Cubic Schrödinger half-wave equation and random initial data

Nicolas Camps, Louise Gassot, Slim Ibrahim

Abstract. We give an overview of results on the cubic Schrödinger-half-wave equation. This equation is motivated by the study of long time behavior of solutions. It acts as a toy model for equations with low dispersion, as there is a lack of dispersion in one of the two spatial variables. In particular, the question of local and global well-posedness is a delicate issue. We present a recent almost-sure local well-posedness result below the energy space. Because of the lack of probabilistic smoothing in the second Picard's iteration, we rely on a refined probabilistic ansatz adapted from the work on Bringmann on the derivative nonlinear wave equation.

Contents

1. Introduction

In this note, we mention some results on the cubic Schrödinger half-wave equation on $\mathbb{R}^2$. Let us denote $|D_y| = \sqrt{-\partial_{yy}^2}$ and $\mu \in \mathbb{R}$, the Schrödinger half-wave equation takes the form

$$i\partial_t u + \left(\partial_{xx}^2 - |D_y|\right) u = \mu \left|u\right|^2 u\,, \quad (t,x,y) \in \mathbb{R}\times\mathbb{R}^2\,. \tag{NLS-HW}$$

This equation is motivated by a mathematical interest on the long time behavior and qualitative properties of solutions. It was first introduced in [36] in the defocusing case ($\mu > 0$) on the wave guide $\mathbb{R}_x \times \mathbb{T}_y$ to evidence weak turbulence in the growth of Sobolev norms. Then, in the focusing case ($\mu < 0$), the ground state standing waves and traveling waves on the wave guide $\mathbb{R}_x \times \mathbb{T}_y$ was first constructed in [2], and the stability properties of some standing waves investigated in [3]. We summarize these results in section 3.

Our main concern is to investigate the local Cauchy problem at low regularity. We recall in section 2 the know deterministic well-posednesss and ill-posedness properties for equation (NLS-HW). The main difficulties are caused by the lack of dispersion in the second spatial variable y, as equation (NLS-HW) can be decomposed as a coupled system of two transport equations in this variable. Hence we are led to consider generic initial

Keywords: Nonlinear Schrödinger equation, probabilistic Cauchy theory.
2020 *Mathematics Subject Classification:* 35B40 primary.

data under the form of random initial data. In this setting, the lack of dispersion causes a lack of probabilistic smoothing of the Duhamel's iterate, hindering the classical approach of Bourgain. However, recent developments enable us to overcome this problem by using a refined quasilinear ansatz, which we explain in section 4.

2. Cauchy problem

Due to the anisotropy of the equation, the relevant regularity spaces are anisotropic Sobolev spaces $\mathcal{H}^s$ defined as

$$\mathcal{H}^s := L^2_x H^s_y \cap H^{2s}_x L^2_y\,, \quad \dot{\mathcal{H}}^s := L^2_x \dot{H}^s_y \cap \dot{H}^{2s}_x L^2_y\,.$$

Note that as a consequence, the exponents in the Sobolev embeddings are the same as the ones in $\mathbb{R}^3$ even if there are only two variables, because the homogeneous dimension is 3. For instance, we have $\mathcal{H}^{\frac{3}{2}+} \hookrightarrow L^\infty$.

Equation (NLS-HW) is a Hamiltonian system, with a formal conserved energy

$$H(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\partial_x u|^2 + \left||D_y|^{\frac{1}{2}}u\right|^2 \mathrm{d}x\,\mathrm{d}y + \frac{\mu}{4}\int_{\mathbb{R}^2} |u|^4 \,\mathrm{d}x\,\mathrm{d}y\,.$$

The mass $\|u\|^2_{L^2}$ is also formally conserved by the flow. There is no known conservation law above regularity $\mathcal{H}^{\frac{1}{2}}$ for equation (NLS-HW). Moreover, a Brezis-Gallouët argument does not appear to be sufficient in order to control the norm of high-regularity solutions, therefore it seems necessary to handle the Cauchy problem below regularity $\mathcal{H}^{\frac{1}{2}}$ in the hope to get global well-posedness.

However, we will see that the flow map cannot be $\mathcal{C}^3$ in $\mathcal{H}^s$ when $s < \frac{1}{2}$, and there is even a norm-inflation mechanism when $s < \frac{1}{4}$. As a consequence, the question of local well-posedness below the energy space is a challenging problem.

One can summarize the state-of-the-art Cauchy theory results for (NLS-HW) in the following diagram.

Figure 1. Deterministic Cauchy theory for equation (NLS-HW).

Let us now detail the properties of the three zones evidenced in this diagram.

2.1. Local well-posedness

When $s > \frac{1}{2}$, semilinear well-posedness is obtained using Strichartz estimates with a derivative loss in [2].

Proposition 2.1 (Local well-posedness above the energy space, [2] Theorem 1.6)**.** *Let $s > \frac{1}{2}$. For every $u_0 \in \mathcal{H}^s(\mathbb{R}^2)$, there exists $T = T(\|u_0\|_{\mathcal{H}^s})$ such that equation* (NLS-HW) *admits a unique local solution in $\mathcal{C}((-T,T),\mathcal{H}^s)$ with initial data u_0.*

It is shown in [2] that local well-posedness actually holds in $L^2_x H^s_y$. The proof follows from a fixed point argument in $L^\infty_t([-T\,,T];L^2_x H^s_y) \cap L^4_t([-T\,,T];L^\infty_{x,y}(\mathbb{R}^2))$, for some $T > 0$ depending on the $\mathcal{H}^s(\mathbb{R}^2)$-norm of the initial data. The use of the Strichartz space $L^4_t([-T\,,T];L^\infty_{x,y}(\mathbb{R}^2))$ requires that $s > \frac{1}{2}$ because of the lack of dispersion in the y-direction, so that one need to use the Sobolev embeddings. Consequently, we have no control on the growth of the $\mathcal{H}^s$-norm of the solution since there we do not have access to the conserved energy, which is at the level of $\mathcal{H}^{\frac{1}{2}}$. Global existence for smooth solutions to (NLS-HW) is an open problem.

2.2. Ill-posedness results

The solutions of equation (NLS-HW) are invariant under the scaling symmetry

$$u \mapsto u_\lambda(t,x,y) = \lambda u(\lambda^2 t, \lambda x, \lambda^2 y)\,. \tag{2.1}$$

This scaling leaves the $\dot{\mathcal{H}}^{\frac{1}{4}}$-norm invariant. As a consequence, when $0 < s < \frac{1}{4}$, there is short-time inflation of the $\dot{\mathcal{H}}^s$-norm for the solutions obtained by regularizing a rough initial data. We call this phenomenon a *low-to-high frequency cascade* or *norm inflation mechanism*. An adaptation of the arguments from [8] implemented in [24] imply the following ill-posedness result.

Theorem 2.2 (Norm inflation [24])**.** *Let $s < \frac{1}{4}$. For every bounded set B of $\mathcal{H}^s$ and $T > 0$, the flow map cannot be extended as a continuous map from B to $\mathcal{C}([-T,T],\mathcal{H}^s)$. More precisely, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers tending to zero and a sequence $(u_n(t))_{n\gg 1}$ of $\mathcal{C}^\infty(\mathbb{R}^2)$ solutions of (NLS-HW) defined for $t \in [0,t_n]$, such that when $n \to +\infty$,*

$$\|u_n(0)\|_{\mathcal{H}^s} \to 0\,,$$
$$\|u_n(t_n)\|_{\mathcal{H}^s} \to +\infty\,.$$

In other words, for arbitrarily short times, there exist sequences of smooth initial data going to zero in $\mathcal{H}^s$, such that the corresponding solutions go to infinity in $\mathcal{H}^s$. Such a norm inflation mechanism was originally exhibited in [25, 26, 27] for the wave equation, then extended to the Schrödinger equation in [14]. Non-uniform continuity of the flow map for $s = \frac{1}{4}$ in equation (NLS-HW) has also been investigated in [24].

To evidence norm inflation in the scaling-supercritical regime, we perform a *small dispersion analysis*. By rescaling an arbitrary compactly supported smooth function, one generates a sequence of smooth initial data $(\psi_n)_n$ going to zero in $\mathcal{H}^s$, and spatially concentrating around a point. Let u_n be the smooth solution to (NLS-HW) with initial data ψ_n. We show that for short times, u_n stays close to the *bubble solution* v_n to the dispersionless ODE

$$\begin{cases} i\partial_t v_n = \sigma |v_n|^{p-1} v_n\,, \\ v_n(0) = \psi_n\,. \end{cases} \tag{2.2}$$

The ODE profile v_n is very oscillating and grows in $\mathcal{H}^s$ at times t_n satisfying $t_n \to 0$. When $0 < s < s_c$, a priori energy estimates up to time t_n imply that uniformly in n,

$$\|u_n(t_n) - v_n(t_n)\|_{H^s} \lesssim 1\,. \tag{2.3}$$

Therefore, the oscillations dominate the dispersion, hence the instability stems from the same frequency cascade from low to high Fourier modes.

The argument from [35] implies that this result presented in Theorem 2.2 is still valid around any initial data $u(0) \in \mathcal{H}^s$ and not just the zero initial data. However, this indicates that allowing for any sequence of smooth functions $(u_n(0))_n$ to approximate a given initial data $u(0)$ in $\mathcal{H}^s$ may not be restrictive enough, so that one may rather need to restrict this study only to the natural approximations of the initial data obtained by convolution by an approximate identity $(\rho_\varepsilon)_{\varepsilon>0}$. More precisely, we consider the sequence of initial data $(\rho_{\varepsilon_n} * u(0))_n$ for some sequence $\varepsilon_n \to 0$. In this case, the zero initial is approximated by the smooth zero initial data $u_n(0) = 0$, so that the corresponding smooth solution is the zero solution which does not present norm inflation.

Using the method of Sun and Tzvetkov [32], adapted to Schrödinger-type equations in [11], we show that the regularization of rough initial data by convolution does not prevent norm inflation in $\mathcal{H}^s$. This result holds for a dense set of initial data, which can be proven to be a dense G_δ set when a sufficient global Cauchy theory is available.

More precisely, we fix $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, valued in $[0,1]$, such that $\int_{\mathbb{R}^2} \rho(x)\,\mathrm{d}x = 1$ and ρ vanishes for $x^2+y^2 \geq \frac{1}{10^4}$. Due to the anisotropy, we define an approximate identity $(\rho_\varepsilon)_{\varepsilon>0}$ of the form

$$\rho_\varepsilon(x,y) := \frac{1}{\varepsilon^3}\rho\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon^2}\right).$$

Theorem 2.3 (Generic ill-posedness for (NLS-HW) [12])**.** *Let $s<\frac{1}{4}$. There exists a dense set $S\subset\mathcal{H}^s$ such that for every $f\in S$, the family of local solutions u^ε of (NLS-HW) with initial data $\rho_\varepsilon * f$ does not converge as $\varepsilon\to 0$. More precisely, there exist $\varepsilon_n\to 0$ and $t_n\to 0$ such that $u^{\varepsilon_n}(t_n)$ is well-defined and*

$$\lim_{n\to\infty}\|u^{\varepsilon_n}(t_n)\|_{\mathcal{H}^s} = +\infty\,.$$

The idea is to superpose an infinite number of *bubble* solutions in the dispersionless ODE (2.2), in other words, we replace the bump initial data ψ_n by an infinite series of bumps with different scales. Given one convolution parameter ε_n, we prove that only the n-th bubble exhibits norm inflation at time t_n. Indeed, the bubbles at bigger scale would need a bigger time than t_n to grow, whereas the bubbles at smaller scale are shrunk because of the convolution. Finally, a perturbative argument implies that the actual solution to (NLS-HW) still satisfies estimate (2.3).

2.3. Semilinear ill-posedness

We have seen that when $s<\frac{1}{4}$, the equation is scaling-subcritical, so that one can evidence some norm-inflation mechanisms of the solutions. Yet, semilinear local well-posedness is only known in $\mathcal{H}^s$ when $\frac{1}{2}<s$. We can show that actually, the flow map cannot be of class $\mathcal{C}^3$ when $\frac{1}{4}<s<\frac{1}{2}$, meaning that (NLS-HW) is *semilinearly ill-posed* for this range of exponents.

Theorem 2.4 (Semilinear ill-posedness [12])**.** *If there exists a local in time flow map on $\mathcal{H}^s$ with regularity $\mathcal{C}^3$ at the origin, then $s\geq\frac{1}{2}$.*

To establish this result, we note that as a corollary of Remark 2.12 in [7], if there exists a $\mathcal{C}^3$ local in time flow map at the vicinity of the origin in the space $\mathcal{H}^s$, then the following Strichartz estimate holds:

$$\|e^{it(\partial_{xx}^2-|D_y|)}\phi\|_{L^4([0,1]\times\mathbb{R}^2)} \lesssim \|\phi\|_{\dot{\mathcal{H}}^{\frac{s}{2}}}\,. \tag{2.4}$$

We consider a one-parameter family constructed from traveling waves for the one-dimensional Szegő equation to invalidate these Strichartz estimates when $s<\frac{1}{2}$. Then, we conclude that the flow map cannot be of class $\mathcal{C}^3$ at the origin when $\frac{1}{4}<s<\frac{1}{2}$. As a consequence, it is not possible to run a contraction mapping argument when $\frac{1}{4}<s<\frac{1}{2}$, since otherwise the flow-map would be analytical.

More precisely, we consider a Gaussian distribution G, a family of traveling waves profiles for the Szegő equation $K_\rho(y)=\frac{1}{y+i\rho}$ for $\rho\in(0\,,+\infty)$, and set

$$\phi(x,y)=G(x)K_\rho(y)\,.$$

Since K_ρ is a traveling wave for the Szegő equation on the line [30], and in particular a traveling wave for equation (NLS-HW), one can see that for every t, there holds

$$\|e^{it|D_y|}K_\rho\|_{L^4(\mathbb{R}_y)} = \|K_\rho\|_{L^4(\mathbb{R}_y)}$$

implying that as $\rho\to 0$,

$$\|e^{it(\partial_{xx}^2-|D_y|)}\phi\|_{L^4([0,1]\times\mathbb{R}^2_{x,y})} = \|e^{it\partial_{xx}}G\|_{L^4([0,1]\times\mathbb{R}_x)}\|K_\rho\|_{L^4(\mathbb{R}_y)} \sim C\rho^{-\frac{3}{4}}\,.$$

Then we show using the independence of the functions G and K_ρ that

$$\|\phi\|_{\dot{\mathcal{H}}^s} \underset{\rho\to 0}{\sim} C'\rho^{-\frac{1}{2}-\frac{s}{2}}\,.$$

Since inequality (2.4) has to hold as $\rho \to 0$, we see that necessarily, $\rho^{-\frac{3}{4}} \lesssim \rho^{-\frac{1}{2}-s}$ as $\rho \to 0$, therefore $\frac{1}{2} \leq s$. This completes the study of deterministic Cauchy theory for equation (NLS-HW).

3. Long-time behavior of solutions

Given the challenges to overcome in order to get a satisfying global well-posedness theory for equation (NLS-HW), the study of long-time behavior of solutions relies on special phenomenon, such as modified scattering in the defocusing case, or special solutions, such as traveling waves in the focusing case. We mention the known results regarding these problem, then we compare them with the one-dimensional cubic half-wave equation and cubic Schrödinger equation that are obtained by considering both space variables separately.

3.1. Modified scattering in the defocusing case

Equation (NLS-HW) was originally introduced by Xu [36] in the defocusing case ($\mu > 0$) on the spatial wave guide $\mathbb{R}_x \times \mathbb{T}_y$ to evidence weak turbulence mechanisms in the growth of Sobolev norms. Global existence and modified scattering are obtained for a class of sufficiently smooth and decaying small initial data on the wave guide $\mathbb{R}_x \times \mathbb{T}_y$. In addition, the author shows that the limiting effective dynamics is governed by the Szegő equation on the torus.

In order to formulate the result, we define two spaces S and S^+ of sufficiently smooth initial data with enough decay in the spatial variable x, mainly, for some fixed $N \geq 13$,

$$\|u_0\|_S = \|u_0\|_{H^N_{x,y}} + \|xu_0\|_{L^2_{x,y}}, \quad \|u_0\|_{S^+} = \|u_0\|_S + \|xu_0\|_S + \|(1-\partial_{xx})^4 u_0\|_S.$$

We also denote by Π_y is the Szegő projector onto nonnegative Fourier frequencies in the variable y.

Theorem 3.1 (Modified scattering [36]). *There exists $\varepsilon = \varepsilon(N) > 0$ such that if the initial data $u_0 \in S^+$ satisfies $\|u_0\|_{S^+} \leq \varepsilon$, then the corresponding solution $u \in \mathcal{C}(\mathbb{R}_+, S)$ exists globally in S. Moreover, there exists $G \in \mathcal{C}(\mathbb{R}_+, S)$ such that*

$$\|u(t) - e^{it(\partial^2_{xx} - |D_y|)} G(\pi \ln(t))\|_S \to 0, \quad t \to +\infty. \tag{3.1}$$

The profile G is solution to the following resonant system. We denote $G_+ = \Pi_y(G)$ and $G_- = G - G_+$. Let us consider the the partial Fourier transform $\widehat{G}_\pm$ of $G_\pm$ in the variable x only, with corresponding Fourier variable ξ. Then

$$\begin{cases} i\partial_t \widehat{G}_+ = \Pi_y(|\widehat{G}_+|^2 \widehat{G}_+)(\xi, y) \\ i\partial_t \widehat{G}_- = (\mathrm{Id} - \Pi_y)(|\widehat{G}_-|^2 \widehat{G}_-)(\xi, y). \end{cases}$$

By considering the Fourier variable ξ as a parameter, the equation for $\widehat{G}_+$ is the cubic Szegő equation on the torus. As a consequence of the work [21] on the growth of Sobolev norms for the Szegő equation, this remark implies the existence of arbitrarily small initial data such that for every $s > \frac{1}{2}$ and $N \geq 1$, the solution u exhibits weak turbulence. It is expected that this result actually holds for a dense G_δ set of such initial data.

Corollary 3.2 (Growth of Sobolev norms [36, 21])**.** *For every $N \geq 13$, for every $\varepsilon > 0$, there exists $u_0 \in S^+$ such that $\|u_0\|_{S^+} \leq \varepsilon$ and the corresponding solution u satisfies:*

$$\forall s > \frac{1}{2}, \quad \limsup_{t\to\infty} \frac{\|u(t)\|_{L^2_x H^s_y}}{\log(t)^N} = \infty, \quad \liminf_{t\to\infty} \|u(t)\|_{L^2_x H^s_y} < \infty .$$

The strategy employed by Xu on $\mathbb{R}_x \times \mathbb{T}_y$ is adapted from the study for the Schrödinger equation on the wave guide $\mathbb{R}_x \times \mathbb{T}^d_y$ from Hani, Pausader, Tzvetkov and Visciglia in [23].

On $\mathbb{R}_x \times \mathbb{R}_y$, modified scattering should also be expected but it would rely on different arguments. Xi [34] constructed wave operators in this setting, and deduced a different type of growth of Sobolev norms in infinite time. Indeed, the resonant behavior is then linked to the cubic Szegő equation on the line, for which there is a transition towards high Fourier frequencies [22]. As a consequence, many solutions have a growth of the following form:

$$\frac{1}{C}\log(t) \leq \|u(t)\|_{L^2_x H^1_y} \leq C\log(t).$$

In comparison, the defocusing Schrödinger equation

$$i\partial_t u + \Delta u = |u|^2 u$$

is completely integrable in one dimension. In particular, on $\mathbb{R}$ or on $\mathbb{T}$, the existence of conservation laws controlling Sobolev norms of arbitrary high regularity imply that the solutions satisfy that for every $s \geq 1$ and $t \in \mathbb{R}$,

$$\|u(t)\|_{H^s} \leq C_s(\|u_0\|_{H^s}).$$

However, as noticed in [23], the Schrödinger equation on the wave guide $\mathbb{R}_x \times \mathbb{T}^d_y$ for $d \geq 2$ exhibits modified scattering. As a consequence of the analysis of the resonant system [15], they prove that for every $s \geq 30$, for every $\varepsilon > 0$, there exists a global solution $u \in \mathcal{C}(\mathbb{R}_+, H^s)$ satisfying $\|u_0\|_{H^s} \leq \varepsilon$ and

$$\limsup_{t\to\infty} \|u(t)\|_{H^s} = +\infty.$$

Concerning the defocusing half-wave equation on the torus ($y \in \mathbb{T}$)

$$i\partial_t u - |D_y| u = |u|^2 u\,, \tag{3.2}$$

we know that for every $s > 1$, there exists a sequence $(u^n)_n$ of solutions and a sequence of times $t^n \to \infty$ satisfying

$$\|u^n_0\|_{H^s} \to 0, \quad \|u^n(t^n)\|_{H^s} \to +\infty.$$

The proof relies on the closeness between solutions of the half-wave equation and the Szegő equation for large times [20, 31] and the growth of Sobolev norms for the Szegő equation [21]. However, the existence of an arbitrary small initial data such that the solution to the half-wave equation satisfies $\limsup_{t\to\infty} \|u(t)\|_{H^s} = \infty$ is an open problem.

3.2. Traveling waves in the focusing case

Subsequently, Bahri, Ibrahim and Kikuchi consider in [2, 3] the focusing case $\mu < 0$ for equation (NLS-HW) on the wave guide $\mathbb{R}_x \times \mathbb{T}_y$. They construct ground state standing waves and traveling waves. Then they obtain orbital stability and transverse instability results for the family of standing waves, conditional to the existence of a good Cauchy theory in the energy space. Unfortunately, such a Cauchy theory is yet to be addressed, since not much is known about the global existence of smooth solutions in Sobolev spaces.

More precisely, on $\mathbb{R}_x \times \mathbb{T}_y$, the authors introduce a family of ground state standing wave solutions u_ω with frequency $\omega > 0$ of the form

$$u_\omega(x,y,t) = e^{i\omega t} Q_\omega(x,y)\,.$$

The ground states are constructed as minimizers of the energy functional

$$\mathcal{S}_\omega(u) = \frac{1}{2}\|\partial_x u\|_{L^2}^2 + \frac{1}{2}\||D_y|^{1/2}u\|_{L^2}^2 + \frac{\omega}{2}\|u\|_{L^2}^2 - \frac{1}{4}\|u\|_{L^4}^4$$

under the constraint $\mathcal{N}_\omega(u) = 0$ with

$$\mathcal{N}_\omega(u) = \|\partial_x u\|_{L^2}^2 + \||D_y|^{1/2}u\|_{L^2}^2 + \omega\|u\|_{L^2}^2 - \|u\|_{L^4}^4.$$

In [3], the authors establish that for small frequencies $0 < \omega < \omega_*$, the ground state Q_ω does not depend on the spatial variable y. As a consequence, Q_ω is equal to the line soliton for the Schrödinger equation, which is known to be orbitally stable on the line [13]. On the wave guide for (NLS-HW), however, the Schrödinger line soliton presents transverse instability properties: it is orbitally stable for small frequencies $0 < \omega < \omega_p$ whereas it is orbitally unstable for $\omega > \omega_p$. After showing that $\omega_p \geq \omega_*$, the authors deduce the orbital stability of ground states for small frequencies.

Theorem 3.3 (Ground state standing waves [3])**.** *There exists $\omega_* > 0$ such that Q_ω does not depend on y when $0 < \omega \leq \omega_*$, but depends on y when $\omega > \omega_*$. Moreover, the ground state standing wave u_ω is orbitally stable when $0 < \omega \leq \omega_*$.*

Their result is actually true for any nonlinearity of order $1 < p < 5$ including the cubic nonlinearity $p = 3$. The strategy employed to study orbital stability relies on the fact that the standing wave only depends in one of the two variables, transferring the problem to the Schrödinger equation on the line. It would be interesting to consider other geometries. For instance one could try to establish some orbital stability or instability property when the two spatial variables lie in $\mathbb{R}_x \times \mathbb{R}_y$, so that the ground state has to depend on both variables, but also on the wave guide $\mathbb{T}_x \times \mathbb{R}_y$ where the dynamics at small frequencies ω would be governed by the half-wave equation on the line rather than the Schrödinger equation.

4. Cauchy problem and random initial data

Given the difficulties to tackle the Cauchy problem due to the lack of dispersion, an alternative approach is to resort to random initial data and study whether a generic well-posedness holds. The probabilistic Cauchy theory goes back to Bourgain in [4] and Burq, Tzvetkov in [9, 10].

4.1. Bourgain's historical approach

Let s_c be a critical threshold under which instabilities are known to occur, or a threshold under which deterministic local well-posedness is unknown. When $s < s_c$, we remark that a *generic* initial data in $\mathcal{H}^s$ has better integrability properties in L^p spaces than expected by the Sobolev embedding. Moreover, in favorable situations, the dispersion is strong enough to give some local energy decay or bilinear estimates, or in the case wave-type equations, the Duhamel formula gains one derivative. In this case, one can exploit the enhanced integrability property of a generic initial data to prove that the Picard's iterations are actually smoother than the initial data or its linear evolution, therefore they fall into the subcritical regime. As a consequence, one can construct strong solutions with initial data distributed according to a non-degenerate probability measure charging any open set in $\mathcal{H}^s$.

We construct *randomized* initial data ϕ^ω from a given function $\phi \in \mathcal{H}^s$ as follows. We consider the unit-scale frequency decomposition $(\phi_n^\omega)_n$ of ϕ in the frequency space. In compact settings, it would also be natural to consider the decomposition of ϕ along a spectral resolution of the dispersive operator (such as the Laplace operator for the

Schrödinger equation). We decouple each mode using a sequence $(g_n(\omega))_n$ of normalized independent Gaussian variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

$$\omega \in \Omega \mapsto \phi^\omega \sim \sum_n g_n(\omega)\phi_n \,, \quad \text{where } \phi \sim \sum_n \phi_n \in \mathcal{H}^s \,.$$

Then, for *many* initial data ϕ^ω in a statistical ensemble $\Sigma \subset \mathcal{H}^s$ which has full measure, one expects to observe a *probabilistic smoothing effect* for the recentered solution around the linear evolution thanks to the combination of space-time oscillations (dispersion) and probabilistic oscillations (randomization). Namely, the goal is to show that there exists $\nu > s_c$ such that for all $\phi^\omega \in \Sigma$,

$$v(t) := u(t) - \mathrm{e}^{it(\partial_{xx} - |D_y|)}\,\phi^\omega \in \mathcal{C}([-T\,,T];\mathcal{H}^\nu)\,. \tag{4.1}$$

Then v solves the original equation perturbed by stochastic terms stemming from the linear evolution $\mathrm{e}^{it(\partial^2_{xx} - |D_y|)}\,\phi^\omega$:

$$i\partial_t v + (\partial^2_{xx} - |D_y|)v = \mu|v + \mathrm{e}^{it(\partial^2_{xx} - |D_y|)}\,\phi^\omega|^2(v + \mathrm{e}^{it(\partial^2_{xx} - |D_y|)}\,\phi^\omega).$$

In this case, if $\nu > s_c$, the recentered solution v is obtained from a fixed point argument at subcritical regularities in $\mathcal{H}^\nu$. This strategy has been successful on the Euclidean and periodic case in many contexts for the Schrödinger equation

$$i\partial_t u + \Delta u = \mu|u|^2 u,$$

where we replace the operator $\partial^2_{xx} - |D_y|$ by the Laplace operator Δ. Unfortunately we will see that the recentered solution v does not gain regularity for equation (NLS-HW), so that the traditional probabilistic method fails.

4.2. Probabilistic smoothing effect and dispersion

Let us understand why probabilistic smoothing does not occur for (NLS-HW). In order to observe a probabilistic smoothing effect, we exploit dispersive properties of the equation to gain decay and Strichartz estimates without trading regularity. Equation (NLS-HW), however, is constructed so that there is no dispersion in the y-direction. Therefore, in the low x-frequency regimes, the Strichartz estimates come with a derivative loss, so that we have neither usable bilinear estimates, nor local smoothing estimates at our disposal. A manifestation of this lack of dispersion is that the second Picard iteration of the randomized initial data does not have a better regularity than the initial data: there is no probabilistic smoothing.

The terms of the equation that prevent probabilistic smoothing are coming from *high-low-low* interactions present in nonlinearity from the second Picard iteration. For simplicity, we assume that there is no dependence in the variable x (in practice, we rather restrict the solution to low x-frequency). In the second Picard iteration applied to the initial data ϕ^ω, the high-low-low type interactions involve the product of ϕ^ω projected at in high y-frequencies $|\eta| \gg 1$, and the square of ϕ^ω projected at low y-frequencies $|\eta| \lesssim 1$: they take on the form

$$\int_0^t \mathrm{e}^{i(t-\tau)|D_y|}\left(e^{-i\tau|D_y|}P_{|\eta|\gg 1}\phi^\omega\right)\left(\overline{e^{-i\tau|D_y|}P_{|\eta|\leq 1}\phi^\omega}\right)\left(e^{-i\tau|D_y|}P_{|\eta|\leq 1}\phi^\omega\right)\,\mathrm{d}\tau\,.$$

In this case, the linear operator $e^{-i\tau|D_y|}$ is not dispersive as it acts as a transport equation on each term $P_{|\eta|\gg 1}\phi^\omega$ and $P_{|\eta|\leq 1}\phi^\omega$ after separating between positive and negative frequencies. Hence derivatives of the high-low-low interaction term can all fall at the same time onto the first term $P_{|\eta|\gg 1}\phi^\omega$. This implies that one can only handle s derivatives for the high-low-low interaction instead of the desired ν derivatives. Similarly, Oh [28] considered the Szegő equation on the circle and proved that the first nontrivial Picard's iterate does not gain regularity compared to the initial data. As a consequence, one needs to resort to a more sophisticated quasilinear scheme than Bourgain's method.

4.3. **A refined probabilistic ansatz**

We have seen that the first nontrivial Picard's iterate has the same regularity as the initial data because of the high-low-low iteractions. As a consequence, the standard probabilistic method is not sufficient. In order to handle this difficulty, Bringmann [5] developed a refined probabilistic ansatz in a quasilinear setting using paracontrolled calculus, to prove probabilistic local well-posedness for a derivative wave equation. In [12], we adapt this strategy to prove almost-sure local well-posedness in the quasilinear regime (i.e. below the energy space) for equation (NLS-HW). The main idea is to refine the classical probablistic ansatz (4.1), where we replace the linear correction $e^{it(\partial_{xx}^2-|D_y|)}\phi^\omega$ by a more sophisticated probabilistic term that incorporates the high-low-low interactions.

We mention that breakthrough papers of Deng, Nahmod and Yue [17, 18], Sun and Tzvetkov [33], and more recently Bringmann, Deng, Nahmod and Yue [6], pushed even further the paracontrolled approach for dispersive PDE in the probabilistic setting. The new developments incorporate tools from random matrix theory and introduce powerful methods such as random averaging operators and random tensors.

Let us explain how the refined probabilistic ansatz applies to equation (NLS-HW). We fix $\phi \in \mathcal{H}^s(\mathbb{R}^2)$. Taking the partial Fourier transform in the y-variable only, we decompose ϕ using Fourier projectors $P_{1,k}$ on an interval of unit length centered around $k \in \mathbb{Z}$:

$$\phi = \sum_{k\in\mathbb{Z}} P_{1,k}\phi, \quad \operatorname{supp}(\mathcal{F}_{y\to\eta}P_{1,k}\phi) \subseteq [k-1, k+1].$$

We do not need to introduce a randomization along the Schrödinger variable x since the Schrödinger equation exhibits dispersion. We consider a sequence of independent normalized Gaussian variables $(g_k(\omega))_{k\in\mathbb{Z}}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and define the Wiener randomization of ϕ as

$$\omega \in \Omega \mapsto \phi^\omega := \sum_{k\in\mathbb{Z}} g_k(\omega)P_{1,k}\phi\,.$$

The relevant probability measure on $\mathcal{H}^s$ is the measure induced by this random variable. For $T > 0$ and $s, \sigma \in \mathbb{R}$, we establish the convergence of approximate local solutions towards a local solution to (NLS-HW) in the space

$$X^s_{T_0} := \mathcal{C}_t\big([-T_0\,, T_0]\,;\, \mathcal{H}^s(\mathbb{R}^2)\big) \cap L^8_t\left([-T_0\,, T_0]\,;\, L^4_x W^{\sigma,\infty}_y(\mathbb{R}^2)\right)\,.$$

We also denote the truncated initial data as

$$P_{\le n}\phi^\omega := \sum_{|k|\le n} g_k(\omega)P_{1,k}\phi\,.$$

The main result from [12] is as follows.

Theorem 4.1 (Probabilistic local well-posedness [12])**.** *Let $s \in (13/28, 1/2]$ and $\phi \in \mathcal{H}^s$. There exist $T_0 > 0$ and a full measure set $\Sigma \subset \mathcal{H}^s$ such that for any $\phi^\omega \in \Sigma$ the following holds. There exists a uniform random time $T^\omega \in (0, T_0]$ such that for all $n \in \mathbb{N}$, there exists a function $u_n \in C([-T_0, T_0], \mathcal{H}^\infty)$ which is the unique solution on $[-T^\omega, T^\omega]$ to (NLS-HW) with smooth initial data $P_{\le n}\phi^\omega$:*

$$\begin{cases} i\partial_t u_n + (\partial^2_{xx} - |D_y|)u_n = |u_n|^2 u_n\,, & (t, x, y) \in [-T^\omega\,, T^\omega] \times \mathbb{R}^2\,, \\ u_n|_{t=0} = P_{\le n}\phi^\omega\,. \end{cases}$$

Moreover, the sequence $(u_n)_{n\ge1}$ converges in $L^2_\omega\left(\Omega\,; X^{s,\sigma}_{T_0}\right)$ for some $0 < \sigma < s$ to a limiting object u which is solution to (NLS-HW) on $[-T^\omega, T^\omega]$ with initial data ϕ^ω.

The idea behind the refined probabilistic ansatz is the following. We first construct the solution u_N for dyadic $N \in 2^{\mathbb{N}}$ only by induction on N. The solution at step N is constructed from the solution at step $\frac{N}{2}$ following the ansatz

$$u_N = u_{\frac{N}{2}} + F_N + w_N \,. \tag{4.2}$$

The probabilistic term F_N isolates the problematic rough *high-low-low* frequency interactions from the equation, and is called *adapted linear evolution.* The nonlinear remainder term w_N will exhibit a nonlinear smoothing effect.

As a consequence of the induction scheme (4.2), the solution at step N can be written as a series

$$u_N = u_{N_0} + \sum_{L=2N_0+1}^{N} (w_L + F_L^{\omega}) \,.$$

We prove that the series of smooth remainder terms $(w_N)_{N\geq N_0}$ converges almost-surely in a subcritical space $\mathcal{C}([-T_0\,,T_0];\mathcal{H}^{\nu}(\mathbb{R}^2))$, for some $\nu > \frac{1}{2}$ and some $0 < T_0 \ll 1$. On the other hand, the series with general term $(F_N)_{N\geq N_0}$ composed of the probabilistic corrections converges almost-surely in the space of rough regularity $\mathcal{C}([-T_0\,,T_0];\mathcal{H}^{s}(\mathbb{R}^2))$. Then, we prove that there exists a random time $T^{\omega} > 0$ such that the limit of $(u_N)_N$ solves (NLS-HW) in $\mathcal{C}\big([-T^{\omega}\,,T^{\omega}];\mathcal{H}^{s}(\mathbb{R}^2)\big)$. Finally, we use an argument from [33] to show that the result for dyadic frequencies N extends to the general approximation with integer frequencies n.

Let $0 < \gamma < 1$ be some parameter. We denote the cubic nonlinear term by

$$\mathcal{N}(u) = |u|^2 u\,, \quad \mathcal{N}(u_1,u_2,u_3) = \overline{u_1}u_2u_3 + u_1\overline{u_2}u_3 + u_1u_2\overline{u_3}\,.$$

The *adapted linear evolution* F_N is solution to the equation

$$\begin{cases} i\partial_t F_N + (\partial_{xx}^2 - |D_y|)F_N = \mathcal{N}(F_N, P_{\leq N^{\gamma}}u_{\frac{N}{2}}, P_{\leq N^{\gamma}}u_{\frac{N}{2}})\,, \\ F_N(0) = P_N\phi^{\omega}\,. \end{cases} \tag{4.3}$$

It encapsulates the *high-low-low* interactions at scale of order N. The high frequencies are carried by the solution F_N, whose initial data is the projection $P_N\phi^{\omega} = P_{\leq N}\phi^{\omega} - P_{\leq \frac{N}{2}}\phi^{\omega}$ of ϕ^{ω} at frequency N, so that we expect this property to stay true at least for small times. The low frequencies are carried by the projection of the solution $u_{\frac{N}{2}}$ at step $\frac{N}{2}$ onto the frequencies $|\eta| \leq N^{\gamma} \ll N$.

The nonlinear remainder w_N is solution to (NLS-HW) with a stochastic forcing term and zero initial condition:

$$\begin{cases} i\partial_t w_N + (\partial_{xx}^2 - |D_y|)w_N = \mathcal{N}\big(u_N\big) - \mathcal{N}\big(u_{\frac{N}{2}}\big) - \mathcal{N}\big(F_N, P_{\leq N^{\gamma}}u_{\frac{N}{2}}, P_{\leq N^{\gamma}}u_{\frac{N}{2}}\big)\,, \\ w_N(0) = 0\,. \end{cases}$$

Since we removed the high-low-low interactions from the stochastic forcing term, we expect that w_N exhibits probabilistic nonlinear smoothing.

The local existence of the smooth solution u_n is guaranteed by the local well-posedness result from Theorem 1.6 in [2]. However, the time of existence in say $\mathcal{H}^{\nu}$, for some $\nu > \frac{1}{2}$, depends on the $\mathcal{H}^{\nu}$-norm of u_n, therefore it depends on n. The strategy is to first modify slightly the equation for u_n to show the convergence of $(u_n)_{n\in\mathbb{N}}$, on a time interval $[-T_0, T_0]$ which does not depend on n. In order to get convergence on a fixed time interval, we follow [5] by making use of the *truncation method* from De Bouard and Debussche [16], which consists in truncating the terms involved in the nonlinearity so that they stay bounded on a long enough time interval. Then, we prove that on some random time interval $[-T^{\omega}, T^{\omega}]$, the limit of $(u_n)_n$ solves (NLS-HW) in $\mathcal{C}\big([-T^{\omega}\,,T^{\omega}];\mathcal{H}^{s}(\mathbb{R}^2)\big)$.

The key idea behind construction (4.3) is a probabilistic independence between F_N, whose initial data $P_N\phi^{\omega}$ only depends on $(g_n)_{\frac{N}{2}<n\leq N}$, and the solution at step $u_{\frac{N}{2}}$, whose

initial data $P_{\leq \frac{N}{2}}\phi^\omega$ only depends on $(g_n)_{n\leq \frac{N}{2}}$. As a consequence, one can show that these two objects remain decoupled for small times. We obtain *probabilistic Strichartz estimates* for small times, controlling the L^∞ norm of F_N with a loss $N^{\frac{\gamma}{2}-\sigma}$ ($\frac{\gamma}{2}-\sigma$ derivatives) instead of the expected loss $N^{\frac{1}{2}}$ ($\frac{1}{2}$ derivatives).

We stress out that we only implement the probabilistic scheme in the half-wave variable y, whereas a traditional deterministic analysis is performed in the Schrödinger variable x. Indeed, since we can exploit the dispersion materialized by the Strichartz estimates for the Schrödinger equation, there is no need to introduce random initial data in the x-variable. Also note that we use mixed Lebesgue spaces and a TT^*-type argument, which seem more adapted for this Schrödinger-type equation (NLS-HW) than the Gronwall inequalities and energy estimates implemented for the wave-type equation in [5].

5. **Perspectives**

To conclude this note, let us mention some open problems linked to equation (NLS-HW).

(1) Modified scattering for defocusing (NLS-HW) on $\mathbb{R}_x \times \mathbb{R}_y$, global solutions for smooth and decaying initial data, following [34].

(2) Growth of Sobolev norms for the defocusing half-wave equation (3.2). In particular it is open whether there exist solutions satisfying $\limsup_{t\to\infty} \|u(t)\|_{H^s} = +\infty$.

(3) Orbital stability or instability property for traveling waves of focusing (NLS-HW) on different geometries than the wave guide $\mathbb{R}_x \times \mathbb{T}_y$, for instance when the two spatial variables lie in $\mathbb{R}_x \times \mathbb{R}_y$, or in the wave guide $\mathbb{T}_x \times \mathbb{R}_y$.

(4) Probabilistic local well-posedness for other non-dispersive PDE.

The refined probabilistic ansatz developed in [12] for equation (NLS-HW) also applies to the half-wave equation on the line (3.2) by removing the variable x all throughout the paper. We believe that this strategy will also be successful to investigate other Schrödinger-type equations which lack dispersion. One example is the Schrödinger equation on the Heisenberg group in the radial case

$$i\partial_t u - \Delta_{\mathbb{H}^1} u = |u|^2 u, \qquad \text{(NLS-}\mathbb{H}^1\text{)}$$

where $\mathbb{H}^1$ is parameterized by three real coordinates $(x, y, s) \in \mathbb{H}^1$ for which the sub-Laplacian for radial functions takes on the from

$$\Delta_{\mathbb{H}^1} = \partial_{xx} + \partial_{yy} + (x^2 + y^2)\partial_{ss}.$$

This equation is a totally non dispersive equation [1]. As a consequence, the study of the Cauchy problem at low regularity is a delicate issue. Properties of the flow map are similar to equation (NLS-HW), and can be summarized in the following diagram in the scale of Sobolev spaces associated to the operator $\Delta_{\mathbb{H}^1}$.

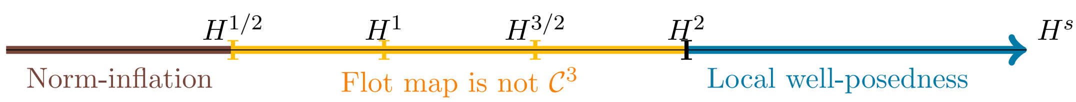

FIGURE 2. Deterministic Cauchy theory for equation (NLS-$\mathbb{H}^1$).

The situation is even less favorable in this context since the energy space, at regularity H^1, is far from the minimal exponent at which deterministic local well-posedness is known to hold. The refined probabilistic method presented in this note would relieve the penalization of the dispersionless direction that was necessary in [19] to tackle local well-posedness properties with random initial data for equation (NLS-$\mathbb{H}^1$).

(5) Reaching a global well-posedness theory for equation (NLS-HW), whether in the deterministic or the probabilistic case.

Using a deterministic approach, two difficulties are the following. First, one cannot use a Yudovich argument to get local existence in the energy space $\mathcal{H}^{\frac{1}{2}}$, because the L^q-norms are not controlled by the energy when $q > 6$. Then, in the defocusing case, the Brezis-Gallouët estimate fails to extend smooth solutions globally in time, since $\mathcal{H}^s$ is not an algebra when $s < \frac{3}{4}$, and there is no conservation law that controls the $\mathcal{H}^{\frac{3}{4}}$-norm. This is in contrast with the half-wave equation or the Szegő equation on the line.

From the point of view of random initial data, it would be challenging but interesting to understand the long-time behavior of the probabilistic solutions to equation (NLS-HW) generated by the paracontrolled decomposition. Indeed, we constructed in Theorem 4.1 a probabilistic solution in the presence of a conserved energy. However, this construction is not sufficient to globalize the solutions, since the probabilistic information propagated for short times is crucial in the iteration scheme. To understand how this information is transported by the flow on longer time scales, one could try to prove quasi-invariance of the probability measure, but the current techniques rely on dispersion properties of the equation, see for instance [29]. Hence, the study of the long-time behavior of solutions provided by the refined probabilistic ansatz would require a combination of energy methods with some measure theory arguments.

Bibliography

[1] H. Bahouri, P. Gérard, and C.-J. Xu. Espaces de Besov et estimations de Strichartz généralisées sur le groupe de Heisenberg. *Journal d'Analyse Mathématique*, 82(1):93–118, 2000.

[2] Y. Bahri, S. Ibrahim, and H. Kikuchi. Remarks on solitary waves and Cauchy problem for Half-wave-Schrödinger equations. *Communications in Contemporary Mathematics*, page 2050058, 2020.

[3] Y. Bahri, S. Ibrahim, and H. Kikuchi. Transverse Stability of Line Soliton and Characterization of Ground State for Wave Guide Schrödinger Equations. *Journal of Dynamics and Differential Equations*, pages 1–43, 2021.

[4] J. Bourgain. Invariant measures for the 2D-defocusing nonlinear Schrödinger equation. *Commun. Math. Phys.*, 176(2):421–445, 1996.

[5] B. Bringmann. Almost sure local well-posedness for a derivative nonlinear wave equation. *Internat. Math. Res. Notices*, 2021(11):8657–8697, 2021.

[6] B. Bringmann, Y. Deng, A. R. Nahmod, and H. Yue. Invariant Gibbs measures for the three dimensional cubic nonlinear wave equation. *arXiv preprint arXiv:2205.03893*, 2022.

[7] N. Burq, P. Gérard, and N. Tzvetkov. Bilinear eigenfunction estimates and the nonlinear Schrödinger equation on surfaces. *Invent. Math.*, 159:187–225, 2005.

[8] N. Burq, P. Gérard, and N. Tzvetkov. Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations. 38(2):255–301, 2005.

[9] N. Burq and N. Tzvetkov. Random data Cauchy theory for supercritical wave equations I: local theory. *Invent. Math.*, 173(3):449–475, 2008.

[10] N. Burq and N. Tzvetkov. Random data Cauchy theory for supercritical wave equations II: a global existence result. *Invent. Math.*, 173(3):477–496, 2008.

[11] N. Camps and L. Gassot. Pathological set of initial data for scaling-supercritical nonlinear Schrödinger equations. *Int. Math. Res. Not.*, 2022.

[12] N. Camps, L. Gassot, and S. Ibrahim. Refined probabilistic local well-posedness for a cubic Schr\" odinger half-wave equation. *arXiv preprint arXiv:2209.14116*, 2022.

[13] T. Cazenave. *Semilinear Schrödinger equations*. Courant lecture notes. American Mathematical Society, Providence, RI, 2003.

[14] M. Christ, J. Colliander, and T. Tao. Ill-posedness for nonlinear Schrodinger and wave equations. *arXiv preprint arXiv:math/0311048*, 2003.

[15] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation. *Inventiones mathematicae*, 181(1):39–113, 2010.

[16] A. de Bouard and A. Debussche. A stochastic nonlinear Schrödinger equation with multiplicative noise. *Comm. Math. Phys.*, 205(1):161–181, 1999.

[17] Y. Deng, A. Nahmod, and H. Yue. Invariant gibbs measures and global strong solutions for nonlinear schrödinger equations in dimension two. *arXiv:1910.08492*, 2019.

[18] Y. Deng, A. R. Nahmod, and H. Yue. Invariant Gibbs measure and global strong solutions for the Hartree NLS equation in dimension three. *J. Math. Phys.*, 62(3), 2021.

[19] L. Gassot and M. Latocca. Probabilistic local well-posedness for the Schrödinger equation posed for the Grushin Laplacian. *arXiv:2103.03560*, 2021.

[20] P. Gérard and S. Grellier. Effective integrable dynamics for a certain nonlinear wave equation. *Analysis and PDE*, 5, 10 2012.

[21] P. Gérard and S. Grellier. The cubic Szegő equation and Hankel operators. *Astérisque, Société mathématique de France, Paris*, 389, 2017.

[22] P. Gérard and A. Pushnitski. Unbounded Hankel operators and the flow of the cubic Szegő equation. *arXiv:2206.11543*, 2022.

[23] Z. Hani, B. Pausader, N. Tzvetkov, and N. Visciglia. Modified scattering for the cubic Schrödinger equation on product spaces and applications. In *Forum of mathematics, Pi*, volume 3. Cambridge University Press, 2015.

[24] I. Kato. Ill-posedness for the Half wave Schrödinger equation. *arXiv:2112.10326*, 2021.

[25] G. Lebeau. Non linear optic and supercritical wave equation. *Bull. Soc. R. Sci. Liege*, 2001.

[26] G. Lebeau. Perte de régularité pour les équations d'ondes sur-critiques. *Bull. SMF*, 133(1):145–157, 2005.

[27] H. Lindblad. A sharp counterexample to the local existence of low-regularity solutions to nonlinear wave equations. *Duke Math. J.*, 72:503–539, 1993.

[28] T. Oh. Remarks on nonlinear smoothing under randomization for the periodic KdV and the cubic Szegö equation. *Funkcialaj Ekvacioj*, 54(3):335–365, 2011.

[29] T. Oh, P. Sosoe, and N. Tzvetkov. An optimal regularity result on the quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation. *Journal de l'Ecole Polytechnique Mathématiques*, 5:793–841, 2018.

[30] O. Pocovnicu. Traveling waves for the cubic Szegő equation on the real line. *Analysis and PDE*, 4(3):379–404, 2011.

[31] O. Pocovnicu. First and second order approximations for a nonlinear wave equation. *Journal of Dynamics and Differential Equations*, 25(2):305–333, 2013.

[32] C. Sun and N. Tzvetkov. Concerning the pathological set in the context of probabilistic well-posedness. *Comptes Rendus. Math.*, 358(9-10):989–999, 2020.

[33] C. Sun and N. Tzvetkov. Refined probabilistic global well-posedness for the weakly dispersive NLS. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 213:91, 2021.

[34] C. Xi. In preparation, 2023.

[35] B. Xia. Generic Ill-posedness for wave equation of power type on three-dimensional torus. *Int. Math. Res. Not.*, 20:15533–15554, 2021.

[36] H. Xu. Unbounded Sobolev trajectories and modified scattering theory for a wave guide nonlinear Schrödinger equation. *Mathematische Zeitschrift*, 286(1):443–489, 2017.

Nicolas Camps, Louise Gassot, Slim Ibrahim
Université Paris-Saclay, Laboratoire de mathématiques d'Orsay, UMR 8628 du CNRS, Bâtiment 307, 91405 Orsay Cedex, France
Departement Mathematik und Informatik, Universität Basel, Spiegelgasse 1, 4051 Basel, Schweiz
Department of Mathematics and Statistics
University of Victoria, 3800 Finnerty Road, Victoria BC V8P 5C2, Canada
Pacific Institute for Mathematical Sciences, 4176-2207 Main Mall, Vancouver, BC V6T 1Z4, Canada
nicolas.camps@universite-paris-saclay.fr
louise.gassot@normalesup.org
ibrahims@uvic.ca