Spherical varieties

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Introduction

Let G be a reductive algebraic group over k an algebraically closed field. Consider $\mathbb{G}$-varieties *i.e.* varieties X with an algebraic action of G. Equivariant birational geometry consists of the study of G-equivariant birational classes of G-varieties or, in a more algebraic perspective, of the study of the function fields $\mathsf{k}(X)$ of G-varieties together with its G-action. This contains the birational classification of varieties (for G trivial) thus it is way too ambitious for this lecture.

From this equivariant birational geometry point of view, spherical varieties correspond to one of the easiest possible situations. Indeed, a normal G-variety X is spherical if and only if $\mathsf{k}(X)^B = \mathsf{k}$ where B is a Borel subgroup of G (Theorem 2.3.9). This point of view is the starting point of the Luna-Vust Theory of embeedings (see **[17]**, **[15]** and **[6]**). Another equivalent definition is to ask that any G-birational model of X has finitely many G-orbits (cf. Theorem 4.0.3). For this reason, spherical varieties have especially nice geometric properties that we review in these lectures.

There are many well known examples of spherical varieties. Let us cite the most famous ones: rational projective homogeneous spaces like projective spaces or Grassmannians, toric varieties and symmetric spaces. One of the goal of a geometric study of spherical varieties is to extend as much as possible the classical geometric results known for the above three classes of spherical varieties to the general case. Another important motivation for the theory comes from the *equivariant compactification problem*: given a G-homogeneous variety $X_0 = G/H$, construct a G-equivariant embedding $X_0 \to X$ such that X is projective (compact) and X_0 is dense in X. Here are few example of such situations:

(1) Let X_0 be the reductive group G itself, what are the possible $G \times G$-equivariant compactifications, where $G \times G$ acts by left and right multiplication?
(2) Let X_0 be the set of non-degenerate quadratic forms on k^n. Then $X_0 = \mathrm{GL}_n(\mathsf{k})/\mathrm{O}_n(\mathsf{k})$, what are the possible compactification which are $\mathrm{GL}_n(\mathsf{k})$-equivariant?
(3) Let X_0 be the space of irreducible plane conis in $\mathbb{P}^2$. Then we have $X_0 = \mathrm{PGL}_3(\mathsf{k})/\mathrm{PO}_3(\mathsf{k})$. What is a good $\mathrm{PGL}_3(\mathsf{k})$-equivariant compactification for X_0? This last problem is related to the famous Steiner's conic problem: how many plane conics are tangent to 5 given conics?[1]

In these lectures, we want to carefully study G-varieties with a special focus on the case of spherical varieties. In particular, we would like to adress the following problems.

[1] Answer: 3264.

(1) Given geometric, group theoretic or representation theoretic characterisations of spherical varieties.
(2) Give a classification of spherical varieties.
(3) Describe the geometry of spherical varieties, in particular:
   - Describe the Picard group of a spherical variety.
   - Compute a canonical divisor.
   - Describe Chow groups and especially duality between curves and divisors.
   - Describe B-orbits and the inclusions between their closures.

We will very partially answer the above questions, reviewing results of many authors on spherical varieties.

Convention, notation and prerequisites

We work over an algebraically closed field k. We assume $\mathrm{char}(\mathsf{k}) \geq 0$ and specify when the assumption $\mathrm{char}(\mathsf{k}) = 0$ is needed but we will in many cases restrict to $\mathrm{char}(\mathsf{k}) = 0$ to simplify the proofs.

All the groups we shall consider will be linear algebraic groups (except in the first chapter). Denote by Γ such a group and use G for connected reductive groups. Denote by $R(\Gamma)$ and $R_u(\Gamma)$ the radical and the unipotent radical of a group Γ. The group of characters of Γ is denoted by $\mathfrak{X}(\Gamma)$.

Denote by T a maximal torus of G and B a Borel subgroup containing T. Denote by R the root system of (G, T), by R^+, respectively R^-, the sets of positive, respectively negative roots. For P a parabolic subgroup of G containing B, denote by P^- the opposite subgroup with respect to T.

Denote by W the Weyl group of G and by W_P the Weyl group of a parabolic subgroup $P \supset B$. Denote by U the maximal unipotent subgroup of B. Denote by $\mathrm{Lie}(G)$ the Lie algebra of G, we shall also use the gothic letter $\mathfrak{g}$. Write $\mathfrak{g}_\alpha$ for the root space associated to the root α and U_α for the one-dimensional unipotent subgroup of G with $\mathrm{Lie}(U_\alpha) = \mathfrak{g}_\alpha$.

We will also assume some familiarities with basics on algebraic geometry and use **[13]** as reference. A k-variety is a reduced and separated k-scheme of finite type over k. In particular a k-variety might be reducible. We will mainly consider k-varieties in these lectures.

Part 1

G-varieties

CHAPTER 1

Algebraic groups, actions and quotients

1. Algebraic groups

We recall few definitions and facts about algebraic groups and actions of algebraic groups. For more details we refer to one of the classical books on the subject [**1**], [**14**] and [**25**].

Definition 1.1.1. An algebraic group is a k-variety Γ which has a group structure such that the multiplication map $\mu : \Gamma \times \Gamma \to \Gamma$ and the inverse map $i : \Gamma \to \Gamma$ are morphisms.

Example 1.1.2. The following are examples of algebraic groups.

(1) Finite groups.
(2) The additive group $(\mathbb{G}_a, +)$ with $\mathbb{G}_a = \mathsf{k}$.
(3) The multiplicative group $(\mathbb{G}_m, \times)$ with $\mathbb{G}_m = \mathsf{k}^\times$.
(4) The group $\mathrm{GL}_n(\mathsf{k})$ of invertible matrices of size n.
(5) The special linear group $\mathrm{SL}_n(\mathsf{k}) = \{A \in \mathrm{GL}_n(\mathsf{k}) \mid \det(A) = 1\}$.
(6) The orthogonal group $\mathrm{O}_n(\mathsf{k})$, the special orthogonal group $\mathrm{SO}_n(\mathsf{k})$ and the symplectic group $\mathrm{Sp}_{2n}(\mathsf{k})$.
(7) The projective linear group $\mathrm{PGL}_n(\mathsf{k})$ obtained as the quotient $\mathrm{GL}_n(\mathsf{k})/\mathbb{G}_m$ where $\mathbb{G}_m$ is the center of $\mathrm{GL}_n(\mathsf{k})$.
(8) Elliptic curves are projective algebraic groups.

In the above list of example, we only consider algebraic groups which are either affine (all cases but the last one) or projective (the first and last cases). This dichotomy is classical because of the big difference between these cases. However, recently mixed cases gained a lot of attention.

Here is a non example in positive characteristic which shows that bad thing can hapend even if we start with algebraic groups defined as above.

Example 1.1.3. Let $\mathrm{char}(\mathsf{k}) = p$. The morphism of k-varieties $f : \mathbb{G}_m \to \mathbb{G}_m$ defined by $f(a) = a^p$ is a group morphism. The fiber (as scheme) of f over 1 is $\mathrm{Spec}(\mathsf{k}[x, x^{-1}]/(x^p - 1)) = \mathrm{Spec}(\mathsf{k}[x]/((x-1)^p)$ and is therefore not reduced. It has a group structure induced by the group structure of $\mathbb{G}_m$ but is not an algebraic group with our definition. The kernel of the above map (the fiber of 1 with reduced structure) is trivial.

The above subgroup is everywhere non-reduced and thus nowhere smooth. This never happends for algebraic groups. In fact, with the above definition, any algebraic group Γ is smooth and we may describe its irreducible components as follows.

Proposition 1.1.4. *Let Γ be an algebraic group, e its neutral element, and Γ^0 the connected component of Γ containing e.*

(1) *Γ^0 is a closed normal subgroup of Γ, and a smooth irreducible variety.*

(2) *The connected components of* Γ *are the cosets* $g.\Gamma^0$ *for* $g \in \Gamma$.
(3) Γ^0 *is the largest closed connected subgroup of finite index of* Γ.

In particular, Γ *is smooth and all its components are isomorphic as varieties. Thus* Γ *is equidimensional.*

DEFINITION 1.1.5. The finite group Γ/Γ_0 is called the group of components of Γ. We denote it by $\pi_0(\Gamma)$.

EXAMPLE 1.1.6. The groups $\mathrm{GL}_n(\mathsf{k})$, $\mathrm{SL}_n(\mathsf{k})$, $\mathrm{SO}_n(\mathsf{k})$ and $\mathrm{Sp}_{2n}(\mathsf{k})$ are connected. We have $\pi_0(\mathrm{O}_n(\mathsf{k})) = \{\pm 1\}$ via the determinant.

We will only consider affine algebraic groups but for completeness we state the following two results.

THEOREM 1.1.7. *A complete connected algebraic group is commutative.*

Complete connected algebraic groups are called *Abelian Varieties*. We refer to the book [**18**] for a proof of the above result and much more on abelian varieties. Furthermore, we have the following structure result on algebraic groups which explains why (up to extensions), we may split the study of algebraic groups into the study of affine algebraic groups and abelian varieties.

THEOREM 1.1.8 (Chevalley Structure Theorem). *Let* Γ *be a connected algebraic group. Then* Γ *has a largest connected affine normal subgroup* Γ_{aff}. *Further, the quotient group* $A := \Gamma/\Gamma_{\mathsf{aff}}$ *is an abelian variety. We thus have an exact sequence*

$$1 \to \Gamma_{\mathsf{aff}} \to \Gamma \to A \to 0.$$

Actually the map $\Gamma \to A$ above in a special instance of the Abel-Jacobi map. We refer to [**3**] for a proof of this result. From now on, we focus on affine algebraic groups. Recall the following classical results on affine algebraic groups.

THEOREM 1.1.9. *Let* Γ *be an affine algebraic group, then there exists a finite dimensional* k*-vector space* V *and a closed embedding* $\Gamma \to \mathrm{GL}(V)$.

2. Actions

Let X be a variety with an action of an affine algebraic group Γ.

DEFINITION 1.2.1. The action of Γ on X is called *rational* or *algebraic* if the map $a : \Gamma \times X \to X$ induced by the action is a morphism.

Since we will only deal with algebraic action, we will drop the word *rational* or *algebraic* when working with group actions. We will use the notation $g.x$ for the image of (g, x) via the action map $a : \Gamma \times X \to X$.

DEFINITION 1.2.2. Let Γ be an affine algebraic group.

(1) A Γ-variety is a variety X equipped with an algebraic action of Γ.
(2) Given two Γ-varieties X,Y and a morphism of varieties $f : X \to Y$, we say that f is *equivariant* (or a Γ-morphism) if $f(g.x) = g.f(x)$ for all $g \in \Gamma$ and $x \in X$.
(3) Let X be a Γ-variety, and $Y \subset X$ be a (locally closed) subvariety. We say that Y is Γ-stable (resp. Γ-fixed) if for all $g \in \Gamma$ and $y \in Y$, we have $g.y \in Y$ (resp. $g.y = y$). The Γ-fixed points X^Γ in X form a closed subvariety.

(4) Let X be a Γ-variety and $x \in X$, the *orbit map of* x is the morphism $a_x : \Gamma \to X$, $g \mapsto g.x$.
(5) The *orbit* $\Gamma.x$ of x is the image of a_x.
(6) The *stabiliser* of x (or *isotropy subgroup*) is $\Gamma_x = \{g \in \Gamma \mid g.x = x\}$ with its reduced structure. It is the reduced fiber of a_x at e the identity element of Γ.
(7) A Γ-variety X is Γ-*homogeneous* if there exists $x_0 \in X$ such that the map $a_{x_0} : G \to X$ is surjective. If this holds, a_x is surjective for any $x \in X$.
(8) A Γ-variety X is *quasi-homogeneous* if there exists $x \in X$ such that $\Gamma.x \subset X$ is a dense open orbit.

REMARK 1.2.3. The stabiliser Γ_x is a closed subgroup of Γ (since the fiber is closed). Furthermore, for $g \in \Gamma$, we have $\Gamma_{g.x} = g\Gamma_x g^{-1}$. Note also that any orbit $\Gamma.x$ is Γ-homogeneous

EXAMPLE 1.2.4. Some homogeneous and quasi-homogeneous spaces.
(1) The projective space $\mathbb{P}^n$ is a homogeneous $\mathrm{GL}_{n+1}(\mathsf{k})$-variety but also a quasi-homogeneous $(\mathbb{G}_m)^{n+1}$-variety where $(\mathbb{G}_m)^{n+1}$ is the subtorus of diagonal matrices in $\mathrm{GL}_{n+1}(\mathsf{k})$.
(2) The grassmannian variety $\mathrm{Gr}(p, n)$ of linear subspaces of dimension p in k^n is a homogeneous Γ-variety with $\Gamma = \mathrm{GL}_n(\mathsf{k})$.

PROPOSITION 1.2.5. *Let X be a Γ-variety and let $x \in X$.*
(1) *The orbit $\Gamma.x$ is smooth and open in its closure (thus locally closed in X).*
(2) *$\Gamma.x$ is equidimensional of dimension $\dim(\Gamma) - \dim(\Gamma_x)$.*
(3) *The boundary $\overline{\Gamma.x} \setminus \Gamma.x$ is a union of orbits of strictly smaller dimension.*
(4) *Every orbit of minimal dimension is closed.*
(5) *The dimension of Γ_x is upper semicontinuous for $x \in X$ while the dimension of $\Gamma.x$ is lower semicontinuous.*
(6) *In particular if X is irreducible, it contains an open dense subset on which $\dim \Gamma.x$ is maximal and $\dim \Gamma_x$ is minimal.*

Proof. (1) The orbit $\Gamma.x$ contains an open subset of its closure as image of the map a_x, the result follows by translation by Γ, the orbit being homogeneous.

(2) By homogeneity, all fibers are isomorphic of the same dimension. The dimension formula follows from general results on dimension of fibers for morphisms.

(3) By (1), the boundary is closed and of strictly smaller dimension. Since it is Γ-stable it is a union of orbits.

(4) Follows from (3).

(5) Consider the map $\Gamma \times X \to X \times X$ defined by $(g, x) \mapsto (g.x, x)$ and let $Y = (\Gamma \times X) \times_{X \times X} \Delta_X$ with $\Delta_X \subset X \times X$ the diagonal. Note that Γ_x identifies with the fiber over (x, x) of the map $Y \to \Delta_x$. The result now follows from the semicontinuity of dimension of fibers of morphisms. The result for orbits follows from this and (2).

(6) Follows from (5). $\square$

3. Quotients

The existence and construction of quotients of a Γ-variety under the action of Γ is often difficult.

EXAMPLE 1.3.1. The following examples show that the usual quotient for abstract goup actions will not have nice geometric properties in general.

(1) Let $\Gamma = \mathbb{G}_m$ be the mutliplicative group of invertible elements in k and let Γ act on $X = \mathsf{k}^2$ via $z.(x, y) = (zx, zy)$. The orbits are the origin $x_0 = \{(0, 0)\}$ and the lines through x_0 (without x_0). So any orbit closure contains x_0 and there is no topology on the quotient X/Γ such that X/Γ is separated and the quotient map $X \to X/\Gamma$ is continuous.

    On the other hand, removing x_0, we can define a nice quotient $(X \setminus x_0)/\Gamma = (\mathsf{k}^2 \setminus \{(0, 0)\})/\Gamma = \mathbb{P}^1$.

(2) Let $\Gamma = \mathbb{G}_a$ be the one-dimensional addivitive group over k, let $X = \mathsf{k}^2$ and let Γ act on X via $t.(x, y) = (x, y + tx)$. The points on the line $x = 0$ are fixed while the other orbits are the lines $x = x_0$ with $x_0 \neq 0$. In particular all orbits are closed.

    Once again there is no topology on the quotient X/Γ such that it is separated and the quotient map $X \to X/\Gamma$ is continuous. Furthermore, there is no structure of variety on X/Γ such that $X \to X/\Gamma$ is a morphism. Indeed, otherwise, we would be able to separate points on X/Γ by rational functions so to separate orbits on X by invariant rational functions. But an invariant rational function on X is a rational function in x and rational functions in x do not separate fixed points.

    Restricting to the open G-invariant subset $X \setminus \{x = 0\}$, there is a nice quotient given by $X \setminus \{x = 0\} \to \mathsf{k}, (x, y) \mapsto x$.

Let us define what we mean by a quotient.

DEFINITION 1.3.2. Let X be a Γ-variety. A geometric quotient of X by Γ is a morphism $\pi : X \to Y$ such that

(1) π is surjective and the fiber of π coincide with the Γ-orbits
(2) π induces and isomorphism $\mathsf{k}(Y) \simeq \mathsf{k}(X)^\Gamma$.

REMARK 1.3.3. T>he first condition above imples that π is Γ-equivariant, that the orbits are closed and that $Y = X/\Gamma$.

The first example is the case of finite groups.

PROPOSITION 1.3.4. *Let Γ be a finite group and let X be a quasi-projective Γ-variety, then there exists a geometric quotient $X \to X/\Gamma$.*

Proof. We only give a sketch of proof. If $X = \mathrm{Spec}(A)$ is affine with A a finitely generated k-algebra, then one checks that A^Γ is a finitely generated as well. Then the geometric quotient is given by $X = \mathrm{Spec}(A) \to X/\Gamma = \mathrm{Spec}(A^\Gamma)$. Furthermore this quotient is uniquely determined.

For X quasi projective, any finite set is contained in an open affine subset. In particular any orbit is contained in an affine open subset and taking the intersection of its translates, any orbit is contained in a Γ-stable affine open subset. We construct the quotient on each of these affine open Γ-stable subsets and glue them together using the fact that the quotient is unique. $\square$

We now give important examples of geometric quotient.

DEFINITION 1.3.5. A subgroup Γ' of Γ is called a closed subgroup if it is a closed subvariety of Γ.

THEOREM 1.3.6. *Let Γ' be a closed subgroup of Γ, then the quotient Γ/Γ' has a unique structure of algebraic variety such that the quotient map $\pi : \Gamma \to \Gamma/\Gamma'$ is a morphism. This map is flat and separable.*

PROPOSITION 1.3.7. *Let $\Gamma' \subset \Gamma$ be a closed subgroup.*

(1) *The quotient map $\Gamma \to \Gamma/\Gamma'$ is a geometric quotient.*
(2) *The quotient Γ/Γ' is quasi-projective.*
(3) *The quotient Γ/Γ' is projective iff Γ' is a parabolic subgroup*[1].
(4) *If Γ' is a normal subgroup, then Γ/Γ' is affine and has a structure of linear algebraic group such that the quotient map is a morphism of algebraic groups.*
(5) *If Γ is solvable, then Γ/Γ' is a affine.*
(6) *If Γ' is reductive, then Γ/Γ' is a affine.*

Proof. (1) and (2) See [**25**, Theorem 5.5.5]. (3) See [**25**, Theorem 6.2.7]. (4) See [**25**, Proposition 5.5.10]. (5) See [**28**, Theorem 3.5]. (6) See [**28**, Theorem 3.7]. □

From a given geometric quotient, one can construct new geometric quotients. We refer to Appendix A for proofs and more details.

PROPOSITION 1.3.8 (See Corollary A.2.11). *Let Γ be a group, $\Gamma' \subset \Gamma$ be a closed subgroup and F be a quasi-projective Γ'-variety. Consider the action of Γ' on $\Gamma \times F$ given by $(g,(x,f)) \mapsto (xg^{-1}, g.f)$.*

Then $\Gamma \times F$ admits a geometric quotient denoted by $\Gamma \times^{\Gamma'} F$ and we have a cartesian diagram

$$\begin{tikzcd} \Gamma \times F \arrow[r] \arrow[d] & \Gamma \times^{\Gamma'} F \arrow[d] \\ \Gamma \arrow[r] & \Gamma/\Gamma'. \end{tikzcd}$$

A very important particular example is the construction of Γ-equivariant vector bundles on homogeneous spaces.

EXAMPLE 1.3.9. Let $\Gamma' \subset \Gamma$ be closed subgroup and let V be a Γ'-representation, we obtain a morphism $\Gamma \times^{\Gamma'} V \to \Gamma/\Gamma'$ whose fibers are isomorphic to V. This is actually a vector bundle on Γ'/Γ as a result of the following generalisation of Hilbert's Theorem 90.

THEOREM 1.3.10. *Any surjective $\mathrm{GL}_n(\mathsf{k})$-invariant morphism $\pi : X \to Y$ with fibers isomorphic to $\mathrm{GL}_n(\mathsf{k})$ is locally trivial for Zariski topology: there exists an Zariski open covering $\{U_i\}$ of Y such that $\pi^{-1}(U_i) \simeq U_i \times \mathrm{GL}_n(\mathsf{k})$.*

As a consequence the above fibration $\Gamma \times^{\Gamma'} V \to \Gamma/\Gamma'$ is locally trivial for Zariski topology. Indeed using the map $\Gamma' \to \mathrm{GL}(V)$, we can write $\Gamma \times^{\Gamma'} V = (\Gamma \times^{\Gamma'} \mathrm{GL}(V)) \times^{\mathrm{GL}(V)} V$. But $\Gamma \times^{\Gamma'} \mathrm{GL}(V) \to \Gamma/\Gamma'$ satisfies the assumptions of the above theorem and is Zariski locally trivial proving the result.

We finish this section on quotients with a result of Rosenlicht [**23**].

THEOREM 1.3.11. *Let Γ be a linear algebraic group and let X be an irreducible Γ-variety. There exists a non empty open Γ-stable subset X_0 such that $X_0 \to X_0/\Gamma$ is a geometric quotient.*

[1] *i.e.* contains a Borel subgroup.

Proof. We first assume that Γ is connected. Replacing X by an open subset, we may assume that all orbits have the same dimension. Since $\mathsf{k}(X)$ is a finitely generated extension of k, the same holds for $\mathsf{k}(X)^\Gamma$. Choose $f_1, \cdots, f_r \in \mathsf{k}(X)^\Gamma$ some generators of this extention. Replacing X by an open subset, we may assume that $f_i \in \mathsf{k}[X]$ for all i. Consider the morphism $f : X \to \mathsf{k}^r$ defined by $x \mapsto (f_1(x), \cdots, f_r(x))$. Replacing X by an open subset, we may assume that the image Y is affine and by generic flatness that f is flat. In particular all fibers have the same dimension.

To prove that f (or the restriction of f to an open subset of X) is a geometric quotient, we are left to prove that the fibers of f are Γ-orbits. Since f is Γ-invariant, the fibers are union of orbits. Since all the orbits have the same dimension, it is enough to prove that generically, for $x \in X$, the orbit $\Gamma.x$ is dense in $f^{-1}(f(x))$.

To prove the above claim, consider the morphism $\phi : \Gamma \times X \to X \times X$ defined by $(g, x) \mapsto (x, g.x)$. Let $Z = \{(x, x') \in X \times X \mid x' \in \Gamma.x\}$ be its image. Since Γ is connected, Z is irreducible. Define $W = X \times_Y X = \{(x, x') \in X \times X \mid f(x) = f(x')\}$. Since f is Γ-invariant, it is constant on orbits therefore we have $Z \subset W$. Note for $p = \mathrm{pr}_2 : W = X \times_Y X \to X$, we have $p^{-1}(x) = f^{-1}(x)$ and $p^{-1}(x) \cap Z = \Gamma.x$. In particular, if Z is dense in W, then W is irreducible and by Lemma 1.3.13, there exists a open subset X_0 of X such that all fibers $p^{-1}(x) \cap Z$ is dense in $p^{-1}(x)$ for $x \in X_0$ proving that the restriction of the map $f : X \to Y$ to X_0 is the desired geometric quotient.

To conclude the proof, we need to prove that Z is dense in W. Consider U and U' two non empty affine open subsets of X and consider the restriction

$$\psi : V = \phi^{-1}(U \times_Y U') \to U \times_Y U'.$$

We prove that ψ is dominant. Note that since X is irreducible, the diagonal Δ_X of $X \times X$ meets $U \times U'$ and therefore V is non empty. Note also that $U \times_Y U'$ is affine so it is sufficent to prove that $\psi^* : \mathsf{k}[U] \otimes_{\mathsf{k}[Y]} \mathsf{k}[U'] \to \mathsf{k}[V]$ is injective. Let $\sum_i u_i \otimes v_i \in \mathrm{Ker}\psi^*$. For all $(g, x) \in \Gamma \times X$, we have

$$\psi^* \left(\sum_i u_i \otimes v_i \right) (g, x) = \sum_i u_i(x) v_i(g.x).$$

For $g \in \Gamma$, define $h_g \in \mathsf{k}(X)$ by $h_g(x) = \sum_i u_i(x) v_i(g.x)$. Then h_g vanishes on $U \cap g^{-1}U'$ so $h_g = 0$. We prove that this implies the vanishing of $\sum_i u_i \otimes v_i$. First we may assume that the v_i are linearly independent over $\mathsf{k}(X)^\Gamma = \mathsf{k}(Y)$. Indeed, if $v_j = \sum_{i \neq j} c_i v_i$, then $\sum_i u_i \otimes v_i = \sum_{i \neq j} (u_i + c_i u_j) \otimes v_i$. The vanishing of $\sum_i u_i \otimes v_i$ now follows from the following lemma.

Lemma 1.3.12. *Let $(u_i)_{i \in [1,s]}$, $(v_i)_{i \in [1,s]}$ be rational functions on X such that the v_i are linearly independent over $\mathsf{k}(X)^\Gamma$. If $h_g = \sum_i u_i(g^{-1}.v_i) = 0$ for all $g \in \Gamma$, then $u_i = 0$ for all i.*

Proof. We proceed by induction on s. If $s = 1$, the result is obvious. Assume $s > 1$ and $u_1 \neq 0$. For all $g, g' \in \Gamma$, we have $\sum_i g'.(u_i u_1^{-1})(g.v_i) = 0$. Since $g'.(u_1 u_1^{-1}) = g'.1 = 1 = u_1 u_1^{-1}$, we get

$$\sum_{i \geq 2} \left(g'.(u_i u_1^{-1}) - (u_i u_1^{-1}) \right) (g.v_i) = 0.$$

By induction hypothesis, it follows that $g'.(u_i u_1^{-1}) = u_i u_1^{-1}$ for all $g' \in \Gamma$, so that $u_i u_1^{-1} \in \mathsf{k}(X)^\Gamma$. We therefore get a linear dependence relation

$$v_1 + \sum_{i\geq 2}(u_i u_1^{-1})v_i = 0$$

contradicting the linear independence of the v_i's. □

This proves that Z is dense in W and finishes the proof for Γ connected. If Γ is not connected, let Γ^0 be its connected component containing the identity. By the above argument, we find an Γ^0-stable open subset X_0 admitting a geometric quotient by Γ^0. The intersection of all $g.X_0$ is open (since Γ/Γ^0 is finite) and Γ-stable. Replacing X by this open subset, we may assume that there is a geometric quotient $X \to X/\Gamma^0$ on which the finite group Γ/Γ^0 acts. The geometric quotient $(X/\Gamma^0)/(\Gamma/\Gamma^0)$ which exists for finite groups is a geometric quotient X/Γ. □

Lemma 1.3.13. *Let W be irreducible and $Z_0 \subset W$ be a dense open subset. Let $p : W \to X$ be a dominant morphism. Then there exists a dense open subset X_0 of X such that $p^{-1}(x) \cap Z_0$ is dense in $p^{-1}(x)$ for all $x \in X_0$.*

Proof. Let $C_1, \cdots, C_s$ be the irreducible components of $W \setminus Z_0$ and reorder them so that $C_1, \cdots, C_r$ map dominantly onto X while $C_r, \cdots, C_s$ do not. Set

$$U = X \setminus \bigcup_{i>r} \overline{p(C_i)}.$$

Let X_0 be the open subset of U such that the following holds:

(1) Any component of the fiber of $p^{-1}(X_0) \to X_0$ has dimension $\dim W - \dim X$,
(2) For all $i \in [1, r]$, any component of the fiber of $C_i \cap p^{-1}(X_0) \to X_0$ has dimension $\dim C_i - \dim X$.

Now for $x \in X_0$, the fiber $p^{-1}(x)$ will not meet C_i for $i > r$ and for $i \in [1, r]$, any irreducible component C of $p^{-1}(x)$ satisfies the following inequalities:

$$\dim(C_i \cap C) = \dim C_i - \dim X < \dim W - \dim X = \dim C.$$

In particular C meets Z_0 and the result follows. □

Exercise 1.3.14. Recall Example 1.3.1.

(1) Let $\Gamma = \mathbb{G}_m$ be the mutliplicative group of invertible elements in k and let Γ act on $X = \mathsf{k}^2$ via $z.(x, y) = (zx, zy)$. Prove that the quotient X/Γ endowed with the quotient topology is not separated.
(2) Let $\Gamma = \mathbb{G}_a$ be the one-dimensional addivitive group over k, let $X = \mathsf{k}^2$ and let Γ act on X via $t.(x, y) = (x, y + tx)$. Prove that the quotient X/Γ endowed with the quotient topology is not separated.

Exercise 1.3.15. Let $G = \mathrm{GL}_n(\mathsf{k})$ act by conjugation on $X = M_n(\mathsf{k})$ the space of square matrices of size n: $g.A = gAg^{-1}$ for $g \in G$ and $A \in X$.

(1) Find an open subset on which we have a geometric quotient.
(2) Does there exist a geometric quotient for X itself?
(3) Find a maximal open subset of X on which there exists a geometric quotient.

CHAPTER 2

Invariants of G-varieties

In this chapter, we focus on varieties with the action of a reductive group G.

1. Rank and Complexity

Let Γ be a linear algebraic group, let G be a reductive linear algebraic group and choose $T \subset B \subset G$ where B is a Borel subgroup and T a maximal torus.

DEFINITION 2.1.1. The complexity $c_\Gamma(X)$ of an irreducible Γ-variety X is the minimal codimension of a Γ-orbit: $c_\Gamma(X) = \min\{\mathrm{codim}(Y) \mid Y \subset X \text{ is a } \Gamma\text{-orbit}\}$. If the group Γ is clear from the context, we write $c_\Gamma(X) = c(X)$.

Let X be an irreducible Γ-variety, the following proposition computes the complexity $c_\Gamma(X)$ using the action of Γ on $\mathsf{k}(X)$, the field of rational functions.

PROPOSITION 2.1.2. *We have* $c_\Gamma(X) = \mathrm{Trdeg}(\mathsf{k}(X)^\Gamma)$.

Proof. Let X_0 be a Γ-stable open subset such that there is a geometric quotient $\pi : X_0 \to X_0/\Gamma$ as in Rosenlicht's Theorem (Theorem 1.3.11). Since the dimension of Γ-orbits is lower semi-continuous the maximal dimension of an orbit is the dimension of the fibers of π thus $c_\Gamma(X) = \dim X_0/\Gamma = \mathrm{Trdeg}(\mathsf{k}(X_0/\Gamma)) = \mathrm{Trdeg}(\mathsf{k}(X)^\Gamma)$. $\square$

DEFINITION 2.1.3. Let X be an irreducible B-variety, the weight lattice of X is the subgroup $\Lambda(X) \subset \mathfrak{X}(B)$ of weights of B occuring in $\mathsf{k}(X)$. This is a free abelian group and its rank $\mathrm{rk}(X)$ is the rank of X.

EXAMPLE 2.1.4. Let $X = \mathbb{P}^n$ be the projective space.

(1) Let $G = \mathrm{PGL}_{n+1}(\mathsf{k})$ and consider X as a G-variety. Choose $B \subset G$ be the Borel subgroup of upper triangular matrices and U its unipotent radical (the subgroup of upper triangular matrices with a 1 on the diagonal). Then U acts with an open orbit on X thus any B-eigenfunction is U-invariant and thus constant. In particular the set $\mathsf{k}(X)^{(B)}$ of B-eigenvectors in $\mathsf{k}(X)$ is given by constant functions and $\Lambda(X)$ is trivial and $\mathrm{rk}(X) = 0$.

(2) Let $G = T \subset \mathrm{PGL}_{n+1}(\mathsf{k})$ be the maximal torus of diagonal matrices. Then $T = B = G$ (the group G is solvable and connected). Furthermore, the coordinate functions x_i/x_j on $\mathbb{P}^n$ have weight $\epsilon_i - \epsilon_j$ and $\Lambda(X)$ is the roots lattice of $\mathrm{PGL}_{n+1}(X)$ (which is also the weight lattice of $T_0 = \{t \in T \mid \det(t) = 1\}$). We thus have $\mathrm{rk}(X) = n$.

(3) Let T be a torus and $X = T.x$ be a T-orbit. Then $X \simeq T/T'$ for some subgroup $T' \subset T$. In particular X is isomorphic to a torus $T_0 \simeq \mathbb{G}_m^{\dim X}$. In an adapted basis, we have $\mathsf{k}(X) \simeq \mathsf{k}(x_1^{\pm 1}, \cdots, x_{\dim X}^{\pm 1})$ where T acts via a basis of character of T_0 on $x_1, \cdots, x_{\dim X}$. We thus have $\mathrm{rk}(X) = \dim X$.

REMARK 2.1.5. Both rank and complexity and birational invariants. Its is clear for the rank since it is defined using the field $\mathsf{k}(X)$ of rational functions, for complexity it comes from the semicontinuity of the dimension of orbits or from Proposition 2.1.2.

For T-varieties we have the following relation between rank and complexity.

PROPOSITION 2.1.6. *Let X be an irreducible T-variety, then* $\mathrm{rk}(X)$ *is the maximal dimension of T-orbits. In other words* $c_T(X) = \dim X - \mathrm{rk}(X)$

Proof. Since the dimension of orbits is lower semi-continuous, we may replace X by an open subset on which we have a geometric quotient $\pi : X \to X/T$ by Theorem 1.3.11. Fix generators $\chi_1, \cdots, \chi_r$ with $r = \mathrm{rk}(X)$ of $\Lambda(X)$. There exists elements $f_1, \cdots, f_r \in \mathsf{k}(X)^{(T)}$ with weights $\chi_1, \cdots, \chi_r$. Choose a general enough fiber F of π such that f_i restricts to a well defined non vanishing rational function on F. Then the χ_i are weight of $\mathsf{k}(F)$ and we have an inclusion $\Lambda(X) \subset \Lambda(F)$ and therefore $\mathrm{rk}(X) \leq \mathrm{rk}(F)$. By Proposition 2.2.7 below, we have $\mathrm{rk}(F) \leq \mathrm{rk}(X)$ and the result follows by Example 2.1.4.(3). The last assertion follows from Proposition 2.1.2. □

EXAMPLE 2.1.7. Beware that rank and complexity depend on the group acting. If we replace T by an arbitrary reductive group G, the above result is false.

(1) For $X = \mathbb{P}^n$ and $G = T \subset \mathrm{PGL}_{n+1}(\mathsf{k})$ a maximal torus, then $c_T(X) = 0$ since T has a dense orbit and $\mathrm{rk}(X) = \dim X$.
(2) However, if $X = \mathbb{P}^n$ and $G = \mathrm{PGL}_{n+1}(\mathsf{k})$, then we have $c_G(X) = 0$ since X is G-homogenous but $\mathrm{rk}(X) = 0$ and $c_G(X) = 0 < n = \dim X - \mathrm{rk}(X)$.
(3) Let $G = \mathrm{SL}_2(\mathsf{k}) = X$ and $T \subset G$ be a maximal torus. As a T-variety, we have $\mathrm{rk}(X) = \dim X - c_T(X) = 3 - 2 = 1$ which is the dimension of any T-orbit in X. It is also easy to check that $\mathrm{rk}(X) = 1$ as G-variety as well. However, we have $c_G(X) = 0$ since X is G-homogeneous so that $c_G(X) = 0 < 2 = \dim(X) - \mathrm{rk}(X)$ as G-variety.

In the previous proposition we use a relation between the rank of X and the rank of G-subvarieties. In the next section, we consider the behaviour of rank and complexity by restriction.

2. Rank and complexity of stable subvarieties

We want to compare rank and complexity of X with those of stable subvarieties. For this we will need to find stable affine open subsets. This is in general not possible as shows the following example.

EXAMPLE 2.2.1. Let $X = \mathbb{P}^n$ and $\Gamma = \mathrm{GL}_{n+1}(\mathsf{k})$. Then X is a homogeneous Γ-variety and admits no affine open Γ-stable subset. So in general there is no Γ-stable affine covering.

This can be solved in two different – both useful – ways. The first one keeps the group Γ unchanged by replaces *affine* by *quasi-projective*. In this direction, we recall the following results of Sumihiro [**26, 27**] enabling in many cases to assume that a Γ-variety is quasi-projective. Let me mention the paper by Brion [**2**] where the second result below is generalised to Γ-varieties defined over any field and with Γ algebraic by not necessarily linear. We refer to Appendix B for some proofs.

THEOREM 2.2.2 (Equivariant Chow-Lemma). *Let Γ be a connected linear algebraic group and X a Γ-variety. There exists a quasi-projective Γ-variety $\widetilde{X}$ and a birational Γ-equivariant projective surjective morphism $f : \widetilde{X} \to X$.*

THEOREM 2.2.3. *Let Γ be a connected linear algebraic group, X a normal Γ-variety and $Y \subset X$ a Γ-orbit.*

(1) *There exists a quasi-projective Γ-invariant open subset containing Y.*
(2) *If X is quasi-projective, there exists a finite dimensional Γ-module V together with a Γ-equivariant embedding $X \to \mathbb{P}(V)$.*
(3) *There exists a Γ-equivariant embedding $X \to \bar{X}$ with $\bar{X}$ normal and proper.*

We prove parts (1) and (2) of the above theorem in Theorem B.3.4.

DEFINITION 2.2.4. A Γ-variety X is called locally linear if it admits a Γ-stable covering $(U_i)_i$ such that for each i there exists a finite fdimensional Γ-module V_i and a Γ-equivariant embedding $U_i \to \mathbb{P}(V_i)$.

Note that by Theorem 2.2.3, any normal Γ-variety is locally linear. We will mainly consider locally linear Γ-varieties.

Another possible direction, is to look for affine open subsets which are stable under a subgroup of the acting group. Borel subgroups will work.

PROPOSITION 2.2.5. *Let X be a normal G-variety and Y a G-stable closed subset. Then there exists an open B-stable affine open subset X_0 of X such that*

(1) $X_0 \cap Y \neq \emptyset$
(2) $\forall f \in \mathsf{k}[X_0 \cap Y]^{(B)}$, $\exists N \in \mathbb{N}$ *and* $\exists f' \in \mathsf{k}[X_0]^{(B)}$ *with* $f'|_{X_0\cap Y} = f^{p^N}$,

where p is the characteristic exponent of k.

Proof. By Sumihiro's Theorem, we may assume that X is equivariantly embedded in $\mathbb{P}(V)$ where V is a finite dimensional G-module. Let $\bar{X}$ and $\bar{Y}$ be the closures of X and Y in $\mathbb{P}(V)$. Set $\partial X = \bar{X} \setminus X$ and let $\widehat{X}$, $\widehat{\partial X}$ and $\widehat{Y}$ be the cones in V over $\bar{X}$, ∂X and $\bar{Y}$.

Note that $\widehat{Y} \not\subset \widehat{\partial X}$ and choose a homogeneous B-eigenfunction $f \in \mathsf{k}[\widehat{Y} \cup \widehat{\partial X}]^{(B)}$ vanishing on $\widehat{\partial X}$ but not on $\widehat{Y}$. We now need the following result from representation theory (see [**8**, Theorems 1.3 and 2.1]):

THEOREM 2.2.6. *Let X be an affine G-variety, $Y \subset X$ a closed G-stable subset and $f \in \mathsf{k}[Y]^{(B)}$. Then there exists $N \in \mathbb{N}$ and $f \in \mathsf{k}[X]^{(B)}$ with $f'|_Y = f^{p^N}$.*

Note that if $\mathrm{char}(\mathsf{k}) = 0$, then we have a surjective map $\mathsf{k}[X] \to \mathsf{k}[Y]$ and a finite dimensional G-module $W \subset \mathsf{k}[Y]$ containing f. Since G is reductive, the G-module W also occurs in $\mathsf{k}[X]$ (here we use that $\mathrm{char}(\mathsf{k}) = 0$) thus there exists $f' \in \mathsf{k}[X]$ which maps onto f via $\mathsf{k}[X] \to \mathsf{k}[Y]$.

Choose f' such that $f'|_{\widehat{Y}} = f^{p^N}$ and set $X_0 = D_X(f') = \{x \in X \mid f'(x) \neq 0\}$. Then $X_0 \subset X$ is affine B-stable and meets Y.

Let $\phi \in \mathsf{k}[X_0 \cap Y]^{(B)}$ be homogeneous. There exists $m \geq 0$ with $\phi f^m \in \mathsf{k}[\widehat{Y}]^{(B)}$. By the above surjectivity (or the same argument if $\mathrm{char}(\mathsf{k}) = 0$), we get $\psi \in \mathsf{k}[\widehat{X}]^{(B)}$ with $\psi|_{\widehat{Y}} = (\phi f^m)^{p^N}$. Then $(\psi f'^{-m}) \in \mathsf{k}[X_0]^{(B)}$ with $(\psi f'^{-m})|_{\widehat{Y}} = \phi^{p^N}$. $\square$

Proposition 2.2.7. *Let X be an irreducible normal G-variety and $Y \subset X$ be irreducible and G-stable, then* $\mathrm{rk}(Y) \leq \mathrm{rk}(X)$.

Proof. Let X_0 be as in Proposition 2.2.5. Any weight of $\Lambda(X)$ is the difference of weights of $\mathsf{k}[X_0]$ and any weight of $\Lambda(Y)$ is the difference of weights of $\mathsf{k}[X_0 \cap Y]$. The result follows from Theorem 2.2.6 for the inclusion $Y \cap X_0 \subset X_0$. □

Remark 2.2.8. If $\mathrm{char}(\mathsf{k}) = 0$, the above proof implies that we have an inclusion $\Lambda(Y) \subset \Lambda(X)$ for $Y \subset X$ is a G-subvariety. In general, this inclusion might not be true but will be true after multiplication by some power of p.

The fact that the rank is decreasing on G-subvarieties fails for complexity in general as shows the following example.

Example 2.2.9. Quasi-homogeneous G-varieties do not always admit finitely many G-orbits. For example, consider $X = \mathbb{P}(M_2(\mathsf{k}))$ the projective space over the space of square matrices of size 2. The group $G = \mathrm{SL}_2(\mathsf{k})$ acts by left multiplication and has a dense orbit: the locus where the determinant is not vanishing therefore $c_G(X) = 0$. However, for $v \in \mathsf{k}^2$ with $v \neq 0$, the variety $Y_v = \{[M] \in X \mid v \in \mathrm{Ker} M\}$ is stable under the G-action so that we must have infinitely many G-orbits.

Let B be the Borel subgroup of upper triangular matrices. The B-complexity is $c_B(X) = 1$. Indeed, the open subset of rank 2 matrices is covered by a 1-dimensional family of 2-dimensional B-orbits:

$$X_{[v]} = \{[M] \in X \mid \mathrm{rk}(M) = 2 \text{ and } [Mv] \text{ is } B\text{-stable}\},$$

with $v \in \mathsf{k}^2$ and $[v] \in \mathbb{P}^1$ its class. We also have the equalities $c_U(X) = 2$ and $\mathrm{rk}(X) = 1$ (see Proposition 2.3.5 below).

To get a nice behaviour, one needs to consider the B-complexity of G-varieties.

Proposition 2.2.10. *Let X be an irreducible normal G-variety and Y be a closed irreducible B-stable subvariety, then* $c_B(Y) \leq c_B(X)$.

Proof. Write c for c_B. Let $Y \subset X$ be closed and B-stable. We prove $c(Y) \leq c(X)$.

We start with a G-orbit Y. Let X_0 be an open affine B-stable subvariety of X with $X_0 \cap Y \neq \emptyset$ as in Proposition 2.2.5. Let $f \in \mathsf{k}(Y)^B$, we can write $f = u/v$ with $u, v \in \mathsf{k}[X_0 \cap Y]^{(B)}$ having the same weight: consider $\{v \in \mathsf{k}[X_0 \cap Y] \mid fv \in \mathsf{k}[X_0 \cap Y]\}$ which is B-stable and thus admits a B-eigenvector v, then $u = fv$ is a B-eigenvector of the same weight. There exist $u', v' \in \mathsf{k}[X_0]^{(B)}$ such that $u'|_Y = u^{p^N}$ and $v'|_Y = v^{p^N}$. We get $(u'/v')|_Y = f^{p^N}$. It follows that the transcendence degree of $\mathsf{k}(X)^B$ is bigger than or equal to the transcendence degree of $\mathsf{k}(Y)^B$ so $c(X) \geq c(Y)$.

Let Y be any closed B-stable subset. We prove that $c(Y) \leq c(GY)$ and conclude by the previous argument. Recall that G is generated by the minimal parabolic subgroups strictly containing B. We therefore only need to prove that $c(Y) \leq c(PY)$ for any minimal parabolic subgroup P.

Consider the contracted product $P \times^B Y$ defined as the quotient of $P \times Y$ by the action of B defined by $b.(p, y) = (pb, b^{-1}y)$. The projection on the first factor induces a morphism $\mathrm{pr}_1 : P \times^B Y \to P/B$ which is P-equivariant, locally trivial for the Zariski tolopogy (it is trivial on the open subset $(U^- \cap P)B/B$ where U^- is the unipotent radical of B^-) with fiber isomorphic to Y. In particular $\dim P \times^B Y = \dim Y + 1$.

The map $\pi : P \times^B Y \to PY, [p, y] \mapsto py$ is surjective. It is also proper since it can be viewed as the restriction of the projection $P \times^B X \to X$. In particular PY is closed and $\dim PY \leq \dim Y + 1$. Assume that $PY \neq Y$ (otherwise we trivially get $c(Y) \leq c(PY)$). Then the map π is generically finite so $c(P \times^B Y) = c(PY)$.

For $p \in P \setminus B$, the orbit BpB/B is dense in P/B and if we set $B_p = B \cap pBp^{-1}$, we have an isomorphism $BpB/B \simeq B/B_p$. Consider the contracted product $B \times^{B_p} p.Y$. We have an embedding $B \times^{B_p} p.Y \to P \times^B Y$ defined by $[b, p.y] \mapsto [bp, y]$. Its image is $\mathrm{pr}_1^{-1}(BpB/B)$ therefore B-invariant and open. In particular $c(P \times^B Y) \geq c(B \times^{B_p} p.Y)$. But any B-orbit in $B \times^{B_p} p.Y$ is of the form $B \times^{B_p} Z$ for Z a B_p-orbit in Y. In particular $c(B \times^{B_p} Y) = c_{B_p}(Y)$ where we write $c_{B_p}(Y)$ for the minimal codimension of a B_p orbit in $p.Y$. On the other hand, we have $c_{B_p}(p.Y) \geq c_{pBp^{-1}}(p.Y) = c(Y)$ so that we get

$$c(PY) \geq c(P \times^B Y) \geq c(B \times^{B_p} Y) = c_{B_p}(p.Y) \geq c(Y).$$

This completes the proof. $\square$

Example 2.2.11. The assumption that X is a G-variety is important. Recall Example 2.2.9 where $X = \mathbb{P}(M_2(\mathsf{k}))$ is the projective space over the vector space of 2×2 matrices. For $v \in \mathsf{k}^2$ with $v \neq 0$, let $Z = \overline{X_{[v]}}$ be the closure of the B-orbit

$$X_{[v]} = \{[M] \in X \mid [Mv] \text{ is } B\text{-stable}\}.$$

Then Z contains the subvariety

$$Y = \left\{ \left[\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right] \in X \mid a, b \in \mathsf{k} \right\}$$

whose points are fixed by B. In particular, we have $c_B(Z) = 0$ and $c_B(Y) = 1$ so $c_B(Y) > c_B(Z)$ even if $Y \subset Z$.

3. Spherical varieties

We define spherical varieties and give their first characterisations.

Definition 2.3.1. A spherical variety is a normal G-variety X with $c_B(X) = 0$.

Remark 2.3.2. A G-variety is spherical iff it is normal with a dense B-orbit.

Example 2.3.3. Projective rational homogeneous spaces are spherical varieties. Indeed such a variety X is of the form $X = G/P$ for G reductive and $P \subset G$ a parabolic subgroup. The Bruhat decomposition shows that X has a dense B-orbit, so we get $c_B(X) = 0$.

Example 2.3.4. Assume that $G = T$ is a torus. Then a spherical G-variety is called a toric variety. Any dense B-orbit in X is a T-orbit and since the quotient of any torus is again a torus, a toric variety is an equivariant partial completion of a torus. By Example 2.1.4.(3), we have $\mathrm{rk}(X) = \dim X$ for toric varieties.

We first prove the following relationship between rank and complexity.

Proposition 2.3.5. *Let X be a B-variety, then $c_U(X) = c_B(X) + \mathrm{rk}(X)$.*

Proof. Replacing X with an open B-stable subset and using Rosenlicht's Theorem, we have morphisms $X \to X/U \to X/B$ with $\dim(X/U) = c_U(X)$ and $\dim(X/B) = c_B(X)$. Note that $\mathrm{rk}(X)$ is the rank of the group of characters of elements in $\mathsf{k}(X)^{(B)} = (\mathsf{k}(X)^U)^{(T)} = \mathsf{k}(X/U)^{(T)}$ so $\mathrm{rk}(X) = \mathrm{rk}(X/U)$. Furthermore $X/B = (X/U)/T$ so by Proposition 2.1.6, we have $\mathrm{rk}(X) = \mathrm{rk}(X/U) = \dim(X/U) - \dim(X/B) = c_U(X) - c_B(X)$. □

Corollary 2.3.6. *For a spherical variety X, we have $\mathrm{rk}(X) = c_U(X)$.*

Example 2.3.7. Let $X = G/P$ be a projective rational homogeneous space. The Bruhat decomposition shows that the dense B-orbit is actually also a U-orbit so $\mathrm{rk}(X) = c_U(X) = 0$, see Example 2.1.4.(1).

Proposition 2.3.8. *A spherical G-variety X has finitely many B-orbits.*

Proof. Assume that X has infinitely many B-orbits and let Y be a B-stable closed subvariety of minimal dimension in X having infinitely many B-orbit. By Proposition 2.2.10, we have $c_B(Y) \leq c_B B(X) = 0$ therefore Y has a dense B-orbit Z. But then $W = Y \setminus Z$ is closed B-stable with $\dim(W) < \dim Y$ and W contains infinitely many B-orbits. A contradiction. □

As a consequence we obtain equivalent definitions of spherical varieties.

Theorem 2.3.9. *Let X be a normal G-variety. The following are equivalent*
(1) *X is spherical.*
(2) *X has finitely many B-orbits.*
(3) $\mathsf{k}(X)^B = \mathsf{k}$.

Proof. (1) $\Rightarrow$ (2) Follows from the previous result.
(2) $\Rightarrow$ (3) Any function $f \in \mathsf{k}(X)^B$ is constant on B-orbits. Since X has finitely many B-orbits, there must be a dense orbit thus f is constant on X.
(3) $\Rightarrow$ (1) We know that $c_B(X) = \mathrm{Trdeg}(\mathsf{k}(X)^B) = \mathrm{Trdeg}(\mathsf{k}) = 0$. □

Exercise 2.3.10. Let $X = \mathbb{P}(M_2(\mathsf{k}))$ and $G = \mathrm{GL}_2(\mathsf{k})$ acting by left multiplication. Let $B \subset G$ be the Borel subgroup of uper triangular matrices. Describe all G-orbits and B-orbits in X. Compute the complexity for G and B of all G-orbit closures and the B-complexity of all B-orbit closures.

Exercise 2.3.11. Let $V = \mathsf{k}^n$ and $S^2V^\vee$ be the space of quadratic forms on V. Let $G = \mathrm{GL}(V)$ act on $S^2V^\vee$ by the action induced by the standard action of G on V. Compute the complexity $c_G(S^2V^\vee)$ and the rank $\mathrm{rk}(S^2V^\vee)$.

CHAPTER 3

Affine G-varieties

Let G be a connected reductive group, we focus on affine G-varieties.

1. Existence of quotients by reductive groups

The results of this section are part of Geometric Invariant Theory (GIT). Most of the results will be proved in J.-B. Bost's lectures so we only state them and refer to the book **[19]** for more details.

THEOREM 3.1.1. *Let A be a finitely generated* k*-algebra with a rational action of* G*. Then* A^G *is finitely generated over* k*.*

DEFINITION 3.1.2. Let X be an affine G-variety, the quotient $X/\!\!/G$ is defined by $X/\!\!/G = \mathrm{Spec}(\mathsf{k}[X]^G)$. Let $\pi : X \to X/\!\!/G$ be the morphism defined by the inclusion $\mathsf{k}[X]^G \to \mathsf{k}[X]$.

The quotient $\pi : X \to X/\!\!/G$ has the following properties.

PROPOSITION 3.1.3. *Let X be an affine G-variety.*

1. *The morphism π is G-invariant (constant on the G-orbits).*
2. *Any G-invariant morphism $X \to Z$ factors through π.*
3. *The variety $X/\!\!/G$ has the quotient topology.*
4. *The fibers of π contain a unique closed orbit.*

REMARK 3.1.4. The map $\pi : X \to X/\!\!/G$ is not a quotient in the classical sense: the fibers of π may contain more that one orbit as the following example shows.

EXAMPLE 3.1.5. Let $X = M_n(\mathsf{k})$ and let $G = \mathrm{GL}_n(\mathsf{k})$ act by conjugation. Then a classical result asserts that $\mathsf{k}[X]^G$ is the polynomial ring generated by the coefficients of the characteristic polynomial. In particular, the map $\pi : X \to X/\!\!/G$ is given by $M \mapsto \chi(M)$ where $\chi(M)$ is the characteristic polynomial of M. In particular $X/\!\!/G \simeq \mathsf{k}^{n-1}$ and the fiber of $\mathbf{0}$ is the nilpotent cone (the set of all nilpotent matrices) which has a unique closed orbit: the zero matrix.

COROLLARY 3.1.6. *Let X be an affine G-variety with a dense orbit, then X has a unique closed orbit.*

Proof. Consider the quotient $\pi : X \to X/\!\!/G$ and recall that π is surjective and that there is a unique closed orbit in each fiber of π. We are therefore left to prove that the quotient is reduced to one point. But π is constant on the G-orbits, therefore it is constant on a dense subset thus π is constant and the result follows. $\square$

The above construction is an example of a categorical quotient.

DEFINITION 3.1.7. A G-invariant morphism $\pi : X \to Y$ is a *categorical quotient* if any G-invariant morphism $p : X \to Z$ uniquely factors through π.

Remark 3.1.8. Categorical quotients are unique.

Proposition 3.1.9. *Geometric quotients and the quotients $X \to X/\!\!/G$ with X affine and G reductive are categorical quotients.*

Proposition 3.1.10. *Let $\pi : X \to Y$ be a categorical quotient. If X is normal, then so is Y.*

Proof. Let $\nu : Y' \to Y$ be the normalisation of Y. Since X is normal, the map π lifts to a G-invariant map $\pi' : X \to Y'$ and thus factors through π. The uniqueness in the definition of categorical quotients implies that ν is an isomorphism. $\square$

GIT is based on the above result and aims at constructing quotients for projective G-varieties (see [**19**] for more details).

2. Unipotent quotients of affine G-varieties

We want to extend the above construction to quotients by unipotent subgroups.

Theorem 3.2.1. *Let X be an affine G-variety, then $\mathsf{k}[X]^U$ is finitely generated.*

We give a proof of this result for $\mathrm{char}(\mathsf{k}) = 0$ in Appendix C for the general case see [**8**]. We will also need the following fact (see Corollary C.1.8).

Fact 3.2.2. *Any G-module M is determined by the T-module M^U.*

Remark 3.2.3. Theorem 3.2.1 is false if we only assume that X is an U-variety. Examples of non-finitely generated invariant rings were first given by Nagata, see [**20**, Theorem 2.45]

Theorem 3.2.1 allows the following definition.

Definition 3.2.4. For X an affine G-variety, define the quotient $\pi : X \to X/\!\!/U$ induced by the map $\mathsf{k}[X]^U \to \mathsf{k}[X]$.

Remark 3.2.5. The above quotient may not be surjective. Indeed, let $X = G = \mathrm{SL}_2$. Then the quotient X/U is isomorphic to $\mathbb{A}^2 \setminus \{0\}$ while $X/\!\!/U \simeq \mathbb{A}^2$. In particular, the quotient $X/\!\!/U$ is not a categorical quotient in general.

Some of the properties of X can be detected on $X/\!\!/U$.

Proposition 3.2.6. *Let G be a reductive group, U a maximal unipotent subgroup and X an irreducible affine G-variety.*

1. *$\mathsf{k}(X)^U$ is the field of fractions of $\mathsf{k}[X]^U$.*
2. *Any element of $\mathsf{k}(X)^{(B)}$ is the quotient of two B-eigenvectors in $\mathsf{k}[X]$.*
3. *If $\mathrm{char}(\mathsf{k}) = 0$, then the variety X is normal if and only if $X/\!\!/U$ is normal.*

Proof. (1) and (2). The fraction field of $\mathsf{k}[X]^U$ is contained in $\mathsf{k}(X)^U$ and the quotient of any two B-eigenvectors of $\mathsf{k}[X]$ is an element of $\mathsf{k}(X)^{(B)}$.

Conversely, let $f \in \mathsf{k}(X)^U$ (resp. in $\mathsf{k}(X)^{(B)}$) and consider the vector space:

$$V_f = \{f' \in k[X] \mid f'f \in \mathsf{k}[X]\}.$$

Since f is U-stable (resp. a B-eigenvector), then V_f is U-stable. Since U is unipotent, there is a U-invariant element f' in V_f (and even a B-eigenvector). This proves the result.

(3) Assume that X is normal, then $\mathsf{k}[X]$ is integrally closed in $\mathsf{k}(X)$. We want to prove that $\mathsf{k}[X]^U$ is integrally closed in its field of fractions which is $\mathsf{k}(X)^U$ by (1).

If $f \in \mathsf{k}(X)^U$ is such that $P(f) = 0$ with P a monic polynomial with coefficients in $\mathsf{k}[X]^U$. Then f is in $\mathsf{k}[X]$ and the result follows.

Conversely, if $X/\!\!/U$ is normal, let $\nu : X' \to X$ be the normalisation of X. We define a G-action on X': the action morphism $a : G \times X \to X$ induces a morphism $a' : G \times X' \to G \times X \to X$ and since $G \times X'$ is normal it factors through X' *i.e.* we have a commutative diagram:

$$\begin{tikzcd} G \times X' \arrow[r] \arrow[d] & X' \arrow[d] \\ G \times X \arrow[r] & X. \end{tikzcd}$$

Because this is an action on an open subset (where ν is an isomorphism) and the varieties are normal, this is an action. Thus we also have a quotient $X'/\!\!/U$ and a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\nu"] \arrow[d] & X \arrow[d] \\ X'/\!\!/U \arrow[r, "\bar{\nu}"] & X/\!\!/U \end{tikzcd}$$

with $X'/\!\!/U$ and $X/\!\!/U$ normal varieties with $\mathsf{k}(X')^U = \mathsf{k}(X)^U$ *i.e.* $\mathsf{k}[X']^U$ and $\mathsf{k}[X]^U$ have the same field of fractions.

The algebra $\mathsf{k}[X']$ is the integral closure of $\mathsf{k}[X]$ in $\mathsf{k}(X)$. Consider the ideal

$$I = \{f \in \mathsf{k}[X] \mid f\mathsf{k}[X'] \subset \mathsf{k}[X]\}.$$

This ideal is stable under the action of G and therefore stable under U and thus contains an U-invariant element $f \in \mathsf{k}[X]^U$. This implies the inclusion $f\mathsf{k}[X']^U \subset \mathsf{k}[X]^U$. The subspace $f\mathsf{k}[X']^U$ is thus an ideal of $\mathsf{k}[X]^U$ and thus a finite $\mathsf{k}[X]^U$-module. Therefore $\mathsf{k}[X']^U$ is also a finite $\mathsf{k}[X]^U$-module but since $X/\!\!/U$ is normal we get the equality $\mathsf{k}[X']^U = \mathsf{k}[X]^U$. Finally since $\mathrm{char}(\mathsf{k}) = 0$, the U-invariants determine the module and we get $\mathsf{k}[X] = \mathsf{k}[X']$. □

For V a G-module and λ a character of B, set $V_\lambda^{(B)} = \{v \in V \mid b.v = \lambda(b)v\}$. Note that $V^U = \bigoplus_\lambda V_\lambda^{(B)}$. For X an affine G-variety, define the monoid $\Lambda(X)^+$.

Definition 3.2.7. Define $\Lambda(X)^+ = \{\lambda \in \mathfrak{X}(T) \mid k[X]_\lambda^{(B)} \neq 0\}$.

Proposition 3.2.8. *For X an affine G-variety, $\Lambda(X)^+$ is finitely generated.*

Proof. Follows from the decomposition $k[X]^U = \bigoplus_\lambda k[X]_\lambda^{(B)}$ and the fact that this algebra is finitely generated by Theorem 3.2.1. □

The following is a direct application of Proposition 3.2.6.(2).

Corollary 3.2.9. *Let X be an affine G-variety. The weight lattice $\Lambda(X)$ of X is the subgroup of $\mathfrak{X}(T)$ generated by $\Lambda(X)^+$.*

3. Characterisation of affine spherical G-varieties

We extend the results of Theorem 2.3.9 for affine spherical varieties.

Definition 3.3.1. A G-module M is *multiplicity free* if any simple G-module occurs in M with multiplicity at most 1.

Proposition 3.3.2. *For X an affine irreducible G-variety, the following are equivalent.*

(1) *The variety X contains a dense B-orbit.*
(2) *Any B-invariant rational function is constant: $\mathsf{k}(X)^B = \mathsf{k}$.*
(3) *The G-module $\mathsf{k}[X]$ is multiplicity free.*

Proof. We proved the equivalence of the first two properties in Theorem 2.3.9. We prove the equivalence of (2) and (3). Assume that the representation $V(\lambda)$ appears with multiplicity at least 2 in $\mathsf{k}[X]$. Then there exists two B-eigenfunctions f and f' with eigenvalue λ. The quotient f/f' is a B-invariant non trivial rational function, proving (2) $\Rightarrow$ (3). Let f be a B-invariant rational function. Then by Proposition 3.2.6 it is the quotient f_1/f_2 of two B-eigenfunctions. Their eigenvalue have to be the same and by assumption f_1 and f_2 must be colinear. This imples that f is constant, proving (3) $\Rightarrow$ (2). $\square$

Definition 3.3.3. A normal irreducible variety X is toric if there exists a torus T acting on X with a dense orbit isomorphic to T.

Definition 3.3.4. Let Λ^+ be a finitely generated monoid.

(1) Define $\Lambda_{\mathbb{Q}} = \Lambda^+ \otimes_{\mathbb{Z}} \mathbb{Q}$.
(2) $\mathrm{Cone}(\Lambda^+)$ is the cone generated by Λ^+ in $\Lambda_{\mathbb{Q}}$.
(3) The saturation of Λ^+ is $\overline{\Lambda}^+ = \mathbb{Z}\Lambda^+ \cap \mathrm{Cone}(\Lambda^+)$.
(4) A finitely generate monoid Λ^+ is called *saturated* if $\overline{\Lambda}^+ = \Lambda^+$.

Lemma 3.3.5. *Let Y be an affine irreducible variety with an action by a torus T such that $\mathsf{k}[Y]$ is multiplicity free. Then the following are equivalent.*

(1) *The variety Y is normal.*
(2) *The monoid $\Lambda(Y)^+$ is saturated.*

Proof. If $\Lambda(Y)^+$ is not saturated, there exists a $\lambda \in \mathbb{Z}\Lambda(Y)^+ \cap \mathrm{Cone}(\Lambda(Y)^+)$ with $\lambda \notin \Lambda(Y)^+$. Since $\lambda \in \mathrm{Cone}(\Lambda(Y)^+)$, we have $n\lambda \in \Lambda(Y)^+$ for $n \in \mathbb{N}$ large enough. Furthermore, since $\lambda \in \mathbb{Z}\Lambda(Y)^+$, there exists $f \in \mathsf{k}(Y)_\lambda \setminus \{0\}$. Let $g \in \mathsf{k}[Y]_{n\lambda}$, then $f^n/g \in \mathsf{k}(Y)^B = \mathsf{k}$ since Y is multiplicity free thus $f^n \in \mathsf{k}g \subset \mathsf{k}[Y]$. In particular f is integral on $\mathsf{k}[Y]$ but not in $\mathsf{k}[Y]$ proving that Y is not normal.

Conversely, assume that $\Lambda(Y)^+$ is saturated and let $f \in \mathsf{k}(Y)$ be integral on $\mathsf{k}[Y]$. Decomposing f in sum of eigenvectors, we may assume that f is an eigenvector of weight λ. Since f is integral on $\mathsf{k}[Y]$, a multiple of λ lies in $\Lambda(Y)^+$. Since the later is saturated, the weight λ already lies in $\Lambda(Y)^+$. Because $\mathsf{k}[Y]$ is multipllicity free we get $f \in \mathsf{k}[Y]$. $\square$

Example 3.3.6. Let $X = \mathrm{Spec}(\mathsf{k}[x,y]/(y^2 - x^3))$ be the cuspidal cubic with action of $G = T = \mathbb{G}_m$ given by $t.(x,y) = (t^2x, t^3y)$. Then $\Lambda(X)^+ = \mathbb{Z}_{\geq 0} \setminus \{1\}$ while $\Lambda_{\mathbb{Q}} = \mathbb{Q}$ and $\overline{\Lambda}^+ = \mathbb{Z}_{\geq 0}$, so that $\Lambda(X)^+$ is not saturated and X is not normal.

Corollary 3.3.7. *Assume* $\mathrm{char}(\mathsf{k}) = 0$ *and let X be an affine toric T-variety and $Y \subset X$ be a closed T-stable subvariety. Then Y is an affine toric variety.*

Proof. Since T has a dense orbit, the complexity of X is 0 and by Proposition 2.2.10, the subvariety Y also has vanishing complexity. We only need to prove that Y is normal. Let $\lambda \in \overline{\Lambda(Y)}^+$ and let $f \in \mathsf{k}(Y)^{(T)}$ of weight λ. Write $f = u/v$ with $u, v \in \mathsf{k}[Y]^{(B)}$. There exist $u', v' \in \mathsf{k}[X]^{(T)}$ such that $u'|_Y = u$ and $v'|_Y = v$.

We get $f' = u'/v'$ with $f' \in \mathsf{k}(X)^{(T)}$ of weight λ and $f'|_Y = f$. Now there exists $n \in \mathbb{Z}_{>0}$ such that $n\lambda \in \Lambda(Y)^+ \subset \Lambda(X)^+$. Since X is normal, the monoid $\Lambda(X)^+$ is saturated thus $\lambda \in \Lambda(X)^+$ and by the multiplicity free property $\varphi \in \mathsf{k}[X]$. This implies $f \in \mathsf{k}[Y]$ and $\lambda \in \Lambda(Y)^+$. The weight monoid of Y is saturated and Y is normal. □

REMARK 3.3.8. Corollary 3.3.7 holds in positive characteristic as well since tori are linearly reductive so that invariants lift without having to take p^{th} powers.

Applying this, we get a characterisation of affine spherical varieties.

PROPOSITION 3.3.9. *Let X be an irreducible affine G-variety. The following conditions are equivalent.*

(1) *X is spherical.*
(2) *$\mathsf{k}[X]$ is multiplicity free and $\Lambda(X)^+$ is saturated.*
(3) *Assume* $\mathrm{char}(\mathsf{k}) = 0$. *The affine T-variety $X/\!\!/U$ is a toric variety.*

Proof. (1)⇒(3) We already know (Proposition 3.2.6) that if X is normal so is $X/\!\!/U$. The T-module $\mathsf{k}[X/\!/U]$ is $\mathsf{k}[X]^U$ and therefore multiplicity free as T-module. Therefore, there is a dense T-orbit in $X/\!\!/U$. This orbit is isomorphic to $T/T' \simeq T''$ which is a torus thus $X/\!\!/U$ is toric.

(3) ⇒(2) If $X/\!\!/U$ is toric then $\mathsf{k}[X]^U$ is multiplicity free as T-module thus $\mathsf{k}[X]$ is multiplicity free as G-module. Since $X/\!\!/U$ is toric its weight monoid $\Lambda(X/\!\!/U)^+$ is saturated but since $\Lambda(X/\!\!/U)^+ = \Lambda(X)^+$, the result follows.

(2) ⇒ (1) By Proposition 3.3.2, we only need to check that X is normal. By Proposition 3.2.6, we only need to prove that $X/\!\!/U$ is normal and this follows from Lemma 3.3.5 and the fact that $\Lambda(X)^+ = \Lambda(X/\!\!/U)^+$ is saturated. □

COROLLARY 3.3.10. *Assume* $\mathrm{char}(\mathsf{k}) = 0$ *and let X be an affine spherical G-variety and $Y \subset X$ be a closed G-stable subvariety. Then Y is G-spherical.*

Proof. By Proposition 2.2.10, we only need to prove that Y is normal. But $Y/\!\!/U$ is a closed T-stable subvariety of $X/\!U$ which is toric. By Corollary 3.3.7, the variety $Y/\!\!/U$ is also toric and thus Y is G-spherical. □

EXAMPLE 3.3.11. In positive characteristic, the above result is false. Let $\mathrm{char}(\mathsf{k}) = p > 0$, let $G = \mathrm{SL}_2(\mathsf{k}) \times \mathbb{G}_m^2$ and let $X = \mathsf{k}^4$. Write (x_1, x_2, y_1, y_2) for the coordinates in X and write $x = (x_1, x_2)$ resp. $y = (y_1, y_2)$. Define a $\mathrm{SL}_2(\mathsf{k})$-action via the usual action $(g, x) \mapsto g.x$ and $(g, y) \mapsto (F(g)^T)^{-1}.y$ where F is the Frobenius map and $F(g)$ is the matrix obtained by applying F to each coefficient. Define the $\mathbb{G}_m^2$ action as $(u, v).(x, y) = (ux, vy)$ for $(u, v) \in \mathbb{G}_m^2$ and $(x, y) \in X$.

Then X is G-spherical. Set $Y = V(x_1^p y_1 + x_2^p y_2)$, then Y is G-stable and closed in X but its singular locus contains $Z = V(x_1, x_2)$ which has codimension 1 therefore Y is not normal along Z and is not G-spherical.

4. The cone of an affine G-variety

DEFINITION 3.4.1. The cone of X is defined as $\mathrm{Cone}(X) = \mathrm{Cone}(\Lambda(X)^+)$.

EXAMPLE 3.4.2. Let $X = G$, then $\Lambda(X) = \mathfrak{X}(T)$ and $\Lambda(X)^+ = \mathfrak{X}(T)^+$. The cone $\mathrm{Cone}(X)$ of X is the cone of dominant characters. The rank $\mathrm{rk}(X)$ is the rank of G as a reductive group.

For a spherical variety, the cone of The cone of X contains many information since $\Lambda(X)^+ = \mathrm{Cone}(X) \cap \Lambda(X)$. Furthermore, the multiplicity free property implies that as G-modules, we have an isomorphism

$$\mathsf{k}[X] = \bigoplus_{\lambda \in \Lambda(X)^+} V(\lambda).$$

However, the algebra structure on $\mathsf{k}[X]$ is not determined by $\Lambda(X)^+$ as the following examples shows.

Example 3.4.3. Let $G = \mathrm{SL}_2(\mathsf{k})$ and consider the irreducible representation $V_2 = \mathsf{k}[x,y]_2$ of homogeneous polynomials of degree 2. The orbits of non-degenerate forms are closed and isomorphic to $X = \mathrm{SL}_2(\mathsf{k})/\mathrm{SO}_2(\mathsf{k})$. The variety of degenerate forms is $X_0 = G.x^2 \cup \{0\}$ (that is the affine cone over $\mathbb{P}^1$ in its second Veronese embedding). Both X and X_0 are spherical with $\Lambda(X) = \Lambda(X_0) = 2\mathbb{Z} \subset \mathbb{Z}$ and $\mathrm{Cone}(X) = \mathrm{Cone}(X_0) = \mathbb{Q}_{\geq 0}$. However X and X_0 are not G-isomorphic since X is smooth while X_0 is singular.

However, for smooth varieties, we have the following result, see **[16]**.

Theorem 3.4.4. *Any two smooth affine spherical G-varieties having the same weight monoid are G-isomorphic.*

Proposition 3.4.5. *Assume* $\mathrm{char}(\mathsf{k}) = 0$. *Let X be a irreducible affine G-variety and let $x \in X$. Then* $\mathrm{Cone}(\overline{G.x}) \subset \mathrm{Cone}(X)$ *with equality for x in a non-empty open subset of X.*

Proof. Let $x \in X$, we have a surjective map $\mathsf{k}[X] \to \mathsf{k}[\overline{G.x}]$. By Theorem 2.2.6 any eigenvector of B in $\mathsf{k}[\overline{G.x}]$ extends to $\mathsf{k}[X]$, proving the inclusion $\mathrm{Cone}(\overline{G.x}) \subset \mathrm{Cone}(X)$.

Since $\Lambda(X)^+$ is finitely generated, so is $\mathrm{Cone}(X)$. Let $(\lambda_i)_{i\in[1,n]}$ be generators of $\Lambda(X)^+$ and for each $i \in [1,n]$, let $f_i \in \mathsf{k}[X]$ be a B-eigenfunction of weight λ_i. Since X is irreducible, there exists $x \in X$ with $f_i(x) \neq 0$ for all $i \in [1,n]$. Then f_i is a non trivial function on $\overline{G.x}$ thus $\lambda_i \in \mathrm{Cone}(\overline{G.x})$. Since $(\lambda_i)_{i\in[1,n]}$ generate $\mathrm{Cone}(X)$ we get the equality $\mathrm{Cone}(\overline{G.x}) = \mathrm{Cone}(X)$. □

Proposition 3.4.6. *Let X be an affine G-variety and $Y \subset X$ be a closed G-stable subvariety, then* $\mathrm{Cone}(Y) \subset \mathrm{Cone}(X)$.

Proof. We have a surjective map $\mathsf{k}[X] \to \mathsf{k}[Y]$. By Theorem 2.2.6 any eigenvector of B in $\mathsf{k}[Y]$ extends to $\mathsf{k}[X]$, proving the inclusion $\mathrm{Cone}(Y) \subset \mathrm{Cone}(X)$. □

Definition 3.4.7. Denote by $\mathrm{Lin}(\mathrm{Cone}(X))$ the linear part of $\mathrm{Cone}(X)$ *i.e.* the maximal vector subspace contained in $\mathrm{Cone}(X)$.

Definition 3.4.8. For H a closed subgroup of G, denote by $\mathfrak{X}(G)^H$ the kernel of the restriction map $\mathfrak{X}(G) \to \mathfrak{X}(H)$ and set $\mathfrak{X}(G)^H_{\mathbb{Q}} = \mathfrak{X}(G)^H \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proposition 3.4.9. *Let $x \in X$ such that $G.x$ is closed and let G_x be the stabiliser of x. Then we have the inclusion* $\mathfrak{X}(G)^{G_x}_{\mathbb{Q}} \subset \mathrm{Lin}(\mathrm{Cone}(X))$ *with equality for some $x \in X$.*

Proof. Since $G.x$ is closed, we have $\mathrm{Cone}(G.x) \subset \mathrm{Cone}(X)$. Let $\lambda \in \mathfrak{X}(G)^{G_x}$, then λ induces a regular function $\lambda : G \to \mathbb{G}_m$ constant on G_x. Since $G.x = G/G_x$, this function descents to a function $\bar{\lambda} : G.x \to \mathsf{k}$ which is a B-eigenvector of weight $-\lambda$: $t.\bar{\lambda}(g.x) = \bar{\lambda}(t^{-1}.g.x) = \lambda(t^{-1})\bar{\lambda}(g.x)$ for any $t \in T$. Thus $\mathfrak{X}(G)^{G_x} \subset \mathrm{Cone}(X)$ and since $\mathfrak{X}(G)^{G_x}$ is a subgroup of $\mathfrak{X}(T)$, we get $\mathfrak{X}(G)^{G_x} \subset \mathrm{Lin}(\mathrm{Cone}(X))$.

For the second assertion, first remark that by Proposition 3.4.5, we may assume that x has a dense orbit in X: replace X by $\overline{G.x}$ with x such that $\mathrm{Cone}(X) = \mathrm{Cone}(\overline{G.x})$. Assume that X has a dense orbit. By Corollary 3.1.6, there is a unique closed orbit in X. Let $x \in X$ such that $G.x$ is the unique closed orbit and let $\lambda \in \mathrm{Lin}(\mathrm{Cone}(X))$. There exists an integer n such that $\pm n\lambda$ are weights of $\mathsf{k}[X]$. Let f and f' be eigenfunctions associated to these weights. The function ff' is again an eigenfunction with weight 0. As the only representation with highest weight 0 is the trivial representation the product ff' is an invariant function for G. It is therefore constant on the dense subset of X and thus on the all of X. Thus f (and f') are non vanishing functions on $G.x = G/G_x$. They come from a non vanishing function $\bar{f}$ on G constant on G_x. Such a non vanishing function on G is a character (modulo a constant scalar: use Lemma B.1.5). The result follows. □

Definition 3.4.10. The group (G, G) is the closed subgroup of G generated by the commutators $(g, h) = ghg^{-1}h^{-1}$ for all $g, h \in G$.

Corollary 3.4.11. *Let X be an affine irreducible G-variety.*

(1) $\mathrm{Cone}(X)$ *contains no non trivial linear subspace if and only if we have the equality $G = (G, G)G_x$ for all $x \in X$ such that $G.x$ is closed.*
(2) $\mathrm{Cone}(X)$ *is a vector space if and only if (G, G) acts trivially and for x in a non empty open subset of X, the orbit $G.x$ is closed.*

Proof. (1) The space $\mathrm{Lin}(\mathrm{Cone}(X))$ is trivial if and only if for any $x \in X$ such that $G.x$ is closed the character group $\mathfrak{X}(G)^{G_x}$ is trivial. Recall that any character is trivial on (G, G) therefore $\mathfrak{X}(G) = \mathfrak{X}(G/(G, G))$. The previous condition thus reads $\mathrm{Ker}(\mathfrak{X}(G/(G, G)) \to \mathfrak{X}(G_x/((G, G) \cap G_x))) = 0$. Since $G/(G, G)$ is a torus, this condition is equivalent to $G_x/((G, G) \cap G_x) = G/(G, G)$ *i.e.* $G = (G, G)G_x$.

(2) If (G, G) acts non trivially on X it also acts non trivially on $\mathsf{k}[X]]$ ahs thus $\mathsf{k}[X]$ has a B-eigenvector of non trivial weight λ. The weight λ is therefore in the dominant cone of (G, G) which forms a stricly convex cone, contradicting the fact that $\mathrm{Cone}(X)$ is a vector space. Therefore (G, G) has to act trivially on X. As $G/(G, G)$ is a torus, we may assume that G is a torus. Choose x be in the open subset such that $\mathrm{Cone}(\overline{G.x}) = \mathrm{Cone}(X)$. For λ a weight in this cone, then $-\lambda$ is in the cone therefore there exists an integer n such that $\pm n\lambda$ are weights of functions on X. Let f and f' in $\mathsf{k}[X]$ be functions of weights $n\lambda$ and $-n\lambda$ respectively. Then ff' is weight 0 thus invariant for G and therefore constant on $\overline{G.x}$. This in particular implies that the ideal of $\overline{G.x} \setminus G.x$ is trivial: any B-eigenfunction f in this ideal is constant on $\overline{G.x}$ and therefore vanishes. By Lie-Kolchin this implies that the ideal is trivial. The orbit $G.x$ is therefore closed.

Conversely, there exists $x \in X$ with $G.x$ closed and $\mathrm{Cone}(\overline{G.x}) = \mathrm{Cone}(X)$. We have $\mathsf{k}[G.x] = \mathsf{k}[G/G_x]$ and $(G, G) \subset G_x$ thus G/G_x is a torus quotient of $G/(G, G)$ and the weights of $\mathsf{k}[G.x]$ form a group, proving the result. □

Exercise 3.4.12. Let $X = G = \mathrm{SL}_2$. Prove that the quotient X/U is isomorphic to $\mathbb{A}^2 \setminus \{0\}$ while $X/\!\!/U$ is isomorphic to $\mathbb{A}^2$.

EXERCISE 3.4.13. Let $T \subset \mathrm{GL}_n(\mathsf{k})$ be the maximal torus of diagonal matrices and let T act on $X = \mathbb{A}^n = \mathsf{k}^n$. Set $Y = \{(x_i)_{i\in[1,n]} \in X \mid x_i \neq 0 \text{ for all } i \in [1,n]\}$. Compute the monoids $\Lambda^+(X)$ and $\Lambda^+(Y)$ and the lattices $\Lambda(X)$ and $\Lambda(Y)$.

EXERCISE 3.4.14. Check the assertions in Example 3.4.3 and Example 3.3.11.

CHAPTER 4

Characterisations of spherical varieties

In this chapter, we prove several new characterisations of spherical varieties.

Definition 4.0.1. Two G-varieties are called G-birational if there exists dense G-stable open subsets which are G-isomorphic.

Definition 4.0.2. Let X be a G-variety. A *G-embedding* of X is a G-morphism $X \to X'$ inducing an isomorphism of X onto a dense open subset of X'.

Theorem 4.0.3. *Let X be a normal quasi-projective G-variety. The following conditions are equivalent.*

(1) *The variety X is spherical.*
(2) *Any G-variety G-birational to X has finitely many G-orbits.*
(3) *For any G-linearised line bundle L on X, the G-module $H^0(X, L)$ is multiplicity free.*

Proof. (1) $\Rightarrow$ (2). A spherical variety has B-complexity 0. As complexity is a birational invariant, the same is true for any G-birational variety. Furthermore, varieties with B-complexity 0 have finitely many B-orbits. In particular finitely many G-orbits.

(2) $\Rightarrow$ (3). Let G/H be an open dense orbit of X. The G-module $H^0(X, L)$ is a submodule of $H^0(G/H, L)$, we may thus assume that $X = G/H$. Then L is of the form $G \times^H \mathsf{k}_\chi$ for some character χ of H. The group $H^0(G/H, L)$ is the group of sections of the map $p : G \times^H \mathsf{k}_\chi \to G/H$ induced by the first projection on $G \times \mathsf{k}_\chi$. In particular $H^0(G/H, L) = \mathsf{k}[G]^{(H)}_{-\chi}$. Let $\widehat{G}$ be the set of irreducible representations of G. We have a decomposition $\mathsf{k}[G] = \oplus_{\lambda \in \widehat{G}} V(\lambda)^\vee \otimes V(\lambda)$(see Corollary C.1.4) implies that the multiplicity of $V(\lambda)^\vee$ in $H^0(G/H, L)$ is $\dim V(\lambda)^{(H)}_{-\chi}$.

Assume that $\dim V(\lambda)^{(H)}_{-\chi} \geq 2$ and let $v, w \in V(\lambda)^{(H)}_{-\chi}$ be two linearly independent vectors. Let $y = [v \oplus w] \in \mathbb{P}(V(\lambda) \oplus V(\lambda))$ and $Y = \overline{G.y} \subset \mathbb{P}(V(\lambda) \oplus V(\lambda))$.

Lemma 4.0.4. *The variety Y has infinitely many closed G-orbits.*

Proof. Let B be a Borel subgroup and η be a (unique up to scalar) B-eigenvector in $V(\lambda)^\vee$. This defines an hyperplane $H_\eta = \mathrm{Ker}\eta$ in $V(\lambda)$. Since $V(\lambda)^\vee$ is simple, the G-orbit of η spans $V(\lambda)^\vee$ thus, there exists $g \in G$ such that $(g.\eta)(v) = 1$. Replacing B by a conjugate we may assume that $\eta(v) = 1$.

Define a rational function f on $\mathbb{P}(V(\lambda) \oplus V(\lambda))$ by

$$f(v_1 \oplus v_2) = \frac{\eta(v_2)}{\eta(v_1)}.$$

This function is defined on y and B-invariant. We claim that f is not constant on Y. Otherwise we would have $z \in \mathsf{k}$ with $\eta(g.w) = z\eta(g.v)$ for all $g \in G$. This in turn implies $(g^{-1}.\eta)(w - zv) = 0$ for all $g \in G$ but since the orbit of η spans $V(\lambda)^\vee$ we get $\nu(w - zv) = 0$ for all $\nu \in V(\lambda)^\vee$ *i.e.* $w = zv$ a contradiction.

The image of f is locally closed in k thefore for any $z \in \mathsf{k}$ except finitely many values, there exists $g_z \in G$ with $\eta(g_z w) = z\eta(g_z v)$. Excluding finitely many more values of z we may even assume that $\eta(g_z v) \neq 0$.

Let T be a maximal torus in B and B^- the opposite Borel subgroup *i.e.* the Borel subgroup of G such that $B \cap B^- = T$. There is a unique B^--highest weight vector t_λ such that $\eta(t_\lambda) = 1$: pick a basis (t_μ) of T-eigenvectors in $V(\lambda)$ and the dual basis $(t_\mu^\vee)$ in $V(\lambda)^\vee$, then $\eta = t_\lambda^\vee$. Because η is the dual basis element corresponding to t_λ, we may then write

$$g_z v = c_z t_\lambda + \sum_\mu v_\mu$$

with v_μ eigenvectors of eigenvalue μ and such that $\lambda - \mu$ is a non negative linear combination of simple roots. Furthermore we have $c_z \neq 0$ for z avoiding our finite set of values. We may also write

$$g_z w = d_z t_\lambda + \sum_\mu w_\mu$$

with w_μ eigenvectors of eigenvalue μ and such that $\lambda - \mu$ is a non negative linear combination of simple roots. We have $d_z = zc_z$.

Let θ be a cocharacter of T such that $\langle \theta, \alpha \rangle > 0$ for any simple root α. Then

$$\begin{aligned} \theta(s) \cdot y &= \Big[c_z s^{\langle \theta, \lambda \rangle} (t_\lambda \oplus z t_\lambda) + \textstyle\sum_\mu s^{\langle \theta, \mu \rangle} (v_\mu \oplus w_\mu) \Big] \\ &= \Big[c_z (t_\lambda \oplus z t_\lambda) + \textstyle\sum_\mu s^{\langle \theta, \mu - \lambda \rangle} (v_\mu \oplus w_\mu). \Big] \end{aligned}$$

In particular we get the limit $\lim\limits_{s \to \infty} \theta(s) \cdot y = [t_\lambda \oplus z t_\lambda]$.

The variety Y therefore contains the G-orbit of $[t_\lambda \oplus z t_\lambda]$ for all $z \in \mathsf{k}$ except maybe for a finite number of values of z. Because $[t_\lambda \oplus z t_\lambda]$ is a highest weight vector for B, the stabiliser of this point contains B and the orbit is therefore projective thus compact and in particular closed. For each z in k except maybe for a finite number of values, we get a closed G-orbit in Y, proving the claim. $\square$

We use Y to construct an embedding of G/H with infinitely many G-orbits. Since H acts on $v \oplus w$ via a character, it acts trivially on y therefore $H \subset G_y$.

Let X' be a compact embedding of $X = G/H$. Denote by x' the element of $G/H = X \subset X'$ corresponding to the identity element $e \in G$. Consider X'' the nomalisation of the closure of $G.(x', y)$ in $X' \times Y$. The G-orbit $G.(x', y)$ is isomorphic to G/H (since $H \subset G_y$) thus X'' is an embedding of X and the projection $X'' \to Y$ is proper since X' is compact. Therefore X'' maps surjectively on Y and contains infinitely many G-orbits.

(3) $\Rightarrow$ (1). In view of Theorem 2.3.9, we only have to check that any B-invariant rational function on X is constant. Let L be a very ample line bundle. Up to taking a high power of L, we may assume that L is G-linearised. Let $f \in \mathsf{k}(X)^B$. Then there exists an integer $n > 0$ and elements $u, v \in H^0(X, L^{\otimes n})$ with $f = u/v$. Looking at the B-module $\{v \in H^0(X, L^{\otimes n}) \mid fv \in H^0(X, L^{\otimes n})\}$ we get by Lie-Kolchin's Theorem the existence of v which is a B-eigenvector and in this case u is also a B-eigenvector for the same weight. By the multiplicity free condition, u and v have to be colinear proving the result. $\square$

CHAPTER 5

Weight lattice, colors and examples

We define some the main combinatorial objects for classifying spherical varieties: the colors and a map from the set of colors to the dual of the weight lattice. We then give some examples of these invariants.

1. Colors

Let X be a spherical G-variety and recall the definition of the weight lattice $\Lambda(X)$. For $f \in \mathsf{k}(X)^{(B)}$, we denote its weight by $\lambda_f \in \Lambda(X)$. Note that for $\lambda \in \Lambda(X)$, there is, up to scalar, a unique rational function $f \in \mathsf{k}(X)^{(B)}$ with $\lambda_f = \lambda$. Indeed, if f and f' have the same weight, then $f/f' \in \mathsf{k}(X)^B = \mathsf{k}$ is constant. In particular we have an exact sequence

$$1 \to \mathsf{k}^\times \to \mathsf{k}(X)^{(B)} \to \Lambda(X) \to 0.$$

Definition 5.1.1. Let $Y \subset X$ be a G-orbit in a spherical G-variety X.

(1) Set $\mathcal{D}(X) = \{B\text{-stable prime divisors}\}$.
(2) Set $\mathcal{D}_Y(X) = \{D \in \mathcal{D}(X) \mid Y \subset D\}$.
(3) A color is a $D \in \mathcal{D}(X)$ such that D is not G-stable.
(4) The set of colors is $\Delta(X) = \{D \in \mathcal{D}(X) \mid D \text{ is not } G\text{-stable}\}$.
(5) Set $\Delta_Y(X) = \Delta(X) \cap \mathcal{D}_Y(X)$.

Definition 5.1.2. Define $\mathcal{Q}(X) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Q}) = \Lambda(X)^\vee_{\mathbb{Q}}$.

For $D \in \Delta(X)$ and $\lambda \in \Lambda(X)$, choose $f \in \mathsf{k}(X)^{(B)}$ with $\lambda_f = \lambda$. Since f is unique up to scalar, the vanishing multiplicity $\nu_D(f)$ of f at D does not depend on the choice of f.

Definition 5.1.3. Define the map $\rho_X : \Delta(X) \to \mathcal{Q}(X)$, $D \mapsto \rho_X(D)$ by the formula $\langle \rho_X(D), \lambda \rangle = \nu_D(f)$, where $f \in \mathsf{k}(X)^{(B)}$ with $\lambda_f = \lambda$.

2. Examples

Most of these examples are taken from Gandini [**6**].

Example 5.2.1 (Projective rational homogeneous spaces). Let $X = G/P$ be a projective rational homogeneous space. Let $B \subset P$ be a Borel subgroup and $U \subset B$ a maximal unipotent subgroup. Recall the Bruhat decomposition

$$G = \coprod_{w \in W} UwB.$$

In particular we have $G = \cup_{w \in W} UwP$ and X is the union of finitely many U-orbits and thus of finitely many B-orbits. This in particular implies that X is spherical. We have $c_B(X) = 0 = c_U(X)$. This imples that $\mathrm{rk}(X) = c_B(X) - c_U(X) = 0$ (see Proposition 2.3.5) thus $\Lambda(X) = 0$. The B-stable divisors are the Schubert divisors and are indexed by simple roots α which are not in the root system of P. So if Δ is the set of simple root of G and Δ_P the subset of simple roots of P we have

$$\Delta(X) = \Delta \setminus \Delta_P = \mathcal{D}(X).$$

There is a unique G-orbit $Y = X$ and we have $\Delta_Y(X) = \Delta(X) = \mathcal{D}(X) = \mathcal{D}_Y(X)$. Note that since $\mathrm{rk}(X) = 0$, the map ρ_X is the zero map. In particular, we see that ρ_X may be non-injective.

Example 5.2.2 (Grassmannian). Grassmannian varieties are special cases of projective rational homogeneous space. Let $E = \mathsf{k}^n$ and let $X = \mathrm{Gr}(p, E) = \{V \subset \mathsf{k}^n \mid V \text{ is a vector subspace of } E \text{ with } \dim(V) = p\}$. Let $G = \mathrm{GL}_n(\mathsf{k})$, we have that G acts transitively on X so there is a unique G-orbit. Let $B \subset G$ be the Borel subgroup of upper triangular matrices. If $(e_i)_{i\in[1,n]}$ is the canonical bases in E, then B is the stabiliser of the flag $(E_i)_{i\in[1,n]}$ where $E_i = \langle e_j \mid j \in [1, i]\rangle$. The group U is the subgroup of B of matrices having all diagonal coefficients equal to 1. The group U has a dense orbit in X given by $U.E^p$ where $E^p = \langle e_j \mid j \in [n+1-p, n]\rangle$. Indeed, it is an easy exercise to check that

$$U.E^p = \{V \in X \mid \dim(V \cap E_i) = \max(0, p + i - n) \text{ for all } i \in [1, n]\}.$$

Since $\dim(V \cap E_i) \geq p + i - n$ for all $i \in [1, n]$, the above set is open proving that $c_U(X) = c_B(X) = 0$. We thus have $\Lambda(X) = 0$. Define $D \subset X$ as follows

$$D = \{V \in X \mid V \cap E_{n-p} \neq 0\}.$$

It is an easy exercise to check that $D \subset X$ is a prime B-stable divisor. There is a unique G-orbit $Y = X$ with $\Delta_Y(X) = \Delta(X) = \mathcal{D}(X) = \mathcal{D}_Y(X) = \{D\}$. Again ρ_X is the zero map.

Example 5.2.3 (Toric varieties). A toric variety is a T-spherical variety for T a torus. In particular X contains a dense T-orbit and $c_B(X) = c_T(X) = 0$. Since U is trivial in this case, we have $c_U(X) = \dim X$ and $\mathrm{rk}(X) = \dim X$ and

$$\Lambda(X) = \mathbb{Z}^{\dim X}.$$

Since $G = B = T$ in this case, we have $\Delta(X) = \emptyset$. There are not colors: every B-stable divisor is G-stable.

Example 5.2.4. Let $G = \mathrm{SL}_2(\mathsf{k})$, let T be the subgroup of diagonal matrices and let B be the subgroup of upper triangular matrices. Denote by α the positive root, by ϖ_α the fundamental weight and by s the simple reflection. An easy computation gives the following decomposition of $SL_2(\mathsf{k})/T$ in B-orbits:

$$\mathrm{SL}_2(\mathsf{k})/T = B/T \cup B\dot{s}/T \cup BuT/T, \text{ with } u = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \text{ and } \dot{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In particular $X = \mathrm{SL}_2(\mathsf{k})/T$ is a spherical 2-dimensional $\mathrm{SL}_2(\mathsf{k})$-variety. There are two B-stable divisors: $D^+ = B/T$ and $D^- = Bs/T$ and

$$\Delta(X) = \{D^+, D^-\} = \mathcal{D}(X).$$

To compute $\Lambda(X)$, assume for simplicity that $\mathrm{char}(\mathsf{k}) = 0$. The decomposition of the algebra $\mathsf{k}[\mathrm{SL}_2(\mathsf{k})] = \oplus_{n\in\mathbb{N}} V(n\varpi_\alpha) \otimes V(n\varpi_\alpha)^\vee$ induces a decomposition

$$\mathsf{k}[X]^T = \bigoplus_{n\in\mathbb{N}} V(n\varpi_\alpha) \otimes (V(n\varpi_\alpha)^\vee)^T.$$

Since $V(n\varpi_\alpha)^\vee$ has a (unique) trivial weight only for $n = 2k$ even, we get

$$\mathsf{k}[X]^T = \bigoplus_{k\in\mathbb{N}} V(2k\varpi_\alpha) = \bigoplus_{k\in\mathbb{N}} V(k\alpha).$$

The set $\Lambda^+(X)$ of weights of $\mathsf{k}[X]$ is therefore $\mathbb{N}\alpha$. Since T is reductive, Proposition 1.3.7.(6), implies that $X = \mathrm{SL}_2(\mathsf{k})/T$ is affine and by Corollary 3.2.9, we get

$$\Lambda(X) = \mathbb{Z}\alpha.$$

We now compute the map $\rho_X : \Delta_X \to \mathcal{Q}(X)$. For $i, j \in \{1, 2\}$, let $a_{i,j} \in \mathsf{k}[\mathrm{SL}_2(\mathsf{k})]$ be the corresponding matrix coefficient. Then the inverse image of B/T in $\mathrm{SL}_2(\mathsf{k})$ is the vanishing locus $a_{2,1}$ while the inverse image of Bs/T is the vanishing locus of $a_{2,2}$. Furthermore $a_{2,1}a_{2,2} \in \mathsf{k}[\mathrm{SL}_2(\mathsf{k})]^T = \mathsf{k}[X]$ and is a B-semiinvariant *i.e.* $f \in \mathsf{k}[X]^{(B)}$ of weight α. Since $\nu_{D^+}(f) = 1 = \nu_{D^-}(f)$ we get $\langle\rho_X(D^+), \alpha\rangle = 1 = \langle\rho_X(D^-), \alpha\rangle$. In particular:

$$\rho_X(D^+) = \frac{1}{2}\alpha^\vee = \rho_X(D^-).$$

Example 5.2.5 (Determinantal varieties). Let $M_{m,n}$ be the space of matrices of size $M \times n$ and let $M^r_{m,n} \subset M_{m,n}$ be the closed subset of matrices of rank at most r. Let $G = \mathrm{GL}_m(\mathsf{k}) \times \mathrm{GL}_n(\mathsf{k})$ act on $M_{m,n}$ via $(g, h).x = gxh^{-1}$. Define

$$x_r = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where I_r is the identity matrix of size r. Then $G.x_r$ is dense in $M^r_{m,n}$. The stabiliser of x_r in G is given by pairs of matrices of the form

$$\left(\begin{pmatrix} A_{r,r} & B_{r,m-r} \\ 0 & C_{m-r,m-r} \end{pmatrix}, \begin{pmatrix} A_{r,r} & 0 \\ D_{n-r,r} & E_{n-r,n-r} \end{pmatrix}\right),$$

where the indices indicate the size of the matrices. In particular this stabiliser has dimension $r^2 + r(m-r) + r(n-r) + (m-r)^2 + (n-r)^2$ thus $\dim M^r_{m,n} = r^2 r(m-r) + r(n-r) = r(m+n-r)$. Let $B^-_m \subset \mathrm{GL}_m(\mathsf{k})$ be the Borel subgroup of lower triangular matrices and $B_n \subset \mathrm{GL}_n(\mathsf{k})$ be the Borel subgroup of upper triangular matrices. Then $B = B^-_m \times B_n$ is a Borel subgroup of G and the stabiliser of x_r in B has dimension $r + (m-r)(m-r+1)/2 + (n-r)(n-r+1)/2$. This proves that $B.x_r$ is dense in $M^r_{m,n}$ thus $c_B(M^r_{m,n}) = 0$.

The G-orbits in $M^r_{m,n}$ are indexed by the rank, there are therefore $r+1$ G-orbits. We compute the weight lattice and weight monoid of $M^r_{m,n}$. Let $U^-_m \subset B^-_m$ and $U_n \subset B_n$ be the unipotent radicals such that $U = U^-_m \times U_n$ is a maximal connected unipotent subgroup in G. We compute $\mathsf{k}[M^r_{m,n}]^U$. Let d_k be the k^{th} principal minor that is the determinant of the upper left $k \times k$ block of a matrix in $M_{m,n}$. It is easy to check that $d_k \in \mathsf{k}[M_{m,n}]^U$ and that $d_k \in \mathsf{k}[M^r_{m,n}]^U$ does not vanish for all $k \in [1, r]$. Let $\epsilon_1, \cdots, \epsilon_m$ be the characters on B^-_m defined by $\epsilon_k(\mathrm{diag}(t_1, \cdots, t_m)) = t_k$ and $\eta_1, \cdots, \eta_n$ be the characters on B_n defined by $\eta_k(\mathrm{diag}(t_1, \cdots, t_n)) = t_k$. Set $\varpi_k = \epsilon_1 + \cdots + \epsilon_k$ and $\omega_k = \eta_1 + \cdots + \eta_k$. The weights $(\varpi_k)_{k\in[1,m]}$ and $(\omega_k)_{k\in[1,n]}$ are the dominant weights of $\mathrm{GL}_m(\mathsf{k})$ and $\mathrm{GL}_n(\mathsf{k})$. Note that $d_k \in \mathsf{k}[M^r_{m,n}]^{(B)}$ is a B-eigenfunction with weight $\varpi_k - \omega_k$.

We prove that $\mathsf{k}[M^r_{m,n}]^U = \mathsf{k}[d_1, \cdots, d_r]$. Let $f \in \mathsf{k}[M^r_{m,n}]^{(B)}$ be of weight

$$\left(\sum_{i=1}^m \lambda_i \epsilon_i, \sum_{j=1}^n \mu_j \eta_j\right).$$

Since $(\mathrm{diag}(t_1, \cdots, t_r, t_{r+1}, \cdots, t_m), \mathrm{diag}(t_1, \cdots, t_r, t'_{r+1}, \cdots, t'_n))$ acts trivially on x_r, we must have $\lambda_i + \mu_i = 0$ for $i \in [1, r]$ and $\lambda_i = \mu_j = 0$ for $i > r$ and $j > r$. In particular the weight of f is a $\mathbb{Z}$-linear combination of $\epsilon_i - \eta_i$ for $i \in [1, r]$. Since the weight of f has to be dominant, we have the inclusion $\Lambda(M^r_{m,n})^+ \subset \mathbb{N}(\varpi_1 - \omega_1) + \cdots + \mathbb{N}(\varpi_r - \omega_r)$. Since d_k has weight $\varpi_k - \omega_k$, we get the equality

$$\Lambda(M^r_{m,n})^+ = \mathbb{N}(\varpi_1 - \omega_1) + \cdots + \mathbb{N}(\varpi_r - \omega_r).$$

This implies

$$\Lambda(M^r_{m,n}) = \mathbb{Z}(\varpi_1 - \omega_1) + \cdots + \mathbb{Z}(\varpi_r - \omega_r) = \mathbb{Z}(\epsilon_1 - \eta_1) + \cdots + \mathbb{Z}(\epsilon_r - \eta_r).$$

In particular $M^r_{m,n}$ has rank r and since the monoid $\Lambda(M^r_{m,n})^+$ is saturated, we get (if $\mathrm{char}(\mathsf{k}) = 0$) that $M^r_{m,n}$ is an affine spherical G-variety. The fact that $M^r_{m,n}$ is normal actually holds true in any characteristic (see [**5**]).

We now focus on colors. First notice that G-orbits are given by the rank so that $M^r_{m,n}$ contains a G-stable divisor if and only if $m = n$. In that case the unique G-stable divisor is $M^{r-1}_{m,n}$. For $k \in [1, r]$, set $D_k = \{x \in M^r_{m,n} \mid d_k(x) = 0\}$. Then D_k is a prime B-stable divisor (D_k is irreducible since d_k is an irreducible polynomial). Note that for $m = n$, we have $D_r = M^{r-1}_{m,n}$. Conversely, let D be a B-stable prime divisor. Let $I_D \subset \mathsf{k}[M^r_{m,n}]$ be its ideal. Then I_D is a B-module thus there exists $f \in I_D^{(B)} \subset \mathsf{k}[M^r_{m,n}]^U$. We thus have $f = \lambda d_1^{a_1} \cdots d_r^{a_r}$ with $\lambda \in \mathsf{k}^\times$ and $a_i \in \mathbb{N}$. In particular D is contained in $D_1 \cup \cdots \cup D_r$ and since it is prime $D = D_k$ for some k. Note that $M^r_{m,n}$ has a unique closed G-orbit: $Y = M^0_{m,n} = \{0\}$. Thus for $m \neq n$, we have $\mathcal{D}_Y(M^r_{m,n}) = \mathcal{D}(M^r_{m,n}) = \{D_1, \cdots, D_r\} = \Delta(M^r_{m,n}) = \Delta_Y(M^r_{m,n})$ and for $m = n$, we have $\mathcal{D}_Y(M^r_{m,n}) = \mathcal{D}(M^r_{m,n}) = \{D_1, \cdots, D_r\} \supset \{D_1, \cdots, D_{r-1}\} = \Delta(M^r_{m,n}) = \Delta_Y(M^r_{m,n})$.

Finally, we compute the map ρ_X. We have $\nu_{D_i}(d_j) = \delta_{i,k}$. Identifying $\Lambda(M^r_{m,n})$ with the character group of $\mathbb{G}^r_m \subset \mathrm{GL}_r(\mathsf{k})$ via

$$\mathrm{diag}(t_1, \cdots, t_r) \mapsto (\mathrm{diag}(t_1, \cdots, t_r, 1, \cdots, 1), I_n),$$

the weight of d_k identifies with the fundamental weight ϖ_k of $\mathrm{GL}_r(\mathsf{k})$ and $\rho_X(D_k)$ identifies with $\alpha_k^\vee$ the corresponding coroot.

Remark 5.2.6. For $m = n = r$ the previous example is a partial compactification of $\mathrm{GL}_n(\mathsf{k})$ as $\mathrm{GL}_n(\mathsf{k}) \times \mathrm{GL}_n(\mathsf{k})$-variety. This example generalises to any reductive group G.

Example 5.2.7 (Reductive groups). Any reductive group G is spherical with respect to the action of $G \times G$. Indeed, the Bruhat decomposition implies that $B^- \times B$ has a dense orbit.

Let $f \in \mathsf{k}[G]^{(B^- \times B)}$ and write $\lambda_f = (\lambda_1, \lambda_2)$ with $\lambda_1, \lambda_2 \in \mathfrak{X}(T)$. Since $\mathrm{diag}(G)$ fixes the identity of G and since $B^- B$ is open in G, we have $\lambda_1 + \lambda_2 = 0$. Conversely, recall that the multiplication induces an isomorphism $B^- B \simeq U^- \times T \times U$ (see [**25**, Theorem 6.3.5 and Lemma 8.3.6]). Therefore, for all $\lambda \in \mathfrak{X}(T)$, we get $f_\lambda \in$

$\mathsf{k}[B^-B]^{(B^-\times B)} \subset \mathsf{k}(G)^{(B^-\times B)}$ of weight $(\lambda, -\lambda)$ by setting $f_\lambda(utu') = \lambda(t)$. We therefore have

$$\Lambda(G) = \{(\lambda, -\lambda) \mid \lambda \in \Lambda(T)\}.$$

We now describe the set of colors $\Delta(G)$. By the Bruhat decomposition, the colors coincide with the Schubert divisors $D_\alpha = B^- s_\alpha B$, where $\alpha \in \Delta$ is a simple root and s_α is the corresponding simple reflection. In particular

$$\Delta(G) = \Delta.$$

Identifying $\Lambda(G)$ with $\Lambda(T)$ via $\Lambda(T) \to \Lambda(G), \lambda \mapsto (\lambda, -\lambda)$, similar arguments to those given in the previous example for $\mathrm{GL}_n(\mathsf{k})$ imply that the map ρ_G is given as follows (see [**6**, Example 3.7] for a proof):

$$\rho_G(D_\alpha) = \alpha^\vee,$$

where we identify $\Lambda(G)^\vee$ with the coroot lattice $\Lambda(T)^\vee$ of G.

Example 5.2.8 (Symmetric matrices). Assume $\mathrm{char}(\mathsf{k}) \neq 2$ and let $X = \mathrm{Sym}_n(\mathsf{k})$ be the space of symmetric $n\times n$-matrices. Let $G = \mathrm{GL}_n(\mathsf{k})$ act by congruence: for $g \in G$ and $A \in X$, then $g.A = ((g^{-1})^T A g^{-1}$. Let $B \subset G$ be the subgroup of upper triangular matrices, then $B.I_n$ is open in X (here I_n is the identity matrix) thus X is G-spherical.

The G-orbits are parametrised by the rank. If I_k is the diagonal matrix with k ones and $n-k$ zeros on the diagonal, then the G-orbits are

$$G.I_k = \{A \in X \mid \mathrm{rk}(A) = k\}.$$

Set $X_k = \overline{G.I_k} = \{A \in X \mid \mathrm{rk}(A) \leq k\}$, we have the inclusions $X_0 \subset \cdots \subset X_n = X$. For $A \in X$, let $d_k(A)$ be the k-principal minor of A *i.e.* the determinant of the upper left square block of order k. Note that $d_k \in \mathsf{k}[X]^{(B)}$ and we have

$$(\mathrm{diag}(t_1, \cdots, t_n).d_k)(A) = d_k(\mathrm{diag}(t_1, \cdots, t_n)^{-1}.A) = t_1^2 \cdots t_k^2 d_k(A).$$

In particular $\lambda_{d_k} = 2\varpi_k$ with ϖ_k the k-th fundamental weight so $2\Lambda(T) \subset \Lambda(X)$ where T is the maximal torus of diagonal matrices in G. We prove that the inclusion is an equality. Indeed, let $f \in \mathsf{k}(X)^{(B)}$ and $t = \mathrm{diag}(1, \cdots, 1, -1, 1 \cdots, 1)$ the matrix with a unique -1 at the k-th position. Since $B.I_n$ is dense, we may assume that f is defined at I_n and that $f(I_n) \neq 0$. we have $tI_n = I_n$ and $\lambda_f(t)f(I_n) = (t.f)(I_n) = f(t^{-1}.I_n) = f(I_n)$. In particular $\lambda_f(t) = 1$ so writing $\lambda_f = \sum_i a_i \varpi_i$ we have $a_k \in 2\mathbb{Z}$. Since this is true for all k, we get

$$\Lambda(X) = 2\Lambda(T).$$

Let $D_k = \{A \in X \mid d_k(A) = 0\}$. Since $d_k \in \mathsf{k}[X]^{(B)}$, we have $D_k \in \mathcal{D}(X)$. Furthermore, one can check that $X \setminus B.I_n = D_1 \cup \cdots \cup D_n$. Furthermore, the only G-stable divisor is $D_n = X_{n-1}$, therefore

$$\Delta(X) = \{D_1, \cdots, D_{n-1}\} \subset \mathcal{D}(X) = \{D_1, \cdots, D_n\}.$$

The above description shows that $\nu_{D_k}(d_l) = \delta_{k,l}$ therefore

$$\rho_X(D_k) = \frac{1}{2}\alpha_k^\vee,$$

where $(\alpha_i^\vee)_{i\in[1,n]}$ is the dual basis of $(\varpi_k)_{k\in[1,n]}$. Note that $\alpha_n^\vee$ is not a coroot since there is no root α_n in $\mathrm{GL}_n(\mathsf{k})$. If $(\epsilon_i)_{i\in[1,n]}$ is the usual orthonormal basis of $\Lambda(T)$

and setting $\epsilon_{n+1} = 0$, we have

$$\varpi_k = \sum_{i=1}^{k} \epsilon_i, \ \alpha_k^\vee = \epsilon_k - \epsilon_{k+1}, \text{ for all } k \in [1, n].$$

CHAPTER 6

Local structure theorems

In this chapter, we assume $\mathrm{char}(\mathsf{k}) = 0$. We start with a structure theorem for G-modules and deduce a structure theorem for G-spherical varieties.

1. Local structure for G-varieties

Let V be a G-module and let Y be a closed orbit of $\mathbb{P}(V)$. The stabiliser of any point in Y is a parabolic subgroup of G because Y is projective. Furthermore, there exists an element $y \in Y$ such that $B.y$ is open and dense in Y.

LEMMA 6.1.1. *Let $v \in V$ such that $[v] = y$.*

(1) *There exists a B-eigenvector $\eta \in (V^\vee)^{(B)}$ such that $\langle \eta, v\rangle = 1$.*
(2) *G_y and G_η are opposite parabolic subgroups and we have $B.y = G_\eta.y$.*

Proof. (1) If for any $\eta \in (V^\vee)^{(B)}$ we have $\langle \eta, v\rangle = 0$, then $\langle \eta, b.v\rangle = 0$ for all $b \in B$ thus $\langle \eta, w\rangle = 0$ for all $w \in Y$ and because Y is a G-orbit we get $\langle g.\eta, w\rangle = 0$ for all $g \in G$ and $w \in Y$ therefore Y is anihilated by $V^\vee$ (since $V^\vee$ is spanned by its highest weight vectors $(V^\vee)^{(B)}$). This implies $Y = \emptyset$, a contradiction.

(2) Let η as in (1) and let $P = G_\eta$ be its stabiliser. The orbit Y is the quotient G/G_y and is projective. The subgroup G_y is therefore a parabolic subgroup and thus contains a Borel subgroup B' of G. Since any two Borel subgroups contain a maximal torus in their intersection, there exists T a maximal torus in $B \cap B'$. But $B.y$ is open and dense in $Y = G/G_y$, thus BG_y is open and dense in G. This implies the decomposition $\mathfrak{g} = \mathfrak{b} + \mathfrak{g}_y$ on the level of Lie algebras and therefore $\mathfrak{g}_y$ contains $\mathfrak{b}^-$ the opposite Borel Lie algebra with respect to the torus T. Thus G_y contains B^- the Borel opposite to B and the vector v with $[v] = y$ is a T-fixed point (since $T \subset B' \subset G_y$). It is therefore a highest weight vector for B^-. Let λ_v be the T-weight of v and consider the decompositions

$$V = \bigoplus_{\lambda \in \hat{G}} V_\lambda^{m_\lambda} \text{ and } V^\vee = \bigoplus_{\lambda \in \hat{G}} (V_\lambda^\vee)^{m_\lambda},$$

where the weights $\lambda \in \hat{G}$ are considered as dominant weight for the Borel B (thus $-\lambda_v$ is dominant). The element η must lie in $(V^\vee_{-\lambda_v})^{m_{-\lambda_v}}$. The stabilisers of y and $[\eta] \in \mathbb{P}(V^\vee)$ are therefore respectively spanned by T and the unipotent subgroups U_α associated to the roots α such that $\langle \alpha^\vee, \lambda_v\rangle \geq 0$, respectively $\langle \alpha^\vee, -\lambda_v\rangle \geq 0$ and are therefore opposite parabolic subgroups.

For the second statement, let us remark that P is spanned by B and the unipotent subgroups U_α with $\langle \alpha^\vee, -\lambda_v\rangle = 0$. But these subgroups U_α are also in G_y proving the result. $\square$

Set $P = G_\eta$ and $L = P \cap G_y$. The group L is reductive and is a maximal reductive subgroup of both P and G_y. Denote by $R_u(P)$ the unipotent radical of P, we have $P = LR_u(P)$. The subset $\mathbb{P}(V)_\eta$ where η does not vanish is open, P-stable and contains $P.y$.

PROPOSITION 6.1.2. *There exists a closed L-subvariety S of $\mathbb{P}(V)_\eta$ containing y such that the morphism*

$$R_u(P) \times S \to \mathbb{P}(V)_\eta$$

defined by $(p, x) \mapsto px$ is a P-equivariant isomorphism.

Proof. We first reduce to the case of simple modules. Denote by $\langle G.v\rangle$ and $\langle G.\eta\rangle$ the G-submodules of V and $V^\vee$ spanned by v and η. Note that $\langle G.v\rangle$ is simple while $\langle G.\eta\rangle$ is isomorphic to its dual. The orthogonal $\langle G.\eta\rangle^\perp$ is therefore of codimension $\dim\langle G.v\rangle$ in V and in direct sum with $\langle G.v\rangle$. We thus get a decomposition $V = \langle G.v\rangle \oplus \langle G.\eta\rangle^\perp$. The projection p from $\langle G.\eta\rangle^\perp$ onto $\mathbb{P}\langle G.v\rangle$ defines a rational G-equivariant morphism $p : \mathbb{P}(V)_\eta \to \mathbb{P}\langle G.v\rangle$. This morphism restricts to the identity on Y since $Y \subset \mathbb{P}\langle G.v\rangle$. If the statement is true for $\langle G.v\rangle$, then there exists S as above and we get the Cartesian diagram

$$\begin{tikzcd}
R_u(P) \times p^{-1}(S) \arrow[r] \arrow[d, "p"'] & \mathbb{P}(V)_\eta \arrow[d] \\
R_u(P) \times S \arrow[r] & \mathbb{P}\langle G.v\rangle.
\end{tikzcd}$$

Since the bottom horizontal arrow is an isomorphism, the same is true for the top horizontal arrow and the result follows.

We are left to prove the result for V simple. Let $T_v = T_v(G.v)$ be the tangent space of $G.v$ at v. We consider T_v as a vector subspace of V. Since v is a G_y-eigenvector, the group G_y acts on T_v. Since the weight of v as an eigenvector is non trivial (otherwise V would be trivial), the space T_v contains the line $\mathsf{k}v$.

The space T_v is thus a sub-L-representation of V and since L is reductive there is a decomposition

$$V = T_v \oplus E$$

with E a representation of L. Define $S = \mathbb{P}(\mathsf{k}v \oplus E)_\eta$. This is a closed subvariety of $\mathbb{P}(V)_\eta$ which is stable under L and contains y. Note that S is isomorphic to the affine space $y + E$ and that S meets Y tranversaly in y: indeed, the tangent spaces of Y and S at y are $T_v/\mathsf{k}v$ and E which are supplementary in $V/\mathsf{k}v$. Also note that the variety $\mathbb{P}(V)_\eta$ has a unique closed T-orbit: the fixed point y. Indeed, since V is simple, the weight λ_v is the smallest weight (for B) of V. Any weight of V is therefore of the form $\lambda_v + \mu$ with μ a non-negative linear combination of simple roots of B. Any element $z = [w] \in \mathbb{P}(V)_\eta$ can be written $w = \sum_\mu w_{\lambda_v+\mu}$ with $w_{\lambda_v+\mu}$ of weight $\lambda_v + \mu$ and $t \in T$ acts via $t.w = \sum_\mu (\lambda_v + \mu)(t) w_{\lambda_v+\mu}$. Let θ be a dominant cocharacter, then

$$\theta(s).w = \sum_\mu s^{\langle\theta,\lambda_v+\mu\rangle} w_{\lambda_v+\mu} = s^{\langle\theta,\lambda_v\rangle}\left(w_{\lambda_v} + \sum_{\mu\neq 0} s^{\langle\theta,\mu\rangle} w_{\lambda_v+\mu}\right)$$

and when s goes to 0 we get $[\theta(s).w] \to [w_{\lambda_v}] = y$.

Consider $R_u(P) \times S$ as a T-variety via the action $t.(p, z) = (tpt^{-1}, t.z)$. By the same argument as before and since (e, y) has lowest weight for the T-action, we also have that $R_u(P) \times S$ has a unique closed T-orbit: the fixed point (e, y).

Consider the multiplication morphism $m : R_u(P) \times S \to \mathbb{P}(V)_\eta$. We want to prove that this morphism is an isomorphism. We first claim, that the differential $d_{(e,y)}m$ is injective.

Let $\mathfrak{r}_u(P)$ be the Lie algebra of $R_u(P)$. We know that $P.y = R_u(P)L.y = R_u(P).y$ is open in $Y = G.y$ therefore the morphism $R_u(P) \times \mathsf{k}v \to Y$ is dominant and its tangent map $\mathfrak{r}_u(P) \times \mathsf{k}v \to T_yY$ is surjective. We get the equality $T_yY = \mathfrak{r}_u(P)v/\mathsf{k}v$. The same argument gives $T_v(G.v) = \mathfrak{r}_u(P)v + \mathsf{k}v$.

Furthermore, since η is fixed by $R_u(P)$, we have $\langle \eta, pv\rangle = \langle p^{-1}\eta, v\rangle = \langle \eta, v\rangle = 1$ for all $p \in R_u(P)$ and therefore η is constant on $R_u(P).y$. This implies by derivation that η vanishes on $\mathfrak{r}_u(P).v$. In particular $\mathfrak{r}_u(P).v$ and $\mathsf{k}v$ are complement and $T_v(G.v) = \mathfrak{r}_u(P).v \oplus \mathsf{k}v$. This also implies the equality $T_vV = \mathsf{k}v \oplus \mathfrak{r}_u(P).v \oplus E$.

These equalities lead to the identifications of S and $\mathbb{P}(V)_\eta$ with the affine spaces $v+E$ and $v+(\mathfrak{r}_u(P).v\oplus E)$. The morphism m is given by $m((p,(v+x)) = p.(v+x)$. We may now compute the differential: $d_{(e,v)}m(\xi, x) = v + \xi.v + x$ for $\xi \in \mathfrak{r}_u(P)$ and $x \in E$. Indeed, the first two terms come from the differentiation of the action of $R_u(P)$ on v while the second term comes from the differential of the action on E which is linear.

We are left to prove that the map $\mathfrak{r}_u(P) \to \mathfrak{r}_u(P).v$ given by the action on v is injective. This is true since the intersection of $R_u(P)$ with the stabiliser G_y of y is trivial thus $R_u(P)$ acts freely on y and v thus by differentiation the same is true at the Lie algebra level.

Let Z be the locus in $R_u(P) \times S$ where the differential of m is not surjective. This is a closed subset of $R_u(P) \times S$. If Z is non empty, then it contains a closed T-orbit which has to be (e, y), a contradiction. The morphism $m : R_u(P) \times S \to \mathbb{P}(V)_\eta$ is therefore open.

Let Z be the complement of the image, then Z is closed and T-stable. If it is non empty, then it contains a closed T-orbit which has to be y, a contradiction. Thus m is surjective.

Thus m is a covering but since both varieties are affine spaces which are simply connected, the map m is an isomorphism. $\square$

Example 6.1.3. Consider $V = M_n(\mathsf{k})$ and $G = \mathrm{GL}_n(\mathsf{k}) \times \mathrm{GL}_n(\mathsf{k})$ acting via $(P, Q).M = PMQ^{-1}$. Consider Y the set of rank 1 matrices. It is the only closed subvariety stable by G and let $y = [M] \in Y$ with

$$M = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

and η the linear form defined by $\eta(a_{i,j}) = a_{1,1}$. If P is the stabiliser of η, we have

$$P = \left\{ \left(\begin{pmatrix} a & 0 \\ C & D \end{pmatrix}, \begin{pmatrix} a' & B' \\ 0 & D' \end{pmatrix} \right) \mid a, a' \in \mathbb{G}_m, D, D' \in \mathrm{GL}_{n-1}(\mathsf{k}), C^T, B' \in M_{1,n-1}(\mathsf{k}) \right\}$$

$$G_y = \left\{ \left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \begin{pmatrix} A' & 0 \\ C' & D' \end{pmatrix} \right) \mid a, a' \in \mathbb{G}_m, D, D' \in \mathrm{GL}_{n-1}(\mathsf{k}), C', B^T \in M_{n-1,1}(\mathsf{k}) \right\}.$$

Setting

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \mid M \in M_{n-1}(\mathsf{k}) \right\} \simeq M_{n-1}(\mathsf{k}),$$

we have the isomorphism $R_u(P) \times S \to \mathbb{P}(V)_\eta$ given by the action

$$\left(\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}, \begin{pmatrix} 1 & B' \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}\right) \mapsto \begin{pmatrix} 1 & -B' \\ C' & M - C'B' \end{pmatrix}.$$

Note that the action of $L = P \cap G_y$ on S is given as follows

$$\left(\begin{pmatrix} a & 0 \\ 0 & D \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & D' \end{pmatrix}\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} = \begin{pmatrix} a/a' & 0 \\ 0 & DM(D')^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{a'}{a} DM(D')^{-1} \end{pmatrix}.$$

To compute the invariants we may therefore use Example 5.2.5.

Remark 6.1.4. The above result enables to replace locally the study of quasi-projective G-varieties to quasi-affine G-varieties and of projective G-varieties to affine G-varieties.

Example 6.1.5. Let us consider V to be the vector space of quadratic forms on k^n *i.e.* $V = (S^2\mathsf{k}^n)^\vee$. There is a unique closed orbit of $G = \mathrm{GL}_n(\mathsf{k})$ in $\mathbb{P}(V)$ given by the quadratic forms of rank one. Pick $y = x_1^2$. Then the stabiliser of y is the stabiliser of the hyperplane given by the last $n-1$ coordinate vectors. The linear form $\eta \in V^\vee$ can be chosen to be $\eta(q) = q(1, 0, \cdots, 0)$. If B is the Borel subgroup of lower triangular matrices in G, we have $\eta \in (V^\vee)^{(B)}$.

A quadratic form q satisfies $\eta(q) \neq 0$ if and only if it can be writen in the form

$$q(x_1, \cdots, x_n) = \lambda(x_1 + a_2x_2 + \cdots + a_nx_n)^2 + q'(x_2, \cdots, x_n)$$

where $\lambda \neq 0$ and q' is a quadratic form on the last $n-1$ variables.

Now $R_u(P)$ maps e_1 to $e_1 + a_2e_2 + \cdots + a_ne_n$ and is the identity on the other vectors, therefore the above Theorem boils down to the fact that there exists a unique $u \in R_u(P)$ such that

$$q = \lambda u(y + q'')$$

with $q'' = u^{-1}q'$ is a quadratic form in the last $n-1$ variables thus here S is the set of quadratic forms in these last $n-1$ variables.

Corollary 6.1.6. *For a G-variety X, the following are equivalent.*

(1) *We have the vanishing* $\mathrm{rk}(X) = 0$.
(2) *Any G-orbit in X is compact.*

Proof. Assume that any G-orbit is compact and let $f \in \mathsf{k}(X)^{(B)}$. Let Y be a G-orbit meeting the locus where f is defined. This orbit is compact it is of the form G/P for some parabolic subgroup P. Thus there is a dense U-orbit in Y. In particular we have $f|_Y \in \mathsf{k}(G/P)^{(B)} = \mathsf{k}(G/P)^B$ thus the weight of f is trivial.

Conversely, assume that $\mathrm{rk}(X) = 0$. Let Y be a G-orbit and let $\overline{Y}$ be its closure in X. Then we know that $\mathrm{rk}(\overline{Y}) \leq \mathrm{rk}(X)$ and since the rank is a birational invariant we get $\mathrm{rk}(Y) = 0$. We may therefore assume that X is homogeneous *i.e.* $X = Y$.

If $X = G/H$, we know that X is quasi-projective and that there exists V a G-module such that G/H is a locally closed subset of $\mathbb{P}(V)$. let $\overline{X}$ be the closure of X in $\mathbb{P}(V)$ and let $x \in \overline{X}$ be an element in a closed G-orbit Y. Then we get a parabolic subgroup P (opposite to $\mathrm{Stab}(x)$) with Levi factor L and a closed subvariety S of

$\mathbb{P}(V)_\eta$ stable under L such that $R_u(P) \times S \to \mathbb{P}(V)_\eta$ is an isomorphism. Let $Z = \overline{X} \cap S$. We get an isomorphism $R_u(P) \times Z \to \overline{X}_\eta$ thus an open immersion

$$R_u(P) \times Z \to \overline{X}.$$

Furthermore Z contains x which is fixed by L.

LEMMA 6.1.7. *We have* $\mathsf{k}(X)^{(B)} = \mathsf{k}(Z)^{(L\cap B)}$.

Proof. If $f \in \mathsf{k}(Z)^{(L\cap B)}$ is of weight λ, then the composition $R_u(P) \times Z \xrightarrow{f} \mathbb{A}^1$ is a rational function $\bar{f}$ on X. Furthermore, for $b \in B$ we may write $b = uc$ with $u \in R_u(P)$ and $c \in L \cap B$ thus we get $(b.\bar{f})(u', z) = \bar{f}(c^{-1}u'cu^{-1}, cz) = f(c^{-1}z) = \lambda(c)f(z) = \lambda(b)\bar{f}(u', z)$ proving that $\bar{f}$ is indeed in $\mathsf{k}(X)^{(B)}$.

Conversely for $f \in k(X)^{(B)}$, then f defines a rational function on $R_u(P) \times Z$ such that if $b = uc$ is a decomposition with $u \in R_u(P)$ and $c \in B \cap L$ we have $b.f(u', z) = f(c^{-1}u'cu^{-1}, c^{-1}z) = \lambda(b)f(u', z)$. In particular for $c = 1$ and $u = u'$ we have $f(u', z) = f(e, z)$. We may define $\tilde{f}(z) = f(e, z) = f(u', z)$ for all $u' \in R_u(P)$ which will therefore be a rational function on Z. We obviously have $\tilde{f} \in \mathsf{k}(Z)^{(L\cap B)}$.

Furthermore one checks that $\tilde{\bar{f}} = f$ and $\bar{\tilde{f}} = f$ proving the result. $\square$

As a consequence we get that $\mathrm{rk}(Z) = 0$ as a L-variety. We get that in $\mathsf{k}[Z]$, any $B \cap L$-eigenfunction has a trivial weight. This implies that $\mathsf{k}[Z]$ is a trivial L-module. But notice that Z is affine thus L acts trivally on Z. Therefore the maximal torus T of G, which is contained in L acts trivially on Z. But Y was closed thus compact and of the form G/P with P parabolic. Thus T only has finitely many fixed points in Y. Therefore Z must be finite and since $R_u(P) \times Z$ is open in $\overline{X}$ is it irreducible thus Z is one point. Then Y is the G-orbit of Z and has to be dense in $\overline{X}$ thus $Y = X$ which is compact. $\square$

2. Local structure for spherical varieties

For a spherical variety X, we describe the local structure not only along projective orbits but along any G-orbit Y. Recall that the following definitions:

$$\begin{aligned}
&\mathcal{D}(X) = \{D \subset X \ B\text{-stable prime divisor}\},\\
&\Delta(X) = \{D \in \mathcal{D}(X) \mid D \text{ is not } G\text{-stable}\},\\
&\mathcal{D}_Y(X) = \{D \in \mathcal{D}(X) \mid Y \subset D\} \text{ and}\\
&\Delta_Y(X) = \{D \in \Delta(X) \mid Y \subset D\} = \Delta(X) \cap \mathcal{D}_Y(X).
\end{aligned}$$

Finally denote the set of B-stable prime divisors containing no G-orbit by

$$\mathring{\Delta}(X) = \Delta(X) \setminus \bigcup_Y \Delta_Y(X) = \mathcal{D}(X) \setminus \bigcup_Y \mathcal{D}_Y(X)$$

where Y runs over the set of G-orbits in X.

DEFINITION 6.2.1. A G-variety is called **simple** if it has a unique closed G-orbit.

Define the G-chart $X_{Y,G} = \{x \in X \mid \overline{G.x} \supset Y\}$.

LEMMA 6.2.2. *The G-chart $X_{Y,G}$ is an open G-stable neighborhood of Y and is a simple G-variety.*

Proof. Indeed $X_{Y,G}$ is G-stable and $Y \subset X_{Y,G}$. If $x \in X \setminus X_{Y,G}$, then $\overline{G.x} \subset X \setminus X_{Y,G}$ thus $X_{Y,G}$ is open. Finally any closed orbit in $X_{Y,G}$ has to contain Y which is therefore the unique closed orbit. □

COROLLARY 6.2.3. *Any spherical variety is covered by finitely many open simple spherical varieties.*

PROPOSITION 6.2.4. *Let X be a simple G-spherical variety with closed orbit Y.*

(1) *Y has a unique dense B-orbit Y_B°.*
(2) *X contains a unique minimal B-stable affine open subset $X_{Y,B}$ meeting Y non trivially.*
(3) *We have $Y_B^\circ = Y \cap X_{Y,B}$.*
(4) *We have*
$$X_{Y,B} = X \setminus \bigcup_{D \in \mathcal{D}(X) \setminus \mathcal{D}_Y(X)} D$$
and any divisor $D \in \mathcal{D}(X) \setminus \mathcal{D}_Y(X)$ is Cartier and globally generated.
(5) *We have $X_{Y,B} = \{x \in X \mid \overline{B.x} \supset Y\}$.*

Proof. (1) Since X contains finitely many B-orbits, so does Y and the result follows.

(2) By Proposition 2.2.5, there exists a B-stable affine open subset X_0 meeting Y. Since X_0 is affine, its complement has codimension 1 and therefore $X \setminus X_0$ is the union of prime B-stable divisors not containing Y, in other words a subset of $\mathcal{D}(X) \setminus \mathcal{D}_Y(X)$. Since this set is finite, so is the set of all such affine open subsets. Since a finite intersection of non-empty affine open subsets is still non-empty affine and open the results follows.

(3) Since the intersection $Y \cap X_{Y,B}$ is non-empty and open, we have the inclusion $Y_B^\circ \subset Y \cap X_{Y,B}$. Conversely, let X_0 as in (1), or as in Proposition 2.2.5, and let $f \in \mathsf{k}[Y \cap X_0]^{(B)}$ such that f vanishes on $(Y \cap X_0) \setminus Y_B^\circ$. Lift f to $f' \in \mathsf{k}[X_0]^{(B)}$, then $X_0 \cap \{x \in X \mid f'(x) \neq 0\}$ is a B-stable affine subset meeting Y non-trivially along Y_B° proving the converse inclusion.

(4) Let $\mathcal{D} = \cup_{D \in \mathcal{D}(X) \setminus \mathcal{D}_Y(X)} D$. By the above, we have $X \setminus \mathcal{D} \subset X_{Y,B}$.

LEMMA 6.2.5. *Then $\mathcal{O}_X(\mathcal{D})$ is Cartier and globally generated. The same is true for any of the sheaves associated to any irreducible component of $\mathcal{D}$*

Proof. Modulo replacing G by a finite cover, we may assume that $\mathcal{O}_{X_{G,Y}}(\mathcal{D})$ is G-linearised. In particular, the non-Cartier locus and the locus where it is non globally generated are G-stable. If these loci are non empty, they must contain the only closed G-orbit: Y. But $\mathcal{O}_X(\mathcal{D})$ is locally free and globally generated outside $\mathcal{D}$ and therefore on an open subset of Y proving the assertion. The same argument works for any irreducible component of $\mathcal{D}$. □

We prove $X_{Y,B} = X \setminus \mathcal{D}$. For this it is enough to produce a affine B-stable open subset meeting Y with trivial intersection with $\mathcal{D}$. Let $\eta \in H^0(X, \mathcal{O}_X(\mathcal{D}))$ be the canonical section defining $\mathcal{D}$. Set $N = \langle G.\eta \rangle$ and $M = N^\vee$. We have a morphism
$$\varphi : X \to \mathbb{P}(M)$$
define by $\varphi(x) = [\sigma \in N \mapsto \sigma(x)]$. This morphism is G-equivariant and is defined everywhere. Indeed if there exists $x \in X$ with $\sigma(x) = 0$ for all $\sigma \in N$, then $g.\eta(x) = 0$ for all $g \in G$ thus $Y \subset \overline{G.x} \subset \mathcal{D}$, a contradiction. By definition of η, we

have $X \setminus \mathcal{D} = \varphi^{-1}(\mathbb{P}(M)_\eta)$. By Theorem 6.1.2, there exists $S \subset X \setminus \mathcal{D}$ closed and stable under a Levi subgroup of $P = \mathrm{Stab}(\eta)$ such that the map

$$R_u(P) \times S \to X \setminus \mathcal{D}$$

is an isomorphism. Since $B = R_u(P)(B \cap L)$ and $B \cap L$ is a Borel subgroup of L, the variety S is L-spherical and meets Y. By Proposition 2.2.5, the variety S contains an open $B \cap L$-stable affine open subset S_0 meeting Y. Then $R_u(P)S_0$ is an open affine B-stable open subset of $X \setminus \mathcal{D}$ meeting Y. This proves the equality $X_{Y,B} = X \setminus \mathcal{D}$.

(5) Note that (4) implies that $X \setminus \mathcal{D}$ is affine and thus S is also affine. Furthermore, the restriction of $R_u(P) \times S \to X \setminus \mathcal{D}$ to Y induces an isomorphism $R_u(P) \times (Y \cap S) \to (X \setminus \mathcal{D}) \cap Y = X_{Y,B} \cap Y = Y_B^\circ$ which is a unique orbit thus $Y \cap S$ is a unique $B \cap L$-orbit.

We prove the equality $X_{Y,B} = \{x \in X \mid \overline{B.x} \supset Y\}$. Note that the right hand side is B-stable, open and contains Y_B°. Let $x \in \mathcal{D}$, then $\overline{B.x} \subset \mathcal{D} \not\supset Y$ thus x is not in the right hand side. This proves the inclusion $\{x \in X \mid \overline{B.x} \supset Y\} \subset X_{Y,B}$. Conversely, let Z be a closed B-orbit in $X_{Y,B}$. Then $Z \cap S$ is a closed $B \cap L$-orbit in S thus $L(Z \cap S)$ is a closed L-orbit in S. Since S is affine and L-spherical, it has a unique closed L-orbit. This $Z \cap S$ and $Y \cap S$ are closed L-orbits in S, we have $Z \cap S = Y \cap S$ and $Z \subset X_{Y,B} \cap Y = Y_B^\circ$. This gives $Z = Y_B^\circ$ and $X_{Y,B} \subset \{x \in X \mid \overline{B.x} \supset Y\}$. □

Corollary 6.2.6. *We have $X_{Y,G} = GX_{Y,B}$.*

Proof. If $x \in X_{Y,B}$, then $\overline{G.x} \supset \overline{B.x} \supset Y$ thus $x \in X_{Y,G}$ and since the later is G-stable, we get $GX_{Y,B} \subset X_{Y,G}$. Let $x \in X_{Y,G}$, then $Y \subset \overline{G.x}$. Pick $x' \in G.x$ such that $\overline{B.x'} = \overline{G.x}$, then $x' \in X_{Y,B}$ and the result follows. □

Remark 6.2.7. In positive characteristic, we will introduce $X_{Y,G}$ and $X_{Y,B}$ in a slightly different way. We will set:

$$X_{Y,B} = X \setminus \bigcup_{D \in \mathcal{D}(X) \setminus \mathcal{D}_Y(X)} D \text{ and } X_{Y,G} = GX_{Y,B}.$$

The equality $X_{Y,G} = \{x \in X \mid \overline{G.x} \supset Y\}$ will still hold true but the equality $X_{Y,B} = \{x \in X \mid \overline{B.x} \supset Y\}$ might fail.

Let P be the stabiliser of $X_{Y,B}$ *i.e.* $P = \{g \in G \mid g.X_{Y,B} = X_{Y,B}\}$. Then P is a parabolic subgroup containing B.

Theorem 6.2.8. *Let X be G-spherical and $Y \subset X$ be a G-orbit.*

1. *There exists $L \subset P$ a Levi subgroup and $S \subset X_{Y,B}$ closed such that:*
   (a) *The variety S is stable under L.*
   (b) *The map $R_u(P) \times S \to X_{Y,B}, (p,x) \mapsto p.x$ is a P-isomorphism.*
2. *The variety S is affine L-spherical and $S \cap Y$ is an L-orbit with isotropy subgroup L_y containing (L,L) the derived subgroup of L for any $y \in S \cap Y$. The subgroup L_y is independent of y. Denote it by L_Y.*
3. *There exists a closed L_Y-stable subvariety $S_Y \subset S$ containing an L_Y-fixed point such that the morphism*

$$L \times^{L_Y} S_Y \to S, [l,y] \mapsto l.y$$

is an L-equivariant isomorphism. The variety S_Y is affine L_Y-spherical of rank $\mathrm{rk}(X) - \mathrm{rk}(Y)$.

Proof. Since $X_{Y,B}$ is contained in $X_{Y,G}$ we may assume that $X = X_{Y,G}$ is simple.

(1) As in the previous proof, let $\mathcal{D} = X \setminus X_{Y,B}$. The divisor $\mathcal{D}$ is Cartier and globally generated. Let $\eta \in H^0(X, \mathcal{O}_X(\mathcal{D})$ be the canonical section, $N = \langle G.\eta \rangle$ and $M = N^\vee$. We have a G-equivariant morphism $\varphi : X \to \mathbb{P}(M)$ defined by $\varphi(x) = [\sigma \in N \mapsto \sigma(x)]$ and $X \setminus \mathcal{D} = \varphi^{-1}(\mathbb{P}(M)_\eta)$. By Theorem 6.1.2, there exists $S \subset X \setminus \mathcal{D}$ closed and stable under a Levi subgroup of $P = \mathrm{Stab}(\eta)$ such that the map $R_u(P) \times S \to X \setminus \mathcal{D}$ is an isomorphism.

(2) We have finitely many B-orbits in $X_{Y,B}$ since X is spherical thus B also has finitely many orbits in $R_u(P) \times S$. Recall that $P = R_u(P)L$ thus $B = R_u(P)(L \cap B)$ and $L \cap B$ is a Borel subgroup of L. Recall also that the action of $P = R_u(P)L$ on $R_u(P) \times S$ is given by $ul.(u', x) = (ulu'l^{-1}, l.x)$ thus $B \cap L$ must have finitely many orbits in S. Since $X_{Y,B}$ is normal as an open subset of X, the variety S is also normal thus S is L-spherical. It is an affine variety as closed subvariety of the affine variety $X_{Y,B}$. The isomorphism in (1) induces an isomorphism $R_u(P) \times (S \cap Y) \to Y \cap X_{Y,B}$. The right hand side is a P-orbit and a B-orbit, thus $S \cap Y$ is an L-orbit and a $(B \cap L)$-orbit as well.

Let $y \in S \cap Y$. We have $S \cap Y = L.y = (B \cap L).y$ thus $(B \cap L)L_y = L$.

LEMMA 6.2.9. *Let H be a closed subgroup of a connected reductive group G such that $G = HB$ for B a Borel subgroup of G, then H contains* (G, G).

Proof. Since (G, G) is connected, we may assume H to be connected. First assume G to be semisimple. Since $G = HB$, we have $G/B = H/(B \cap H)$. On the one hand this implies $\mathrm{rk}(G) = \mathrm{rk}(\mathrm{Pic}(G/B)) = \mathrm{rk}(\mathrm{Pic}(H/(H \cap B))) \leq \mathrm{rk}(H)$. On the other hand $\dim U_G = \dim G/B = \dim H/(H \cap B) \leq \dim U_H$ where U_G and U_H are maximal unipotent subgroups of G and H. We deduce $\dim H \geq \dim G$ and $H = G$.

For a general G, let $\pi : G \to G' = G/R(G)$ be the quotient of G by its radical. We have $G' = H'B'$ with $H' = \pi(H)$ and $B' = \pi(B)$. Since G' is semisimple, we deduce $G' = (G', G') \subset H'$. Thus $\pi|_H$ is surjective and $\pi((H, H)) = (G', G') = G'$. Since $(H, H) \cap R(G) \subset (G, G) \cap R(G)$ is finite, we get $\dim(H, H) = \dim G' = \dim(G, G)$ thus $(G, G) = (H, H) \subset H$. $\square$

We deduce that L_y contains (L, L). This implies that L_y does not depend on y. Indeed, for $l \in L$, then $L_{l.y} = lL_yl^{-1}$. For any $h \in L_y$ we have $lhl^{-1}h^{-1} \in (L, L) \subset L_y$ thus $lhl^{-1} \in L_y$ and $L_{l.y} \subset L_y$ and by symmetry $L_{l.y} = L_y$. Denote by L_Y this stabiliser. We have $(L, L) \subset L_Y$ thus L/L_Y is a quotient of $L/(L, L)$ which is a torus.

(3) The orbit $S \cap Y = L.y$ is thus isomorphic to the torus L/L_Y. Let $(\chi_1, \cdots, \chi_n)$ a basis of the group of characters of L/L_Y. Since $S \cap Y$ is closed in S which is affine, we can extend these functions to functions $(f_1, \cdots, f_n)$ on S. We may furthermore assume that these functions are $(L \cap B)$-eigenfunctions of weights $(\chi_1, \cdots, \chi_n)$. These functions do not vanish on the closed orbit $S \cap Y$ in S thus they do not vanish at all on S. These functions therefore define an L-equivariant morphism $\psi : S \to (\mathbb{G}_m)^n \simeq L/L_Y$. Let S_Y be the fiber over the identity element of this morphism. The natural map defined by the action: $L \times S_Y \to S$ factors through $L \times^{L_Y} S_Y \to S$. This map is bijective. Indeed, if $s \in S$, then there exists $l \in L$ such that $\bar{l} = \psi(s)$ and if l' satisfies the same condition, then $l' = lh$ with $h \in L_Y$.

Define $s \mapsto [l, l^{-1}s] \in L \times^{L_Y} S_Y$. This is well defined and an inverse map. But since S is normal, this morphism must be an isomorphism.

Finally $\mathrm{rk}(X) - \mathrm{rk}(Y) = \mathrm{rk}(S) - \mathrm{rk}(S \cap Y) = \mathrm{rk}(S) - \dim(L/L_Y) = \mathrm{rk}(S_Y)$. □

COROLLARY 6.2.10. *Let X be a spherical G-variety, then any closed G-stable subvariety Y is again a spherical G-variety.*

Proof. We only have to prove that Y is normal. By the local structure Theorem, we may assume X affine, the result follows from Corollary 3.3.10. □

REMARK 6.2.11. Recall from Example 3.3.11 that the previous result fails in positive characteristic. In characteristic 0, spherical varieties enjoy even stronger properties: any spherical variety is Cohen-Macaulay and has rational singularities (see [**22**, Corollary 2.3.4]).

EXAMPLE 6.2.12. Let us consider again the case of quadratic forms: V is the vector space of quadratic forms on k^n *i.e.* $V = (S^2\mathsf{k}^n)^\vee$. We have seen that the G-orbits for $G = \mathrm{GL}_n(\mathsf{k})$ in $\mathbb{P}(V)$ are given by the rank. Pick $y = x_1^2 + \cdots + x_i^2$ and let Y be its G-orbit. Then $X_{Y,B}$ is the set of quadratic forms such that the first i principal minors are non zero. Let P be the stabiliser of $\langle e_1 \rangle, \cdots, \langle e_1, \cdots, e_i \rangle$, this is a parabolic subgroup and is the stabiliser of $X_{Y,B}$.

Let S be the set of quadratic forms of the form $a_1x_1^2 + \cdots + a_ix_i^2 + q'(x_{i+1}, \cdots, x_n)$ with $a_k \in \mathbb{G}_m$. Then S is stable under $L = \mathbb{G}_m^i \times \mathrm{GL}_{n-i}(\mathsf{k})$ which is a Levi subgroup of P. We see that the natural map

$$R_u(P) \times S \to X_{Y,B}$$

is an isomorphism.

The intersection $S \cap Y$ is the set of quadratic forms of the form $a_1x_1^2 + \cdots + a_ix_i^2$ with $a_k \in \mathbb{G}_m$. This is an L-orbit and the stabiliser of y is $L_Y = \{\pm 1\} \times \mathrm{GL}_{n-i}(\mathsf{k})$ which contains $(L, L) = \mathrm{SL}_{n-i}(\mathsf{k})$.

If S_Y is the set of elements of the form $x_1^2 + \cdots + x_i^2 + q'(x_{i+1}, \cdots, x_n)$ we see that S_Y is stable under L_Y and meets Y in the unique point $y = x_1^2 + \cdots + x_i^2$. Furthermore, we have $L \times^{L_Y} S_Y \simeq S$.

3. Structure Theorem for toroidal varieties

We define an important class of spherical varieties.

DEFINITION 6.3.1. A a spherical variety is **toroidal** if $\mathring{\Delta}(X) = \Delta(X)$.

For X toroidal, we set $\Delta_X = \cup_{D \in \Delta(X)} D$. If $\mathring{X}_G$ is the dense G-orbit and $\mathring{X}_B$ is the dense B-orbit, we have

$$\Delta_X = \overline{\mathring{X}_G \setminus \mathring{X}_B}.$$

Let P_X be the stabiliser of $\mathring{X}_B$. It is also the stabiliser of Δ_X and $B \subset P_X$.

REMARK 6.3.2. The group P_X is a birational invariant of X.

THEOREM 6.3.3. *Let X be spherical. The following conditions are equivalent.*

(1) *The variety X is toroidal.*

(2) *There exists a Levi subgroup $L \subset P_X$ only depending on the open G-orbit of X and a closed L-stable subvariety $Z \subset X \setminus \Delta_X$ such that the map*

$$R_u(P_X) \times Z \to X \setminus \Delta_X$$

is a P_X-isomorphism, (L, L) acts trivially on Z which is a toric variety for a quotient of $L/(L, L)$. Any G-orbit meets Z along a unique L-orbit.

Proof. Assume that X is toroidal. First remark that Δ_X is Cartier and globally generated. Indeed this is a local condition and can be checked on the simple spherical varieties $X_{Y,G}$ for Y any G-orbit. Then the restriction of Δ_X to $X_{Y,G}$ is Cartier and globally generated by Lemma 6.2.5.

Now proceed as in the proof of Theorem 6.2.8 replacing $\mathcal{D}$ with Δ_X, to obtain a variety Z (called S in the proof of Theorem 6.2.8).

The intersection of $\mathring{X}_G$ the dense G-orbit in X with $X \setminus \Delta_X$ is the dense B-orbit $\mathring{X}_B$. Thus $\mathring{X}_G \cap Z = \mathring{X}_B \cap Z$. It is a $B \cap L$-orbit and also a L-orbit. By Lemma 6.2.9 we get that (L, L) acts trivially on Z which has to be a toric variety under the action of a quotient of $L/(L, L)$.

Let Y be a G-orbit in X. Then Y is not contained in Δ_X therefore $Y \cap (X \setminus \Delta_X)$ is dense in Y and $R_u(P_X)(Z \cap Y)$ is also dense in Y. Since Y is a G-orbit, there is a P_X-orbit $Y' \subset Y$ dense in Y and $Z' = Y' \cap Z$ is an L-orbit dense in $Z \cap Y$. The variety Z is toric for some torus T_Z. The structure Theorem of spherical varieties applied to toric varieties gives a $T_{Z'}$-variety $S_{Z'}$ and an isomorphism

$$T \times^{T_{Z'}} S_{Z'} \to Z.$$

The orbit Z' therefore corresponds to a $T_{Z'}$-fixed point s in $S_{Z'}$. Consider the cone $\mathcal{C}_s(S_{Z'})$ associated to s (see for example [**6**, Section 6]) and choose a basis $(\rho(\nu_D))$ (over $\mathbb{Q}$ of this cone) given by $L_{Z'}$-stable divisors D. The affine chart gives that s is the intersection of these divisors therefore Z' is the intersection of divisors $D_1, \cdots D_r$ of Z with $r = \mathrm{codim}_Z(Z')$. Then each divisor $X_i = \overline{R_u(P_X)D_i}$ is irreducible B-stable and does not meet the dense G-orbit (this is true since D_i does not meets the dense L-orbit of Z). This implies that X_i is G-stable. Now consider $X' = \overline{R_u(P_X)Z'}$. It is a subvariety of codimension r in X which is contained in the intersection of the X_i. Since Δ_X contains no closed G-orbit the intersection of the X_i has a dense open subset given by $(\bigcap_i X_i) \cap (X \setminus \Delta_X)$. The variety X' has to be an irreducible components of the intersection of the X_i and is thus G-stable. We get $X' = \overline{Y}$ and $Y \cap (X \setminus \Delta_X) = R_u(P_X)Z'$ thus $Y \cap Z = Z'$ proving the result.

Conversely, any G-orbit of X meets Z and thus is not contained in Δ_X and therefore is not contained in any B-stable but not G-stable divisor. □

Remark 6.3.4. Note that $G(X \setminus \Delta_X) = X$.

Corollary 6.3.5. *The irreducible G-stable subvarieties of a smooth toroidal variety are smooth, toroidal and transverse intersections of G-stable divisors.*

Proof. We need to consider the closure Y of a G-orbit $G \cdot y$. The above Structure Theorem for toroidal varieties gives an isomorphism $X \setminus \Delta_X \simeq R_u(P_X) \times Z$ and any G-orbit meets Z along a unique L-orbit. Furthermore, Z is toric and smooth therefore Y meets Z along a smooth toric subvariety Y' (see Lemma **??** for a smoothness criterion for toric varieties). Then $R_u(P_X) \times Y'$ is an open subset of Y with $G \cdot Y' = Y$ thus Y is smooth and toroidal.

For the last assertion, Y' being a toric subvariety of the smooth toric variety Z, it is a complete intersection of toric divisors and we get the result. $\square$

Part 2

Classification of spherical varieties

CHAPTER 7

Invariant valuations

1. Valuations and existence of invariant valuations

Definition 7.1.1. Let X be a normal variety. A *valuation* of X is a map $\nu : \mathsf{k}(X) \to \mathbb{Q} \cup \{\infty\}$ satisfying the following four properties for $f_1, f_2 \in \mathsf{k}(X)$:

(V1) $\nu(0) = \infty$ (V2) $\nu(f_1 + f_2) \geq \min(\nu(f_1), \nu(f_2))$
(V3) $\nu(f_1 f_2) = \nu(f_1) + \nu(f_2)$ (V4) $\nu|_{\mathsf{k}^\times} = 0$.

If X is a G-variety and $\nu(g.f) = \nu(f)$, then the valuation is called *invariant*. We denote by $\mathcal{V}(X)$ the set of invariant valuations on X.

Example 7.1.2. The following are classical examples of valuations.

1. The trivial valuation is defined by $\nu(\mathsf{k}(X)^*) = 0$ and $\nu(0) = \infty$.
2. A prime divisor $D \subset X$ defines a valuation ν_D: the local ring R at the generic point of D is a discrete valuation ring with fraction field $\mathsf{k}(X)$. For $f \in \mathsf{k}(X)$, write $f = u/v$ with $u, v \in R$ and $u = z^a u'$, $v = z^b v'$ with z an uniformising element of R not divising u' and v'. Set $\nu_D(f) = a - b$.
3. If ν is a valuation and $\lambda \in \mathbb{Q}_{\geq 0}$, then $\lambda\nu$ is also a valuation.

The following is a classical fact on valuations (see [**7**, Proposition B.69]).

Fact 7.1.3. *Let K be a field extension of k and L be an extension of K. If $\nu : K \to \mathbb{Q}$ is a valuation, then there exists a valuation $\nu' : L \to \mathbb{Q}$ with $\nu'|_K = \nu$.*

Definition 7.1.4. For ν a valuation of $\mathsf{k}(X)$, set $R_\nu = \{f \in \mathsf{k}(X) \mid \nu(f) \geq 0\}$ and $\mathfrak{m}_\nu = \{f \in \mathsf{k}(X) \mid \nu(f) > 0\}$. A *center* for ν is a subvariety Z of X such that $\mathcal{O}_{X,Z} \subset R_\nu$ and $\mathfrak{m}_{X,Z} \subset \mathfrak{m}_\nu$.

Remark 7.1.5. If Z is the center of a valuation ν, then Z is the center of $\lambda\nu$ for any $\lambda \in \mathbb{Q}_{>0}$.

Lemma 7.1.6. *Let ν be a valuation on a normal variety X.*

1. *If X is affine, then ν has a center if and only if ν is non negative on $\mathsf{k}[X]$. In that case its center is defined by the ideal $\mathfrak{m}_\nu \cap \mathsf{k}[X]$.*
2. *A center, if it exists, is unique.*
3. *Every subvariety $Y \subset X$ is the center of a valuation.*
4. *Assume that X is a G-variety. If $\nu \in \mathcal{V}(X)$ has a center, then the center is G-stable. Conversely every G-stable subvariety Y is the center of a G-invariant valuation.*

Proof. (1) If Z is a center, then $\mathsf{k}[X] \subset \mathcal{O}_{X,Z} \subset \mathcal{O}_\nu$ and ν is non negative on $\mathsf{k}[X]$. Conversely, define Z by its ideal $I(Z) = \mathsf{k}[X] \cap \mathfrak{m}_\nu$. This is indeed a center.

(2) Taking an affine cover, it is enough to check the affine case so assume that X is affine. Let Z be the center defined by $I(Z) = \mathsf{k}[X] \cap \mathfrak{m}_\nu$. and let Y be another

center for ν. Then $I(Y) \subset \mathsf{k}[X] \cap \mathfrak{m}_\nu = I(Z)$ thus $Z \subset Y$. Let $f \in I(Z) \setminus I(Y)$, then f has an inverse in $\mathcal{O}_{X,Y} \subset \mathcal{O}_\nu$ but $f \in \mathfrak{m}_\nu$ thus cannot be invertible in $\mathcal{O}_\nu$. A contradiction thus $I(Z) = I(Y)$ and $Y = Z$.

(3) Let Y be a subvariety and let E be a component of the exceptional divisor of the normalisation of the blow-up of X along Y. The valuation ν_E is a valuation on $\mathsf{k}(X)$ with center Y.

(4) Let Y be the center. Since $\nu \in \mathcal{V}(X)$, for any $g \in G$, both Y and $g.Y$ are centers therefore $Y = g.Y$. Conversely, since G is connected, the construction in 3, can be done equivariantly giving a G-invariant valuation ν_E. $\square$

REMARK 7.1.7. Let $D \subset X$ be a prime divisor and ν_D the associated valuation, then ν_D is the unique valuation in $\mathbb{Q}_{\geq 0}\nu_D$ such that $\nu_D(\mathsf{k}(X)^\times) = \mathbb{Z}$. Indeed, since ν_D is the valuation of a valuation ring with quotient field $\mathsf{k}(X)$, we have $\nu_D(\mathsf{k}(X)^\times) = \mathbb{Z}$.

LEMMA 7.1.8. *Let ν be a valuation of $\mathsf{k}(G)$, there exists a unique invariant valuation $\bar{\nu}$ of $\mathsf{k}(G)$ such that $\bar{\nu}(f) = \nu(g.f)$ for any $f \in \mathsf{k}(G)$ and all g in a non empty open subset U_f of G.*

Proof. We claim that for $f \in \mathsf{k}(G)$, there exists an open subset $U_f \subset G$ such that $\nu(g.f)$ is constant for $G \in U_f$. If the claim holds, define $\bar{\nu}(f) = \nu(g.f)$ for $g \in U_f$ and $\bar{\nu}(0) = \infty$. By definition $\bar{\nu}$ satisfies (V1) and for $f, f' \in \mathsf{k}(G)$, let $g \in U_f \cap U_{f'} \cap U_{f+f'}$. We have

$$\begin{aligned}
&\bar{\nu}(f+f') = \nu(g.f + g.f') \geq \min(\nu(g.f), \nu(g.f')) = \min(\bar{\nu}(f), \bar{\nu}(f')) \text{ and}\\
&\bar{\nu}(ff') = \nu((g.f)(g.f')) = \nu(g.f)\nu(g.f') = \bar{\nu}(f)\bar{\nu}(f'),
\end{aligned}$$

proving (V2) and (V3) for $\bar{\nu}$. The condition (V4) is obviously satisfied. Furthermore for $h \in U_f \cap U_{g.f}g$, we have $\bar{\nu}(f) = \nu(h.f) = \nu(hg^{-1}.(g.f)) = \bar{\nu}(g.f)$ thus $\bar{\nu} \in \mathcal{V}(G)$.

We are left to prove the claim. Since G is affine, we may assume that $f \in \mathsf{k}[G]$. For $q \in \mathbb{Q}$, let $V_q = \{f' \in \mathsf{k}[G] \mid \nu(f') \geq q\}$. This is a linear subspace of $\mathsf{k}[G]$. Let V be a finite dimensional subrepresentation of $\mathsf{k}[G]$ containing f. Let $f_1, \cdots, f_r$ a basis and set $q_0 = \min_i \nu(f_i)$. Then $V \subset V_{q_0}$. Define $V' = \{f' \in V \mid \nu(f') > q_0\}$. This is a proper subspace of V and set $U_f = \{g \in G \mid g.f \notin V'\}$ is open in G. For $g \in U_f$, we have $g.f \in V \setminus V'$ therefore $\nu(g.f) = q_0$. $\square$

COROLLARY 7.1.9. *Let H be a closed subgroup of G. We have a surjective restriction map* $\mathrm{res} : \mathcal{V}(G) \to \mathcal{V}(G/H)$.

Proof. We have $\mathsf{k}(G/H) = \mathsf{k}(G)^H \subset \mathsf{k}(G)$ and we define res as the restriction of the valuation. Let $\nu' \in \mathcal{V}(G/H)$. By Fact 7.1.3, we can lift ν' to a valuation $\bar{\nu}$ of $\mathsf{k}(G)$ and by the previous lemma, we can find an invariant valuation $\nu \in \mathcal{V}(G)$ with $\nu(f) = \bar{\nu}(g.f)$ for g in some open subset U_f of G. For $f \in \mathsf{k}(G/H) = \mathsf{k}(G)^H$ and $g \in U_f$, we have $\nu(f) = \nu(g.f) = \bar{\nu}(g.f) = \nu'(g.f) = \nu'(f)$ (note that the left action of G commutes with the right action of H so that if f is H-invariant for the right action, so is $g.f$ for all $g \in G$). $\square$

2. Relation to weights of rational functions

For $V \subset \mathsf{k}[G]$ a vector subspace and n an integer, we define V^n to be the vector space spanned by all the products of n elements of V: $V^n = \langle f_1 \cdots f_n | f_i \in V\rangle$

Proposition 7.2.1. *Let $\nu \in \mathcal{V}(G/H)$ and let $f \in \mathsf{k}(G/H)$. Assume that there exists $f_0 \in \mathsf{k}(G)^{(B\times H)}$ such that $ff_0 \in \mathsf{k}[G]$, and let $V = \langle G.ff_0\rangle$. Then*

(1) *For all $n \in \mathbb{N}$, $V^n f_0^{-n} \subset \mathsf{k}(G/H)$.*
(2) *$\nu(f) = \min\{\frac{1}{n}\nu(f'/f_0^n) \mid n \in \mathbb{N}, f' \in (V^n)^{(B)}\}$.*

Proof. (1) Let λ be the H-character of f_0. The left G-action commutes with the right H-action therefore all the elements in V^n are H-eigenfunctions with eigenvalue $n\lambda$. We thus have $V^n f_0^{-n} \subset \mathsf{k}(G)^H = \mathsf{k}(G/H)$.

(2) Let $\nu' \in \mathcal{V}(G)$ be a lifting of ν and consider $V_q = \{f' \in \mathsf{k}[G] \mid \nu'(f') \geq q\}$. Since ν' is G-invariant, this is a G-stable vector subspace of $\mathsf{k}[G]$. For $f' \in V^n$, we have $\nu'(f') \geq n\nu'(ff_0) = n(\nu(f) + \nu'(f_0))$ and $\nu(f) \leq \frac{1}{n}\nu'(f') - \nu'(f_0) = \frac{1}{n}\nu'(f'/f_0^n) = \frac{1}{n}\nu(f'/f_0^n)$ by (1). This proves that the left hand side is smaller that the right hand side in (2). Define $R = \oplus_{n\geq 0} V^n$. This is a graded integral k-algebra. For $r \in R$, denote by r_n its n-th graded part and for $r \in R$, define

$$\nu''(r) = \inf_{n\in\mathbb{N}} \{\nu'(r_n) - n\nu'(ff_0)\}.$$

Then ν'' is a G-invariant valuation with $\nu''|_R \geq 0$. Let $I = \{r \in R \mid \nu''(r) > 0\}$. This is a G-invariant prime homogeneous ideal. Note that $\nu''(ff_0) = 0$ thus the quotient R/I contains non-trivial elements of positive degree. There exists therefore a B-eigenvector in $(R/I)^{(B)}$ of positive degree, a power of which can be lifted to a B-eigenvector say f' in $R^{(B)}$. Taking graded parts, we may even choose $f' \in (V^n)^{(B)}$. We get $\nu''(f') = 0$ therefore $\nu'(f') = n\nu'(ff_0)$ concluding the proof. □

Let $x_0 \in G/H$ and B a Borel subgroup of G such that $B.x_0$ is dense in G/H.

Lemma 7.2.2. *Assume that G is simply connected and let $B \times H$ act on G via $(b,h).g = bgh^{-1}$. If $D \subset G$ is a $B \times H$-stable divisor, there exists $f \in \mathsf{k}(G)^{(B\times H)}$ such that $D = div(f)$.*

Proof. Since G is simply connected, we have $\mathrm{Pic}(G) = 0$ (see Corollary B.2.7). In particular if D is a divisor on G, then $D = \mathrm{div}(f)$ for some $f \in \mathsf{k}(G)$. Since D is $B \times H$-stable, the function f is defined and non vanishing on the dense open subset $BH \subset G$. In particular, we may furthermore assume $f(1) = 1$. Define $\varphi \in \mathsf{k}(B \times H)$ by $\varphi(b,h) = f(bh)$. We have $\varphi \in \mathsf{k}[B \times H]^\times$ and by Lemma B.1.5, there exists $\lambda \in \mathsf{k}[B]^\times$ and $\mu \in \mathsf{k}[H]^\times$ such that $\varphi = \lambda\mu$ and by Exercise B.1.6 the functions λ and μ can be chosen to be characters of B and H. For $b, b' \in B$ and $h, h' \in H$, we thus have $f(bb'h'h) = \varphi(bb'h'h) = \lambda(bb')\mu(hh') = \lambda(b)\lambda(b')\mu(h)\mu(h') = \lambda(b)\mu(h)\varphi(b'h') = \lambda(b)\mu(h)f(b'h')$. Since BH is dense in G, we get $f(bgh) = \lambda(b)\mu(h)f(g)$ proving the result. □

Corollary 7.2.3. *Let $f \in \mathsf{k}[B.x_0]$ and $\nu_0 \in \mathcal{V}(G/H)$. There exists $n \geq 0$ and $f' \in \mathsf{k}(G/H)^{(B)}$ such that the following three conditions hold:*

(1) *$\nu_0(f') = \nu_0(f^n)$;*
(2) *$\nu(f') \geq \nu(f^n)$ for all $\nu \in \mathcal{V}(G/H)$;*
(3) *$\nu_D(f') \geq \nu_D(f^n)$ for all $D \in \Delta(G/H)$.*

Proof. Replacing G by a finite cover, we may assume that G is simply connected. Let δ be the B-stable part of $\mathrm{div}(1/f)$ and let $D = \pi^*(\delta)$ with $\pi : G \to G/H$. Then by the previous lemma, $D = \mathrm{div}(f_0)$ for some $f_0 \in \mathsf{k}(G)^{(B\times H)}$ Note that ff_0 is defined on BH and on all the B-stable divisors of G by definition of f_0. Since

$G \setminus BH$ contains only B-stable divisors, ff_0 is defined on a open subset whose complement has codimension at least 2, by normality of G, $ff_0 \in \mathsf{k}[G]$.

By Proposition 7.2.1, there exist $n \in \mathbb{N}$ and $\varphi \in \mathsf{k}[G]^{(B)}$ such that $\varphi/f_0^n \in \mathsf{k}(G/H)^{(B)}$ with $\nu_0(\varphi/f_0^n) = \nu_0(f)$ and $\nu(\varphi/f_0^n) \geq \nu(f^n)$ for any $\nu \in \mathcal{V}(G/H)$, proving (1) and (2). For $D \in \Delta(G/H)$ a B-stable divisor and D' any component of $\pi^{-1}(D)$, we have

$$\nu_D(\varphi/f_0^n) = \nu_{D'}(\varphi/f_0^n) \geq \nu_{D'}(1/f_0^n) = \nu_{D'}(f^n) = \nu_D(f^n).$$

Setting $f' = \varphi/f_0^n$, this concludes the proof. □

Any valuation $\nu \in \mathcal{V}(G/H)$ induces a morphism $\rho_\nu : \mathsf{k}(G/H)^{(B)} \to \mathbb{Q}$ defined by $\rho_\nu(f) = \nu(f)$. Therefore we have a map

$$\rho : \mathcal{V}(G/H) \to \mathcal{Q}(X).$$

Corollary 7.2.4. *The map ρ is injective.*

Proof. Let $\nu \neq \nu'$ in $\mathcal{V}(G/H)$. Since $BH/H \simeq B/B \cap H$ and B is solvable, the quotient BH/H is affine. In particular $\mathsf{k}(G/H) = \mathsf{k}(BH/H) = \mathrm{Frac}(\mathsf{k}[BH/H])$ and any valuation is determined by its value on $\mathsf{k}[BH/H]$. Therefore, there exists $f \in \mathsf{k}[BH/H]$ with $\nu(f) < \nu'(f)$. By the previous corollary, there exists $f' \in \mathsf{k}(G/H)^{(B)}$ with $\nu(f') = \nu(f^n) < \nu'(f^n) \leq \nu'(f')$ proving the injectivity. □

For X a G-spherical variety and $Y \subset X$ a G-orbit, we may now define the B-chart and G-charts of Y in any characteristics.

Definition 7.2.5. Let X be a G-spherical variety. Define the B-chart $X_{Y,B}$ of Y and the G-chart $X_{Y,G}$ of Y and as follows:

$$X_{Y,B} = X \setminus \bigcup_{D \in \mathcal{D}(X) \setminus \mathcal{D}_Y(X)} D \text{ and } X_{Y,G} = G.X_{Y,B}.$$

Proposition 7.2.6. *Let X be a G-spherical variety and $Y \subset X$ a G-orbit.*

1. *$X_{Y,B}$ is the unique minimal B-stable affine open subset of X meeting Y non trivially.*
2. *We have $X_{G,B} = \{x \in X \mid \overline{G.x} \supset Y\}$.*
3. *$Y \cap X_{Y,B}$ is a B-orbit.*
4. *Any $D \in \Delta(X_{Y,G}) \setminus \Delta_Y(X_{Y,G})$ is Cartier and globally generated. We have*

$$X_{Y,G} \setminus X_{Y,B} = \bigcup_{D \in \Delta(X_{Y,G}) \setminus \Delta_Y(X_{Y,G})} D.$$

Proof. Let X_0 be an affine open subset as in Proposition 2.2.5, let ν_0 be a valuation with center $\overline{Y}$. Let $f \in \mathsf{k}[X_0]$ be a function not vanishing on Y (thus $\nu_0(f) = 0$) but vanishing on all $D \in \mathcal{D}(X) \setminus \mathcal{D}_Y(X)$ (thus $\nu_D(f) > 0$ for all such D). By Corollary 7.2.3, we may find $f_0 \in \mathsf{k}(X)^{(B)}$ with the same properties but since f_0 is defined on Y (because $\nu_0(f_0) = 0$) and vanishes on all $D \in \mathcal{D}(X) \setminus \mathcal{D}_Y(X)$ (because $\nu_D(f_0) = 0$), we get that f_0 is defined on all B-stable divisors of X_0 and by B-semiinvariance of f_0 and normality of X that $f_0 \in \mathsf{k}[X_0]^{(B)}$.

(1) Any B-stable affine open subset meeting Y has to contain $X_{Y,B}$ since the complement of an affine open subset is of pure codimension 1. Thus $X_{Y,B} \subset X_0$ and we get $X_{Y,B} = \{x \in X_0 \mid f_0(x) \neq 0\}$ thus $X_{Y,B}$ is affine, meets Y and is therefore minimal for this property.

(2) Let $Z \subset X$ be a G-orbit and assume that $Y \not\subset \overline{Z}$. Let ν_Z be an invariant valuation centered at Z, then in the construction of f_0, we may assume that $\nu_Z(f_0) > 0$ thus $X_{Y,B} \cap Z = \emptyset$ and Y is the unique closed orbit in $X_{Y,G} = G.X_{Y,B}$. This gives the inclusion $X_{Y,G} \subset \{x \in X \mid \overline{G.x} \supset Y\}$. Conversely, let $x \in X$ such that $\overline{G.x} \supset Y$ and assume that $G.x \cap X_{Y,B} = \emptyset$. Then $G.x \subset D$ for some $D \in \mathcal{D}(X) \setminus \mathcal{D}_Y(X)$ and $Y \subset \overline{G.x} \subset D \not\supset Y$, a contradiction.

(3) It is enough to prove that any $f \in \mathsf{k}[X_{Y,B} \cap Y]^{(B)}$ is invertible. Indeed, let $Z \subsetneq X_{Y,B} \cap Y$ be a closed B-orbit and let $I(Z) \subset \mathsf{k}[X_{Y,B} \cap Y]$ be its ideal. Then any element in $I(Z)^{(B)}$ would be invertible and $Z = \emptyset$.

Let $f \in \mathsf{k}[X_{Y,B} \cap Y]^{(B)}$, then there exists $d \in \mathbb{N}$ such that $ff_0^d \in \mathsf{k}[X_0 \cap Y]^{(B)}$ thus there exists n and $f' \in \mathsf{k}[X_0]^{(B)}$ such that $f'|_Y = (ff_0)^n$. Now $f'|_{X_{Y,B}}$ is invertible since the only divisors on which it can vanish are the divisors $D \in \mathcal{D}_Y(X)$ containing Y and f' does not vanish on Y. This implies that f is invertible and finishes the proof of (3).

(4). Since $X_{Y,B}$ is affine and B-stable, its complement in $X_{Y,G}$ is a union of prime B-stable divisors non containing Y. By definition of $X_{Y,G}$, these divisors are not G-stable proving the equality.

Let D be the union of all the above divisors and let $X^{\mathrm{sm}} \to X$ be the inclusion of the smooth locus in X. Let $L^{\mathrm{sm}} = \mathcal{O}_{X^{\mathrm{sm}}}(D \cap X^{\mathrm{sm}})$. This is an invertible sheaf on X. Replacing G by a finite cover, we may assume that L^{sm} is G-linearised. The group G acts on $L = i_*L^{\mathrm{sm}}$. Since X is normal we get $L = \mathcal{O}_X(D)$. The locus in X where L is not locally free is a closed G-stable subset contained in D. It has to contain a closed G-orbit. Its intersection with $X_{Y,G}$ has to contain a closed G-orbit as well but the only closed G-orbit is Y which is not contained in D thus D is Cartier.

Let s be the canonical section of $L = \mathcal{O}_X(D)$. The group G acts on the sections of L thus gs is again a section and the locus where all these section vanish is a closed G-orbit therefore empty in $X_{Y,G}$. $\square$

CHAPTER 8

Simple embeddings

1. Classification of simple spherical embeddings

Definition 8.1.1. An embedding of an homogeneous space G/H is a normal G-variety with an open G-orbit isomorphic to G/H.

Remark 8.1.2. A divisor $D \in \mathcal{D}(X) \setminus \Delta(X)$ is G-stable and defines a G-invariant valuation $\nu_D \in \mathcal{V}(X)$ and we recover D from ν_D as its unique center.

Definition 8.1.3. For a G-spherical variety X with dense orbit G/H and $Y \subset X$ a G-orbit, define $\mathcal{V}_Y(X) = \{\nu_D \in \mathcal{V}(G/H) \mid D \in \mathcal{D}_Y(X) \setminus \Delta_Y(X)\}$. Since $\rho : \mathcal{V}(X) \to \mathcal{Q}(X)$ is injective, we may view $\mathcal{V}_Y(X)$ as a subset of $\mathcal{Q}(X)$.

Theorem 8.1.4. *A simple spherical embedding X of G/H with closed orbit Y is completely determined by the pair $(\mathcal{V}_Y(X), \Delta_Y(X))$.*

Proof. Let X' be another simple embedding of G/H with closed orbit Y' and with $(\mathcal{V}_{Y'}(X'), \Delta_{Y'}(X')) = (\mathcal{V}_Y(X), \Delta_Y(X))$. Let $X_{Y,B}$ and $X'_{Y',B}$ the corresponding B-stable affine subsets and define X_0 which is an open subset of both X and X' as follows:

$$X_0 = G/H \setminus \bigcup_{D \in \mathcal{D}(G/H) \setminus \Delta_Y(X)} .$$

By normality of X and X', we have equalities

$$\mathsf{k}[X_{Y,B}] = \{f \in \mathsf{k}[X_0] \mid \nu(f) \geq 0 \text{ for all } \nu \in \mathcal{V}_Y(X)\} = \mathsf{k}[X'_{Y',B}].$$

Therefore, the G-birational isomorphism between X and X' induces an isomorphism $X_{Y,B} \simeq X'_{Y',B}$ and therefore an isomorphism $X = X_{Y,G} \simeq X'_{Y',G} = X'$. $\square$

We now switch to convex geometry.

Definition 8.1.5. Let V be a $\mathbb{Q}$-vector space and $C \subset V$.

(1) C is a cone if it stable by addition and by multiplication with $\mathbb{Q}_{\geq 0}$.
(2) The dual of a cone $C \subset V$ is $C^\vee = \{f \in V^\vee \mid f(v) \geq 0 \text{ for all } v \in C\}$.
(3) A cone is strictly convex if $C \cap -C = 0$, equivalently C contains no line.
(4) A cone is polyedral if C can be written $C = \mathbb{Q}_{\geq 0}v_1 + \cdots + \mathbb{Q}_{\geq 0}v_n$.
(5) A face of a cone C is a subset $F_f = \{v \in C \mid f(v) = 0\}$ for some $f \in C^\vee$.
(6) The dimension of a cone is the dimension of its linear span.
(7) An extremal ray is a face of dimension one.
(8) The relative interior C° of C is the complement of all proper faces of C.

Definition 8.1.6. Let X be a spherical G-variety and $Y \subset X$ a G-orbit. Define the cone $\mathcal{C}_Y(X) \subset \mathcal{Q}(X)$ as the cone generated by $\mathcal{V}_Y(X)$ and $\rho_X(\Delta_Y(X))$.

Lemma 8.1.7. *The sets $\mathbb{Q}_{\geq 0}\nu$ with $\nu \in \mathcal{V}_Y(X)$ are exactly the extremal rays of $\mathcal{C}_Y(X)$ which do not contain any element of $\rho(\Delta_Y(X))$.*

Proof. Let $D \in \mathcal{D}_Y(X) \setminus \Delta_Y(X)$ and let ν_D be the corresponding element in $\mathcal{V}_Y(X)$. Consider the affine chart $X_{Y,B}$ and let $\mathcal{D} = \cup_{D' \in \mathcal{D}_Y(X), D' \neq D} D'$. There exists $f' \in \mathsf{k}[X_{Y,B}]$ such that f' vanishes on $\mathcal{D}$ and not on D. By Corollary 7.2.3 there exists a function $f \in k(G/H)^{(B)}$ vanishing on all $D' \in \mathcal{D}_Y(X)$ except D. The face F_{λ_f} defined by the weight λ_f of f is therefore an extremal ray of $\mathcal{C}_Y(X)$ containing D and not containing any element of $\rho(\Delta_Y(X))$. The converse is obvious by definition of $\mathcal{C}_Y(X)$. $\square$

Corollary 8.1.8. *A simple spherical embedding X of G/H with closed orbit Y is completely determined by the pair $(\mathcal{C}_Y(X), \Delta_Y(X))$.*

Proof. It suffices to prove that we can recover $\mathcal{V}_Y(X)$ from $(\mathcal{C}_Y(X), \Delta_Y(X))$. By the above lemma we recover the ray $\mathbb{Q}_{\geq 0}\nu_D$ for all $\nu_D \in \mathcal{V}_Y(X)$. Now ν_D is determined by the fact that $\nu_D(\Lambda(X)) = \mathbb{Z}$. $\square$

Proposition 8.1.9. *Let X be a simple G-spherical embedding of G/H with closed orbit Y and let $\nu \in \mathcal{V}(G/H)$.*

(1) *We have the equality $\mathsf{k}[X_{Y,B}]^{(B)} = \{f \in \mathsf{k}(G/H)^{(B)} \mid \lambda_f \in \mathcal{C}_Y(X)^\vee\}$.*
(2) *The center of ν exists if and only if $\nu \in \mathcal{C}_Y(X)$.*
(3) *The center of ν is Y if and only if $\nu \in \mathcal{C}_Y(X)^\circ$.*

Proof. (1) Let $f \in \mathsf{k}[X_{Y,B}]^{(B)}$. Then $f \in \mathsf{k}(G/H)^{(B)}$ and is defined on the divisors $D \in \mathcal{D}_Y(X)$ thus λ_f is non negative on $\mathcal{C}_Y(X)$. Conversely, a function $f \in \mathsf{k}(G/H)^{(B)}$ non negative on $\mathcal{C}_Y(X)$ is defined on BH/B and on all $D \in \mathcal{D}_Y(X)$ therefore on an open subset of $X_{Y,B}$ whose complement has codimension at least 2. By normality $f \in \mathsf{k}[X_{Y,B}]$.

(2) Since $X_{Y,B}$ is affine, ν is G-invariant and $X = GX_{Y,B}$, the valuation ν has a center if and only if ν is non negative on $\mathsf{k}[X_{Y,B}]$. By (1), this is equivalent to $\nu \in \mathcal{C}_Y(X)$.

(3) If Y is the center of ν, then any $f \in \mathsf{k}[X_{Y,B}]^{(B)}$ with $\nu(f) = 0$ will not vanish on Y and therefore will not vanish at all (since the zero locus of f is a B-stable divisor of $X_{Y,B}$ thus has to contain Y). Thus for any $\rho(\nu') \in \mathcal{C}_Y(X)$, we have $\nu'(f) = 0$ and the weight of f is in the vertex of $\mathcal{C}_Y(X)^\vee$. This proves that ν is in the interior of $\mathcal{C}_Y(X)$.

Now let $\nu' \in \mathcal{C}_Y(X)$ with center $Z \supsetneq Y$. Let ν_Y be the valuation associated to Y. There exists $f \in \mathsf{k}[X_{Y,B}]$ with $\nu'(f) = 0$ and $\nu_Y(f) > 0$. By Corollary 7.2.3 we may assume f to be a B-eigenfunction and we get that ν' is not in the interior of the cone. $\square$

Definition 8.1.10. A colored cone for G/H, is a pair $(\mathcal{C}, \mathcal{F})$ with $\mathcal{C} \subset \mathbb{Q}(G/H)$ and $\mathcal{F} \subset \Delta(G/H)$ having the following properties

(CC1) $\mathcal{C}$ is a cone generated by $\rho(\mathcal{F})$ and finitely many elements of $\mathcal{V}(G/H)$;
(CC2) The intersection $\mathcal{C}^\circ \cap \mathcal{V}(G/H)$ is non empty.

The colored cone is called strictly convex if the following condition holds.

(SCC) The cone $\mathcal{C}$ is strictly convex and $0 \notin \rho(\mathcal{F})$.

Theorem 8.1.11. *The map $X \mapsto (\mathcal{C}_Y(X), \Delta_Y(X))$ is a bijection between the isomorphism classes of simple spherical embeddings X of G/H with closed orbit Y and strictly convex colored cones.*

Proof. We first check that the map is well defined. For such a simple embedding X, we already know that (CC1) is satisfied and by the previous proposition, since Y induces an invariant valuation in the interior of the cone, we know that (CC2) is also satisfied. Finaly, to prove that the cone is strictly convex we need to prove that there is no linear subspace in it. Let $\mathcal{D}$ the union of B-stable divisors containing Y, these are exactly the B-stable divisors meeting $X_{Y,B}$. There exists $f \in \mathsf{k}[X_{Y,B}]$ vanishing on $\mathcal{D}$. By Corollary 7.2.3, we may choose $f \in \mathsf{k}[X_{Y,B}]^{(B)}$ and for any ν_D with $D \in \mathcal{D}_Y(X)$ we have $\nu_D(f) > 0$. Therefore $\mathcal{C}_Y(X)$ is in the half-space with non negative value on f proving (SCC).

The injectivity of the map follows from Theorem 8.1.4. Let us prove the surjectivity. Let $(\mathcal{C}, \mathcal{F})$ be a colored cone with the above three properties (CC1), (CC2) and (SCC). By (CC1) and Gordan's Lemma (see [**21**, Page 3]), there exists a finite set of elements $g_1, \cdots, g_n \in \mathsf{k}(G/H)^{(B)} \subset \mathsf{k}(G)^H$ such that the weights λ_{g_i} span $\mathcal{C}^\vee \cap \Lambda(G/H)$ as a monoid. Denote by $\pi : G \to G/H$ the quotient map and let $\mathcal{D}_0$ be the union of all divisors $D \in \Delta(G/H) \setminus \mathcal{F}$. Note that the poles of the g_i are contained in $\mathcal{D}_0$. Since $\mathcal{D}_0$ is a B-stable divisor of G/H, there exists an element $f_0 \in \mathsf{k}[G]^{(B \times H)}$ vanishing on $\mathcal{D}_0$ and with $f_i = f_0 g_i \in \mathsf{k}[G]$ for all $i \in [1, n]$. Let W be the G-submodule of $\mathsf{k}[G]$ spanned by the $(f_i)_{i \in [0,n]}$. Since the g_i are H-invariants while f_0 is a H-eigenfunction of weight say $\chi \in \mathcal{X}(H)$, we get that all the elements in W are H-eigenfunctions of weight χ. Therefore we have a G-equivariant morphism

$$\varphi : G/H \to \mathbb{P}(W^\vee)$$

defined by $x \mapsto [f_0(x) : \cdots : f_n(x)]$. Let $D(f_0)$ be the open subset of $\mathbb{P}(W^\vee)$ defined by the non vanishing of f_0 and define X_0' and X' by

$$X_0' = \overline{\varphi(G/H)} \cap D(f_0) \text{ and } X' = GX_0'.$$

Note that $\varphi(G/H)$ is dense in X' therefore X' has a dense B-orbit and that X_0' is a B-stable dense open affine subset in X' containing the dense B-orbit.

LEMMA 8.1.12. *We have* $\mathsf{k}[X_0']^{(B)} = \{f \in \mathsf{k}(G/H)^{(B)} \mid \lambda_f \in \mathcal{C}^\vee\}$.

Proof. Let $f \in \mathsf{k}(G/H)^{(B)}$ with $\lambda_f \in \mathcal{C}^\vee$. Write $\lambda_f = \sum_i a_i \lambda_{g_i}$ with $a_i \geq 0$ for all i. Define $F = \prod_{i=1}^n g_i^{a_i} = \prod_{i=1}^n (f_i/f_0)^{a_i} \in \mathsf{k}[X_0']^{(B)}$. Then $\lambda_f = \lambda_F$ thus $f/F \in \mathsf{k}(G/H)^B = \mathsf{k}$ is constant. Therefore f is a multiple of $F \in \mathsf{k}[X_0']^{(B)}$.

Conversely, let $f \in \mathsf{k}[X_0']^{(B)}$ and let v be in an extremal ray of $\mathcal{C}$. If $v = \rho(\nu_D)$ with $D \in \mathcal{F}$, since $\varphi(D)$ meets non trivially $D(f_0)$, then f is defined on an open subset of D thus $\nu_D(f) \geq 0$. If $v = \nu \in \mathcal{V}(G/H)$, first note that $\nu(f_i/f_0) = \nu(g_i) \geq 0$ by definition of the g_i. For a general $f \in \mathsf{k}[X_0']$, write $f = f'/f_0^n$ with $f' \in S^n W$ and extend ν to an invariant valuation $\bar{\nu} \in \mathcal{V}(G)$. Write $f' = \prod_i \sum_j a_{i,j} g_{i,j} . f_{k_{i,j}}$, with $a_{i,j} \in \mathsf{k}$ and $g_{i,j} \in G$. We then have

$$\nu(f) \geq \sum_i \min_j \{\bar{\nu}(f_{k_{i,j}}) - \bar{\nu}(f_0)\} = \sum_i \min_j \{\nu(g_{k_{i,j}})\} \geq 0,$$

proving the result. □

Define $\Upsilon = \{\lambda_f \mid f \in \mathsf{k}[X_0']^{(B)}\}$. The previous lemma implies that $\mathrm{Cone}(\Upsilon) = \mathcal{C}^\vee$. We also have that Υ spans $\Lambda(G/H)$. Indeed, the orthogonal of the linear span of Υ has to be contained in $\mathcal{C}$. As $\mathcal{C}$ contains no linear subspace by (SCC), this orthogonal is trivial. Since X_0' is an affine open subset of X' we get that $\Lambda(X') = \Lambda(G/H)$.

LEMMA 8.1.13. *The fibers of φ are finite.*

Proof. We prove that the fibers are affine and proper. We first prove that the fibers are affine. Let $D \in \mathcal{F}$ be a B-stable divisor. Then $\nu_D(f) \geq 0$ for all $f \in \mathsf{k}[X'_0]^{(B)}$ and by (SCC) there exists f with $\nu_D(f) > 0$. Thus $D \subset \{x \in G/H \mid f(x) = 0\}$ and $\varphi|_D$ is not dominant on X'. If $D \in \mathcal{D}(G/H)\backslash\mathcal{F}$, then $\varphi(D) \subset \{x \in G/H \mid f_0(x) = 0\}$ and again $\varphi|_D$ is not dominant on X'. Since φ is equivariant, this implies that the proper B-orbits of G/H are mapped to proper B-orbits in X' thus $\varphi^{-1}(\varphi(BH/H)) = BH/H$. This B-orbit and its image are affine thus the restriction of φ is a morphism between affine varieties and the fibers are therefore affine.

Let us now prove that the fibers are proper. For this consider $\phi : G/H \to X''$ a proper embedding of G/H (for example embedded G/H equivariantly in a projective space and take the normalisation of its closure). Set $x = \varphi(H/H)$, $x' = \varphi(x)$ and $x'' = \phi(x)$ and define X''' as the normalisation of $\overline{G(x', x'')} \subset X' \times X''$. This is an embedding of G/H since $\mathrm{Stab}(x') \supset H = \mathrm{Stab}(x'')$. Furthermore, the projection $p : X''' \to X'$ is proper since X'' is proper. We claim that $p^{-1}(\varphi(G/H)) = G/H$. This proves that the fibers of φ are equal to those of p and therefore proper. To prove the claim, assume that there exists a closed G-stable subset Z in X''', disjoint from G/H such that $p|_Z$ is dominant on X'. Let $\nu_Z \in \mathcal{V}(G/H)$ be the corresponding valuation. The map p induces an injection of fields $\mathsf{k}(X') \to \mathsf{k}(X''') = \mathsf{k}(G/H)$. Let $f \in \mathsf{k}(X')$ a non trivial element, the corresponding function induced on Z is also non trivial since $p|_Z$ is dominant on X'. In particular $\nu_Z(f) = 0$. So for $f \in \mathsf{k}[X'_0]^{(B)}$, we have $\nu_Z(f) = 0$ and $\nu_Z \in \Upsilon^\perp$. By (SCC), ν_Z must be trivial, a contradition. $\square$

Let X be the normalisation of X' in $\mathsf{k}(G/H)$. The morphism φ factors as follows

$$\begin{tikzcd} G/H \arrow[r] \arrow[dr, "\varphi"'] & X \arrow[d, "\psi"] \\ & X' \end{tikzcd}$$

where all maps are G-equivariant, the horizontal map is an open embedding while the map ψ is finite. Let $X_0 = \psi^{-1}(X'_0)$. Since ψ is finite, X_0 is an affine B-stable open subset of X. Note that X_0 is the normalisation of X'_0 in $\mathsf{k}(G/H)$ and $X' = GX'_0$ proving that $GX_0 = X$. The following lemma concludes the proof. $\square$

LEMMA 8.1.14. *X is a simple embedding with colored cone* $(\mathcal{C}, \mathcal{F})$.

Proof. Since $X_0 \to X'_0$ is finite, we have $\mathsf{k}[X'_0]^{(B)} \subset \mathsf{k}[X_0]^{(B)}$ and for $f \in \mathsf{k}[X_0]^{(B)}$, there exists $n \in \mathbb{Z}_{>0}$ such that $\lambda_f = n\lambda f'$ with $f' \in \mathsf{k}[X'_0]^{(B)}$. Recall from Lemma 8.1.12 the equality

$$\Upsilon = \{\lambda_{f'} \mid f' \in \mathsf{k}[X'_0]^{(B)}\} = \{\lambda_{f'} \mid f' \in \mathsf{k}(G/H)^{(B)} \text{ and } \lambda_{f'} \in \mathcal{C}^\vee\}.$$

In particular, for $f \in \mathsf{k}(G/H)^{(B)}$ with $n\lambda_f \in \Upsilon$, then $\lambda_f \in \Upsilon$. We deduce that $\mathsf{k}[X_0]^{(B)} = \mathsf{k}[X'_0]^{(B)}$.

By (CC2), there exists $\nu \in \mathcal{C}^\circ \cap \mathcal{V}(G/H)$. By the above ν is non negative on $\mathsf{k}[X_0]^{(B)}$. By Corollary 7.2.3 it is non negative on $\mathsf{k}[X_0]$. Therefore it has a center Y_0 in X_0 and since ν is G-invariant it has a G-stable center Y in X. Let $Z \subset X$ be a G-stable closed subvariety such that $Y \not\subset Z$. Since $X = GX_0$, we have $X_0 \cap Z \neq \emptyset$ and ν_Z is non negative on $\mathsf{k}[X_0]$. In particular $\nu_Z \in \mathcal{C}$. On the other hand, there exists a function $f \in \mathsf{k}[X_0]$ with $\nu_Z(f) > 0$ and $\nu(f) = 0$. By Corollary 7.2.3, we may assume $f \in \mathsf{k}[X_0]^{(B)}$ so $\lambda_f \in \mathcal{C}^\vee$. Since $\nu \in \mathcal{C}^\circ$ and $\nu_Z \in \mathcal{C}$, the vanishing $\langle \lambda_f, \rho(\nu)\rangle = \nu(f) = 0$ implies $\lambda_f = 0$ and therefore $\nu_Z(f) = 0$. A contradiction. Thus Y is the only closed orbit of G in X and X is a simple embedding with closed orbit Y. The same argument works replacing Z by $D \in \mathcal{F}$, so Y is contained in any B-stable divisor of X_0.

Since X is normal and by the above $X_0 = X_{Y,B}$, we get that $\mathcal{C}_Y(X)^\vee \cap \Lambda(G/H)$ is the set of weights of $\mathsf{k}[X_0]$ and is therefore is equal to $\mathcal{C}^\vee \cap \Lambda(G/H)$ by Proposition 8.1.9. This implies $\mathcal{C}_Y(X) = \mathcal{C}$. By the above, we also have the inclusion $\mathcal{F} \subset \Delta_Y(X)$. Conversely, for $D \in \Delta(G/H) \setminus \mathcal{F}$, we have $\varphi(D) \subset \mathcal{D}_0$ so D is not a divisor of X_0 and therefore not in $\Delta_Y(X)$. □

Remark 8.1.15. In characteristic 0, the variety X' in the above proof is already the simple embedding with colored cone $(\mathcal{C}, \mathcal{F})$: indeed with notations as in the above proof $\mathbb{Z}\Upsilon = \Lambda(G/H)$ thus $\Lambda(X') = \Lambda(G/H)$ and, in characteristic 0, this implies that $G/H \to X'$ is injective (prove this as an exercise).

Here is a proof, *a posteriori*, that $X = X'$ using the above construction of X and X'. Using Theorem 6.2.8 and the fact that $P = \mathrm{Stab}_G(X_0) = \mathrm{Stab}_G(X'_0)$, we can write $X \simeq P \times^L S_Y$ and $X' = P \times^L S'_{Y'}$ where $L \subset P$ is a Levi subgroup, $Y' = \psi(Y)$ and where $S_Y \subset X_0$ and $S'_{Y'} \subset X'$ are closed L-stable subsets which are multiplicity-free. The map ψ restricts to an L-equivariant finite morphism $\psi : S_Y \to S'_{Y'}$ but $\Lambda^+(S_Y) = \Lambda(G/H) \cap \mathcal{C}^\vee = \Lambda(S'_{Y'})$ and decomposing decomposing the coordinate rings into irreducible$L$-module, we get $\mathsf{k}[S_Y] = \mathsf{k}[S'_{Y'}]$, thus $S_Y \simeq S'_{Y'}$ and $X_0 \simeq X_0$ proving $X \simeq X'$.

Definition 8.1.16. Let $(\mathcal{C}, \mathcal{F})$ be a colored cone. A colored cone $(\mathcal{C}_0, \mathcal{F}_0)$ is called a colored face of $(\mathcal{C}, \mathcal{F})$ if the following conditions are satisfied:

(CF1) $\mathcal{C}_0$ is a face of the cone $\mathcal{C}$;
(CF2) $\mathcal{F}_0 = \mathcal{F} \cap \rho^{-1}(\mathcal{C}_0)$.

Remark 8.1.17. A colored face $(\mathcal{C}_0, \mathcal{F}_0)$ of a colored cone $(\mathcal{C}, \mathcal{F})$ is completely determined by the face $\mathcal{C}_0$. The colored faces of a colored cone $(\mathcal{C}, \mathcal{F})$ are the faces of $\mathcal{C}$ whose interior meets $\mathcal{V}(G/H)$.

Lemma 8.1.18. *Let X be a G-spherical embedding of G/H and let $Y \subset X$ a G-orbit. There is a bijection $Z \mapsto (\mathcal{C}_Z(X), \Delta_Z(X))$ between the set of G-orbits in X with $\overline{Z} \supset Y$ and the set of faces of $(\mathcal{C}_Y(X), \Delta_Y(X))$.*

Proof. If Z is a G-orbit with $\overline{Z} \supset Y$, then $\Delta_Z(Y) \subset \Delta_Y(X)$, $\mathcal{D}_Z(X) \subset \mathcal{D}_Y(X)$. We get $\mathcal{C}_Z(X) \subset \mathcal{C}_Y(X)$. We also get $X_{Z,B} \subset X_{Y,B}$ corresponding to the localisation morphism $\mathsf{k}[X_{Y,B}] \to \mathsf{k}[X_{Z,B}]$. Let $D \in \mathcal{D}_Y(X) \setminus \mathcal{D}_Z(X)$ and $f \in \mathsf{k}[X_{Y,B}]$ such that $\nu_D(f) = 0$ and $\nu_{D'}(f) > 0$ for $D' \in D_Z(X)$. By Corollary 7.2.3, we may assume that $f \in \mathsf{k}[X_{Y,B}]^{(B)}$. Then $\mathcal{C}_Z(X)$ is in the face defined by f while D is not. As we can do this for any $D \in \mathcal{D}_Y(X) \setminus \mathcal{D}_Z(X)$, we get that $\mathcal{C}_Z(X)$ is the intersection of all these faces. Furthermore ν_Z is in the interior of $\mathcal{C}_Z(X)$ thus $(\mathcal{C}_Z(X), \Delta_Z(X))$ is indeed a colored face of $(\mathcal{C}_Y(X), \Delta_Y(X))$.

If $(\mathcal{C}_0, \mathcal{F}_0)$ is a colored face of $(\mathcal{C}_Y(X), \Delta_Y(X))$, pick $\nu \in \mathcal{C}_0^\circ$. By Proposition 8.1.9, it has a G-stable center. Since X is spherical, this center is the closure of a G-orbit Z and $Y \subset \overline{Z}$. We already proved that $(\mathcal{C}_Z(X), \Delta_Z(X))$ is a colored face of $(\mathcal{C}_Y(X), \Delta_Y(X))$. Proposition 8.1.9 implies that $\nu = \nu_Z \in \mathcal{C}_Z(X)^\circ$. Since $\nu \in \mathcal{C}^\circ$, we must have $\mathcal{C} = \mathcal{C}_Z(X)$ and therefore $\mathcal{F} = \Delta_Z(X)$. □

EXAMPLE 8.1.19. Consider the simplest simple embedding: $X = G/H$. The unique closed orbit is G/H. The open subset $X_{Y,B}$ is the dense B-orbit BH/H. In this case the cones $\mathcal{C}_Y(X)$ and $\mathcal{C}_Y(X)^\vee$ are trivial. The center of the trivial valuation is X and $\mathcal{V}(G/H) = \{0\}$.

EXAMPLE 8.1.20. Let $G = \mathrm{SL}_2(\mathsf{k})$ and let U be the maximal unipotent subgroup of unipotent upper-triangular matrices. The standard action of G on $\mathbb{A}^2$ realises an isomorphism $G/U \simeq \mathbb{A}^2 \setminus \{0\}$. Set $X = \mathbb{A}^2$. The G-orbits are $\mathbb{A}^2 \setminus \{0\}$ and $Y = \{0\}$. The latter is the unique closed orbit. The B-orbits are

- $B.(1,0) = \{(a,b) \in \mathbb{A}^2 \mid a \neq 0\}$,
- $B.(0,1) = \{(a,b) \in \mathbb{A}^2 \mid a = 0 \text{ and } b \neq 0\}$ and
- $B.(0,0) = \{0\}$.

There is a unique B-stable divisor $D = \{(a,b) \in \mathbb{A}^2 \mid a = 0\}$ and $Y \subset D$. We thus have $\Delta_Y(X) = \Delta(X) = \{D\} = \mathcal{D}(X) = \mathcal{D}_Y(X)$ and $X_{Y,B} = X$.

We also have $\mathsf{k}(G/H)^{(B)} = \mathsf{k}(\mathbb{A}^2)^{(B)} = \mathsf{k}(a,b)^{(B)} = \{a^n \mid n \in \mathbb{Z}\}$ and $\Lambda(X) = \mathbb{Z}\lambda_a = \mathbb{Z}\varpi_\alpha$. The cone $\mathcal{C}_Y(X) = \mathbb{Z}_{\geq 0}\rho(\nu_a)$ is one dimensional generated by the image valuation ν_a with respect to a. We have $\nu_Y = \nu_a \in \mathcal{C}_Y(X)^\circ$ and $\rho(D) = \alpha^\vee$. We have $\mathcal{V}(G/H) = \Lambda(G/H)$.

EXAMPLE 8.1.21. Consider $X = \mathbb{A}^2$ as a spherical variety for $G = B = T = \mathbb{G}_m^2$. The G-orbits and the B-orbits coincide and are described by:

- $\{(a,b) \in \mathbb{A}^2 \mid a \neq 0,\ b \neq 0\}$,
- $\{(a,b) \in \mathbb{A}^2 \mid a \neq 0,\ b = 0\}$,
- $\{(a,b) \in \mathbb{A}^2 \mid a = 0,\ b \neq 0\}$ and
- $\{(a,b) \in \mathbb{A}^2 \mid a = b = 0\} = Y$.

Then Y is the unique closed orbit., there are two B-stable divisors and $X_{Y,B} = X$.

On the level of invariants we have $\mathsf{k}(G/H)^{(B)} = \mathsf{k}(\mathbb{A}^2)^{(T)} = \{a^n b^m \mid n, m \in \mathbb{Z}\}$. The cone $\mathcal{C}_Y(X) = \mathbb{Z}_{\geq 0}\rho(\nu_a) \oplus \mathbb{Z}_{\geq 0}\rho(\nu_b)$ is of dimension 2. In this case we again have the equality $\mathcal{V}(G/H) = \Lambda(G/H)$.

EXAMPLE 8.1.22 (Symmetric matrices). Recall the notation of Example 5.2.8. Then X has a unique closed orbit $Y = \{0\}$. We have $\Delta_Y(X) = \Delta(X)$ and $\mathcal{D}_Y(X) = \mathcal{D}(X)$. We get $X_{Y,B} = X$. Furthermore, with the notation of Example 5.2.8, one can prove that $\mathsf{k}[X]^U = \mathsf{k}[d_1, \cdots, d_n]$. By Proposition 8.1.9, we get

$$\mathcal{C}_Y(X) = \mathrm{Cone}\left(\frac{1}{2}\alpha_1^\vee, \cdots, \frac{1}{2}\alpha_n^\vee\right).$$

There are $n-1$ colors on $\mathcal{C}_Y(X)$. Note that not all the faces are colored faces, otherwise there would be 2^n irreducible closed G-stable subvarieties. This comes from the fact that the cone $\mathcal{V}(G/H)$ is strictly contained in $\mathcal{Q}(G/H)$ and even in $\mathcal{C}_Y(X)$: The only G-orbits are given by the rank and the only irreducible closed G-stable subvarieties are their closure $(X_k)_{k\in[0,n]}$. One can actually prove that the following holds:

$$\mathcal{V}(G/H) = \mathrm{Cone}\left(\varpi_1^\vee, \cdots, \varpi_n^\vee\right),$$

where $(\varpi_i^\vee)_{i\in[1,n]}$ is the dual basis of $(\alpha_i)_{i\in[1,n]}$. Explicitly

$$\varpi_k^\vee = \sum_{i=k+1}^{n} \epsilon_i \text{ with } \alpha_k = \epsilon_k - \epsilon_{k+1}.$$

CHAPTER 9

Classification of spherical embeddings

1. Colored fans

Definition 9.1.1. A colored fan $\mathbb{F}$ is a finite collection of colored cones $(\mathcal{C},\mathcal{F})$ satisfying the following properties:

(CF1) Every colored face of a colored cone $(\mathcal{C},\mathcal{F})$ of $\mathfrak{F}$ is in $\mathfrak{F}$.
(CF2) For every $\nu \in \mathcal{V}(G/H)$ there is at most one colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}$ such that $\nu \in \mathcal{C}^\circ$.

A colored cone is called stricly convex if any cone of $\mathfrak{F}$ is strictly convex. This is equivalent to the fact that $(0,\emptyset)$ is in $\mathfrak{F}$.

Definition 9.1.2. Let X be an embedding of G/H we define $\mathfrak{F}(X)$ to be the set of all colored cones $(\mathcal{C}_Y(X),\Delta_Y(X))$ for Y a G-orbit in X.

Theorem 9.1.3. *The map $X \mapsto \mathfrak{F}(X)$ is a bijection between isomorphism classes of embedding and strictly colored fans.*

Proof. If X is a spherical embedding of G/H, we prove that $\mathfrak{F}(X)$ is a colored fan. By Lemma 8.1.18, we know that any face of a cone of $\mathfrak{F}(X)$ is again a cone of $\mathfrak{F}(X)$. If Y_1 and Y_2 are such that a valuation $\nu \in \mathcal{V}(G/H)$ lies in $\mathcal{C}_{Y_1}(X)^\circ \cap \mathcal{C}_{Y_2}(X)^\circ$, then their closure are the center of ν and must be equal. Thus $Y_1 = Y_2$. Finally, all the colored cones are strictly convex.

Conversely, let $\mathfrak{F}$ be a colored fan. Then for any cone $(\mathcal{C},\mathcal{F})$, there exists a simple spherical embedding $X(\mathcal{C},\mathcal{F})$. These embedding are isomorphic on the smaller simple spherical embedding given by colored faces. We can therefore glue these embedding along their intersection to get X. This is a priori not a separated scheme. If we prove that it is separated, then it will be a spherical embedding with colored fan $\mathfrak{F}$. So we are left to prove that X is indeed separated. By definition we need to prove that the diagonal embedding $X \to X \times X$ is closed. Let Y be an orbit of the closure of the diagonal, we want to prove that Y is contained in the diagonal. We may assume that Y is contained in a product $X_1 \times X_2$ with $X = X(\mathcal{C},\mathcal{F})$ and $X_2 = X(\mathcal{C}',\mathcal{F}')$ and choose $(\mathcal{C},\mathcal{F})$ and $(\mathcal{C}',\mathcal{F}')$ minimal for this property. Let X_3 be the normalisation of the closure of the diagonal embedding of G/H in $X_1 \times X_2$. The variety X_3 is a spherical embedding. Let Y_3 be an orbit in X_3 mapping onto Y. Let Y_1 and Y_2 be the orbits images of Y under the projections to X_1 and X_2. By minimality, Y_1 and Y_2 are the closed orbits in X_1 and X_2. Let $\nu \in \mathcal{V}(G/H)$ with center $\overline{Y_3}$. Then ν has Y_1 and Y_2 for center therefore $\nu \in \mathcal{C}_{Y_1}(X)^\circ \cap \mathcal{C}_{Y_2}(X)^\circ \cap \mathcal{V}(G/H) = \mathcal{C}^\circ \cap (\mathcal{C}')^\circ \cap \mathcal{V}(G/H)$. By (CF2) we get $(\mathcal{C},\mathcal{F}) = (\mathcal{C}',\mathcal{F}')$, thus $X_1 = X_2$ and Y is in the diagonal. $\square$

2. Morphisms

Let $\varphi : G/H \to G/H'$ a dominant (surjective) G-equivariant morphism between homogeneous spherical varieties. This morphism induces a field extension $\mathsf{k}(G/H') \to \mathsf{k}(G/H)$ which in turn induces an injection $\mathsf{k}(G/H')^{(B)} \to \mathsf{k}(G/H)^{(B)}$. Taking weight leads to the injection

$$\varphi^* : \Lambda(G/H') \longrightarrow \Lambda(G/H).$$

Taking duals induces a surjection

$$\varphi_* : \mathcal{Q}(G/H) \longrightarrow \mathcal{Q}(G/H').$$

FACT 9.2.1. *We have the equality* $\varphi_*(\mathcal{V}(G/H)) = \mathcal{V}(G/H')$.

Proof. Any invariant valuation is mapped to an invariant valuation proving the inclusion $\varphi_*(\mathcal{V}(G/H)) \subset \mathcal{V}(G/H')$. Conversely, for a valuation $\nu \in \mathcal{V}(G/H')$, then lift the valuation to an invariant valuation of $\mathsf{k}(G)$ and restrict it to $\mathsf{k}(G/H) = \mathsf{k}(G)^H$. □

DEFINITION 9.2.2. Let $\varphi : G/H \to G/H'$ be a G-equivariant morphism.

1. Define $\mathcal{F}_\varphi = \{D \in \mathcal{D}(G/H) \mid \varphi$ maps D dominantly onto $G/H'\}$.
2. Define $\varphi_* : \Delta(G/H) \setminus \mathcal{F}_\varphi \to \Delta(G/H')$ by $D \mapsto \varphi(D)$.

DEFINITION 9.2.3. Let $\varphi : G/H \to G/H'$ be a G-equivariant morphism.

1. Let $(\mathcal{C}, \mathcal{F})$ and $(\mathcal{C}', \mathcal{F}')$ be colored cones of G/H and G/H' respectively. We say that $(\mathcal{C}, \mathcal{F})$ maps to $(\mathcal{C}', \mathcal{F}')$ is the following conditions holds:

(CM1) $\varphi_*(\mathcal{C}) \subset \mathcal{C}'$;
(CM2) $\varphi_*(\mathcal{F} \setminus \mathcal{F}_\varphi) \subset \mathcal{F}'$.

2. Let $\mathfrak{F}$ and $\mathfrak{F}'$ be colored fans of embeddings of G/H and G/H' respectively. The colored fan $\mathfrak{F}$ maps to $\mathfrak{F}'$ if every colored cone of $\mathfrak{F}$ is mapped to a colored cone of $\mathfrak{F}'$.

THEOREM 9.2.4. *Let $\varphi : G/H \to G/H'$ be a surjective morphism between spherical homogeneous spaces. Let X and X' be embeddings of G/H and G/H' respectively.*

Then φ extends to a morphism $X \to X'$ if and only if $\mathfrak{F}(X)$ maps to $\mathfrak{F}(X')$.

Proof. We may assume that X and X' are simple embedding. Assume that φ extends to such a morphism and let Y be the closed orbit in X. It is mapped to an orbit Y' in X'. This is the closed orbit: if Z is the closed orbit, then $Y \subset \varphi^{-1}(Z)$ thus $Y' \subset Z$ and both are orbits proving the equality. Let $D \in \Delta_Y(X)$. If $\varphi(D)$ is not dense, we have $\overline{\varphi(D)} \in \Delta_{Y'}(X')$. This proves (CM2). We now compare $X_{Y,B}$ and $X'_{Y',B}$. We have the equalities

$$X_{Y,B} = X \setminus \bigcup_{D \in \mathcal{D}(X) \setminus \mathcal{D}_Y(X)} D \text{ and } X'_{Y',B} = X' \setminus \bigcup_{D' \in \mathcal{D}(X') \setminus \mathcal{D}_{Y'}(X')} D'.$$

For $x \in X$, if $\varphi(x) \notin X'_{Y',B}$, then there exists $D' \in \mathcal{D}(X') \setminus \mathcal{D}_{Y'}(X')$ such that $\varphi(x) \in D'$. Let D be an irreducible component of $\varphi^{-1}(D')$ containing x. Then $Y \not\subset D$ so $D \in \mathcal{D}(X) \setminus \mathcal{D}_Y(X)$ and finally $x \notin X_{Y,B}$. This proves that $\varphi(X_{Y,B}) \subset X'_{Y',B}$. In particular, for $f' \in \mathsf{k}[X'_{Y',B}]$, we have $\varphi^* f' \in \mathsf{k}[X_{Y,B}]$. We get $\varphi^*\Lambda^+(X'_{Y',B}) \subset \Lambda^+(X_{Y,B})$ and thus $\varphi_*\mathcal{C}_Y(X) \subset \mathcal{C}_{Y'}(X')$. In particular (CM1) holds.

Conversely, assume that (CM1) and (CM2) hold and consider the open subsets of G/H and G/H' defined by

$$X_0 = G/H \setminus \bigcup_{D \in \mathcal{D}(X) \setminus \mathcal{D}_Y(X)} D \text{ and } X'_0 = G/H' \setminus \bigcup_{D' \in \mathcal{D}(X') \setminus \mathcal{D}_{Y'}(X')} D'.$$

Since $\varphi_*(\mathcal{F} \setminus \mathcal{F}_\varphi) \subset \mathcal{F}'$ and proceeding as above, we get that φ maps X_0 to X'_0. On the other hand we proved the following description:

$$\begin{aligned} &\mathsf{k}[X_{Y,B}] = \{f \in \mathsf{k}[X_0] \mid \nu(f) \geq 0 \text{ for } \nu \in \mathcal{V}(G/H) \cap \mathcal{C}_Y(X)\} \text{ and} \\ &\mathsf{k}[X'_{Y',B}] = \{f' \in \mathsf{k}[X'_0] \mid \nu'(f') \geq 0 \text{ for } \nu' \in \mathcal{V}(G/H') \cap \mathcal{C}_{Y'}(X')\}. \end{aligned}$$

By (CM1), if $f' \in \mathsf{k}[X'_{Y',B}]$, then $\varphi^*(f') \in \mathsf{k}[X_0]$ with $\nu(\varphi^*(f')) = \varphi_*(\nu)(f') \geq 0$ for $\nu \in \mathcal{V}(G/H) \cap \mathcal{C}_Y(X)$. Therefore the map φ extends on these affine subspaces and by G-invariance, it extends to $X = GX_{Y,B}$. □

Definition 9.2.5. The support $\mathrm{Supp}(\mathfrak{F})$ of a colored fan $\mathfrak{F}$ is defined as follows:

$$\mathrm{Supp}(\mathfrak{F}) = \mathcal{V}(G/H) \bigcap \left(\bigcup_{(\mathcal{C},\mathcal{F}) \in \mathfrak{F}} \mathcal{C} \right).$$

Remark 9.2.6. Note that $\mathrm{Supp}(\mathfrak{F}(X)) \subset \varphi_*^{-1}(\mathrm{Supp}(\mathfrak{F}(X'))) \cap \mathcal{V}(G/H)$.

Theorem 9.2.7. *Let $\varphi : X \to X'$ be a dominant morphism extending a surjective morphism $G/H \to G/H'$ between spherical embeddings. Then φ is proper if and only if $\mathrm{Supp}(\mathfrak{F}(X)) = \varphi_*^{-1}(\mathrm{Supp}(\mathfrak{F}(X'))) \cap \mathcal{V}(G/H)$.*

In particular X is proper if and only if $\mathrm{Supp}(\mathfrak{F}(X)) = \mathcal{V}(G/H)$.

Proof. Recall the valuative criterion of properness (see for example [**13**, Theorem II.4.7]). A morphism $\varphi : X \to X'$ is proper if and only if for every valuation ring R with field of fractions K and for every commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "\varphi"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed, "\psi"] & X', \end{tikzcd}$$

we can complete the diagram with a unique morphism $\psi : \mathrm{Spec}(R) \to X$. Note that to prove properness we may assume that the map $\mathrm{Spec}(K) \to X$ is dominant.

Assume that $\nu \in \mathcal{V}(G/H) \setminus \mathrm{Supp}(\mathfrak{F}(X))$ but $\varphi_*\nu \in \mathrm{Supp}(\mathfrak{F}(X'))$. Then $\mathfrak{F}(X), (\mathbb{Q}_{\geq 0}\nu, \emptyset)$ is a colored fan associated to a spherical variety $X_\star$. Furthermore, we have a factorisation

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & X_\star \arrow[d, "\varphi_\star"] \\ & X' \end{tikzcd}$$

Then $X_\star \setminus X = Y_\star$ which is a closed orbit associated to the maximal cone $(\mathbb{Q}_{\geq 0}\nu, \emptyset)$ and $Y' = \varphi_\star(Y_\star)$. Then Y' is not in the image of varphi (since it is the center of $\varphi_*\nu$) thus the image $\varphi(Y)$ is not closed (since its closure contains Y'). □

3. The cone of valuations and Toroidal embedding

We prove that the set of invariant valuations is a convex polyhedral cone. For this, we define a special class of embedding which have a simple combinatorial description and are useful in many other geometric situations.

Let $f_1, \cdots, f_s \in \mathsf{k}[G]^{(H)}$ and let $M_i \subset \mathsf{k}[G]^{(H)}$, the submodule generated by f_i for all $i \in [1, s]$. Denote by $M_1 \cdots M_s \subset \mathsf{k}[G]$ the G-submodule generated by all products of s elements respectively in $M_1, \cdots, M_s$. Any element in M_i is of the form $\sum_j g_j \cdot f_i$ with $g_j \in G$.

Any $f \in M_1 \cdots M_s$ is a linear combinaison of product $\prod_i \sum_j g_j \cdot f_i$. Thus $f \in \mathsf{k}[G]^{(H)}$ has the same H-weight as the product $f_1 \cdots f_s$. Therefore $f_1 \cdot f_s f^{-1} \in \mathsf{k}(G)^H = \mathsf{k}(G/H)$. By the above description of f, we have for $\bar{\nu} \in \mathcal{V}(G)$, the inequality $\bar{\nu}(f) \geq \sum_i \bar{\nu}(f_i)$. Since any $\nu \in \mathcal{V}(G/H)$ can be extended to $\bar{\nu} \in \mathcal{V}(G)$, we have

$$\nu(f_1 \cdots f_s f^{-1}) \leq 0,$$

for all $f_1, \cdots, f_s \in \mathsf{k}[G]^{(H)}$ and all $f \in M1 \cdots M_s$.

Definition 9.3.1. In the above situation, assume furthermore that $f_i \in \mathsf{k}[G]^{(B \times H)}$ for all $i \in [1, s]$ and $f \in (M_1 \cdots M_s)^{(B)}$. We have $f_1 \cdots f_s f^{-1} \in \mathsf{k}(G/H)^{(B)}$

1. Define the weight $\gamma(f_1, \cdots, f_s, f) = \lambda_{f_1} + \cdots + \lambda_{f_s} - \lambda_f$.

2. Let $\Gamma_0^+ \subset \Lambda(X)$ be the set of all weights $\gamma(f_1, \cdots, f_s, f)$ where $f_1, \cdots, f_s \in \mathsf{k}[G]^{(B \times H)}$ run over all possible choices as above.

The root lattice Γ of G/H is the sublattice of $\Lambda(G/H)$ generated by Γ_0^+. The root monoid Γ^+ of G/H is the monoid $\Gamma \cap \mathrm{Cone}(\Gamma_0^+)$.

Lemma 9.3.2. *As a subset of $\Lambda(T)$, the root monoid Γ^+ is contained in the monoid generated by the simple roots.*

Proof. Consider $\gamma = \gamma(f_1, \cdots, f_s, f)$, with $f_1, \cdots, f_s, f \in \mathsf{k}[G]^{(B)}$. Then f_i is a highest weight of M_i and for any T-weight λ of M_i, the difference $\lambda_{f_i} - \lambda$ is a sum of simple roots. The result follows. □

Proposition 9.3.3. *The set of invariant valuations $\mathcal{V}(G/H)$ is a cone. More precisely, we have*

$$\mathcal{V}(G/H) = \{v \in \mathcal{Q}(G/H) \mid \langle v, \gamma \rangle \leq 0 \textit{ for all } \gamma \in \Gamma^+\}.$$

Proof. By the above discussion, the inclusion from the left hand side in the right hand side holds. Let $v \in \mathcal{Q}(G/H) \setminus \{0\}$ such that $\langle v, \gamma \rangle \leq 0$ for all $\gamma \in \Gamma^+$. Define $(\mathcal{C}, \mathcal{F}) = (\mathbb{Q}_{\geq 0} v, \emptyset)$. Then $\mathcal{C}^\vee$ is a polyhedral cone generated by finitely many weights $\lambda_1, \cdots, \lambda_s$ and for all i, we choose $g_1, \cdots, g_s \in \mathsf{k}(G/H)^{(B)}$ such that $\lambda_{g_i} = \lambda_i$. Let $f_0 \in \mathsf{k}[G]^{(B \times H)}$ vanishing on all divisors $D \in \mathcal{D}(G/H)$ and such that $f_i = f_0 g_i \in \mathsf{k}[G]$. With notation as above, let M be the G-submodule of $\mathsf{k}[G]$ generated by the $(f_i)_{i \in [0,s]}$. We have a morphism $\varphi : G/H \to \mathbb{P}(M^\vee)$. Let $X_0' = \{x' \in \overline{\varphi(G/H)} \mid f_0(x') \neq 0\}$ and $X' = G X_0'$.

Let $f' \in \mathsf{k}(G/H)^{(B)}$ such that $\langle \lambda_{f'}, v \rangle \geq 0$. Write $\lambda_{f'} = \sum_i a_i \lambda_{g_i}$ with $a_i \geq 0$ for all i and define $F = \prod_{i=1}^n g_i^{a_i} = \prod_{i=1}^n (f_i/f_0)^{a_i} \in \mathsf{k}[X_0']^{(B)}$. Then $\lambda_{f'} = \lambda_F$ thus $f'/F \in \mathsf{k}(G/H)^B = \mathsf{k}$ is constant. Therefore f' is a multiple of $F \in \mathsf{k}[X_0']^{(B)}$ and $\{f' \in \mathsf{k}(G/H)^{(B)} \mid \langle \lambda_{f'}, v \rangle \geq 0\} \subset \mathsf{k}[X_0']$. Conversely, let $f' \in \mathsf{k}[X_0']^{(B)}$. Then

$f' = f/f_0^n$ with $f \in (M_0^{n_0} M_1^{n_1} \cdots M_s^{n_s})^{(B)}$ and $n = \sum_i n_i$. We have

$$\begin{aligned} n_1\lambda_1 + \cdots + n_s\lambda_s - \lambda_{f'} &= n_1(\lambda_{f_1} - \lambda_{f_0}) + \cdots + n_s(\lambda_{f_s} - \lambda_{f_0}) - \lambda_{f'} \\ &= n_0\lambda_{f_0} + n_1\lambda_{f_1} + \cdots + n_s\lambda_{f_s} - \lambda_f \in \Gamma^+. \end{aligned}$$

In particular $\langle v, \lambda_{f'} \rangle \geq \langle v, \lambda_1 \rangle + \cdots + \langle v, \lambda_s \rangle \geq 0$ therefore we have the equality $\mathsf{k}[X_0'] = \{f' \in \mathsf{k}(G/H)^{(B)} \mid \langle \lambda_{f'}, v \rangle \geq 0\}$.

The same proof as in Lemma 8.1.13 shows that the fibers of φ are finite. We therefore get a normal G-variety X such that φ factors through a spherical embedding $G/H \to X$ and a finite morphism $\psi : X \to X'$. Set $X_0 = \psi^{-1}(X_0')$. This is an affine B-stable open subset of X. By the above argument $\mathsf{k}[X_0'] = \{f' \in \mathsf{k}(G/H)^{(B)} \mid \langle \lambda_{f'}, v \rangle \geq 0\} \subset \mathsf{k}[X_0]^{(B)}$ and the same argument as in Lemma 8.1.14 gives the equality $\mathsf{k}[X_0]^{(B)} = \mathsf{k}[X_0']^{(B)}$. For $D \in \mathcal{D}(G.H)$, we have $\psi(D) \cap X_0' = \emptyset$ therefore $X_0 \cap G/H = BH/H$. Now $\mathsf{k}[X_0]^{(B)} \subsetneq \mathsf{k}(G/H)^{(B)} \subset \mathsf{k}[BH/H]^{(B)}$. In particular X_0 meets a non dense G-orbit $Y \subset X$. Let $\nu_Y \in \mathcal{V}(G/H)$ be a valuation with center Y. Then $\nu_Y|_{\mathsf{k}[X_0]^{(B)}} \geq 0$ therefore $\nu \subset \mathbb{Q}_{\geq 0}\nu_Y$ proving the result. $\square$

We will see that this cone is actually a polyhedral cone.

Definition 9.3.4. A spherical variety X is called toroidal if $\Delta_Y(X) = \emptyset$ for all G-orbit $Y \subset X$.

Remark 9.3.5. Define the set of B-stable prime divisors containing no G-orbit by

$$\mathring{\Delta}(X) = \Delta(X) \setminus \bigcup_Y \Delta_Y(X) = \mathcal{D}(X) \setminus \bigcup_Y \mathcal{D}_Y(X)$$

where Y runs in the set of G-orbits in X. The embedding $G/H \to X$ is toroidal if and only if $\Delta(X) = \mathring{\Delta}(X)$.

Suppose that $G/H \to X$ is a toroidal embedding and let $Y \subset X$ be a closed G-orbit. By definition we have $\mathcal{C}_Y(X) \subset \mathcal{V}(G/H)$. If moreover X is complete and toroidal, then $\mathcal{V}(/G) = \cup_Y \mathcal{C}_Y(X)$, where Y runs in the set of closed orbits of X. In particular, every complete toroidal embeddings corresponds to a subdivision of $\mathcal{V}(G/H)$ into strictly convex cones, and a simple complete toroidal embedding exists if and only if $\mathcal{V}(G/H)$ is a strictly convex cone. In general such an embedding needs not to exists, on the other hand complete toroidal embeddings always do exist.

Proposition 9.3.6. *Let G/H be a spherical homogeneous space, then G/H admits a complete toroidal embedding.*

Proof. Let $\pi : G \to G/H$ be the projection. Let $f_0 \in \mathsf{k}[G]^{(B\times H)}$ be a function vanishing on $\pi^{-1}(D)$ for all $D \in \Delta(G/H)$ and let $V \subset \mathsf{k}[G]$ be the G-module generated by f_0. This induces a G-equivariant map $G/H \to \mathbb{P}(V^\vee)$. Let X' be the closure of the image and $\psi : X \to X'$ the induced map. If $Z \subset X'$ is the locus defined by the vanishing of f_0, then $\psi(D) \subset Z$ for all $D \in \Delta(G/H)$. If $Y \subset Z$ is a G-orbit, then $f_0|_Y = 0$ and also $(g \cdot f_0)|_Y = 0$ for all $g \in G$ and finally $f|_Y = 0$ for all $f \in V$. This implies $Y = \emptyset$, a contradiction.

Let $G/H \to \bar{X}$ be a G-equivariant completion of G/H and define X as the normalisation of the closure of the image of the diagonal map $G/H \to X' \times \bar{X}$. Since both X' and $\bar{X}$ are proper, so is X. Let $D \in \Delta(X)$, then the image of D in X' under the first projection is contained in Z and therefore contains no G-orbit. So X is a toroidal embedding. $\square$

COROLLARY 9.3.7. *The set of invariant valuations $\mathcal{V}(G/H)$ is a polyhedral cone.*

Proof. Let X be a toroidal proper embedding of G/H. Then $\mathcal{V}(G/H)$ is the union of the cones of $\mathfrak{F}(X)$. As there are finitely many such cones all of which are polyedral the result follows. $\square$

EXAMPLE 9.3.8. Recall Example 5.2.4. Let $G = \mathrm{SL}_2(\mathsf{k})$, let $H = T$ be the subgroup of diagonal matrices and let B be the subgroup of upper triangular matrices. Denote by α the positive root, by ϖ_α the fundamental weight. We have $\Lambda(X) = \mathbb{Z}\alpha$ and $\mathcal{Q}(G/H) = \mathbb{Q}\alpha^\vee$. We also have $\Delta(G/H) = \{D^+, D^-\}$ with $\rho(D^+) = \rho(D^-) = \frac{1}{2}\alpha^\vee$.

Note that G/H has the following two embeddings:

(1) G/H the homogeneous space.
(2) $X = \mathbb{P}^1 \times \mathbb{P}^1$ with diagonal action of $\mathrm{SL}_2(\mathsf{k})$.

We will prove that these are the only possible embedding.

Consider X given by embedding (2). The dense orbit is $G.([1:0],[0:1])$ and X has a unique closed orbit $Y = G.([1:0],[1:0])$ which is the diagonal in X. The variety X has only two G-orbits Y and $X \setminus Y = G/H$. Note in particular that X is simple and no color contains Y (since Y is a divisor). This means that $(\mathcal{C}_Y(X), \Delta_Y(X)) = (\mathcal{V}(G/H), \emptyset)$.

We use this to compute $\mathcal{V}(G/H)$. One easily checks that the open affine B-chart is given by $X_{Y,B} = X \setminus (\{0\} \times \mathbb{P}^1 \cup \mathbb{P}^1 \times \{0\}) = \mathrm{Spec}\left(\mathsf{k}\left[\frac{a}{b}, \frac{c}{d}\right]\right)$ with coordinates $([a:b],[c:d])$ in X. Furthermore $\lambda_{\frac{a}{b}} = \lambda_{\frac{c}{d}} = -\alpha$. Therefore, the weights occuring in $\mathsf{k}[X_{Y,B}]^{(B)}$ form the cone $\mathbb{Q}_{\geq 0}(-\alpha)$. We deduce that $\mathcal{V}(G/H)$ is the dual of this cone thus $\mathcal{V}(G/H) = \mathbb{Q}_{\geq 0}(-\alpha^\vee)$.

It is now easy to use the definition of colored cones and fans to see that there are only two possible colored cones :

(1) $(0, \emptyset)$ and
(2) $(\mathbb{Q}_{\geq 0}(-\alpha), \emptyset)$

These two colored cones correspond to the above two embeddings and there is no other embedding.

EXAMPLE 9.3.9. Recall Example 8.1.20 with $G = \mathrm{SL}_2(\mathsf{k})$ and $H = U$ the maximal unipotent subgroup of unipotent upper-triangular matrices. We have $\Lambda(G/H) = \mathbb{Z}\varpi_\alpha$ and $\mathcal{Q}(G/H) = \mathbb{Q}\alpha^\vee$. We also have $\Delta(G/H) = \{D\}$ with $\rho(D) = \alpha^\vee$. Finally, we have $\mathcal{V}(G/H) = \mathcal{Q}(G/H)$. We list the possible colored fans and the associated embeddings.

Colored fans:

(1) $(0, \emptyset)$
(2) $(\mathbb{Q}_{\geq 0}(\alpha), \{D\})$
(3) $(\mathbb{Q}_{\geq 0}(-\alpha), \emptyset)$
(4) $(\mathbb{Q}_{\geq 0}(\alpha), \emptyset)$
(5) $((\mathbb{Q}_{\geq 0}(-\alpha), \emptyset), (\mathbb{Q}_{\geq 0}(\alpha), \{D\}))$
(6) $((\mathbb{Q}_{\geq 0}(-\alpha), \emptyset), (\mathbb{Q}_{\geq 0}(\alpha), \emptyset))$

Associated embeddings:

(1) $G/H = \mathbb{A}^2 \setminus \{0\}$
(2) $\mathbb{A}^2$

(3) $\mathbb{P}^2 \setminus \{0\}$
(4) $\mathrm{Bl}_0(\mathbb{A}^2)$
(5) $\mathbb{P}^2$
(6) $\mathrm{Bl}_0(\mathbb{P}^2)$

where $\mathrm{Bl}_0(\mathbb{A}^2)$ and $\mathrm{Bl}_0(\mathbb{P}^2)$ are the blow-ups of $\mathbb{A}^2$ and $\mathbb{P}^2$ at the origin 0. It is also easy to check on the colored fans that we have maps as follows between the embeddings:

$$\begin{tikzcd}
\mathbb{A}^2 \setminus \{0\} \arrow[r] \arrow[d] & \mathbb{P}^2 \setminus \{0\} \arrow[d] \\
\mathbb{A}^2 \arrow[r] & \mathbb{P}^2 \\
\mathrm{Bl}_0(\mathbb{A}^2) \arrow[r] \arrow[u] & \mathrm{Bl}_0(\mathbb{P}^2). \arrow[u]
\end{tikzcd}$$

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APPENDIX A

Principal bundles

In this chapter be explain some constructions used in the text especially contracted products. We start with finite groups.

1. Galois and unramified coverings

1.1. Existence of quotients by finite groups.

LEMMA A.1.1. *Let A be a finitely generated* k*-algebra and G a finite group acting on A. Then A^G is finitely generated.*

Proof. The ring A is integrally closed over A^G. Indeed, for $a \in A$, we have the equation

$$\prod_{g\in G}(a - g\cdot a) = 0.$$

Let $a_1, \cdots a_n$ be generators of A as an algebra and let P_i be equations for the a_i over B. Let C be the subalgebra of B spanned by the coefficients of the P_i. The elements a_i are integral over C thus A is a finite module over C. But B is a sub-C-module of A and C is noetherian (because $C = \mathsf{k}[\text{coefficients of the } P_i]$. Thus B is also finite over C and $B = C[b_1, \cdots, b_k]$, the result follows. □

PROPOSITION A.1.2. *Let $X = \operatorname{Spec}(A)$ be affine and G be a finite group acting on X. Let $X/G = \operatorname{Spec}(A^G)$ and $\pi : X \to X/G$ be the induced morphism.*

1. *The morphism π is constant on the orbits of G and any morphism $X \to Z$ constant on the G-orbits factors through X/G.*
2. *The morphism π is finite.*
3. *The variety X/G has the quotient topology.*
4. *The fibers of π contain a unique closed orbit.*

REMARK A.1.3. Note that as any orbit if finite and therefore closed, the last condition implies that X/G is the set of orbits justifying *a posteriori* the notation.

COROLLARY A.1.4. *Assume that X is a variety such that any finite set is contained in an affine open subset (for example X is quasi-projective). Then there exists an algebraic variety X/G with a morphism $\pi : X \to X/G$ constant on the G orbits and such that for any morphism $\phi : X \to Z$ constant on the G-orbits, there exists a morphism $\psi : X/G \to Z$ with $\phi = \psi \circ \pi$.*

Proof. The condition tells us that X can be covered by affine subsets (U_i) stable under the action of X. The quotient exists on U_i and by the universal property is unique. Therefore on $U_i \cap U_j$ the two quotients coming from U_i and U_j are isomorphic. We can glue them to get the quotient of X. □

REMARK A.1.5. If we replace G by a reductive group, then it is harder to find a mild condition to replace the condition that any finite subset of X is contained in an affine open subset which ensured the existence of an affine covering stable under the action. This is where the so called stability conditions are needed. We shall not enter this subject. For more details, see for example [**19**].

1.2. Unramified covers.

DEFINITION A.1.6. A morphism $f : X \to Y$ is called unramified if the condition $\Omega^1_{X/Y} = 0$ is satisfied. Unramified finite morphism are called an unramified cover.

REMARK A.1.7. Few remarks on unramified morphisms.

(1) There are several equivalent conditions for a morphism to be unramified (see for example [**11**, IV_4 Théorème 17.4.1]). A map $f : X \to Y$ is unramified if and only if one of the following equivalent conditions are satisfied
   (a) the diagonal morphism $X \to X \times_Y X$ is open;
   (b) for any point y in Y, the fiber of π over y is a disjoint union of reduced points (here we assume that k is algebraic closed).
(2) A finite cover is not necessarily étale. To get the étale property, we need to add the assumption that the morphism is flat. If $f : X \to Y$ is a non ramified cover between irreducible varieties, then there is an open subset where f is flat and therefore étale.
(3) For $f : X \to Y$ a separable morphism between irreducible varieties (*i.e.* such that $\mathsf{k}(X)/\mathsf{k}(Y)$ is separable) then set of point in Y such that the morphism is separable is open and dense.

Let us state the following two facts that we shall use without proof.

FACT A.1.8. [**11**, IV_4 Proposition 17.3.3] *Unramified covers are stable under base change.*

FACT A.1.9. [**24**, Section 1.4] *If $\pi : X \to X/K$ is the quotient of a variety X by a finite groups K, then π is an unramified cover if and only if K acts freely on X. These unramified covers are called Galois covers.*

LEMMA A.1.10. *Let $f : X \to Y$ be an unramified cover. There exists a Galois cover $\pi : Z \to Y$ such that X is a partial quotient of Z.*

Proof. Let n be the degree of f and consider X^n_Y the n-fold fibered product of X over Y. Remove the (open because of the unramification and closed) subset of points fixed by at least one non trivial element in $\mathfrak{S}_n$ acting by permuting the points. Then the complement Z maps to Y and this map $Z \to Y$ is an unramified covering. We have $X = Z/\mathfrak{S}_{n-1}$ and $Y = Z/\mathfrak{S}_n$ (the degree of the maps $Z/\mathfrak{S}_{n-1} \to X$ and $Z/\mathfrak{S}_n \to Y$ are both 1). □

LEMMA A.1.11. *Let $\pi : X \to X/K$ be a Galois cover of group K and let $f : Y \to X/K$ be a morphism, then the base change morphism $X \times_{X/K} Y \to Y$ is again a galois cover.*

Proof. Define the action on $X \times_{X/K} Y$ by $\sigma \cdot (x, y) = (\sigma(x), y)$ and let Z be the quotient. The map $X \times_{X/K} Y \to Y$ is constant on the K-orbits thus we have a morphism $Z \to Y$ and therefore a commutative diagram:

$$\begin{tikzcd} X \times_{X/K} Y \arrow[r] \arrow[d] & Y \\ Z. \arrow[ur] & \end{tikzcd}$$

Both maps starting from $X \times_{X/K} Y$ are of degre $|K|$ thus $Z \to Y$ is of degree 1 and therefore an isomorphism. $\square$

2. Principal bundles

2.1. Isotrivial bundles and special groups.

DEFINITION A.2.1. A principal bundle of group G over X is a morphism $f : P \to X$ with a faithful right action of G on P such that f is G-equivariant for the trivial action of G on X.

DEFINITION A.2.2. Let $f : P \to X$ be a G-principal bundle.

(1) The fibration is called trivial if there is an isomorphism $P \simeq X \times G$ such that f is the first projection.
(2) The fibration is called isotrivial if there exists an unramified cover $X' \to X$ such that the pull-back of the fibration to X' (obtained by base change) is trivial.
(3) A fibration is called locally trivial if there exists a open covering $(U_i)_{i \in I}$ of X (for the Zariski topology) such that the restriction of the fibration to U_i is trivial for all $i \in I$.
(4) A fibration is called locally isotrivial if there exists a open covering $(U_i)_{i \in I}$ (for the Zariski topology) and unramified maps $U_i' \to U_i$ such that pull-back to U_i' of the restriction of the fibration to U_i is trivial for all $i \in I$.

REMARK A.2.3. It can be proved, see **[12]** that if G is a linear algebraic group, then any principal bundle is locally isotrivial.

LEMMA A.2.4. *Let $f : P \to X$ be a principal G-bundle and let $\pi : X' \to X$ be a Galois cover of group K.*

(1) *Assume that the pull-back $X' \times_X P$ is trivial over X', then the action of K on $X' \times G$ is given by morphisms $f_\sigma : X' \to G$ for $\sigma \in K$ such that*
$$\sigma \cdot (x, g) = (\sigma(x), f_\sigma(x)g).$$
(2) *Furthermore, the principal bundle $f : P \to X$ is trivial if (and only if) there is a morphism $a : X' \to G$ such that*
$$f_\sigma(x) = a(\sigma(x))^{-1}a(x).$$

REMARK A.2.5. The classes of families $(f_\sigma)_{\sigma \in K}$ such that the above formula gives an action modulo the classes of functions of the form $f_\sigma(x) = a(\sigma(x))^{-1}a(x)$ is a pointed set usually denoted by $H^1(K, \mathrm{Hom}(X', G))$.

Proof. (1) We know that the base change of the Galois cover is again a Galois cover thus we have an action of K on $X' \times G$. This action has to induce an equivariant map $X' \times G \to X'$ thus $\sigma(x, g) = (\sigma(x), a_\sigma(x, g))$. Furthermore, the action has to respect the G-action *i.e.*
$$\sigma(x, gh) = \sigma(x, g)h.$$

In particular $\sigma(x,g) = \sigma(x,e)g$ therefore $\sigma(x,g) = (\sigma(x), a_\sigma(x,e)g)$ proving (1) by setting $f_\sigma(x) = a_\sigma(x,e)$. Note that the associativity of the action gives the cocycle condition $f_{\sigma\tau} = f_\tau \circ \sigma \cdot f_\sigma$.

(2) Consider the composition $X' \times G \to X' \times G \to (X \times G)/K$ whose first map is given by $(x,g) \mapsto (x, a(x)^{-1}g)$ and second map is given by the quotient of the action given by $f_\sigma(x) = a(\sigma(x))^{-1}a(x)$. This map is constant on the orbits of the action $\sigma \cdot (x,g) = (\sigma(x), g)$ and therefore factors through $X/K \times G$. The same argument gives the inverse map. □

Exercise A.2.6. With the assumptions of the previous lemma.

(1) Prove the converse statement of (2) in the previous lemma.
(2) Prove that if we define an action on $X' \times G$ as in the above lemma with the cocycle condition $f_{\sigma\tau} = f_\tau \circ \sigma \cdot f_\sigma$, then the quotient is a principal G-bundle over X.

Definition A.2.7. A group G is called special if any isotrivial principal bundle of group G is locally trivial.

Remark A.2.8. We have the following results on special groups, see **[24, 10]**.

(1) Any special group is connected and linear.
(2) Connected solvable groups are special.
(3) The groups GL, SL or Sp are special.
(4) The groups PGL, SO or Spin are not special.
(5) A subgroup G of GL is special if and only if the fibration $\mathrm{GL} \to \mathrm{GL}/G$ is locally trivial.
(6) There is a complete classification of special groups, see **[10]**.

Let us now prove that GL is special (Hilbert's Theorem 90).

Theorem A.2.9. *Any isotrivial* GL*-principal fibration is locally trivial.*

Proof. Let $P \to X$ be a locally isotrivial principal GL fibration. We thus have an unramified covering $\pi : X' \to X$ such that $X' \times_X P$ is trivial *i.e.* isomorphic to $X' \times \mathrm{GL}$. We want to prove that P is trivial. It is enough to prove this for $X' \to X$ a Galois covering in view of Lemma A.1.10.

Let $X' \to X$ be a Galois covering trivialising P *i.e.* $X' \times_X P \simeq X' \times \mathrm{GL}$. We need to prove that locally the *cocycle* $(\varphi_\sigma)_{\sigma \in K} \in \mathrm{Hom}(X', \mathrm{GL})^K$ defining the action of K on $X' \times \mathrm{GL}$ comes from a boundary *i.e.* is of the form $\varphi_\sigma(x) = a(\sigma(x))^{-1}a(x)$ for $a \in \mathrm{Hom}(X', \mathrm{GL})$.

For this let $x \in X$, we will work locally around x. Consider the scheme $\pi^{-1}(x)$. This is a discrete disjoint union of 0-dimensional irreducible schemes. Let $A(x)$ be the (semi)local ring $\mathcal{O}_{X',\pi^{-1}(x)}$. Let x' be a point in $\pi^{-1}(x)$ and pick an element h in $\mathrm{GL}(A(x))$ with $h(x') = \mathrm{id}$ and $h(y) = 0$ for $y \in \pi^{-1}(x)$ and $y \neq x'$. Define

$$a = \sum_{\sigma \in K} h \circ \sigma \cdot \varphi_\sigma \in \mathrm{GL}(A(x)).$$

We can now check the following equalities:

$$a \circ \sigma \cdot \varphi_\sigma = \sum_{\tau \in K} h \circ \tau \circ \sigma \cdot \varphi_\tau \circ \sigma \cdot \varphi_\sigma = \sum_{\tau \in K} h \circ \tau \circ \sigma \cdot \varphi_{\tau\sigma} = a$$

the second equality coming from the cocycle condition. □

2.2. Existence of some quotients. Let G be an algebraic group and let H be a closed subgroup.

PROPOSITION A.2.10. *The quotient morphism $\pi : G \to G/H$ is a locally isotrivial H-principal bundle.*

In other words, there exists a covering of G/H by open subsets $(U_i)_{i\in I}$ and unramified coverings $\varphi_i : U_i' \to U_i$ such that the map $\pi : G \to G/H$ trivialises when pulled-back to U_i'.

Proof. Because the morphism is equivariant and G/H homogeneous, it is enough to check that there exists a non trivial open subset U of G/H with an unramified covering $\varphi : U' \to U$ such that the fibration π trivialises on U'.

Let G^0 be the connected component of G and let $H_0 = H \cap G^0$. The variety G^0/H_0 is irreducible and it is the connected component of G/H at the image of e the unit of G. Because π is separable, the extension $k(G^0) \to k(G^0/H_0)$ is separable. Consider the map on local rings $(\mathcal{O}_{G/H,\bar{e}}, \mathfrak{m}_{G/H,\bar{e}}) \to (\mathcal{O}_{G,e}, \mathfrak{m}_{G,e})$. Because the morphism is separable, the corresponding map $\mathfrak{m}_{G/H,\bar{e}}/\mathfrak{m}^2_{G/H,\bar{e}} \to \mathfrak{m}_{G,e}/\mathfrak{m}^2_{G,e}$ is injective. Pick a subspace $\mathfrak{n}$ of $\mathfrak{m}_{G,e}$ such that its image in the quotient is a supplementary of the image of this injection. Let I be the ideal in $\mathcal{O}_{G,e}$ spanned by $\mathfrak{n}$. Then the local ring $(\mathcal{O}_{G,e}/I, \mathfrak{m}_{G,e}/I)$ is the local ring of a subvariety X in G containing e whose tangent space is supplementary to that of H. The map $\pi : X \to G/H$ is therefore separable at e and $\dim X = \dim G/H$. Thus the map is quasi-finite and this implies that there exist an open dense subset U of G/H such that if we set $U' = X \cap \pi^{-1}(U)$, the morphism $\varphi = \pi|_{U'} : U' \to U$ is finite and thus an unramified covering (see last semester lecture Theorem 6.2.25).

We now only need to check that π trivialises when restricted to U'. We look at the pull-back diagram

$$\begin{tikzcd} U' \times_{G/H} G \arrow[r] \arrow[d] & G \arrow[d, "\pi"] \\ U' \arrow[r, "\varphi"] & U \subset G/H. \end{tikzcd}$$

We want to prove that $U' \times H$ is isomorphic to $U' \times_{G/H} G$. For this we check the universal property of the product. We have a natural map $\phi : U' \to G$ (the inclusion) such that $\varphi \circ \pi = \mathrm{id}_U$. We may thus define maps $U' \times H \to G$ and $U' \times H \to U'$ by $(u, h) \mapsto \phi(u)h$ and $(u, h) \mapsto u$. This map obviously factors through the fibered product. If we have maps $a : Z \to G$ and $b : Z \to U'$ with $\pi \circ a = \varphi \circ b$ then we define $Z \to U' \times H$ by $z \mapsto (a(z), a(z)^{-1}\phi(b(z)))$. This concludes the proof. $\square$

COROLLARY A.2.11. *Let H be a closed subgroup of an algebraic group G and let X be a variety with a left action of H. Assume furthermore that any finite set of points in X is contained in an affine open subset (for example X quasi-projective). Let us define a right action of H on $G \times X$ by $h \cdot (g, x) = (gh, h^{-1}x)$.*

(1) *Then there exists an unique structure of algebraic variety on the set $G \times^H X$ of H-classes in $G \times X$. The morphism $G \times X \to G \times^H X$ is flat and separable.*
(2) *There is an action of G on $G \times^H X$.*

(3) *There is a G-equivariant morphism $G \times^H X \to G/H$ which is isotrivial with fibers isomorphic to X.*

Proof. The quotient being the solution of an universal problem. If it exists it is unique therefore we only need to construct it locally. By uniqueness the resulting quotients will glue together.

Since the map $\pi : G \to G/H$ it is locally isotrivial, we first consider an open subset U and an unramified covering $\varphi : U' \to U$ such that we have a trivialisation $\pi^{-1}(U') \simeq U' \times H$ and thus we get an isomorphism $\pi^{-1}(U') \times X = U' \times H \times X$. Furthermore, the action is given by $h \cdot (u, h', x) = (u, h'h, h^{-1}x)$. In particular on this open set, there is a quotient isomorphic to $U' \times X$. Indeed, we have a morphism $\phi : U' \times H \times X \to U' \times X$ defined by $(u, h, x) \mapsto (u, hx)$. This morphism is contant on the H-orbits. Furthermore, for any morphism $\psi : U' \times H \times X \to Z$ which is constant on the H-orbits, we may define $\bar{\psi} : U' \times X \to Z$ simply by composition with the map $U' \times X \to U' \times H \times X$ given by $(u, x) \mapsto (u, e, x)$. This map is a section of the quotient map ϕ thus $\bar{\psi}$ factorises ψ.

To prove the existence of the quotient on U, we only need to *descent* from U' to U. But the morphism $\varphi : U' \to U$ is an unramified cover. By taking another covering, we may assume that $U = U'/K$ with K a finite group (see Lemma A.1.10). We may thus assume that $U' \to U$ is given as the quotient by a finite group K. Therefore the pull-back $U' \times H \to \pi^{-1}(U)$ is also given by a quotient of an action of K. Because of the compatibility with the first projection and the action of H, the action is given by $\sigma \cdot (u, h) = (\sigma(u), f_\sigma(u)h)$ with $f_\sigma : U' \to H$ a morphism. We may therefore define an action of K on $U' \times X$ by $\sigma \cdot (u, x) = (\sigma(u), f_\sigma(u) \cdot x)$. By our assumption on X there is a quotient of $U' \times X$ by K. For this quotient we have the diagram

$$\begin{tikzcd}
U' \times H \times X \arrow[r] \arrow[d] & \pi^{-1}(U) \times X \arrow[d, dashed] \\
U' \times X \arrow[r] \arrow[d] & (U' \times X)/K \arrow[d, dashed] \\
U' \arrow[r] & U
\end{tikzcd}$$

that we want to complete with the dashed arrows to get a commutative diagram. But the composition morphism $U' \times H \times X \to U' \times X \to (U' \times X)/K$ is constant on the K-orbits thus factors through $(U' \times H \times X)/K = \pi^{-1}(U) \times X$. This gives the first right vertical arrow. Now because the top square is commutative we get that the map $\pi^{-1}(U) \times X$ is constant on the H-orbits. We need to check that it satisfies the universal property of the quotient. If $\psi : \pi^{-1}(U) \times X \to Z$ is constant on the H-orbits, then the composition $U' \times H \times X \to \pi^{-1}(U) \times X \to Z$ is constant on the H-orbits and thus factors through $U' \times X$. Furthermore the above composition and therefore the induced map $U' \times X \to Z$ is constant on the K-orbits thus it factors through $(U' \times X)/K$. The existence of the last dashed arrow comes from the universal property of the quotient $U = U'/K$.

Again, because of the universal property of the quotient, the quotients on open subsets with trivialisation on an unramified covering will patch together to give a global quotient $G \times^H X$ which furthermore has a morphism to G/H (because it is the case locally) which is locally isotrivial.

Note that the morphism $G \times G \times X \to G \times X$ defined by left multiplication on G is equivariant under the H-action thus by the same construction for the H-action on $G \times G \times X$, this induces a morphism $G \times G \times^H X \to G \times^H X$ and it is easy to check that this morphism defines an action. $\square$

Remark A.2.12. With the above assumptions.

(1) If H is a parabolic subgroup, then the map $G \to G/H$ is locally trivial for the Zariski topology and the result is even easier.
(2) This result is a special case of faithfully flat descent (see [**12**]): indeed the map $G \to G/H$ is faithfully flat and there is a locally trivial fibration with fiber isomorphic to X over G: the trivial fibration $G \times X \to G$ therefore by faithfully flat descent, there exists a fibration $G \times^H X \to G/H$ with fibers isomorphic to X such that the following diagram is Cartesian:

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d] & G \arrow[d] \\ G \times^H X \arrow[r] & G/H. \end{tikzcd}$$

Corollary A.2.13. *Let $X' \to X$ be a Galois covering of Galois group K and let $\rho : K \to \mathrm{GL}(V)$ be a representation of K. Consider the action of K on $X' \times V$ defined by $\sigma(x, v) = (\sigma(x), \rho(\sigma)(v))$.*

Then the quotient $X' \times^K V := (X' \times V)/K$ is a vector bundle over $X'/K = X$ i.e. *locally trivial.*

Proof. Consider the trivial principal $\mathrm{GL}(V)$ bundle $X' \times \mathrm{GL}(V)$ and the action of K on it induced by the representation ρ. The quotient $X' \times^K \mathrm{GL}(V)$ has a morphism to $X'/K = X$ and is an isotrivial principal $\mathrm{GL}(V)$ bundle. By the above result, we may assume that this principal bundle is trivial over X (by restriction to an open subset). The above fibration $X' \times^K V \to X'/K$ is obtained from $X' \times^K \mathrm{GL}(V) \to X'/K$ as follows:

$$X' \times^K V = (X' \times^K \mathrm{GL}(V)) \times^{\mathrm{GL(V)}} V \simeq (X'/K) \times (\mathrm{GL}(V) \times^{\mathrm{GL(V)}} V) \simeq X'/K \times V.$$

Proving the result. $\square$

Example A.2.14. A very special case of the above construction is the following. Let V be a linear representation of H, then $G \times^H V \to G/H$ is a vector bundle over G/H with fibers isomorphic to V. This is the very first example of linearised vector bundle.

Note that if the action of H on V extends to an action of G, then the bundle is trivial. Indeed, we have the trivialisation morphisms given by $(\bar{g}, v) \mapsto \overline{(g, g^{-1} \cdot v)}$ and $\overline{(g, v)} \mapsto (\bar{g}, g \cdot v)$.

Note also that we only proved that the fibration $G \times^H V \to G/H$ is isotrivial. But as GL is special it is locally trivial and thus a vector bundle.

APPENDIX B

Linearisation of line bundles

1. First definitions

Let G be a linear algebraic group and let X be a variety acted on by G.

Definition B.1.1. A G-linearisation of a vector (line) bundle $\pi : L \to X$ is a G-action on L given by $\Phi : G \times L \to L$ such that

(1) the morphism $\pi : M \to X$ is G-equivariant and
(2) the action of G on the fibers is linear *i.e.* for all $x \in X$ and $g \in G$, the map $\phi_{g,x} : L_x \to L_{gx}$ is linear.

We shall mainly consider G-linearised line bundles but in the following lemma we get G-linearised vector bundles as well.

Lemma B.1.2. *Let G be a linear algebraic group*

(1) *Let V be a representation of H, then $G \times^H V \to G/H$ is a G-linearised vector bundle.*
(2) *In particular for $\chi \in X^*(H)$ a character of H we get a linearised line bundle $L_\chi = G \times^H \mathsf{k}$ with action $h.(g,z) = (gh, \chi(h)^{-1}z)$. Any linearised line bundle is of that form. We thus have a group morphism*

$$X^*(H) \to \mathrm{Pic}(G/H)$$

whose image is the subgroup $\mathrm{Pic}_G(G/H)$ of linearised line bundles.

Proof. (1) As observed in the construction of $G \times^H V$, this variety has a G-equivariant map to G/H whose fibers are isomorphic to V and furthermore the local trivialisation shows that the action is linear on the fibers.

(2) Assume conversely that $\pi : L \to G/H$ is a G-linearised line bundle. Then H acts linearly on the fiber L_e over the class of the identity element e. In particular, we get a character of H via the action map $\chi : H \to \mathrm{GL}(L_e)$ defined by $h.l = \chi(h)l$ for $l \in L_e$. Consider the morphism $G \times L_e \to L$ defined by $(g,l) \mapsto g.l$ and the action of H on $G \times L_e$ defined by $h.(g,l) \mapsto (gh, \chi^{-1}(h)l)$. The map is then constant on the H-orbits, thus factors through $L_\chi \to L$. The morphism $G \times L_e \to L$ being surjective, so is $L_\chi \to L$ and therefore this is an isomorphism of line bundles. $\square$

We denote by p_X and p_G the projections from $G \times X$ to X and G respectively and by $\varphi : G \times X \to X$ and $\Phi : G \times L \to L$ the action of G on X and a linearisation of this action on a line bundle L.

LEMMA B.1.3. *For $\Phi : G \times L \to L$ a linearisation of a line bundle, there is a commutative diagram:*

$$\begin{tikzcd} G \times L \arrow[r, "\Phi"] \arrow[d, "id\times\pi"'] & L \arrow[d, "\pi"] \\ G \times X \arrow[r, "\varphi"] & X \end{tikzcd}$$

which is furthermore cartesian. In otherwords, we have an isomorphism of line bundles

$$p_X^*(L) \simeq \varphi^*(L).$$

The restriction of Φ to $\{e\} \times L$ is the identity.

Proof. The commutativity of the diagram is equivalent to the fact that $\pi : L \to X$ is equivariant. To prove that the diagram is cartesian, let us check the universal property of the product. Let $\alpha : Z \to L$ and $\beta : Z \to G \times X$ such that $\pi \circ \alpha = \varphi \circ \beta$. We define $\gamma : Z \to G \times L$ by $\gamma = (p_G \circ \beta, \Phi(i(p_G \circ \beta), \alpha))$. We need to check the equalities $(\mathrm{id} \times \pi) \circ \gamma = \beta$ and $\Phi \circ \gamma = \alpha$. We compute $\pi(\Phi(i(p_G \circ \beta), \alpha)) = \varphi(\pi(\Phi(i(p_G \circ \beta), \alpha))) = \varphi(i(p_G \circ \beta), \varphi(p_G \circ \beta, p_X \circ \beta)) = p_X \circ \beta$ giving the first equality. The second equality is obvious.

The last assertion is obvious. □

PROPOSITION B.1.4. *Conversely, assume that L is a line bundle together with a morphism $\Phi : G \times L \to L$ satisfying the following two conditions:*

(1) *There is a commutative diagram:*

$$\begin{tikzcd} G \times L \arrow[r, "\Phi"] \arrow[d, "id\times\pi"'] & L \arrow[d, "\pi"] \\ G \times X \arrow[r, "\varphi"] & X \end{tikzcd}$$

which is furthermore cartesian. In otherwords, we have an isomorphism of line bundles

$$p_X^*(L) \simeq \varphi^*(L).$$

(2) *The restriction of Φ to $\{e\} \times L$ is the identity and $\Phi(g, \cdot) : L \to L$ maps the zero section to itself for all $g \in G$.*

Then Φ is a linearisation of L.

Proof. We only need to check that this defines an action which is linear on the fibers. For $g \in G$, the morphism $\Phi(g, \cdot) : L_x \to L_{gx}$ is bijective and map 0 to 0. It is therefore a linear isomorphism. We thus has a function $f : G \times G \times L \to \mathbb{G}_m$ such that for all $g\, h \in G$ and $z \in L$ we have the equality

$$\Phi(gh, z) = f(g, h, z)\Phi(g, \Phi(h, z)).$$

But looking at trivialisations, we easily see that this function is regular.

LEMMA B.1.5. *The map $\mathsf{k}[X]^\times \times \mathsf{k}[Y]^\times \to \mathsf{k}[X \times Y]^\times$ is surjective, for X and Y irreducible varieties.*

Proof. Let x_0 and y_0 be normal points on X and Y and let $f \in \mathsf{k}[X \times Y]^\times$. We may define the function $F : X \times Y \to \mathbb{G}_m$ by $F(x, y) = f(x_0, y_0)^{-1} f(x, y_0) f(x_0, y)$. We only need to prove that $f = F$. For this it is sufficient to prove that these functions coincide in a neighbourhood $U \times V$ of (x_0, y_0). We may therefore assume that X and Y are affine and normal.

Let $\bar{X}$ and $\bar{Y}$ be normal projective compactifications of X and Y, in particular X and Y are dense open subsets in these compactifications. Consider f and F as rational functions on $\bar{X} \times \bar{Y}$. The support of the divisor $\mathrm{div}(f/F)$ is contained in $((\bar{X} \setminus X) \times \bar{Y}) \cup (\bar{X} \times (\bar{Y} \setminus Y))$. It is therefore a sum of divisors of the form $D \times \bar{Y}$ and $\bar{X} \times D'$ with D and D' irreducible components of the boundary of X and Y. If f/F has a zero on a divisor $D \times \bar{Y}$, then it is regular on an open set meeting $D \times \{y_0\}$. But $f(x, y_0) = F(x, y_0)$ for all $x \in X$ and therefore also on $\bar{X}$ leading to a contradiction. The same argument prove that f/F has no pole and is therefore in $\mathsf{k}[\bar{X} \times \bar{Y}]^\times$. It has to be a constant and the value at (x_0, y_0) proves that this constant is 1. □

Exercise B.1.6. Prove the following consequence of this lemma: any invertible function $f \in \mathsf{k}[G]^\times$ over a group G with $f(e) = 1$ is a character.

This lemma implies that f above has the form $f(g, h, z) = r(g)r(h)t(z)$ for some functions $r \in \mathsf{k}[G]^\times$, $s \in \mathsf{k}[G]^\times$ and $t \in \mathsf{k}[L]^\times$. Now the equality $\Phi(e, z) = z$ gives the equalities

$$r(e)s(h)t(z) = 1 \text{ and } r(g)s(e)t(z) = 1.$$

for all $g, h \in G$ and $z \in L$. We then get the equalities

$$\begin{aligned} f(g,h,z) = \quad & r(g)s(h)t(z) = (r(g)s(h)t(z))(r(e)s(e)t(z)) \\ & (r(g)s(e)t(z))(r(e)s(h)t(z)) = 1. \end{aligned}$$

The result follows. □

Corollary B.1.7. *A line bundle L over X with a G-action $\varphi : G \times X \to X$ is linearisable is and only if there exists an isomorphism $\varphi^*(L) \simeq p_X^*(L)$.*

Proof. The former Lemma implies that if L is linearisable, then such an isomorphism exists. Conversely such an isomorphism induces a pull-back diagram

$$\begin{tikzcd} G \times L \arrow[r, "\Phi"] \arrow[d, "\mathrm{id}\times\pi"'] & L \arrow[d, "\pi"] \\ G \times X \arrow[r, "\varphi"] & X \end{tikzcd}$$

such that for all $g \in G$ the map $\Phi(g, \cdot)$ sends the zero section to itself (because it is a pull-back diagram). Furthermore, the restriction of ϕ to $\{e\} \times L$ is an isomorphism of L. Therefore there is a regular function $\lambda : X \to \mathbb{G}_m$ defined by $\lambda(\pi(z)).z = \Phi(e, z)$. Replacing Φ by $\lambda^{-1}\Phi$ we obtain a morphism satisfying the conditions of the previous proposition and the result follows. □

2. The Picard group of homogeneous spaces

Let us recall the following fact on reductive algebraic groups.

Fact B.2.1. *A reductive algebraic group contains an open dense affine subset isomorphic to $\mathbb{G}_a^p \times \mathbb{G}_m^{2q}$.*

Proof. Use Bruhat decomposition to write the dense open cell as UTU^- where T is a maximal torus and U a maximal unipotent subgroup with U^- its opposite. Then we have seen that $T \simeq \mathbb{G}_m^q$ while $U \simeq \mathbb{G}_a^q \simeq U^-$. □

REMARK B.2.2. The fact that an open subset of G is isomorphic to a product of $\mathbb{G}_a$ and $\mathbb{G}_m$ is true for any connected algebraic group: by a result of Grothendieck **[9]**, if G is connected then as variety we have $G \simeq R(G) \times (G/R(G))$ and $G/R(G)$ is reductive. The group $R(G)$ is unipotent and one can prove that it is isomorphic to $\mathbb{G}_a^s$.

Let us prove the following result.

LEMMA B.2.3. *Let X be a normal variety with an action of G and let L be a line bundle on $G \times X$. Then we have an isomorphism*

$$L \simeq p_G^*(L|_{G\times\{x_0\}}) \otimes p_X^*(L|_{\{e\}\times X}),$$

for some $x_0 \in X$.

Proof. Let $M = L^{-1} \otimes p_G^*(L|_{G\times\{x_0\}}) \otimes p_X^*(L|_{\{e\}\times X})$.

Let us assume first that X is smooth. The Picard group of $G \times X$ is then isomorphic to the group of Weil divisors $\mathrm{Cl}(G \times X)$ (cf. **[13**, Chapter II, Section 6**]**). By *loc. cit.* Proposition II.6.6, the pull-back gives identifications $\mathrm{Cl}(\mathbb{G}_a \times X) \simeq \mathrm{Cl}(X)$ and $\mathrm{Cl}(\mathbb{G}_m \times X) \simeq \mathrm{Cl}(X)$. Therefore on an affine open space U of G, we have $M|_U \simeq \mathcal{O}_U$.

Therefore the divisor class corresponding to M is represented by a divisor D supported in $(G \setminus U) \times X$. Therefore we have $D = p_G^{-1}(D')$ with D' a divisor in G. We thus have

$$M \simeq p_G^*(M_{G\times\{x_0\}})$$

but $M_{G\times\{x_0\}}$ is trivial therefore so is M.

If X is normal but not necessarily smooth, then X^{sm} the smooth locus of X has complementary in codimension 2 and every function defined on X^{sm} extends to a regular function on X. By the previous argument $M|_{X^{\mathrm{sm}}}$ is trivial therefore so is M. □

PROPOSITION B.2.4. *Let L be a line bundle on G and denote by $L^\times$ the complement of the zero section. Then $L^\times$ has a structure of a linear algebraic group such that the following two conditions hold.*

(1) *The projection $p : L \to G$ induces a group morphism $L^\times \to G$ with kernel central in $L^\times$ and isomorphic to $\mathbb{G}_m$.*
(2) *The line bundle L is $L^\times$-linearisable.*

Proof. We need to define the multiplication map $\mu : L^\times \to L^\times$. Let us denote by $m : G \times G \to G$ the multiplication in G and by p_1 and p_2 the two projections on $G \times G$. We know by the previous lemma that there is an isomorphism $\psi : p_1^*(L) \otimes p_2^*(L) \to m^*(L)$. We construct via ψ a morphism $\mu : L \times L \to L$ as the composition:

$$\begin{tikzcd}
L \times L \arrow[r] \arrow[d, "p\times p"'] & p_1^*(L) \otimes p_2^*(L) \arrow[r, "\psi"] \arrow[d] & m^*(L) \arrow[r] \arrow[d] & L \arrow[d, "p"] \\
G \times G \arrow[r, "\mathrm{id}"] & G \times G \arrow[r, "\mathrm{id}"] & G \times G \arrow[r, "m"] & G.
\end{tikzcd}$$

If M is the locally free $k[G]$-module corresponding to L, then this map is given as follows

$$M \xrightarrow{m^\sharp} M \otimes^m_{k[G]} k[G\times G] \xrightarrow{\psi^\sharp} M \otimes^{p_1}_{k[G]} k[G\times G] \otimes_{k[G\times G]} M \otimes^{p_2}_{k[G]} k[G\times G] \longrightarrow M \otimes_k M$$

where $\otimes^f_{k[G]} k[G\times G]$ is the tensor product using the map $f : G\times G \to G$ and where $m^\sharp$ is given by the formula $m^\sharp((m\otimes(a\otimes b))\otimes(m'\otimes(a'\otimes b'))) = aa'm\otimes bb'm'$.

We want to modify ψ so that μ will induce the desired multiplication map. Let us fix an identification $L_e \simeq \mathbb{G}_a$ and fix $1 \in L_e$ be the element corresponding to the unit in $\mathbb{G}_a$. The composition $L \to L\times\{1\} \xrightarrow{\mu} L$ is an isomorphism (check on the modules!) inducing the identity on G *i.e.* an isomorphism of vector bundles. Therefore there is an invertible function $r \in k[G]^\times$ with

$$\mu(l,1) = r(p(l))l$$

for all $l \in L$. The same argument gives an invertible function $s \in k[G]^\times$ with

$$\mu(1,l') = s(p(l'))l'$$

for all $l' \in L$. Let us replace ψ by $\psi\circ(r^{-1}\otimes s^{-1})$ and denote by $\Delta : L\times L \to L$ the corresponding morphism. Then $1 \in L_e$ is a unit for this morphism:

$$\Delta(l,1) = \mu(r^{-1}(p(l))l,1) = l \text{ and } \Delta(1,l') = \mu(1,s^{-1}(p(l'))l') = l'.$$

Let us now prove that Δ is associative. Indeed, by linearity (use the same arguments as in Proposition B.1.4) of the maps, there is an invertible function $t \in k[G\times G\times G]^\times$ with

$$\Delta(\mathrm{id}\times\Delta)(l,l',l'') = t(p(l),p(l'),p(l''))\Delta(\Delta\times\mathrm{id}(l,l',l'')).$$

As usual we can write $t(g,g',g'') = u(g)v(g')w(g'')$ with $u,v,w \in k[G]^\times$. We have (because 1 is a unit) the equality $t(e,e,e) = 1$ therefore we may assume $u(e) = v(e) = w(e)$ (replace u by $u(e)^{-1}u$ and do the same for v and w). Because 1 is a unit we have $t(g,e,e) = t(e,g',e) = t(e,e,g'') = 1$. We obtain the equalities $u(g) = v(g') = w(g'') = 1$ for all $g,g',g'' \in G$ therefore Δ is associative. Furthermore the morphism Δ being bilinear, the subset $L^\times$ is contained in the locus of invertible elements this proves the existence of the group structure on $L^\times$.

Furthermore by construction the map $p : L \to G$ induces a group morphism $L^\times \to G$. The kernel of this map is $L_e^\times \simeq \mathbb{G}_m$. This groups acts by scalar multiplication on the fibers and is therefore in the center of $L^\times$.

Finally, the restriction of Δ give a group action $L^\times \times L \to L$ which is a linearisation of the action of $L^\times$ on G ($L^\times$ acts on G via G and the map $L^\times \to G$), in other words the central kernel $L_e^\times \simeq \mathbb{G}_m$ acts trivially on G. □

COROLLARY B.2.5. *Let G be a linear algebraic group and let $L \in \mathrm{Pic}(G)$. There exists a finite covering $\pi : G' \to G$ such that π^*L is trivial.*

Proof. We may assume G to be connected since all the connected components of G are isomorphic.

We consider $L^\times$ as a linear algebraic group and denote by $L_e^\times$ the kernel of the map $L^\times \to G$. Choose a representation V of $L^\times$ such that $L_e^\times$ does not act trivially (for example take a faithful representation, see [**25**, Theorem 2.3.7]). Replacing V by a submodule W, we may assume that $L_e^\times$ acts by a nontrivial scalar on W (take an eigenspace V_χ of V with χ a non trivial character of $L_e^\times \simeq \mathbb{G}_m$, because $L_e^\times$ is central, this is again a sub-$L^\times$-module). We may furthermore assume that the representation V is also faithful on the Lie algebra level (*i.e.* $d_e\rho : \mathrm{Lie}(L^\times) \to \mathfrak{gl}(V)$ is injective, see [**25**, Lemma 5.5.1]). The character χ corresponds to an integer n, the action is $t.v = \chi(t)v = t^n v$. By the condition on the Lie algebra, the integer n has to be prime to $p = \mathrm{char}(k)$. Denote by $\rho : L^\times \to \mathrm{GL}(W)$ this new representation.

Let G' be the identity component of $\rho^{-1}(\mathrm{SL}(W))$. Then the restriction $\pi : G' \to G$ of $p : L^\times \to G$ is surjective and with finite fibers (the dimension of G' is $\dim L^\times - 1 = \dim G$ and G is connected). The map π is quasi-finite and affine thus finite. Note that the kernel of the map $\pi : G' \to G$ is isomorphic to the intersection $\mathbb{G}_m \cap \rho^{-1}(\mathrm{SL}(W))$ and therefore is isomorphic to the finite group of n-th root of the unit and therefore is a *reduced* finite group K and the map π is unramified.

Now the restriction of the action of $L^\times$ to G' induces a G'-linearisation of L. Therefore the line bundle π^*L is G'-linearised on G' and by Lemma B.1.2 it is trivial. $\square$

Corollary B.2.6. *The Picard group* $\mathrm{Pic}(G)$ *is finite.*

Proof. Let $L \in \mathrm{Pic}(G)$ and let $\pi : G' \to G$ be a covering such that L is linearised and $\pi^*(L)$ is trivial. If K is the kernel of π, then K acts on L and if n is the order of K, the action of K on $L^{\otimes n}$ is trivial. Therefore G acts on $L^{\otimes n}$ and thus $L^{\otimes n}$ is G-linearisable. As above we get that $L^{\otimes n}$ is trivial. Therefore $\mathrm{Pic}(G)$ is a torsion group.

We are left to prove that $\mathrm{Pic}(G)$ is of finite type. But we have seen that there is an open subset U of G isomorphic to $\mathbb{G}_a^p \times \mathbb{G}_m^q$. We thus have $\mathrm{Pic}(U) = 0$ and the non trivial elements in $\mathrm{Pic}(G)$ are supported by divisors corresponding to irreducible components of $G \setminus U$. There are only finitely many of them concluding the proof (note that we use here the fact that G is smooth and thus that $\mathrm{Pic}(G)$ coincides with $\mathrm{Cl}(G)$ the group of Weil divisors. $\square$

Corollary B.2.7. *There is a finite covering* $\pi : G' \to G$ *such that* $\mathrm{Pic}(G') = 0$. *In particular if* G *is simply connected, then* $\mathrm{Pic}(G) = 0$.

Proof. Again we may assume that G (and G') are connected.

By what we proved, it is enough to check that if $\pi : G' \to G$ is a finite covering, then the morphism $\pi^* : \mathrm{Pic}(G) \to \mathrm{Pic}(G')$ is surjective. Let $L' \in \mathrm{Pic}(G')$ and let $\phi : G'' \to G'$ be a finite covering such that L' is G''-linearisable and $\phi^*(L')$ is trivial. Let K be the kernel of ϕ. As L' is G''-linearisable, there exists a representation k_χ of K such that $L' \simeq G'' \times^K k_\chi$.

Let K' be the kernel of the composition $\pi \circ \phi : G'' \to G$. Then K is a subgroup of K'. But K' is finite and abelian, thus we may extend the representation of K in k_χ in a K'-representation k_η (we act by roots of the unit). We may set $L = G'' \times^{K'} k_\eta$. Then $L' = \pi^*L$ and the result follows.

If G is simply connected, then there are only trivial finite coverings. $\square$

Remark B.2.8. Note that we used here the fact that any non trivial covering of G comes from an abelian kernel or equivalentely that the fundamental group of G is abelian. Here is a proof for $\mathrm{char}(k) = 0$.

Let $f : [0,1] \to G$ and $g : [0,1] \to G$ be loops in G with $f(1) = g(1) = e$. Define the product $f \cdot g$ of loops by $(f \cdot g)(t) = f(t)g(t)$ and the concatenation of loops by:

$$(f \tilde{\cdot} g)(e^{2\pi i x}) := \begin{cases} f(e^{4\pi i x}) & , \quad 0 \leq x \leq \frac{1}{2} \\ g(e^{4\pi i x}) & , \quad \frac{1}{2} \leq x \leq 1 \ . \end{cases}$$

We construct a homotopy of loops $f \tilde{\cdot} g \approx f \cdot g \approx g \tilde{\cdot} f$. For each $-1 \leq \epsilon \leq 1$, let $p_\epsilon : [0,1] \to [0,1] \times [0,1]$ be a path in the unit square starting at $(0,0)$ and ending at $(1,1)$, such that p_{-1} goes along the left and top boundaries, p_0 goes along the diagonal, and p_1 goes along the bottom and right boundaries. Define $H : [0,1] \times [0,1] \to K$ by $H(x_1, x_2) := f(e^{2\pi i x_1})\, g(e^{2\pi i x_2})$. Then defining $h_\epsilon : S^1 \to K$ by $h_\epsilon(e^{2\pi i x}) := H(p_\epsilon(x))$ gives a continuous family of loops with $h_{-1} = f \tilde{\cdot} g$, $h_0 = f \cdot g$, and $h_1 = g \tilde{\cdot} f$. $\square$

Proposition B.2.9. *Let G be a connected algebraic group and let H be a closed subgroup and denote by $\pi : G \to G/H$ the quotient map. Let us also denote by $\psi : X^*(H) \to \mathrm{Pic}(G/H)$ the group morphism defined in Lemma B.1.2. Then we have an exact sequence*

$$X^*(G) \xrightarrow{\mathrm{res}} X^*(H) \xrightarrow{\psi} \mathrm{Pic}(G/H) \xrightarrow{\pi^*} \mathrm{Pic}(G).$$

Recall that the image of ψ is the subgroup of linearisable line bundles.

Proof. Let us start with the exactness at $X^*(H)$. If χ is a character of G, then we have already seen that the line bundle $L_\chi = G \times^H k_\chi$ is trivial (see Example A.2.14). Conversely, if χ is a character of H such that $L_\chi = G \times^H k_\chi$ is trivial, then we have a trivialisation $\psi : G/H \times k \simeq L_\chi$ but G acts on $G \times^H k_\chi$ therefore it acts on k and this action extends the action of H.

Consider the exactness at $\mathrm{Pic}(G/H)$. Let $L \in \mathrm{Pic}(G/H)$ and denote by $\varphi : G \times G/H \to G/H$ the action of G. By Lemma B.2.3 we have an isomorphism $\varphi^* L \simeq p_G^* M \otimes p_{G/H}^* N$ with $M = \varphi^* L|_{G \times \{\bar{e}\}} = \pi^* L$ and $N = \varphi^* L|_{\{e\} \times G/H} = L$. In other words, we have an isomorphism

$$\varphi^* L \simeq p_G^* \pi^*(L) \otimes p_{G/H}^* L.$$

The image of ψ is composed of the G-linearisable line bundles. We know that L is linearisable if and only if there is an isomorphism $\varphi^* L \simeq p_{G/H}^* L$ which in turn is equivalent to the fact that $\pi^*(L)$ is trivial. $\square$

Corollary B.2.10. *Assume that G is semisimple and simply connected. Let P be a parabolic subgroup of G, then we have $\mathrm{Pic}(G/P) \simeq X^*(P)$.*

In particular $\mathrm{rank}(\mathrm{Pic}(G/B)) = \mathrm{rank}(G)$ for B a Borel subgroup of G.

Proof. If G is semisimple, then $X^*(G) = 0$ (we have $G = D(G)$ for example) and if it is semisimple then $\mathrm{Pic}(G) = 0$. The result follows from the above exact sequence. $\square$

3. Existence of linearisations and a result of Sumihiro

In this section we present a result of Sumuhiro **[26]** and **[27]**.

Proposition B.3.1. *Let L be a line bundle on a normal G-variety. There exists a positive integer n such that $L^{\otimes n}$ is G-linearisable.*

Proof. By Lemma B.2.3 we have an isomorphism $\varphi^* L \simeq p_X^* M \otimes p_G^* N$ (this uses the normality assumption). Note that we have $M = \varphi^* L|_{\{e\} \times X} = L$.

But the Picard group of G is finite therefore there exists a positive integer n such that $N^{\otimes n}$ is trivial. We get an isomorphism $\varphi^*(L^{\otimes n}) \simeq p_X^*(M^{\otimes n}) \otimes p_G^*(M^{\otimes n}) \simeq p_G^*(L^{\otimes n})$. Therefore $L^{\otimes n}$ is linearisable. $\square$

LEMMA B.3.2. *Let L be a G-linearisable line bundle on X. Then G acts on $H^0(X, L)$ via*

$$g.\sigma(x) = g(\sigma(g^{-1}.x))$$

for all $g \in G$, $\sigma \in H^0(X, L)$ and $x \in X$. Furthermore this representation is locally finite and rational.

Recall that a representation V of G is locally finite and rational if for all $v \in V$, there is a finite dimensional G-subspace W of V containing v such that the action is given by an algebraic group morphism $G \to \mathrm{GL}(W)$.

Proof. Note that there is an isomorphism $H^0(G \times X, p_X^* L) \simeq k[G] \otimes H^0(X, L)$ defined by $s \mapsto (f, \sigma)$ with $\sigma(x) = s(e, x)$ and $s(g, x) = f(g)s(e, x)$ (such an f exists because we take the pull-back of a line bundle on X). The inverse is defined by $f \otimes \sigma \mapsto [(g, x) \mapsto (g, f(g)\sigma(x))]$.

But because L is G-linearisable, the linearisation $\Phi : G \times L \to L$ proves that $p_X^* L$ is also isomorphic to $\varphi^* L$. Pulling back sections, we get a morphism

$$\Phi^* : H^0(X, L) \to H^0(G \times X, \varphi^* L) \simeq H^0(G \times X, p_X^*(L)) \simeq k[G] \otimes H^0(X, L)$$

defined by $\sigma \mapsto s$ with $s(g, x) = g^{-1}.\sigma(gx) = \Phi(g^{-1}, \sigma(gx))$. We may write $\Phi^*(\sigma) = \sum_i f_i \otimes \sigma_i$ with $f_i \in k[G]$ and $\sigma_i \in H^0(X, L)$, the sum being finite. We get $g.\sigma = \sum_i f_i(g^{-1})\sigma_i$ and the result follows. $\square$

DEFINITION B.3.3. (i) A G-variety is called linear if there exists a representation V of G and a G-equivariant isomorphism of X to a G-stable locally closed subvariety of $\mathbb{P}(V)$.

(ii) A G-variety is called locally linear if there exists a covering of X by linear G-stable open subsets.

The next result proves that normal G-varieties are locally linear.

THEOREM B.3.4 (Sumihiro's Theorem). *Let X be a normal variety with an action of an algebraic group G. Let Y be a G-orbit in X.*

There exists a finite dimensional representation V of G and a G-stable neigbourhood U of Y in X such that U is G-equivariantly isomorphic to a G-stable locally closed subvariety in $\mathbb{P}(V)$.

Proof. Let U_0 be an affine open subset in X meeting Y non trivially. Consider the divisor $D = X \setminus U_0$ and the invertible sheaf $\mathcal{O}_X(mD)$. Recall that $\mathcal{O}_X(mD)$ is the sheaf of rational functions with pole of order at most m at D.

Let $f_0 = 1, f_1, \cdots, f_n$ be generators of the algebra $k[U_0] \subset k(X)$ and let N be the linear span of these elements in $k(X)$. Then for some $m \geq 0$, we have the inclusion $N \subset H^0(X, \mathcal{O}_X(mD))$.

Now there exists an integer $n \geq m$ such that $\mathcal{O}_X(nD)$ is linearisable. Therefore we have a locally finite and rational action of G on $H^0(X, \mathcal{O}_X(nD))$. The space N is contained in $H^0(X, \mathcal{O}_X(nD))$ and we denote by W the (finite dimensional) subspace spaned by all the G-translates of N. We get a rational map

$$\psi : X \dashrightarrow \mathbb{P}(W^\vee)$$

defined by $x \mapsto [\ell_x]$ with $\ell_x(s) = s(x)$ and where $[\ell_x]$ is the class in the projective space of ℓ_x. This map is G-equivariant. Indeed, we have $(g.\ell_x)(s) = \ell_x(g^{-1}.s) = \ell_x(g^{-1}sg) = g^{-1}s(gx) = g^{-1}\ell_{gx}(s)$. But g acts by scalar multiplication thus there exists an invertible function (it is even a character) $f \in \mathsf{k}[G]^\times$ with $g.\ell_x = f(g)\ell_{gx}(s)$ therefore $[g.\ell_x] = [\ell_{gx}]$.

The map ψ induces an isomorphism on U_0 and by G-equivariance on $U = GU_0$ concluding the proof. $\square$

REMARK B.3.5. The normality assumption is important as shows the following example. Consider X a plane nodal cubic. It has a $\mathbb{G}_m$ action with 2 orbits: the node and the complement of the node.

But the closure of any non trivial $\mathbb{G}_m$-orbit in a projective space is isomorphic to $\mathbb{P}^1$ with 3 orbits. Therefore X does not satisfy the conclusion of the previous proposition.

APPENDIX C

Finite generation of U-invariants

In this appendix, we want to prove for $\mathrm{char}(\mathsf{k}) = 0$ the following result proved in full generality in [**8**]. Recall that G is a reductive group. Denote by B a Borel subgroup of G and by $U \subset B$ its unipotent radical. The subgroup U is a maximal unipotent closed connected subgroup in G.

THEOREM C.0.1. *Let X be an affine G-variety, then $\mathsf{k}[X]^U$ is finitely generated.*

1. Isotypical decomposition

DEFINITION C.1.1. Let G be a reductive group.

(1) For V and W two G-module. Denote by $\mathrm{Hom}_G(V, W)$ the group of morphisms of G-module from V to W.
(2) Let X and Y be G-varieties. Denote by $\mathrm{Mor}_G(X, Y)$ the set of G-equivariant morphisms from X to Y.

FACT C.1.2. *For X a G-variety and V a G-module, we have isomorphisms* $\mathrm{Hom}_G(V, \mathsf{k}[X]) \simeq (\mathsf{k}[X] \otimes V^\vee)^G \simeq \mathrm{Mor}_G(X, V^\vee)$.

Proof. Let $(e_i)_{i\in[1,n]}$ be a basis of V and $(e_i^\vee)_{i\in[1,n]}$ be the dual basis. Define the map $f \mapsto \sum_i f(e_i) \otimes e_i^\vee$. One easily checks that this does not depend on the choice of the base. The action on the tensor product is given by the diagonal action: $g.\sum_i f(e_i)\otimes e_i^\vee = \sum_i (g.f)(e_i)\otimes(g.e_i^\vee) = \sum_i f(g.e_i)\otimes(g.e_i^\vee) = \sum_i f(e_i)\otimes e_i^\vee$ thus we get an invariant. The converse map is $\sum_i a_i \otimes l_i \mapsto f$ with $f(x) = \sum_i a_i l_i(x)$ (complete the l_i in a basis of $V^\vee$ and take the dual basis to get the expression as $\sum_i f(e_i) \otimes e_i^\vee$). This proves the first isomorphism.

For the second map, define $\mathrm{Hom}_G(V, \mathsf{k}[X]) \to \mathrm{Mor}_G(X, V^\vee)$ by $f \mapsto \phi_f$ with $\phi_f(x) = (v \mapsto f(v)(x))$ and the converse map $\phi \mapsto f_\phi$ with $f_\phi(v)(x) = \phi(x)(v)$. One easily checks the compatibility of the actions. $\square$

Note that the group $\mathrm{Hom}_G(V, \mathsf{k}[X])$ is a $\mathsf{k}[X]^G$-module with action given by $(\phi.f)(v) = \phi.[f(v)]$. Let $\widehat{G}$ be the set of irreducible representations of G.

LEMMA C.1.3. *Let M be a G-module M. The assigment $f \otimes v \mapsto f(v)$ induces a G-module isomorphism*

$$\bigoplus_{V\in\widehat{G}} \mathrm{Hom}_G(V, M) \otimes V \simeq M$$

In particular for any G-variety X, we have a G-module isomorphism

$$\mathsf{k}[X] \simeq \bigoplus_{V\in\widehat{G}} \mathrm{Mor}_G(X, V^\vee) \otimes V$$

and each of the $\mathsf{k}[X]^G$-modules $\mathrm{Mor}_G(X, V^\vee)$ are finitely generated.

Proof. Because M is a rational representation, it has to be the direct sum of its irreducible finite dimensional sub-G-modules. We therefore only have to deal with M a simple module. The first assertion is then easy to verify.

For the second assertion, recall the isomorphism $\mathrm{Mor}_G(X, V^\vee) \simeq (\mathsf{k}[X] \otimes V^\vee)^G$. But the algebra $\mathsf{k}[X \times V]^G$ is isomorphic to

$$\mathsf{k}[X \times V]^G \simeq \bigoplus_{n \geq 0} (\mathsf{k}[X] \otimes S^n V^\vee)^G.$$

Therefore this algebra is finitely generated as a k-algebra and has a grading. Let $f_1, \cdots, f_n$ be generators of this algebra. The algebra $\mathsf{k}[X]^G$ is generated by the elements f_i of degree 0 while the $\mathsf{k}[X]^G$-module $(\mathsf{k}[X] \otimes V^\vee)^G = (\mathsf{k}[X \times V]^G)_1$ is generated by the elements of degree 1. $\square$

Corollary C.1.4. *There is a canonical decomposition as $G \times G$-module*

$$\mathsf{k}[G] \simeq \bigoplus_{V \in \widehat{G}} V \otimes V^\vee = \bigoplus_{V \in \widehat{G}} \mathrm{End}_{\mathsf{k}}(V).$$

Proof. Lemma C.1.3 gives the $G \times G$-equivariant decomposition

$$\mathsf{k}[G] = \bigoplus_{V \in \widehat{G}} \mathrm{Mor}_G(G, V^\vee) \otimes V.$$

We claim that there is a G-module isomorphism $\mathrm{Mor}_G(G, V^\vee) \simeq V^\vee$. Indeed, the direct map is given by $\phi \mapsto \phi(e_G)$ while the inverse map is defined by $f \mapsto (g \mapsto g \cdot f)$. Both maps are G-equivariant. $\square$

Assume that G is connected and let B be a Borel subgroup, T a maximal torus in B and U be the unipotent part of B. We can write $B = TU$ and U being normal we have an exact sequence

$$1 \to U \to B \to T \to 1$$

giving on the level of characters the identification $\mathfrak{X}(B) = \mathfrak{X}(T)$ since U has no nontrivial character (being unipotent). Denote by R the root system associated to T and by R^+ the set of positive roots associated to B. Recall the following result from the representation theory of G.

Theorem C.1.5. *Let G be a reductive group.*

(1) *If V be a simple G-module, then V^U is of dimension 1 on which B acts by a character $\lambda \in \mathfrak{X}(T)$ and V is uniquely determined by λ.*
(2) *The set $\mathfrak{X}(T)^+$ of all possible characters λ as above for simple G-module is the set of dominant characters:*

$$\mathfrak{X}(T)^+ = \{\lambda \in \mathfrak{X}(T) \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in R^+\}.$$

(3) *In particular $\mathfrak{X}(T)^+$ is a finitely generated monoid.*

Definition C.1.6. Let $\lambda \in \mathfrak{X}(T)^+$.

(1) Denote by $V(\lambda)$ the simple G-module of highest weight λ .
(2) For M a G-module, denote by $M_\lambda^{(B)}$ the subspace of semi-B-invariants where B acts by λ: $M_\lambda^{(B)} = \{m \in M \mid b.m = \lambda(b)m \text{ for all } b \in B\}$.

Corollary C.1.7. *Let M be a G-module.*

(1) *We have a G-equivariant isomorphism $M \simeq \bigoplus_{\lambda \in \mathfrak{X}(T)^+} M_\lambda^{(B)} \otimes V(\lambda)$.*

(2) *We have isomorphisms* $\mathrm{Hom}_G(V(\lambda), M) \simeq M_\lambda^{(B)}$ *for all* $\lambda \in X^*(G)^+$.

COROLLARY C.1.8. *The G-module M is determined by the T-module M^U.*

2. U-invariants

We are now in position to prove the following result.

THEOREM C.2.1. *For X an affine G-variety, $\mathsf{k}[X]^U$ is finitely generated.*

Proof. We first reduce this problem to the case where $X = G$. Indeed, consider the principal U-bundle $\pi : G \to G/U$ and the action of G on $X \times G/U$ defined by $h.(x, [g]) = (h.x, [hg])$. We claim that there is an isomorphism

$$\mathsf{k}[X]^U \simeq \mathsf{k}[X \times G/U]^G.$$

Indeed, define the map $f \mapsto \varphi_f$ by $\varphi_f(x, [g]) = f(g^{-1}x)$. It is well defined since f is U-invariant thus $(x, g) \mapsto f(g^{-1}x)$ is constant on U-orbits. The converse map is defined by $\varphi \mapsto f_\varphi$ with $f_\varphi(x) = \varphi(x, [e])$.

By Theorem 3.1.1, we only need to check that $\mathsf{k}[X \times G/U]$ is finitely generated *i.e.* that $k[G/U]$ is finitely generated or that $\mathsf{k}[G]^U$ is finitely generated. Recall the decomposition $\mathsf{k}[G] = \oplus_{V \in \widehat{G}} V \otimes V^\vee$ as $G \times G$-module. This induces the following decomposition

$$\mathsf{k}[G]^U = \bigoplus_{V \in \widehat{G}} V^\vee \simeq \bigoplus_{V \in \widehat{G}} V = \bigoplus_{\lambda \in \mathfrak{X}(T)^+} V(\lambda).$$

But the monoid $\mathfrak{X}(T)^+$ of dominant character is finitely generated, this concludes the proof because the span of $V(\lambda)V(\mu)$ is a G-module contained in $V(\lambda + \mu)$ and therefore equal to that module. $\square$