# Probabilistic approaches for NLS in $\mathbb{R}^d$

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MIT PDE/Analysis seminar

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... with random data

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$(g_n)_n$: independent normalized Gaussian variables, with complex values.

$$g_n(\omega) = \frac{1}{\sqrt{2}}(g_{n,1} + i g_{n,2})\,, \quad g_{n,i} \sim \mathcal{N}_{\mathbb{R}}(0,1)\,.$$

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- At the linear level:

$$e^{it\Delta} \, u_0^\omega = \sum_n e^{it\lambda_n^2} g_n(\omega) c_n \, e^{inx} \quad \rightsquigarrow \quad \mathcal{L}(e^{it\Delta} \, u_0^\omega) = \mathcal{L}(u_0^\omega)\,.$$

$\rightsquigarrow$ evolution of probability measures *v.s.* individual trajectories

* To understand the transport of Gaussian measures by nonlinear flows
* To go beyond deterministic obstructions
* Study of wave turbulence

* Hamiltonian system

$$H(u) = \frac{1}{2} \sum_{n \in \mathbb{Z}^2} (1 + |n|^2) |\widehat{u}_n|^2 + \frac{\lambda}{4} \int_{\mathbb{T}^2} \Big| \sum_{n \in \mathbb{Z}^2} \widehat{u}_n e^{inx} \Big|^4 dx \,.$$

* Gaussian free field

$$d\mu(u) = Z^{-1} \exp\Big(-\sum_{n \in \mathbb{Z}^2} (1+|n|^2) |\widehat{u}_n|^2\Big) du = Z^{-1} \prod_{n \in \mathbb{Z}^2} \exp(-(1+|n|^2)|\widehat{u}_n|^2) d\widehat{u}_n$$

* It is induced by

$$\omega \in \Omega \mapsto \phi^\omega(x) = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle} e^{in \cdot x} \in L^2_\omega(\Omega; H^{0-}(\mathbb{T}^2)) \,, \quad H^{0-} = \Big(\bigcap_{\epsilon > 0} H^{-\epsilon}\Big) \setminus L^2 \,.$$

* Gibbs measure $\rho$

$$d\rho(u) = Z^{-1} \exp(-H(u)) du = Z^{-1} \exp\Big(-\lambda \int_{\mathbb{T}^2} |u|^4 dx\Big) d\mu \,.$$

Bourgain ('96): Existence of sol. and invariance of the Gibbs meas. for NLS on $\mathbb{T}^2$.

Let $s$ be small enough such that instabilities are known to occur for *some* initial data in $H^s$.

∗ Burq, Tzvetkov ('08): General framework for random data in singular regimes.

| | | |
|---|---|---|
| *Dispersion* | $\rightsquigarrow$ *time oscillations* | $\rightsquigarrow$ *dispersive estimates* |
| *Random initial data* | $\rightsquigarrow$ *probabilistic oscillations* | $\rightsquigarrow$ *improved estimates* |

$\rightsquigarrow$ Many developments in a variety of contexts.

I. Cauchy problem in supercritical regimes

*i. Pathological set of initial data*

*ii. Probabilistic semi-linear Cauchy theory in* $\mathbb{R}^d$

II. Probabilistic non-perturbative schemes adapted to weakly-dispersive equations

# Cauchy problem in supercritical regimes

* Perturbative viewpoint $\rightsquigarrow$ the Duhamel formulation:

$$(*) \qquad u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-\tau)\Delta} |u(\tau)|^2 u(\tau) d\tau \,.$$

* Given $B \subset H^s(\mathcal{M})$ bounded perform a contraction mapping-argument, for some $0 < T \ll 1$ in a suitable functional space

$$u \in X_T \subset L^\infty([-T, T]; H^s) \,.$$

* When such a strategy succeeds, the *flow-map* $\Phi_t$ is uniformly continuous on $B$.

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**Def: Well-posedness in $H^s$, $s \in \mathbb{R}$.**

The Cauchy pbm is well-posed in $H^s(\mathcal{M})$ if for all $B \subset H^s(\mathcal{M})$, there is $T > 0$ and $X_T$ such that $(*)$ has a unique solution in $X_T$. Moreover,

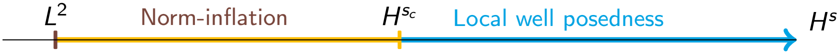
* $\Phi : u_0 \mapsto u \in \mathcal{C}([-T, T], H^s)$ is uniformly continuous.
* higher regularity is propagated.

# Scaling-supercritical regimes

$$\begin{cases} i\partial_t u + \Delta u = \mu |u|^2 u\,, & \mathcal{M} = \mathbb{T}^d \text{ or } \mathbb{R}^d\,. \\ u(0,\cdot) = u_0 \in H^s(\mathcal{M})\,, \end{cases}$$

* Scaling invariance: $s_c = \frac{d}{2} - 1$

$$u_\lambda(t,x) = \lambda^{-1} u(\lambda^{-2} t, \lambda^{-1} x) \rightsquigarrow \|u_{0,\lambda}\|_{\dot{H}^s} = \lambda^{s_c - s} \|u_{0,\lambda}\|_{\dot{H}^s}\,.$$




* Norm inflation mechanism in supercritical regimes ($0 < s < s_c$)

**Thm:** (Christ-Colliander-Tao '03) $0 < s < s_c \rightsquigarrow$ norm inflation around any $u_0 = 0$

*There exist* $(u_{n,0})$ *in* $\mathcal{C}_c^\infty(\mathcal{M})^{\mathbb{N}}$ *and times* $t_n \to 0$ *s.t.*

$$\lim_{n\to\infty} \|u_{n,0}\|_{H^s} = 0\,, \qquad \lim_{n\to\infty} \|\Phi_{t_n}(u_{n,0})\|_{H^s} = +\infty\,.$$

## Scaling-supercritical regimes

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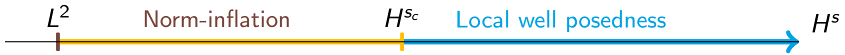


* Norm inflation mechanism in supercritical regimes ($0 < s < s_c$)

**Thm: (Xia '21) $0 < s < s_c \rightsquigarrow$ norm inflation around any $u_0$**

*For all $u_0 \in H^s$, there exist $(u_{n,0})$ in $C_c^\infty(\mathcal{M})^{\mathbb{N}}$ and times $t_n \to 0$ s.t.*

$$\lim_{n\to\infty} \|u_{n,0} - u_0\|_{H^s} = 0\,, \quad \lim_{n\to\infty} \|\Phi_{t_n}(u_{n,0})\|_{H^s} = +\infty\,.$$

## Scaling-supercritical regimes

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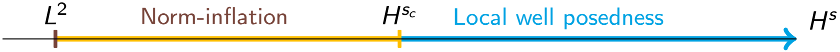


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* The flow-map is nowhere continuous in $H^s$ due to bad approximations $(u_n)$.
* What about $u_n = \rho_{\epsilon_n} * u_0$?

**Can we describe the class of $u_0$ s.t, for proper regularization, $u_n$ converges?**

approximate identity: Set $\rho \in \mathcal{C}_c^\infty$, and $\rho_\epsilon := \epsilon^{-d}\rho(\epsilon^{-1}) \rightsquigarrow u_{0,\epsilon} = \rho_\epsilon * u_0$.

Thm: Probabilistic well-posedness (à la Burq-Tzvetkov '08), for $s_0 < s \leq s_c$

*There exists a full measure set* $\Sigma \subset H^s(\mathcal{M})$, *s.t.* $\forall u_0 \in \Sigma\,, \exists T > 0$

$$\exists\ u \in \mathcal{C}([0,T]; H^s) \quad \lim_{\epsilon \to 0} \|\Phi_t(u_{0,\epsilon}) - u(t)\|_{L^\infty([0,T],H^s)} = 0\,.$$

*Moreover, the limit u is solution to NLS.*

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- Do we have $\Sigma = H^s$?
- What happens outside the *statistical set* $\Sigma$?

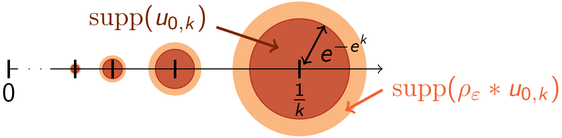
Pathological set of data where *strong norm inflation* occurs

$$\mathcal{P} := \left\{ u_0 \in H^s\,,\ \limsup_{T\,,\,\epsilon\to 0} \|\Phi_t(u_0 * \rho_\epsilon)\|_{L^\infty([0,T],H^s)} = \infty \right\}.$$

**Thm: (Sun-Tzvetkov NLW '19, C.-Gassot '22 NLS) Generic norm-inflation**

The pathological set $\mathcal{P}$ of data contains a dense $G$-delta set.

$\rightsquigarrow$ The *Tanghuru* construction (Lebeau'01)




- For NLW the proof from Sun-Tzvetkov strongly relies on finite propagation speed
- It turns out that the mechanism is deeper (no need of propagation speed nor global existence of smooth solutions).
- In particular, it occurs on general geometries and likely for any Schrödinger-type equation.

Can one find a set $E \subset [0,1]$ such that

1. $\text{mes}(\Sigma) = 1$
2. $[0,1] \setminus \Sigma$ contains a $G_\delta$-dense set

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Let's move on to the **probabilistic Cauchy theory**:

$\rightsquigarrow$ how to construct the statistical set $\Sigma$ of initial data that lead to strong solutions below $H^{s_c}$ ?

$$u_0^\omega = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{\langle n\rangle} e^{in\cdot x} \in H^{0-}(\mathbb{T}^2).$$

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∗ Nonlinear smoothing:

$$u(t) - \underbrace{\mathrm{e}^{it\Delta} u_0^\omega}_{:=f(t)\in H^{0-}(\mathbb{T}^2)} = -i \underbrace{\int_0^t \mathrm{e}^{i(t-t')\Delta}\left(\left|\mathrm{e}^{it'\Delta} u_0^\omega\right|^2 \mathrm{e}^{it'\Delta} u_0^\omega\right) \mathrm{d}\, t'}_{\text{second Picard's iteration in } H^{\frac{1}{2}-}(\mathbb{T}^2)\text{, counting estimates}} + \dots \in H^{0+}(\mathbb{T}^2)\,.$$

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$\rightsquigarrow$ Consider *semi-linear* decomposition

$$u(t) = f(t) + v(t)\,.$$

$\rightsquigarrow$ $v \in H^{0+}(\mathbb{T}^2)$ solves a perturbed version of NLS, in a subcritical regime:

$$i\partial_t v + \Delta v = |v|^2 v + |f|^2 f + \underbrace{\mathcal{O}(v^2 f + v f^2)}_{\text{need random matrices}}\,, \quad \text{in } H^{0+}(\mathbb{T}^2)\,, \quad v(0) = 0\,.$$

Moreover,

$$i\partial_t + \Delta v_n = |v_n|^2 v_n\,, \quad v_n(0) = \Pi_{\leq n} u_0^\omega \rightsquigarrow v = \lim v_n \in C([0,T], H^{+0}(\mathbb{T}^d))\,.$$

* Dispersion: (*time oscillations*) Dispersive Strichartz estimates: If $\frac{p}{2} + \frac{d}{q} = \frac{d}{2}$,

$$\| e^{it\Delta} u_0 \|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^d)} \lesssim \| u_0 \|_{L_x^2} .$$

* Probabilistic decoupling: (Paley-Zygmund 1930): (*random oscillations*)

$$\mathbb{E}\Big[\Big| \sum_n g_n(\omega) c_n \Big|^\ell\Big]^{\frac{1}{\ell}} \lesssim \sqrt{\ell}\Big( \sum_n |c_n|^2 \Big|\Big)^{\frac{1}{2}}, \quad \ell \geq 2 .$$

$\rightsquigarrow$ *Littlewood-Paley square function:* $P_N$ Fourier proj. onto $\frac{N}{2} < |\xi| \leq 2N$,

$$\|u\|_{L^p(\mathbb{R}^d)} \sim \|( \sum_{N\in 2^{\mathbb{Z}}} |P_N u(x)|^2 )^{\frac{1}{2}} \|_{L^p(\mathbb{R}^d)} .$$

* Bernstein Estimate:

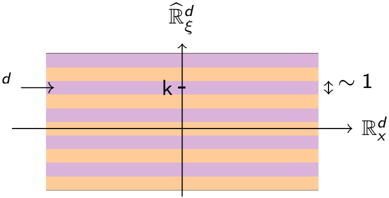
$$\operatorname{supp} \widehat{\varphi} \subset E \implies \|\varphi\|_{L_x^q(\mathbb{R}^d)} \leq C|E|^{\frac{1}{2}-\frac{1}{q}} \|\varphi\|_{L_x^2(\mathbb{R}^d)} .$$

* To randomize, pick an orth. basis of $L^2(\mathbb{R}^d)$ from a tiling of $\widehat{\mathbb{R}^d}$ with cubes of volume $|E| = 1$.

$\rightsquigarrow$ *"A typical function in $L_x^2(\mathbb{R}^d)$ is in $L_x^q(\mathbb{R}^d)$, for every $q > 2$"*

$$u_0 \in H^s(\mathbb{R}^d) \;\rightsquigarrow\; u_0 = \sum_{k\in\mathbb{Z}^d} Q_k u_0 \;\text{(Wiener cubes)}\quad \|Q_k u_0\|_{L^q_x} \lesssim \|Q_k u_0\|_{L^2_x}\,,\; q \geq 2\,.$$

supp $\widehat{Q_k u_0}(\xi) \subset k + [-1\,,1]^d$ $\longrightarrow$




Given $(g_k)$ indep. normalized Gaussian variables on $(\Omega, \mathcal{A}, \mathbb{P})$, set

$$\omega \in \Omega \mapsto u_0^\omega = \sum_{k\in\mathbb{Z}^d} g_k(\omega) Q_k u_0 \quad \in L^2_\omega(\Omega; H^s(\mathbb{R}^d))\,.$$

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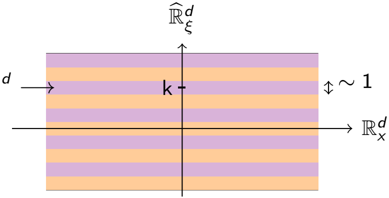


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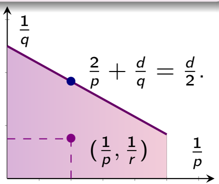
**Probabilistic Strichartz estimates (Benyi Oh Pocovnicu '15)**

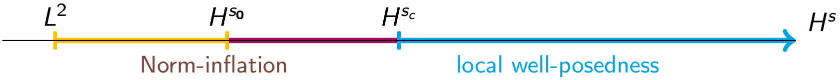
Let $(p,q)$ admissible, and $r \geq q$: $\exists C, c > 0, \forall \lambda > 0$

$$\mathbb{P}\left(\| e^{it\Delta} u_0^\omega \|_{L^p_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \leq \lambda \|u_0\|_{L^2_x(\mathbb{R}^d)}\right) \geq 1 - C e^{-c\lambda^2}\,.$$



# Almost-sure local existence in $\mathbb{R}^d$ (Benyi-Oh-Pocovnicu '15)

Fix $s_0 < s \leq s_c$, $f(t) = e^{it\Delta} u_0^\omega \in H^s(\mathbb{R}^d) \quad \rightsquigarrow \quad u(t) = f(t) + v(t)$




* Ingredients: probabilistic Strichartz & dispersive estimates:
  - *Strichartz estimates*
  - *bilinear estimates*
  - *local smoothing effect*

**Thm: (Benyi-Oh-Pocovnicu '15) Probabilistic local well-posedness when $s > s_0 = \frac{1}{4}$.**

*For almost-every $u_0^\omega$, there exists $T > 0$ and a unique $v \in X_T^{2s-}$ s.t.*

$$u = f + v \in \mathcal{C}([-T, T], H^s) \quad \text{solves NLS with} \quad u(0) = u_0^\omega .$$

* Global well-posedness and dynamics (for NLS and NLW): perturbative energy methods for $v$:

$$i\partial_t v + \Delta v = \mu |v|^2 v + \mu |f|^2 f + \mathcal{O}(v^2 f + v f^2)$$

* Global existence:
  - Colliander, Oh ('12) NLS $\mathbb{T}$
  - Burq, Tzvetkov ('14) NLW $\mathbb{T}^2$
  - Lürhmann Mendelson ('14) NLW $\mathbb{R}^3$
  - Bényi, Oh, Pocovnicu ('15) NLS $\mathbb{R}^d$
  - Nahmod, Staffilani ('15) NLS $\mathbb{T}^3$
* Scattering:
  - Dodson, Lührmann, Mendelson ('17-'19) NLS, NLW $\mathbb{R}^4$
  - Killip, Murphy, Visan ('17) NLS $\mathbb{R}^4$
  - Bringmann ('19) NLW $\mathbb{R}^4$
  - N.C ('21), Shun, Soffer, Wu ('21) NLS $\mathbb{R}^3$
* Asymptotic stab.:
  - Kenig, Mendelson ('19)
  - N.C ('20) NLS with confining short-range potential $\rightsquigarrow$ small ground states.
  - Bringmann, wave maps ('22)
* stab. of blow-up profile:
  - Bringmann ('20) NLW
  - Fan, Mendelson ('20) NLS

# III. Quasilinear probabilistic Cauchy theory.

The case of weakly dispersive equations

Bourgain Does there exist global solutions whose $H^s(\mathbb{T}^d)$-norm $s > 1$ grows:

$$\limsup_{t\to\infty} \|u(t)\|_{H^s(\mathbb{T}^d)} = +\infty\,?$$

* CKSTT'10: NLS on $\mathbb{T}^d$ with $d \geq 2$: $\forall\delta > 0, K > 0$, $\exists u_0 \in H^s$, $T > 0$,

$$\|u_0\|_{H^s} \leq \delta\,, \quad \|u(T)\|_{H^s} \geq K\,.$$

* (Han.-Pau.-Tzv.-Vis.'15) NLS on $\mathbb{R}\times\mathbb{T}^d$, for $d \geq 2$ and $s > 30$, $\epsilon > 0, \exists u_0$

$$\|u_0\|_{H^s(\mathbb{R}\times\mathbb{T}^d)} \leq \epsilon\,, \quad \exists t_k \to \infty \ : \ \|u(t_k)\|_{H^s(\mathbb{R}\times\mathbb{T}^d)} \gtrsim (\log\log(t_k))^C\,.$$

**Other models on $\mathbb{T}$?**

**Bourgain** Does there exist global solutions whose $H^s(\mathbb{T}^d)$-norm $s > 1$ grows:

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**Other models on $\mathbb{T}$?**

(Szegő) $$i\partial_t u = \Pi_+(|u|^2 u) \, .$$

(HW) $$i\partial_t u + |D_x| u = |u|^2 u \, .$$

* (Pocovnicu'11,Gérard-Grellier'12) *Weak turbulence* for Szegő on $\mathbb{R}$ and $\mathbb{T}$

Observation:

- The critical scaling is $s_c = 0$.
- Though, the flow map cannot be extended uniformly continuously below $s = \frac{1}{2}$.

## Can statistical approach overcome the lack of uniform continuity?

* Random data approach in $H^s$ for $s < 1/2$? ($s_c = 0$)

$$u(t) = e^{it|D_x|}\, u_0^\omega + v(t)\,, \quad v(t) \in \mathcal{C}\left([0,T]; H^s_+(\mathbb{R})\right) \text{ for } s \geq \frac{1}{2}\,?$$

* Lack of regularization for the Picard's iteration (see (Oh '11) for Szegő on $\mathbb{T}$)

$$u_0^\omega \notin H^{s+\epsilon} \implies \int_0^t S(t-\tau) P_{\text{hi}} S(\tau) u_0^\omega |P_{\text{low}} S(\tau) u_0^\omega|^2 d\tau \quad \notin H^{s+\epsilon}\,.$$

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∗ Random data approach in $H^s$ for $s < 1/2$? ($s_c = 0$)

$$u(t) = e^{it|D_x|}\, u_0^\omega + v(t)\,, \quad v(t) \in \mathcal{C}\left([0,T]; H^s_+(\mathbb{R})\right) \ \textit{for } s \geq \frac{1}{2}\,?$$

∗ Lack of regularization for the Picard's iteration (see (Oh '11) for Szegő on $\mathbb{T}$)

$$u_0^\omega \notin H^{s+\epsilon} \implies \int_0^t S(t-\tau) P_{\mathrm{hi}} S(\tau) u_0^\omega |P_{\mathrm{low}} S(\tau) u_0^\omega|^2 d\tau \quad \notin H^{s+\epsilon}\,.$$

∗ Random data in "non-perturbative" regimes ?

- (Bringmann '19) developed a quasilinear iteration scheme for derivative NLW
- (NC, Gassot, Ibrahim '22) adapted the scheme to general weakly dispersive equ.

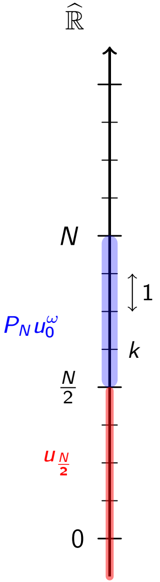
**Thm: (C.-Gassot-Ibrahim'22) Probabilistic local well-posedness, $s \in (s_0, 1/2]$**

For $f \in H^s(\mathbb{R})$, there is $T_0 > 0$ and a seq. $(u_n)_{n \geq 1} \in \mathcal{C}([-T_0, T_0]\,;\, H^\infty(\mathbb{R}))^{\mathbb{N}}$

$$\lim_{n \to \infty} \mathbb{E}\left[ \| u_n - u \|^2_{L^\infty([0, T_0], H^s(\mathbb{R}))} \right] = 0\,,$$

s.t. almost surely in $\omega$, $u_n$ and $u$ solve (HW) on $[-T^\omega, T^\omega]$, with data

$$u_n(0) = P_{\leq n} f_0^\omega\,, \quad u(0) = f_0^\omega\,.$$




* Set $N \in 2^{\mathbb{N}}$, and $u_N \in H^\infty$ s.t.

$$\begin{cases} i\partial_t u_N + \mathcal{L} u_N = \mathcal{N}(u_N)\,, \\ u_N(0) = P_{\leq N} u_0^\omega\,. \end{cases}$$

$$u_N = u_{\frac{N}{2}} + F_N + w_N\,.$$

* Refined ansatz: for some $0 < \gamma < 1$, $P_{\text{low}} := P_{\leq N^\gamma}$

- $F_N$: **paracontrolled** linear evolution: $( \sim f^\omega = e^{it\mathcal{L}} u_0^\omega )$

$$\begin{cases} i\partial_t F_N + \mathcal{L} F_N = \mathcal{N}(F_N, P_{\text{low}} u_{\frac{N}{2}}, P_{\text{low}} u_{\frac{N}{2}})\,. \\ F_N(0) = P_N u_0^\omega\,. \end{cases}$$

- Obs: proba. indep. ( $P_N u_0^\omega \perp u_{\frac{N}{2}}$ ) $\rightsquigarrow$ good control in $L_x^\infty(\mathbb{R})$
- $w_n$: solution in $H^{\frac{1}{2}+}(\mathbb{R})$ to

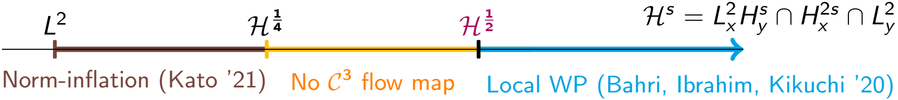
$$\begin{cases} i\partial_t w_N + \mathcal{L} w_N = \mathcal{N}(u_N) - \mathcal{N}(u_{\frac{N}{2}}) - \mathcal{N}(F_N, P_{\text{low}} u_{\frac{N}{2}}, P_{\text{low}} u_{\frac{N}{2}})\,. \\ w_N(0) = 0\,. \end{cases}$$

## Another model to evidence growth of Sobolev norms

(NLS-HW) $\quad i\partial_t u + (\partial_{xx}^2 - |D_y|)u = \mu |u|^2 u\,, \quad (x,y) \in \mathbb{R} \times \mathcal{M}\,, \quad \mathcal{M} = \mathbb{T},\ \mathbb{R}\,.$

* (Xu'15) NLS-HW on $\mathbb{R} \times \mathbb{T}$: modified scattering $\rightsquigarrow$ unbounded $H^s$-norms.
* (Bahri-Ibrahim-Kikuchi '19) Transverse instability (assuming LWP in $\mathcal{H}^{\frac{1}{2}}$)

**Cauchy theory?** $\rightsquigarrow$ unknown in the energy space $\mathcal{H}^{\frac{1}{2}}$!



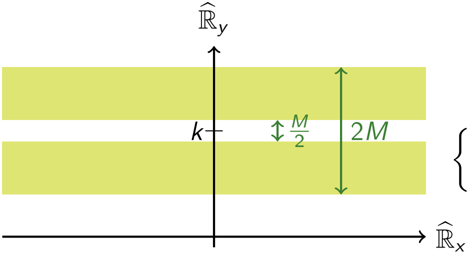
- Unit-scale randomization in $y$ (dispersion in $x$)

$$u_0^\omega = \sum_k g_k(\omega) Q_k u_0$$

- Bernstein-Strichartz estimates in mixed Lebesgue spaces:

$$\|S(t)\mathbf{1}_E(|D_y|)u_0\|_{L_t^p L_x^q L_y^r} \lesssim |E|^{\frac{1}{2}-\frac{1}{r}} \|\mathbf{1}_E(|D_y|)u_0\|_{L^2_{x,y}}, \quad \frac{2}{p}+\frac{1}{q}=\frac{1}{2}, \; r \geq 2.$$

- Structure of $F_N$ captured by centered Besov spaces




$P_{M;k}$ : *proj. onto*

$$\left\{(\widehat{x},\widehat{y}) \in \widehat{\mathbb{R}}_x \times \widehat{\mathbb{R}}_y \; : \; \frac{M}{2} < |\widehat{y}| \leq M\right\}.$$

- Also works for half-wave, Szegő (take $x$ constant)
- On degenerate geometries with subelliptic Laplacian: Grusin, Heisenberg,...

## Deterministic instabilities *vs* probabilistic oscillations

- Euclidean space: full-measure set of local solutions vs *pathological* dense $G_\delta$ set.

⇝ Bourgain's semilinear scheme:

$$u = e^{it\Delta} u_0^\omega + v\,, \quad v \in H^s(M) \quad \text{with } s \geq s_c\,, \quad \textit{(semi-group structure)}$$

- Iterate the local structure to get long-time existence and dynamics

⇝ Probabilistic quasilinear schemes: Deng-Nahmod-Yue, Bringmann, Sun-Tzvetkov...

$$\lim_{t\to\infty} u_n = u \text{ in } C([-T^\omega, T^\omega], H^s)\,, \quad u_n, u \text{ solution}\,, \quad u_n(0) = P_{\leq n} u_0^\omega\,.$$

- **Global existence?**
- To iterate the local well-posedness, one needs to propagate some *probabilistic* information
- Evolution of the measure under the nonlinear flow? (invariance, quasi-invariance...) Tzvetkov, Oh, Thomann, Visciglia, Sun, Deng, Tolomeo, Sosoe, Genovese, Lucà...

Answer to the question: by (D. Eceizabarrena): Yes!

$$\mathcal{D} = \bigcap_{\delta_n > 0} \left\{ x \in [0,1] \ : \ \exists \text{ finitely many } (p,q) \ : \ |x - \frac{p}{q}| \leq \frac{1}{q^{2+\delta_n}} \right\}.$$

the set of the Diophantine numbers.

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# Thank you