c & d \\ -b & a \end{pmatrix}$$

whose determinant is 1, we may view the second cone as the cone generated by $(Av_2, Av_3) = (e_1, de_1 + ae_2)$. Using the matrix U_x again, we may replace this basis by $(e_1, -a'e_1 + ae_2)$ with $0 \leq a' < a < b$. Repeating this process we end up to the basis (e_1, e_2) which gives a smooth cone. Note that since we only changed the singular cones, the constructed map is an isomorphism on the smooth locus. □

2.3. General case. The general case is more complicated. We will procees in two steps. The first step produces a refinement of any fan Δ which is simplicial in the sense of Definition . We get his way a variety which is locally $\mathbb{Q}$-factorial *i.e.* every Weil divisor has a multiple which is Cartier. The second step constructs a refinement resolving these last singularities.

For $A \subset V$ any subset, write $\mathrm{Cone}(A)$ for the cone generated by A.

Definition 5.2.4. Let $v \in |\Delta| \cap N \setminus \{0\}$ such that v in minimal in $\mathbb{R}_{\geq 0} v \cap N$. Define $\Delta^*(v)$, the star subdivision of Δ with respect to v as follows:

$$\Delta^*(v) = \{\sigma \in \Delta \mid v \notin \sigma\} \cup \{\mathrm{Cone}(\tau \cup \{v\}) \mid v \in \sigma \in \Delta, \tau \prec \sigma \text{ with } v \notin \tau\}.$$

Lemma 5.2.5. *Let $v \in \sigma \in \Delta$ and $\tau \prec \sigma$ such that $v \notin \tau$, then $v \notin \langle \tau \rangle$ and*

$$\dim \mathrm{Cone}(\tau \cup \{v\}) = \dim \tau + 1.$$

Proof. Since $\tau \prec \sigma$, we have $\tau = \langle \tau \rangle \cap \sigma$. Thus $v \in \sigma$ and $v \notin \tau$ imply $v \notin \langle \tau \rangle$. □

Proposition 5.2.6. *$\Delta^*(v)$ is a refinement of Δ.*

Proof. We start by proving that $\Delta^*(v)$ is a fan. Let $\sigma' \in \Delta^*(v)$ and $\tau' \prec \sigma'$.
Case 1. If $v \notin \sigma'$, then $\sigma' \in \Delta$ and $v \notin \tau'$ thus $\tau' \in \Delta^*(v)$.
Case 2. If $v \in \sigma'$, then $\sigma' = \mathrm{Cone}(\tau \cup \{v\})$ with $\tau \prec \sigma \in \Delta$ and $v \in \sigma \setminus \tau$. Let $m \in \mathrm{Cone}(\tau \cup \{v\})^\vee$ such that $\tau' = \sigma' \cap m^\perp$. We have $m \in \tau^\vee$ and $\langle m, v \rangle \geq 0$, in particular $\gamma := \tau \cap m^\perp$ is a face of τ. If $\langle m, v \rangle = 0$, then $\tau' = \sigma' \cap m^\perp = \mathrm{Cone}(\tau \cup \{v\}) \cap m^\perp = \mathrm{Cone}((\tau \cap m^\perp) \cup \{v\}) = \mathrm{Cone}(\gamma \cup \{v\}) \in \Delta^*(v)$. If $\langle m, v \rangle > 0$, then $\tau' = \tau \cap m^\perp = \gamma \in \Delta^*(v)$.

Let $\sigma', \sigma'' \in \Delta^*(v)$ and set $\tau' = \sigma' \cap \sigma''$.
Case 1. If $v \notin \sigma'$ and $v \notin \sigma'$, then $\sigma', \sigma'' \in \Delta$, $\tau' \in \Delta$ is a face of both and $v \notin \sigma' \cap \sigma'' \in \Delta$. Thus $\tau'' \in \Delta^*(v)$ is a face of σ' and σ''.
Case 2. If $v \notin \sigma$ and $v \in \sigma'$, then $\sigma' = \sigma \in \Delta$ and $\sigma' = \mathrm{Cone}(\tau \cup \{v\})$ for some $\tau \in \Delta$ with $v \notin \tau$ (and there exists $\gamma \in \Delta$ with $\tau \prec \gamma$ and $v \in \gamma$). We prove that $\tau' = \sigma \cap \mathrm{Cone}(\tau \cup \{v\}) = \sigma \cap \tau$. This implies that $\tau' \in \Delta^*(v)$ since $v \notin \sigma \cap \tau$ and is a face of both σ' and σ'' since it is a face of $\sigma = \sigma'$ and of $\tau \prec \mathrm{Cone}(\tau, \{v\}) = \sigma''$. We claim that $\sigma \cap \mathrm{Cone}(\tau \cup \{v\}) = \sigma \cap \tau$. We clearly have the inclusion $\sigma \cap \tau \subset \sigma \cap \mathrm{Cone}(\tau \cup \{v\})$. Conversely, let $v' \in \sigma \cap \mathrm{Cone}(\tau \cup \{v\})$ and write $v' = v'' + \lambda v$ with $v'' \in \tau$ and $\lambda \in \mathbb{R}_{\geq 0}$. Then $v' = v'' + \lambda v \in \sigma \cap \gamma$ (recall that $\gamma \in \Delta$ contains both τ and v) which is a face of γ thus $v'', \lambda v \in \sigma \cap \gamma$. But since $v \notin \sigma$, we get $\lambda = 0$ proving the claim.
Case 3. Assume now that $v \notin \sigma'$ and $v \notin \sigma''$. Then $\sigma' = \mathrm{Cone}(\tau_1 \cup \{v\})$ and $\sigma'' = \mathrm{Cone}(\tau_2 \cup \{v\})$ with $\tau_1, \tau_2 \in \Delta$ not containing v. We claim that $\gamma = \mathrm{Cone}((\tau_1 \cap \tau_2) \cup \{v\})$. This implies that $\gamma \in \Delta^*(v)$ and by Lemma 5.2.5 that γ is a face of σ' and σ''. Again we clearly have $\mathrm{Cone}((\tau_1 \cap \tau_2) \cup \{v\}) \subset \mathrm{Cone}(\tau_1 \cup \{v\}) \cap \mathrm{Cone}(\tau_2 \cup \{v\})$. Let $v' \in \mathrm{Cone}(\tau_1 \cup \{v\}) \cap \mathrm{Cone}(\tau_2 \cup \{v\})$, then $v' = v_1 + \lambda_1 v = v_2 + \lambda_2 v$ with $v_i \in \tau_i$ and $\lambda_i \in \mathbb{R}_{\geq 0}$ for $i \in \{1, 2\}$. Assume that $\lambda_1 \geq \lambda_2$, then $v_2 = v_1 + (\lambda_1 - \lambda_2) \in \mathrm{Cone}(\tau_1 \cup \{v\}) \cap \tau_2 = \tau_1 \cap \tau_2$ (the last equality follows from the previous claim applied to $\tau = \tau_1$ and $\sigma = \tau_2$). We thus get $v = v_2 + \lambda_2 v \in \mathrm{Cone}(\tau_1 \cap \tau_2 \cup \{v\})$ as claimed.

We now prove that $\Delta^*(v)$ is a refinement of Δ. By definition any cone in $\Delta^*(v)$ is contained in a cone of Δ thus $|\Delta^*(v)| \subset |\Delta|$. Conversely, let $\sigma \in \Delta$. If $v \notin \sigma$, then $\sigma \in \Delta^*(v)$ and $\sigma \subset |\Delta^*(v)|$. If $v \notin \sigma$, let $v' \in \sigma$ and let $m_1, \cdots, m_r$ be generators of $\sigma^\vee$. We have $\langle m_i, v\rangle \geq 0$ for all i. Furthermore, since $v \neq 0$ and σ is strictly convex, there exists at least one i with $\langle m_i, v\rangle > 0$. Set

$$\lambda = \max_{i,\ \langle m_i,v\rangle>0} \frac{\langle m_i, v'\rangle}{\langle m_i, v\rangle},$$

and let $m = \sum_i \mu_i m_i$ with $\mu_i \geq 0$ be any element in $\sigma^\vee$. We compute

$$\langle m, v' - \lambda v\rangle = \sum_i \mu_i \langle m_i, v'\rangle + \sum_{i,\ \langle m_i,v\rangle>0} \mu_i\langle m_i, v\rangle \left(\frac{\langle m_i, v'\rangle}{\langle m_i, v\rangle} - \lambda\right)$$

and all terms on the right hand side are non-negative thus $v' - \lambda v \in (\sigma^\vee)^\vee = \sigma$. Let i_0 such that $\langle m_{i_0}, v\rangle > 0$ and $\lambda = \frac{\langle m_{i_0}, v'\rangle}{\langle m_{i_0}, v\rangle}$. Then $\langle m_{i_0}, v' - \lambda v\rangle = 0$ thus $v' - \lambda v \in \tau$ the face of σ defined by m_{i_0}. Furthermore $v \notin \tau$ since $\langle m_{i_0}, v\rangle > 0$. Since $\lambda \geq 0$, we have $v' = (v' - \lambda v) + \lambda v \in \mathrm{Cone}(\tau \cup \{v\}) \in \Delta^*(v)$. □

Proposition 5.2.7. *Set $\rho_v = \mathbb{R}_{\geq 0} v$, we have*

1. $\Delta^*(v)(1) = \Delta(1) \cup \{\rho_v\}$
2. D_{ρ_v} *is $\mathbb{Q}$-Cartier (*i.e. $\exists k \in \mathbb{Z}_{\geq 0}, kD_{\rho_v} \in \mathrm{Ca}_T(X)$*).*

Proof. (1) Let $\tau \in \Delta(1)$. If $v \notin \tau$, then $\tau \in \Delta^*(v)(1)$. Otherwise, $\tau = \rho_v$ and $\rho_v = \mathrm{Cone}(\{0\}\cup\{v\}) \in \Delta^*(v)(1)$. If $\rho_v \notin \Delta$, then $\rho_v = \mathrm{Cone}(\{0\}\cup\{v\}) \in \Delta^*(v)(1)$. Conversely, let $\tau' \in \Delta^*(v)(1)$ with $\tau' \neq \rho_v$. Then $v \notin \tau'$ and $\tau' \in \Delta(1)$.

(2) We construct a support function ψ_v for D_{ρ_v}. For $\sigma \in \Delta^*(v) \cap \Delta$, set $(\psi_v)|_\sigma = 0$. For $\sigma \in \Delta^*(v) \setminus \Delta$, then $\sigma = \mathrm{Cone}(\tau \cup \{v\})$ with $v \notin \tau \in \Delta$. By Lemma 5.2.5, we have that $v \notin \langle\tau\rangle$ thus there exists a linear map f on $\langle \tau \cup \{v\}\rangle$ such that $f(\tau) = 0$ and $f(v) = 1$. Set $(\psi_v)|_\sigma = f|_\sigma$. We need to check that this is well defined. Let $\tau' = \sigma' \cap \sigma''$ with $\sigma', \sigma'' \in \Delta^*(v)$. If $v \notin \sigma'$ and $v \notin \sigma''$, then $(\psi_v)|_{\sigma'} = 0 = (\psi_v)|_{\sigma''}$. If $v \notin \sigma'$ and $v \in \sigma''$, then $\sigma'' = \mathrm{Cone}(\tau \cup \{v\})$ for some $\tau \in \Delta$ and $\tau' = \sigma' \cap \tau$ and $(\psi_v)|_{\tau'} = 0$. Finally if $v \in \sigma' \cap \sigma''$, then $\sigma' = \mathrm{Cone}(\tau_1 \cup \{v\})$ and $\sigma'' = \mathrm{Cone}(\tau_1 \cup \{v\})$ for $\tau_1, \tau_2 \in \Delta$ and $\tau' = \mathrm{Cone}((\tau_1 \cap \tau_2) \cup \{v\})$. We get that $(\psi_v)|_{\tau'}$ is the linear map vanishing on $\tau_1 \cap \tau_2$ and with value 1 on v. From the definition, we clearly have that $(\psi_v)(N) \subset \mathbb{Q}$ (however, we do not necessarily have $(\psi_v)(N) \subset \mathbb{Z}$, see Exazmple 5.2.8 below) thus there exists $k \in \mathbb{Z}_{\geq 0}$ such that $k\psi_v$ is integral. In particular the divisor

$$D = -\sum_{\tau \in \Delta^*(v)(1)} k(\psi_v)(v_\tau) D_\tau$$

lies in $\mathrm{Ca}_T(X(\Delta^*(v)))$. Now for $\tau \in \Delta^*(v)(1) \setminus \{\rho_v\}$, we have $\psi_v(v_\tau) = 0$ while $\psi_v(v) = 1$. This implies that $D_{\rho_v} = -D \in \mathrm{Ca}_T(X(\Delta^*(v)) \otimes_{\mathbb{Z}} \mathbb{Q}$. □

Example 5.2.8. Take $N = \mathbb{Z}^2$ with canonical basis (e_1, e_2) and dual basis $(e_1^\vee, e_2^\vee)$. Let $\sigma = \mathrm{Cone}(e_1, e_2)$ and $v = e_1 + 2e_2$, then ψ_v is equal to $e_1^\vee$ on $\mathrm{Cone}(e_2, v)$ and to $\frac{1}{2}e_2^\vee$ on $\mathrm{Cone}(e_1, v)$.

Remark 5.2.9. In Proposition 5.2.7, the morphism $X(\Delta^*(v)) \to X$ induced by the refinement is projective. To prove this, it is enough to check that for any $\sigma \in \Delta$, the function ψ_v is strictly convex with respect to the fan $\{\sigma' \in \Delta^*(v) \mid \sigma' \subset \sigma\}$.

THEOREM 5.2.10. *Any fan* Δ *has a refinement* Δ' *such that*

(1) Δ' *is simplicial.*
(2) $\Delta'(1) = \Delta(1)$.
(3) Δ' *contains every simplicial cone of* Δ.
(4) Δ' *is obtained by a sequence of star subdivision of* Δ.

REMARK 5.2.11. By Remark 5.2.9, the induced map $X(\Delta') \to X$ is projective.

Proof. We first prove that if (2) holds, then so does (3). Let $\sigma \in \Delta$ be a simplicial cone and let $\sigma' \in \Delta'$ such that $\sigma' \subset \sigma$. If (2) holds, then $\sigma'(1) \subset \sigma(1)$, thus σ' is a face of σ (since σ is simplicial). But since Δ' is a refinement of Δ, we have

$$\sigma = \bigcup_{\sigma' \in \Delta', \sigma' \subset \sigma} \sigma'$$

and since this is a union of faces of σ, one of them must be equal to σ.

We are left to construction, by a sequence of star subdivision, a refinement Δ' of Δ satisfying (1) and (2). The idea is to do a star subdivision for each ray associated to a non-$\mathbb{Q}$-Cartier divisor on X. We proceed by induction on

$$r = |\{\tau \in \Delta(1) \mid D_\tau \text{ is not } \mathbb{Q}\text{-Cartier in } X\}|.$$

If $r = 0$, then we clearly have that X is $\mathbb{Q}$-factorial and Δ is simplicial. Assume that $r > 0$ and let $\tau \in \Delta(1)$ such that D_τ is not $\mathbb{Q}$-Cartier. Let $v = v_\tau$ be the minimal generator of the ray τ. We have $\Delta^*(v)(1) = \Delta(1) \cup \{\rho_v\} = \Delta(1) \cup \{\tau\} = \Delta(1)$ and D_τ is $\mathbb{Q}$-Cartier on $X(\Delta^*(v))$. Furthermore, for $\tau' \in \Delta(1)$, if $D_{\tau'}$ is $\mathbb{Q}$-Cartier on X, then $kD_{\tau'} = \sum_{\rho \in \Delta(1)} \psi(v_\rho) D_\rho$ for some integral piecewise linear with respect to Δ function ψ and since Δ' is a refinement of Δ, the function ψ is piecewise linear with respect to Δ' thus $D_{\tau'}$ is $\mathbb{Q}$-Cartier on $X(\Delta')$ proving that r decreases by at least 1 in this process. The result follows by induction. $\square$

We are left with resolutions of $\mathbb{Q}$-factorial toric varieties. For that we will need a measure of singularities which in terms of convex geometry should be a measure of how far a basis of V is to a basis of N.

DEFINITION 5.2.12. Let σ be a simplicial cone. The multiplicity of σ is

$$\mathrm{mult}(\sigma) = \left| N_\sigma / \sum_{\tau \in \sigma(1)} \mathbb{Z} v_\tau \right|$$

where $N_\sigma = N \cap \langle \sigma \rangle$.

PROPOSITION 5.2.13. *Let* σ *be simplicial.*

(1) σ *is smooth if and only if* $\mathrm{mult}(\sigma) = 1$.
(2) $\mathrm{mult}(\sigma) = |P_\sigma \cap N|$ *with* $P_\sigma = \{\sum_{\tau \in \sigma(1)} \lambda_\tau v_\tau \mid \lambda_\tau \in [0,1]\}$.
(3) *If* $(e_\tau)_{\tau \in \sigma(1)}$ *is a basis of* N_σ, *write* $u_\tau = \sum_{\gamma \in \sigma(1)} a_{\tau,\gamma} e_\gamma$, *then*
$$\mathrm{mult}(\sigma) = |\det(a_{\tau,\gamma})_{\tau,\gamma}|.$$
(4) *For* $\tau \preceq \sigma$, *we have* $\mathrm{mult}(\tau) \leq \mathrm{mult}(\sigma)$.

Proof. (1) The cone σ is smooth if and only if $(v_\tau)_{\tau \in \sigma(1)}$ is a basis of N_σ.

(2) We have a bijection $P_\sigma \cap N \to N_\sigma / \sum_{\tau \in \sigma(1)} \mathbb{Z} v_\tau$ given by $v \mapsto [v]$.

(3) Follows from the structure result on finitely generated groups.

(4) Follows from the inclusion $P_\tau \subset P_\sigma$ and (2). $\square$

THEOREM 5.2.14. *Any fan Δ has a refinement Δ' such that*

(1) *Δ' is smooth.*
(2) *Δ' contains all the smooth cones of Δ.*
(3) *Δ' is obtained from Δ by star subdivisions.*
(4) *The induced morphism $X(\Delta') \to X$ is a resolution of singularities.*

REMARK 5.2.15. By Remark 5.2.9, the map $X(\Delta') \to X$ is a projective resolution of singularities.

Proof. Note that (4) follows from (1), (2) and (3). We construct a refinement satisfying (1), (2) and (3). Using Theorem 5.2.10, we may replace Δ with a refinement for which X is $\mathbb{Q}$-factorial. Now the combinatorics give use a measure of the singularity of X. This is the key for the process (and what is missing in full generality for the existence of resolutions of singularities). Define

$$\mathrm{mult}(\Delta) = \max_{\sigma \in \Delta} \mathrm{mult}(\sigma).$$

Note that $\mathrm{mult}(\Delta) = 1$ if and only if X is smooth. It is therefore enough to construct a sequence of star-subdivisions $\Delta^*(v)$ satisfying (2) above such that either $\mathrm{mult}(\Delta^*(v)) < \mathrm{mult}(\Delta)$ or $\mathrm{mult}(\Delta^*(v) = \mathrm{mult}(\Delta)$ but $\Delta^*(v)$ has fewer cones of maximal multiplicity than Δ.

Assume that $\mathrm{mult}(\Delta) > 1$ and let $\sigma_0 \in \Delta$ with $\mathrm{mult}(\sigma_0) = \mathrm{mult}(\Delta)$. Then $P_\sigma \cap N \neq \{0\}$ so we may choose $v \in P_\sigma \cap N \setminus \{0\}$ primitive and consider $\Delta^*(v)$. Recall the descritption of cones in $\Delta^*(v)$: for $\sigma' \in \Delta^*(v)$, if $v \notin \sigma'$, then $\sigma' \in \Delta$ and if $v \in \sigma'$, then $\sigma' = \mathrm{Cone}(\tau \cup \{v\})$ where $\tau \prec \sigma \in \Delta$ and $v \in \sigma \setminus \tau$.

Let $\gamma_0 \in \Delta$ be the unique cone such that $v \in \mathring{\gamma}_0$. Then $v = \sum_{\tau \in \gamma(1)} \lambda_\tau v_\tau$ with $\lambda_\tau \in]0,1[$ for all $\tau \in \gamma(1)$ and we have $\gamma_0 \preceq \sigma_0$ thus $v \in P_{\gamma_0} \subset P_{\sigma_0}$. In particular $\mathrm{mult}(\gamma_0) > 1$. Furthermore, for any $\sigma \in \Delta$ with $v \in \sigma$, we have $\gamma_0 \preceq \sigma$ thus $\mathrm{mult}(\sigma) \geq \mathrm{mult}(\gamma_0) > 1$. This implies that for any $\sigma \in \Delta$ smooth, we have $v \notin \sigma$ and $\sigma \in \Delta^*(v)$. In particular $\Delta^*(v)$ satisfies (2).

We now check that $\Delta^*(v)$ satisfies the above decreasing condition on the multiplicity. In particular, let $\sigma' \in \Delta$ with $v \in \sigma'$, so that $\sigma' = \mathrm{Cone}(\tau \cup \{v\})$ where $\tau \prec \sigma \in \Delta$ and $v \in \sigma \setminus \tau$. We prove that

$$\mathrm{mult}(\sigma') = \mathrm{mult}(\mathrm{Cone}(\tau \cup \{v\}) < \mathrm{mult}(\sigma).$$

Note that $v \in \sigma$ implies $\gamma_0 \preceq \sigma$ and that $\tau \cup \{v\} \subset \sigma$. Furthermore, since $v \notin \tau$, there exists $\tau_0 \in \sigma(1)$ such that $v_{\tau_0} \notin \tau$. In particular the set $\{v_\delta \mid \delta \in \sigma(1) \setminus \{\tau_0\}\} \cup \{v\}$ is a basis of $(N_\sigma)_\mathbb{R}$. We have $\mathrm{Cone}(\tau \cup \{v\}) \preceq \mathrm{Cone}(\{v_\delta \mid \delta \in \sigma(1) \setminus \{\tau_0\}\} \cup \{v\})$. Let $(e_\delta)_{\delta \in \sigma(1)}$ be a basis of N_σ, we compute all determinants with respect to this basis. We get (using $v = \lambda_{\tau_0} v_{\tau_0} + \sum_{\delta \in \sigma(1) \setminus \{\tau_0\}} \lambda_\delta v_\delta$ on the third line)

$$\begin{aligned}
\mathrm{mult}(\mathrm{Cone}(\tau \cup \{v\})) &\leq \mathrm{mult}(\mathrm{Cone}(\{v_\delta \mid \delta \in \sigma(1) \setminus \{\tau_0\}\} \cup \{v\})) \\
&= |\det(v, (v_\delta)_{\delta \in \sigma(1) \setminus \{\tau_0\}})| \\
&= |\det(\lambda_{\tau_0} v_{\tau_0}, (v_\delta)_{\delta \in \sigma(1) \setminus \{\tau_0\}})| \\
&= \lambda_{\tau_0} |\det((v_\delta)_{\delta \in \sigma(1)})| \\
&= \lambda_{\tau_0} \mathrm{mult}(\sigma) < \mathrm{mult}(\sigma).
\end{aligned}$$

Now for a cone $\sigma' \in \Delta^*(v)$, we have the alternative $\sigma' \in \Delta$ or $\mathrm{mult}(\sigma') < \mathrm{mult}(\Delta)$. Since $v \in \sigma_0$, we have $\sigma_0 \notin \Delta^*(v)$ thus we have the alternative $\mathrm{mult}(\Delta^*(v)) < \mathrm{mult}(\Delta)$ or $\mathrm{mult}(\Delta^*(v) = \mathrm{mult}(\Delta)$ but $\Delta^*(v)$ has fewer cones of maximal multiplicity than Δ. This prove by induction that the result holds. □

We finish this chapter by a description of the exceptional locus.

PROPOSITION 5.2.16. *Let $f : X(\Delta') \to X$ be the morphism associated to the refinement Δ' of Δ and let $\sigma \in \Delta$. Then we have*

$$f^{-1}(V(\sigma)) = \bigcup_{i=1}^{r} V(\sigma'_i),$$

where $(\sigma'_i)_{i\in[1,r]}$ is the family of minimal cones $\sigma' \in \Delta'$ such that $\sigma' \cap \mathring{\sigma} \neq \emptyset$.

Proof. We star with the following general fact. For $\sigma' \in \Delta'$, let $\sigma \in \Delta$ be the smallest cone containing σ'. Then $\sigma' \cap \mathring{\sigma} \neq \emptyset$. We claim that $f(p_{\sigma'}) = p_\sigma$. Indeed, $p_{\sigma'}$ is given by the morphism of monoids $\gamma' : S_{\sigma'} \to \mathsf{k}$ such that $\gamma'(m) = \delta_{m\in(\sigma')^\perp}$. The image $f(p_{\sigma'}$ is contained in U_σ and is given by the morphism of monoids γ such that $\gamma(m) = 1$ if and only if $m \in (\sigma')^\perp$ and $\gamma(m) = 0$ else. But the condition $m \in (\sigma')^\perp$ for $m \in S_\sigma$ is equivalent to $m \in \sigma^\perp$ since $\sigma' \cap \mathring{\sigma} \neq \emptyset$. This proves our claim. Thus, for $\sigma \in \Delta$ and $\sigma' \in \Delta'$, we have $f(O(\sigma')) \subset O(\sigma)$ if and only if $\sigma' \cap \mathring{\sigma} \neq \emptyset$.

Using this we compute all orbits $O(\sigma') \subset X(\Delta')$ that are mapped into $V(\sigma)$. Recall that $V(\sigma) = \cup_{\tau\succeq\sigma} O(\tau)$, thus for $\sigma' \in \Delta'$, we have $f(O(\sigma')) \subset V(\sigma)$ if and only if there exists $\tau \succeq \sigma$ such that $\sigma' \cap \mathring{\tau} \neq \emptyset$. Now the irreducible components of $f^{-1}(V(\sigma))$ are the closure of the biggest orbit of this kind therefore corresponding to the smallest cones σ' satisfying this condition. The result follows from this. $\square$

CHAPTER 6

Canonical divisor and Fano varieties

In this chapter, we will (almost) only consider smooth varieties.

1. Differentials and canonical divisor

Recall the definition nof Kähler differentials.

Definition 6.1.1. Let A be a ring and B be an A-algebra. The module of Kähler differentials of B over A $\Omega^1_{B/A}$ is the B-module obtained as the quotient of the free B-module over the symbols $(db)_{b\in B}$ modulo the relations

(1) $d(ab+b') = adb + d(b')$ for $b, b' \in B$ and $a \in A$,
(2) $d(bb') = bd(b') + b'db$ for $b, b' \in B$.

Remark 6.1.2. We have $d(1) = d(1.1) = d1 + d1$ thus $d1 = 0$. In particular $da = ad1 = 0$ for all $a \in A$.

Example 6.1.3. For $B = A[x_1, \cdots, x_n]$, then it is easy to check that $\Omega^1_{B/A} \simeq \bigoplus_{i=1}^n Bdx_i$. Indeed, for any polynomial $P \in A[x_1, \cdots, x_n]$, we have

$$dP = \sum_{i=1}^n \frac{\partial P}{\partial x_i} dx_i.$$

The following results will be useful for computing Kähler differentials.

Proposition 6.1.4. *Consider the following commutative diagram of rings:*

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B'. \end{array}$$

(1) *Then we have a morphism of B'-modules $\Omega^1_{B/A} \otimes_B B' \to \Omega^1_{B'/A'}$.*
(2) *If $B' = B \otimes_A A'$, then $\Omega^1_{B'/A'} \simeq \Omega^1_{B/A} \otimes_B B'$.*
(3) *If $B' = S^{-1}B$ is a localisation, then $\Omega^1_{B'/A} \simeq \Omega^1_{B/A} \otimes_B B' \simeq S^{-1}\Omega^1_{B/A}$.*

Example 6.1.5. We will use the part on localisation. Consider a lattice N.

(1) If $\sigma = \mathrm{Cone}(e_1, \cdots, e_r)$ is a cone generated by a part of a basis of N, then $U_\sigma \simeq \mathbb{A}^r_{\mathsf{k}} \times (\mathbb{G}_m)^{n-r}$ and $\Omega^1_{\mathsf{k}[U_\sigma]/\mathsf{k}} \simeq \bigoplus_{i=1}^n \mathsf{k}[U_\sigma]dx_i$.
(2) In particular, if $\sigma = 0$ *i.e.* $U_\sigma = T$ the torus, then $\Omega^1_{\mathsf{k}[T]/\mathsf{k}} \simeq M \otimes_{\mathbb{Z}} \mathsf{k}[T]$ with isomorphism given by $d\chi^m \mapsto m \otimes \chi^m$.

Definition 6.1.6. Let X be an irreducible variety, define the sheaf Ω^1_X by $\Omega^1_X(U) = \Omega^1_{\mathcal{O}_X(U)/\mathsf{k}}$. The localisation property implies that this is indeed a sheaf.

For a smooth toric variety, Example 6.1.5 implies that Ω^1_X is locally free.

Definition 6.1.7. The tangent sheaf T_X of X is the dual of Ω^1_X *i.e.*

$$T_X = \mathscr{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X).$$

Definition 6.1.8. For X smooth, both Ω^1_X and T_X are locally free of rank $n = \dim X$. Define the canonical sheaf as $\omega_X = \bigwedge^n \Omega^1_X$.

Example 6.1.9. Let $X = \mathbb{P}^2_{\mathsf{k}}$ given by the complete fan Δ such that the one dimensional cone are generated by e_1, e_2 and $e_0 = -(e_1 + e_2)$ where (e_1, e_2) is the canonical basis of $N = \mathbb{Z}^2$. Let $\sigma_i = \text{Cone}(e_j \mid j \neq i)$, $\tau_i = \text{Cone}(e_i)$ and $D_i = D_{\tau_i}$. Then we have

$$\mathsf{k}[U_{\sigma_0}] = \mathsf{k}[x, y], \quad \mathsf{k}[U_{\sigma_1}] = \mathsf{k}\left[\frac{1}{x}, \frac{y}{x}\right] \text{ and } \mathsf{k}[U_{\sigma_2}] = \mathsf{k}\left[\frac{1}{y}, \frac{x}{y}\right].$$

We get

$$\omega_X|_{U_{\sigma_0}} = dx \wedge dy, \quad \omega_X|_{U_{\sigma_0}} = -\frac{1}{x^3} dx \wedge dy \text{ and } \omega_X|_{U_{\sigma_0}} = \frac{1}{y^3} dx \wedge dy.$$

We see that the line bundle ω_X corresponds to the Cartier divisor given by 1, $-1/x^3$ and $1/y^3$ on U_{σ_0}, U_{σ_1} and U_{σ_2} respectively. We get that $\omega_X = \mathcal{O}_X(-3D_0)$.

Let X be a smooth toric variety. For each cone $\sigma \in \Delta$, since Ω^1_X is locally free, we have an isomorphism $\Omega^1_X|_{U_\sigma} \simeq \mathsf{k}[U_\sigma]^n$. Actually, there is a uniform such map given as follows:

$$\varphi_\sigma : \Omega^1_X(U_\sigma) \to M \otimes_{\mathbb{Z}} \mathsf{k}[U_\sigma], d(\chi^m) \mapsto m \otimes \chi^m.$$

Proposition 6.1.10. *Let $r \leq n$ and $(e_1, \cdots, e_n)$ a basis of N such that $\sigma =$* Cone$(e_1, \cdots, e_r)$ *and let $D_i = D_{\text{Cone}(e_i)}$. Let $(e_1^\vee, \cdots, e_n^\vee)$ be the dual basis in M*

(1) *The map φ_σ is injective.*
(2) *We have* $\text{Coker}(\varphi_\sigma) = \oplus_{i=1}^r \mathsf{k}[D_i]$.

Proof. We have $U_\sigma \simeq \mathbb{A}^r_{\mathsf{k}} \times (\mathbb{G}_m)^{n-r}$, $\mathsf{k}[U_\sigma] = \mathsf{k}[x_1, \cdots, x_r, x_{r+1}^{\pm 1}, \cdots, x_n^{\pm 1}]$ and $\Omega^1_{\mathsf{k}[U_\sigma]} = \oplus_{i=1}^n \mathsf{k}[U_\sigma] dx_i$. Under this identification, the map φ_σ is given by

$$\varphi\left(\sum_{i=1}^n f_i dx_i\right) = \sum_{i=1}^n e_i^\vee \otimes f_i x_i.$$

(1) If $\sum_i f_i dx_i \in \text{Ker}(\varphi_\sigma)$, then $f_i x_i = 0$ for all i thus $f_i = 0$ for all i.

(2) If $i > r$, then $\varphi_\sigma((1/x_i)dx_i) = e_i^\vee$ while for $i \leq r$, $\varphi(f_i dx_i) \in (x_i)e_i^\vee$. This proves that $\text{Im}(\varphi_\sigma) = (x_1)e_1^\vee \oplus \cdots \oplus (x_r)e_r^\vee \oplus \mathsf{k}[U_\sigma]e_{r+1}^\vee \oplus \cdots \oplus \mathsf{k}[U_\sigma]e_n^\vee$. The result follows from this. □

These map therefore glue together and we get a injective morphism of sheaves $\varphi : \Omega^1_X \to M \otimes_{\mathbb{Z}} \mathcal{O}_X$. We define another morphism of sheaves.

$$\psi : M \otimes_{\mathbb{Z}} \mathcal{O}_X \to \bigoplus_{\tau \in \Delta(1)} \mathcal{O}_{D_\tau}, \psi(m \otimes f) = (\langle m, v_\tau \rangle f|_{D_\tau})_{\tau \in \Delta(1)}.$$

Corollary 6.1.11. *We have an exact sequence*

$$0 \to \Omega^1_X \to M \otimes_Z \mathcal{O}_X \to \bigoplus_{\tau \in \Delta(1)} \mathcal{O}_{D_\tau} \to 0.$$

Proof. We already know that φ is injective. Let $\sigma \in \Delta$ be a cone, then the restriction to U_σ of the right hand side of the exact sequence is $\oplus_{\tau\in\sigma(1)}\mathcal{O}_{D_\tau}$. Furthermore, Proposition 6.1.10 implies that the cokernel of φ_σ is exactly $\oplus_{\tau\in\sigma(1)}\mathcal{O}_{D_\tau}$ proving the result on each affine chart and thus the result. □

THEOREM 6.1.12. *Let X we a smooth toric variety. We have an exact sequence*

$$0 \to \Omega^1_X \to \bigoplus_{\tau\in\Delta(1)} \mathcal{O}_X(-D_\tau) \to \mathrm{Pic}(X)\otimes_{\mathbb{Z}} \mathcal{O}_X \to 0.$$

Proof. Fro X smooth, recall the exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha} \oplus_{\tau\in\Delta(1)}\mathbb{Z}D_\tau \longrightarrow \mathrm{Pic}(X) \longrightarrow 0\ .$$

We thus get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & M\otimes_Z \mathcal{O}_X & = & M\otimes_Z \mathcal{O}_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \alpha & & \downarrow \psi & & \downarrow \\ 0 & \longrightarrow & \bigoplus_{\tau\in\Delta(1)} \mathcal{O}_X(-D_\tau) & \longrightarrow & \bigoplus_{\tau\in\Delta(1)} \mathcal{O}_X & \longrightarrow & \bigoplus_{\tau\in\Delta(1)} \mathcal{O}_{D_\tau} & \longrightarrow & 0, \end{array}$$

such that the cokernel of α is $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathcal{O}_X$ and the kernel of ψ is Ω^1_X. Applying Snake Lemma, we get the desired exact sequence. □

EXAMPLE 6.1.13. Let $X = \mathbb{P}^n_{\mathsf{k}}$. Then we get the Euler exact sequence

$$0 \to \Omega^1_{\mathbb{P}^n_{\mathsf{k}}} \to \mathcal{O}_{\mathbb{P}^n_{\mathsf{k}}}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n_{\mathsf{k}}} \to 0.$$

DEFINITION 6.1.14. Let X be a toric variety, set $\partial X = \sum_{\tau\in\Delta(1)} D_\tau$.

THEOREM 6.1.15. *Let X be a smooth toric variety, then $\omega_X = \mathcal{O}_X(-\partial X)$.*

Proof. For $d = |\Delta(1)|$, the d-th exterior power of the sequence in Theorem 6.1.12 gives $\mathcal{O}_X(-\partial X) = \bigwedge^d(\oplus_{\tau\in\Delta(1)}\mathcal{O}_X(-D)) \simeq \bigwedge^n \Omega^1_X \otimes \bigwedge^{d-n}(\mathrm{Pic}(X)\otimes_Z \mathcal{O}_X) = \omega_X \otimes \mathcal{O}_X = \omega_X$, proving the result. □

2. Toric Fano varieties

We give a description of smooth toric Fano varieties in terms of polytopes.

DEFINITION 6.2.1. A smooth projective variety X is called Fano if ω_X^{-1} is ample.

Recall the following on polytopes. Let X be a smooth projective toric variety. Then Proposition 4.7.9 implies that $\Delta = \Delta_P$ for some rational polytope P (actually for any polytope $P = P_D$ where $D \in \mathrm{Ca}_T(X)$ is ample).

We prove a converse statement. Let $P \subset W$ be a rational polytope of maximal dimension such that 0 is an interior point of P. Let Δ_P be the associated fan. Let P° be the dual polytope. The construction of Δ_P and Proposition 3.5.10 imply that there is a bijection between vertices of P and maximal cones in Δ_P. For $\sigma \in \Delta(n)$, denote by $m_\sigma \in P\cap M$ the corresponding vertex. The same arguments give bijections between $\Delta_P(1)$, facets of P and vertices of P°. For $\tau \in \Delta(1)$, denote by $F_\tau \subset P$ the corresponding facet and v_τ the minimal element in $\tau\cap N$. For each $\tau \in \Delta_P(1)$, there exists $a_\tau \in \mathbb{R}$ such that we have

$$P = \{w \in W \mid \langle w, v_\tau\rangle \geq -a_\tau \text{ for all } \tau \in \Delta(1)\}.$$

The facet F_τ is given by $F_\tau = \{w \in P \mid \langle w, v_\tau\rangle = -a_\tau\}$. Note that for any vertex $m_\sigma \in F_\tau$, we have $a_\tau = -\langle m_\sigma, v_\tau\rangle \in \mathbb{Z}$ since m_σ is a lattice point. Furthermore, since 0 is an interior point of P, we have $0 = \langle 0, v_\tau\rangle > -a_\tau$ for any $\tau \in \Delta_P(1)$ thus $a_\tau \in \mathbb{Z}_{>0}$ for any $\tau \in \Delta_P(1)$. Finally, note that this gives an explicit description of the vertices of P° which are given by $(v_\tau/a_\tau)_{\tau\in\Delta_P(1)}$.

PROPOSITION 6.2.2. *Let P as above and $X = X(\Delta_P)$.*

(1) *X is projective.*
(2) *The divisor $D_P = \sum_{\tau\in\Delta_P(1)} a_\tau D_\tau$ is ample.*

Proof. (1) The fact that X is complete follows from Propositions 3.5.14 and 3.5.16. The result will thus follow from (2). To prove (2) it is enough to construct a function ψ adapted to Δ_P which is integral, piececewise linear and strictly convex whose associated divisor is D_P. Set $\psi|_\sigma = m_\sigma|_\sigma$. By definition, we have $\sigma^\vee = \mathrm{Cone}(m_{\sigma'} - m_\sigma \mid \sigma' \in \Delta_P(n))$, thus $m_{\sigma'}|_\sigma \geq \psi|_\sigma$. This gives $(m_{\sigma'} - m_\sigma)|_\sigma \geq 0$ and by symmetry $(m_{\sigma'} - m_\sigma)|_{\sigma'} \leq 0$. Thus m_σ and $m_{\sigma'}$ agree on $\sigma \cap \sigma'$ so that ψ is well defined. Since $m_\sigma \neq m_{\sigma'}$ for $\sigma \neq \sigma'$, Lemma 4.8.5 implies that ψ is stricly convex. Finally, since for any σ such that $\tau \in \sigma(1)$, we have $m_\sigma \in F_\tau$ and thus $\langle m_\sigma, v_\tau\rangle = -a_\tau$, the divisor associated to ψ is $D = -\sum_{\tau\in\Delta_P(1)} \psi(v_\tau)D_\tau = \sum_{\tau\in\Delta_P(1)} a_\tau D_\tau = D_P$. □

DEFINITION 6.2.3. A rational polytope P is reflexive if P° is also rational.

PROPOSITION 6.2.4. *Let P be a rational polytope in W.*

(1) *P is reflexive iff for each facet F of P, there exist $v_F \in N$ such that*
$$P = \{w \in W \mid \langle w, v_F\rangle \geq -1, \text{ for } F \text{ a facet of } P\}.$$
(2) *If P is reflexive, then 0 is the only lattice interior point in P.*

Proof. (1) Let $(v_F)_F$ be the vertices of P° (with F a facet of P using the bijection given in Proposition 3.5.10). We have $P = P^{\circ\circ} = \{w \in W \mid \langle w, v_F\rangle \geq -1, \text{ for } F \text{ a facet of } P\}$. Thus P is reflexive iff $v_F \in N$ for all F facet of P.

(2) Note that the interior of P is given by
$$\mathrm{Int}(P) = \{w \in W \mid \langle w, v_F\rangle > -1, \text{ for } F \text{ a facet of } P\},$$
thus 0 is an interior point in P. If $m \in M$ is in the interior of P, then $\langle m, v_F\rangle > -1$ for all facet F, thus $\langle m, v_F\rangle = 0$ for all facet F. Since P is a polytope, it is bounded thus $(v_F)_F$ spans V as a cone, we get $m = 0$. □

THEOREM 6.2.5. *X is Fano iff there exists a reflexive polytope P with $\Delta = \Delta_P$.*

Proof. If X is Fano, then $-K_X$ is ample and $\Delta = \Delta_P$ for $P = P_{-K_X} = \{w \in W \mid \langle w, v_\tau\rangle \geq -1, \forall \tau \in \Delta(1)\}$, thus P is reflexive. Conversely, if P is reflexive and $\Delta = \Delta_P$, then $-K_X = \partial X = D_P$ which is ample by Proposition 6.2.2. □

Actually there is a more general statement.

DEFINITION 6.2.6. A variety X is called Gorenstein if its canonical divisor $K_X \in \mathrm{Weil}(X)$ defined by $\omega_X = \mathcal{O}_X(K_X)$ is Cartier (*i.e.* $K_X \in \mathrm{Ca}(X)$). A Gorenstein variety is called Fano if K_X is ample.

Let X be a Gorenstein projective toric variety.

THEOREM 6.2.7. *X is Fano iff there exists a reflexive polytope P with $\Delta = \Delta_P$.*

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APPENDIX A

Tori

1. Algebraic groups

We recall few definitions and facts about algebraic groups and actions of algebraic groups and then focus on tori. For more details we refer to one of the classical books on the subject [1], [6] and [9].

Definition A.1.1. An algebraic group is a k-variety Γ which has a group structure such that the multiplication map $\mu : \Gamma \times \Gamma \to \Gamma$ and the inverse map $i : \Gamma \to \Gamma$ are morphisms.

Example A.1.2. The following are examples of algebraic groups.

(1) Finite groups.
(2) The additive group $(\mathbb{G}_a, +)$ with $\mathbb{G}_a = \mathsf{k}$.
(3) The multiplicative group $(\mathbb{G}_m, \times)$ with $\mathbb{G}_m = \mathsf{k}^\times$.
(4) The group $\mathrm{GL}_n(\mathsf{k})$ of invertible matrices of size n.
(5) The special linear group $\mathrm{SL}_n(\mathsf{k}) = \{A \in \mathrm{GL}_n(\mathsf{k}) \mid \det(A) = 1\}$.
(6) Elliptic curves are projective algebraic groups.

Here is a non-example in positive characteristic which shows that bad thing can happen even if we start with algebraic groups defined as above.

Example A.1.3. Let $\mathrm{char}(\mathsf{k}) = p$. The morphism of k-varieties $f : \mathbb{G}_m \to \mathbb{G}_m$ defined by $f(a) = a^p$ is a group morphism. The fiber (as scheme) of f over 1 is $\mathrm{Spec}(\mathsf{k}[x, x^{-1}]/(x^p - 1)) = \mathrm{Spec}(\mathsf{k}[x]/((x-1)^p)$ and is therefore not reduced. It has a group structure induced by the group structure of $\mathbb{G}_m$ but is not an algebraic group with our definition. The kernel of the above map (the fiber of 1 with reduced structure) is trivial.

The above subgroup is everywhere non-reduced and thus nowhere smooth. This never happens for algebraic groups. In fact, with the above definition, any algebraic group Γ is smooth and we may describe its irreducible components as follows.

Proposition A.1.4. *Let Γ be an algebraic group, e its neutral element, and Γ^0 the connected component of Γ containing e.*

(1) *Γ^0 is a closed normal subgroup of Γ, and a smooth irreducible variety.*
(2) *The connected components of Γ are the cosets $g.\Gamma^0$ for $g \in \Gamma$.*
(3) *Γ^0 is the largest closed connected subgroup of finite index of Γ.*

In particular, Γ is smooth and all its components are isomorphic as varieties. Thus Γ is equidimensional.

Definition A.1.5. The group of components of Γ is $\pi_0(\Gamma) := \Gamma/\Gamma_0$. It is finite.

Example A.1.6. The groups $\mathrm{GL}_n(\mathsf{k})$, $\mathrm{SL}_n(\mathsf{k})$, $\mathrm{SO}_n(\mathsf{k})$ and $\mathrm{Sp}_{2n}(\mathsf{k})$ are connected. We have $\pi_0(\mathrm{O}_n(\mathsf{k})) = \{\pm 1\}$ via the determinant.

We will only consider (very special) affine algebraic groups but for completeness we state the following two results.

THEOREM A.1.7. *A complete connected algebraic group is commutative.*

Complete connected algebraic groups are called *Abelian Varieties*. We refer to the book [**7**] for a proof of the above result and much more on abelian varieties. Furthermore, we have the following structure result on algebraic groups which explains why (up to extensions), we may split the study of algebraic groups into the study of affine algebraic groups and abelian varieties.

THEOREM A.1.8 (Chevalley Structure Theorem). *Let Γ be a connected algebraic group. Then Γ has a largest connected affine normal subgroup Γ_{aff}. Further, the quotient group $A := \Gamma/\Gamma_{\mathrm{aff}}$ is an abelian variety. We thus have an exact sequence*

$$1 \to \Gamma_{\mathrm{aff}} \to \Gamma \to A \to 0.$$

Actually the map $\Gamma \to A$ above in a special instance of the Abel-Jacobi map. We refer to [**3**] for a proof of this result. From now on, we focus on affine algebraic groups. Recall the following classical results on affine algebraic groups.

THEOREM A.1.9. *Let Γ be an affine algebraic group, then there exists a finite dimensional k-vector space V and a closed embedding $\Gamma \to \mathrm{GL}(V)$.*

2. Characters and diagonalisable groups

Recall the definition of character from Definition 1.3.1. We prove the following result in a special case, see [**6**, Chapter 16, Exercise 12] for the general case.

PROPOSITION A.2.1. *$\mathfrak{X}^*(G)$ is a finitely generated abelian group.*

DEFINITION A.2.2. G is called diagonalisable if $\mathfrak{X}^*(G)$ spans $\mathsf{k}[G]$.

This definition is different from the one given in the many part (see Definition 1.3.12) but we shall see that both definitions are equivalent.

EXAMPLE A.2.3. Recall that $D_n \subset \mathrm{GL}_n(\mathsf{k})$ is the subgroup of diagonal matrices. We have $\mathsf{k}[D_n] = \mathsf{k}\lfloor\mathfrak{X}^*(D_n)\rfloor$. More explicitely, we have $\mathfrak{X}^*(D_n) \simeq \mathbb{Z}^n$ and $\mathsf{k}[D_n] = \mathsf{k}[T_1^{\pm 1}, \cdot, T_n^{\pm 1}]$. In particular $D_n \simeq (\mathbb{G}_m)^n$ is diagonalisable.

PROPOSITION A.2.4. *Let G be diagonalisable group.*

1. *Any closed subgroup of G is also diagonalisable.*
2. *Any closed subgroup of G is the intersection of kernels of some characters.*
3. *The group G is a closed subgroup of $D_n \simeq (\mathbb{G}_m)^n$ for some n.*
4. *$\mathfrak{X}^*(G)$ is finitely generated.*

Proof. Let $H \subset G$ be a closed subgroup.

(1) Then the map $\mathsf{k}[G] \to \mathsf{k}[H]$ is surjective and maps an element of $\mathfrak{X}^*(G)$ to an element of $\mathfrak{X}^*(H)$. In particular $\mathfrak{X}^*(H)$ spans $\mathsf{k}[H]$ and H is diagonalisable.

(2) Let $f \in \mathrm{Ker}(\mathsf{k}[G] \to \mathsf{k}[H])$. Then f is a linear combination of elements in $\mathfrak{X}^*(G)$ thus $f = \sum_i \lambda_i \chi_i$ and we may assume that $\lambda_i \neq 0$ for all i. Restricting to H, we get $0 = \sum_i \lambda_i \chi_i|_H$ and since the $\chi_i|_H \in \mathfrak{X}^*(H)$ are characters thus linearly independent if non trivial, we must have $\chi_i|_H = 0$ for all i. In particular, the kernel of the map $\mathsf{k}[G] \to \mathsf{k}[H]$ is spanned by characters, proving the claim.

(3) Since $\mathsf{k}[G]$ is finitely generated and $\mathfrak{X}^*(G)$ spans $\mathsf{k}[G]$ as vector space, there exist $\chi_1, \cdots, \chi_n \in \mathfrak{X}^*(G)$ which generate $\mathsf{k}[G]$ as k-algebra. Define $\varphi : G \to (\mathbb{G}_m)^n$

by $\varphi(g) = (\chi_1(g), \cdots, \chi_n(g))$. This is a morphism of (affine) algebraic groups and since $\mathfrak{X}^*(G)$ generate $\mathsf{k}[G]$ the corresponding k-algebra morphism $\mathsf{k}[(\mathbb{G}_m)^n] \to \mathsf{k}[G]$ is surjective. In particular the map is injective and G is a closed subgroup of $D_n \simeq (\mathbb{G}_m)^n$ (this map may not be an isomorphism in positive characteristics). □

Corollary A.2.5. *G is diagonalisable iff it is a closed subgroup of D_n.*

With what we proved above, we leave the following result as an exercise.

Theorem A.2.6. *The contravariant functor $G \mapsto \mathfrak{X}^*(G)$ is a fully faithful from the category of diagonalisable group to the category of finitely generated groups.*

Remark A.2.7. The above functor is not an equivalence of category in positive characteristic since $\mathfrak{X}^*(G)$ has no p-torsion for $p = \mathrm{char}(\mathsf{k})$.

3. Actions and representations

Let X be a variety with an action of an affine algebraic group Γ.

Definition A.3.1. The action of Γ on X is called *rational* or *algebraic* if the map $a : \Gamma \times X \to X$ induced by the action is a morphism.

Since we will only deal with algebraic action, we will drop the word *rational* or *algebraic* when working with group actions. We will use the notation $g.x$ for the image of (g, x) via the action map $a : \Gamma \times X \to X$.

Definition A.3.2. Let Γ be an affine algebraic group.

(1) A Γ-variety is a variety X equipped with an algebraic action of Γ.
(2) Given two Γ-varieties X,Y and a morphism of varieties $f : X \to Y$, we say that f is *equivariant* (or a Γ-morphism) if $f(g.x) = g.f(x)$ for all $g \in \Gamma$ and $x \in X$.
(3) Let X be a Γ-variety, and $Y \subset X$ be a (locally closed) subvariety. We say that Y is Γ-stable (resp. Γ-fixed) if for all $g \in \Gamma$ and $y \in Y$, we have $g.y \in Y$ (resp. $g.y = y$). The Γ-fixed points X^Γ in X form a closed subvariety.
(4) Let X be a Γ-variety and $x \in X$, the *orbit map of x* is the morphism $a_x : \Gamma \to X$, $g \mapsto g.x$.
(5) The *orbit* $\Gamma.x$ of x is the image of a_x.
(6) The *stabiliser* of x (or *isotropy subgroup*) is $\Gamma_x = \{g \in \Gamma \mid g.x = x\}$ with its reduced structure. It is the reduced fiber of a_x over the unit of Γ.
(7) A Γ-variety X is Γ*-homogeneous* if there exists $x_0 \in X$ such that the map $a_{x_0} : G \to X$ is surjective. If this holds, a_x is surjective for any $x \in X$.
(8) A Γ-variety X is *quasi-homogeneous* if there exists $x \in X$ such that $\Gamma.x \subset X$ is a dense open orbit.

Remark A.3.3. The stabiliser Γ_x is a closed subgroup of Γ (since the fiber is closed). Furthermore, for $g \in \Gamma$, we have $\Gamma_{g.x} = g\Gamma_x g^{-1}$. Note also that any orbit $\Gamma.x$ is Γ-homogeneous

Example A.3.4. Some homogeneous and quasi-homogeneous spaces.

(1) The projective space $\mathbb{P}^n$ is a homogeneous $\mathrm{GL}_{n+1}(\mathsf{k})$-variety but also a quasi-homogeneous $(\mathbb{G}_m)^{n+1}$-variety where $(\mathbb{G}_m)^{n+1}$ is the subtorus of diagonal matrices in $\mathrm{GL}_{n+1}(\mathsf{k})$.

(2) The grassmannian variety $\mathrm{Gr}(p,n)$ of linear subspaces of dimension p in k^n is a homogeneous Γ-variety with $\Gamma = \mathrm{GL}_n(\mathsf{k})$.

PROPOSITION A.3.5. *Let X be a Γ-variety and let $x \in X$.*

(1) *The orbit $\Gamma.x$ is smooth and open in its closure (thus locally closed in X).*
(2) *$\Gamma.x$ is equidimensional of dimension $\dim(\Gamma) - \dim(\Gamma_x)$.*
(3) *The boundary $\overline{\Gamma.x} \setminus \Gamma.x$ is a union of orbits of strictly smaller dimension.*
(4) *Every orbit of minimal dimension is closed.*
(5) *The dimension of Γ_x is upper semicontinuous for $x \in X$ while the dimension of $\Gamma.x$ is lower semicontinuous.*
(6) *In particular if X is irreducible, it contains an open dense subset on which $\dim \Gamma.x$ is maximal and $\dim \Gamma_x$ is minimal.*

Proof. (1) The orbit $\Gamma.x$ contains an open subset of its closure as image of the map a_x, the result follows by translation by Γ, the orbit being homogeneous.

(2) By homogeneity, all fibers are isomorphic of the same dimension. The dimension formula follows from general results on dimension of fibers for morphisms.

(3) By (1), the boundary is closed and of strictly smaller dimension. Since it is Γ-stable it is a union of orbits.

(4) Follows from (3).

(5) Consider the map $\Gamma \times X \to X \times X$ defined by $(g,x) \mapsto (g.x,x)$ and let $Y = (\Gamma \times X) \times_{X \times X} \Delta_X$ with $\Delta_X \subset X \times X$ the diagonal. Note that Γ_x identifies with the fiber over (x,x) of the map $Y \to \Delta_x$. The result now follows from the semicontinuity of dimension of fibers of morphisms. The result for orbits follows from this and (2).

(6) Follows from (5). □

Representations are very simples and important examples of actions.

DEFINITION A.3.6. Let Γ be an algebraic group.

(1) A rational or algebraic representation of Γ (or a Γ-module) is a k-vector space V such that Γ acts on V by linear maps. Equivalently, such a representation is given by a morphism of algebraic groups $\Gamma \to \mathrm{GL}(V)$.
(2) For $\chi \in \mathfrak{X}^*(\Gamma)$ and V a Γ-module, set $V_\chi = \{v \in V \mid \Gamma.v = \chi(\gamma)v, \forall \gamma \in \Gamma\}$.
(3) A character χ is a weight of the Γ-module V if $V_\chi \neq 0$.

Diagonalisable groups have very simple representations.

THEOREM A.3.7. *G is diagonalisable iff for any G-module V, we have*

$$V = \oplus_{\chi \in \mathfrak{X}^*(G)} V_\chi.$$

Proof. See [9, Theorem 3.2.3]. □

DEFINITION A.3.8. Let V be a vector space and G an algebraic group.

(1) An element $f \in \mathrm{End}(V)$ is called semisimple if its minimal polynomial has simple roots.
(2) An element $g \in G$ is called semisimple if its image in any G-module is semisimple.

The following is a characterisation of diagonalisable groups.

PROPOSITION A.3.9. *A algebraic group G is diagionalisable if and only if G is commutative and every element in G is semisimple.*