Journées *Équations aux dérivées partielles*
, -1
RT AEDP (CNRS)

Stability results for the nonlinear Schrödinger equation on Diophantine tori

Nicolas Camps

Résultats de stabilité pour l'équation de Schrödinger non linéaire sur des tores diophantiens

Résumé

Nous présentons deux résultats de stabilité en temps long pour l'équation de Schrödinger non linéaire posée sur des tores diophantiens. Ce proceeding pour la conférence "Journées Équations aux dérivées partielles" offre l'occasion de présenter des méthodes modernes à l'interface des équations aux dérivées partielles (EDP) et de la mécanique classique, actuellement en cours de développement pour étudier la dynamique d'EDP hamiltoniennes résonantes.

Abstract

We present two results of enhanced long-time stability for the nonlinear Schrödinger equation posed on rescaled tori with Diophantine properties. This proceeding for the conference "Journées Équations aux dérivées partielles" is the opportunity to give a glimpse on modern methods at the interface of PDEs and classical mechanics that are currently being developed to study the long-time dynamics of Hamiltonian resonant PDEs.

1. Background and formulation of the problem

Nonlinear dispersive equations exhibit intricate dynamics on compact surfaces, where dispersion is weaker and nonlinear resonances can generate small-scale of oscillations. We consider the cubic nonlinear Schrödinger equation (NLS), a fundamental model in the study of nonlinear waves and a rich mathematical object:

$$i\partial_t u + \Delta u = |u|^2 u\,. \tag{NLS}$$

Here, Δ is the Laplace–Beltrami operator associated with the underlying geometry. Solutions have infinite propagation speed, allowing waves to explore the space instantly. The interaction of the waves with geometry is governed by the spectrum of $-\Delta$ and the concentration properties of its eigenfunctions [15, 41].

We study the possible energy transfer towards small scales in the periodic setting, when NLS is posed on $\mathbb{T}^d$ with $d \geq 2$. Known examples of energy cascade stem from *nonlinear resonant interactions*. The analysis involves methods from PDEs and classical mechanics, including perturbation theory of integrable Hamiltonian systems, Arnold diffusion, and invariant measures. The fundamental question raised by Bourgain [13] is the following:

Does there exist a global solution $u \in C(\mathbb{R}; H^s(\mathbb{T}^d))$ to (NLS), for some $s > 1$, that experiences an infinite cascade in the sense that

$$\|u(0)\|_{H^s} \ll 1\,, \qquad \limsup_{t\to+\infty} \|u(t)\|_{H^s} = +\infty \qquad ?$$

2020 *Mathematics Subject Classification:* 35B34, 35B35, 35Q55, 37K45, 37K55.

In the compact setting, we are not able to answer this question at this day. While significant progress has been made over the past decades, the mechanisms driving the growth of high Sobolev norms $H^s(\mathbb{T}^d)$ — preserved by the linear evolution — are still not fully understood.

We would like to understand whether cascade mechanisms depend on the periodic boundary condition, and we consider the NLS equation posed on rescaled tori — in a sense made precise below. In these models, the underlying geometry is still Euclidean, and the eigenfunctions of the Laplace operator are still plane waves, convenient for nonlinear analysis. However, the frequencies are not necessarily integer-valued, and the resonance function exhibit different behaviors.

The two results we present, taken from [7, 16], illustrate modern techniques and show that the arithmetic properties of the tori parameters play a crucial role in the long-time dynamics of the solutions.

1.1. The cubic Schrödinger on $\mathbb{T}^d$

Let us set up the formalism and notations. Expanding the solution at a given time t in a Fourier series

$$u(t,x) = \sum_{n\mathbb{Z}^d} u_n(t)\, \mathrm{e}^{in\cdot x}\,, \quad u_n(t) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \mathrm{e}^{in\cdot x}\, u(t,x)\mathrm{d}x\,,$$

and identifying it with the sequence of its Fourier coefficients $u(t) = (u_n(t))_{n\in\mathbb{Z}^d}$, equation (NLS) has the Hamiltonian formulation

$$\dot{u}_n = i\frac{\partial H}{\partial \overline{u_n}}(u,\overline{u})$$

where the Hamiltonian functional is

$$\begin{aligned} H(u,\overline{u}) &= \frac{1}{2}\sum_n \lambda_n^2|u_n|^2 + \frac{1}{4}\int_{\mathbb{T}^d}|u|^4\mathrm{d}x \\ &= \frac{1}{2}\sum_n \lambda_n^2|u_n|^2 + \frac{1}{4}\sum_{n_1-n_2+n_3-n_4=0} u_{n_1}\overline{u_{n_2}}u_{n_3}\overline{u_{n_4}}\,. \end{aligned} \tag{1.1}$$

In the periodic case — when the torus $\mathbb{T}^d$ is not rescaled — the frequencies are

$$\lambda_n^2 = \|n\|^2 = n^\intercal n\,.$$

To better exploit dispersion, it is convenient to pull back the solution by the linear flow and adopt the following ansatz:

$$u(t,x) = \sum_{n\in\mathbb{Z}^d} a_n(t)\, \mathrm{e}^{-it\lambda_n^2 + in\cdot x}\,, \quad a_n(t) = \mathrm{e}^{it\lambda_n^2}\, u_n(t)\,.$$

We see here the direct connection between the time oscillations — the dispersion — and the eigenvalues of the Laplace operator. Equation (NLS) reduces to a system of time-dependent ODEs for the coefficients (a_n):

$$i\dot{a}_n(t) = \sum_{n_1-n_2+n_3=n} \mathrm{e}^{it\Omega(\vec{n})}\, a_{n_1}(t)\overline{a_{n_2}}(t)a_{n_3}(t)\,, \quad n\in\mathbb{Z}^d\,. \tag{1.2}$$

For a multi-index $\vec{n} = (n_1,n_2,n_3,n)\in(\mathbb{Z}^d)^4$, we define the resonance function by

$$\Omega(\vec{n}) = \lambda_{n_1}^2 - \lambda_{n_2}^2 + \lambda_{n_3}^2 - \lambda_n^2 = \|n_1\|^2 - \|n_2\|^2 + \|n_3\|^2 - \|n\|^2\,.$$

Under the zero momentum condition $n_1 - n_2 + n_3 = n$ it factorizes into

$$\Omega(\vec{n}) = (n_1-n_2)\cdot(n_2-n_3)\,.$$

The four-wave resonant set is the zero level set of the resonance function, constrained to the zero momentum condition. It corresponds to the set of rectangles in $\mathbb{Z}^d$:

$$\begin{aligned} \Gamma_0 &= \Big\{\vec{n} = (n_1,n_2,n_3,n_4)\in(\mathbb{Z}^d)^4 \ :\ n_1-n_2+n_3-n_4 = (n_1-n_2)\cdot(n_2-n_3) = 0\Big\} \\ &= \Big\{\text{rectangles in } \mathbb{Z}^d\Big\}\,. \end{aligned}$$

This set of frequency interactions governs the known energy cascade mechanisms, as detailed below. Outside the resonant set, the resonance function — which, in this particular case, takes integer values — does not vanish. Consequently, its magnitude is at least one, and the corresponding interactions oscillate in time. In principle, resonances dictate the dynamics — at least over finite time intervals. Several techniques exist to handle the non-resonant interactions that oscillate in time, and the following two normal form methods are discussed in this document:

- The Poincaré–Dulac normal form relies on integration by parts in time — or more precisely, derivation by parts. It allows gaining two orders by the Leibniz rule, when $\frac{d}{dt}$ hits $a_{n_1}\overline{a_{n_2}}a_{n_3}$ in (1.2): a quintic interaction replaces a cubic one. This is particularly useful in the small-data regime or when amplitudes decay in time, for instance on the unbounded domain $\mathbb{R}\times\mathbb{T}^d$ discussed in Section 2. The method is also relevant for constructing modified energies, with applications to the study of quasi-invariance [39] — transport of out of equilibrium Gaussian measures. This method can be seen as a mild version of Birkhoff normal form techniques.

- At the Hamiltonian level, Birkhoff normal forms eliminate non-resonant interactions through symplectic transformations of phase space, up to higher-order terms. Compared to the Poincaré–Dulac normal form, it has the advantage of preserving the Hamiltonian structure of the equation. This is crucial, as it retains more nonlinear properties beyond mere degree considerations. It enables the derivation of conserved quantities and slow variables — such as actions (3.1) or super-actions (2.4) — directly from the structure of the Hamiltonian functional.

1.2. Rescaled Diophantine tori

In the general case of rescaled tori, the situation is very different. When the ratio of lengths is irrational, the resonant system is much smaller and does not, by itself, allow energy to flow from large to small scales. Meanwhile, the resonance function, which is no longer integer-valued, can be arbitrarily close to zero without vanishing — these are the quasi-resonances, the main obstacles in developing normal form methods. To some extent, they encapsulate the complexity of the resonant system, which shifts towards a quasi-resonant system when the square torus is rescaled into general tori.

Given a symmetric definite positive matrix $\mathrm{A}\in S_d^{++}(\mathbb{R})$ we set

$$i\partial_t u + \operatorname{div}(\mathrm{A}\nabla)u = |u|^2 u\,. \tag{A-NLS}$$

The matrix A encodes a general dispersion relation, and the eigenvalues of $-\operatorname{div}(\mathrm{A}\nabla)$ are

$$\lambda_n^2 = n^{\intercal}\mathrm{A}n\,,\quad n\in\mathbb{Z}^d\,.$$

Given a multi-index $\vec{n}=(n_1,n_2,n_3,n_4)\in(\mathbb{Z}^d)^4$ satisfying the zero-momentum condition, the resonance function factorizes into

$$\Omega_{\mathrm{A}}(\vec{n}) = (n_1-n_2)^{\intercal}\mathrm{A}(n_2-n_3)\,.$$

This generalizes the case of $-\Delta$, where $\mathrm{A}=\mathrm{Id}$. Note that changing the dispersion amounts to rescaling $\mathbb{T}^d$, and that diagonal matrices A correspond to *rectangular tori*. For irrational rectangular tori — when (λ_n^2) are rationally independent — Staffilani and Wilson [42] observed that the resonant system is made of rectangles in $\mathbb{Z}^d$ only with sides parallel to the axes. The corresponding dynamics preserve Sobolev norms. See also [33] for numerical evidence of this enhanced stability property on irrational tori.

However, quasi-resonances can still lead to the amplification of high Sobolev norms, as shown by Giuliani–Guardia [27] when the coefficients are well approximated by rational numbers.

In [7], we study non-rectangular tori satisfying Diophantine conditions, ensuring that the resonance function cannot decay to zero faster than polynomially.

Definition 1 (Admissible tori)**.** A matrix $\mathrm{A}\in S_n^{++}(\mathbb{R})$ is admissible with a parameter $\tau>0$ if there exists $c>0$ such that for all $(a,b)\in(\mathbb{Z}^d\setminus\{0\})^2$,

$$|a^{\intercal}\mathrm{A}b| \geq \frac{c}{\|a\|^{\tau}\|b\|^{\tau}}\,. \tag{1.3}$$

When $\tau>\frac{d(d+1)}{2}$, the set of admissible tori has full Lebesgue measure. Under such a generic dispersion relation, there is less interference, and waves take longer to refocus compared to the square torus [24, 25]. Studying these models has applications in proving upper polynomial bounds for the growth of Sobolev norms [22, 23], as well as in the derivation of the kinetic wave equation [20]. In our framework, the key observation is that the 4-waves resonant set is trivial on admissible tori:

$$\Omega_{\mathrm{A}}(\vec{n}) = 0 \qquad \Longrightarrow \qquad \{n_1, n_3\} = \{n_2, n_4\}\,.$$

Hence, the 4-waves resonant system is integrable and does preserves the amplitudes $|u_n|^2$:

$$R_4(u) = \sum_{\vec{n}\in\Gamma_0} u_{n_1}\overline{u_{n_2}}u_{n_3}\overline{u_n} = 2\Big(\sum_{n\in\mathbb{Z}^d}|u_n|^2\Big)^2 - \sum_{n\in\mathbb{Z}^d}|u_n|^4 = 2\|u\|_{L^2(\mathbb{T}^d)}^4 - \sum_{n\in\mathbb{Z}^d}|u_n|^4\,.$$

By Gauging out the conserved L^2-mass, we are able to reduce the Hamiltonian functional to

$$H(u) = \sum_{n\in\mathbb{Z}^d}\lambda_n^2|u_n|^2 - \sum_{n\in\mathbb{Z}^d}|u_n|^4 + \text{ non-resonant terms of order } 4\,. \tag{1.4}$$

Since the resonant system preserves the amplitudes of individual Fourier modes — the *actions* (3.1) for the linear system, energy transfer can only occur through *quasi-resonant* interactions. The main challenge in the works [7, 16] presented in this proceeding is precisely to analyze these quasi-resonances. A key ingredient is a *frequency separation property* which is clear when $d = 1$: for all distinct $n_1, n_2 \in \mathbb{Z}$,

$$|n_1 - n_2| + |n_1^2 - n_2^2| \geq |n_1| + |n_2|\,. \tag{1.5}$$

This property no longer holds for pairwise frequencies when $d \geq 2$, but it does for frequencies organized according to a decomposition into clusters.

Lemme 2 (Frequency separation, Theorem 2.1 in [9])**.** *There exist* $c(d) \in (0, 2]$ *and a partition of* $\mathbb{Z}^d$ *into dyadic clusters*

$$\mathbb{Z}^d = \bigcup_{\alpha\geq 0}\mathscr{C}_\alpha\,,$$

with $\max_{n\in\mathscr{C}_\alpha}|n| \leq 2\min_{n\in\mathscr{C}_\alpha}|n|$ *when* $\alpha \geq 1$, *such that for all* $\alpha \neq \beta$,

$$(n_1, n_2) \in \mathscr{C}_\nu \times \mathscr{C}_\beta \qquad \Longrightarrow \qquad |n_1 - n_2| + |\lambda_{n_1}^2 - \lambda_{n_2}^2| \geq (|n_1| + |n_2|)^{c(d)}\,.$$

We will see that interactions in which the two highest frequencies belong to a same cluster govern the asymptotic dynamics. The existence of this cluster decomposition was proved by Bourgain for the square torus [11] and later generalized to all tori by Berti-Maspero [9]. It has applications to stability results for linear systems [21], and for NLS with a convolution potential [4].

1.3. Forward energy cascades

While Arnold diffusion mechanisms [1, 37] have been extensively studied in finite-dimensional Hamiltonian systems, energy cascades for nonlinear Schrödinger equations remain much less understood. Let us state two of the most striking known results. The first one concerns norm amplification.

Theorem 3 (Norm amplification on $\mathbb{T}^d$ [19])**.** *For all* $d \geq 2$, $s > 1$, $M > 0$, *and* $\varepsilon > 0$, *there exist* $T > 0$ *and* $u(0) \in H^s(\mathbb{T}^d)$ *such that*

$$\|u(0)\|_{H^s(\mathbb{T}^d)} \leq \varepsilon\,, \quad \|u(T)\|_{H^s(\mathbb{T}^d)} \geq M\,.$$

The proof consists of two main steps. First, constructing the so-called *Toy model* — see also [17, 30] — based on resonant interactions, which exhibits infinite cascades. Second, demonstrating that this system governs the dynamics of the full PDE at least up to the amplification time T. The key difficulty to construct the cascade in infinite time lies in understanding the behavior of u beyond this time. Later, [29] obtained quantitative upper bounds on T in terms of ε and M.

On unbounded waveguides $\mathbb{R}\times\mathbb{T}^d$, dispersion is much stronger and ensures that the effective dynamics is governed by the resonant system in infinite time.

Theorem 4 (Infinite cascade on waveguides $\mathbb{R}\times\mathbb{T}^d$ [31])**.** *For* $s \gg 1$ *and* $\varepsilon \ll 1$, *there exists* $u(0) \in H^s(\mathbb{R}\times\mathbb{T}^d)$ *with* $d \in \{2, 3, 4\}$ *such that the corresponding solution is global in* $H^s(\mathbb{R}\times\mathbb{T}^d)$ *and*

$$\|u(0)\|_{H^s} \leq \varepsilon\,, \quad \limsup_{t\to\infty}\|u(t)\|_{H^s} = +\infty\,.$$

In Section 2, we discuss Theorem 5, which shows that this behavior is unstable under domain perturbation. In Section 3, we present Theorem 6, which establishes that for typical data of size ε, there is no norm amplification over time scales of order ε^{-r} for any fixed $r \gg 1$.

The motivations are twofold. First, to determine whether cascade mechanisms persist when modifying the dispersion relation and when the complexity of the system is concentrated in quasi-resonances rather than exact resonances. Second, to develop refined techniques leveraging the Hamiltonian structure of the PDE for stability analysis, which remains the key challenge in addressing Bourgain's question.

2. Modified scattering on waveguides

The model is the NLS equation on the rescaled product space $(x, y) \in \mathbb{R} \times \mathbb{T}^d$

$$i\partial_t u + \Delta_{\mathrm{A}} u = |u|^2 u\,, \quad \Delta_{\mathrm{A}} := \partial_x^2 + \operatorname{div}_y(\mathrm{A}\nabla_y)\,, \tag{2.1}$$

with A admissible in the sense of Definition 1. Recall that Bourgain's question is about the possible growth of solutions with small amplitude at time zero. The small parameter $\varepsilon > 0$ is the size of the initial data:

$$\|u(0)\|_{H^s_{x,y}} + \|xu(0)\|_{L^2_{x,y}} \leq \varepsilon\,.$$

We consider the nonlinear profile:

$$f(t, x, y) = \mathrm{e}^{-it\Delta_{\mathrm{A}}}\, u(t, x, y)\,,$$

which is a solution to

$$i\partial_t f(t) = \mathrm{e}^{-it\Delta_{\mathrm{A}}}\left(\mathrm{e}^{it\Delta_{\mathrm{A}}}\, f(t)\, \mathrm{e}^{-it\Delta_{\mathrm{A}}}\, \overline{f(t)}\, \mathrm{e}^{it\Delta_{\mathrm{A}}}\, f(t)\right).$$

We do not expect the dispersion on $\mathbb{R} \times \mathbb{T}^d$ to be stronger than on the real line $\mathbb{R}$, where modified scattering to an explicit integrable dynamics is known [32, 35]. In this setting, solutions exhibit long-time decay similar to the free evolution:

$$\|u(t)\|_{L^\infty_x} \leq \frac{C(u(0))}{(1 + |t|)^{\frac{1}{2}}}\,.$$

Substituting this bound into the cubic interaction suggests that the interaction is long-range and scattering fails due to a logarithmic growth.

2.1. Modified scattering and infinite cascades on $\mathbb{R} \times \mathbb{T}^d$.

2.1.1. Space-time resonances

To isolate the effective dynamics, one must carefully analyze the trilinear interactions to identify the terms responsible for the logarithmic growth. Below, we retain only the *space-time resonant* terms that contribute to the effective dynamics, while denoting all other interactions by the generic symbol:

$$\mathcal{O}\left(\frac{1}{t^{1+}}\right).$$

Of course, obtaining such a decay factor requires detailed analysis and the propagation of suitable bounds. We only mention that the decaying mechanisms stem from time or space oscillations, and we refer to [26] for an introduction to the general method.

The strategy consists in capturing oscillations in the equation satisfied by the nonlinear profile in Fourier space. Given a function $f \in L^2(\mathbb{R} \times \mathbb{T}^d)$, we denote its full Fourier transform by $\widehat{f}_n(\xi)$, where $\xi \in \mathbb{R}$ and $n \in \mathbb{Z}^d$. At fixed t, linear changes of variables yield

$$i\partial_t \widehat{f}_n(t, \xi) = \sum_{n_1 - n_2 + n_3 = n} \int_{\mathbb{R}^2} \mathrm{e}^{it\Phi(\vec{n}, \sigma, \eta)}\, \widehat{f}_{n_1}(t, \xi - \sigma)\overline{\widehat{f}_{n_2}}(t, \xi - \eta - \sigma)\widehat{f}_{n_3}(t, \xi - \eta)\mathrm{d}\sigma\mathrm{d}\eta\,,$$

where the phase function is $\Phi(\vec{n}, \eta, \sigma) = \Omega(\vec{n}) + 2\sigma \cdot \eta$. For fixed $\vec{n} = (n_1, n_2, n_3, n_4) \in (\mathbb{Z}^d)^4$ the *space-resonant* set is

$$\left\{(\sigma, \eta) \in \mathbb{R}^2 \,:\, \nabla_{\sigma,\eta}\Phi(\vec{n}, \sigma, \eta) = 0\right\} = \left\{(0, 0)\right\}.$$

To handle space non-resonant interactions, the analysis largely follows [34]. See [16, Section 3.1] for the details. This reduces (2.1) to a Hamiltonian system in the transverse direction:

$$i\partial_t \widehat{f}_n(t,\xi) = \frac{\pi}{t} \sum_{n_1-n_2+n_3=n} e^{it\Omega(\vec{n})}\, \widehat{f}_{n_1}(t,\xi)\overline{\widehat{f}_{n_2}}(t,\xi)\widehat{f}_{n_3}(t,\xi) + \mathcal{O}\left(\frac{1}{t^{1+}}\right).$$

Note that the $\frac{1}{t}$ factor in front of the nonlinearity barely misses the integrability threshold: any additional decay $\frac{1}{t^{\delta}}$ would suffice to ensure integrability and convergence of the nonlinear profile, leading to scattering. To achieve this extra decay for certain interactions, we analyze the *time-resonances*:

$$\Gamma_0 = \left\{\vec{n} \in (\mathbb{Z})^4 \ : \ \Omega(\vec{n}) = 0\right\}.$$

In the case of the square torus — corresponding to $\mathrm{A} = \mathrm{Id}$ — the resonance function is integer valued. In particular, its amplitude is greater than one when it does not vanish and there is no small divisor issues. An elementary integration by parts — or a Poincaré–Dulac normal form — suffices to change the cubic nonlinearity into a quintic one, and to transform the time-oscillations into decay. Under some bootstrap conditions, this leads to

$$i\partial_t \widehat{f}_n(t,\xi) = \frac{1}{t} \sum_{(n_1,n_2,n_3)\in\Gamma_0(n)} \widehat{f}_{n_1}(t,\xi)\overline{\widehat{f}_{n_2}}(t,\xi)\widehat{f}_{n_3}(t,\xi) + \mathcal{O}\left(\frac{1}{t^{1+}}\right). \tag{2.2}$$

The first term captures the effective system, introducing a nonlinear contribution to the asymptotic dynamics. Notably, the effective system depends on ξ and, after a time rescaling to absorb the $\frac{1}{t}$ factor, reduces to the resonant part of NLS in the transverse direction $\mathbb{T}^d$.

2.1.2. Norms

The proof relies on the *Z-method*, which involves two norms. The weak norm Z, typically at lower regularity, is a conservation law for the effective system. The strong norm S, at the regularity of the initial data, may grow over time with the effective dynamics. However, modified scattering still holds in this norm: while the solution itself may grow, it converges in the strong norm to an asymptotic state satisfying the effective system. The conservation of the weak norm provides *global* control and a priori bounds.

2.1.3. Effective dynamics and stability

By constructing wave operators[1] one can transfer dynamics occurring in the resonant system in the transverse direction to the full equation.

As explained in the introduction, when $d \geq 2$, the resonant system provides complex dynamics such as energy cascades. By doing so, Hani–Pausader–Tzvetkov–Visciglia were able to deduce Theorem 4 from the Toy model in [19].

Theorem 4 provides the only known infinite cascade mechanism for the Schrödinger equation. Subsequent works followed the same strategy but with different models such as weakly dispersive[2] equations [18, 43] or systems that display beating effect [40]. For NLS on $\mathbb{R}\times\mathbb{T}$ or with a convolution potential [28] the resonant system is integrable and quasi-resonances are easy to absorb thanks to a separation property like (1.5). In such a case stability directly follows from the framework of [31].

2.2. The case of Diophantine waveguides

Let us now turn to NLS equation (2.1) posed on a waveguide whose transverse direction is an admissible Diophantine torus, as defined in Definition 1. In this case, quasi-resonant interactions play a role in the effective dynamics. We recall that the resonant system is integrable and the corresponding dynamics preserves the amplitudes of the Fourier coefficients. We proved that there is no norm amplification, in contrasts with Theorem 4.

[1] That is, an operator mapping a given state u_f to an initial datum $u(0)$, such that the solution starting from $u(0)$ asymptotically follows the dynamics of the effective system, which itself evolves from u_f as $t \to +\infty$.

[2] In these cases, only the construction of wave operators is achieved. Proving modified scattering via the Z-method, which requires propagating weak norms at the level of the conserved quantity $H^{\frac{1}{2}}$, remains a significant challenge.

Theorem 5 (Stability on Diophantine waveguides, [16]). *When* A *is admissible, there exists* $C > 0$ *such that for all small* $u(0)$*, the solution* u *to* (2.1) *is global and satisfies*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^s(\mathbb{R}\times\mathbb{T}^d)} \le C\|u(0)\|_{H^s(\mathbb{R}\times\mathbb{T}^d)}\,.$$

To establish this result, one must understand the role of *quasi-resonant* interactions. We now attempt a naive H^s-energy estimate for the quasi-resonant system to demonstrate that a Poincaré–Dulac normal form fails for certain interactions, thereby ruling out the possibility of applying the general strategy of [31] outlined in the previous section. For simplicity, we consider only a_n solutions to the quasi-resonant system in the transverse direction, namely those 4-wave interactions with

$$\Gamma_{\mathrm{q-res}} := \Big\{ \vec{n} \in (\mathbb{Z}^d)^4 \ : \ 0 < |\Omega_{\mathrm{A}}(\vec{n})| < 1\,, \quad n_1 - n_2 + n_3 - n = 0 \Big\}\,.$$

The time derivative of the $H^s(\mathbb{T}^d)$-norm of a solution to the quasi-resonant system is

$$\frac{1}{2}\sum_{n\in\mathbb{Z}^d} |n|^{2s}\frac{\mathrm{d}}{\mathrm{d}t}|a_n(t)|^2 = \frac{1}{4t}\,\mathrm{Im}\left(\sum_{\vec{n}\in\Gamma_{\mathrm{q-res}}} \psi_{2s}(\vec{n})\,\mathrm{e}^{it\Omega_{\mathrm{A}}(\vec{n})}\,a_{n_1}(t)\overline{a_{n_2}}(t)a_{n_3}(t)\overline{a_n}(t)\right), \tag{2.3}$$

with $\psi_{2s}(\vec{n}) := |n_1|^{2s} - |n_2|^{2s} + |n_3|^{2s} - |n|^{2s}$. Then we assume that the solutions to the above system are a priori uniformly bounded by $\mathcal{O}(\varepsilon)$ in $H^s(\mathbb{T}^d)$. In addition we impose that $a^{(i)}$ has frequency support localized around dyadic numbers N_i. For $i \in \{0,1,2,3\}$ and t,

$$\|a_n^{(i)}(t)\|_{\ell^2_n} \lesssim \varepsilon N_i^{-s}\,.$$

We can assume that $N_0 \sim N_1 \ge N_2 \ge N_3$ and deduce from assumption (1.3) and the mean-value theorem that

$$\vec{n} \in \Gamma_{\mathrm{q-res}} \quad \Longrightarrow \quad |\psi_{2s}(\vec{n})\Omega_{\mathrm{A}}(\vec{n})^{-1}| \lesssim N_1^{2(s-1)+2\tau}N_2^2\,.$$

The Poncaré–Dulac normal form, which converts oscillations into decay, consists in writing

$$\begin{aligned}\mathrm{e}^{it\Omega_{\mathrm{A}}(\vec{n})}\, & a_{n_1}^{(1)}(t)\overline{a_{n_2}}^{(2)}(t)a_{n_3}^{(3)}(t)\overline{a_n}^{(0)}(t)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{e}^{it\Omega_{\mathrm{A}}(\vec{n})}}{i\Omega_{\mathrm{A}}(\vec{n})}a_{n_1}^{(1)}\overline{a_{n_2}}^{(2)}a_{n_3}^{(3)}\overline{a_n}^{(0)}\right) - \frac{\mathrm{e}^{it\Omega_{\mathrm{A}}(\vec{n})}}{i\Omega_{\mathrm{A}}(\vec{n})}\frac{\mathrm{d}}{\mathrm{d}t}\left(a_{n_1}^{(1)}\overline{a_{n_2}}^{(2)}a_{n_3}^{(3)}\overline{a_n}^{(0)}\right)\end{aligned}$$

In the second term on the right-hand side, when the time derivative falls on $a_{n_1}^{(1)}\overline{a_{n_2}}^{(2)}a_{n_3}^{(3)}\overline{a_n}^{(0)}$, we use the equation and obtain six terms instead of four. The second term is a boundary term that requires controlling quantities of the form:

$$\begin{aligned}\sum_{\vec{n}\in\Gamma_{\mathrm{q-res}}} \frac{|\psi_{2s}(\vec{n})|}{|\Omega_{\mathrm{A}}(\vec{n})|}|a_{n_1}^{(1)}\overline{a_{n_2}}^{(2)}a_{n_3}^{(3)}\overline{a_n}^{(0)}| &\lesssim N_1^{2(s-1)+2\tau}N_2^2(N_0N_1)^{-s}(N_2N_3)^{\frac{d}{2}-s}\\ &\lesssim N_1^{2(\tau-1)}N_2^2(N_2N_3)^{\frac{d}{2}-s}\,.\end{aligned}$$

To close an energy estimate we need to get a negative power of N_1, which fails for high×low×low→low interactions: $N_0 \sim N_1 \gg N_2 \ge N_3$ even if $s \gg \frac{d}{2}$.[3]

The analysis needs to be more subtle. By combining the zero-momentum condition and the assumption $|\Omega_{\mathrm{A}}(\vec{n})| < 1$, one can prove that the high×low×low→low problematic case is excluded when the two highest modes satisfy a separation property like (1.5). Consequently, we decompose the interactions according to the cluster decomposition stated in Lemma 2. The Poincaré–Dulac normal form we have just described applies to the interactions where this separation holds.

The remaining high×low×low→low interactions, in which the two highest frequencies lie within the same cluster and only one of them carries a conjugation bar, govern the effective dynamics:

$$\Lambda := \left\{\vec{n} \in (\mathbb{Z}^d)^4 \ : \ \begin{array}{ll} n_1 - n_2 + n_3 = n\,, & |\Omega_{\mathrm{A}}(\vec{n})| < 1\,, \\ (n_1, n) \text{ in a same cluster } \mathscr{C}_\alpha\,, & n_2, n_3 \ll \min\limits_{m\in\mathscr{C}_\alpha} |m| \end{array}\right\}.$$

It is not difficult to prove that the system restricted to the frequency configurations in Λ preserves the following *super-actions*:

$$\mathcal{I}_\alpha(\widehat{u}(\xi)) := \sum_{n\in\mathscr{C}_\alpha} |u_n(\xi)|^2\,. \tag{2.4}$$

[3] Refinements using Strichartz inequality are possible, but can only gain with respect to the low frequencies N_2 and N_3.

Thanks to the dyadic structure of the clusters, it turns out that these constants of the motion can be combined to form a conserved quantity equivalent to the $H^s(\mathbb{T}^d)$-norm:

$$\|\widehat{u}(\xi)\|^2_{H^s(\mathbb{T}^d)} \sim \sum_{\alpha\geq 0} K_\alpha^{2s}\mathcal{I}_\alpha(\widehat{u}(\xi)), \quad K_\alpha := \min_{n\in\mathscr{C}_\alpha} 1+|n|$$

Note that the existence of a Z-norm — a conserved quantity for the effective dynamics — at regularity $H^s(\mathbb{T}^d)$ simplifies the stability analysis compared to the case of the square torus [31], where the Z-norm is at the level of $H^1(\mathbb{T}^d)$.

This analysis ultimately leads to a modified scattering result for small amplitude solutions [16, Theorem 1.4], from which Theorem 5 follows.

3. A Nekhoroshev stability result on Diophantine tori

We now return to the compact setting of (A-NLS) posed on $\mathbb{T}^d$. Due to the lack of dispersion, time oscillations do not generate sufficient decay to ensure stability over infinite time. However, it is still possible to establish stability results over *relatively long time scales*, depending on a small parameter ε — which may represent the size of the initial data or a coefficient in front of the nonlinearity. In fact, these time scales extend well beyond the *linear time scale* ε^{-2} before which nonlinear effects remain negligible. Such results fall under the framework of Nekhoroshev stability, named after fundamental work [38] on finite-dimensional Hamiltonian systems.

3.1. Perturbation Theory of integrable Hamiltonian systems

The linear Schrödiner equation, associated with the Hamiltonian

$$H_{\mathrm{Lin}}(u) = \sum_{n\in\mathbb{Z}^d} \lambda_n^2|u_n|^2,$$

is an integrable system corresponding to an infinite superposition of decoupled harmonic oscillators. The *actions* defined by

$$I_n := |u_n|^2, \tag{3.1}$$

are constant of the motion: for all $n\in\mathbb{Z}^d$ and t,

$$I_n(t) = I_n(0).$$

The angles $(\theta_n)_n$ — namely the phase of $u_n = |u_n|^2\,\mathrm{e}^{i\theta_n}$ — verify[4]

$$\theta_n(t) = \theta_n(0) + t\lambda_n^2,$$

so that the solutions to the linear system are quasi-periodic. Such a view-point makes a direct connection between PDE's and perturbation theory of integrable Hamiltonian systems. The nonlinear Schrödinger equation (A-NLS), whose Hamiltonian formulation is written in (1.1), is a perturbation of H_{Lin} in the small data regime. In our situation, A is admissible and the resonant system is integrable, so we can expect stability results over longer time-scales, and the amplitudes $|u_n(0)|^2$ to be slow variables. Nevertheless, due to the complexity of the quasi-resonant system materialized by resonant higher order terms in the Birkhoff normal form, diffusion mechanisms might occur for some initial data.

For *typical initial data* of size ε in $H^s(\mathbb{T}^d)$ — in a sense that will be made precise — we establish stability results over time scales ε^{-r}, for arbitrary r.

Theorem 6 (Nekhoroshev stability [7])**.** *Let $r\geq 2$ and $s\gtrsim r^2$. For sufficiently small $\varepsilon>0$, and for typical initial data $u(0)$ in $H^s(\mathbb{T}^d)$ with*

$$\|u(0)\|_{H^s} \leq \varepsilon,$$

the corresponding solution u to (A-NLS) with an admissible dispersion relation A remains in $H^s(\mathbb{T}^d)$ at least until a certain time $T_\varepsilon \gtrsim \varepsilon^{-r}$. Moreover, no norm amplification occurs:

$$\sup_{|t|\leq T_\varepsilon} \|u(t)\|_{H^s} \leq 2\|u(0)\|_{H^s}. \tag{3.2}$$

[4] In our framework we do not use the action-angle coordinates to avoid possible problems for vanishing Fourier modes

This rather informal statement of the main result in [7] deserves some explanations. The analysis involves a crucial frequency truncation parameter $N_\varepsilon = N(\varepsilon, r, s) \to +\infty$ as $\varepsilon \to 0$ with

$$N_\varepsilon \sim \varepsilon^{-\frac{r}{s}} .$$

We will see that small divisors induce losses of size $N_\varepsilon^s \sim \varepsilon^{-r}$ — which match the time scales we aim to reach — that are unacceptable, and will appeal for a very careful analysis.

Under the admissibility condition on A, we leverage the frequency separation property induced by the cluster structure to establish that the *high-frequency super-actions* remain stable until a time $T \gtrsim \varepsilon^{-r}$. This preliminary reduction to a finite-dimensional system via frequency truncation at scales $\leq N_\varepsilon$ is written in [7, Section 2].

The core of the analysis on the reduced system aims to establish that the *low-frequency actions* evolve as slow variables:

$$\sup_{|t|\leq T_\varepsilon} \sum_{|n|\leq N_\varepsilon} \langle n \rangle^{2s} ||u_n(t)|^2 - |u_n(0)|^2| \ll \varepsilon^2 .$$

Then, inequality (3.2) follows from the stability of the high-frequency super-actions. The proof of the above bound is done by *amplitude-frequency modulation* and an induction scheme on the frequency scales, which we outline in the next subsection.

The set of *typical initial data* leading to stable dynamics is an open set with large measure in an appropriate sense. Denoting the Fourier projector

$$\Pi_{N_\varepsilon} u(x) := \sum_{|n|\leq N_\varepsilon} u_n \, \mathrm{e}^{in\cdot x} ,$$

there exists an open set $\Theta_\varepsilon \subset \Pi_{N_\varepsilon} B_s(0, \varepsilon)$ with asymptotic full Lebesgue measure

$$\mathrm{meas}(\Theta_\varepsilon) \geq (1 - \varepsilon^{\frac{1}{40}}) \, \mathrm{meas}(\Pi_{N_\varepsilon} B_s(0, \varepsilon)) , \tag{3.3}$$

such that Theorem 6 holds for the initial data that satisfy

$$u(0) \in \Pi_{N_\varepsilon}^{-1} \Theta_\varepsilon .$$

There are no restrictions on the high Fourier modes of the initial data, but only on the modes below N_ε. This allows us to use the Lebesgue measure on a finite dimensional space to mimic the notion of *sets with an asymptotic full measure*, which is the standard way to state similar result for ODE systems. The restriction arises from the need to tune the initial data to satisfy some non-resonance conditions.

The Theorem 6 can be understood as an extension to an infinite dimensional setting of Nekhoroshev's [38] Theorem for a strongly resonant PDE. In finite dimension, this theorem states that stability holds for all initial data, over exponential time scales. Nevertheless, in the limit as the dimension d tends to $+\infty$ with $d \sim \log(\frac{1}{\varepsilon})$ Bourgain–Kaloshin [14] have constructed examples of Hamiltonian perturbations that lead to Arnold diffusion. This strongly suggests that the restriction to typical initial data in the PDE setting is necessary, as slow diffusion mechanisms might occur.

We conclude this discussion by highlighting the remarkable fact that certain completely integrable PDEs, such as the cubic NLS on $\mathbb{T}$ [2] or Benjamin–Ono [5], exhibit stability over exponential time scales for any data initially close to a finite-gap state.

3.2. Sketch of the proof: amplitude-frequency modulation

The analysis is at the level of the Hamiltonian — and not of the PDE – and consists in symplectic transformations which, in contrast with other methods, preserve the Hamiltonian structure. This is of particular importance when tracking refined constant of the motion: the actions — amplitudes of the Fourier coefficients — for integrable Hamiltonian, and super-actions — sum of the amplitudes of all the Fourier coefficient supported within a given cluster — for Hamiltonian that preserve the cluster structure [5]. Consider the projection $H_{N_\varepsilon} := H \circ \Pi_{N_\varepsilon}$ of the Hamiltonian formulation (1.4)

[5] Namely, when the monomials $u_{k_1} \cdots u_{k_p} \overline{u_{\ell_1}} \cdots \overline{u_{\ell_p}}$ contain as many terms with and without the bar in each cluster: for all $\alpha \geq 0$, $\mathrm{Card}\{j \in \{1, \cdots, p\} : k_j \in \mathscr{C}_\alpha\} = \mathrm{Card}\{j \in \{1, \cdots, p\} : \ell_j \in \mathscr{C}_\alpha\}$

of (A-NLS):

$$H_{N_\varepsilon}(u) = \sum_{|n|\le N_\varepsilon} \lambda_n^2 |u_n|^2 - \sum_{|n|\le N_\varepsilon} |u_n|^4 \quad + \quad \text{non-resonant terms of order } 4\,.$$

We use the Diophantine condition (1.3) on A to perform a preliminary Birkhoff normal form in finite dimension. This consist in constructing a symplectic transformation, close to the identity, such that in the new coordinates, the non-resonant terms of order 4 are transformed — up to a remainder of size $\mathcal{O}(\varepsilon^r)$ — into terms of order greater than or equal to 6 that are possibly resonant. In the new coordinates, the Hamiltonian takes the form:

$$\widetilde{H}_{N\varepsilon}(u) = \sum_{|n|\le N_\varepsilon} \lambda_n^2 |u_n|^2 - \sum_{|n|\le N_\varepsilon} |u_n|^4 \quad + \quad \text{terms of order } \ge 6\,. \tag{3.4}$$

This operation gains a power ε^2 in terms of homogeneity and provides a way to prove stability beyond the linear time scale. This procedure is the exact analog, in a Hamiltonian framework, of the Poincaré–Dulac normal form presented in Section 2.

At the heart of the Birkhoff normal form procedure lies the construction of an auxiliary Hamiltonian, whose associated flow-map at time 1 defines the symplectic change of coordinates. This auxiliary Hamiltonian χ is solution to a homological equation:

$$\{Z_2, \chi\} = Q\,,$$

where $Z_2(u) = \sum_{|n|\le N_\varepsilon} \lambda_n^2 |u_n|^2$ is the quadratic part of the Hamiltonian and $Q(u)$ is a term we want to remove. Since the Poisson bracket with Z_2 acts on the monomials like a diagonal operator:

$$\{Z_2, u_{n_1}\overline{u_{n_2}}\cdots\overline{u_{n_{2q}}}\} = 2i\Omega_{\mathrm{A}}(\vec{n}) u_{n_1}\overline{u_{n_2}}\cdots\overline{u_{n_{2q}}}\,,$$

the formal solution to the homological equation is given by

$$\chi(u) = \sum_{\vec{n}} \frac{Q_{\vec{n}}}{2i\Omega_{\mathrm{A}}(\vec{n})} u_{n_1}\overline{u_{n_2}}\cdots\overline{u_{n_{2q}}} \quad \text{when} \quad Q(u) = \sum_{\vec{n}} Q_{\vec{n}} u_{n_1}\overline{u_{n_2}}\cdots\overline{u_{n_{2q}}}\,.$$

This procedure involves the inverse of the resonance function, and small divisor problems arise. In particular, it is not possible to eliminate the resonant terms, which will stay in the normal form. Returning to our problem, it is *a priori* not possible to gain another order due to potential resonances of degree 6.

To reach higher orders, we must modulate — or perturb — the frequencies (λ_n) to make them non-resonant. A classical approach is to introduce and adjust external parameters in the system: the mass for Klein–Gordon equations [3], the capillarity for water-wave equations [10], and a Fourier multiplier $V * u$ for Schrödinger equations, where no natural parameter arises. By properly choosing V, one not only removes resonances but also ensures acceptable small divisor bounds. We refer to [6] for the case of the square torus and to the recent work [4] for general dispersion relations, where the cluster structure was also utilized.

We discuss alternative approaches — much less developed — that allow for proving long-time stability results for the original equation without introducing and tuning external parameters. These methods date back to the works of Kuksin–Pöschel [36] and Bourgain in [12] for NLS on the line, who refers to the strategy as *amplitude frequency modulation*.

Starting from the normal form (3.4) at order four, this method consists in recentering the amplitudes of the Fourier coefficients — expected to be slow variables — around their initial values:

$$\xi_n = |u_n(0)|^2.$$

This re-centering, or linearization, of the integrable quartic terms has the effect of introducing new quadratic terms that modulate the frequencies of the system:

$$|u_n|^4 = (|u_n|^2 - \xi_n)^2 - 2\xi_n |u_n|^2 + \xi_n^2\,,$$

so that the new quadratic term takes the form

$$\widetilde{Z}_2(u) = \sum_{|n|\le N} \omega_n(\xi)(|u_n|^2 - \xi_n)\,, \qquad \omega_n(\xi) = \lambda_n^2 - 2\xi_n\,.$$

This shows how to modulate the frequencies by using the initial data together with the nonlinearity. Nevertheless, the correction of λ_n^2 is of magnitude $|n|^{-2s}\varepsilon^2$ since $u(0)$ is of size ε in $H^s(\mathbb{T}^d)$. Given a multi-index $\vec{n}$, one needs to tune ξ_{n_0} with n_0 the smallest index to get the largest possible correction. For *typical* choices of ξ we expect that the modulated resonance function satisfies

$$|\Omega_{\mathrm{A}}(\xi;\vec{n})| \gtrsim \varepsilon^2 |\min(\vec{n})|^{-2s} .$$

This leads to very small divisor issue when $\min(\vec{n}) \sim N$, in which case the loss is $\sim \varepsilon^{-2r}$: the system is *strongly resonant.*

A different strategy, developed in [8] for NLS on $\mathbb{T}$ to reprove Bourgain's result [12], is the *rational normal form.* This approach exploits the explicit expression of integrable terms of degree six to obtain large corrections through non-local terms, which always involve ξ_{n_0} with a small index n_0, leading to acceptable small divisor losses. While effective in the 1D case, this method is less natural for general *strongly resonant* systems, where such corrections are not readily accessible.

Instead, we have developed and extended Bourgain's *amplitude-frequency modulation* approach from [12]. This method is particularly flexible and well-suited for handling strongly resonant systems with small divisors, as in (3.5), without relying on the explicit structure of integrable higher-order terms in the Birkhoff normal form.

3.3. Normal form by induction on the frequency scales

Let us introduce some notations to keep track of the terms under the form of an action in the Hamiltonian. To a monomial $u_{n_1}\overline{u_{n_2}}\cdots\overline{u_{n_{2q}}}$ corresponds a unique sequence $\mathbf{n} = (\mathbf{k},\mathbf{l},\mathbf{m}) \in (\mathbb{N}^{\mathbb{Z}^d})^3$ such that, for $n \in \mathbb{Z}^d$, $\mathbf{n}(n) := (k_n, \ell_n, m_n)$ are the respective multiplicities of $u_n, \overline{u_n}$ and $|u_n|^2$ in the monomial, with the restrictions that

$$\sum_{n\in\mathbb{Z}^d} k_n = \sum_{n\in\mathbb{Z}^d} \ell_n \lesssim r\,, \qquad \sum_{n\in\mathbb{Z}^d} nk_n = \sum_{n\in\mathbb{Z}^d} n\ell_n\,, \qquad k_n\ell_n = 0 \quad \text{for all} \quad n\,.$$

The first restriction encodes the reality condition, and the fact that we only consider monomials of order $\lesssim r$, and the second condition the zero momentum condition. The third one imposes the important non-pairing condition.

We also impose the frequency truncation assumption: $k_n = \ell_n = m_n = 0$ whenever $|n| > N$. Then, $\mathcal{N}$ denotes the set of all sequences $\mathbf{n}$ satisfying the above conditions, and we consider polynomial written under the form

$$H(u) = \sum_{n\in\mathcal{N}} H_{\mathbf{n}} z_{\mathbf{n}}(u,|u|^2)\,,$$

where, given $\mathbf{n}\in\mathcal{N}$,

$$z_{\mathbf{n}}(u,|u|^2) := \prod_{|n|\le N} u_n^{k_n}\overline{u_n}^{\ell_n}(|u_n|^2)^{m_n}\,.$$

Recentering the actions around the parameter $\xi = (\xi_n)$ yields

$$H(u) = \sum_{n\in\mathcal{N}} H_{\mathbf{n}}[\xi] z_{\mathbf{n}}(u,|u|^2-\xi)\,.$$

Given a vector of modulated frequencies $\omega(\xi) = (\omega_n(\xi))_{n\in\mathbb{Z}^d}$, the resonance function can be reformulated by:

$$\Omega_{\mathbf{n}}(\omega(\xi)) = \sum_{n\in\mathbb{Z}^d} \omega_n(\xi)(k_n - \ell_n) = \omega(\xi)\cdot(\mathbf{k}-\mathbf{l})\,.$$

We suppose that ξ is tuned to ensure the expected small-divisor bound

$$|\Omega_{\mathbf{n}}(\omega(\xi))| \gtrsim \varepsilon^2(\mathbf{n}_-)^{2s}\,, \tag{3.5}$$

where $\mathbf{n}_- := \min\{|n| \,:\, k_n\ell_n \ne 0\}$ is the size of the smallest non-paired index in the monomial. The small divisor problem occurs for monomials with high frequencies $\mathbf{n}_- \sim N_\varepsilon$.

For these monomials, however, the dynamics is stable over time scales ε^{-r}. Indeed, if a monomial $Q(u) = u_{n_1}\overline{u_{n_2}}u_{n_3}\overline{u_{n_4}}u_{n_5}\overline{u_{n_6}}$ — possibly resonant — comes with only high-frequencies $n_i \sim N_\varepsilon$, then one has the vector field estimate:

$$\|\nabla Q(u)\|_{H^s} \lesssim N_\varepsilon^{-4s}\|u\|_{H^s}^5 \lesssim \varepsilon^r N_\varepsilon^{-2s}\|u\|_{H^s}^5\,.$$

These heuristics suggest that monomials with Fourier support at high frequencies do not contribute to norm amplification over time scales ε^{-r}, whereas monomial with small $\mathbf{n}_- = \mathcal{O}(1)$ — at least one u_n with $n = \mathcal{O}(1)$ does not come under the form of an action — are easier to remove. It is therefore natural to proceed *by induction on the frequency scales*: consider a sequence of frequency scales: $N_1 < \cdots < N_\alpha < N_{\alpha+1} < \cdots < N_\beta$ with

$$N_1 = \mathcal{O}(1)\,, \quad \left(\frac{N_{\alpha+1}}{N_\alpha}\right)^{2s} = \varepsilon^{-\frac{1}{100}}\,, \quad N_\beta \sim N_\varepsilon^{1-}\,.$$

For fixed ξ, we construct by induction on $\alpha \in \{1, \cdots, \beta\}$ a symplectic change of coordinates Φ_ξ^α, such that in the new coordinates

$$H \circ \Phi_\xi^\alpha(u) = \sum_{|n| \le N} \omega_n^\alpha(\xi)(|u_n|^2 - \xi_n) + Z_4^\alpha(\xi; u) + Q^\alpha(\xi; u) + R^\alpha(\xi; u)\,.$$

In the above *normal form at scale* α, the modulated frequencies are of the form

$$\omega_n^\alpha(\xi) = \lambda_n + \xi_n + h_n^\alpha(\xi)\,, \quad |\nabla_\xi h_n^\alpha(\xi)| = \mathcal{O}(\varepsilon).$$

The term Z_4^α collects the quartic integrable terms and does not contribute to the dynamics of the amplitudes $|u_n|^2$. The Hamiltonian Q^α collects monomials of order greater than six once re-centered that are either integrable, or associated to multi-indices in which the unpaired indices are larger than N_α. In the later case they *operates at scale* α:

$$\|\nabla Q^\alpha(u)\|_{H^s} \lesssim N_\alpha^{-4s}\|u\|_{H^s}^5. \tag{3.6}$$

We emphasize the crucial property that no non-integrable term of order 4 is generated during the induction. Note also that the construction is global in ξ within a ball of radius ε. However, the last term is a remainder $R^\alpha(\xi; u) = \mathcal{O}(\varepsilon^r)$ only when ξ is non-resonant — when (3.5) holds — and when u lies in a suitable neighborhood depending on ξ.

In the proof, we define appropriate norms on the coefficients $H_{\mathbf{n}}[\xi]$. These norms are designed to quantitatively encode all the key properties mentioned informally.

When passing from scale N_α to $N_{\alpha+1}$, we must propagate the bounds to the newly generated terms in the Hamiltonian. These terms can be expressed as iterated Poisson brackets between the auxiliary Hamiltonian χ and the Hamiltonian in the old coordinates.

Let us consider an important exemple. Given a monomial $z_{\mathbf{n}}$ that we remove thanks to the auxiliary Hamiltonian $\chi = \frac{1}{2i\Omega_{\mathbf{n}}(\xi)} z_{\mathbf{n}}$, the Poisson bracket with a quartic integrable term produces the new term

$$\begin{aligned}&\{\chi_{\mathbf{n}}(\xi;\cdot), (|u_{n_1}|^2 - \xi_{n_1})(|u_{n_2}|^2 - \xi_{n_2})\}(u)\\ &\qquad = (k_{n_1} - \ell_{n_1})(|u_{n_2}|^2 - \xi_{n_2})\chi_{\mathbf{n}}(\xi; u) + (k_{n_2} - \ell_{n_2})(|u_{n_1}|^2 - \xi_{n_1})\chi_{\mathbf{n}}(\xi; u)\,.\end{aligned}$$

Suppose that $\mathbf{n}_- \sim N$, $|n_2| = \mathcal{O}(1)$ and $k_{n_1} = 1$ and $k_{n_2} = \ell_{n_2} = \ell_{n_1} = 0$. The new term reduces to

$$\{\chi_{\mathbf{n}}(\xi;\cdot), (|u_{n_1}|^2 - \xi_{n_1})(|u_{n_2}|^2 - \xi_{n_2})\}(u) = \chi_{\mathbf{n}}(\xi; u)(|u_{n_2}|^2 - \xi_{n_2})\,.$$

Without further information, we expect $||u_{n_2}|^2 - \xi_{n_2}| \sim \varepsilon^2$. Hence, the coefficient in front of the monomial $z_{\mathbf{n}}$ to remove is of magnitude

$$\frac{1}{|\Omega_{\mathbf{n}}(\xi)|}||u_{n_2}|^2 - \xi_{n_2}| \sim \varepsilon^{-2}\mathbf{n}_-^{2s} \times \varepsilon^2 \sim N_\varepsilon^{2s}\,.$$

This is way too large. Instead, in order to get a convergent scheme, the strategy found by Bourgain in [12] is to propagate a *slow-variable condition*. Given a scale N_α and a fixed parameter ξ, we restrict the analysis to solutions u in the neighborhood

$$\mathcal{V}_{\alpha,s}(\varepsilon,\xi) = \left\{u \in \Pi_{N_\varepsilon} B_s(\varepsilon)\ :\ \sum_n \langle n\rangle^{2s}||u_n|^2 - \xi_n| \le \varepsilon^{2+\frac{1}{5}} N_\alpha^{-2s}\right\}\,.$$

The above condition ensures the uniform bound $||u_n|^2 - \xi_n| \le N_\alpha^{-2s}\varepsilon^2$, even if $|n| \ll N_\alpha$ is a low index. Note that if a Hamiltonian operates at scale N_α in the sense of (3.6) then its corresponding flow preserves $\mathcal{V}_{\alpha,s}(\varepsilon,\xi)$ over the time scales ε^{-r} we aim to reach. In the above example, if $\mathbf{n}_- = N_{\alpha+1}$ we get

$$\frac{1}{|\Omega_{\mathbf{n}}(\omega^\alpha(\xi))|}||u_{n_2}|^2 - \xi_{n_2}| \sim \varepsilon^{-2}\left(\frac{N_{\alpha+1}}{N_\alpha}\right)^{2s}\varepsilon^{2+\frac{1}{5}} \lesssim \varepsilon^{\frac{1}{10}}\,,$$

which provides a gain $\varepsilon^{\frac{1}{10}}$ in front of $z_{\mathbf{n}}$. Iterating this procedure until the gain in homogeneity reaches $\mathcal{O}(\varepsilon^r)$ allows to eventually remove the contribution of $z_{\mathbf{n}}$ to the dynamics until time $\sim \varepsilon^{-r}$.

3.4. Description of the set made of stable initial data

At each step α of the induction, we impose additional non-resonance conditions on the (ξ_n)'s. This yields an open set of asymptotic full measure consisting of non-resonant parameters for which we obtain the desired normal form. However, the dynamics associated with the normal form — in the new coordinates — can only be described in the small neighborhood $\mathcal{V}_{\beta,s}(\varepsilon,\xi)$ of size N_α^{-2s}, which is much smaller than the precision of the symplectic transformation. As a consequence, ξ_n, which should be close to $|v_n(0)|^2$ where v is the solution in the new coordinates, is not necessarily close to $|u_n(0)|^2$.

Without additional ingredients, the admissible set of initial data $u(0)$ remains difficult to characterize beyond the fact that it is open and non-empty. We overcome this limitation by propagating Lipschitz regularity with respect to ξ on the transformations and the Hamiltonian functionals. This allows us to construct a bi-Lipschitz transformation close to the identity, which maps the well-understood set of parameters ξ to the set of admissible initial data Θ_ε, leading to the measure estimate (3.3).

For precise statements and proofs, we refer to [7, Section 7]. Clarifying and quantifying this aspect of the result represents a significant refinement compared to Bourgain's original work [12].

References

[1] V. I. Arnol'd, "Instability of dynamical systems with several degrees of freedom", *Sov. Math., Dokl.* **5** (1964), p. 581-585.

[2] D. Bambusi, "Nekhoroshev theorem for small amplitude solutions in nonlinear Schrödinger equations", *Math. Z.* **230** (1999), no. 2, p. 345-387.

[3] D. Bambusi, J.-M. Delort, B. Grébert & J. Szeftel, "Almost global existence for Hamiltonian semilinear Klein-Gordon equations with small Cauchy data on Zoll manifolds", *Commun. Pure Appl. Math.* **60** (2007), no. 11, p. 1665-1690.

[4] D. Bambusi, R. Feola & R. Montalto, "Almost global existence for some Hamiltonian PDEs with small Cauchy data on general tori", *Commun. Math. Phys.* **405** (2024), no. 1, article no. 15 (50 pages).

[5] D. Bambusi & P. Gérard, "Derivation of the homogeneous kinetic wave equation: longer time scales", 2023, https://arxiv.org/abs/2312.02833.

[6] D. Bambusi & B. Grébert, "Birkhoff normal form for partial differential equations with tame modulus", *Duke Math. J.* **135** (2006), p. 507-567.

[7] J. Bernier & N. Camps, "Long time stability for cubic nonlinear Schrödinger equations on non-rectangular flat tori", 2024, https://arxiv.org/abs/2402.04122.

[8] J. Bernier, E. Faou & B. Grébert, "Rational normal forms and stability of small solutions to nonlinear Schrödinger equations", *Ann. PDE* **6** (2020), no. 2, article no. 14 (64 pages).

[9] M. Berti & A. Maspero, "Long time dynamics of Schrödinger and wave equations on flat tori", *J. Differ. Equations* **267** (2019), no. 2, p. 1167-1200.

[10] M. Berti, A. Maspero & F. Murgante, "Hamiltonian Birkhoff normal form for gravity-capillary water waves with constant vorticity: almost global existence", *Ann. PDE* **10** (2024), no. 2, article no. 22 (150 pages).

[11] J. Bourgain, "On growth of Sobolev norms in linear Schrödinger equations with smooth time dependent potential", *J. Anal. Math.* **77** (1999), p. 315-348.

[12] ———, "On diffusion in high-dimensional Hamiltonian systems and PDE", *J. Anal. Math.* **80** (2000), p. 1-35.

[13] ———, "Problems in Hamiltonian PDE's", in *GAFA 2000. Visions in mathematics—Towards 2000. Proceedings of a meeting, Tel Aviv, Israel, August 25–September 3, 1999. Part I.*, Birkhäuser, 2000, p. 32-56.

[14] J. Bourgain & V. Kaloshin, "On diffusion in high-dimensional Hamiltonian systems", *J. Funct. Anal.* **229** (2005), no. 1, p. 1-61.

[15] N. Burq, P. Gérard & N. Tzvetkov, "Bilinear eigenfunction estimates and the nonlinear Schrödinger equation on surfaces", *Invent. Math.* **159** (2005), no. 1, p. 187-223.

[16] N. Camps & G. Staffilani, "Modified scattering for the cubic Schrödinger equation on diophantine waveguides", 2024, https://arxiv.org/abs/2404.16817.

[17] R. Carles & E. Faou, "Energy cascades for NLS on the torus", *Discrete Contin. Dyn. Syst.* **32** (2012), no. 6, p. 2063-2077.

[18] X. Chen, "Existence of modified wave operators and infinite cascade result for a half wave Schrödinger equation on the plane", *J. Funct. Anal.* **286** (2024), no. 2, article no. 110222 (74 pages).

[19] J. E. Colliander, M. Keel, G. Staffilani, H. Takaoka & T. Tao, "Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation", *Invent. Math.* **181** (2010), no. 1, p. 39-113.

[20] C. Collot & P. Germain, "Derivation of the homogeneous kinetic wave equation: longer time scales", 2020, https://arxiv.org/abs/2007.03508.

[21] J.-M. Delort, "Growth of Sobolev norms for solutions of time dependent Schrödinger operators with harmonic oscillator potential", *Commun. Partial Differ. Equations* **39** (2014), no. 1, p. 1-33.

[22] Y. Deng, "On growth of Sobolev norms for energy critical NLS on irrational tori: small energy case", *Commun. Pure Appl. Math.* **72** (2019), no. 4, p. 801-834.

[23] Y. Deng & P. Germain, "Growth of solutions to NLS on irrational tori", *Int. Math. Res. Not.* **2019** (2019), no. 9, p. 2919-2950.

[24] Y. Deng, P. Germain & L. Guth, "Strichartz estimates for the Schrödinger equation on irrational tori", *J. Funct. Anal.* **273** (2017), no. 9, p. 2846-2869.

[25] Y. Deng, P. Germain, L. Guth & S. L. R. Myerson, "Strichartz estimates for the Schrödinger equation on non-rectangular two-dimensional tori", *Am. J. Math.* **144** (2022), no. 3, p. 701-745.

[26] P. Germain, "Space-time resonances", *Journ. Équ. Dériv. Partielles* (2010), article no. 8 (10 pages).

[27] F. Giuliani & M. Guardia, "Sobolev norms explosion for the cubic NLS on irrational tori", *Nonlinear Anal., Theory Methods Appl.* **220** (2022), article no. 112865 (25 pages).

[28] B. Grébert, E. Paturel & L. Thomann, "Modified scattering for the cubic Schrödinger equation on product spaces: the nonresonant case", *Math. Res. Lett.* **23** (2016), no. 3, p. 841-861.

[29] M. Guardia & V. Kaloshin, "Growth of Sobolev norms in the cubic defocusing nonlinear Schrödinger equation", *J. Eur. Math. Soc.* **17** (2015), no. 1, p. 71-149.

[30] Z. Hani, “Long-time instability and unbounded Sobolev orbits for some periodic nonlinear Schrödinger equations”, *Arch. Ration. Mech. Anal.* **211** (2014), no. 3, p. 929-964.

[31] Z. Hani, B. Pausader, N. Tzvetkov & N. Visciglia, “Modified scattering for the cubic Schrödinger equation on product spaces and applications”, *Forum Math. Pi* **3** (2015), article no. e4 (63 pages).

[32] N. Hayashi & P. I. Naumkin, “Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations”, *Am. J. Math.* **120** (1998), no. 2, p. 369-389.

[33] A. Hrabski, Y. Pan, G. Staffilani & B. Wilson, “Energy transfer for solutions to the nonlinear Schrödinger equation on irrational tori”, 2021, https://arxiv.org/abs/2107.01459.

[34] B. B. Kadomtsev & V. I. Petviashvili, “On the Stability of Solitary Waves in Weakly Dispersing Media”, *Sov. Phys., Dokl.* **15** (1970), p. 539-541.

[35] J. Kato & F. G. Pusateri, “A new proof of long-range scattering for critical nonlinear Schrödinger equations”, *Differ. Integral Equ.* **24** (2011), no. 9-10, p. 923-940.

[36] S. Kuksin & J. Pöschel, “Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation”, *Ann. Math. (2)* **143** (1996), no. 1, p. 149-179.

[37] P. Lochak, “Arnold diffusion; a compendium of remarks and questions”, in *Hamiltonian systems with three or more degrees of freedom. Proceedings of the NATO Advanced Study Institute, S’Agaró, Spain, June 19–30, 1995*, Kluwer Academic Publishers, 1999, p. 168-183.

[38] N. N. Nekhoroshev, “An exponential estimate of the time of stability of nearly-integrable Hamiltonian systems”, *Russ. Math. Surv.* **32** (1977), no. 6, p. 1-65.

[39] T. Oh, P. Sosoe & N. Tzvetkov, “An optimal regularity result on the quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation”, *J. Éc. Polytech., Math.* **5** (2018), p. 793-841.

[40] V. V. D. Rocha, “Modified scattering and beating effect for coupled Schrödinger systems on product spaces with small initial data”, *Trans. Am. Math. Soc.* **371** (2019), no. 7, p. 4743-4768.

[41] C. D. Sogge, “Oscillatory integrals and spherical harmonics”, *Duke Math. J.* **53** (1986), p. 43-65.

[42] G. Staffilani & B. Wilson, “Stability of the cubic nonlinear Schrodinger equation on an irrational torus”, *SIAM J. Math. Anal.* **52** (2020), no. 2, p. 1318-1342.

[43] H. Xu, “Unbounded Sobolev trajectories and modified scattering theory for a wave guide nonlinear Schrödinger equation”, *Math. Z.* **286** (2017), no. 1-2, p. 443-489.

Nicolas Camps
IRMAR - UMR CNRS 6625
Univ Rennes
F-35000 Rennes
FRANCE
nicolas.camps@univ-rennes.fr