Minimal rational curves on complete symmetric varieties

Michel Brion[*], Shin-young Kim[†] and Nicolas Perrin[‡]

Abstract

We describe the families of minimal rational curves on any complete symmetric variety, and the corresponding varieties of minimal rational tangents (VMRT). In particular, we prove that these varieties are homogeneous and that for non-exceptional irreducible wonderful varieties, there is a unique family of minimal rational curves, and hence a unique VMRT. We relate these results to the restricted root system of the associated symmetric space.

1 Introduction

Let X be a projective uniruled variety over the field of complex numbers. An irreducible family $\mathcal{K}$ of rational curves on X is called a *covering family* if there is a member of $\mathcal{K}$ passing through a general point $x \in X$. If in addition the subfamily $\mathcal{K}_x$ of curves in $\mathcal{K}$ passing through x is proper, then $\mathcal{K}$ is called a *minimal family of rational curves.*

These curves play a prominent role in the study of the variety X. There is a rational map $\tau_x : \mathcal{K}_x \dashrightarrow \mathbb{P}(T_xX)$ sending a curve to its tangent direction at x and its image $\mathcal{C}_x \subset \mathbb{P}(T_xX)$ is an important invariant of X called *the variety of minimal rational tangents* or VMRT of X, see [Hw01], [HM04] and references therein.

The VMRT of projective rational homogeneous spaces G/P for G reductive and P a parabolic subgroup are well understood. For example, if G/P has Picard rank 1, then there is a unique VMRT which characterizes G/P and was used to prove its rigidity in [HM05], with the unique exception of B_3/P_2 which admits an explicit degeneration constructed in [PP10]. If the Picard number of G/P is greater than 1, there are several minimal families of rational curves.

In [BF15], the authors consider another case where X has a large Picard group, namely the wonderful compactifications of adjoint semisimple groups. Suprisingly, they prove that there is a unique minimal family of rational curves for any such wonderful compactification

[*]Université Grenoble Alpes, Institut Fourier, CS 40700, 38058 Grenoble Cedex 9, France, michel.brion@univ-grenoble-alpes.fr

[†]Center for Geometry and Physics, Institute for Basic Science, Pohang 37673, Korea, shinyoungkim@ibs.re.kr

[‡]École Polytechnique, Centre de Mathématiques Laurent Schwartz, 91128 Palaiseau Cedex, France, nicolas.perrin.cmls@polytechnique.edu

and also that the corresponding VMRT is a rational homogeneous variety. These results were used in [FL20] to prove the rigidity of wonderful compactifications of groups, under the condition that the special fiber is Fano.

In this paper we generalize the results of [BF15] and describe the minimal families of rational curves on any complete symmetric variety. Rigidity of symmetric varieties of Picard number 1 has already attracted some attention (see [KP19] and [CFL22]), we hope that our results will open new directions for higher Picard numbers.

To state our main results, we recall basic definitions and properties of complete symmetric varieties. Let G be a connected reductive group and let σ be a group involution of G. A symmetric subgroup is a closed subgroup $H \subset G$ such that $G^{\sigma,0} \subset H \subset G^{\sigma}$. The homogeneous space G/H is a *symmetric space*. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H. Note the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ as H-representations, where $\mathfrak{h} = \mathfrak{g}^{\sigma}$ and $\mathfrak{p} = \mathfrak{g}^{-\sigma}$.

Consider the normalizer $N = \mathrm{N}_G(H)$; the quotient G/N is the *adjoint homogeneous space* of the symmetric space G/H. This last space admits by [deCP83] a unique wonderful compactification X_{ad}. This is a smooth projective G-variety having an open dense orbit $G \cdot x_{\mathrm{ad}} = X^0_{\mathrm{ad}} \simeq G/N$, such that the boundary $\partial X_{\mathrm{ad}} = X_{\mathrm{ad}} \setminus X^0_{\mathrm{ad}}$ is a simple normal crossing divisor: $\partial X_{\mathrm{ad}} = X^1_{\mathrm{ad}} \cup \cdots \cup X^r_{\mathrm{ad}}$ where X^i_{ad} is a prime G-stable divisor for all $i \in [1, r]$. Furthermore for any $y, z \in X_{\mathrm{ad}}$ we have $G \cdot y = G \cdot z$ if and only if $\{i \mid y \in X^i_{\mathrm{ad}}\} = \{i \mid z \in X^i_{\mathrm{ad}}\}$. The integer r is *the rank* of G/N. A *complete symmetric variety* is a smooth proper G-variety X having a dense orbit $G \cdot x = X^0 \simeq G/H$ such that the natural map $G/H \to G/N \subset X_{\mathrm{ad}}$ extends to a G-equivariant morphism $\pi : X \to X_{\mathrm{ad}}$. The boundary $\partial X = X \setminus X^0$ is also a simple normal crossing divisor with G-stable prime components.

Let X be a complete symmetric variety with base point x and map $\pi : X \to X_{\mathrm{ad}}$, and let $\mathcal{K}$ be a minimal family of rational curves on X. We will prove the following results.

Theorem 1.1 (Theorem 5.1)**.** *$\mathcal{K}_x$ is smooth and $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ is an isomorphism.*

In particular, understanding the VMRT as an abstract variety is equivalent to understanding $\mathcal{K}_x$. If the map π contracts curves of the family $\mathcal{K}$, then the description of the VMRT follows easily from the case of toric varieties treated in [CFH14], see Lemma 3.4.

Theorem 1.2. *If π contracts a curve of $\mathcal{K}$, then $\mathcal{C}_x$ is a linear subspace of $\mathbb{P}(\mathfrak{p})$.*

We are therefore left to consider curves not contracted by π.

Theorem 1.3 (Proposition 3.6)**.** *Assume that π contracts no curve in $\mathcal{K}$ and let $C \in \mathcal{K}$.*

1. *There exists a unique minimal family of rational curves $\mathcal{L}$ in X_{ad} such that π maps curves of $\mathcal{K}$ to curves of $\mathcal{L}$.*

2. *We have $1 \leq \partial X \cdot C \leq \partial X_{\mathrm{ad}} \cdot \pi(C) \leq 2$.*

3. *If $\partial X \cdot C = \partial X_{\mathrm{ad}} \cdot \pi(C)$ for some $C \in \mathcal{K}$, then any component of $\mathcal{K}_x$ is isomorphic to a component of $\mathcal{L}_{x_{\mathrm{ad}}}$.*

One of the key ingredients for proving the above results is what we call highest weight curves (see Subsection 2.6). Given a Borel subgroup B_H of H, the Borel Fixed Point Theorem implies that any irreducible component of $\mathcal{K}_x$ contains a B_H-fixed point C. Moreover, if C is not contracted by π, then C is mapped to a B_H-fixed point C_{ad} in $\mathcal{L}_{x_{\mathrm{ad}}}$ that determines the associated component of $\mathcal{L}_{x_{\mathrm{ad}}}$. Furthermore, the tangent space at x_{ad} of the highest weight curve C_{ad} in $\mathcal{L}_{x_{\mathrm{ad}}}$ is a highest weight line in $T_{x_{\mathrm{ad}}}(X_{\mathrm{ad}})$.

In view of the above results, we focus on wonderful compactifications of adjoint symmetric spaces. Decomposing G into a product of irreducible σ-stable factors, we obtain a decomposition of G/N into a product of *irreducible symmetric spaces*. There are three possible types for these irreducible factors (since N contains the center of G, we may assume that G is of adjoint type for this list, see Subsection 2.5 for more details):

1. *Group type*: $(H \times H)/\mathrm{diag}(H)$, where H is simple adjoint.

2. *Hermitian type*: $G/\,\mathrm{N}_G(L)$, where G is simple adjoint and $L \subset G$ is a Levi subgroup.

3. *Simple type*: G/H, where G is simple adjoint and H^0 is simple.

Given a highest weight curve C on X, we prove that there is a unique irreducible factor X_C of X_{ad} such that the composition of $\pi : X \to X_{\mathrm{ad}}$ with the projection $X_{\mathrm{ad}} \to X_C$ sends C isomorphically to its image. We may thus replace X_{ad} by X_C and assume that X_{ad} is irreducible. In particular G/N is as in one of the above three cases. To understand the geometry of the irreducible factors, we use the *restricted root system*.

There exists a maximal torus T_{s}, called of *split type*, such that T_{s} is σ-stable and $S = \{t \in T_{\mathrm{s}} \mid \sigma(t) = t^{-1}\}^0$ has maximal dimension. The root system R of (G, T_{s}) is stable under the action of σ and there is a basis Δ of R such that, for $\alpha \in \Delta$, either $\sigma(\alpha) = \alpha$ or $\sigma(\alpha) < 0$. Set $\Delta_1 = \{\alpha \in \Delta \mid \sigma(\alpha) < 0\}$ and $\overline{\alpha} = \alpha - \sigma(\alpha)$. The set $\overline{R} = \{\overline{\alpha} \mid \alpha \in R\}$ is a (possibly non-reduced) root system with basis $\overline{\Delta} = \{\overline{\alpha} \mid \alpha \in \Delta_1\}$ called the *restricted root system* of the symmetric space. The rank of $\overline{R}$ is the rank r of G/H. In Subsection 4.2, we relate curves and divisors in X_{ad} to the restricted root system (the results are probably well known to the experts but we could not find a convenient reference). Let $\overline{R}^\vee$ be the dual root system of $\overline{R}$ with coroot lattice $\mathbb{Z}\overline{\Delta}^\vee$ and denote by $A_1(X)$ the Chow group of curves modulo rational equivalence. We prove the following result.

Proposition 1.4 (Proposition 4.19)**.** *There is a surjective $\mathbb{Z}$-linear map $\psi : A_1(X_{\mathrm{ad}}) \to \mathbb{Z}\overline{\Delta}^\vee$ such that:*

1. *The image of the monoid of effective curves is the monoid spanned by coroots.*

2. *The image of the monoid of curves having non-negative intersection with any component of ∂X_{ad} is the intersection of $\mathbb{Z}\overline{\Delta}^\vee$ with the monoid of dominant cocharacters.*

If the map ψ is not injective, then X_{ad} is called *exceptional.* Since the class of a curve C in a covering family $\mathcal{L}$ of rational curves is effective and has non-negative intersection with any component of ∂X_{ad}, it has to be contained in the intersection of the monoids spanned by coroots and by the dominant cocharacters. There is a unique minimal such coroot $\overline{\Theta}^\vee$, the coroot of the highest root $\overline{\Theta} \in \overline{R}$. This gives a very natural candidate for classes of minimal families. Indeed we prove the following result.

Theorem 1.5 (Corollary 4.20). *Assume that X_{ad} is irreducible.*

1. *If X_{ad} is not exceptional, there is a unique minimal family $\mathcal{L}$ and the class of any $C_{\mathrm{ad}} \in \mathcal{L}$ satisfies $\psi([C_{\mathrm{ad}}]) = \overline{\Theta}^{\vee}$.*

2. *If X_{ad} is exceptional, then there are exactly two minimal families $\mathcal{L}^+$ and $\mathcal{L}^-$ and the class of any $C^{\pm}_{\mathrm{ad}} \in \mathcal{L}^{\pm}$ satifies $\psi([C^{\pm}_{\mathrm{ad}}]) = \overline{\Theta}^{\vee}$.*

Note that if C_{ad} is a highest weight curve in $\mathcal{L}_{x_{\mathrm{ad}}}$, then its H-orbit $H \cdot C_{\mathrm{ad}}$ is contained in $\mathcal{L}_{x_{\mathrm{ad}}}$. We describe the family $\mathcal{L}_{x_{\mathrm{ad}}}$ by comparing the dimension of this orbit with the dimension of the family $\mathcal{L}_{x_{\mathrm{ad}}}$ of curves whose class is described by the previous result. To compute the dimension of the H-orbits, we prove that the tangent line $T_{x_{\mathrm{ad}}} C_{\mathrm{ad}}$ lies in very specific nilpotent orbits in $\mathfrak{g}$. Let $\mathcal{O}_{\min}$ be the minimal non-zero nilpotent orbit in $\mathfrak{g}$ and $\mathcal{O}_{\mathrm{sum},\sigma}$ be the nilpotent orbit of $e_\Theta - \sigma(e_\Theta)$, where Θ is the highest root of G and $e_\Theta \in \mathfrak{g}_\Theta \setminus \{0\}$. Let $m \in T_{x_{\mathrm{ad}}} C_{\mathrm{ad}} \setminus \{0\}$, we prove the following in Corollary 4.25.

Proposition 1.6. *We have $m \in \mathcal{O}_{\min}$ if $\sigma(\Theta) = -\Theta$ and $m \in \mathcal{O}_{\mathrm{sum},\sigma}$ otherwise.*

Using results of Kostant and Rallis [KR71] we prove that the orbit $H \cdot m$ is Lagrangian in the nilpotent orbit $G \cdot m$ (equipped with the Kirillov-Kostant-Souriau invariant symplectic structure) and we are able to compute the dimension of these orbits. We obtain:

Theorem 1.7 (Theorem 4.43). *If the restricted root system $\overline{R}$ is not of type A_r, then $\partial X_{\mathrm{ad}} \cdot C_{\mathrm{ad}} = 1$ and $\dim H \cdot C_{\mathrm{ad}} = \dim \mathcal{L}_{x_{\mathrm{ad}}}$. Otherwise, we have $\partial X_{\mathrm{ad}} \cdot C_{\mathrm{ad}} = 2$ and $\dim H \cdot C_{\mathrm{ad}} = \dim \mathcal{L}_{x_{\mathrm{ad}}} - 1$.*

1. *If $\overline{R}$ is not of type A_r, then $\mathcal{L}_{x_{\mathrm{ad}}} = H \cdot C_{\mathrm{ad}}$. Furthermore, $\mathcal{L}_{x_{\mathrm{ad}}}$ has two components if X is Hermitian non-exceptional and is irreducible otherwise.*

2. *If $\overline{R}$ is of type A_1, then $\mathcal{L}_{x_{\mathrm{ad}}} \simeq \mathbb{P}(\mathfrak{p})$.*

3. *If $\overline{R}$ is of type A_r with $r \geq 2$, then there exists a G-equivariant birational morphism $X_{\mathrm{ad}} \to \mathbb{P}(V)$, for some irreducible G-representation V, and $\mathcal{L}_{x_{\mathrm{ad}}}$ is isomorphic to the closed G-orbit in $\mathbb{P}(V)$. The orbit $H \cdot C_{\mathrm{ad}}$ is a prime divisor in $\mathcal{L}_{x_{\mathrm{ad}}}$.*

4. *The orbits $H \cdot C_{\mathrm{ad}}$ and the variety $\mathcal{L}_{x_{\mathrm{ad}}}$ are described in Table 1.*

By results of Ruzzi [Ru12], the wonderful compactifications of irreducible symmetric spaces are weak Fano varieties and most of them are Fano, with exceptions classified in loc. cit., Table 2 (see also §6.3). As a consequence, the wonderful compactifications of all Hermitian non-exceptional symmetric spaces are Fano, except in type CI. We thus obtain the following result.

Corollary 1.8. *Let X_{ad} be the wonderful compactification of a Hermitian non-exceptional symmetric space not of type* CI *and whose restricted root system is not of type A_r. Then X_{ad} is Fano and its* VMRT *has two irreducible components.*

The assumptions of the above corollary hold for four types in the classification: AIII, BDI, DIII and EVII. This yields examples of Fano varieties with reducible VMRT of positive dimension, thereby giving a negative answer to a question of Hwang, see [Hw01, Section 5, Question 2]. Note that these Fano varieties have Picard number at least 2, whereas there are examples of Fano varieties with Picard group $\mathbb{Z}$ and reducible VMRT of positive dimension, see [IM05, Proposition 3.15] and [MOS14, Remark A.9].

In Table 1, we also give the embedding of $\mathcal{L}_{\mathrm{ad}} \simeq \mathrm{VMRT}(X_{\mathrm{ad}})$ in $\mathbb{P}(\mathfrak{p})$. All VMRT are disjoint unions of projective rational homogeneous varieties, which are in turn products of homogeneous varieties of Picard rank one. In most cases, the embedding is the minimal embedding. For two cases (types AI and CI), the embedding is twice the minimal embedding. There is also a mixed case in type G.

From these results and Theorem 1.3, we obtain a full description of the VMRT of any complete symmetric variety X. We refer to Theorem 5.2 for more details.

Contents

Acknowledgements. The authors thank the Institut Fourier for hosting several meetings that led to this work. They also thank the Tsinghua Sanya International Mathematics Forum (TSIMF) and Baohua Fu for his invitation. The stay of the authors in TSIMF was very influential for this work. They thank Baohua Fu and Jun-Muk Hwang for helpful comments, in particular for pointing out the references [IM05] and [MOS14]. They finally thank the Centre Mathématique Laurent Schwartz (CMLS) at École Polytechnique for hosting the final meeting which led to this work. S.K. was partially supported by NRF Korea (2018R1A6A3A03012791) and the Institute for Basic Science (IBS-R003-D1). N.P. was partially supported by ANR Catore (ANR-18-CE40-0024) and ANR FanoHK (ANR-20-CE40-0023).

2 Rational curves and symmetric spaces

In this section, we recall basic results on rational curves on uniruled varieties and then specialise to the case of almost homogeneous varieties. We also introduce symmetric homogeneous spaces and their adjoint symmetric space, and we obtain the existence of highest weight curves and their basic properties.

2.1 Families of rational curves

In this subsection, we recall some notions and results on rational curves, after [Ko96, II.2.2, II.2.3] and [BK21, §2.1, §2.2].

Let X be a smooth projective variety. Consider the scheme of morphisms $\mathrm{Hom}(\mathbb{P}^1, X)$ and the open subscheme $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$ consisting of morphisms which are birational onto their image. The (normalized) *space of rational curves* $\mathrm{RatCurves}(X)$ is the quotient of the normalization $\mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X)$ by the free action of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$ via reparametrization. We have a universal family

$$\rho : \mathrm{Univ}(X) \longrightarrow \mathrm{RatCurves}(X)$$

which is a $\mathbb{P}^1$-bundle, and an evaluation map

$$\mu : \mathrm{Univ}(X) \longrightarrow X$$

such that the morphism $\rho \times \mu : \mathrm{Univ}(X) \to \mathrm{RatCurves}(X) \times X$ is finite.

Let $f \in \mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$ with image $C \subset X$. We say that C is *free* if the pull-back $f^*(T_X)$ is globally generated, where T_X denotes the tangent bundle. Every free morphism yields a smooth point of $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$, and hence of $\mathrm{RatCurves}(X)$. Also, we say that C is *embedded* if f is an immersion. The free (resp. embedded free) curves form smooth open subschemes $\mathrm{RatCurves}_{\mathrm{emfr}}(X) \subset \mathrm{RatCurves}_{\mathrm{fr}}(X)$ of the space of rational curves.

A *family of rational curves* on X is a component $\mathcal{K}$ of $\mathrm{RatCurves}(X)$. We then have a universal family $\rho : \mathcal{U} = \rho^{-1}(\mathcal{K}) \to \mathcal{K}$ which is again a $\mathbb{P}^1$-bundle, and an evaluation map $\mu : \mathcal{U} \to X$. For any $x \in X$, let $\mathcal{U}_x = \mu^{-1}(x)$ and $\mathcal{K}_x = \rho(\mathcal{U}_x)$; then $\mathcal{K}_x$ is the subfamily of curves through x. The restriction $\rho_x : \mathcal{U}_x \to \mathcal{K}_x$ is finite, and is an isomorphism above the smooth open subset of embedded free curves (see [BK21, Lem. 2.1]).

The family $\mathcal{K}$ is *covering* if $\mathcal{K}_x$ is non-empty for x general. If in addition $\mathcal{K}_x$ is projective for x general, we say that $\mathcal{K}$ is *minimal.*

By sending every embedded free curve in $\mathcal{K}_x$ to its tangent direction at x, we obtain a morphism $\tau_x : \mathcal{K}_{\mathrm{emfr},x} \to \mathbb{P}(T_xX)$, where $\mathbb{P}(T_xX)$ denotes the projectivization of the tangent space. We will view τ as a rational map $\mathcal{K}_x \dashrightarrow \mathbb{P}(T_xX)$, defined at every curve which is smooth at x. The closure of the image of τ is denoted by $\mathcal{C}_x$ and called the *variety of tangents* of $\mathcal{K}$ at x.

Let $\mathcal{K}$ be a minimal family of rational curves on X. By [Ke02, Thm. 3.3], for a general point x, there are only finitely many curves in $\mathcal{K}_x$ which are singular at x. Thus, τ_x is defined along every positive-dimensional irreducible component of $\mathcal{K}_x$. In view of [Ke02, Thm. 3.4], τ_x extends to a finite morphism

$$\tau_x^{\mathrm{n}} : \mathcal{K}_x^{\mathrm{n}} \longrightarrow \mathbb{P}(T_xX),$$

where $\mathcal{K}_x^{\mathrm{n}}$ denotes the normalization. Moreover, τ_x^{n} is birational onto its image by [HM04, Thm. 1]. The image $\mathcal{C}_x$ is called the *variety of minimal tangents* of $\mathcal{K}$ at x (VMRT).

Next, we consider covariance properties of families under a morphism of smooth projective varieties $\pi : X \to Y$. Let $\mathcal{K}$ be a family of rational curves on X. Assume that some $C \in \mathcal{K}$ is represented by a free morphism $f : \mathbb{P}^1 \to X$ which is birational onto its image, and such that the composition $\pi \circ f : \mathbb{P}^1 \to Y$ is free and birational onto its image as well. Let D be the corresponding rational curve in Y, and $\mathcal{L}$ the family on Y containing the free rational curve D. Finally, let $x = f(0) \in C$ and $y = \pi(x) \in D$.

Lemma 2.1. *With the above notation and assumptions, the morphism $\pi : X \to Y$ induces rational maps*

$$\pi_* : \mathcal{K} \dashrightarrow \mathcal{L}, \quad \pi_{*,x} : \mathcal{K}_x \dashrightarrow \mathcal{L}_y$$

which are defined at C and send C to D. If the differential $d\pi_x : T_xX \to T_yY$ is injective, then so is the differential of $\pi_{*,x}$ at C.

Proof. Composing by π yields a morphism $\mathrm{Hom}(\mathbb{P}^1, X) \to \mathrm{Hom}(\mathbb{P}^1, Y)$ which is $\mathrm{Aut}(\mathbb{P}^1)$-equivariant, and hence an equivariant rational map $\mathrm{Hom}_{\mathrm{fr}}(\mathbb{P}^1, X) \dashrightarrow \mathrm{Hom}_{\mathrm{fr}}(\mathbb{P}^1, Y)$ which is defined at f. This readily yields the rational map π_*. The rational map $\pi_{*,x}$ is obtained from the analogous morphism $\mathrm{Hom}(\mathbb{P}^1, X; 0 \mapsto x) \to \mathrm{Hom}(\mathbb{P}^1, Y; 0 \mapsto y)$ with the notation of [Ko96, II.1]. By loc. cit., II.2.3, the differential of the above morphism at f is identified with the natural map $H^0(\mathbb{P}^1, (f^*T_X)(-1)) \to H^0(\mathbb{P}^1, (f^*\pi^*T_Y)(-1))$. This implies the final assertion as $d\pi$ is injective on an open dense subset of X. $\square$

In the opposite direction, assume that π *contracts* a curve $C \in \mathcal{K}$, i.e., the composition $\rho^{-1}(C) \xrightarrow{\mu} X \xrightarrow{\pi} Y$ is constant; then π contracts all the curves in $\mathcal{K}$. With this terminology, we may recall a useful observation ([BK21, Lem. 2.3]):

Lemma 2.2. *Consider two smooth projective varieties Y, Z, and let $X := Y \times Z$ with projections $p : X \to Y$, $q : X \to Z$.*

1. *The pull-back map $p^* : \mathrm{Hom}(\mathbb{P}^1, Y) \times Z \to \mathrm{Hom}(\mathbb{P}^1, X)$, $(f, z) \to (t \mapsto (f(t), z))$ induces a closed immersion $\mathrm{RatCurves}(Y) \times Z \to \mathrm{RatCurves}(X)$ with image a union of components.*

2. *The map p^* sends covering (resp. minimal) families to covering (resp. minimal) families.*

3. *A family of rational curves $\mathcal{K}$ on X is the pull-back of a family on Y if and only if q contracts some curve in $\mathcal{K}$.*

4. *Every family of minimal rational curves on X is the pull-back of a unique family of minimal rational curves on Y or Z.*

2.2 Almost homogeneous varieties

We now assume that X is *almost homogeneous*, i.e., it is equipped with an action of a connected linear algebraic group G, and contains an open G-orbit X^0. We recall and slightly generalize results from [BF15, §2] and [BK21, §2.3].

Choose a base point $x \in X^0$, and denote by $H = G_x$ its isotropy group. Then the orbit $X^0 = G \cdot x$ is identified with the homogeneous space G/H, and the pair (X, x), with an equivariant embedding of this homogeneous space. Denoting by $\mathfrak{g}$ (resp. $\mathfrak{h}$) the Lie algebra of G (resp. H), the tangent space $T_x X$ is identified with the quotient $\mathfrak{g}/\mathfrak{h}$ as a representation of H (the *isotropy representation*).

Since G is a rational variety, X is unirational; as a consequence, covering families exist. Also, G acts on $\mathrm{RatCurves}(X)$ and on $\mathrm{Univ}(X)$ so that ρ and μ are equivariant. Since G is connected, it stabilizes every family $\mathcal{K}$, as well as the open subset $\mathcal{K}^0$ consisting of curves which meet X^0. Every such curve is free (see e.g. [BF15, Lem. 2.1(i)]); thus, $\mathcal{K}^0$ is smooth. The subgroup $H \subset G$ acts compatibly on $\mathcal{U}_x$, $\mathcal{K}_x$, $\mathbb{P}(T_x X)$ and $\mathcal{C}_x$.

We now obtain a variant of [BK21, Lem. 2.4]:

Lemma 2.3. *A family of rational curves $\mathcal{K}$ on X is covering if and only if $\mathcal{U}_x$ is non-empty; equivalently, $\mathcal{K}_x$ is non-empty. Under these assumptions, $\mathcal{U}_x$ is smooth and its components are permuted transitively by H.*

Proof. The morphism μ restricts to a G-equivariant morphism

$$\mu^0 : \mathcal{U}^0 = \mu^{-1}(X^0) \longrightarrow X^0 = G/H$$

with fiber at x being $\mathcal{U}_x$. This yields an isomorphism $\mathcal{U}^0 \simeq G \times^H \mathcal{U}_x$, where the right-hand side denotes the quotient of $G \times \mathcal{U}_x$ by the H-action via $h \cdot (g, z) = (gh^{-1}, h \cdot z)$. Since $\mathcal{K}^0$ is smooth, so are $\mathcal{U}^0$ and hence $\mathcal{U}_x$. Also, $\mathcal{U}^0$ is irreducible; thus, H acts transitively on the components of $\mathcal{U}_x$. □

Next, let $\pi : X \to Y$ be a surjective morphism, where Y is a smooth projective variety. Assume that Y is equipped with a G-action such that π is equivariant. Let $y = \pi(x)$ and $Y^0 = G \cdot y$; then $Y^0 = \pi(X^0)$ is open in Y. We now have the following variants of [BK21, Lem. 2.6, Rem. 2.7]:

Lemma 2.4. *Keep the above notation and assumptions, and consider a covering family of rational curves $\mathcal{K}$ on X. Assume that there exists $C \in \mathcal{K}^0$ such that $\pi|_C$ is birational onto its image D. Then:*

1. *$D \in \mathcal{L}$ for a unique covering family $\mathcal{L}$ of rational curves on Y.*

2. *π induces a G-equivariant rational map $\pi_* : \mathcal{K} \dashrightarrow \mathcal{L}$, which is defined at C and satisfies $\pi_*(C) = D$, and an H-equivariant rational map*

$$\pi_{*,x} : \mathcal{K}_x \dashrightarrow \mathcal{L}_y, \quad C \longmapsto D.$$

3. *We have a commutative diagram of H-equivariant rational maps*

$$\begin{tikzcd} \mathcal{K}_x \arrow[r, dashed, "\pi_{*,x}"] \arrow[d, dashed, "\tau_x"'] & \mathcal{L}_y \arrow[d, dashed, "\tau_y"] \\ \mathbb{P}(T_xX) \arrow[r, dashed, "d\pi_x"] & \mathbb{P}(T_yY). \end{tikzcd}$$

Proof. (1) Replacing C with a translate $g \cdot C$ for some $g \in G$, we may assume that $x \in C$. Then the assertion follows from Lemma 2.3.

(2) This is a consequence of Lemma 2.1, except for the equivariance assertions which are easily checked.

(3) This follows readily from the definitions. □

Remark 2.5. If π is birational (equivalently, it induces an isomorphism $X^0 \to Y^0$), then the assumptions of Lemma 2.4 hold and moreover $\pi_{*,x}$ is an immersion. Indeed, $\pi_{*,x}$ is clearly an injective morphism. Moreover, the differential of $\pi_{*,x}$ at every $C \in \mathcal{K}_x$ is injective by Lemma 2.1.

Still considering a covering family of rational curves $\mathcal{K}$ on X, we now assume that π contracts some curve in $\mathcal{K}$, and hence all curves in $\mathcal{K}$. Let

$$X \xrightarrow{\pi'} Y' \xrightarrow{\eta} Y$$

be the Stein factorization of π, where Y' is a normal projective variety (possibly singular), π' is a contraction (that is, $\pi'_*(\mathcal{O}_X) = \mathcal{O}_{Y'}$), and η is finite surjective. Then there is a unique action of G on Y' such that π' and η are equivariant. Let $y' = \pi'(x)$ and $I = G_{y'}$; then $H \subset I \subset G$ and the orbit $G \cdot y' \simeq G/I$ is open in Y'. Also, let $F = \pi'^{-1}(y')$; then F is the connected component of x in the fiber $\pi^{-1}(y)$, and hence is a smooth projective variety (by generic smoothness). Moreover, F is stable by I and contains $I \cdot x$ as its open orbit. Clearly, every curve in $\mathcal{K}_x$ is contained in F.

Lemma 2.6. *Keep the above notation and assumptions, and assume that I normalizes H. Then $\mathcal{K}_x$ is irreducible and there exists a unique covering family of rational curves $\mathcal{L}$ on F such that $\mathcal{K}_x = \mathcal{L}_x$. Moreover, $\mathcal{K}^0 = G \cdot \mathcal{L}^0$.*

Proof. Note that H acts trivially on I/H, since $H \triangleleft I$. Thus, H acts trivially on F, and hence on $\mathcal{U}_x$. As $\mathcal{U}^0 \simeq G \times^H \mathcal{U}_x$, we obtain $\mathcal{U}^0 \simeq G/H \times \mathcal{U}_x$. Since $\mathcal{U}^0$ is irreducible, so are $\mathcal{U}_x$ and $\mathcal{K}_x$.

The inclusion $\iota : F \to X$ induces compatible immersions

$$\mathrm{RatCurves}_{\mathrm{fr}}(F) \longrightarrow \mathrm{RatCurves}_{\mathrm{fr}}(X), \quad \mathrm{Univ}_{\mathrm{fr}}(F) \longrightarrow \mathrm{Univ}_{\mathrm{fr}}(X),$$

since they are injective and their differentials are injective as well. It follows that $\mathcal{K}_x$ is an irreducible component of $\mu_F^{-1}(x)$, where $\mu_F : \mathrm{Univ}(F) \to F$ denotes the evaluation map. So $\mathcal{K}_x$ is an irreducible component of $\mathcal{L}_x$ for a unique family of rational curves $\mathcal{L}$ on F. Since (F, x) is an equivariant embedding of the homogeneous space $I^0/I^0 \cap H$ and $I^0 \cap H \triangleleft I^0$, we see that $\mathcal{L}_x$ is irreducible. Thus, $\mathcal{K}_x = \mathcal{L}_x$ and $\mathcal{L}^0 = I^0 \cdot \mathcal{L}_x = I^0 \cdot \mathcal{K}_x$, so that $\mathcal{K}^0 = G \cdot \mathcal{K}_x = G \cdot \mathcal{L}_x = G \cdot \mathcal{L}^0$. □

Remark 2.7. The above assignement $\mathcal{K} \mapsto \mathcal{L}$ yields a bijection between covering families of rational curves on X which are contracted by π, and covering families of rational curves on F. This restricts to a bijection between minimal families.

2.3 Symmetric spaces

In this subsection, we recall some basic facts on symmetric spaces, after [Ti11, §26] and its references. We begin with some notation and conventions which will be used throughout the sequel.

Let G be a connected reductive algebraic group. Let $T \subset G$ be a maximal torus, and $B \subset G$ a Borel subgroup containing T. We denote the character group of T by $\mathfrak{X} = \mathfrak{X}(T)$, and the root system of (G, T) by $R = R(G, T) \subset \mathfrak{X}$. The roots of (B, T) form the set of positive roots R^+, with basis Δ (the set of simple roots). The Weyl group of (G, T) is denoted by W.

Recall the decomposition of Lie algebras $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. For any $\alpha \in R$, we denote by U_α the closed subgroup of G with Lie algebra $\mathfrak{g}_\alpha$, and by G_α the subgroup of G generated by U_α and $U_{-\alpha}$. Then G_α is a closed subgroup of G, isomorphic to SL_2 or PSL_2.

Next, let σ be a group involution of G. Denote by G^σ the fixed point subgroup, and by $G^{\sigma,0}$ its neutral component. Let H be a subgroup of G such that $G^{\sigma,0} \subset H \subset G^\sigma$; then we say that H is a *symmetric subgroup* of G, and the homogeneous space G/H is a *symmetric space.*

The group H is reductive; equivalently, the variety G/H is affine. Also, σ induces an involution of $\mathfrak{g}$, still denoted by σ for simplicity. The Lie algebra of H satisfies $\mathfrak{h} = \mathfrak{g}^\sigma$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where

$$\mathfrak{p} = \mathfrak{g}^{-\sigma} = \{x \in \mathfrak{g} \mid \sigma(x) = -x\}$$

is a G^σ-stable complement of $\mathfrak{h}$ in $\mathfrak{g}$.

Recall that σ stabilizes a maximal torus T of G. Thus, σ acts on the character group $\mathfrak{X}$ and stabilizes the root system R; it also acts on the Weyl group W by conjugation. We may choose a scalar product $(-, -)$ on the real vector space $\mathfrak{X}_\mathbb{R} = \mathfrak{X} \otimes_\mathbb{Z} \mathbb{R}$ which is invariant under W and σ.

Definition 2.8. For $\alpha \in R$, one of the following cases occurs:

1. $\sigma(\alpha) = \alpha$ and σ fixes pointwise $\mathfrak{g}_\alpha$. Then α is called a *compact imaginary* root.

2. $\sigma(\alpha) = \alpha$ and σ acts on $\mathfrak{g}_\alpha$ by -1. Then α is *non-compact imaginary.*

3. $\sigma(\alpha) = -\alpha$. Then α is *real.*

4. $\sigma(\alpha) \neq \pm\alpha$. Then α is *complex.*

Recall that any two maximal tori of G are conjugate and hence $\mathfrak{X}$, R and W are independent of the choice of T. But the action of σ on these objects depends on the choice of the σ-stable torus T, up to conjugacy by H. We now consider two special conjugacy classes of σ-stable maximal tori, that we call of fixed (resp. split) type. These are constructed as follows.

Maximal tori of fixed type. Choose a maximal torus T_H of H; then its centralizer $T = \mathrm{C}_G(T_H)$ is a σ-stable maximal torus of G and we have $T_H = T^{\sigma,0}$. Moreover, T is contained in a σ-stable Borel subgroup B of G; then $B_H = B^{\sigma,0}$ is a Borel subgroup of H (see [Ti11, Lem. 26.7]). Thus, $B_H = U_H T_H$, where $U_H = U \cap H$ is a maximal unipotent subgroup of H.

The action of σ on the root system R stabilizes R^+. In particular, there are no real roots. The subset of simple roots Δ is σ-stable as well.

Clearly, the maximal tori obtained in this way are exactly those containing a maximal σ-fixed torus (i.e., a subtorus $S \subset G$ such that $\sigma(s) = s$ for all $s \in S$); they are all conjugate under $H^0 = G^{\sigma,0}$. We call every such maximal torus *of fixed type* and denote it by T_{f}.

Maximal tori of split type. In the opposite direction, a subtorus $S \subset G$ is called *σ-split* if $\sigma(s) = s^{-1}$ for all $s \in S$. Choose such a torus S maximal for this property; then $L = \mathrm{C}_G(S)$ satisfies $[L, L] \subset H^0$. As a consequence, every maximal torus of G containing S is σ-stable. We call every such maximal torus *of split type* and denote it by T_{s}. Also, L is a Levi subgroup of a minimal σ-split parabolic subgroup P, that is, P is a parabolic subgroup of G which is opposite to $\sigma(P)$, and minimal for this property. Denote by $\mathrm{R}_u(P)$ the unipotent radical of P; then by the Iwasawa decomposition, the morphism

$$\iota : \mathrm{R}_u(P) \times S/S \cap H \longrightarrow G/H, \quad (g, z) \longmapsto g \cdot z$$

is an open immersion with image $P/P \cap H$, the P-orbit of the base point. Moreover, ι is P-equivariant, where $P = \mathrm{R}_u(P) \rtimes L$ acts on $\mathrm{R}_u(P) \times S/S \cap H$ via $(u, l) \cdot (g, z) = (ulgl^{-1}, l \cdot z)$, and on G/H via left multiplication. Finally, recall that the maximal split tori are all conjugate under H^0, as well as the minimal split parabolic subgroups and the maximal tori of split type.

The maximal tori of fixed type will be used in the rest of this section and in Section 3. Those of split type, and the corresponding restricted root system, feature prominently in the subsequent sections.

2.4 The normalizer of a symmetric subgroup

We keep the notation of §2.3 and denote by $Z = Z(G)$ the center of G, with Lie algebra $\mathfrak{z}$. Since Z is σ-stable, we have $\mathfrak{z} = (\mathfrak{z} \cap \mathfrak{h}) \oplus (\mathfrak{z} \cap \mathfrak{p})$. Also, we denote by $N = \mathrm{N}_G(G^\sigma)$ the normalizer of G^σ in G and by $G_{\mathrm{ad}} = G/Z$ the adjoint group.

Lemma 2.9. *1.* $N = \{g \in G \mid \sigma(g) g^{-1} \in Z\}$.

2. $N = \mathrm{N}_G(G^{\sigma,0}) = \mathrm{N}_G(\mathfrak{h})$.

3. $N^0 = Z^0 H^0$.

4. $Z^0 T_H$ *is a maximal torus of* N *for any maximal torus* T_H *of* H.

Proof. (1) This is obtained in [Vu90, Lem. 1] (see also [deCP83, I.7]); we recall the argument for completeness.

Let $g \in G$ such that $\sigma(g)g^{-1} \in Z$. For any $h \in G^\sigma$, we have $\sigma(ghg^{-1}) = \sigma(g)h\sigma(g)^{-1} = ghg^{-1}$, that is, $ghg^{-1} \in G^\sigma$. So $g \in N$. For the converse, observe that N is reductive and normalized by σ. The corresponding semi-direct product $N \rtimes \langle\sigma\rangle$ is a reductive algebraic group, which acts linearly on $\mathfrak{g}$ and stabilizes $\mathfrak{h}$. Thus, $\mathfrak{h}$ has an $N \rtimes \langle\sigma\rangle$-stable complement, which must be $\mathfrak{p}$. In particular, $\mathfrak{p}$ is N-stable; thus, $\mathrm{Ad}(N)$ commutes with σ. So $\mathrm{Ad}(\sigma(g)g^{-1}) = \sigma \,\mathrm{Ad}(g)\sigma^{-1}\,\mathrm{Ad}(g)^{-1} = \mathrm{id}$ for any $g \in N$, that is, $\sigma(g)g^{-1} \in Z$.

(2) Clearly, we have $N = \mathrm{N}_G(G^\sigma) \subset \mathrm{N}_G(G^{\sigma,0}) = \mathrm{N}_G(\mathfrak{h})$. Moreover, $\mathrm{N}_G(\mathfrak{h})$ is reductive and normalized by σ. Arguing as in the proof of (1), it follows that $\sigma(g)g^{-1} \in Z$ for any $g \in \mathrm{N}_G(\mathfrak{h})$, and hence $g \in N$.

(3) Denote by $\mathfrak{n}$ the Lie algebra of N. Then (1) yields that $\mathfrak{n} = \{x \in \mathfrak{g} \mid \sigma(x) - x \in \mathfrak{z}\}$. Using the σ-stable decomposition $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g},\mathfrak{g}]$, it follows that $\mathfrak{n} = \mathfrak{z} \oplus [\mathfrak{g},\mathfrak{g}]^\sigma = \mathfrak{z} + \mathfrak{h}$. This yields the assertion.

(4) This follows readily from (3). □

Lemma 2.10. *Let* S *be a maximal* σ*-split torus of* G*.*

1. $N = G^{\sigma,0}(N \cap S)$.

2. $H = G^{\sigma,0}(H \cap S)$ *and* $H \cap S$ *is an elementary abelian* 2*-group.*

3. $N = \mathrm{N}_G(H)$ *and* $N/H \simeq N \cap S/H \cap S$*. In particular,* N/H *is diagonalizable.*

Proof. (1) Let P be a minimal σ-split parabolic subgroup of G containing S. Then $P_{\mathrm{ad}} = P/Z$ is a minimal σ-split parabolic subgroup of G_{ad}, containing $S_{\mathrm{ad}} = S/S \cap Z$ which is a maximal σ-split torus of G_{ad}. By the Iwasawa decomposition, the morphism

$$\mathrm{R}_u(P_{\mathrm{ad}}) \times S_{\mathrm{ad}}/S_{\mathrm{ad}}^\sigma \longrightarrow G_{\mathrm{ad}}/G_{\mathrm{ad}}^\sigma, \quad (g,z) \longmapsto g \cdot z$$

is an open immersion. As a consequence, the multiplication map $\mathrm{R}_u(P) \times SN \to G$ is an open immersion as well. Its image is $\mathrm{R}_u(P)SN = PN$, the open orbit of $P \times N$ in G. Likewise, $PG^{\sigma,0}$ is the open orbit of $P \times G^{\sigma,0}$ in G. Let $g \in N$; then the orbit $PgG^{\sigma,0} = PG^{\sigma,0}g$ is open in G. So $g \in PG^{\sigma,0}$, and hence $PN = PG^{\sigma,0}$. It follows that $SN = SG^{\sigma,0}$; this yields the assertion.

(2) The first assertion follows readily from (1). For the second assertion, just note that every $g \in S \cap H$ satisfies $g^{-1} = \sigma(g) = g$.

(3) Clearly, we have $\mathrm{N}_G(H) \subset \mathrm{N}_G(H^0) = \mathrm{N}_G(G^{\sigma,0})$. Moreover, $\mathrm{N}_G(G^{\sigma,0}) = N$ in view of Lemma 2.9.2. Also, by combining (1) and (2) above, we see that N normalizes H, since $N \cap S$ normalizes $G^{\sigma,0}$ and centralizes $H \cap S$. Thus, $\mathrm{N}_G(H) = N$. By (1) again, we have $N = H(N \cap S)$, and hence $N/H \simeq (N \cap S)/(H \cap S)$ is diagonalizable. □

2.5 The adjoint symmetric space

Recall that $G_{\mathrm{ad}} = G/Z$ is the adjoint group, and let

$$q : G \longrightarrow G_{\mathrm{ad}}$$

be the quotient homomorphism. Then σ induces an involution of G_{ad}, still denoted by σ for simplicity. The homogeneous space $G_{\mathrm{ad}}/G_{\mathrm{ad}}^{\sigma}$ is called an *adjoint symmetric space*. By Lemma 2.9, we have $N = q^{-1}(G_{\mathrm{ad}}^{\sigma})$; this yields an isomorphism

$$G/N \simeq G_{\mathrm{ad}}/G_{\mathrm{ad}}^{\sigma}.$$

Moreover, G_{ad}^{σ} is its own normalizer in G_{ad}.

By decomposing G_{ad} into a product of indecomposable σ-stable factors, we obtain a decomposition of G/N into a product of *irreducible symmetric spaces*. These fall into three types:

1. (group) $(H \times H)/\mathrm{diag}(H)$, where H is simple adjoint.
2. (Hermitian) $G/\mathrm{N}_G(L)$, where G is simple adjoint and $L \subset G$ is a Levi subgroup.
3. (simple) G/H, where G is simple adjoint and H^0 is simple.

In type (1), we have $\sigma(x, y) = (y, x)$ for all $x, y \in H$. Thus, G/N is just the group H on which $H \times H$ by left and right multiplication. The isotropy representation $\mathfrak{p}$ is the adjoint representation of H in $\mathfrak{h}$. This is an irreducible representation with highest weight the highest root Θ.

In type (2), we have $L = P \cap Q$, where P and Q are opposite maximal parabolic subgroups of G. Moreover, σ is the conjugation $\mathrm{Int}(c)$, where $c \in Z(L)$ and $c^2 \in Z(G)$. Denote by α the unique simple root which is not a root of L; then α has coefficient 1 in the expansion of the highest root Θ as a linear combination of simple roots. We have $\mathfrak{p} = \mathfrak{u}_P \oplus \mathfrak{u}_Q$, where $\mathfrak{u}_P$ (resp. $\mathfrak{u}_Q$) denotes the Lie algebra of $\mathrm{R}_u(P)$ (resp. $\mathrm{R}_u(Q)$). Moreover, the representations $\mathfrak{u}_P$, $\mathfrak{u}_Q$ of L are irreducible and dual to each other (see e.g. [RRS92, §5.5] for these results). Their highest weights relative to L are Θ, $-\alpha$; they are linearly independent unless $G = \mathrm{PSL}_2$.

We say that $G/\mathrm{N}_G(L)$ is (Hermitian) *exceptional* if P and Q are not conjugate in G. Then $\mathrm{N}_G(L) = L$, and hence $G/\mathrm{N}_G(L)$ may be identified with the open G-orbit in $G/P \times G/Q$ on which G acts diagonally. In the non-exceptional case, where P and Q are conjugate in G, the group $\mathrm{N}_G(L)/L$ has order 2 and exchanges P and Q. Moreover, $G/\mathrm{N}_G(L)$ may be identified with the open G-orbit in the symmetric square $(G/P)^{(2)}$ (the quotient of $G/P \times G/P$ by the involution $(y, z) \mapsto (z, y)$).

In type (3), $\mathfrak{p}$ is irreducible as a representation of H^0, with a non-zero highest weight.

2.6 Highest weight curves

Throughout this subsection, we choose a maximal torus $T_H \subset H$ and a Borel subgroup $B_H \subset H$ containing T_H. Recall that $T = \mathrm{C}_G(T_H)$ is a maximal torus of fixed type of G. We first obtain a generalization of [BF15, Lem. 2.2]:

Lemma 2.11. *Let C be an irreducible B_H-stable curve in G/H through the base point x.*

1. *Either C is contained in $Z^0 \cdot x$, or B_H acts non-trivially on C.*
2. *In the latter case, C is smooth and B_H-equivariantly isomorphic to its tangent line at x, which is the T_H-weight space $\mathfrak{p}_\lambda$ for a unique non-zero highest weight λ of $\mathfrak{p}$. Moreover, λ determines C uniquely, and the stabilizer of C in H equals the stabilizer of the weight space $\mathfrak{p}_\lambda$.*

Proof. (1) Assume that C is fixed pointwise by B_H. Then the orbit $H^0 \cdot y$ is complete for any $y \in C$. Since G/H is affine, this orbit must be a point, i.e., y is fixed pointwise by H^0. Let $g \in G$ such that $y = g \cdot x$, then $g^{-1}H^0 g \cdot x = x$, i.e., $g^{-1}H^0 g \subset H$. So $g \in \mathrm{N}_G(H^0) = N$ (Lemma 2.9). Thus, $C \subset N \cdot x$. As C is connected and contains x, it follows that $C \subset N^0 \cdot x$. But $N^0 \cdot x = Z^0 \cdot x$ by Lemma 2.9 again; this yields the assertion.

(2) This is obtained by arguing as in the proof of [BF15, Lem. 2.2(i)]. We provide details for the reader's convenience.

Since C is not fixed pointwise by B_H, it contains an open orbit $B_H \cdot y$, where $y \neq x$. Thus, the isotropy group $B_{H,y}$ has codimension 1 in B_H. We thus have $B_{H,y}^0 = U_{H,y} \rtimes S$ for some subtorus S of B_H. Replacing y with a B_H-translate, we may assume that $S \subset T_H$.

If $S = T_H$, then $B_H \cdot y = U_H \cdot y$ is closed in G/H, since the latter is an affine variety. But $x \in \overline{B_H \cdot y} \setminus \{y\}$, a contradiction. For dimension reasons, it follows that S is a subtorus of codimension 1 of T_H, and $U_H \subset B_{H,y}$. As a consequence, C is fixed pointwise by U_H, since the latter is a normal subgroup of B_H. Thus, $T_H \cdot y$ is open in C.

In particular, $x \in \overline{H \cdot y}$. So C is contained in the fiber at x of the geometric invariant theory quotient $G/H \to H\backslash\!\backslash G/H$ of the smooth affine H-variety G/H. By a corollary of Luna's slice theorem (see [Lu73, II.1, III.1]), this fiber is H-equivariantly isomorphic to the nilcone $\mathcal{N}$ of $\mathfrak{p}$ (the fiber at 0 of the quotient $\mathfrak{p} \to \mathfrak{p}/\!\!/H$; it consists of the points $z \in \mathfrak{p}$ such that $0 \in \overline{H \cdot z}$). Thus, C is B_H-equivariantly isomorphic to a B_H-stable curve D in $\mathcal{N}$. Moreover, C and D have the same stabilizer in H.

As U_H fixes D pointwise, we have $D \subset \mathcal{N} \cap \mathfrak{p}^{U_H}$. Also, $\mathfrak{p} = (\mathfrak{p} \cap \mathfrak{z}) \oplus (\mathfrak{p} \cap [\mathfrak{g}, \mathfrak{g}])$ and the projection $\mathfrak{p} \to \mathfrak{p} \cap \mathfrak{z}$ is H-invariant, hence sends $\mathcal{N}$ to 0. it follows that $D \subset (\mathfrak{p} \cap [\mathfrak{g}, \mathfrak{g}])^{U_H}$.

So we may assume that G/H is an adjoint symmetric space. Using the product decomposition of these spaces, we see that D is a highest weight line from a unique indecomposable factor of G/H, and is uniquely determined by its weight. □

We say that a curve C as in Lemma 2.11.2 is a *highest weight curve*. The corresponding highest weight λ satisfies $\lambda = \alpha|_{T_H}$ for some root α, since the non-zero weights of T_H in $\mathfrak{p}$ are restrictions of non-zero weights of T in $\mathfrak{g}$. Let $S = \mathrm{Ker}(\lambda)^0 = (\mathrm{Ker}(\alpha) \cap T_H)^0$; then S is a subtorus of codimension 1 of T_H, and fixes C pointwise. Thus, the centralizer $\mathrm{C}_G(S)$ is a σ-stable connected reductive subgroup of G containing T. Moreover, $\mathrm{C}_H(S)$ is a symmetric subgroup of $\mathrm{C}_G(S)$ containing T_H, and C is a highest weight curve of the symmetric space $\mathrm{C}_G(S)/\mathrm{C}_H(S)$.

Recall the following easy result (see [Sp83, §2] and [Br99, Lem. 2.5]):

Lemma 2.12. *With the above notation and assumptions, the adjoint symmetric space of $\mathrm{C}_G(S)/\mathrm{C}_H(S)$ is one of the following:*

(A_1) PSL_2/N, *where* N *denotes the normalizer of the diagonal torus in* PSL_2.

($\mathrm{A}_1 \times \mathrm{A}_1$) $(\mathrm{PSL}_2 \times \mathrm{PSL}_2)/\mathrm{diag}(\mathrm{PSL}_2)$. *Then* $\sigma(\alpha)$ *is strongly orthogonal to* α.

(A_2) $\mathrm{PSL}_3/\mathrm{SO}_3$, *where* SO_3 *denotes the special orthogonal group. Then* $\alpha + \sigma(\alpha) \in R$.

Moreover, α *is non-compact imaginary in types* (A_1) *and* (A_2). *If the involution* σ *is inner (equivalently,* T_H *is a maximal torus of* G*), then only type* (A_1) *occurs.*

Still considering a highest curve C of weight λ, we now obtain a description of C and its tangent line $T_xC \subset T_xG/H$ (using the identifications $T_xG/H = \mathfrak{g}/\mathfrak{h} = \mathfrak{p}$) in the above three types.

Proposition 2.13. *In type* (A_1), *there is a unique root* α *such that* $\lambda = \alpha|_{T_H}$. *Moreover,* $C = U_\alpha \cdot x$ *and* $T_xC = \mathfrak{g}_\alpha$.

In type ($\mathrm{A}_1 \times \mathrm{A}_1$), *there are exactly two roots* α, β *such that* $\lambda = \alpha|_{T_H} = \beta|_{T_H}$. *Moreover,* α *and* $\beta = \sigma(\alpha)$ *are the simple roots of* $(\mathrm{C}_G(S), T)$. *We have* $C = U_\alpha \cdot x = U_\beta \cdot x$ *and* $T_xC = \mathbb{C}(e_\alpha - \sigma(e_\alpha))$, *where* $e_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$.

In type (A_2), *there is a unique root* α *such that* $\lambda = \alpha|_{T_H}$. *Moreover,* $\alpha = \alpha_1 + \alpha_2$, *where* α_1 *and* $\alpha_2 = \sigma(\alpha_1)$ *are the simple roots of* $(\mathrm{C}_G(S), T)$. *Also,* $C = U_\alpha \cdot x$ *and* $T_xC = \mathfrak{g}_\alpha$.

Proof. In type (A_1), there are two highest weight curves in $\mathrm{C}_G(S)/\mathrm{C}_H(S)$, namely, $U_\alpha \cdot x$ and $U_{-\alpha} \cdot x$.

In type ($\mathrm{A}_1 \times \mathrm{A}_1$), recall that the adjoint symmetric space of $\mathrm{C}_G(S)/\mathrm{C}_H(S)$ is the group $(\mathrm{PSL}_2 \times \mathrm{PSL}_2)/\mathrm{diag}(\mathrm{PSL}_2) = \mathrm{PSL}_2$. So the roots of $(\mathrm{C}_G(S), T)$ are $\pm\alpha$, $\pm\sigma(\alpha)$. Moreover, $U_\alpha \cdot x$ is the unique highest weight curve; it is identified with the standard unipotent subgroup $U \subset \mathrm{PSL}_2$, and likewise for $U_{\sigma(\alpha)} \cdot x$.

In type (A_2), the adjoint symmetric space of $\mathrm{C}_G(S)/\mathrm{C}_H(S)$ is $\mathrm{PSL}_3/\mathrm{SO}_3$; one checks that the highest weight of its isotropy representation is $(\alpha_1+\alpha_2)|_{T_H}$, where α_1, α_2 are the simple roots of PSL_3. It follows that $\alpha = \alpha_1 + \alpha_2$, and $\sigma(\alpha_1) = \alpha_2$. Finally, $U_\alpha \cdot x$ is an irreducible curve in G/H, stable by T_H and fixed by U_H (since the latter commutes with U_α and fixes x). Thus, $U_\alpha \cdot x$ is the highest weight curve in $\mathrm{C}_G(S)/\mathrm{C}_H(S)$.

This yields the assertions on roots and highest weight curves. Those on their tangent lines are readily verified. □

Corollary 2.14. *Let* C *be a highest weight curve of weight* λ. *Then there exists* $\alpha \in R$ *such that* $\lambda = \alpha|_{T_H}$ *and* $C = U_\alpha \cdot x$.

Corollary 2.15. *We have the following alternative for a simple group* G:

1. $T_xC = \mathfrak{g}_\alpha$ *for a long root* α, *or*
2. $T_xC = \mathfrak{g}_\alpha$ *for a short root* α, *or*
3. T_xC *is spanned by* $e_\alpha - \sigma(e_\alpha)$, *where* $\alpha \in R$ *is strongly orthogonal to* $\sigma(\alpha)$. *Moreover,* G *is simply-laced.*

Proof. In view of Proposition 2.13, we only have to show that G is simply laced in case (3). Then $\sigma(\alpha) \neq \alpha$, and hence σ acts non-trivially on R. As σ stabilizes the set Δ of simple roots, it induces a non-trivial automorphism of the Dynkin diagram. But this only occurs for G simply-laced. $\square$

Remark 2.16. The three cases in Corollary 2.15 do occur (see Table 1 for the notation on types):

1. In type AI with $G = \mathrm{SL}_n$ and $H = \mathrm{SO}_n$, then $T_xC = \mathfrak{g}_\Theta$ with Θ the highest root.

2. In type BII with $G = \mathrm{SO}_{2n+1}$ and $H = S(\mathrm{O}_1 \times \mathrm{O}_{2n})$, then $T_xC = \mathfrak{g}_\theta$ with θ the highest short root.

3. In type AII with $G = \mathrm{SL}_{2n}$ and $H = \mathrm{Sp}_{2n}$, then T_xC is spanned by $e_{\Theta-\alpha_1} - e_{\Theta-\alpha_{2n-1}}$ where Θ is the highest root and α_1 and α_{2n-1} are simple roots labeled as in [Bo68].

4. Clearly, case (2) does not occur for simply-laced groups.

5. In type G_2, none of cases (2) and (3) do occur. Indeed $\sigma(\alpha) = \alpha$ for any root $\alpha \in R$. Therefore, every root is imaginary. An easy computation shows that the highest non-compact root must be long.

Next, we assume that G/H is irreducible; in particular, G is simple or $G = H \times H$ with H simple. We show that T_xC is contained in a nilpotent orbit of a very special type, defined as follows:

Definition 2.17. Let G/H be an irreducible symmetric space.

1. If $G = H \times H$, then set $\mathcal{O}_{\min} = G \cdot (e, -e) \subset \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$ where $e \in \mathfrak{h}$ is a highest weight vector for H.

2. If G is simple, define a *nilpotent orbit* $\mathcal{O}_{\min}$ and a *type of nilpotent orbits* $\mathcal{O}_{\mathrm{sum}}$ in $\mathfrak{g}$ as follows.

   (a) $\mathcal{O}_{\min} = G \cdot e$ where e is a highest weight vector in $\mathfrak{g}$.

   (b) A nilpotent orbit $\mathcal{O}$ is of type $\mathcal{O}_{\mathrm{sum}}$ if $\mathcal{O} = G \cdot (e_1 + e_2)$, where $e_i \in \mathfrak{g}_{\alpha_i}$ is a root vector with α_1 and α_2 two strongly orthogonal long roots.

Remark 2.18. There is a unique nilpotent orbit of type $\mathcal{O}_{\mathrm{sum}}$ except for G of type B_n or D_n, in which case there are two possible nilpotent orbits.

Proposition 2.19. *With the above notation, $T_xC \setminus \{0\}$ is contained in $\mathcal{O}_{\min}$ or in a nilpotent orbit of type $\mathcal{O}_{\mathrm{sum}}$.*

This follows by combining Corollary 2.15, Remark 2.16.5 and the next result.

Lemma 2.20. *Assume that G is not simply-laced and not of type G_2. Let $e_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$ with α a short root. Then e_α belongs to a nilpotent orbit of type $\mathcal{O}_{\mathrm{sum}}$.*

Proof. Since all short roots are in the same orbit under the action of the Weyl group, we may assume that α is the highest short root θ.

There exists a simple root β such that $\theta+\beta$ is a root; then $\theta+\beta$ must be a long root. We claim that $\langle\beta^\vee, \theta+\beta\rangle = 2$. Indeed, since θ is a dominant weight, we have $\langle\beta^\vee, \theta\rangle \geq 0$ and we get $\langle\beta^\vee, \theta+\beta\rangle \geq \langle\beta^\vee, \beta\rangle = 2$. Since G is not of type G_2, we must have equality: $\langle\beta^\vee, \theta+\beta\rangle = 2$.

We get $s_\beta(\theta+\beta) = \theta-\beta$; thus, $\theta-\beta$ is a long root. Denote by $(-,-)$ a scalar product on $\mathfrak{X}_\mathbb{R}$ which is invariant under W and σ, and such that long roots have length 2. We have $(\theta-\beta, \theta-\beta) = 2 = (\theta+\beta, \theta+\beta)$. This gives $(\theta, \beta) = 0$ and $(\theta, \theta) = 1 = (\beta, \beta)$, so both θ and β are short roots. We also get $(\theta-\beta, \theta+\beta) = (\theta, \theta) - (\beta, \beta) = 0$, so $\theta-\beta$ and $\theta+\beta$ are long orthogonal roots. We check that they are strongly orthogonal: their sum is $\theta-\beta+\theta+\beta = 2\theta$ which is not a root, and their difference $\theta+\beta-(\theta-\beta) = 2\beta$ is not a root either.

We are left to prove that e_θ and $e_{\theta-\beta} + e_{\theta+\beta}$ are in the same G-orbit in $\mathfrak{g}$. The group G_β (generated by $U_{\pm\beta}$) acts on $\mathfrak{g}$ and stabilizes the subspace $V = \mathfrak{g}_{\theta-\beta} \oplus \mathfrak{g}_\theta \oplus \mathfrak{g}_{\theta+\beta}$ on which it acts via the adjoint representation. Moreover, G_β acts with two orbits in the projective space $\mathbb{P}(V)$: the minimal orbit and its complement. The point $[e_\theta]$ is in this last orbit, which also contains $[e_{\theta-\beta} + e_{\theta+\beta}]$. This yields the assertion, since nilpotent orbits are stable under non-trivial homotheties. $\square$

Remark 2.21. In Proposition 4.23 and Corollary 4.25, we will give a more precise statement describing the nilpotent orbit containing T_xC for $C \in \mathcal{K}_x$ with $\mathcal{K}$ a minimal family.

3 Complete symmetric varieties

In this section, we recall the definitions of wonderful symmetric varieties and complete symmetric varieties and we describe their relations, especially how to compare their respective minimal families. We then recall the result of [BF15] on minimal families on wonderful compactifications of groups. We end the section by a description of the minimal families on complete symmetric varieties in the group type, in the Hermitian type, and in some cases of simple type.

3.1 Wonderful and complete symmetric varieties

We use the notation of Subsections 2.3 and 2.4. In particular, G denotes a connected reductive group, H a symmetric subgroup relative to an involution σ, and N the normalizer of H in G. We denote by x (resp. x_{ad}) the base point of the homogeneous space G/H (resp. G/N). The natural morphism

$$\pi : G/H \longrightarrow G/N, \quad x \longmapsto x_{\mathrm{ad}}$$

is a principal bundle under N/H. Moreover, N/H is diagonalizable by Lemma 2.10. We have the "Stein factorization" of π as

$$G/H \xrightarrow{\pi'} G/N^0H \xrightarrow{\eta} G/N,$$

where π' is a principal bundle under the torus $N^0H/H \simeq Z^0/H \cap Z^0$ (Lemma 2.9), and η is a principal bundle under N/N^0H, a finite abelian group (Lemma 2.10).

By [deCP83], the adjoint symmetric space $G/N = G_{\mathrm{ad}}/G_{\mathrm{ad}}^\sigma$ admits a wonderful equivariant embedding that we denote by X_{ad}, with base point x_{ad}. We say that X_{ad} is a *wonderful symmetric variety.*

We now recall from [LP90, §3.3] how to obtain X_{ad} from the wonderful $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$-equivariant embedding $\overline{G_{\mathrm{ad}}}$ of $G_{\mathrm{ad}} = (G_{\mathrm{ad}} \times G_{\mathrm{ad}})/\mathrm{diag}(G_{\mathrm{ad}})$. We begin with a general construction: the morphism

$$G \longrightarrow G, \quad g \longmapsto \sigma(g)g^{-1}$$

factors through a closed immersion $\iota : G/G^\sigma \to G$ which sends the base point x to the neutral element e. The image of ι is a connected component of the fixed locus $G^{-\sigma}$, where $-\sigma$ denotes the involution $g \mapsto \sigma(g^{-1})$ of G (viewed as a variety). Note that ι is equivariant for the natural action of G on G/G^σ, and the G-action on itself via twisted conjugation, defined by $g_1 \cdot g_2 := \sigma(g_1)g_2g_1^{-1}$. Also, the differential of ι at x is identified with the inclusion $\mathfrak{p} \hookrightarrow \mathfrak{g}$.

This construction applies to the involution σ of G_{ad}; moreover, $-\sigma$ extends uniquely to an involution of $\overline{G_{\mathrm{ad}}}$ that we still denote by $-\sigma$, and ι extends uniquely to a closed immersion

$$\bar{\iota} : X_{\mathrm{ad}} \hookrightarrow \overline{G_{\mathrm{ad}}}$$

which identifies X_{ad} with a component of $(\overline{G_{\mathrm{ad}}})^{-\sigma}$.

Definition 3.1. A *complete symmetric variety* is a smooth projective equivariant embedding (X, x) of G/H such that the morphism $\pi : G/H \to G/N$ extends to a morphism $X \to X_{\mathrm{ad}}$; equivalently, X is toroidal in the sense of [Ti11, §29]).

We still denote by $\pi : X \to X_{\mathrm{ad}}$ this extension, which is of course unique and hence G-equivariant. The boundary $\partial X = X \setminus X^0$ is a divisor with simple normal crossings. We will use the following relation between the canonical divisors of X and X_{ad}:

Lemma 3.2. *With the above notation, we have the equality of divisor classes*

$$K_X + \partial X = \pi^*(K_{X_{\mathrm{ad}}} + \partial X_{\mathrm{ad}}).$$

Proof. Recall from [deCP83] that X_{ad} is isomorphic to the G-orbit closure of $\mathfrak{h}$ in the Grassmannian of subspaces of $\mathfrak{g}$. Moreover, $K_{X_{\mathrm{ad}}} + \partial X_{\mathrm{ad}}$ is the hyperplane class H in the corresponding Plücker embedding (see [Ti11, Prop. 30.8]). Also, $K_X + \partial X = \pi^*(H)$ by loc. cit. □

We will also use the following description of the general fibers of π; by equivariance, it suffices to describe the fiber at x. In view of the Stein factorization, π is the composition of a contraction $\pi' : X \to X'$ as discussed after Remark 2.5, and a finite surjective equivariant morphism $\eta : X' \to X_{\mathrm{ad}}$. The pair $(X', x' := \pi'(x))$ is a normal projective equivariant embedding (possibly singular) of $G/N^0H = G/Z^0H$ (a symmetric space under G/Z^0). Moreover, the fiber of π at x is isomorphic to the associated bundle $N \times^{N^0H} F$, where F denotes the fiber of π' at x. The group N^0H acts on F via its quotient torus $N^0H/H \simeq N^0/H \cap N^0 \simeq Z^0/H \cap Z^0$ (where the second isomorphism follows from Lemma 2.9), and F is a smooth projective toric variety under that torus.

3.2 Relation between minimal families

In this subsection, we consider a complete symmetric variety X with base point x. We will reduce somehow the description of minimal families of rational curves on X to the cases where X is a smooth projective toric variety or a wonderful symmetric variety. A key notion is that of a *highest weight curve*, i.e., an irreducible curve $C \subset X$ through x which is stable under the Borel subgroup B_H; equivalently, $C \cap X^0$ is a highest weight curve in the sense of §2.6.

By Corollary 2.14, we have $C = \overline{U_\alpha \cdot x}$ for some root α. In view of [BF15, Lem. 2.1 (i), Lem. 2.4], this yields:

Lemma 3.3. *Let C be a highest weight curve. Then C is an embedded free rational curve.*

We now obtain an alternative for minimal families:

Lemma 3.4. *Let $\mathcal{K}$ be a family of minimal rational curves on X.*

1. *Either $\mathcal{K}$ is contracted by π, or $\mathcal{K}_x$ contains a highest weight curve.*

2. *In the former case, $\mathcal{K}_x$ is a minimal family of rational curves on the toric variety F. Moreover, the tangent map τ_x is an isomorphism of $\mathcal{K}_x$ with a linear subspace of $\mathbb{P}(\mathfrak{p} \cap \mathfrak{z})$.*

Proof. (1) This follows from Lemma 2.11 together with Borel's fixed point theorem, which yields the existence of a B_H-fixed point in $\mathcal{K}_x$.

(2) The first assertion is a consequence of Lemma 2.6. The second assertion follows from [CFH14, Cor. 2.5]. □

Proposition 3.5. *Let $X = X_{\mathrm{ad}}$ be a wonderful symmetric variety, and $\mathcal{K}$ a family of rational curves on X. Then $\mathcal{K}$ is minimal if and only if $\mathcal{K}_x$ contains a highest weight curve.*

If X is irreducible, then it has a unique minimal family unless X is Hermitian exceptional; in that case, there are two minimal families.

Proof. If $\mathcal{K}$ is minimal, then $\mathcal{K}_x$ contains a highest weight curve by Lemma 3.4.

For the converse, using Lemma 2.2 and the structure of adjoint symmetric spaces (§2.5), we may assume that X is irreducible.

For X not Hermitian, there is a unique highest weight curve C. Let $\mathcal{L}$ be a minimal family of rational curves on X. Then $C \in \mathcal{L}$ by the above step, and hence $\mathcal{K} = \mathcal{L}$.

For X Hermitian, there are two highest weight curves, with highest weights Θ and $-\alpha$. Consider a Chevalley involution of (G, T), i.e., an involution τ of G such that $\tau(t) = t^{-1}$ for all $t \in T$. Then τ commutes with $\sigma = \mathrm{Int}(c)$, since $\tau(c) = c^{-1} = cz$ for some $z \in Z$. Thus, τ induces an involution ι of X fixing x. Also, τ sends every root to its opposite; in particular, $\tau(\Theta) = -\Theta$. Choose a representative $g \in \mathrm{N}_L(T)$ of the longest element of the Weyl group of (L, T). Then $g \circ \iota$ is an automorphism of X which fixes x and exchanges the two highest weight curves. So each of these curves is contained in a minimal family, and hence $\mathcal{K}$ is minimal.

This completes the proof of the first assertion. For the second assertion, it only remains to show that there are two minimal families if X is Hermitian exceptional. But then the open immersion $G/N \to G/P \times G/Q$ extends to a morphism $X \to G/P \times G/Q$, and the resulting projection $X \to G/P$ contracts the highest weight curve with weight Θ but not the other one. So these two curves cannot be in the same family. (In the non-exceptional case, they are exchanged by any element of $\mathrm{N}_G(L) \setminus L$). $\square$

Proposition 3.6. *Let $\mathcal{K}$ be a minimal family of rational curves on X containing a highest weight curve C.*

1. *$\mathcal{K}_x$ consists of embedded free curves; it is smooth and equidimensional, of dimension $-K_X \cdot C - 2$.*

2. *There is a unique minimal family of rational curves $\mathcal{L}$ on X_{ad} and a commutative diagram of H-equivariant rational maps*

$$\begin{tikzcd} \mathcal{K}_x \arrow[r, "\pi_{*,x}"] \arrow[d, "\tau_x"'] & \mathcal{L}_{x_{\mathrm{ad}}} \arrow[d, "\tau_{x_{\mathrm{ad}}}"] \\ \mathbb{P}(T_x X) \arrow[r, dashed, "d\pi_x"] & \mathbb{P}(T_{x_{\mathrm{ad}}} X_{\mathrm{ad}}) \end{tikzcd}$$

   where τ_x and $\tau_{x_{\mathrm{ad}}}$ are finite and birational onto their image, and $\pi_{*,x}$ is a finite morphism. If π is birational, then $\pi_{*,x}$ is finite and birational onto its image as well.

3. *We have $1 \leq \partial X \cdot C \leq \partial X_{\mathrm{ad}} \cdot \pi(C)$. Moreover, $\partial X \cdot C = \partial X_{\mathrm{ad}} \cdot \pi(C)$ if and only if the image of $\pi_{*,x}$ is a union of components of $\mathcal{L}_{x_{\mathrm{ad}}}$.*

4. *If each connected component of $\mathcal{L}_{x_{\mathrm{ad}}}$ is a unique N^0-orbit, then $\pi_{*,x}$ sends each component of $\mathcal{K}_x$ isomorphically to a component of $\mathcal{L}_{x_{\mathrm{ad}}}$.*

Proof. (1) The open subset $\mathcal{K}_{\mathrm{emfr},x}$ is B_H-stable, and contains every B_H-fixed point by Lemma 2.11.2 and Lemma 3.3. Using Borel's fixed point theorem, it follows that $\mathcal{K}_{\mathrm{emfr},x}$ is the whole $\mathcal{K}_x$. Thus, $\mathcal{K}_x$ is smooth; it is equidimensional by Lemma 2.3. The assertion on its dimension follows from [Ko96, II.3.2].

(2) By Lemma 2.11.2 again, $\pi|_C$ is birational to its image D. In view of Lemma 2.1, this yields a commutative diagram of rational maps

$$\begin{tikzcd} \mathcal{K}_x \arrow[r, dashed, "\pi_{*,x}"] \arrow[d, dashed, "\tau_x"'] & \mathcal{L}_{x_{\mathrm{ad}}} \arrow[d, dashed, "\tau_{x_{\mathrm{ad}}}"] \\ \mathbb{P}(T_x X) \arrow[r, dashed, "d\pi_x"] & \mathbb{P}(T_{x_{\mathrm{ad}}} X_{\mathrm{ad}}) \end{tikzcd}$$

for a unique covering family of rational maps $\mathcal{L}$ on X_{ad}. Since $\mathcal{K}_x$ is smooth, τ_x is finite and birational onto its image. Also, $\pi_{*,x}$ is a morphism since it is B_H-equivariant and defined at every B_H-fixed point. Moreover, $\mathcal{L}$ contains the highest weight curve D, and hence is minimal by Proposition 3.5. Thus, $\tau_{x_{\mathrm{ad}}}$ is also a morphism, and is finite and birational onto its image as well.

The rational map $d\pi_x : \mathbb{P}(T_xX) \dashrightarrow \mathbb{P}(T_{x_{\mathrm{ad}}}X_{\mathrm{ad}})$ is a linear projection, and hence yields an affine morphism on its domain of definition. As a consequence, the fibers of $\pi_{*,x}$ are affine; thus, $\pi_{*,x}$ is a finite morphism.

The final assertion follows from Remark 2.5.

(3) Since X^0 is affine, C intersects ∂X and hence $\partial X \cdot C \geq 1$.

By (1), we have $\dim(\mathcal{K}_x) = -K_X \cdot C - 2$ and $\dim(\mathcal{L}_{x_{\mathrm{ad}}}) = -K_{X_{\mathrm{ad}}} \cdot \pi(C) - 2$. Since $\pi_{*,x} : \mathcal{K}_x \to \mathcal{L}_{x_{\mathrm{ad}}}$ is finite, it follows that

$$K_X \cdot C \geq K_{X_{\mathrm{ad}}} \cdot \pi(C),$$

with equality if and only if the image of $\pi_{*,x}$ is a union of components of $\mathcal{L}_{x_{\mathrm{ad}}}$. Moreover,

$$(K_X + \partial X) \cdot C = (K_{X_{\mathrm{ad}}} + \partial X_{\mathrm{ad}}) \cdot \pi(C)$$

by Lemma 3.2 and the projection formula. This yields the remaining statements.

(4) By assumption, each component of $\mathcal{L}_{x_{\mathrm{ad}}}$ is homogeneous under N^0, and hence under H^0 (Lemma 2.9). The corresponding isotropy group is a parabolic subgroup of H^0; thus, it is connected. As $\pi_{*,x}$ is finite and H^0-equivariant, this yields the assertion. □

Example 3.7. Let $G = \mathrm{SO}_n$, where $n \geq 3$, and let σ be the conjugation by $c = \mathrm{diag}(1, \ldots, 1, -1) \in \mathrm{O}_n$. Let $H = G^{\sigma,0} = \mathrm{SO}_{n-1}$. Then $N = G^\sigma = \mathrm{O}_{n-1}$ embedded in SO_n via $g \mapsto (g, \det(g))$, and $\mathfrak{p} = \mathbb{C}^{n-1}$ on which O_{n-1} acts via its standard representation. Also, G/H has a unique smooth projective equivariant embedding: the quadric $\mathbb{Q}_{n-1} \subset \mathbb{P}^n = \mathbb{P}(\mathbb{C}^n \oplus \mathbb{C})$, where SO_n acts on $\mathbb{P}(\mathbb{C}^n \oplus \mathbb{C})$ via its standard representation on $\mathbb{C}^n$. Moreover, $X_{\mathrm{ad}} = \mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$ and $\pi : X \to X_{\mathrm{ad}}$ is a ramified double cover induced by the linear projection $\mathbb{P}(\mathbb{C}^n \oplus \mathbb{C}) \dashrightarrow \mathbb{P}(\mathbb{C}^n)$.

If $n \geq 4$ then X has a unique minimal family of rational curves $\mathcal{K}$; it consists of the lines in $\mathbb{Q}_{n-1}$. Moreover, π_* sends $\mathcal{K}$ to the family $\mathcal{L}$ of lines in $\mathbb{P}^{n-1}$, and $\pi_{*;x} : \mathcal{K}_x \to \mathcal{L}_{x_{\mathrm{ad}}}$ is identified with the inclusion $\mathbb{Q}_{n-3} \subset \mathbb{P}^{n-2} = \mathbb{P}(\mathfrak{p})$, compatibly with the action of $\mathrm{O}_{n-1} = N$.

If $n = 3$ then $X = \mathbb{Q}^2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ has two minimal families of rational curves, the fibers of the two projections to $\mathbb{P}^1$. For both families, $\pi_{*,x}$ identifies $\mathcal{K}_x$ with a point in $\mathbb{P}^1$.

3.3 Minimal families of group and Hermitian types

We still consider a complete symmetric variety X with base point x, and a minimal family of rational curves $\mathcal{K}$ on X; we assume that $\mathcal{K}$ contains a highest weight curve C. As explained above, there is a unique indecomposable factor X_C of X_{ad} such that the composition of $\pi : X \to X_{\mathrm{ad}}$ with the projection $X_{\mathrm{ad}} \to X_C$ sends C isomorphically to its image. In this subsection, we will handle in details the cases where X_C is of group or Hermitian types.

We first handle the group type, where X_C is the wonderful completion of an adjoint simple group H_C. The Lie algebra of H_C is denoted by $\mathfrak{h}_C$, and we still denote by C the highest weight curve in X_C. By the main result of [BF15], X_C has a unique minimal family of rational curves $\mathcal{L}$. Moreover, the tangent map $\tau_{x_C} : \mathcal{L}_{x_C} \to \mathbb{P}(\mathfrak{h}_C)$ is an H_C-equivariant isomorphism to its image $\mathcal{C}_{x_C}$. If H_C is of type A_r with $r \geq 2$, i.e., $H_C \simeq \mathrm{PGL}(V)$ where

V is a vector space of dimension $r+1 \geq 3$, then $\mathfrak{h}_C \simeq \mathrm{End}(V)/\mathbb{C}\mathrm{id}$ and $\mathcal{C}_{x_C}$ is isomorphic to $\mathbb{P}(V) \times \mathbb{P}(V^\vee)$ embedded in $\mathbb{P}(\mathfrak{h}_C)$ via the Segre embedding

$$\mathbb{P}(V) \times \mathbb{P}(V^\vee) \hookrightarrow \mathbb{P}(V \otimes V^\vee) = \mathbb{P}(\mathrm{End}(V))$$

followed by the linear projection $\mathbb{P}(\mathrm{End}(V)) \dashrightarrow \mathbb{P}(\mathrm{End}(V)/\mathbb{C}\mathrm{id})$. In all other types, we have $\mathcal{C}_{x_C} = \mathbb{P}(\mathcal{O}_{C,\min})$, the projectivization of the minimal nilpotent orbit in $\mathbb{P}(\mathfrak{h}_C)$.

Proposition 3.8. *If X_C is of group type, then $\partial X \cdot C$ equals 1 or 2. In the former case, every component of $\mathcal{K}_x$ is isomorphic to $\mathbb{P}(\mathcal{O}_{C,\min})$. In the latter case, we have $H_C \simeq \mathrm{PGL}(V)$, where $\dim(V) \geq 3$, and every component of $\mathcal{K}_x$ is isomorphic to $\mathbb{P}(V) \times \mathbb{P}(V^\vee)$.*

Proof. Recall from Proposition 3.6 that $1 \leq \partial X \cdot C \leq \partial X_{\mathrm{ad}} \cdot \pi(C)$. Moreover, the line bundle on X_{ad} associated with the divisor ∂X_{ad} equals $\mathcal{L}_{X_{\mathrm{ad}}}(\alpha_1 + \cdots + \alpha_r)$ with the notation of [BF15, §3]. By combining [BF15, Lem. 3.3, Lem. 3.4], it follows that $\partial X_{\mathrm{ad}} \cdot \pi(C) = 2$ if H_C is of type A_r where $r \geq 2$; otherwise, $\partial X_{\mathrm{ad}} \cdot \pi(C) = 1$.

In the latter case, we must have $\partial X \cdot C = \partial X_{\mathrm{ad}} \cdot \pi(C)$. So every component of $\mathcal{K}_x$ is isomorphic to the orbit $H_C \cdot C$, by Proposition 3.6 again. Moreover, $H_C \cdot C = \mathbb{P}(\mathcal{O}_{C,\min})$.

In the former case, $\mathcal{L}_{x_C} = \mathbb{P}(V) \times \mathbb{P}(V^\vee)$ consists of two orbits of $H_C = \mathrm{PGL}(V)$: a closed orbit of codimension 1 (the incidence variety, isomorphic to $\mathbb{P}(\mathcal{O}_{C,\min})$), and an open orbit isomorphic to $\mathrm{SL}_{r+1}/\mathrm{GL}_r$, and hence simply connected.

If $\partial X \cdot C = 2$, then by Proposition 3.6 again, we get a finite surjective H-equivariant morphism $\pi_{*,x} : \mathcal{K}_x \to \mathbb{P}(V) \times \mathbb{P}(V^\vee)$. Since the open orbit in the right-hand side is simply connected, it follows that $\pi_{*,x}$ is birational on each component, and hence an isomorphism in view of Zariski's main theorem.

On the other hand, if $\partial X \cdot C = 1$, then the image of $\pi_{*,x}$ is the closed orbit and we conclude as above. $\square$

Example 3.9. Assume that $\pi : X \to X_{\mathrm{ad}}$ is birational and X_{ad} is the wonderful completion of $\mathrm{PGL}(V)$, where $\dim(V) \geq 3$. Then the highest weight curve $C_{\mathrm{ad}} \in X_{\mathrm{ad}}$ intersects a unique $\mathrm{PGL}(V) \times \mathrm{PGL}(V)$-orbit $\mathcal{O}_{1,r}$ of codimension 2 in X_{ad} (see [BF15, Lem.3.4]).

If π is an isomorphism over $\mathcal{O}_{1,r}$, then the minimal family $\mathcal{K}$ on X satisfies $\partial X \cdot C = 2$ and $\mathcal{K}_x = \mathbb{P}(V) \times \mathbb{P}(V^\vee)$. Indeed, π is an isomorphism over an open neighborhood of $\mathcal{O}_{1,r}$ in X_{ad}, stable by $\mathrm{PGL}(V) \times \mathrm{PGL}(V)$, and every curve in $\mathcal{L}_{x_{\mathrm{ad}}}$ intersects such a neighborhood.

On the other hand, if π is not an isomorphism over $\mathcal{O}_{1,r}$, then $\partial X \cdot C = 1$ and $\mathcal{K}_x$ is the incidence variety $\mathbb{P}(\mathcal{O}_{\min})$; moreover, we have $\partial X_{\mathrm{ad}} \cdot C_{\mathrm{ad}} = 2$ and $\mathcal{L}_{x_{\mathrm{ad}}} = \mathbb{P}(V) \times \mathbb{P}(V^\vee)$. Indeed, π factors through the blow-up $\varphi : X' \to X_{\mathrm{ad}}$ of $\overline{\mathcal{O}_{1,r}}$ in X_{ad}. Using Proposition 3.6, we may thus assume that $X = X'$. Then $K_X = \pi^*(K_{X_{\mathrm{ad}}}) + E$, where E denotes the exceptional divisor. Thus,

$$K_X \cdot C = K_{X_{\mathrm{ad}}} \cdot C_{\mathrm{ad}} + E \cdot C > K_{X_{\mathrm{ad}}} \cdot C_{\mathrm{ad}},$$

since C intersects E. It follows that $\dim(\mathcal{K}_x) < \dim(\mathcal{L}_{x_{\mathrm{ad}}})$, and we conclude by Proposition 3.6 again.

Next, we handle the Hermitian type, where X_C is the wonderful completion of the symmetric space $G_C/\mathrm{N}_{G_C}(L_C)$ for a simple factor G_C of G_{ad} with a Levi subgroup $L_C \subset G_C$. We may then assume that $G = G_C$.

Proposition 3.10. *If X_C is of Hermitian type but not of type* PGL_2/N*, then $\partial X \cdot C = 1$ and every component of $\mathcal{K}_x$ is isomorphic to the orbit $L_C \cdot C$.*

Proof. By Proposition 3.6, we may assume that $X = X_C$. We now view X as a subvariety of $\overline{G}$ (as recalled in §3.1), and use the description of minimal rational curves in $\overline{G}$, as in the proof of Proposition 3.8.

With the notation of §2.6, the highest weight curves are $C_\Theta := \overline{U_\Theta \cdot x}$ and $C_{-\alpha} := \overline{U_{-\alpha} \cdot x}$ (indeed, these curves are irreducible, stable by B_L and distinct). As observed in the proof of Proposition 3.5, these curves are exchanged by an automorphism of X fixing x; thus, we may assume that $C = C_\Theta$.

By [RRS92, §5.5], σ is the inner involution $\mathrm{Int}(c)$, where $c \in T$ satisfies $\alpha(c) = -1$ and $\beta(c) = 1$ for all simple roots $\beta \neq \alpha$. In particular, the roots Θ and $-\alpha$ are non-compact imaginary. Thus, the closed immersion

$$\iota : G/\mathrm{N}_G(L) \longrightarrow G, \quad g\,\mathrm{N}_G(L) \longmapsto \sigma(g)g^{-1}$$

induces isomorphisms

$$U_\Theta \cdot x \xrightarrow{\sim} U_\Theta, \quad U_{-\alpha} \cdot x \xrightarrow{\sim} U_{-\alpha}.$$

So $\bar\iota : X \to \overline{G}$ sends C_Θ, $C_{-\alpha}$ isomorphically to the corresponding root curves considered in [BF15, §3]. Since Θ and $-\alpha$ are long roots, these root curves are minimal; hence $\bar\iota$ sends $\mathcal{K}_x$ to the unique family $\mathcal{L}_{\overline{G},e}$ of minimal rational curves through e in $\overline{G}$. Moreover, $\bar\iota(\mathcal{K}_x)$ is contained in the fixed locus $\mathcal{L}_{\overline{G},e}^{-\sigma}$.

If G is not of type A_r, where $r \geq 2$, then the tangent map τ_e identifies $\mathcal{L}_{\overline{G},e}$ with $\mathbb{P}(\mathcal{O}_{\min})$. Since $d\iota_x$ identifies T_xX with $\mathfrak{p}$, we see that

$$\bar\iota(\mathcal{K}_x) \subset \mathbb{P}(\mathcal{O}_{\min} \cap \mathfrak{p}) \subset \mathbb{P}(\mathcal{O}_{\min})^\sigma.$$

By [Ri82, Thm. A], the right-hand side is a finite union of closed orbits of $G^{\sigma,0} = L$. We conclude that the component of C in $\mathcal{K}_x$ is $L \cdot C$.

Otherwise, $G = \mathrm{PGL}(V)$ where $\dim(V) = r + 1$, and τ_e yields an isomorphism

$$\mathcal{L}_{\overline{G},e} \simeq \mathbb{P}(V) \times \mathbb{P}(V^\vee) \subset \mathbb{P}(\mathrm{End}(V)/\mathbb{C}\mathrm{id}) = \mathbb{P}(\mathfrak{g}),$$

equivariantly for the action of $-\sigma$. Consider the σ-eigenspace decomposition $V = V_1 \oplus V_{-1}$. Then L is the image of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_{-1})$ in $\mathrm{PGL}(V)$; also, $\mathbb{P}(V)^\sigma = \mathbb{P}(V_1) \sqcup \mathbb{P}(V_{-1})$ and likewise for $\mathbb{P}(V^\vee)^\sigma$. Moreover, the image of $\iota(\mathcal{C})$ under τ_e is contained in

$$\mathbb{P}(\mathfrak{p}) = \mathbb{P}(\mathrm{Hom}(V_1, V_{-1}) \oplus \mathrm{Hom}(V_{-1}, V_1)).$$

It follows that $\iota(\mathcal{K}_x)$ is contained in $(\mathbb{P}(V_1^\vee) \times \mathbb{P}(V_{-1})) \sqcup (\mathbb{P}(V_{-1}^\vee) \times \mathbb{P}(V_1))$. As a consequence, the component of C in $\mathcal{K}_x$ is $L \cdot C$ in this case, too.

We now show that $\partial X \cdot C = 1$. Consider first the exceptional case, where $H = L = P \cap Q$. Then we have a G-equivariant birational morphism $\varphi : X \to G/P \times G/Q =: Y$ which sends x to the base point $y = (P, Q)$. Since P is a maximal parabolic subgroup of G associated with a long root, G/P has a unique family of minimal rational curves $\mathcal{L}$. Moreover, denoting by P the base point of the homogeneous space G/P and by D the Schubert line in that space (i.e., the unique irreducible B-stable curve), we have that $\mathcal{L}_P =$

$L \cdot D$ (see e.g. [BK21, Prop. 3.3]). The projection $p : X \to G/P$ sends C isomorphically to D, and yields an isomorphism $p_* : L \cdot C \to L \cdot D$ which identifies $L \cdot C$ with the variety of lines in G/P through its base point. Since $\dim(L \cdot C) = \dim(\mathcal{K}_x) = -K_X \cdot C - 2$ and $\dim(L \cdot D) = -K_{G/P} \cdot D - 2$, we obtain

$$K_X \cdot C = K_{G/P} \cdot D = K_Y \cdot \varphi(C) = \varphi^*(K_Y) \cdot C$$

by using the projection formula. On the other hand, we have $K_X = \varphi^*(K_Y) + \sum_i a_i E_i$, where the E_i are the exceptional divisors of φ and the a_i are positive integers. Since C is not contained in any E_i, it follows that $E_i \cdot C = 0$ for all i. Also, the boundary of Y is an irreducible divisor E, and $\partial X = E' + \sum_i E_i$, where E' denotes the strict transform of E. This yields $\partial X \cdot C = E' \cdot C = E \cdot D$ by the projection formula again. Since $D \subset G/P \times \{Q\}$, where we still denote by Q the base point of G/Q, and $E \cap (G/P \times \{Q\})$ is identified with the Schubert divisor in G/P, we obtain $E \cdot D = 1$. This yields the assertion in that case.

Next, we consider the non-exceptional case, where $H = \mathrm{N}_G(L)$ contains L as a subgroup of index 2. Then there exists a smooth toroidal equivariant embedding X' of G/L such that the natural map $G/L \to G/H$ extends to a morphism $\psi : X' \to X$. By Lemma 3.2, we have $K_{X'} + \partial X' = \psi^*(K_X + \partial X)$. Moreover, C lifts uniquely to a highest weight curve $C' \subset X'$, and the corresponding minimal families have isomorphic components by Proposition 3.6. Taking dimensions, we obtain $K_{X'} \cdot C' = K_X \cdot C = \psi^*(K_X) \cdot C'$. As a consequence, we have $\partial X' \cdot C' = \psi^*(\partial X) \cdot C' = \partial X \cdot C = 1$. $\square$

Proposition 3.10 leaves out the case of type PGL_2/N, which is easily treated:

Lemma 3.11. *If X_C is of type PGL_2/N, then $\partial X \cdot C$ equals 1 or 2. In the former case, $\mathcal{K}_x$ is finite. In the latter case, every component of $\mathcal{K}_x$ is a projective line.*

Proof. Note that PGL_2/N has a unique projective equivariant embedding, namely, $\mathbb{P}^2$ on which PGL_2 acts via the projectivization of its adjoint representation. Thus, ∂X_C is a conic, with C as a tangent line so that $\partial X_C \cdot C = 2$. Also, the minimal rational curves on X_C are just lines, and those through a given point form a $\mathbb{P}^1$. This yields the statement by using Proposition 3.6 as in the proof of the above proposition. $\square$

Corollary 3.12. *Let G be a simple adjoint group, and X the wonderful embedding of a Hermitian symmetric space $G/\mathrm{N}_G(L)$. Denote by C_Θ and $C_{-\alpha}$ the highest weight curves in X, indexed by their weight.*

1. *If X is exceptional, then it has two minimal families of rational curves $\mathcal{K}^+$, $\mathcal{K}^-$. Moreover, $\mathcal{K}^+_x = L \cdot C_\Theta$ and $\mathcal{K}^-_x = L \cdot C_{-\alpha}$.*

2. *If X is non-exceptional, then it has a unique minimal family $\mathcal{K}$. Moreover, $\mathcal{K}_x = \mathrm{N}_G(L) \cdot C_\Theta = L \cdot C_\Theta \sqcup L \cdot C_{-\alpha}$ unless $G/\mathrm{N}_G(L) = \mathrm{PGL}_2/N$.*

3.4 Some cases of simple type

In this subsection we use previous techniques and the results in the group case to briefly describe the unique minimal family in some of the cases of simple type. We refer to Table

1 for the different cases of the classification. A different approach, working in all cases is developed in Section 4.

Consider a highest weight curve C of simple type, and denote by λ its weight relative to B_H. Then λ is the highest weight of the representation of H^0 in $\mathfrak{p}$, and hence is the restriction to T_H of some $\alpha \in R^+$ (not necessarily unique). Moreover, $S := \ker(\lambda)^0$ is a subtorus of codimension 1 in T_H, and C is an irreducible curve in $(G/H)^S$ through x, stable by the Borel subgroup $\mathrm{C}_{B_H}(S)$ of $\mathrm{C}_H(S)$. So C is a highest weight curve of the symmetric space $\mathrm{C}_G(S)/\mathrm{C}_H(S) = (\mathrm{C}_G(S)/S)/(\mathrm{C}_H(S)/S)$, where $\mathrm{C}_H(S)/S$ has rank 1. We now apply Lemma 2.12: the adjoint symmetric space of $\mathrm{C}_G(S)/S$ is of type (A_1), $(\mathrm{A}_1 \times \mathrm{A}_1)$ or (A_2). If σ is inner, then only type (A_1) may occur.

Lemma 3.13. *Assume that $\lambda = \alpha|_{T_H}$ for a unique root α (i.e., type $(\mathrm{A}_1\times\mathrm{A}_1)$ is excluded), and α is long. Then the component of $\mathcal{K}_x$ containing C admits a finite equivariant morphism to $\mathcal{K}^{-\sigma}_{\overline{G_{\mathrm{ad}}},\mathrm{id}}$.*

Proof. By assumption, $C = \overline{U_\alpha \cdot x}$, where α is non-compact imaginary. Thus, the image of C under the morphism $\psi : X \to \overline{G_{\mathrm{ad}}}$, obtained by composing $\phi : X \to X_{\mathrm{ad}}$ with $\iota : X_{\mathrm{ad}} \to \overline{G_{\mathrm{ad}}}$, is just the closure of U_α; since α is long, this is a minimal rational curve on $\overline{G_{\mathrm{ad}}}$. This yields the assertion by arguing as in the proof of Proposition 3.6. □

The assumptions of the lemma hold if and only if $T_xC \setminus \{0\}$ is contained in $\mathcal{O}_{\min}$ (as follows by combining Proposition 2.13, Corollary 2.15 and Lemma 2.20).

Proposition 3.14. *Assume that $T_xC \setminus \{0\} \subset \mathcal{O}_{\min}$.*

1. *If $\partial X \cdot C = 1$, then the component of $\mathcal{K}_x$ containing C is $H^0 \cdot C$.*
2. *If $\partial X \cdot C = 2$, then X is of type* AI *with $G = \mathrm{PGL}_{r+1}$ and $\mathcal{K}_x \simeq \mathbb{P}^r$.*

Proof. If G is not of type A_r or if X is Hermitian (but not of type PGL_2/N), we may argue as in the proof of Proposition 3.10 proving that the component of $\mathcal{K}_x$ containing C is $H^0 \cdot C$ and that $\partial X \cdot C = 1$.

If G is of type A_r and X is not Hermitian, then X is of type AI, with $G = \mathrm{PGL}(V)$ such that $\dim V = r+1$, $r \geq 2$ and $\sigma(g) = (g^t)^{-1}$. In this case, the minimal family $\mathcal{K}_{\overline{G_{\mathrm{ad}}},\mathrm{id}}$ identifies with $\mathbb{P}(V) \times \mathbb{P}(V^\vee)$ and the involution $-\sigma$ acts via $(-\sigma)([v],[H]) = ([H^\perp],[v^\perp])$ where the orthogonality is taken with respect to the standard scalar product. We thus have $\mathcal{K}^{-\sigma}_{\overline{G_{\mathrm{ad}}},\mathrm{id}} \simeq \mathbb{P}(V)$. If $\partial X \cdot C = 1$, then $\dim \mathcal{K}_x = \dim \mathbb{P}(V) - 1 = \dim H \cdot C$ and the result follows as above. If $\partial X \cdot C = 2$, then $\dim \mathcal{K}_x = \dim \mathbb{P}(V)$ proving the result. □

Remark 3.15. In Table 1, we list the nilpotent orbits containing $T_xC \setminus \{0\}$ (see the column "$\sigma(\Theta) = -\Theta$", the condition $T_xC \setminus \{0\} \subset \mathcal{O}_{\min}$ being equivalent to $\sigma(\Theta) = -\Theta$ by Corollary 4.25). In particular, the above proposition settles all cases except the following symmetric spaces: AII, BII, CII, DII, EIV and FII. We will deal with all cases in the next section.

4 Minimal families on wonderful symmetric varieties

In this section we deal with wonderful embeddings of adjoint irreducible symmetric spaces. We may therefore assume that G is semisimple and replacing G by its universal cover, we may further assume that G is simply connected. Note that in this case G^σ is connected, so that the group H satisfies $G^\sigma \subset H \subset \mathrm{N}_G(G^\sigma)$ and $\mathrm{N}_G(G^\sigma)/G^\sigma$ is finite (Lemma 2.9). Set $N = \mathrm{N}_G(G^\sigma)$; then $N = \mathrm{N}_G(H)$ by Lemma 2.10.

Since G/H is irreducible, σ acts transitively on the simple factors of G. Therefore, G/N is of group, Hermitian, or simple type. We will consider its wonderful embedding $X = X_{\mathrm{ad}}$.

We start with reminders on restricted root systems (Subsection 4.1) and their connection to curves and divisors on X (Subsection 4.2). Many results on these topics are well known but we could not find a good reference, so we included proofs for the convenience of the reader. From this we obtain an explicit description of the classes of curves in minimal families on X (Subsection 4.3). We then compute the dimension of these minimal families $\mathcal{K}$ using the contact structures on projectivised nilpotent orbits (Subsection 4.4). It turns out that in all cases except for X of restricted type A_r, the family $\mathcal{K}_x$ has the same dimension as the orbit $N \cdot C$ where $C \in \mathcal{K}_x$ is a highest weight curve, which in turn implies that $\mathcal{K}_x = N \cdot C$. We deal with X of restricted type A_r separately (Subsection 4.5). We conclude with a full description of $\mathcal{K}_x$ (Subsection 4.6).

4.1 Restricted root system

Let us first recall a few facts on the restricted root system; we refer to [Vu90] and [Ti11] for details. Let T_{s} be a maximal torus of split type and $S \subset T_{\mathrm{s}}$ be its maximal split subtorus: $S = \{t \in T_{\mathrm{s}} \mid \sigma(t) = t^{-1}\}^0$. Let R be the root system associated to the pair (G, T_{s}). Then σ acts on R. Set $\overline{S} = S/S^\sigma$, $\overline{\mathfrak{X}} = \mathfrak{X}(\overline{S})$ and $\overline{\chi} = \chi - \sigma(\chi)$ for $\chi \in \mathfrak{X}(T_{\mathrm{s}})$. We have an identification $\overline{\mathfrak{X}} = \{\overline{\chi} \mid \chi \in \mathfrak{X}(T_{\mathrm{s}})\}$. Define the subset $\overline{R} \subset \overline{\mathfrak{X}}$ via

$$\overline{R} = \{\overline{\alpha} \mid \alpha \in R\}.$$

Then $\overline{R}$ is an irreducible root system called the *restricted root system*. It may be non-reduced (see Remark 4.1 below).

Recall that $L = \mathrm{C}_G(S)$ is the Levi subgroup containing T_{s} of a parabolic subgroup $P \subset G$ and that $\sigma(P)$ is the opposite parabolic subgroup to P with common Levi subgroup L. Let $B_{\mathrm{s}} \subset P$ be a Borel subgroup and let $\Delta \subset R^+ \subset R$ be the sets of simple roots and positive roots defined by B_{s}. Then for $\alpha \in R^+$, we have $\sigma(\alpha) = \alpha$ if and only if α is a root of L; moreover, if $\sigma(\alpha) \neq \alpha$, then $\sigma(\alpha) < 0$. Set $\Delta_1 = \{\alpha \in \Delta \mid \sigma(\alpha) < 0\}$ and $\Delta_0 = \Delta \setminus \Delta_1$. Then $\sigma(\alpha) = \alpha$ for any $\alpha \in \Delta_0$. Define $\overline{\Delta} \subset \overline{\mathfrak{X}}$ via

$$\overline{\Delta} = \{\overline{\alpha} \mid \alpha \in \Delta_1\}.$$

Then $\overline{\Delta}$ is a basis of $\overline{R} \subset \overline{\mathfrak{X}}$. In particular $|\overline{\Delta}| = \mathrm{rk}(\overline{\mathfrak{X}}) = \dim T_{\mathrm{s}} = r$ is the rank of X. Furthermore, there exists a length-preserving involution $\overline{\sigma}$ on Δ, preserving Δ_1 and acting trivially on Δ_0, such that for any $\alpha \in \Delta_1$, we have

$$\sigma(\alpha) + \overline{\sigma}(\alpha) = -\sum_{\beta \in \Delta_0} c_\beta \beta$$

with $c_\beta \in \mathbb{Z}_{\geq 0}$. In particular, if $\overline{\sigma}(\alpha) \neq \alpha$, then $\langle \alpha^\vee, \sigma(\alpha) \rangle \geq 0$. Note that for $\alpha, \beta \in \Delta_1$, we have $\overline{\alpha} = \overline{\beta} \Leftrightarrow (\beta = \alpha \text{ or } \beta = \overline{\sigma}(\alpha))$.

A root $\alpha \in \Delta_1$ such that $\sigma(\alpha) \neq -\alpha$ and $\langle \alpha^\vee, \sigma(\alpha) \rangle \neq 0$ is called *exceptional*. If α is exceptional, then $\beta = \overline{\sigma}(\alpha) \neq \alpha$ is also exceptional and one of the following two conditions is satisfied: either $\sigma(\alpha) \neq -\beta$, or $\sigma(\alpha) = -\beta$ and $\langle \alpha^\vee, \beta \rangle \neq 0$ (see [deCS99, Lemma 4.3]). If there exists an exceptional root, then $\overline{R}$ and X are called *exceptional*. This definition is equivalent to the one given in Subsection 2.5, see for example [deCS99, Lemma 4.7]. Note that by [deCS99, Lemma 4.7], there are at most two exceptional roots (thus, of the form α and $\overline{\sigma}(\alpha)$).

Remark 4.1. If $\overline{R}$ is exceptional, it is non-reduced. In fact, for α exceptional, we have $\langle \alpha^\vee, \sigma(\alpha) \rangle \neq 0$. As α and $\overline{\sigma}(\alpha)$ are different but of the same length, we have $\langle \alpha^\vee, \sigma(\alpha) \rangle = 1$. Thus, $\gamma = \alpha - \sigma(\alpha) \in R$ and $\overline{\gamma} = 2\overline{\alpha}$. In particular, $\overline{\alpha}, 2\overline{\alpha} \in \overline{R}$ and $\overline{R}$ is non-reduced.

Example 4.2. There are non-reduced restricted root systems which are non-exceptional (actually only two families: types CII and FII, see Appendix). For example, if $G = \mathrm{Sp}_6$, there exists an involution σ such that $G^\sigma = \mathrm{Sp}_2 \times \mathrm{Sp}_4$ and with the labeling of the simple roots as in Bourbaki [Bo68], we have $\Delta_0 = \{\alpha_1, \alpha_3\}$ and $\Delta_1 = \{\alpha_2\}$. Set $\alpha = \alpha_2$, then $\overline{\sigma}(\alpha) = \alpha$, $\sigma(\alpha) = -(\alpha_1 + \alpha_2 + \alpha_3)$ and $\overline{\alpha} = \alpha_1 + 2\alpha_2 + \alpha_3 = \gamma \in R$. We have $\overline{\gamma} = 2\overline{\alpha}$; thus, $\overline{R} = \{-2\overline{\alpha}, -\overline{\alpha}, \overline{\alpha}, 2\overline{\alpha}\}$ but G/G^σ is not exceptional.

Let $\alpha \in R^+$ such that $\sigma(\alpha) < 0$. The roots α and $\sigma(\alpha)$ have the same length. As explained in [Vu90, Lemme 2.3], three cases occur and the coroot $\overline{\alpha}^\vee$ is defined accordingly:

1. If $\sigma(\alpha) = -\alpha$, then $\overline{\alpha}^\vee = \frac{1}{2}\alpha^\vee$.
2. If $\langle \alpha^\vee, \sigma(\alpha) \rangle = 0$, then $\overline{\alpha}^\vee = \frac{1}{2}(\alpha^\vee - \sigma(\alpha)^\vee)$.
3. If $\langle \alpha^\vee, \sigma(\alpha) \rangle = 1$, then $\overline{\alpha}^\vee = \alpha^\vee - \sigma(\alpha)^\vee$.

Case (3) above actually occurs if and only if $\overline{R}$ is non-reduced, see Proposition 4.3.5 below. In the next proposition, we summarise the results on restricted root systems needed for the study of curves and divisors on X. These results might be well known to the experts but we could not find a good reference, so we included a proof and further results on restricted root systems in Subsection 6.1 in the Appendix.

Proposition 4.3. *Let Θ be the highest root of R and $w_0 \in W$ be the longest element.*

1. *$\overline{\Theta}$ is the highest root of $\overline{R}$, the actions of σ and w_0 on roots commute, and we have $w_0(\overline{\Theta}) = -\overline{\Theta}$.*
2. *If $\sigma(\Theta) \neq -\Theta$, then Θ and $\sigma(\Theta)$ are strongly orthogonal long roots.*
3. *If $\alpha \in \Delta_1$ is exceptional, its coefficient in the expansion of Θ in simple roots is 1.*
4. *If $\overline{R}$ is not of type A_1, then there exists $\overline{\alpha} \in \overline{\Delta}$ with $\langle \overline{\Theta}^\vee, \overline{\alpha} \rangle = 1$ and $2\overline{\Theta}^\vee$ is an indivisible cocharacter of $\overline{S}$.*
5. *For $\alpha \in \Delta_1$, we have the equivalence: $\overline{\alpha}, 2\overline{\alpha} \in \overline{R} \Leftrightarrow \langle \alpha^\vee, \sigma(\alpha) \rangle = 1$.*

Proof. (1) This is Lemma 6.1, Lemma 6.2 and Corollary 6.3. (2) This is Proposition 6.12.4. (3) This is the last statement in Corollary 6.9. (4) This is Proposition 6.12.2-3. (5) This is Proposition 6.4. $\square$

We end this subsection with a piece of notation. For $\overline{\alpha} \in \overline{\Delta}$, we denote by $\widehat{\alpha}^\vee$ the simple root of $\overline{R}^\vee$ colinear to $\overline{\alpha}^\vee$. Note that if $2\overline{\alpha} \notin \overline{R}$, then $\widehat{\alpha}^\vee = \overline{\alpha}^\vee$ but if $2\overline{\alpha} \in \overline{R}$, we have $\widehat{\alpha}^\vee = \frac{1}{2}\overline{\alpha}^\vee = (2\overline{\alpha})^\vee$. In particular, for $\alpha \in \Delta_1$, Proposition 4.3.5 above implies that

$$\widehat{\alpha}^\vee = \begin{cases} \overline{\alpha}^\vee & \text{if } \langle \alpha^\vee, \sigma(\alpha)\rangle \neq 1, \\ \frac{1}{2}\overline{\alpha}^\vee & \text{if } \langle \alpha^\vee, \sigma(\alpha)\rangle = 1. \end{cases}$$

We will also need the following result proved in Lemma 6.11.

Lemma 4.4. *Assume that $\alpha \in \Delta_1$ is an exceptional root. Then the coefficient of $\widehat{\alpha}^\vee$ in the expansion of $\overline{\Theta}^\vee$ in terms of simple coroots of $\overline{R}$ is equal to* 1.

4.2 Divisors and restricted root system

We relate the Picard group of X (viewed as the group of divisors up to linear equivalence), to the restricted root system $\overline{R}$. We will need some definitions from the theory of spherical varieties, we refer to [Pe14] for further details. The variety X is spherical: it is a normal G-variety such that $B_{\rm s}$ has a dense orbit. This implies that $B_{\rm s}$ acts on X with finitely many orbits. In particular there are finitely many prime $B_{\rm s}$-stable divisors in X. The boundary $\partial X = X_1 \cup \cdots \cup X_r$ with r the rank of X is the union of the prime G-stable divisors. The prime $B_{\rm s}$-stable divisors which are not G-stable are called colors. We denote by $\mathcal{D}_X$ the set of colors and by $\mathcal{V}_X = \{X_1, \dots, X_r\}$ the set of prime G-stable divisors.

We start with a description of prime G-stable divisors. Let $i : Y \to X$ be the inclusion of the closed G-orbit in X. Recall that $Y = G/P$ and that for any character λ of P, we have a homogeneous line bundle $\mathcal{L}_Y(\lambda) = G \times^P \mathbb{C}_{-\lambda}$ on G/P, where $\mathbb{C}_{-\lambda}$ is the 1-dimensional P-representation of weight $-\lambda$. Let $B_{\rm s}^-$ be the Borel subgroup containing $T_{\rm s}$ opposite to $B_{\rm s}$ and let $z \in Y$ be the unique $B_{\rm s}^-$-stable point in Y. By [deCP83, Proposition 8.1 and Corollary 8.2], we have the following result.

Proposition 4.5.

1. *The map $i^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is injective.*
2. *For any $i \in [1, r]$, the torus $T_{\rm s}$ acts on T_zX/T_zX_i with weight $\overline{\alpha}_i \in \overline{\Delta}$.*
3. *We have $\mathcal{O}_X(X_i)|_Y = \mathcal{L}_Y(\overline{\alpha}_i)$.*
4. *The map $\mathcal{V}_X \to \overline{\Delta}, X_i \mapsto \overline{\alpha}_i$ is bijective.*

Remark 4.6. We set $X_{\overline{\alpha}_i} := X_i$ for $i \in [1, r]$ so that $X_{\overline{\beta}}$ is well defined for $\overline{\beta} \in \overline{\Delta}$.

Next we want to relate colors and restricted roots. This is more difficult, since there may be more colors than restricted roots as Propositions 4.7 and 4.8 below show. The following proposition holds for any projective spherical variety with a unique closed orbit (see [Pe14, Theorem 3.2.4])

Proposition 4.7. *We have* $\mathrm{Pic}(X) = \bigoplus_{D \in \mathcal{D}_X} \mathbb{Z}[D]$.

The rank of the Picard group is determined as follows (see [deCP83, Theorem 7.6]).

Proposition 4.8. *We have* $\mathrm{Pic}(X) = \mathbb{Z}^{r+s}$ *where* r *is the rank of* X *and* s *is the number of restricted simple roots* $\overline{\gamma} \in \overline{\Delta}$ *such that there exists a pair of exceptional simple roots* $\alpha, \beta = \overline{\sigma}(\alpha)$ *with* $\overline{\alpha} = \overline{\gamma} = \overline{\beta}$.

Remark 4.9. In particular we have the following formula:

$$\mathrm{rk}(\mathrm{Pic}(X)) = \begin{cases} r & \text{if } X \text{ is not exceptional,} \\ r+1 & \text{if } X \text{ is exceptional.} \end{cases}$$

The proof of Proposition 4.8 in [deCP83, Theorem 7.6]) suggests a correspondence between colors and restricted roots. We give a description of this using results of Luna. For $\alpha \in \Delta$, recall that G_α denotes the subgroup of G generated by U_α and $U_{-\alpha}$, and set $\mathcal{D}_X(\alpha) = \{D \in \mathcal{D}_X \mid G_\alpha \cdot D \neq D\}$. Note that if $\sigma(\alpha) = \alpha$, then $\mathcal{D}_X(\alpha) = \emptyset$ (see [Lu01]).

Proposition 4.10. *For any* $\alpha \in \Delta_1$*, the set* $\mathcal{D}_X(\alpha)$ *consists of a unique element. Moreover, for* $\alpha, \beta \in \Delta_1$*, we have* $\mathcal{D}_X(\alpha) = \mathcal{D}_X(\beta)$ *only if* $\langle \alpha^\vee, \beta \rangle = 0$ *and* $\sigma(\alpha) = -\beta$.

Proof. This is a consequence of a result of Luna which holds true for any wonderful variety (see [Lu01, Section 1.4]). Luna proves that three cases, called (a), (a') and (b), occur. In cases (a') and (b), the set $\mathcal{D}_X(\alpha)$ consists of a unique element, while in case (a) the set $\mathcal{D}_X(\alpha)$ consists of two elements. We prove that case (a) does not occur: in this case, we have $\alpha = \overline{\gamma} = \gamma - \sigma(\gamma)$ for some $\gamma \in \Delta$. Thus, $\sigma(\alpha) = -\alpha$ and $\overline{\alpha} = 2\alpha = 2\overline{\gamma} \in \overline{\Delta}$. In particular, $\overline{\gamma}, 2\overline{\gamma} \in \overline{\Delta}$, which contradicts the fact that $\overline{\Delta}$ is a basis of $\overline{R}$.

For $\alpha, \beta \in \Delta_1$, there are, according to [Lu01, Proposition 3.2], the following possibilities to have $\mathcal{D}_X(\alpha) \cap \mathcal{D}_X(\beta) \neq \emptyset$:

- Both α and β are in $\overline{\Delta}$, in which case we may have $|\mathcal{D}_X(\alpha) \cup \mathcal{D}_X(\beta)| = 3$.
- $\langle \alpha^\vee, \beta \rangle = 0$ and $\alpha + \beta \in \overline{\Delta}$ or $\frac{1}{2}(\alpha + \beta) \in \overline{\Delta}$.

The first case does not occur by the above argument. If $\langle \alpha^\vee, \beta \rangle = 0$ and $\frac{1}{2}(\alpha + \beta) \in \overline{\Delta}$ or $\alpha + \beta \in \overline{\Delta}$, then there exists $\gamma \in \Delta_1$ such that $\overline{\gamma} = \gamma - \sigma(\gamma) = \frac{1}{2}(\alpha + \beta)$ or $\overline{\gamma} = \gamma - \sigma(\gamma) = \alpha + \beta$. Write

$$\sigma(\gamma) + \overline{\sigma}(\gamma) = -\sum_{\delta \in \Delta_0} c_\delta \delta.$$

Then, we have

$$\frac{1}{2}(\alpha + \beta) = \gamma + \overline{\sigma}(\gamma) + \sum_{\delta \in \Delta_0} c_\delta \delta \quad \text{or} \quad \alpha + \beta = \gamma + \overline{\sigma}(\gamma) + \sum_{\delta \in \Delta_0} c_\delta \delta.$$

In the first case, this implies $\alpha = \beta$ and $\gamma + \overline{\sigma}(\gamma) \leq \alpha$, which is impossible. In the second case, we get that γ equals α or β. Assume for example that $\gamma = \alpha$, then we have $\overline{\sigma}(\alpha) = \overline{\sigma}(\gamma) = \beta$ and $c_\delta = 0$ for all $\delta \in \Delta_0$. We thus have $\sigma(\alpha) = \sigma(\gamma) = -\overline{\sigma}(\gamma) = -\beta$. Note that $\overline{\alpha} = \overline{\beta}$. $\square$

Remark 4.11. If X is not the wonderful compactification of an adjoint symmetric space, then there may be some simple roots $\alpha \in \Delta_1$ with $|\mathcal{D}_X(\alpha)| = 2$. A typical example is the case $G = \mathrm{SL}_2$ and $H = T$ a maximal torus. Then $N = N_G(T)$ is the normalizer of the torus. There is a unique projective compactification of G/H given by $\mathbb{P}^1 \times \mathbb{P}^1$ with two B-stable divisors D^+ and D^- and both are such that $\mathrm{SL}_2 \cdot D^{\pm} = X$. We thus have $\mathcal{D}_X(\alpha) = \{D^+, D^-\}$, where α is the unique simple root of G. The wonderful compactification X_{ad} of G/N is the quotient of X by the involution exchanging the two factors and is isomorphic to $\mathbb{P}^2$ with a unique B-stable divisor D, so that $\mathcal{D}_{X_{\mathrm{ad}}}(\alpha) = \{D\}$.

The restricted root system is $\overline{R} = \{-\overline{\alpha}, \overline{\alpha}\}$ in both cases. But for a maximal split torus T_{s}, we have $\overline{\alpha} \in \mathfrak{X}(T_{\mathrm{s}}/H \cap T_{\mathrm{s}})$ while $\overline{\alpha} \notin \mathfrak{X}(T_{\mathrm{s}}/N \cap T_{\mathrm{s}})$ but $2\overline{\alpha} \in \mathfrak{X}(T_{\mathrm{s}}/N \cap T_{\mathrm{s}})$.

By Proposition 4.10, we may define a map $\zeta : \Delta_1 \to \mathcal{D}_X$ by $\zeta(\alpha) = D$ with $D \in \mathcal{D}_X(\alpha)$.

Lemma 4.12. *The map $\zeta : \Delta_1 \to \mathcal{D}_X$ is surjective.*

Proof. For $D \in \mathcal{D}_X$, there exists $\alpha \in \Delta_1$ such that $D \in \mathcal{D}_X(\alpha)$. Indeed, since G is generated by the G_α for $\alpha \in \Delta$ and since D is not G-stable, there exists at least one $\alpha \in \Delta$ with $G_\alpha \cdot D \neq D$. Furthermore, by [Lu01], we have $\alpha \notin \Delta_0$; thus, $\alpha \in \Delta_1$. $\square$

Corollary 4.13. *The map $\tau : \mathcal{D}_X \to \overline{\Delta}, D \mapsto \overline{\alpha}$, with $\alpha \in \Delta_1$ such that $D \in \mathcal{D}_X(\alpha)$, is well defined and surjective.*

Proof. The restricted root $\overline{\alpha}$ does not depend on the choice of α with $D \in \mathcal{D}_X(\alpha)$: if $D \in \mathcal{D}_X(\alpha) \cap \mathcal{D}_X(\beta)$, then $\beta = -\sigma(\alpha)$ and $\overline{\beta} = \overline{\alpha}$ by Proposition 4.10. For the surjectivity, note that the composition $\tau \circ \zeta$ is surjective since $\tau \circ \zeta(\alpha) = \overline{\alpha}$. $\square$

Proposition 4.14. *The map τ is injective except if X is exceptional, in which case the only non-trivial fiber is $\tau^{-1}(\overline{\alpha}) = \{\zeta(\alpha), \zeta(\beta)\}$, where α, β are the exceptional roots.*

Proof. Assume first that α is exceptional and set $\beta = \overline{\sigma}(\alpha)$. Then $\tau(\zeta(\alpha)) = \overline{\alpha} = \overline{\beta} = \tau(\zeta(\beta))$. If $\zeta(\alpha) = \zeta(\beta)$, then $\mathcal{D}_X(\alpha) \cap \mathcal{D}_X(\beta) \neq \emptyset$. We thus have $\beta = -\sigma(\alpha)$ and $\langle \alpha^\vee, \sigma(\alpha) \rangle = -\langle \alpha^\vee, \beta \rangle = 0$, a contradiction with the fact that α is exceptional. Therefore, $\zeta(\alpha) \neq \zeta(\beta)$ and τ is not injective.

On the other hand, the map τ is surjective and $|\mathcal{D}_X| = \mathrm{rk}(\mathrm{Pic}(X)) = r + s = |\overline{\Delta}| + s$ with s the number of pairs of exceptional roots $(\alpha, \overline{\sigma}(\alpha))$. The result follows from this. $\square$

Corollary 4.15. *Let $\alpha, \beta \in \Delta_1$ with $\alpha \neq \beta$. Then $\zeta(\alpha) = \zeta(\beta)$ if and only if $\beta = -\sigma(\alpha)$ and $\langle \alpha^\vee, \beta \rangle = 0$.*

Proof. Assume that $\zeta(\alpha) = \zeta(\beta)$, then $\mathcal{D}_X(\alpha) \cap \mathcal{D}_X(\beta) \neq \emptyset$ and the result follows from Proposition 4.10. Conversely, if $\beta = -\sigma(\alpha)$ and $\langle \alpha^\vee, \beta \rangle = 0$, then $\overline{\alpha} = \overline{\beta}$. If $\zeta(\alpha) \neq \zeta(\beta)$, then (α, β) is a pair of exceptional roots. In particular, we would have $0 \neq \langle \alpha^\vee, \sigma(\alpha) \rangle = -\langle \alpha^\vee, \beta \rangle = 0$, a contradiction. Thus $\zeta(\alpha) = \zeta(\beta)$. $\square$

Remark 4.16. We get a characterisation of simple roots having the same restricted root: For $\alpha, \beta \in \Delta_1$ with $\alpha \neq \beta$, we have $\overline{\alpha} = \overline{\beta}$ if and only if either ((α, β) is a pair of exceptional roots with $\overline{\sigma}(\alpha) = \beta$), or ($\beta = -\sigma(\alpha)$ and $\langle \alpha^\vee, \beta \rangle = 0$).

We now compute the restrictions $i^*\mathcal{O}_X(D)$ for $D \in \mathcal{D}_X$. The next proposition is a direct application of results in [Lu97]. For $\alpha \in \Delta$, let ϖ_α be the fundamental weight associated to α.

Proposition 4.17. *Let $\alpha \in \Delta_1$ and let $\lambda_\alpha \in \mathfrak{X}(T_{\mathrm{s}})$ be such that $i^*\mathcal{O}_X(\zeta(\alpha)) = \mathcal{L}_Y(\lambda_\alpha)$. Then we have*

$$\lambda_\alpha = \begin{cases} 2\varpi_\alpha & \text{if } \sigma(\alpha) = -\alpha, \\ \varpi_\alpha + \varpi_{-\sigma(\alpha)} & \text{if } \sigma(\alpha) = -\overline{\sigma}(\alpha) \text{ and } \langle \alpha^\vee, \sigma(\alpha)\rangle = 0, \\ \varpi_\alpha & \text{otherwise.} \end{cases}$$

Proof. For $\beta \in \Delta_1$, let $Y_\beta \subset Y$ be the Schubert curve dual to $\mathcal{O}_Y(\varpi_\beta)$. Then by [Lu97, Lemma 3.1.1 and Lemma 3.1.2], we have

$$Y_\beta \cdot i^*\zeta(\alpha) = \begin{cases} 2\delta_{\alpha,\beta} & \text{if } \sigma(\alpha) = -\alpha, \\ \delta_{\zeta(\alpha),\zeta(\beta)} & \text{otherwise.} \end{cases}$$

The result follows from this and from the facts that if $|\zeta^{-1}(\zeta(\alpha))| > 1$, then $\zeta^{-1}(\zeta(\alpha)) = \{\alpha, \overline{\sigma}(\alpha)\}$ and that, by Corollary 4.15, this occurs if and only if $\sigma(\alpha) = -\overline{\sigma}(\alpha)$ and $\langle \alpha^\vee, \sigma(\alpha)\rangle = 0$. □

The above proof implicitly uses the fact that there exists a family of irreducible B-stable curves $(C_D)_{D\in\mathcal{D}_X}$ such that the classes $[C_D] \in A_1(X)$ form the dual basis to the basis $([D])_{D\in\mathcal{D}_X}$ of $\mathrm{Pic}(X)$ (see [Lu97, Lemma 3.1.2]). Recall the definition of $\widehat{\alpha}$ for $\overline{\alpha} \in \overline{\Delta}$ and the notation $X_{\overline{\beta}}$ for $\overline{\beta} \in \overline{\Delta}$ from Remark 4.6.

Corollary 4.18. *We have $X_{\overline{\beta}} \cdot C_{\zeta(\alpha)} = \langle \widehat{\alpha}^\vee, \overline{\beta}\rangle$ for all $\alpha, \beta \in \Delta_1$.*

Proof. Recall that $i^*\mathcal{O}_X(X_{\overline{\beta}}) = \mathcal{L}_Y(\overline{\beta})$. Note that we have $\overline{\beta} = \sum_{\gamma\in\Delta}\langle\gamma^\vee, \overline{\beta}\rangle\varpi_\gamma$. We get

$$X_{\overline{\beta}} \cdot C_{\zeta(\alpha)} = \begin{cases} \frac{1}{2}\langle \alpha^\vee, \overline{\beta}\rangle & \text{if } \sigma(\alpha) = -\alpha, \\ \langle \alpha^\vee, \overline{\beta}\rangle & \text{otherwise.} \end{cases}$$

Note that $\langle \alpha^\vee, \overline{\beta}\rangle = \langle -\sigma(\alpha)^\vee, \overline{\beta}\rangle$ so that if $\sigma(\alpha) = -\overline{\sigma}(\alpha)$ and $\langle \alpha^\vee, \sigma(\alpha)\rangle = 0$, we have $\langle \alpha^\vee, \overline{\beta}\rangle\varpi_\alpha + \langle -\sigma(\alpha)^\vee, \overline{\beta}\rangle\varpi_{-\sigma(\alpha)} = \langle \alpha^\vee, \overline{\beta}\rangle\lambda_\alpha$.

We now compare the above values to $\langle \widehat{\alpha}^\vee, \overline{\beta}\rangle$. If $\sigma(\alpha) = -\alpha$, then $\widehat{\alpha}^\vee = \overline{\alpha}^\vee = \frac{1}{2}\alpha^\vee$ proving the first case. If $\sigma(\alpha) \neq -\alpha$, we have two possibilities: either $\langle \alpha^\vee, \sigma(\alpha)\rangle = 0$ or $\langle \alpha^\vee, \sigma(\alpha)\rangle = 1$. In the former case, we have $\widehat{\alpha}^\vee = \overline{\alpha}^\vee = \frac{1}{2}(\alpha^\vee - \sigma(\alpha)^\vee)$; thus, $\langle \widehat{\alpha}^\vee, \overline{\beta}\rangle = \frac{1}{2}(\langle \alpha^\vee, \overline{\beta}\rangle + \langle -\sigma(\alpha)^\vee, \overline{\beta}\rangle) = \langle \alpha^\vee, \overline{\beta}\rangle$. Finally, if $\langle \alpha, \sigma(\alpha)\rangle = 1$, then $\widehat{\alpha}^\vee = \frac{1}{2}\overline{\alpha}^\vee = \frac{1}{2}(\alpha^\vee - \sigma(\alpha)^\vee)$ and the result follows as before. □

Recall that $\mathrm{Pic}(X) = \oplus_{D\in\mathcal{D}_X}\mathbb{Z}[D]$. Furthermore, by [Pe14, Theorem 3.2.9] the monoid $\mathrm{Nef}(X)$ of nef divisors is given by $\mathrm{Nef}(X) = \oplus_{D\in\mathcal{D}_X}\mathbb{Z}_{\geq 0}[D]$. It coincides with the monoid of globally generated divisors. On the curve side, we have $A_1(X) = \oplus_{D\in\mathcal{D}_X}\mathbb{Z}[C_D]$ and the monoid of effective classes of curves $\mathrm{NE}(X)$ is given by $\mathrm{NE}(X) = \oplus_{D\in\mathcal{D}_X}\mathbb{Z}_{\geq 0}[C_D]$ (see [Pe18] for more on curves on spherical varieties). Furthermore, we have a $\mathbb{Z}$-linear map

$$\psi : A_1(X) \to \mathbb{Z}\overline{\Delta}^\vee$$

defined by $\psi(C_D) = \widehat{\alpha}^\vee$, for $D \in \mathcal{D}_X(\alpha)$. By Corollary 4.18, we have $X_{\overline{\beta}} \cdot C = \langle \psi(C), \overline{\beta}\rangle$ for all $[C] \in \mathrm{NE}(X)$.

Proposition 4.19. *The map $\psi : A_1(X) \to \mathbb{Z}\overline{\Delta}^\vee$ is surjective.*

1. *The image of the monoid of effective curves is the monoid spanned by coroots.*

2. *The image of the monoid of curves having non-negative intersection with any component of ∂X_{ad} is the intersection of $\mathbb{Z}\overline{\Delta}^\vee$ with the monoid of dominant cocharacters.*

Proof. The surjectivity follows from the surjectivity of τ. The monoid of effective curves is spanned by the set $\{C_D \mid D \in \mathcal{D}_X\}$ whose image by ψ is $\overline{\Delta}^\vee$; this proves (1). Part (2) follows from Corollary 4.18 $\square$

Recall that a curve class $\gamma \in A_1(X)$ is covering if there exists a curve C of class γ passing through a general point $x \in X$. Note that this implies that $\gamma \cdot X_{\overline{\beta}} \geq 0$ for all $\overline{\beta} \in \overline{\Delta}$. We call a class $\gamma \in \mathrm{NE}(X)$ *virtually covering* if $X_{\overline{\beta}} \cdot \gamma \geq 0$ for all $\overline{\beta} \in \overline{\Delta}$.

Corollary 4.20. 1. *If X is non-exceptional, then there is a unique virtually covering curve class $\gamma_0 \in \mathrm{NE}(X)$ which is minimal in this monoid. Moreover, we have $\psi(\gamma) = \overline{\Theta}^\vee$.*

2. *If X is exceptional, then there are exactly two minimal virtually covering curve classes $\gamma_0^+, \gamma_0^- \in \mathrm{NE}(X)$ and we have $\psi(\gamma_0^+) = \overline{\Theta}^\vee = \psi(\gamma_0^-)$.*

Proof. The image by ψ of an effective and virtually covering curve class is in the intersection of the monoid generated by coroots in $\overline{R}$ and the dominant chamber. There is a unique minimal such element: the coroot of the highest root of $\overline{R}$. Since $\overline{\Theta}$ is the highest root of $\overline{R}$ by Proposition 4.3.1, the element $\overline{\Theta}^\vee$ is the smallest possible image by ψ of an effective and virtually covering curve class.

If X is non-exceptional, then ψ is injective and this proves (1). To prove (2), we are left to prove that if X is exceptional, there are exactly two classes γ^+ and γ^- in $\mathrm{NE}(X)$ such that $\psi(\gamma^+) = \overline{\Theta}^\vee = \psi(\gamma^-)$. But the kernel of ψ is $\mathbb{Z}([C_{D_\alpha}] - [C_{D_{\overline{\sigma}(\alpha)}}])$ with α an exceptional root. Since the coefficient of $\widehat{\alpha}^\vee$ in $\overline{\Theta}^\vee$ is 1 by Lemma 4.4, there are exactly two classes in $\mathrm{NE}(X)$ that are mapped to $\overline{\Theta}^\vee$ via ψ, namely, γ^+ with coefficient 1 in C_α and 0 on $C_{\overline{\sigma}(\alpha)}$, and γ^- with coefficient 0 in C_α and 1 on $C_{\overline{\sigma}(\alpha)}$. $\square$

4.3 Curves classes of the minimal families

In this subsection, we prove that the curve classes γ_0, γ_0^+ and γ_0^- are covering and are therefore the classes of minimal rational curves on X.

We will need a few more results on X. Recall that $x \in X^0$ is our base point and that r is the rank of X. The G-orbits in X are indexed by the subsets $I \subset [1, r]$ via $\mathcal{O}_I = \{x' \in X \mid x' \in X_i \Leftrightarrow i \in I\}$.

The local structure theorem associated to the closed orbit Y gives the following: there exists an affine P-stable open subset $X_{Y,B} \subset X$ containing x with $X_{Y,B} \cap Y \neq \emptyset$ and a P-equivariant isomorphism $X_{Y,B} \simeq \mathrm{R}_u(P) \times \mathbb{A}^r$. The closure of $T_{\mathrm{s}} \cdot x = S \cdot x$ in $X_{Y,B}$ is T_{s}-equivariantly isomorphic to $\mathbb{A}^r$, where the torus T_{s} acts linearly with weights $\overline{\Delta} = (\overline{\alpha}_i)_{i \in [1,r]}$. The prime G-divisor $X_{\overline{\alpha}_i}$ is defined in $X_{Y,B}$ by the vanishing of the coordinate with weight $\overline{\alpha}_i$ in $\mathbb{A}^r$.

Recall that $i : Y \to X$ denotes the inclusion of the closed G-orbit and that the map $i^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is injective. Let

$$\mathfrak{X}_X(T_{\mathrm{s}}) = \{\lambda \in \mathfrak{X}(T_{\mathrm{s}}) \mid \mathcal{L}_Y(\lambda) \in \mathrm{Im}\, i^*\mathrm{Pic}(X) \subset \mathrm{Pic}(Y)\}.$$

For $\lambda \in \mathfrak{X}_X(T_{\mathrm{s}})$, we write $\mathcal{L}_X(\lambda)$ for the line bundle such that $i^*\mathcal{L}_X(\lambda) = \mathcal{L}_Y(\lambda)$ (see [deCP83, End of 8.1] for these results).

Given a cocharacter $\eta^\vee : \mathbb{G}_m \to \overline{S} = S/S^\sigma$, we say that $\eta^\vee$ is dominant if $\langle \eta^\vee, \overline{\alpha} \rangle \geq 0$ for all $\alpha \in \Delta$. A cocharacter $\eta^\vee$ defines a map $\mathbb{C}^\times \to X, t \mapsto \eta^\vee(t) \cdot x$. This map extends to a morphism $\eta^\vee : \mathbb{P}^1 \to X$.

The following lemma generalizes [BF15, Lemma 3.1] to the case of wonderful compactifications of adjoint irreducible symmetric spaces.

Lemma 4.21. *Let $\eta^\vee : \mathbb{G}_m \to S$ be a dominant cocharacter, $\eta^\vee : \mathbb{P}^1 \to X$ the corresponding morphism, and $C_{\eta^\vee}$ its image.*

1. *We have $\eta^\vee(0) \in \mathcal{O}_I$, where $I := \{i \in [1, r] \mid \langle \eta^\vee, \overline{\alpha}_i \rangle \neq 0\}$.*
2. *We have $\eta^\vee(\infty) \in \mathcal{O}_J$, where $J := \{j \in [1, r] \mid \langle \eta^\vee, w_0(\overline{\alpha}_j) \rangle \neq 0\}$.*
3. *The morphism $\eta^\vee : \mathbb{P}^1 \to C_{\eta^\vee}$ is an isomorphism if and only if there exists $i \in [1, r]$ such that $\langle \eta^\vee, \overline{\alpha}_i \rangle = 1$.*
4. *For $\lambda \in \mathfrak{X}_X(T_{\mathrm{s}})$, we have $\deg(\eta^\vee)^*\mathcal{L}_X(\lambda) = \langle \eta^\vee, \lambda - w_0\lambda \rangle$.*

Proof. (1) Since $\eta^\vee$ is dominant, its extends to a morphism $\mathbb{A}^1 \to X_{Y,B} \cap \overline{T}$ defined by $t \mapsto (t^{\langle \eta^\vee, \overline{\alpha}_i \rangle})_{i \in [1,r]}$, where $X_{Y,B} \cap \overline{T}$ is identified with $\mathbb{A}^r$ as above. In particular, $\eta^\vee(0) \in X_{Y,B}$ and vanishes on the coodinates with indices in $\{i \in [1, r] \mid \langle \eta^\vee, \overline{\alpha}_i \rangle \neq 0\}$. Moreover, the morphism $\eta^\vee : \mathbb{P}^1 \to C_{\eta^\vee}$ is a local isomorphism at $\eta^\vee(0)$ if and only if there exists i such that $\langle \eta^\vee, \overline{\alpha}_i \rangle = 1$.

(2) Consider the open affine subset $w_0 \cdot X_{Y,B}$ of X. It is isomorphic to $\mathrm{R}_u(P)^{w_0} \times \mathbb{A}^r$ with a linear action of T_{s} on $\mathbb{A}^r$ with weights $w_0(\overline{\Delta})$. All these weights are non-negative linear combinations of negative roots. In particular, the one-parameter subgroup $-\eta^\vee$ acts with non-negative weights on $\mathbb{A}^r$, and hence extends to a morphism $\mathbb{A}^1 \to w_0 \cdot X_{Y,B}$, $t \mapsto (t^{\langle \eta^\vee, -w_0(\overline{\alpha}_i) \rangle})_{i \in [1,r]}$. It follows that $\eta^\vee(\infty) = (-\eta^\vee)(0) \in w_0 \cdot X_{Y,B}$ and as above, $\eta^\vee(\infty) \in \mathcal{O}_J$. Moreover, the morphism $\eta^\vee : \mathbb{P}^1 \to C_{\eta^\vee}$ is a local isomorphism at $\eta^\vee(\infty)$ if and only if there exists i such that $\langle \eta^\vee, -w_0(\overline{\alpha}_i) \rangle = 1$. Note that $-w_0(\overline{\alpha}_i) = -w_0(\alpha_i) + w_0(\sigma(\alpha_i)) = \overline{-w_0(\alpha_i)}$ and since $-w_0$ permutes Δ_1, by Proposition 4.3.1, the previous condition is true if and only if there exists i such that $\langle \eta^\vee, \overline{\alpha}_i \rangle = 1$.

3. Follows from the above conditions at $\eta^\vee(0)$ and $\eta^\vee(\infty)$.

4. The pull-back of $\mathcal{L}_X(\lambda)$ to $X_{Y,B}$ (resp. $w_0 \cdot X_{Y,B}$) has a trivializing section of weight λ (resp. $w_0(\lambda)$). As a consequence, the line bundle $(\eta^\vee)^*\mathcal{L}_X(\lambda)$ is a $\mathbb{G}_m$-linearized line bundle on $\mathbb{P}^1$ with weights $\langle \eta^\vee, \lambda \rangle$ at 0 and $\langle \eta^\vee, w_0\lambda \rangle$ at ∞. Since the degree of such a line bundle is the difference of its weights, this yields our assertion. $\square$

We now apply the above result to $\eta^\vee = \overline{\Theta}^\vee$.

Corollary 4.22. *Consider the morphism $\overline{\Theta}^\vee : \mathbb{P}^1 \to X$.*

1. *If $\overline{R}$ is not of type* A_1*, then $\overline{\Theta}^\vee$ is an isomorphism onto its image.*

2. *If $\overline{R}$ is of type* A_1*, then $\overline{\Theta}^\vee$ has degree 2 over its image.*

3. *The push-forward class is given as follows:*

$$\overline{\Theta}^\vee_*[\mathbb{P}^1] = \begin{cases} 2\gamma_0 & \text{if } X \text{ is non-exceptional,} \\ \gamma_0^+ + \gamma_0^- & \text{if } X \text{ is exceptional.} \end{cases}$$

Proof. (1) If $\overline{R}$ is not of type A_1, then Proposition 4.3.4 implies that there exists a simple root $\overline{\alpha} \in \overline{\Delta}$ such that $\langle \overline{\Theta}^\vee, \overline{\alpha} \rangle = 1$. Therefore, $\overline{\Theta}^\vee : \mathbb{P}^1 \to X$ is an isomorphism onto its image.

(2) If $\overline{R}$ is of type A_1, then $\overline{\Theta}^\vee$ induces a map $\mathbb{G}_m \to X_{Y,B} \cap \overline{T} = \mathbb{A}^1, t \mapsto t^2$ which is of degree 2 onto its image.

(3) For $\lambda \in \mathfrak{X}_X(T_\mathrm{s})$, we have $[\mathcal{L}_X(\lambda)] \cdot \overline{\Theta}^\vee_*[\mathbb{P}^1] = \deg(\overline{\Theta}^\vee)^*\mathcal{L}_X(\lambda) = \langle \overline{\Theta}^\vee, \lambda - w_0(\lambda) \rangle = \langle \overline{\Theta}^\vee, \lambda \rangle - \langle w_0(\overline{\Theta})^\vee, \lambda \rangle = 2\langle \overline{\Theta}^\vee, \lambda \rangle$ by Proposition 4.3.1. This proves the result for X non-exceptional, since $X_{\overline{\beta}} \cdot C_0 = \langle \overline{\Theta}^\vee, \overline{\beta} \rangle$, $\mathcal{O}_X(X_{\overline{\beta}}) = \mathcal{L}_X(\overline{\beta})$ and $(\overline{\beta})_{\overline{\beta} \in \overline{\Delta}}$ generates $\mathfrak{X}_X(T_\mathrm{s}) \otimes_\mathbb{Z} \mathbb{Q}$.

If X is exceptional, then by the same argument, we have that the class of the image and the class $\gamma_0^+ + \gamma_0^-$ agree on all boundary divisors. We therefore only need to check that they agree on D_α for α an exceptional simple root. Assume that γ_0^+ is dual to D_α while γ_0^- is dual to $D_{\overline{\sigma}(\alpha)}$. We get $2\langle \overline{\Theta}^\vee, \lambda_\alpha \rangle = \langle \Theta^\vee, \lambda_\alpha \rangle = 1 = D_\alpha \cdot (\gamma_0^+ + \gamma_0^-)$ by Proposition 4.3.3. Similarly, we have $2\langle \overline{\Theta}^\vee, \lambda_{\overline{\sigma}(\alpha)} \rangle = \langle \Theta^\vee, \lambda_{\overline{\sigma}(\alpha)} \rangle = 1 = D_{\overline{\sigma}(\alpha)} \cdot (\gamma_0^+ + \gamma_0^-)$. ☐

Recall the definitions of the nilpotent orbits $\mathcal{O}_{\min}$ and of type $\mathcal{O}_{\mathrm{sum}}$ from Definition 2.17. For G simple with maximal torus T_s of split type such that $\sigma(\Theta) \neq -\Theta$, define the nilpotent orbit $\mathcal{O}_{\mathrm{sum},\sigma}$ by $\mathcal{O}_{\mathrm{sum},\sigma} = G \cdot (e_\Theta - \sigma(e_\Theta))$ with $e_\Theta \in \mathfrak{g}_\Theta \setminus \{0\}$. Note that by Proposition 4.3.3, the nilpotent orbit $\mathcal{O}_{\mathrm{sum},\sigma}$ is indeed of type $\mathcal{O}_{\mathrm{sum}}$.

Proposition 4.23. *There exists a smooth rational curve C in X such that $x \in C$ and $[C] = \gamma$, γ_0^+ or γ_0^-.*

Furthermore, we have $T_xC \setminus \{0\} \subset \mathcal{O}_{\min}$ if $\sigma(\Theta) = -\Theta$, and $T_xC \setminus \{0\} \subset \mathcal{O}_{\mathrm{sum},\sigma}$ otherwise.

Proof. If $\sigma(\Theta) = -\Theta$, pick $e = e_\Theta \in \mathfrak{g}_\Theta \setminus \{0\}$. If $\sigma(\Theta) \neq -\Theta$, pick $e = e_\Theta - \sigma(e_\Theta)$ with e_Θ as above. Note that $e \in \mathcal{O}_{\min}$ for $\sigma(\Theta) = -\Theta$ and that, by Proposition 4.3.2, e is in a nilpotent orbit of type $\mathcal{O}_{\mathrm{sum}}$ for $\sigma(\Theta) \neq -\Theta$. Set $f = \sigma(e)$. We may choose e so that $h = [e, f] = 2\overline{\Theta}^\vee$. Then (e, h, f) is a $\mathfrak{sl}_2$-triple. The cocharacter h induces a morphism $h : \mathbb{P}^1 \to X$ which factors through $\overline{\Theta}^\vee : \mathbb{P}^1 \to X$

$$\begin{tikzcd} \mathbb{P}^1 \arrow[d, "2:1"'] \arrow[dr, "h"] & \\ \mathbb{P}^1 \arrow[r, "\overline{\Theta}^\vee"] & X \end{tikzcd}$$

so that the vertical map is a double cover. Note in particular that both maps h and $\overline{\Theta}^\vee$ have the same image C' in X.

Denote by $G(h)$ the closed subgroup of G with Lie algebra $\langle e, h, f\rangle$. Then $G(h)$ is isomorphic to SL_2 or PGL_2, and σ acts non-trivially on $G(h)$. In particular, we have an isogeny $\mathrm{SL}_2 \to G(h)$ and σ lifts to a unique involution on SL_2. Let T' be a maximal torus of SL_2 fixed pointwise by σ and let X' be the closure of $G(h)\cdot x$ in X. Then X' is either isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (the unique proper embedding of SL_2/T') or to $\mathbb{P}^2$ (the unique proper embedding of $\mathrm{SL}_2/N_{\mathrm{SL}_2}(T')$). Note that in any case $x \in X'$; thus, $X' \cap X^0$ is non-empty. In the first case, the curve C' is linearly equivalent to the diagonal curve D. In the second case, the curve C' is a line in $\mathbb{P}^2$. In both cases, there exists a line L through x in X' so that T_xL is equal to $\langle e\rangle$, as a subset of T_xX identified to $\mathfrak{p}$.

If $\overline{R}$ is of type A_1, then X is non-exceptional and $\overline{\Theta}^\vee$ has degree 2 onto its image C'. In particular $[C'] = \frac{1}{2}\overline{\Theta}^\vee_*[\mathbb{P}^1] = \gamma$ and by minimality, γ_0 has to be the class of a line in X'. Note that this implies that we are in the case $X' = \mathbb{P}^2$.

Assume that $\overline{R}$ is not of type A_1. Then $\overline{\Theta}^\vee : \mathbb{P}^1 \to X$ is an isomorphism onto its image C'. Note also that $h \in \mathfrak{X}^\vee$ is indivisible as a cocharacter of T_{s} by Proposition 4.3.4; thus, $h(-1)$ is non-central in G. Since $h(-1)$ is central in $G(h)$, this group is isomorphic to SL_2 and since $h(-1)$ is non-central in G, we have $G(h)^\sigma = T'$ and $X' = \mathbb{P}^1 \times \mathbb{P}^1$. Therefore, $[C'] = [D]$. Define $[C^+]$ and $[C^-]$ to be the classes of the two rulings in $\mathbb{P}^1 \times \mathbb{P}^1 \subset X$. Then $[C^+] + [C^-] = [D] = [C']$. Since any ruling meets $X' \cap X^0 = X' \setminus D$, the classes $[C^+]$ and $[C^-]$ are virtually covering. This implies that $[C^+] = [C^-] = \gamma_0$ if X is non-exceptional and (up to exchanging the two rulings) $[C^+] = \gamma_0^+$ and $[C^-] = \gamma_0^-$ if X is exceptional. $\square$

We may now complete Proposition 3.5 by computing the classes of curves in minimal families.

Corollary 4.24. 1. *If X is non-exceptional, there exists a unique family $\mathcal{K}$ of minimal rational curves and for $C \in \mathcal{K}$, we have $[C] = \gamma_0$.*

2. *If X is exceptional, there exists two families $\mathcal{K}^+$ and $\mathcal{K}^-$ of minimal rational curves and for $C \in \mathcal{K}^\pm$, we have $[C] = \gamma_0^\pm$.*

Proof. Follows from Proposition 3.5, Corollary 4.20 and Proposition 4.23. $\square$

Corollary 4.25. *Let $\mathcal{K}$ be a minimal family, and $C \in \mathcal{K}_x$. If $\sigma(\Theta) = -\Theta$, then we have $T_xC \setminus \{0\} \subset \mathcal{O}_{\min}$. Otherwise, $T_xC \setminus \{0\} \subset \mathcal{O}_{\mathrm{sum},\sigma}$.*

We now compute the dimensions of the minimal families $\mathcal{K}$ and $\mathcal{K}^\pm$ and of the nilpotent orbits $\mathcal{O}_{\min}$ and $\mathcal{O}_{\mathrm{sum},\sigma}$. Let ρ be the half-sum of positive roots in G and ρ_P be the half-sum of positive roots in P. Set $\kappa = 2\rho - 2\rho_P$. We have $\kappa = \sum_{\alpha \in R^+, \sigma(\alpha)<0} \alpha$. Let $\Sigma = \sum_{\overline{\alpha} \in \overline{\Delta}} \overline{\alpha}$ be the sum of all restricted simple roots.

Theorem 4.26. *Let $\mathcal{K}$ be a minimal family, let $C \in \mathcal{K}_x$ and $m \in T_xC \setminus \{0\}$. We have*

$$\dim \mathcal{K}_x = \langle \overline{\Theta}^\vee, \kappa + \Sigma\rangle - 2 \text{ and } \dim G\cdot m = 2\langle \overline{\Theta}^\vee, \kappa\rangle.$$

In particular, $\dim \mathcal{K}_x = \frac{1}{2}\dim G\cdot m - 1 + (\langle \overline{\Theta}^\vee, \Sigma\rangle - 1)$.

Remark 4.27. Note that the value of $\langle \overline{\Theta}^\vee, \Sigma\rangle$ depends on the type of $\overline{R}$ as follows:

$$\partial X \cdot C = \langle \overline{\Theta}^\vee, \Sigma\rangle = \begin{cases} 2 & \text{if } \overline{R} \text{ is of type } \mathrm{A}_r \text{ with } r \geq 1, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Recall from Proposition 3.6 that $\dim \mathcal{K}_x = -K_X \cdot C - 2$. By adjunction formula, we have $(-K_X)|_Y = -K_Y + \partial X|_Y$. Since $\mathcal{L}_Y(-K_Y) = \mathcal{L}_Y(\kappa)$ and $i^*\mathcal{L}_X(\partial X) = \mathcal{L}_Y(\Sigma)$, we get $i^*\mathcal{L}_X(-K_X) = \mathcal{L}_Y(\kappa + \Sigma)$ and $\dim \mathcal{K}_x = \langle \overline{\Theta}^\vee, \kappa + \Sigma\rangle - 2$.

We now compute the dimension of $G \cdot m$. Assume first that $\sigma(\Theta) = -\Theta$, then $G \cdot m = \mathcal{O}_{\min}$. It is a classical result that $\dim \mathcal{O}_{\min} = 2\langle \Theta^\vee, \rho\rangle$ (see for example [CP11] or [BF15, Lemma 4.1]). On the other hand, we have $\overline{\Theta}^\vee = \frac{1}{2}\Theta^\vee$. Thus, $\langle \overline{\Theta}^\vee, \kappa\rangle = \langle \Theta^\vee, \rho\rangle - \langle \Theta^\vee, \rho_P\rangle$. Since $\sigma(\rho_P) = \rho_P$, we get $\langle \Theta^\vee, \rho_P\rangle = \langle \sigma(\Theta), \sigma(\rho_P)\rangle = -\langle \Theta^\vee, \rho_P\rangle$ thus $\langle \Theta^\vee, \rho_P\rangle = 0$ and $\dim G \cdot m = 2\langle \Theta^\vee, \rho\rangle = 2\langle \overline{\Theta}^\vee, \kappa\rangle$.

Assume now that $\sigma(\Theta) \neq -\Theta$, then we have $G \cdot m = \mathcal{O}_{\mathrm{sum},\sigma}$. To compute the dimension of the orbit $\mathcal{O}_{\mathrm{sum},\sigma}$, we recall some facts on nilpotent elements obtained as sums of weight vectors (see [Pa99] and [FR08]). We may assume $m = e \in \mathfrak{g}$ is a nilpotent element which can be written as a sum $e = e_\Theta + e_{-\sigma(\Theta)}$ with $e_\Theta \in \mathfrak{g}_\Theta \setminus \{0\}$ and $e_{-\sigma(\Theta)} \in \mathfrak{g}_{-\sigma(\Theta)} \setminus \{0\}$. Recall that Θ and $-\sigma(\Theta)$ are strongly orthogonal long roots. Let $f, h \in \mathfrak{g}$ such that (e, h, f) is a $\mathfrak{sl}_2$-triple. We may assume that $h = \Theta^\vee - \sigma(\Theta)^\vee = 2\overline{\Theta}^\vee$ and $f = \sigma(e)$. Then h induces a grading on $\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$, where $f \in \mathfrak{g}(-2)$ and $e \in \mathfrak{g}(2)$. Furthermore, if Q is the subgroup of G with Lie algebra $\mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$, then Q is a parabolic subgroup and if L is the subgroup of G with Lie algebra $\mathfrak{g}(0)$, then L is a Levi subgroup of Q and the orbit $L \cdot e$ is dense in $\mathfrak{g}(2)$. In addition, we have $G \cdot e = G \times^Q (L \cdot e)$ so that $\dim G \cdot e = \dim(G/Q) + \dim \mathfrak{g}(2) = \dim \mathfrak{g}(1) + 2\dim \mathfrak{g}(2)$.

If $\sigma(\alpha) = \alpha$, then $\langle \Theta^\vee - \sigma(\Theta^\vee), \alpha\rangle = \langle \Theta^\vee, \alpha\rangle - \langle \Theta^\vee, \sigma(\alpha)\rangle = 0$ and $\langle \Theta^\vee - \sigma(\Theta^\vee), \rho_P\rangle = 0$. If $\alpha \in R^+$ is such that $\mathfrak{g}_\alpha \subset \mathfrak{g}(i)$ for $i \geq 0$, then $\langle \Theta^\vee - \sigma(\Theta^\vee), \alpha\rangle = i$, thus

$$\langle \Theta^\vee - \sigma(\Theta^\vee), \kappa\rangle = \langle \Theta^\vee - \sigma(\Theta^\vee), 2\rho\rangle = \sum_{i=0}^{2} \sum_{\alpha \in R^+,\ \mathfrak{g}_\alpha \subset \mathfrak{g}(i)} i = \dim \mathfrak{g}(1) + 2\dim \mathfrak{g}(2).$$

This proves the result, since $\langle \Theta^\vee - \sigma(\Theta^\vee), \kappa\rangle = 2\langle \overline{\Theta}^\vee, \kappa\rangle$. □

4.4 Contact structure

In this subsection, we compute the dimension of $H \cdot m$ for $m \in T_xC \setminus \{0\}$, $C \in \mathcal{K}_x$ and $\mathcal{K}$ a minimal family. We first gather some facts on orbits associated with symmetric spaces, and in particular prove that orbits of symmetric subgroups of G are Lagrangian subvarieties in nilpotent G-orbits. Recall the following general definitions.

Definition 4.28. Let $\widehat{M}$ be a smooth complex variety of dimension $2n+2$ and let M be a smooth complex variety of dimension $2n+1$.

1. *A symplectic structure* on $\widehat{M}$ is a closed symplectic form $\omega : T\widehat{M} \times T\widehat{M} \to \widehat{M} \times \mathbb{C}$.

2. *A contact structure on* M is an everywhere non-vanishing map $\eta : TM \to \mathcal{L}$, where $\mathcal{L}$ is a line bundle, such that the bilinear form $\theta_\eta : D \times D \to TM/D$ defined by $(u, v) \mapsto [u, v]$ (mod D) on $D := \operatorname{Ker} \eta$ is non-degenerate for all $m \in M$.

If $\eta : TM \to \mathcal{L}$ is a contact structure on M, then there is a natural symplectic structure ω defined by $\omega = d(p^*\eta)$ on $\widehat{M} = \mathcal{L}^\times$, where $\mathcal{L}^\times$ is the $\mathbb{C}^\times$-bundle over M with structure map $p : \widehat{M} \to M$, associated to $\mathcal{L}$.

Definition 4.29. A symplectic structure ω on $\widehat{M}$ is induced by a contact structure $\eta : TM \to \mathcal{L}$ on M if $\widehat{M} = \mathcal{L}^\times$ and $\omega = d(p^*\eta)$.

The most famous examples of the above structures are given by coadjoint orbits in the dual $\mathfrak{g}^\vee$ of the Lie algebra $\mathfrak{g}$ of a connected reductive group G. For later purposes, we present a (non-canonical) version of *Kostant-Kirillov form* which takes place in $\mathfrak{g}$ the Lie algebra and not $\mathfrak{g}^\vee$. If $\mathfrak{g}$ is semisimple, the Killing form identifies $\mathfrak{g}$ with $\mathfrak{g}^\vee$ and the construction is canonical.

Example 4.30. Choose an invariant non-degenerate bilinear form B on $\mathfrak{g}$. Let m be a non-zero element in $\mathfrak{g}$ and let $\widehat{M_m} = G \cdot m$ and $M_m = G \cdot [m] \subset \mathbb{P}(\mathfrak{g})$ be the orbits of $m \in \mathfrak{g}$ and of $[m] \in \mathbb{P}(\mathfrak{g})$ under the adjoint action. Let G_m be the isotropy subgroup of G at m, with Lie algebra $\mathfrak{g}_m$. Define the anti-symmetric bilinear form B_m on $\mathfrak{g}$ by $B_m(y, z) = B(m, [y, z])$. We have $\operatorname{Ker} B_m = \mathfrak{g}_m$; thus, B_m descends to a symplectic form $\omega_m \colon \mathfrak{g}/\mathfrak{g}_m \times \mathfrak{g}/\mathfrak{g}_m \to \mathbb{C}$ at $m \in \mathfrak{g}$. By the Jacobi identity, the form ω_m is closed.

If m is such that the orbit $\widehat{M_m} = G.m$ is the cone in $\mathfrak{g}$ over $M_m = G \cdot [m] \subset \mathbb{P}(\mathfrak{g})$ (i.e., the affine cone minus the origin), then the arguments in [Be98, Proposition 2.2] adapt verbatim and yield a contact structure η on M_m which induces the symplectic form ω_m. In particular, if m is a nilpotent element in $\mathfrak{g}$, then the existence of an $\mathfrak{sl}_2$-triple containing m ensures that $\widehat{M_m}$ is the cone over M_m (see [Be98, Paragraph (2.4)]).

Given a symplectic structure on a variety $\widehat{M}$ or a contact structure on a variety M, it is natural to ask for Lagrangian or Legendrian subvarieties; we recall their definitions. A Lagrangian subspace in a symplectic vector space V of dimension $2m$ is an isotropic subspace of maximal dimension, *i.e.*, of dimension m.

Definition 4.31. Let $\widehat{M}$ have a symplectic structure ω. A smooth subvariety $\widehat{L} \subset \widehat{M}$ is called *Lagrangian* if, for all $m \in \widehat{L}$, the subspace $T_m\widehat{L} \subset T_m\widehat{M}$ is Lagrangian for the symplectic form ω_m on $T_m\widehat{M}$.

Definition 4.32. Let M have a contact structure η and let $p : \widehat{M} \to M$ be the $\mathbb{C}^\times$-bundle $\mathcal{L}^\times$ associated to the line bundle $\mathcal{L}$ with symplectic form $\omega = d(p^*\eta)$. A smooth subvariety $L \subset M$ is called *Legendrian* if $\widehat{L} = p^{-1}(L)$ is Lagrangian in $\widehat{M}$.

Example 4.33. Let G be simple and $\mathfrak{g}$ its Lie algebra. Let $m \in \mathfrak{g}$ be a highest weight vector. Set $\mathcal{O}_{\min} = G \cdot m$ and $\mathbb{P}(\mathcal{O}_{\min}) = G \cdot [m] \subset \mathbb{P}(\mathfrak{g})$. The latter is called the *adjoint variety* of G. It is the unique closed orbit of G in $\mathbb{P}(\mathfrak{g})$ under the adjoint action. Let $\mathbb{L}_G$ be the set of lines passing through a given point of $\mathbb{P}(\mathcal{O}_{\min})$. Then $\mathbb{L}_G$ is a smooth Legendrian variety in its linear span and is homogeneous under the isotropy subgroup G_m (see [LM07]). This Legendrian variety $\mathbb{L}_G$ is called *the subadjoint variety.* Note that in type C_n we have $\mathbb{L}_G = \emptyset$: the subadjoint variety is empty, since $\mathbb{P}(\mathcal{O}_{\min})$ is the second Veronese embedding of $\mathbb{P}^{2n-1}$ and hence contains no line. We will see in §6.3 that $\mathbb{L}_G$ can be recovered as the VMRT of a specific wonderful adjoint symmetric variety for G.

Let $H \subset G$ be a symmetric subgroup with group involution σ. The following result is well known, we include a proof for the convenience of the reader.

Proposition 4.34 ([KR71, Proposition 5])**.** *Let $m \in \mathfrak{p}$. Set $\widehat{L}_m := H \cdot m$ and $L_m := H \cdot [m]$. Then the variety $\widehat{L}_m$ is Lagrangian in $\widehat{M}_m$; in particular,* $\dim G \cdot m = 2 \dim H \cdot m$.

If $\widehat{L}_m$ is the cone over L_m (or equivalently is stable by non-trivial homotheties), then the variety L_m is Legendrian in M_m, in particular $\dim G \cdot [m] = 2 \dim H \cdot [m] + 1$.

Proof. Since $\sigma(m) = -m$, the action of σ on $\mathfrak{g}$ restricts to an action on $\mathfrak{g}_m$. Since σ is semisimple, we thus have $\mathfrak{g}_m = \mathfrak{h}_m \oplus \mathfrak{p}_m$ and $\mathfrak{g}/\mathfrak{g}_m = \mathfrak{h}/\mathfrak{h}_m \oplus \mathfrak{p}/\mathfrak{p}_m$ with $\mathfrak{h}_m = \mathfrak{h} \cap \mathfrak{g}_m$ and $\mathfrak{p}_m = \mathfrak{p} \cap \mathfrak{g}_m$. Let $u, v \in \mathfrak{h}/\mathfrak{h}_m$ (resp. $u, v \in \mathfrak{p}/\mathfrak{p}_m$) and let y and z in $\mathfrak{h}$ (resp. in $\mathfrak{p}$) be representatives of u and v. In Example 4.30, the form B can be chosen to be σ-invariant so that we get

$$\begin{aligned} \omega_m(u,v) &= \omega_m(d\sigma(u), d\sigma(v)) \\ &= B(m, [d\sigma(y), d\sigma(z)]) \\ &= -B(d\sigma(m), [d\sigma(y), d\sigma(z)]) \\ &= -B(d\sigma(m), d\sigma[y,z]) \\ &= -B(m, [y,z]) \\ &= -\omega_m(u,v). \end{aligned}$$

Hence $\omega_m(u,v) = 0$ and both $\mathfrak{h}/\mathfrak{h}_m$ and $\mathfrak{p}/\mathfrak{p}_m$ are isotropic and therefore Lagrangian in $\mathfrak{g}/\mathfrak{g}_m$. This proves the first part. If $H \cdot m$ is stable under nontrivial homotheties, then the same holds true for $G \cdot m$. The result follows from this and the first part. $\square$

Corollary 4.35. *Let $\mathcal{K}$ be a minimal family, let $C \in \mathcal{K}_x$ and let $m \in T_xC \setminus \{0\}$. Then $\widehat{L}_m$ is the cone over L_m, in particular* $\dim H \cdot [m] = \frac{1}{2} \dim G \cdot m - 1$.

Proof. Since the H-weight of m is non-trivial, the orbit $H \cdot m$ is the cone over $H \cdot [m]$. The result follows. $\square$

Assume that $\overline{R}$ is not of type A_r and let $C \in \mathcal{K}_x$ with $\mathcal{K}$ a minimal family. We obtain the following description of $\mathcal{K}_x$.

Theorem 4.36. *We have $\mathcal{K}_x = H \cdot C$. Furthermore, if X is Hermitian non-exceptional, then $\mathcal{K}_x$ has two components. Otherwise, $\mathcal{K}_x$ is irreducible.*

Proof. Since $\overline{R}$ is not of type A_r, we have $\partial X \cdot C = 1$ (see Remark 4.27). By Theorem 4.26 and Corollary 4.35, we have $\dim H \cdot C = \dim \mathcal{K}_x$. If C is not Hermitian or Hermitian exceptional, then there exists a unique highest weight curve and $\mathcal{K}_x$ is irreducible, proving the result. If X is Hermitian non-exceptional, then $\mathcal{K}_x$ contains two highest weight curves which are exchanged by H; the result follows. $\square$

In Proposition 4.34, the condition that $\widehat{L}_m$ is the cone over L_m is non-empty.

Example 4.37. There are non-nilpotent elements $m \in \mathfrak{p}$ for which $H \cdot m$ is not invariant under non-trivial homotheties. For example, let $G = \mathrm{SL}_n$ and $\sigma(g) = {}^t g^{-1}$. Then $\mathfrak{g} = \mathfrak{sl}_n$ is the Lie algebra of traceless matrices, $\mathfrak{h}$ is the Lie subalgebra of antisymmetric matrices, and $\mathfrak{p}$ is the subspace of traceless symmetric matrices. Define $m \in \mathfrak{p}$ as block-matrix as follows:

$$m = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \text{ with } A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2.$$

For $\lambda \in \mathbb{C}^\times$, if $\lambda m \in H \cdot m$, then $\lambda = \pm 1$. Thus, $H \cdot m$ is not the cone over $H \cdot [m]$.

We conclude this subsection by the following related result, which follows from [Ri82, Theorem A]; we provide a direct proof for the reader's convenience.

Lemma 4.38. *The orbit $H \cdot m$ (resp. $H \cdot [m]$) is open in $(G \cdot m)^{-\sigma}$ (resp. $(G \cdot [m])^{\sigma}$). In particular, $H \cdot m$ (resp. $H \cdot [m]$) is a union of connected components of $(G \cdot m)^{-\sigma}$ (resp. $(G \cdot [m])^{\sigma}$). Moreover,* $\dim(H \cdot m) = \dim(G \cdot m)^{-\sigma}$ *and* $\dim(H \cdot [m]) = \dim(G \cdot [m])^{\sigma}$.

Proof. Note that m is fixed for $-\sigma$; therefore, $\sigma([m]) = [m]$. We thus have inclusions $H \cdot m \subset (G \cdot m)^{-\sigma}$ and $H \cdot [m] \subset (G \cdot [m])^{\sigma}$. To prove the openness, we only need to check that the tangent spaces agree. We deal with $H \cdot [m]$, the other case works in a similar way. Since σ is semisimple, we have $T_m(G \cdot [m])^{\sigma} = (\mathfrak{g} \cdot [m])^{\sigma} = \mathfrak{g}^{\sigma} \cdot [m] = \mathfrak{h} \cdot [m] = T_m(H \cdot [m])$.

Since every H-orbit in $(G \cdot m)^{-\sigma}$ is open, there are only finitely many such orbits and these orbits are also closed, proving the last statement. $\square$

4.5 Wonderful symmetric varieties of type A_r

As the above discussion shows, the case of symmetric spaces whose restricted root system is of type A_r with $r \geq 1$ will present a different feature: the family $\mathcal{K}_x$ has dimension one more than the orbit $H \cdot C$ for $C \in \mathcal{K}_x$. In this section we prove that $\mathcal{K}_x$ is a rational projective homogeneous space.

Assume that the restricted root system $\overline{R}$ is of type A_r and let $(\overline{\alpha}_i)_{i \in [1,r]}$ be the simple roots of $\overline{R}$ (labeled as in Bourbaki [Bo68]). For $i \in [1, r]$, let $\alpha_i \in R$ be a simple root such that $\alpha_i - \sigma(\alpha_i) = \overline{\alpha}_i$. For $\beta \in \Delta$, let ϖ_β be the associated fundamental weight of R. For each $i \in [1, r]$, recall the definition of the dominant weight $\lambda_i := \lambda_{\alpha_i}$ from Proposition 4.17:

$$\lambda_i = \begin{cases} 2\varpi_{\alpha_i} & \text{if } \sigma(\alpha_i) = -\alpha_i, \\ \varpi_{\alpha_i} + \varpi_{\overline{\sigma}(\alpha_i)} & \text{if } \sigma(\alpha_i) = -\overline{\sigma}(\alpha_i) \text{ and } \langle \alpha_i^\vee, \sigma(\alpha_i) \rangle = 0, \\ \varpi_{\alpha_i} & \text{otherwise.} \end{cases}$$

We now list the different symmetric spaces (up to finite cover) whose restricted root system is of type A_r, the corresponding dominant weights λ_1, the irreducible G-representations V_{λ_1} (which will feature prominently in the rest of this section), and the corresponding H-representations $\mathfrak{p}$.

G/H	Rank	λ_1	V_{λ_1}	$\mathfrak{p}$
$\mathrm{SL}_{r+1} \times \mathrm{SL}_{r+1} / \mathrm{SL}_{r+1}$	r	$(\varpi_1, 0) + (0, \varpi_r)$	$\mathrm{End}(\mathbb{C}^{r+1})$	$\mathfrak{sl}_{r+1}$
$\mathrm{SL}_{r+1} / \mathrm{SO}_{r+1}$	r	$2\varpi_1$	$S^2(\mathbb{C}^{r+1})$	$S^2(\mathbb{C}^{r+1})_0$
$\mathrm{SL}_{2r+2} / \mathrm{Sp}_{2r+2}$	r	ϖ_2	$\Lambda^2(\mathbb{C}^{2r+2})$	$\Lambda^2(\mathbb{C}^{2r+2})_0$
$\mathrm{SO}_n / \mathrm{S}(\mathrm{O}_1 \times \mathrm{O}_{n-1})$	1	ϖ_1	$\mathbb{C}^n$	$\mathbb{C}^{n-1}$
E_6/F_4	2	ϖ_1	$\mathbb{C}^{27}$	$\mathbb{C}^{26}$

Here $S^2(\mathbb{C}^{r+1})_0$ denotes the SO_{r+1}-stable complement of $\mathbb{C}q$ in $S^2(\mathbb{C}^{r+1})$ with q being the standard quadratic form, and $\Lambda^2(\mathbb{C}^{2r+2})_0$ denotes the Sp_{2r+2}-stable complement of $\mathbb{C}\omega$ in $\Lambda^2(\mathbb{C}^{2r+2})$ with ω being the standard symplectic form. As a consequence of this classification, we see that $V_{\lambda_1} = \mathfrak{p} \oplus \mathbb{C}$ as H-representations.

Note that since $\overline{R}$ is reduced, none of the symmetric spaces we consider is exceptional. Let $\overline{R}^\vee$ be the dual root system, let $(\overline{\alpha}_i^\vee)_{i \in [1,r]}$ be the simple coroots and let $C^\vee$ be the dominant chamber of $\overline{R}^{\vee}$. Then the weights $(\lambda_i)_{i\in[1,r]}$ are the fundamental weights of $\overline{R}$. The cone $C^{\vee}$ is thus generated by the fundamental coweights $(\lambda_i^{\vee})_{i\in[1,r]}$.

Next, using the above list, we obtain a geometric construction of the wonderful compactification:

Proposition 4.39. *Let G/H be an adjoint irreducible symmetric space with restricted root system of type A_r and let λ_1 be as above.*

1. *The group G acts on $\mathbb{P}(V_{\lambda_1})$ with $r+1$ orbits whose closures $(Z_i)_{i\in[1,r+1]}$ satisfy the following inclusions: $Z_1 \subsetneq Z_2 \subsetneq \cdots \subsetneq Z_{r+1} = \mathbb{P}(V_{\lambda_1})$. The open orbit is isomorphic to G/H.*

2. *The join $J(Z_1, Z_i)$ (i.e., the union of lines joining Z_1 and Z_i) equals Z_{i+1} for all i.*

3. *The wonderful compactification X is equipped with a birational G-equivariant morphism $f : X \to \mathbb{P}(V_{\lambda_1})$. If $r = 1$, then f is an isomorphism. If $r \geq 2$, then f is the composition of the blow-ups of the strict transforms of the orbit closures $Z_1, \ldots, Z_r$ in this order. Moreover, these strict transforms are smooth.*

Proof. (1) and (2) In all cases except the last one, the G-orbits are given by the rank of matrices (plain matrices, symmetric matrices or skew-symmetric matrices) and the assertions follows from this. The case of E_6/F_4 is a classical result (see e.g. [LM01, Proposition 4.1]).

(3) This is again a classical result in the first three cases, see [Va84, Theorem 1] for $\mathrm{SL}_{r+1} \times \mathrm{SL}_{r+1} / \mathrm{SL}_{r+1}$ (then X is the moduli space of complete collineations) and [Th99, Theorems 10.1, 11.1] for $\mathrm{SL}_{r+1} / \mathrm{SO}_{r+1}$ and $\mathrm{SL}_{2r+2} / \mathrm{Sp}_{2r+2}$ (complete quadrics and complete skew forms). The next case of $\mathrm{SO}_n / \mathrm{S}(\mathrm{O}_1 \times \mathrm{O}_{n-1})$ is easy, as we then have $X = \mathbb{P}^{n-1} = \mathbb{P}(V_{\lambda_1})$.

For E_6/F_4, we have $Z_1 \subset Z_2 \subset Z_3 = \mathbb{P}(V_{\lambda_1})$, where Z_1 is smooth and Z_2 is a prime divisor. Denote by $\varphi : X' \to \mathbb{P}(V_{\lambda_1})$ the blow-up along Z_1. Then X' is a smooth projective equivariant compactification of G/H, and its boundary is the union of two prime divisors: the exceptional divisor X_1' and the strict transform X_2' of Z_2. Moreover, $X_2' \setminus X_1' = Z_2 \setminus Z_1$ is a unique G-orbit. It suffices to show that $X_1' \setminus X_2'$ and $X_1' \cap X_2'$ are G-orbits as well: then X' is a smooth projective embedding of G/H with a unique closed orbit of codimension 2, and hence is isomorphic to X by the classification of embeddings of G/H. We identify Z_1 to G/P_1, where P_1 is the maximal parabolic subgroup of G associated with the fundamental weight $\lambda_1 = \varpi_1$. Denote by M the normal space to Z_1 in $\mathbb{P}(V_{\lambda_1})$ at the base point of G/P_1. Then M is a representation of P_1, and the G-variety X_1' is isomorphic to the homogeneous projective bundle $G \times^{P_1} \mathbb{P}(M)$. Thus, the G-orbits in $\mathbb{P}(V_{\lambda_1})$ correspond bijectively to the P-orbits in $\mathbb{P}(M)$. So it suffices in turn to show that P_1 acts on $\mathbb{P}(M)$ with two orbits. But the Levi subgroup L_1 of P_1 is isomorphic to $\mathrm{SO}_{10} \times \mathbb{C}^*$ up to finite cover, and $M = \mathbb{C}^{10}$ where SO_{10} acts via its standard representation and $\mathbb{C}^*$ acts by scalar multiplication. Therefore, L_1 acts on $\mathbb{P}(M)$ with two orbits: a quadric and its complement. As P_1 does not act transitively on $\mathbb{P}(M)$, it acts with two orbits as well. $\square$

Remark 4.40. The above statements (1) and (2) can be proved in a uniform way using Jordan algebras: the representation V_{λ_1} has the structure of a Jordan algebra with structure group G and the stabiliser of the unit element is H. The above orbit structure is then explained by the notion of rank for elements in a Jordan algebra. We refer to [Sp98] and [BP22] for more on Jordan algebras. We were however not able to fully relate symmetric spaces with restricted root systems of type A_r to Jordan algebras without using a case by case check, so we refrained from using them.

Theorem 4.41. *Let X be the wonderful compactification of an adjoint irreducible symmetric space with restricted root system of type A_r.*

1. *There is a unique minimal family $\mathcal{K}$.*
2. *The tangent map $\mathcal{K}_x \to \mathcal{C}_x$ is an isomorphism.*
3. *If $r = 1$, then $\mathcal{C}_x \simeq \mathbb{P}(\mathfrak{p})$.*
4. *If $r \geq 2$, then $\mathcal{K}_x$ is isomorphic to the closed G-orbit in $\mathbb{P}(V_{\lambda_1})$.*

Proof. If $r = 1$, then $X = \mathbb{P}(V_{\lambda_1})$ with $V_{\lambda_1} = \mathfrak{p} \oplus \mathbb{C}$ and $x = [\mathbb{C}]$. Thus there is a unique minimal covering family $\mathcal{K}$ and it consists of lines in X. The result follows in this case.

If $r \geq 2$, because of the description of X as a successive blow-up in Proposition 4.39 and arguing as in [BF15, Proposition 5.1] for the group case of type A_r, the result follows from Proposition 9.7 in [FH12]: the family $\mathcal{K}_x$ is the set of lines in $\mathbb{P}(V_{\lambda_1})$ passing through a general point and meeting the closed orbit. The tangent map $\mathcal{K}_x \to \mathcal{C}_x$ is an isomorphism and the VMRT is therefore isomorphic to the closed orbit in X. □

4.6 Minimal families on wonderful compactifications

We summarise our results. Let X be the wonderful compactification of an adjoint irreducible symmetric space G/H with base point x and let $\mathcal{K}$ be a minimal family.

Theorem 4.42. *The tangent map $\mathcal{K}_x \to \mathcal{C}_x$ is an isomorphism.*

Theorem 4.43.

1. *Any irreducible component of $\mathcal{K}_x$ contains a unique highest weight curve C. Moreover, $\mathcal{K}_x$ is equidimensional, of dimension $\langle \overline{\Theta}^\vee, \kappa \rangle + \partial X \cdot C - 1 = \dim H \cdot C + \partial X \cdot C - 1$.*
2. *We have $\partial X \cdot C \in \{1, 2\}$. Moreover, $\partial X \cdot C = 2$ if and only if the restricted root system is of type A_r.*
3. *Assume that $\partial X \cdot C = 1$. Then $\mathcal{K}_x = H \cdot C$. Furthermore, if X is Hermitian non-exceptional, then $\mathcal{K}_x$ has two components. Otherwise, $\mathcal{K}_x$ is irreducible.*
4. *Assume that $\partial X \cdot C = 2$, so that the restricted root system of X is of type A_r.*
   (a) *If $r = 1$, then $\mathcal{K}_x \simeq \mathbb{P}(\mathfrak{p})$.*
   (b) *If $r \geq 2$, then there exists a G-equivariant birational morphism $X \to \mathbb{P}(V)$ for some irreducible G-representation V and $\mathcal{K}_x$ is isomorphic to the closed G-orbit in $\mathbb{P}(V)$. The orbit $H \cdot C$ is a prime divisor in $\mathcal{K}_x$.*

5. *The orbits $H \cdot C$ are described in Table 1.*

Proof. We start with Theorem 4.43.

(1) Follows from Theorem 4.26.

(2) Follows from Remark 4.27.

(3) Follows from Theorem 4.36.

(4) Follows from (2) and Theorem 4.41.

Next, we prove Theorem 4.42. This result follows from (4) for $\overline{R}$ of type A_r. Assume that $\overline{R}$ is not of type A_r. By Proposition 3.6, the tangent map $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ is H-equivariant, finite and birational. Furthermore by (3), the variety $\mathcal{K}_x$ is H-homogeneous. Thus, $\mathcal{C}_x$ is H-homogeneous as well, and τ_x is bijective. □

5 Minimal families on complete symmetric varieties

We are now in position to prove our main results. Let X be a complete symmetric variety and let $\mathcal{K}$ be a family of minimal rational curves on X. Let $\pi : X \to X_{\mathrm{ad}}$ be the map from X to the wonderful compactification of the associated adjoint symmetric space.

Theorem 5.1. *The tangent map $\mathcal{K}_x \to \mathcal{C}_x$ is an isomorphism and $\mathcal{K}_x$ is smoth.*

Theorem 5.2. *Let $C \in \mathcal{K}_x$.*

1. *If C is contracted by π, then $\mathcal{K}_x$ is isomorphic to a linear subspace of $\mathbb{P}(\mathfrak{p} \cap \mathfrak{z})$.*

Assume that π does not contract C.

2. *The map π induces an isomorphism between C and its image $D := \pi(C)$ and there exists a unique indecomposable factor X_C of X_{ad} such that the composition map $\pi_C : X \to X_{\mathrm{ad}} \to X_C$ does not contract C.*

3. *We have $\partial X_C \cdot C \in \{1, 2\}$.*

4. *If $\partial X \cdot C = 1$, then $\mathcal{K}_x = H \cdot C$. Moroever, the components of $\mathcal{K}_x$ are isomorphic to the components of $H \cdot D$.*

5. *If $\partial X \cdot C = 2$, then the restricted root system of X_C is of type A_r.*

   (a) *If $r = 1$, then $\mathcal{K}_x \simeq \mathbb{P}(\mathfrak{p}_C)$.*

   (b) *If $r \geq 2$, then there is a G-equivariant birational morphism $X_C \to \mathbb{P}(V)$ for some irreducible G-representation V and $\mathcal{K}_x$ is isomorphic to the closed G-orbit in $\mathbb{P}(V)$. The orbit $H \cdot C$ is a prime divisor in $\mathcal{K}_x$.*

6. *The orbits $H \cdot D$ are described in Table 1.*

Proof. We start with the proof of Theorem 5.2.

(1) Follows from Lemma 3.4.

(2) Follows from Proposition 3.6.

(3) Follows from Proposition 3.6 and Theorem 4.43.

(4) If $\partial X \cdot C = 1$ and X_C is not of type A_r, then $\partial X \cdot C = \partial X_C \cdot D$. Moreover, there is a unique minimal family of rational curves $\mathcal{L}$ containing $D = \pi(C)$ and a finite H-equivariant map $\pi_{*,x} : \mathcal{K}_x \to \mathcal{L}_{\pi_C(x)}$ (Proposition 3.6 again). By Theorem 4.43, we have $\mathcal{L}_{\pi_C(x)} = H \cdot D$; in particular, every component of $\mathcal{L}_{\pi_C(x)}$ is homogeneous under H^0. Using Proposition 3.6 once more, it follows that $\pi_{*,x}$ induces an isomorphism on components.

If X_C is of type A_r, then $\partial X_C \cdot D = 2$. Thus, the image of $\mathcal{K}_x$ in $\mathcal{L}_{\pi_C(x)}$ has codimension 1 and must be equal to $H \cdot D$. The result follows from this by a similar argument as in the previous case.

(5) If $\partial X \cdot C = 2$, then the restricted root system of X_C is of type A_r and $\partial X \cdot C = \partial X_C \cdot D$. Again, there is a unique minimal family of rational curves $\mathcal{L}$ containing D, and a finite H-equivariant map $\pi_{*,x} : \mathcal{K}_x \to \mathcal{L}_{\pi_C(x)}$. Moreover, $\mathcal{L}_{\pi_C(x)}$ is irreducible and has the same dimension as $\mathcal{K}_x$; thus, $\pi_{*,x}$ is surjective. As X_C is not Hermitian, $\mathcal{K}_x$ is irreducible as well. So $\pi_{*,x}$ is an isomorphism.

Proof of Theorem 5.1. By Proposition 3.6, $\mathcal{K}_x$ is smooth and the tangent map $\mathcal{K}_x \to \mathcal{C}_x$ is finite, birational and H-equivariant, therefore an isomorphism if $\mathcal{K}_x = H \cdot C$. If $\mathcal{K}_x$ is not H-homogeneous, then X_C is of type A_r and the result follows from (5). ☐

6 Appendix

The goal of this appendix is twofold: we first prove basic results on restricted root systems used to describe curves and divisors on wonderful compactifications. We also obtain characterisations of exceptional wonderful varieties useful to establish Table 1. We then give an easy way to describe, using the Kac diagram of the symmetric space, the components of the H-orbit $H \cdot C$ in $\mathcal{K}_x$, where C is a highest weight curve. Finally, in Table 1, we give a list, based on the classification of symmetric spaces, of minimal families and VMRT of wonderful symmetric varieties.

6.1 Restricted root systems

In this subsection we prove useful results on restricted root systems that might be well known to experts, but for which we could not find a good reference.

Lemma 6.1. *The restricted root $\overline{\Theta}$ is the highest root of $\overline{R}$.*

Proof. For $\alpha \in R$, write $\alpha = \sum_{\beta\in\Delta} c_\beta \beta$ with all the c_β of the same sign. We have $\overline{\alpha} = \alpha - \sigma(\alpha) = \sum_{\beta\in\Delta_1} c_\beta(\beta - \sigma(\beta)) = \sum_{\beta\in\Delta_1} c_\beta \overline{\beta}$ and the result follows from this. ☐

Lemma 6.2. *Let $w_0 \in W$ be the longest element, then the actions of σ and w_0 on roots commute. In particular, $\sigma(w_0) = w_0$.*

Proof. Note that $-w_0$ is an involution and preserves R^+ and thus Δ. Furthermore, if $-w_0 \neq \mathrm{Id}$, then $\overline{\sigma}$ is either equal to Id or to $-w_0$ since there is at most one non-trivial such involution. Therefore, in any case, $\overline{\sigma}$ and $-w_0$ commute; thus, $\overline{\sigma}$ and w_0 commute. In particular $w_0(\Delta_0) = -\Delta_0$ and $w_0(L) = L$.

Recall that $\sigma = -w_L\overline{\sigma}$ where w_L is the longest element in the Weyl group W_L of the pair (L, T_{s}), for this see [Ti11, Page 149]. Therefore $w_0\sigma = -w_0 w_L \overline{\sigma} = -w_{w_0(L)} w_0 \overline{\sigma} = -w_L \overline{\sigma} w_0 = \sigma w_0$. The result follows. ☐

Corollary 6.3. *We have $w_0(\overline{\Theta}) = -\overline{\Theta}$.*

Proof. We have $w_0(\overline{\Theta}) = w_0(\Theta) - w_0(\sigma(\Theta)) = w_0(\Theta) - \sigma(w_0(\Theta)) = -\Theta + \sigma(\Theta) = -\overline{\Theta}$. $\square$

We now prove a characterisation of non-reduced restricted root systems.

Proposition 6.4. *Let $\alpha \in \Delta_1$. We have the equivalence: $\overline{\alpha}, 2\overline{\alpha} \in \overline{R} \Leftrightarrow \langle \alpha^\vee, \sigma(\alpha)\rangle = 1$.*

Proof. (1) If $\langle \alpha^\vee, \sigma(\alpha)\rangle = 1$, then $\beta = \alpha - \sigma(\alpha) = \overline{\alpha}$ is a root of R and $\sigma(\beta) = -\beta$. Thus, $\overline{\beta} = 2\beta = 2\overline{\alpha} \in \overline{R}$.

Conversely, assume that $\overline{\alpha}, 2\overline{\alpha} \in \overline{R}$. As above, there are three possibilities: $\sigma(\alpha) = -\alpha$, $\langle \alpha^\vee, \sigma(\alpha)\rangle = 0$ or $\langle \alpha^\vee, \sigma(\alpha)\rangle = 1$. We rule out the first two cases.

If $\sigma(\alpha) = -\alpha$, then $\overline{\alpha} = 2\alpha$ and $2\overline{\alpha} = 4\alpha \in \overline{R}$. In particular, we should have a root $\gamma \in R$ with $\gamma - \sigma(\gamma) = \overline{\gamma} = 2\overline{\alpha} = 4\alpha$. This implies that $\gamma = \lambda\alpha + \sum_{\beta\in\Delta_0} c_\beta\beta$ with $\lambda > 0$ and $c_\beta \geq 0$. But $\sigma(\gamma) = -\lambda\alpha + \sum_{\beta\in\Delta_0} c_\beta\beta$ and since γ is a root, we have $c_\beta = 0$ for all β and $\gamma = 2\alpha \notin R$, a contradiction.

Assume that $\langle \alpha^\vee, \sigma(\alpha)\rangle = 0$ and write $\sigma(\alpha) = -\overline{\sigma}(\alpha) - \sum_{\beta\in\Delta_0} c_\beta\beta$. Then we have

$$0 = \langle \alpha^\vee, \sigma(\alpha)\rangle = -\langle \alpha^\vee, \overline{\sigma}(\alpha)\rangle - \sum_{\beta\in\Delta_0} c_\beta\langle \alpha^\vee, \beta\rangle.$$

If $\overline{\sigma}(\alpha) \neq \alpha$, then all terms of the RHS of the above equation are non-negative, thus vanish. In particular, $\langle \alpha^\vee, \beta\rangle = 0$ for all $\beta \in \Delta_0$ with $c_\beta \neq 0$. We get $\langle \overline{\sigma}(\alpha)^\vee, \beta\rangle = 0$ and $\langle \sigma(\alpha)^\vee, \beta\rangle = 0$ for all β with $c_\beta \neq 0$. Denoting by $(-,-)$ a σ-invariant scalar product, we have $(\sigma(\alpha) + \overline{\sigma}(\alpha), \sigma(\alpha) + \overline{\sigma}(\alpha)) = (\sigma(\alpha) + \overline{\sigma}(\alpha), -\sum_\beta c_\beta\beta) = 0$; thus, $\sigma(\alpha) = -\overline{\sigma}(\alpha)$. Let $\gamma \in R$ such that $\overline{\gamma} = 2\overline{\alpha}$. Then $\gamma - \sigma(\gamma) = 2(\alpha + \overline{\sigma}(\alpha))$. Therefore, $\gamma = \lambda\alpha + \mu\overline{\sigma}(\alpha) + \sum_{\beta\in\Delta_0} d_\beta\beta$ with $\lambda, \mu, d_\beta \geq 0$ and $\lambda + \mu > 0$. Then $\sigma(\gamma) = -\lambda\overline{\sigma}(\alpha) - \mu\alpha + \sum_{\beta\in\Delta_0} d_\beta\beta$ is a root; thus, $d_\beta = 0$ for all $\beta \in \Delta_0$ and $\gamma = \lambda\alpha + \mu\overline{\sigma}(\alpha)$ with $\lambda + \mu = 2$. However there is no such root γ, since $\langle \alpha^\vee, \overline{\sigma}(\alpha)\rangle = -\langle \alpha^\vee, \sigma(\alpha)\rangle = 0$, thus, α and $\overline{\sigma}(\alpha)$ are not adjacent in the Dynkin diagram and the support of γ is therefore disconnected.

Finally, assume that $\overline{\sigma}(\alpha) = \alpha$. Then $\sigma(\alpha) = -\alpha - \sum_{\beta\in\Delta_0} c_\beta\beta \neq -\alpha$. Again, let $\gamma \in R$ with $\overline{\gamma} = 2\overline{\alpha}$. Then $\gamma - \sigma(\gamma) = 2(\alpha - \sigma(\alpha))$. We have $(\overline{\gamma}, \overline{\gamma}) = 4(\overline{\alpha}, \overline{\alpha}) = 8(\alpha, \alpha)$. On the other hand, we have

$$(\overline{\gamma}, \overline{\gamma}) = \begin{cases} 4(\gamma, \gamma) & \text{if } \sigma(\gamma) = -\gamma, \\ 2(\gamma, \gamma) & \text{if } \langle \gamma^\vee, \sigma(\gamma)\rangle = 0, \\ (\gamma, \gamma) & \text{if } \langle \gamma^\vee, \sigma(\gamma)\rangle = 1. \end{cases}$$

We get

$$(\gamma, \gamma) = \begin{cases} 2(\alpha, \alpha) & \text{if } \sigma(\gamma) = -\gamma, \\ 4(\alpha, \alpha) & \text{if } \langle \gamma^\vee, \sigma(\gamma)\rangle = 0, \\ 8(\alpha, \alpha) & \text{if } \langle \gamma^\vee, \sigma(\gamma)\rangle = 1. \end{cases}$$

Since R is a reduced root system, only the first case can occur; thus, $\sigma(\gamma) = -\gamma$, the root γ is long and α is short. We thus have $2\gamma = \overline{\gamma} = 2\overline{\alpha}$; thus, $\gamma = \overline{\alpha}$. By Lemma 6.7 below, the root γ is dominant on its support and α is non-orthogonal to γ. Furthermore, γ is long and α is short, so the support of γ generates a non-simply laced root system. Therefore, γ is the highest root of a root system of type B_n, C_n, F_4 or G_2. Since α is short and is the unique simple root which is non-orthogonal to γ, the above root system must be of type C_n with $\alpha = \alpha_1$ and $\sigma(\alpha_1) = -(\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n)$. Note that $\sigma(\alpha_i) = \alpha_i$ for $i > 1$. However, if $\delta = \alpha_1 + \cdots + \alpha_n$, then $\sigma(\delta) = -(\alpha_1 + \cdots + \alpha_{n-1})$ and $\delta + \sigma(\delta) = \alpha_n$. Let $x_\delta \in \mathfrak{g}_\delta$ be non-zero and set $x = [x_\delta, x_{\sigma(\delta)}]$. Then $x \in \mathfrak{g}_{x_{\alpha_n}}$ is non-zero and $\sigma(x) = -x$, which is impossible since T_{s} is of split type. $\square$

Corollary 6.5. *We have the equivalences:*

$$\overline{R} \textit{ is non-reduced } \Leftrightarrow \exists \alpha \in \Delta_1, \langle \alpha^\vee, \sigma(\alpha) \rangle = 1 \Leftrightarrow \exists \alpha \in R, \langle \alpha^\vee, \sigma(\alpha) \rangle = 1.$$

Proof. If $\overline{R}$ is non-reduced then it is of type BC_r and there exists $\alpha \in \Delta_1$ with $\alpha, 2\alpha \in \overline{R}$. The second implication from left to right is clear. If $\alpha \in R$ is such that $\langle \alpha^\vee, \sigma(\alpha) \rangle = 1$, then $\overline{\alpha} = \alpha - \sigma(\alpha) = \gamma$ is a root and $\overline{\gamma} = 2\gamma = 2\overline{\alpha} \in \overline{R}$; thus, $\overline{R}$ is non-reduced. $\square$

Corollary 6.6. *Let $\alpha \in \Delta_1$ such that $\overline{\alpha}, 2\overline{\alpha} \in \overline{R}$.*

1. *The root $\overline{\alpha}$ is the unique root of $\overline{\Delta}$ with $\overline{\alpha}, 2\overline{\alpha} \in \overline{R}$.*

2. *The variety X is exceptional if and only if $\overline{\sigma}(\alpha) \neq \alpha$.*

Proof. (1) Assume that $\alpha \in \Delta_1$ is such that $\overline{\alpha}, 2\overline{\alpha} \in \overline{R}$. Then $\overline{\alpha} \in \overline{\Delta}$ is the unique simple root whose double is a root in the root system BC_r and is therefore unique.

(2) If X is exceptional, then $\overline{\sigma}(\alpha) \neq \alpha$ by definition. If $\overline{\sigma}(\alpha) \neq \alpha$, then since $\langle \alpha^\vee, \sigma(\alpha) \rangle = 1$ by Proposition 6.4, the result follows from [deCS99, Lemma 4.3].

If X is exceptional then for $\alpha \in \Delta_1$ an exceptional root, we have $\overline{\sigma}(\alpha) \neq \alpha$ by definition and $\overline{\alpha}, 2\overline{\alpha} \in \overline{R}$ by Remark 4.1. If $\alpha \in \Delta_1$ is such that $\overline{\alpha}, 2\overline{\alpha} \in \overline{R}$, then $\langle \alpha^\vee, \sigma(\alpha) \rangle = 1$. If furthermore $\overline{\sigma}(\alpha) \neq \alpha$ then α is exceptional by [deCS99, Lemma 4.3] again. $\square$

For $\alpha \in \Delta_1$, write $\overline{\alpha} = \alpha - \sigma(\alpha) = \alpha + \overline{\sigma}(\alpha) + \sum_{\beta \in \Delta_0} c_\beta \beta$ and define the support of $\overline{\alpha}$ by $\mathrm{Supp}(\overline{\alpha}) = \{\alpha, \overline{\sigma}(\alpha), \beta \mid \beta \in \Delta_0 \text{ with } c_\beta > 0\}$.

Lemma 6.7. *Let $\alpha \in \Delta_1$, then $\overline{\alpha}$ is dominant on* $\mathrm{Supp}(\overline{\alpha})$. *More precisely, we have*

$$\langle \alpha^\vee, \overline{\alpha} \rangle > 0, \langle \overline{\sigma}(\alpha)^\vee, \overline{\alpha} \rangle > 0 \textit{ and } \langle \beta^\vee, \overline{\alpha} \rangle = 0 \textit{ for } \beta \in \mathrm{Supp}(\overline{\alpha}) \cap \Delta_0.$$

Proof. For $\beta \in \Delta_0$, we have $\langle \beta^\vee, \overline{\alpha} \rangle = \langle \sigma(\beta)^\vee, \sigma(\overline{\alpha}) \rangle = \langle \beta^\vee, -\overline{\alpha} \rangle$; thus, $\langle \beta^\vee, \overline{\alpha} \rangle = 0$. We have $\langle \alpha^\vee, \overline{\alpha} \rangle = \langle \alpha^\vee, \alpha - \sigma(\alpha) \rangle = 2 - \langle \alpha^\vee, \sigma(\alpha) \rangle$ and since α and $\sigma(\alpha)$ have the same length, we have $\langle \alpha^\vee, \sigma(\alpha) \rangle \leq 1$ proving the result. $\square$

Recall that Θ denotes the highest root of R, and θ the highest short root (if R is simply laced, then $\Theta = \theta$ and all roots are long and short).

Lemma 6.8. *We have the following equivalences*

1. $\sigma(\Theta) = -\Theta \Leftrightarrow$ *there exists a long root α with $\sigma(\alpha) = -\alpha$.*

2. $\sigma(\theta) = -\theta \Leftrightarrow$ *there exists a short root α with $\sigma(\alpha) = -\alpha$.*

Proof. The implications from left to right in (1) and (2) are clear. We prove the converse implications. The proofs in both cases are similar.

We first prove that the converse implications in (1) and (2) are implied by the following claim: if $\sigma(\alpha) = -\alpha$ and there exists $\beta \in \Delta$ with $s_\beta(\alpha) > \alpha$, then there exists $w \in W$ with $\sigma(w) = w$ such that $w(\alpha) > \alpha$ (recall that σ acts on W by conjugaison).

Assume that the above claim is true. We prove the converse implications in (1) and (2) by induction on roots for the natural order on roots: if α is a positive root such that $\sigma(\alpha) = -\alpha$ and α is not maximal, we produce a root $\alpha' > \alpha$ with the same length as α and such that $\sigma(\alpha') = -\alpha'$. Indeed, if α is not maximal, then there exists $\beta \in \Delta$ with $s_\beta(\alpha) > \alpha$. By the above claim, there exists $w \in W$ with $\sigma(w) = w$ and $w(\alpha) > \alpha$. We thus have $\sigma(w(\alpha)) = \sigma(w)(\sigma(\alpha)) = w(-\alpha) = -w(\alpha)$ with $w(\alpha) > \alpha$. Therefore, if the claim is true, we get by induction that $\sigma(\alpha') = -\alpha'$ for α' the highest short root with the same length as α, proving the implication from right to left of (1) and (2).

We now prove our claim, so let $\alpha \in R$ such that $\sigma(\alpha) = -\alpha$ and $\beta \in \Delta$ with $s_\beta(\alpha) > \alpha$. In particular $\langle \beta^\vee, \alpha\rangle < 0$. We have four possible cases: $\sigma(\beta) = \beta$, $\sigma(\beta) = -\beta$, $\langle \beta^\vee, \sigma(\beta)\rangle = 0$ or $\langle \beta^\vee, \sigma(\beta)\rangle = 1$.

If $\sigma(\beta) = \beta$, then $\langle \beta^\vee, \alpha\rangle = \langle \sigma(\beta)^\vee, \sigma(\alpha)\rangle = -\langle \beta^\vee, \alpha\rangle$; thus, $\langle \beta^\vee, \alpha\rangle = 0$ a contradiction, so this case does not occur.

If $\sigma(\beta) = -\beta$, then $w = s_\beta$ works since $\sigma(w) = w$.

If $\langle \beta^\vee, \sigma(\beta)\rangle = 0$, then set $w = s_\beta s_{\sigma(\beta)}$. Since s_β and $s_{\sigma(\beta)}$ commute, we have $\sigma(w) = w$. Furthermore, we have $w(\alpha) = \alpha - \langle \beta^\vee, \alpha\rangle\beta - \langle \sigma(\beta)^\vee, \alpha\rangle\sigma(\beta) = \alpha - \langle \beta^\vee, \alpha\rangle\beta + \langle \beta^\vee, \alpha\rangle\sigma(\beta)$. But since $\sigma(\beta) \neq \beta$, we have $\sigma(\beta) < 0$ and $w(\alpha) > \alpha$.

Finally, if $\langle \beta^\vee, \sigma(\beta)\rangle = 1$, define $\delta = s_{\sigma(\beta)}(\beta) = \beta - \sigma(\beta) = \overline{\beta}$; then δ is a root. Let $w = s_\delta$. We have $\sigma(\delta) = -\delta$; thus, $\sigma(w) = w$. Furthermore, we have $w(\alpha) = \alpha - \langle \delta^\vee, \alpha\rangle\delta = \alpha - \langle s_{\sigma(\beta)}(\beta)^\vee, \alpha\rangle(\beta - \sigma(\beta))$ and $\langle s_{\sigma(\beta)}(\beta)^\vee, \alpha\rangle = \langle \beta^\vee, s_{\sigma(\beta)}(\alpha)\rangle = \langle \beta^\vee, \alpha - \langle \sigma(\beta)^\vee, \alpha\rangle\sigma(\beta)\rangle = \langle \beta^\vee, \alpha\rangle - \langle \sigma(\beta)^\vee, \alpha\rangle = 2\langle \beta^\vee, \alpha\rangle$. Thus, $w(\alpha) = \alpha - 2\langle \beta^\vee, \alpha\rangle(\beta - \sigma(\beta))$ and since $\sigma(\beta) < 0$, we get $w(\alpha) > \alpha$. ☐

Corollary 6.9. *Assume that $\overline{R}$ is non-reduced. We have the equivalences:*

$$X \text{ is exceptional} \Leftrightarrow \sigma(\Theta) = -\Theta \Leftrightarrow R \text{ is simply laced.}$$

Furthermore, if X is exceptional and α is an exceptional root, then its coefficient in the expansion of Θ as a linear combination of simple roots is equal to 1.

Proof. Assume that $\overline{R}$ is non-reduced and let $\alpha \in \Delta_1$ such that $\overline{\alpha}, 2\overline{\alpha} \in \overline{R}$. Then $\langle \alpha^\vee, \sigma(\alpha)\rangle = 1$ and $\gamma = \alpha - \sigma(\alpha) = \overline{\alpha}$ is a root such that $\sigma(\gamma) = -\gamma$. Therefore we either have $\sigma(\Theta) = -\Theta$ or $\sigma(\theta) = -\theta$.

If X is exceptional, then $\overline{\sigma}(\alpha) \neq \alpha$ by Corollary 6.6 thereforen $\overline{\sigma}$ is a non-trivial involution of the Dynkin diagram and this implies that R is simply laced. In particular $\sigma(\Theta) = -\Theta$ (since $\Theta = \theta$). If $\alpha \in \Delta_1$ is exceptional, then $\overline{\sigma}(\alpha) \neq \alpha$ and the coefficients of such roots in Θ are always equal to 1.

On the other hand if X is non-exceptional, then $\overline{\sigma}(\alpha) = \alpha$ and $\gamma = \alpha - \sigma(\alpha)$ is dominant on its support and bigger than 2α. If γ is long then it is the highest root of $\mathrm{Supp}(\gamma)$, but this is impossible by the discussion on pages 150–151 in [Ti11]. Therefore, γ is short and R is not simply laced. Assume that $\sigma(\Theta) = -\Theta$ and let $(-,-)$ be a (W, σ)-invariant scalar product on $\mathfrak{X}_{\mathbb{R}}$ such that long roots have length 2. We have $(\overline{\Theta},\overline{\Theta}) = 4(\Theta,\Theta) = 8$ and $\overline{\gamma} = 2\gamma = 2\overline{\alpha}$ is such that $(\overline{\gamma},\overline{\gamma}) = 4(\gamma,\gamma) = 4$. A contradiction since in $\overline{R}$ all roots which are the double of another root have the same length. Therefore, $\sigma(\Theta) \neq -\Theta$. □

Corollary 6.10. *G/H is exceptional if and only if G is simply laced and $\overline{R}$ is non-reduced.*

Recall the definition of $\widehat{\alpha}$ from the end of Subsection 4.1.

Lemma 6.11. *Assume that $\alpha \in \Delta_1$ is an exceptional root, then the coefficient of $\widehat{\alpha}^\vee$ in the expansion of $\overline{\Theta}^\vee$ is terms of simple coroots of $\overline{R}$ is equal to 1.*

Proof. Note that R is simply laced, so α and Θ have the same length. Since α is exceptional, we have $\overline{\alpha}, 2\overline{\alpha} \in \overline{R}$ and $\widehat{\alpha}^\vee = \frac{1}{2}\overline{\alpha}^\vee = \frac{1}{2}(\alpha^\vee - \sigma(\alpha)^\vee)$. On the other hand, since X is exceptional, we have $\sigma(\Theta) = -\Theta$ so that $\overline{\Theta}^\vee = \frac{1}{2}\Theta^\vee$. Since the coefficient of α in the expansion of Θ in terms of simple roots is equal to 1, then same is true for the coefficient of $\widehat{\alpha}^\vee$ in the expansion of $\overline{\Theta}^\vee$ in terms of simple coroots. □

Note that if R is not of type A_1, there always exists a simple root $\alpha_{\mathrm{adj}} \in \Delta$ such that $\langle \Theta^\vee, \alpha_{\mathrm{adj}} \rangle = 1$. Such a simple root α_{adj} is unique if R is not of type A_r. In type A_r with $r \geq 2$, there are two such simple roots: α_1 and α_r with simple roots labeled as in [Bo68].

Proposition 6.12. *Assume that R is not of type A_1 and let $\alpha_{\mathrm{adj}} \in \Delta$ be any simple root such that $\langle \Theta^\vee, \alpha_{\mathrm{adj}} \rangle = 1$.*

1. *We have the equivalences: $\sigma(\Theta) \neq -\Theta \Leftrightarrow \sigma(\alpha_{\mathrm{adj}}) = \alpha_{\mathrm{adj}} \Leftrightarrow \langle \Theta^\vee, \sigma(\Theta) \rangle = 0$.*
2. *If $\overline{R}$ is not of type A_1, there exists a simple root $\overline{\alpha} \in \overline{\Delta}$ such that $\langle \overline{\Theta}^\vee, \overline{\alpha} \rangle = 1$.*
3. *If $\overline{R}$ is not of type A_1, then $2\overline{\Theta}^\vee$ is indivisible as a cocharacter of T_{s}.*
4. *If $\sigma(\Theta) \neq -\Theta$, then $-\sigma(\Theta)$ is the highest root of a connected component of the subsystem $R^\perp$ of R generated by simple roots orthogonal to Θ.*

Proof. (1) Note that since Θ is dominant and since $\sigma(\Theta) < 0$ (otherwise $P = G$ and σ is trivial), we have $\langle \Theta^\vee, \sigma(\Theta) \rangle \leq 0$ and therefore either $\sigma(\Theta) = -\Theta$ or $\langle \Theta^\vee, \sigma(\Theta) \rangle = 0$. We therefore only need to prove the first equivalence. If $\sigma(\Theta) = -\Theta$, then $\langle \Theta^\vee, \sigma(\alpha_{\mathrm{adj}}) \rangle = \langle \sigma(\Theta^\vee), \alpha_{\mathrm{adj}} \rangle = -\langle \Theta^\vee, \alpha_{\mathrm{adj}} \rangle = -1$ therefore $\sigma(\alpha_{\mathrm{adj}}) < 0$. Conversely, if $\langle \Theta^\vee, \sigma(\Theta) \rangle = 0$, then α_{adj} does not occur in the support of $\sigma(\Theta)$. Since $\sigma(\Theta)$ is a negative root we have $\langle \alpha_{\mathrm{adj}}^\vee, \sigma(\Theta) \rangle \geq 0$ and thus $\langle \sigma(\Theta)^\vee, \alpha_{\mathrm{adj}} \rangle \geq 0$. We get $\langle \Theta^\vee, \sigma(\alpha_{\mathrm{adj}}) \rangle = \langle \sigma(\Theta)^\vee, \alpha_{\mathrm{adj}} \rangle \geq 0$; thus, $\sigma(\alpha_{\mathrm{adj}}) > 0$ and $\sigma(\alpha_{\mathrm{adj}}) = \alpha_{\mathrm{adj}}$.

(2) If $\overline{R}$ is reduced then the result follows, since $\overline{\Theta}$ is the highest root of $\overline{R}$: take $\overline{\alpha} = \overline{\alpha}_{\mathrm{adj}} \in \overline{\Delta}$ a simple root such that $\langle \overline{\Theta}^\vee, \overline{\alpha}_{\mathrm{adj}} \rangle = 1$.

If $\overline{R}$ is non-reduced, then $\overline{\Theta}$ is the highest root; therefore, there exists a root $\overline{\beta}$ such that $\overline{\Theta} = 2\overline{\beta}$. We have $\langle \overline{\Theta}^\vee, \overline{\beta} \rangle = \frac{1}{2}\langle \overline{\Theta}^\vee, \overline{\Theta} \rangle = 1$. Since $\overline{\Theta}^\vee$ is dominant, this implies the result.

(3) If $\sigma(\Theta) = -\Theta$, then $2\overline{\Theta}^\vee = \Theta^\vee$ and we have $\langle 2\overline{\Theta}^\vee, \alpha_{\mathrm{adj}} \rangle = 1$. If $\langle \Theta^\vee, \sigma(\Theta) \rangle = 0$, then $\langle 2\overline{\Theta}^\vee, \alpha_{\mathrm{adj}} \rangle = \langle \Theta^\vee - \sigma(\Theta)^\vee, \alpha_{\mathrm{adj}} \rangle = \langle \Theta^\vee, \alpha_{\mathrm{adj}} \rangle = 1$ (we use (4) below for the last equality).

(4) If $\sigma(\Theta) \neq -\Theta$, then $\langle \Theta^\vee, \sigma(\Theta) \rangle = 0$ and $-\sigma(\Theta) \in R^\perp$ (the subsystem generated by simple roots orthogonal to Θ). Let $\alpha \in R^\perp$. If $\sigma(\alpha) = \alpha$, then $\langle -\sigma(\Theta)^\vee, \alpha \rangle = -\langle \Theta^\vee, \alpha \rangle = 0$. If $\alpha \in \Delta_1$, then $\sigma(\alpha) < 0$ and $\langle -\sigma(\Theta)^\vee, \alpha \rangle = -\langle \Theta^\vee, \sigma(\alpha) \rangle \geq 0$; thus, $-\sigma(\Theta)$ is dominant in $R^\perp$ and the result follows, since $-\sigma(\Theta)$ and Θ are long roots. $\square$

6.2 Marked Kac diagrams

Our description of the components of $H \cdot C$ is based on the fact that $H \cdot C \simeq H \cdot [m] \subset \mathbb{P}(\mathfrak{p})$ for $m \in T_x C \setminus \{0\}$ (Lemma 2.11) together with the following result.

Lemma 6.13 (Lemma 26.8 of [Ti11])**.** *The simple roots of H^0 and the lowest weights of $\mathfrak{p}$ with respect to the H^0-representation form a affine simple root system.*

Furthermore, the lowest weights of $\mathfrak{p}$ together with the Dynkin diagram of H^0 can be encoded in the so-called Kac diagram of G/H. We refer to [Ti11, Sections 26.3 and 26.5] for more on these diagrams.

Proposition 6.14. *Let X be the wonderful compactification of an adjoint irreducible symmetric space. The irreducible components of the orbits $H \cdot C$, where C runs over the highest weight curves on X, are exactly the homogeneous spaces H^0/Q_δ, where δ is a white node in the Kac diagram, and Q_δ denotes the parabolic subgroup of H^0 associated to the set of simple roots of H not adjacent to δ.*

Proof. The result follows from the fact that m is a highest weight vector of $\mathfrak{p}$, because this highest weight is conjugate in H^0 to a lowest weight of $\mathfrak{p}$ corresponding to a white node δ. $\square$

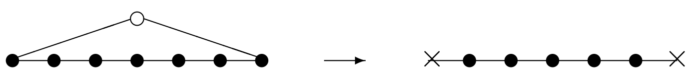
Remark 6.15. We make the following observations.

1. There are two white nodes in the Kac diagram if and only if X is Hermitian.
2. If X is Hermitian, the two corresponding parabolic subgroups are conjugated by an automorphism of G. This automorphism is an outer automorphism if and only if X is exceptional.

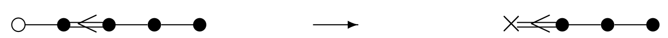
We call a Kac diagram with a marked white node a *Marked Kac Diagram.*

Example 6.16. We illustrate the above proposition by a few examples. We picture the Kac diagram on the left and on the right we picture the Dynkin diagram of H^0 with the simple roots that are not roots of H^0_C crossed.

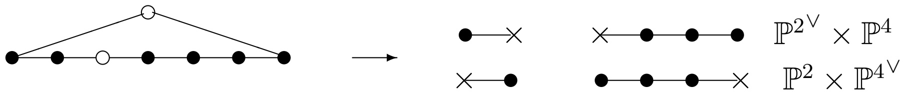
1. $G/H = \mathrm{SL}_8 \times \mathrm{SL}_8 /(Z(G) \cdot \mathrm{SL}_8)$ and $H^0/H^0_C \simeq \mathrm{Flag}(1,6)$ as H^0-varieties.



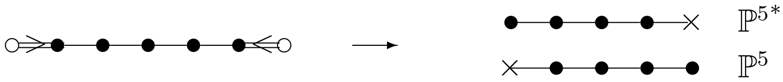
2. $G/H = F_4/B_4$ and $H^0/H^0_C \simeq \mathrm{OG}(4,9)$ as H^0-varieties.



3. $G/H = \mathrm{SL}_8/S(\mathrm{GL}_3 \times \mathrm{GL}_5)$ and $H^0/H^0_C \simeq \mathbb{P}^2 \times \mathbb{P}^4$ as varieties..



4. $G/H = \mathrm{Sp}_{12}/\mathrm{GL}_6$ and $H^0/H^0_C \simeq \mathbb{P}^5$ as varieties.



6.3 Some examples

We describe some families of examples. Recall that $G_{\mathrm{ad}} = G/Z(G)$.

Hermitian types. Assume that G/H is of Hermitian type. The involution σ is given on G_{ad} by conjugation with respect to $\varpi_\alpha^\vee(-1)$, where α is a simple cominuscule root (appearing with coefficient 1 in Θ). In this case, $\sigma(\Theta) = -\Theta$ and any irreducible component of $H \cdot C$ is a smooth irreducible Schubert variety in $\mathbb{P}(\mathcal{O}_{\min})$, of dimension $\frac{1}{2}(\dim \mathcal{O}_{\min} - 1)$. The exceptional cases correspond to the simple cominuscule roots α which are sent to different simple cominuscule roots by a Dynkin diagram involution.

Subadjoint case. Let $\aleph = \{\alpha \in \Delta \mid \langle \Theta^\vee, \alpha\rangle \neq 0\}$. Then $|\aleph| = 1$ except in type A_r with $r \geq 2$, where $|\aleph| = 2$. Let $\varpi^\vee(-1) = \prod_{\alpha\in\aleph} \varpi_\alpha^\vee(-1)$. Define the involution σ on G_{ad} by conjugation by $\varpi^\vee(-1)$. Note that G/G^σ is not Hermitian, except in type A_r. We have $\mathfrak{h} = \mathfrak{g}^\sigma = \mathfrak{g}_\Theta \oplus \mathfrak{k}$ where $\mathfrak{g}_\Theta = \langle e, h, f\rangle$ with $e \in \mathfrak{g}_\Theta \setminus \{0\}$, $f \in \mathfrak{g}_{-\Theta} \setminus \{0\}$ and $h = [e, f]$ (in particular, $\mathfrak{g}_\Theta \simeq \mathfrak{sl}_2$), and $\mathfrak{k}$ is a reductive Lie subalgebra of $\mathfrak{g}$. Let G_Θ and K be the closed connected subgroups of G with Lie algebras $\mathfrak{g}_\Theta$ and $\mathfrak{k}$. Then G_Θ is isomorphic to SL_2 or PSL_2; moreover, $G^{\sigma,0} = G_\Theta K$ and $G_\Theta \cap K$ is finite. Furthermore, we have $\mathfrak{p} = \mathbb{C}^2 \otimes V_K$, where $\mathbb{C}^2$ is the standard representation of $\mathfrak{sl}_2$, and V_K is a K-representation which is irreducible in all cases except in type A_r. In type A_r, we have $V_K = V_K^+ \oplus V_K^-$ which are dual irreducible representations. Let $u^+, u^- \in \mathbb{C}^2$ be a highest and a lowest weight vector for G_Θ and let $v \in V_K$ be a highest weight for K. We identify V_K to the subspace $\langle u^-\rangle \otimes V_K$ of $\mathfrak{p}$.

We have $\mathcal{O}_{\min} = G \cdot e$. We will use the following isomorphism of T-representations: $T_e\mathcal{O}_{\min} = \langle f, h\rangle \oplus V_K \subset \mathfrak{g}$. Note that the symplectic form ω_e on $T_e\mathcal{O}_{\min}$ restricts to orthogonal symplectic forms on $\langle f, h\rangle$ and V_K. Recall the definition of the subadjoint variety $\mathbb{L}_G$ as the set of lines in $\mathbb{P}(\mathcal{O}_{\min})$ passing through $[e]$. In particular, $\mathbb{L}_G \subset \mathbb{P}(T_e\mathcal{O}_{\min})$. We have $\mathbb{L}_G = \mathbb{P}(\mathcal{O}_{\min}) \cap \mathbb{P}(V)$. In particular $\mathbb{L}_G = \emptyset$ in type C_r, since $\mathbb{P}(\mathcal{O}_{\min}) \cap \mathbb{P}(\mathfrak{p}) = \emptyset$ in this case. In the other cases, $\mathbb{L}_G = K \cdot [v]$ is the closed K-orbit in $\mathbb{P}(V_K)$, and spans this projective space. Note that in type A_r, the variety $\mathbb{L}_G$ has two connected components given by the closed K-orbits in $\mathbb{P}(V_K^+)$ and $\mathbb{P}(V_K^-)$. Let $\mathfrak{l}_G = T_{[v]}\mathbb{L}_G$, then $\mathfrak{l}_G$ is a Lagrangian subspace in V_K. We will recover this fact using the VMRT of X_{ad}, the wonderful compactification of the adjoint symmetric space $G/N_G(G^\sigma)$.

Assume that G is not of type C_r. Set $M = \mathrm{VMRT}(X_{\mathrm{ad}})$ and let $\widehat{M}$ be the cone over M in $\mathfrak{p} \subset \mathfrak{g}$. We have $M = \mathbb{P}(\mathcal{O}_{\min}) \cap \mathbb{P}(\mathfrak{p})$: the highest weight $m^+ = u^+ \otimes v$ of $\mathfrak{p}$ lies in $\mathcal{O}_{\min}$ and $M = (G_\Theta \times K) \cdot [m] = \mathbb{P}^1 \times \mathbb{L}_G$. We get that $\mathbb{L}_G = M \cap \mathbb{P}(V_K)$.

Let $m = u^- \otimes v \in V$. Note that $[m] \in \mathbb{L}_G \subset M$. There exists a Weyl group element $s \in W$ such that $[m] = s \cdot [e]$ and $s(V_K) = V_K$. Thus, $T_m\mathcal{O}_{\min} = \langle s(f), s(h) \rangle \oplus V_K$. Furthermore, $\omega_{s(e)}$ restricts to orthogonal symplectic forms on $\langle s(f), s(h) \rangle$ and V_K. We have $T_m\widehat{M} = T_m\mathcal{O}_{\min} \cap \mathfrak{p}$ and this space is Lagrangian for $\omega_{s(e)}$. On the other hand, we have $T_m\widehat{M} = \langle s(f) \rangle \oplus \mathfrak{l}_G$. This implies that $\mathfrak{l}_G$ is a Lagrangian subspace of V_K.

Non-Fano cases. The wonderful compactifications X_{ad} of adjoint irreducible symmetric spaces are not always Fano. The Fano and non-Fano cases have been classified in [Ru12, Theorem 2.1, Table 2]. We summarise the results here: The types for which X_{ad} is not Fano are CI, DI, EI, EV, EVIII, FI and G. An easy way to find them is to use both the restricted root sytem and the Satake diagram (see [Ti11, Table 26.3]): the non-Fano cases are those for which the restricted root system is not of type A nor of type B and the Satake diagram has only white nodes and no arrow.

6.4 Classification table

We list all symmetric spaces G/H (up to finite coverings) with X_{ad} irreducible, their varieties of minimal rational tangents $\mathcal{C}_x$ and the restriction of $\mathcal{O}_{\mathbb{P}(\mathfrak{p})}(1)$ to the VMRT giving the embedding $\mathcal{C}_x \subset \mathbb{P}(\mathfrak{p})$. For $\mathcal{C}_1 \bigsqcup \mathcal{C}_2$, the notation $\mathcal{O}(1)$ corresponds to the embedding in $\mathbb{P}(H^0(\mathcal{C}_1, \mathcal{O}_{\mathcal{C}_1}(1)) \oplus H^0(\mathcal{C}_2, \mathcal{O}_{\mathcal{C}_2}(1)))$. The penultimate column describes the orbit $G \cdot m$ for $m \in T_xC \setminus \{0\}$ with $C \in \mathcal{K}_x$ and $\mathcal{K}$ a minimal family.

Some notations. H.n.e = Hermitian non-exceptional. H.e = Hermitian exceptional. Q_n = smooth quadric of dimension n. $\mathrm{Gr}(a, b)$ = Grassmannian of vector subspaces of dimension a in $\mathbb{C}^b$. $\mathrm{OG}(a, b)$ = closed subset of $\mathrm{Gr}(a, b)$ of isotropic subspaces for a non-degenerate quadratic form on $\mathbb{C}^b$ (with $a < 2b$). $\mathrm{OG}(b, 2b)$ = a connected component of the Grassmannian of maximal isotropic subspaces in $\mathbb{C}^{2b}$ for a non-degenerate quadratic form. $\mathrm{IG}(a, 2b)$ = closed subset of $\mathrm{Gr}(a, b)$ of isotropic subspaces for a non-degenerate symplectic form on $\mathbb{C}^{2b}$. $\mathrm{LG}(b, 2b)$ = Grassmannian of maximal isotropic subspaces in $\mathbb{C}^{2a}$ for a non-degenerate symplectic form. $\mathrm{Flag}(1, r)$ = nested subspaces of dimension 1 and r in $\mathbb{C}^{r+1}$.

References

[Be98] A. Beauville, *Fano contact manifolds and nilpotent orbits*, Comment. Math. Helv. **73** (1998) 566–583.

[BP22] V. Benedetti, N. Perrin, *Cohomology of hyperplane sections of (co)adjoint varieties*, arXiv:2207.02089.

[Bo68] N. Bourbaki, *Éléments de mathématique. Groupes et algèbres de Lie. Chapitre IV, V, VI.* (French). Hermann, Paris 1968.

[Br99] M. Brion, *Rational smoothness and fixed points of torus actions*, Transform. Groups **4** (1999), no. 2-3, 127–156.

[BF15] M. Brion, B. Fu, *Minimal rational curves on wonderful group compactifications*, J. Éc. polytech. Math. **2** (2015), 153–170.

[BK21] M. Brion, S. S. Kannan, *Minimal rational curves on generalized Bott-Samelson varieties*, Compositio Math. **157** (2021), no. 1, 122–153.

[CP11] P.-E. Chaput, N. Perrin, *On the quantum cohomology of adjoint varieties*, Proc. Lond. Math. Soc. (3) **103** (2011), no. 2, 294–330.

[CFH14] Y. Chen, B. Fu, J-M. Hwang, *Minimal rational curves on toric varieties and applications*, Proc. Edinb. Math. Soc. (2) **57** (2014), no. 1, 111–123.

[CFL22] Y. Chen, B. Fu, Q. Li, *Rigidity of projective symmetric manifolds of Picard number 1 associated to composition algebras*, arXiv:2212.02799.

[deCP83] C. de Concini, C. Procesi, *Complete symmetric varieties.* Invariant theory (Montecatini, 1982), 1–44, Lecture Notes in Math., 996, Springer, Berlin, 1983.

[deCS99] C. de Concini, T.A. Springer, *Compactification of symmetric varieties.* Dedicated to the memory of C. Chevalley. Transform. Groups **4** (1999), no. 2-3, 273–300.

[FR08] R. Fowler, G. Röhrle, *Spherical nilpotent orbits in positive characteristic*, Pacific Journal of Math., **237** (2008), no. 2, 241–286.

[FH12] B. Fu, J.-M. Hwang, *Classification of non-degenerate projective varieties with non-zero prolongation and application to target rigidity*, Invent. Math. **189** (2012), no. 2, 457–513.

[FL20] B. Fu, Q. Li, *Rigidity of wonderful group compactifications under Fano deformations*, arXiv:2007.00275.

[Hw01] J.-M. Hwang, *Geometry of minimal rational curves on Fano manifolds*, 335–393, ICTP Lect. Notes **6**, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001.

[HM04] J.-M. Hwang, N. Mok, *Birationality of the tangent map for minimal rational curves*, Asian J. Math. **8** (2004), no. 1, 51–63.

[HM05] J.-M. Hwang, N. Mok, *Prolongations of infinitesimal linear automorphisms of projective varieties and rigidity of rational homogeneous spaces of Picard number 1 under Kähler deformation.* Invent. Math. **160** (2005), no. 3, 591–645.

[IM05] A. Iliev, L. Manivel, *Severi varieties and their varieties of reductions*, J. Reine Angew. Math. **585** (2005), 93–139.

[Ke02] S. Kebekus, *Families of singular rational curves*, J. Algebraic Geom. **11** (2002), no. 2, 245–256.

[KP19] S-Y. Kim, K-D. Park: *On the deformation rigidity of smooth projective symmetric varieties with Picard number 1*, C. R. Math. Acad. Sci. Paris **357** (2019), no. 11–12, 889–896.

[Ko96] J. Kollár, *Rational curves on algebraic varieties*, Ergeb. Math. Grenzgeb. (3) **32**, Springer-Verlag, 1996.

[KR71] B. Kostant, S. Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math. **93** (1971), 753–809.

[LM01] J.M. Landsberg, L. Manivel, *The projective geometry of Freudenthal's magic square*, J. of Algebra **239** (2001), 477–512.

[LM07] J.M. Landsberg, L. Manivel, *Legendrian varieties*, Asian J. of Math. **11** (2007), no. 3, 341–360.

[LP90] P. Littelmann, C. Procesi, *Equivariant cohomology of wonderful compactifications*, in : Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), 219–262, Progr. Math. **92**, Birkhäuser Boston, Boston, MA, 1990.

[Lu73] D. Luna, *Slices étales*, Mémoire Soc. Math. France **33** (1973), 81–105.

[Lu97] D. Luna, *Grosses cellules pour les variétés sphériques.* Algebraic Groups and Lie Groups **9** (1997), 267–280.

[Lu01] D. Luna, *Luna, D. Variétés sphériques de type A.* Publ. Math. Inst. Hautes Études Sci. No. **94** (2001), 161–226.

[MOS14] R. Muñoz, G. Ochetta, L. Solà Conde, *A classification theorem on Fano bundles*, Ann. Inst. Fourier **64** (2014), 341–373.

[Pa99] D. I. Panyushev, *On spherical nilpotent orbits and beyond*, Ann. Inst. Fourier **49** (1999), no. 5, 1453–1476.

[PP10] B. Pasquier, N. Perrin, *Local rigidity of quasi-regular varieties.* Math. Z. **265** (2010), no. 3, 589–600.

[Pe14] N. Perrin, *On the geometry of spherical varieties.* Transform. Groups **19** (2014), no. 1, 171–223.

[Pe18] N. Perrin, *Sanya lectures: geometry of spherical varieties.* Acta Math. Sin. (Engl. Ser.) **34** (2018), no. 3, 371–416.

[Ri82] R. Richardson, *On orbits of algebraic groups and Lie groups*, Bull. Austral. Math. Soc. **25** (1982), 1–28;

[RRS92] R. Richardson, G. Röhrle, R. Steinberg, *Parabolic subgroups with abelian unipotent radical*, Invent. Math. **110** (1992), no. 3, 649–671.

[Ru12] A. Ruzzi, *Fano symmetric varieties with low rank*, Publ. RIMS Kyoto Univ. **48** (2012), 235—278.

[Sp83] T. A. Springer, *Some results on algebraic groups with involutions*, in: Algebraic groups and related topics (Kyoto/Nagoya, 1983), 525–543, Adv. Stud. Pure Math. **6**, North-Holland, Amsterdam, 1985.

[Sp98] T. A. Springer, *Jordan algebras and algebraic groups.* Classics in Mathematics. Springer-Verlag, Berlin, 1998.

[Th99] M. Thaddeus, *Complete collineations revisited*, Math. Ann. **315** (1999), 469–495.

[Ti11] D. A. Timashev, *Homogeneous spaces and equivariant embeddings*, Encyclopaedia Math. Sci. **138**, Springer, Heidelberg, 2011.

[Va84] I. Vainsencher, *Complete collineations and blowing up determinantal ideals*, Math. Ann. **267** (1984), 417–432.

[Vu90] T. Vust, *Plongements d'espaces homogènes algébriques : une classification*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **17** (1990), 165–195.

Type	G/H	Condition	$\overline{R}$	$H \cdot C$	VMRT	$\mathcal{O}_{\mathbb{P}(\mathfrak{p})}(1)$	$\sigma(\Theta) = -\Theta$	Herm/Exc	Fano
Group	$H \times H/H$	$\mathrm{Type}(H) \neq \mathrm{A}_r$	Type of H	$\mathbb{P}(\mathcal{O}_{\min,H})$	$H \cdot C$	$\mathcal{O}(1)$	yes		yes
Group	$\mathrm{SL}_{r+1} \times \mathrm{SL}_{r+1} / \mathrm{SL}_{r+1}$	$r \geq 2$	A_r	$\mathrm{Flag}(1, r)$	$\mathbb{P}^r \times \mathbb{P}^r$	$\mathcal{O}(1,1)$	yes		yes
Group	$\mathrm{SL}_2 \times \mathrm{SL}_2 / \mathrm{SL}_2$		A_1	$\mathbb{P}^1$	$\mathbb{P}^2$	$\mathcal{O}(1)$	yes		yes
A I	$\mathrm{SL}_{r+1} / \mathrm{SO}_{r+1}$	$r \geq 2$	A_r	Q_{r-1}	$\mathbb{P}^r$	$\mathcal{O}(2)$	yes		yes
A I	$\mathrm{SL}_2 / \mathrm{SO}_2$		A_1	$\{\mathrm{pt}\} \bigsqcup \{\mathrm{pt}\}$	$\mathbb{P}^1$	$\mathcal{O}(1)$	yes	H.n.e	yes
A II	$\mathrm{SL}_{2r+2} / \mathrm{Sp}_{2r+2}$	$r \geq 2$	A_r	$\mathrm{IG}(2, 2r+2)$	$\mathrm{Gr}(2, 2r+2)$	$\mathcal{O}(1)$	no		yes
A III	$\mathrm{SL}_n / S(\mathrm{GL}_r \times \mathrm{GL}_{n-r})$	$1 \leq r < n/2$	BC_r	$\mathbb{P}^{r-1} \times \mathbb{P}^{n-r-1}$	$H \cdot C$	$\mathcal{O}(1,1)$	yes	H.e	yes
A III	$\mathrm{SL}_{2r} / S(\mathrm{GL}_r \times \mathrm{GL}_r)$		C_r	$(\mathbb{P}^{r-1})^2 \bigsqcup (\mathbb{P}^{r-1})^2$	$H \cdot C$	$\mathcal{O}(1,1)$	yes	H.n.e	yes
BD I	$\mathrm{SO}_n / S(\mathrm{O}_r \times \mathrm{O}_{n-r})$	$3 \leq r \leq \frac{n-1}{2}$	B_r	$Q_{r-2} \times Q_{n-r-2}$	$H \cdot C$	$\mathcal{O}(1,1)$	yes		yes
BD I	$\mathrm{SO}_n / S(\mathrm{O}_2 \times \mathrm{O}_{n-2})$		B_2	$Q_{n-4} \bigsqcup Q_{n-4}$	$H \cdot C$	$\mathcal{O}(1)$	yes	H.n.e	yes
BD II	$\mathrm{SO}_n / S(\mathrm{O}_1 \times \mathrm{O}_{n-1})$		A_1	Q_{n-3}	$\mathbb{P}^{n-2}$	$\mathcal{O}(1)$	no		yes
C I	$\mathrm{Sp}_{2r}/\mathrm{GL}_r$	$r \geq 3$	C_r	$\mathbb{P}^{r-1} \bigsqcup \mathbb{P}^{r-1}$	$H \cdot C$	$\mathcal{O}(2)$	yes	H.n.e	no
C II	$\mathrm{Sp}_{2n}/\mathrm{Sp}_{2r} \times \mathrm{Sp}_{2n-2r}$	$1 \leq r \leq \frac{(n-1)}{2}$	BC_r	$\mathbb{P}^{2r-1} \times \mathbb{P}^{2n-2r-1}$	$H \cdot C$	$\mathcal{O}(1,1)$	no		yes
C II	$\mathrm{Sp}_{4r}/\mathrm{Sp}_{2r} \times \mathrm{Sp}_{2r}$	$r \geq 2$	C_r	$\mathbb{P}^{2r-1} \times \mathbb{P}^{2r-1}$	$H \cdot C$	$\mathcal{O}(1,1)$	no		yes
D I	$\mathrm{SO}_{2r} / S(\mathrm{O}_r \times \mathrm{O}_r)$	$r \geq 4$	D_r	$Q_{r-2} \times Q_{r-2}$	$H \cdot C$	$\mathcal{O}(1,1)$	yes		no
D III	$\mathrm{SO}_{4r} / \mathrm{GL}_{2r}$		C_r	$\mathrm{Gr}(2, 2r) \bigsqcup \mathrm{Gr}(2, 2r)$	$H \cdot C$	$\mathcal{O}(1)$	yes	H.n.e	yes
D III	$\mathrm{SO}_{4r+2} / \mathrm{GL}_{2r+1}$		BC_r	$\mathrm{Gr}(2, 2r+1)$	$H \cdot C$	$\mathcal{O}(1)$	yes	H.e.	yes
E I	E_6/C_4		E_6	$\mathrm{LG}(4, 8)$	$H \cdot C$	$\mathcal{O}(1)$	yes		no
E II	$E_6/A_5 \times A_1$		F_4	$\mathrm{Gr}(3, 6) \times \mathbb{P}^1$	$H \cdot C$	$\mathcal{O}(1,1)$	yes		yes
E III	$E_6/D_5 \times \mathbb{C}^*$		BC_2	$\mathrm{OG}(5, 10)$	$H \cdot C$	$\mathcal{O}(1)$	yes	H.e.	yes
E IV	E_6/F_4		A_2	F_4/P_4	E_6/P_6	$\mathcal{O}(1)$	no		yes
E V	E_7/A_7		E_7	$\mathrm{Gr}(4, 8)$	$H \cdot C$	$\mathcal{O}(1)$	yes		no
E VI	$E_7/D_6 \times A_1$		F_4	$\mathrm{OG}(6, 12) \times \mathbb{P}^1$	$H \cdot C$	$\mathcal{O}(1,1)$	yes		yes
E VII	$E_7/E_6 \times \mathbb{C}^*$		C_3	$E_6/P_1 \bigsqcup E_6/P_6$	$H \cdot C$	$\mathcal{O}(1)$	yes	H.n.e.	yes
E VIII	E_8/D_8		E_8	$\mathrm{OG}(8, 16)$	$H \cdot C$	$\mathcal{O}(1)$	yes		no
E IX	$E_8/E_7 \times A_1$		F_4	$E_7/P_7 \times \mathbb{P}^1$	$H \cdot C$	$\mathcal{O}(1,1)$	yes		yes
F I	$F_4/C_3 \times A_1$		F_4	$\mathrm{LG}(3, 6) \times \mathbb{P}^1$	$H \cdot C$	$\mathcal{O}(1,1)$	yes		no
F II	F_4/B_4		BC_1	$\mathrm{OG}(4, 9)$	$H \cdot C$	$\mathcal{O}(1)$	no		yes
G	$G_2/A_1 \times A_1$		G_2	$\mathbb{P}^1 \times \mathbb{P}^1$	$H \cdot C$	$\mathcal{O}(1,3)$	yes		no

Table 1: Wonderful compactifications and their VMRT.