TROPICAL WEIERSTRASS POINTS AND WEIERSTRASS WEIGHTS

OMID AMINI, LUCAS GIERCZAK, AND HARRY RICHMAN

Abstract. In this paper, we study tropical Weierstrass points. These are the analogues for tropical curves of ramification points of line bundles on algebraic curves.

For a divisor on a tropical curve, we associate intrinsic weights to the connected components of the locus of tropical Weierstrass points. This is obtained by analyzing the slopes of rational functions in the complete linear series of the divisor. We prove that for a divisor D of degree d and rank r on a genus g tropical curve, the sum of weights is equal to $d-r+rg$. We establish analogous statements for tropical linear series.

In the case D comes from the tropicalization of a divisor, these weights control the number of Weierstrass points which are tropicalized to each component. Our results provide answers to open questions originating from the work of Baker on specialization of divisors from curves to graphs.

We conclude with multiple examples which illustrate interesting features appearing in the study of tropical Weierstrass points, and raise several open questions.

Contents

1. Overview

Weierstrass points have a rich history in the development of algebraic geometry as they provide an important tool for the study of smooth algebraic curves and their moduli spaces. It is natural to ask how their theory can be extended to stable curves, which correspond to boundary points in the Deligne–Mumford compactification $\overline{\mathscr{M}}_g$ of the moduli space of genus g smooth curves. One strategy is to take the *limit Weierstrass points* induced by a one-parameter family $(X_t)_{t\neq 0}$ of smooth curves degenerating to a stable curve X_0; there will be g^3-g limit Weierstrass points on X_0 when counted with appropriate weights. However, the limit points generally depend on the chosen family, and a stable curve X_0 has many possible smoothings corresponding to paths in $\overline{\mathscr{M}}_g$ that end at the point representing X_0.

Tropical geometry provides a new perspective on degeneration methods in algebraic geometry by enriching it with polyhedral geometry. Given the successes of tropical methods in the past two decades in the study of algebraic curves and their moduli spaces, it is natural to ask whether tropical geometry can be used to gain insight about the limiting behavior of Weierstrass points on degenerating families of curves. In the tropical perspective, the data of a stable curve X_0 is replaced by the data of its dual graph. The collection of all stable curves having the same dual graph forms a stratum of $\overline{\mathscr{M}}_g$. This gives a correspondence between the strata of $\overline{\mathscr{M}}_g$ and the set of stable graphs of genus g [Cap15].

The prototype of what we can expect to address using tropical techniques is the following natural question.

Question 1.1. *Given a stratum of $\overline{\mathscr{M}}_g$, and a log-tangent direction of approaching that stratum, what can be said about the limit Weierstrass points of a smooth family $(X_t)_{t\neq 0}$ degenerating to a stable curve in that stratum along the chosen direction?*

The arithmetic geometric version of the above question is the following.

Question 1.2. *Given a smooth proper curve over the field $\mathbb{Q}_p$ of p-adic numbers with stable reduction lying in a given stratum of $\overline{\mathscr{M}}_g$ (over the algebraic closure of the residue field $\mathbb{F}_p$), what can be said about the specialization of the Weierstrass points?*

Previously, there has been much work making incremental progress on the first question [EH87a, EM02, ES07, Dia85, Ami14, Gen21] and on the second question [Ogg78, LN64, Atk67, AP03, Bak08]; see Section 1.4 for a more thorough discussion.

Our aim in this paper is to provide an answer to the above questions from the point of view of tropical geometry. This is done by introducing new tools which allow us to solve problems related to the tropical geometry of curves, whose origin goes back to the beginning of the use of tropical methods in the study of algebraic curves.

Our answer to Question 1.1 can be summarized as follows: we can specify how many Weierstrass points degenerate to each component and to each node of a stable curve X_0 lying in the given stratum. This is done without specifying their precise position within each irreducible component, giving instead a more precise location of those degenerating to a node by specifying their position on the dual metric graph of the family (X_t). Our result also applies to limits of ramification points of arbitrary line bundles, in addition to the case of the canonical bundle.

Similarly, we answer Question 1.2 by specifying where Weierstrass points specialize when reducing modulo p.

Moreover, these results lead to an effective way of locating the limit Weierstrass points.

We next give an overview of our results.

1.1. **Tropical perspective.** The central concept studied in this paper is that of tropical Weierstrass points. The definition is based on divisor theory on metric graphs, and we refer to the survey paper [BJ16] and the references there for more details.

Let Γ be a metric graph, and let D be a divisor of degree d and rank r on Γ.

Definition 1.3 (Weierstrass points)**.** A point x in Γ is called a *Weierstrass point*, or *ramification point*, for D if there exists an effective divisor E in the linear system of D whose coefficient at x is at least $r+1$. The *(tropical) Weierstrass locus* of D, denoted by $L_W(D)$, is the set of all such points in Γ. ◇

The set $L_W(D)$ is a closed subset of Γ which can be infinite, in contrast with the classical setting of algebraic curves. In this regard, Baker comments in [Bak08, Remark 4.14], regarding the canonical divisor, that "it is not clear if there is an analogue for metric graphs of the classical fact that the total weight of all Weierstrass points on a smooth curve of genus g is $g^3 - g$." More generally, we can ask the following question.

Question 1.4. *Is it possible to associate intrinsic tropical weights to the connected components of $L_W(D)$? What is the total sum of weights associated to these components?*

The following question is a special case.

Question 1.5. *Assume the locus of Weierstrass points of D is finite. What is the total weight of these points?*

Our aim in this paper is to provide answers to the above questions. In order to streamline the presentation which follows, we first discuss our results in the case of non-augmented metric graphs. From the geometric perspective, this corresponds to the situation of a *totally degenerate* stable curve, that is, a stable curve whose irreducible components are all projective lines. This is the same as requiring that the arithmetic genus of the stable curve is equal to the genus of the dual graph. We have an analogue of these statements for augmented metric graphs (respectively, arbitrary stable curves), see the discussion which follows below.

In order to solve Question 1.4, we make the following definition.

Definition 1.6 (Intrinsic Weierstrass weight of a connected component)**.** Let D be a divisor of rank r, and let C be a connected component of the Weierstrass locus $L_W(D)$. We define the *tropical Weierstrass weight* of C as

$$\mu_W(C; D) \coloneqq \deg\left(D|_C\right) + (g(C) - 1)\, r - \sum_{\nu \in \partial^{\text{out}} C} s_0^{\nu}(D) \tag{1}$$

where

- $\deg\left(D|_C\right)$ is the total degree of D in C, defined by $\deg\left(D|_C\right) = \sum_{x \in C} D(x)$;
- $g(C)$ is the genus of C, i.e., its first Betti number $\dim H_1(C, \mathbb{R})$;
- $\partial^{\text{out}} C$ is the set of outgoing unit tangent directions from C; and
- $s_0^{\nu}(D)$ is the minimum slope along tangent direction ν of any rational function f on Γ with $\mathrm{div}(f) + D \geq 0$.

We abbreviate $\mu_W(C; D)$ simply as $\mu_W(C)$ if D is understood from the context. ◇

Although it is not obvious from the definition, we will show in Theorem 3.6 that the tropical Weierstrass weight of any component is positive. Note as well that a connected component of $L_W(D)$ is always a metric subgraph of Γ, see Proposition 3.1.

We say that D is *Weierstrass finite* or simply *W-finite* if the tropical Weierstrass locus $L_W(D)$ has finite cardinality. In this case, connected components of $L_W(D)$ are isolated points in Γ, and we define the *tropical Weierstrass divisor* $W(D)$ as the effective divisor

$$W(D) \coloneqq \sum_{x \in L_W(D)} \mu_W(x)\,(x)$$

where $\mu_W(x) \coloneqq \mu_W(\{x\})$. The support $|W(D)|$ of the tropical Weierstrass divisor is exactly the tropical Weierstrass locus $L_W(D)$. The tropical Weierstrass weight of x can be identified as $\mu_W(x) = D_x(x) - r$, with D_x denoting the unique x-reduced divisor in the linear system of D, see Remark 3.3.

This gives the following geometric meaning to the Weierstrass weights, in the spirit of the classical definition on algebraic curves. The coefficient of the reduced divisor at a point $x \in \Gamma$ corresponds precisely to the maximum order of vanishing at x of any global section of the line bundle $\mathcal{O}(D)$ defined by the divisor. The Weierstrass weight of the point x is thus obtained by comparing this quantity to r, which would be the expected minimum value, over points $y \in \Gamma$, of the largest order of vanishing of global sections at y. (Note, however, that r is not always equal to the actual minimum largest order of vanishing, as examples in Section 6.5 show.) That being said, the definition differs from the algebraic setting, where we need to take into account *all* the orders of vanishing of global sections of the line bundle at a given point (and then compare them with the standard sequence, the one obtained for a point in general position on the curve).

The following theorem answers Questions 1.4 and 1.5, and is proved in Section 3.3.

Theorem 1.7 (Total weight of the Weierstrass locus)**.** *Let Γ be a metric graph of genus g, and let D be an effective divisor of degree d and rank r on Γ. Then, the total sum of weights of the connected components of $L_W(D)$ is equal to $d - r + rg$. In particular, if D is W-finite, then we have $\deg(W(D)) = d - r + rg$.*

The proof of this theorem will imply in particular the following result, proved in Section 3.4.

Theorem 1.8. *If the rank r of D is at least one, then every cycle in Γ intersects the tropical Weierstrass locus $L_W(D)$. In particular, if Γ has genus at least two, then every cycle intersects the Weierstrass locus of the canonical divisor K.*

In [Bak08], Baker proves that the tropical Weierstrass locus of the canonical divisor is nonempty if Γ has genus at least two. This earlier tropical result is obtained as a consequence of the analogous algebraic statement, using the specialization lemma. In contrast, our theorem above states that tropical Weierstrass points obey a stronger "local" existence condition, which has seemingly no algebraic analogue. In the case that the canonical divisor of Γ is W-finite, our result implies that for an arbitrary family $(X_t)_{t\neq 0}$ of smooth curves tropicalizing to Γ, every cycle in Γ contains a limit Weierstrass point of the family.

To prove Theorem 1.7, we will show that in fact (1) defines a consistent notion of Weierstrass weight when applied to any connected, closed subset of Γ whose boundary points are not in the interior of $L_W(D)$; see Theorem 3.9. To do so, we retrieve information about the slopes of rational functions in the linear series $\mathrm{Rat}(D)$ along tangent directions in Γ. We have the following description, proved in Section 2.

Theorem 1.9. *Let D be a divisor of rank r on Γ. We take a model for Γ whose vertex set contains the support of D. Let $x \in \Gamma$ be a point and $\nu \in \mathrm{T}_x(\Gamma)$ be a tangent direction.*

(a) If the open interval $(x, x + \varepsilon\nu)$ is disjoint from $L_W(D)$ for some $\varepsilon > 0$, then the set of slopes $\{\mathrm{sl}_\nu f(x) : f \in \mathrm{Rat}(D)\}$ consists of $r+1$ consecutive integers $\{s_0^\nu, s_0^\nu+1, \ldots, s_0^\nu+r\}$.

(b) If the open interval $(x, x + \varepsilon\nu)$ is contained in $L_W(D)$, then the set of slopes $\{\mathrm{sl}_\nu f(x) : f \in \mathrm{Rat}(D)\}$ is a set of consecutive integers of size at least $r + 2$.

1.2. **Comparison results and extensions.** We further justify our definition of weights by making a precise link to tropicalizations of Weierstrass points on algebraic curves.

Suppose that Γ and D come from geometry; that is, let X be a smooth proper curve of genus g over an algebraically closed non-Archimedean field $\mathbb{K}$ of characteristic zero with a non-trivial valuation and a residue field of arbitrary characteristic. Let $\mathcal{L} = \mathcal{O}(\mathcal{D})$ be a line bundle of degree d on X. Assume that Γ is a skeleton of the Berkovich analytification X^{an}

of X. Denote by τ the tropicalization map from X to Γ, and suppose that $D = \tau_*(\mathcal{D})$ is the tropicalization of $\mathcal{D}$ on Γ where $\tau_* \colon \mathrm{Div}(X) \to \mathrm{Div}(\Gamma)$ the induced map on divisors.

Denote by $\mathcal{W}(\mathcal{D})$ the Weierstrass divisor of $\mathcal{D}$ on X, and by $\tau_*(\mathcal{W}(\mathcal{D}))$ its tropicalization on Γ. The following result, proved in Section 5.3, uses the notion of $L_W(D)$-measurable set, for which the connected components of $L_W(D)$ form the atoms, and the natural counting measure $\hat{\mu}_W$ on such sets, induced by Weierstrass weights (see Section 3.3 for more details).

Theorem 1.10 (Algebraic versus tropical Weierstrass weights)**.** *Assume that D and $\mathcal{D}$ have the same rank, and let $A \subset \Gamma$ be a closed, connected subset which is $L_W(D)$-measurable. Then, the total weight of Weierstrass points of $\mathcal{W}(\mathcal{D})$ tropicalizing to points in A is precisely $(r+1)\,\hat{\mu}_W(A; D)$; that is,*

$$\deg\left(\mathcal{W}(\mathcal{D})|_{\tau^{-1}(A)}\right) = (r+1)\left(\deg\left(D|_A\right) + r\,(g(A)-1) - \sum_{\nu\in\partial^{\mathrm{out}}A} s_0^{\nu}(D)\right).$$

In particular, if D is W-finite, then we have the equality

$$\tau_*(\mathcal{W}(\mathcal{D})) = (r+1)\,W(\tau_*(\mathcal{D})).$$

This statement, which involves the metric of Γ in a crucial way, gives an essentially complete description of the behavior of Weierstrass points in the tropical limit. In particular, if the limit divisor is W-finite, then for every family $(X_t)_{t\neq 0}$ of smooth proper curves approaching a stable curve with dual metric graph Γ, the limit Weierstrass points are precisely described by the tropical Weierstrass divisor. This rigidity type theorem on the limiting behavior of Weierstrass points allows us to give a precise count of the number of Weierstrass points going to the nodes or to the smooth parts of a limit stable curve X_0 on the given stratum of $\overline{\mathscr{M}}_g$ along the given log-tangent direction from which the family $(X_t)_{t\neq 0}$ approaches X_0. Moreover, as a special case, the theorem also applies in the context of arithmetic geometry in which the curve X is defined over a finite extension of $\mathbb{Q}_p$. As we will show in Section 5.3, this theorem holds as well over a field $\mathbb{K}$ of positive characteristic provided that the gap sequence of $\mathcal{L}$, defined as the sequence of orders of vanishing of the global sections of $\mathcal{L}$ at a general point of X, is the standard sequence $0, 1, \dots, r$. (In this case, $\mathcal{L}$ is called classical [Lak81, Nee84].)

We provide natural extensions and refinements of the above results to the setting of augmented metric graphs, which, from the degeneration perspective, corresponds to the situation where the limit stable curve has irreducible components of possibly positive genus. Since a given vertex of positive genus hides information about the geometry of the corresponding component, it turns out that there will be an ambiguity when talking about the Weierstrass locus of a divisor D. In fact, the right setup in this context is a divisor D endowed with the data of a closed sub-semimodule M of $\mathrm{Rat}(D)$, which plays the role of a (not necessarily complete) linear series on the augmented metric graph.

In this regard, first, we use the weights defined in Definition 1.6 with a relevant notion of divisorial rank associated to the sub-semimodule which we further modify by including the data of the genus function. We get Theorem 4.12, which provides a global count of weights in this setting.

To the question of *whether it is still possible to associate a natural Weierstrass locus to a divisor in the augmented setting*, we provide an answer by introducing two special classes of semimodules, the *generic semimodule* associated to any divisor (see Section 4.2), and the *canonical semimodule* associated to the canonical divisor on an augmented metric graph (see

Section 4.3). Both of them require some level of genericity, which we properly justify in Section 4.4 using the framework of metrized complexes.

The case of the canonical divisor on an augmented metric graph is particularly interesting as it reveals new facets of divisor theory in the augmented setting. We associate a canonical linear series to any augmented metric graph, show that it has the appropriate rank, and study its Weierstrass locus. To justify the definition and prove these results, we use the setting of metrized complexes and their divisor theory from [AB15]. Using that framework, we show that the canonical linear series on an augmented metric graph is the tropical part of the canonical linear series on any metrized complex with that underlying augmented metric graph, provided that the markings associated to edges on the curves of the metrized complex are in general position. It is interesting to note that this is the assumption made in the works by Esteves and coauthors [EM02, ES07], and our results here complement these works by developing the tropical part of the story in greater generality.

As we show in Theorem 5.5, the statement of Theorem 1.10 remains valid in these settings (when including the genera of points of A on the right-hand side of the stated equality). The following theorem is a direct application of our results on Weierstrass weights for an augmented metric graph. We use the setting of tropicalization preceding Theorem 1.10.

Theorem 1.11. *Suppose $\mathcal{D}$ is a divisor on an algebraic curve X over an algebraically closed non-Archimedean field $\mathbb{K}$ of characteristic zero with a non-trivial valuation and a residue field of arbitrary characteristic. Let $(\Gamma, \mathfrak{g})$ be an (augmented) skeleton of X^{an}. Let H be a vector space of global sections of $\mathcal{O}(\mathcal{D})$ of rank r and denote by $\mathcal{W}(H)$ the Weierstrass divisor of H. Let M be the tropicalization of H. Then, for any connected, closed subset $A \subset \Gamma$ which is $L_W(M, \mathfrak{g})$-measurable, we have the bound*

$$\deg\left(\mathcal{W}(H)|_{\tau^{-1}(A)}\right) \geq \left(r^2 + r\right)\left(g(A) + \sum_{x\in A}\mathfrak{g}(x)\right).$$

The proof of this theorem will be given in Section 5.3. As in the case of Theorem 1.10, the statement holds as well over a field $\mathbb{K}$ of positive characteristic provided the gap sequence of H is the standard sequence.

In the case $L_W(M, \mathfrak{g})$ is finite, this inequality holds for any closed subset $A \subset \Gamma$. In particular, we have the following application to stable curves: suppose X_0 is a stable curve with dual augmented graph $(G, \mathfrak{g})$, and suppose (X_t) is a family degenerating to X_0 with tropicalization $(\Gamma, \mathfrak{g})$. If the locus of canonical Weierstrass points of $(\Gamma, \mathfrak{g})$ is finite, then for every connected subgraph A of G, the number of limit Weierstrass points lying on components and nodes of X_0 which correspond to vertices and edges of A, respectively, is at least $(g^2 - g)\left(g(A) + \sum_{v\in A}\mathfrak{g}(v)\right)$.

Semimodules inside $\mathrm{Rat}(D)$ that come from the tropicalization of linear series verify an extra set of properties. These are thoroughly studied in recent works [AG22] and [JP22] that develop a combinatorial theory of (limit) linear series. In particular, such a semimodule M of rank r satisfies the following:

($\star$) For each point x in Γ and any unit tangent direction $\nu \in \mathrm{T}_x(\Gamma)$, the set of slopes taken by functions in M has size $r + 1$.

(We refer to Section A.3 for more details.)

In Section 5, we associate a refined notion of Weierstrass divisor to any divisor D and any closed sub-semimodule $M \subset \mathrm{Rat}(D)$ that verifies the above property. The definition takes into account the higher orders of vanishing of the combinatorial limit linear series, and is closer to the spirit of the algebraic definition of Weierstrass weights on curves.

Using this together with the results proved in Appendix A, discussed below, we provide a proof of Theorem 1.10 and its extensions to the augmented and incomplete settings.

1.3. **Tropicalization of Weierstrass divisors.** The proof of our comparison results, Theorem 1.10 and its extension Theorem 5.5, makes use of the results proved by the first named author in Appendix A. An earlier version of these results was written around 2014.

Let X be a smooth proper curve defined over $\mathbb{K}$. Let $\mathcal{D}$ be a divisor of degree d on X and let $\mathcal{L} = \mathcal{O}(\mathcal{D})$ be the corresponding line bundle. Let $H \subset H^0(X, \mathcal{L})$ be a space of sections of dimension $r+1$ and denote by $\mathcal{W} = \mathcal{W}(\mathcal{D}, H)$ the corresponding Weierstrass divisor. We assume that the gap sequence of H is the sequence $0, 1, \dots, r$, that is, for a general point $x \in X(\mathbb{K})$, the orders of vanishing of sections of $\mathcal{L}$ in H are $0, 1, \dots, r$. Let τ be the tropicalization map from X to Γ. We describe the tropicalization $W = \tau_*(\mathcal{W})$. The divisor $\mathcal{W}$ is equal to $(r+1)\mathcal{D} + \mathrm{div}(\mathrm{Wr}_{\mathcal{F}})$ for a section $\mathrm{Wr}_{\mathcal{F}}$ of the sheaf $\omega_X^{\otimes r(r+1)/2}$ called the Wronskian. The sheaf $\omega_X^{\otimes r(r+1)/2}$ admits a natural norm; using this norm, we can tropicalize the section $\mathrm{Wr}_{\mathcal{F}}$, and define a rational function $F = \mathrm{trop}(\mathrm{Wr}_{\mathcal{F}}) \colon \Gamma \to \mathbb{R}$. Denote by K the canonical divisor of $(\Gamma, \mathfrak{g})$. Using the slope formula for sections of powers of the canonical sheaf, Lemma A.1, it is shown in Theorem A.2 that for any $x \in \Gamma$, we have

$$W(x) = (r+1)D(x) + \frac{r(r+1)}{2}K(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} \mathrm{sl}_\nu F.$$

It follows from the results proved in Section A.7, that for a point $x \in \Gamma$ and $\nu \in \mathrm{T}_x(\Gamma)$, if

- either, the residue field κ is of characteristic zero,
- or, the sequence $s_0^\nu, \dots, s_r^\nu$ forms an interval, that is, $s_j^\nu = s_0^\nu + j$,

then the slope $\mathrm{sl}_\nu F$ is given by the sum $s_0^\nu + \dots + s_r^\nu$, see Proposition A.4 and Theorem A.2. This result is needed to prove in Section 5 our comparison results between tropical and algebraic Weierstrass loci.

Note that over a field $\mathbb{K}$ of equicharacteristic zero, the first item in the above condition is verified, and we get all the coefficients $W(x)$,

$$W(x) = (r+1)D(x) + \frac{r(r+1)}{2}K(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} \sum_{i=0}^{r} s_i^\nu,$$

see Theorem A.5.

1.4. **Previous work.** The study of Weierstrass points from a tropical perspective was initiated by Baker [Bak08, Section 4]. Baker defines Weierstrass points for graphs and metric graphs, and uses his Specialization Lemma [Bak08, Lemma 2.8] to prove an essential compatibility with Weierstrass points on stable curves—namely, that the tropicalization of the algebraic Weierstrass locus is a subset of the tropical Weierstrass locus. To be more precise, for a divisor $\mathcal{D}$ on a non-Archimedean curve with Weierstrass divisor $\mathcal{W}(\mathcal{D})$, if $\tau_*(\mathcal{D})$ has the same rank as $\mathcal{D}$ and $\mathcal{D}$ has classical gap sequence, then we have an inclusion

$$|\tau_*(\mathcal{W}(\mathcal{D}))| \subseteq L_W(\tau_*(\mathcal{D})),$$

which may be strict in general. (This is stated for the canonical divisor in *loc. cit.*, but the proof works in greater generality.) This statement has strong implications for the behavior of Weierstrass points on a family of degenerating Riemann surfaces, and for p-adic reduction of curves over $\mathbb{Q}_p$, discussed earlier in the introduction. Indeed, Baker motivates his study of Weierstrass points on graphs with several results from the arithmetic geometry of modular curves, in particular, as a way to decide whether certain cusps are Weierstrass points, c.f. [Ogg78, LN64, Atk67, AP03].

The question of how to determine the tropicalization of Weierstrass points on a non-Archimedean curve was settled in [Ami14]; these results appear in Appendix A and are used to prove our comparison results. The question of determining tropical Weierstrass loci and their weights, and the way to properly count them in the tropical setting remained however open. The work [Ric24] by the third-named author studies Weierstrass points on tropical curves. Although the tropical Weierstrass locus may be infinite in general, [Ric24] shows that for a generic divisor class (i.e., lying in a nonempty open subset of Pic^d), this locus is finite, and moreover computes its cardinality. It is worth mentioning that important divisor classes such as the canonical divisor are non-generic, so they are not covered by the methods of [Ric24]. The way tropical Weierstrass points distribute when the degree of divisor classes tend to infinity is studied in [Ami14, Ric24]. For an extended discussion of how divisor theory on graphs is connected to the degeneration of smooth curves to nodal curves, with various applications, see the survey by Baker–Jensen [BJ16], in particular Section 12.

For an extensive and informative survey describing the history and applications of Weierstrass points, starting with Weierstrass and Hürwitz [Wei67, Hur92] in the 1800s, see Del Centina [DC08]. The study of Weierstrass points on stable curves was initiated by Eisenbud and Harris [EH87a], who proved results on nodal curves of *compact type*, i.e., curves whose dual graph is a tree. This work served as an application of their newly-developed theory of limit linear series [EH86]. They moreover raised the question of constructing a moduli space parametrizing all possible limit Weierstrass divisors of a given stable curve, a problem which has been widely open since then.

Moving beyond stable curves of compact type, Lax [Lax87] studied Weierstrass points on stable curves consisting of one rational component with nodes; in this case, the dual graph is a single vertex with self-loops. (The term *tree-like* is used in the literature to describe curves whose dual graph consists of a tree after removing self-loops.) A further breakthrough came with Esteves–Medeiros [EM02] who worked with stable curves with two components, i.e., curves whose dual graph is a dipole graph. (We refer to Section 6.8 for a discussion of our results applied to dipole graphs and the connection to [EM02].) Esteves–Salyehan [ES07] studied further cases of nodal curves, including when the dual graph is a complete graph. Cumino–Esteves–Gatto [CEG08] studied limits of *special* Weierstrass points on certain stable curves, i.e., Weierstrass points with weight at least two. The problem of describing limits of Weierstrass points away from the nodes in a given one-parameter family in characteristic zero is addressed in [Est98].

Although not directly related to the results of this paper, we mention that other works treat the case of irreducible Gorenstein curves, and associate Weierstrass weights to their singular points, see e.g. [LW90, dCS94, GL95, BG95] and the references there. It might be possible to use tropical geometry to describe these weights.

Weierstrass points have appeared in other interesting work on moduli spaces of curves. Arbarello [Arb74] studied subvarieties of the moduli space of curves cut out by Weierstrass points; further results were found in Lax [Lax75] and Diaz [Dia85]. Eisenbud–Harris [EH87b] showed that the moduli space of curves has positive Kodaira dimension, using loci of Weierstrass points as part of their argument. Cukierman(–Fong) [Cuk89, CF91] found the coefficients for the Weierstrass locus in the universal curve $\mathscr{C}_g$ of genus g, in a standard basis for the Picard group of $\mathscr{C}_g$. We discuss the behavior of tropical Weierstrass loci over the moduli space of tropical curves in Section 7.1.

1.5. **Organization of the text.** The paper is organized as follows. We first treat the case of non-augmented metric graphs, and then provide refinements. This choice has the advantage of making the presentation less technical, and we hope this will add to readability.

We define slope sets and prove Theorem 1.9 in Section 2.

In Section 3, we study Weierstrass weights and the Weierstrass measure they define on a metric graph. We state and prove Theorem 3.9, which provides a description of the Weierstrass measure using the slopes, from which we deduce Theorem 1.7 and other interesting consequences. This section contains the proof of positivity of Weierstrass weights as well, and a discussion of the case of combinatorial graphs. The case of the canonical Weierstrass locus on non-augmented metric graphs is treated in Section 3.7.

Section 4 provides several refinements and generalizations of the previous sections. The setting is extended in two ways. First, complete linear series $\mathrm{Rat}(D)$ are replaced with incomplete linear series, by taking closed sub-semimodules of $\mathrm{Rat}(D)$. Second, metric graphs are replaced with augmented metric graphs. We provide justification for our definitions in the augmented setting and provide the corresponding generalizations of Theorem 3.9 on the Weierstrass measure and of Theorem 1.7.

In Section 5, we explain how to associate Weierstrass divisors to combinatorial limit linear series. This is particularly interesting in the case where the locus of Weierstrass points associated to the underlying divisor becomes infinite after forgetting the slopes. We show the compatibility of the definitions appearing in this section with the previous ones.

Using the above materials, we establish a precise link between the tropical Weierstrass divisors with tropicalizations of Weierstrass divisors on smooth curves. This includes the proof of Theorem 1.10, and its generalizations.

In the last two Sections 6 and 7, we provide several examples with the aim of clarifying the concepts introduced in previous sections, and discuss other interesting results related to them. We also raise several open questions.

Appendix A proves the results we need on the tropicalization of Weierstrass divisors.

Acknowledgments. This material is based upon research supported by the Transatlantic Research Partnership of the Embassy of France in the United States and the FACE Foundation. We thank them, as well as the Cultural Services of the French Embassy in the U.S., and the administrations of the CNRS and the University of Washington at Seattle, for support.

O. A. thanks Math+, the Berlin Mathematics Research Center and TU Berlin, where this research was carried out. L. G. thanks the Mathematical Foundation Jacques Hadamard (FMJH) for financial support.

1.6. **Basic notations.** A *(combinatorial) graph* $G = (V, E)$ is defined by a set of vertices V and a set of edges E between certain vertices. In the current paper, graphs will always be taken to be finite and connected. Moreover, they will allow loops and multiple edges.

A *metric graph* is a compact, connected metric space Γ verifying the following properties:

(i) For every point $x \in \Gamma$, there exist a positive integer n_x and a real number $r_x > 0$ such that the r_x-neighborhood of x is isometric to the star of radius r_x with n_x branches.

(ii) The metric on Γ is given by the path metric, i.e., for points x and y in Γ, the distance between x and y is the infimum (in fact minimum) length of any path from x to y.

The integer n_x above is called the *valence* of x and is denoted by $\mathrm{val}(x)$.

Given a graph $G = (V, E)$ and a length function $\ell\colon E \to (0, +\infty)$ assigning to every edge of G a positive length, we can build from this data a metric graph Γ by gluing a closed interval of length $\ell(e)$ between the two endpoints of the edge e, for every $e \in E$, and endowing Γ with the path metric. The space Γ is then called the *geometric realization* of the pair (G, ℓ).

A *model* of a metric graph Γ is a pair (G, ℓ) consisting of a graph $G = (V, E)$ and a length function $\ell\colon E \to (0, +\infty)$ such that Γ is isometric to the geometric realization of (G, ℓ). By an abuse of notation, we also call G a model of Γ.

For a metric graph Γ and a point $x \in \Gamma$, the tangent space $\mathrm{T}_x(\Gamma)$ is defined as the set of all unit outgoing tangent vectors to Γ at x. This is a finite set of cardinality $\mathrm{val}(x)$. If $G = (V, E)$ is a loopless model for Γ such that $x \in V$, then $\mathrm{T}_x(\Gamma)$ is in one-to-one correspondence with the edges of G incident to x. Through this natural bijection, a tangent direction ν is said to be *supported by* the corresponding edge $e \in E$.

Each edge e supports two tangent directions, which belong to either endpoint of e, respectively. If ν is one of those tangent directions, the opposite direction is denoted by $\overline{\nu}$. For $\nu \in \mathrm{T}(\Gamma)$, we denote by x_ν the point x with $\nu \in \mathrm{T}_x(\Gamma)$.

In this paper, all the semimodules will be assumed to be nonempty.

2. Slope sets

In this section, we prove Theorem 1.9. We first recall some terminology for divisors and functions on metric graphs.

Given a metric graph Γ, let $\mathrm{Div}(\Gamma)$ denote the *group of divisors* of Γ, which is the free abelian group generated by points $x \in \Gamma$. Let $\mathrm{Rat}(\Gamma)$ denote the set of real-valued piecewise linear functions on Γ whose slopes are all integers. Given a function $f \in \mathrm{Rat}(\Gamma)$, let $\mathrm{div}(f)$ denote the *principal divisor* of f, defined as

$$\mathrm{div}(f) \coloneqq \sum_{x\in\Gamma} a_x(x) \qquad \text{where} \qquad a_x = -\sum_{\nu\in\mathrm{T}_x(\Gamma)} \mathrm{sl}_\nu f(x).$$

Let D be a divisor of rank r on Γ. Let $\mathrm{Rat}(D)$ denote the set of *rational functions in the complete linear series of* D defined as

$$\mathrm{Rat}(D) \coloneqq \{f \in \mathrm{Rat}(\Gamma) : D + \mathrm{div}(f) \geq 0\}.$$

Given a point $x \in D$, there is a unique representative f_x of the linear series of D defined by

$$f_x \coloneqq \min_{\substack{f\in\mathrm{Rat}(D)\\ f(x)=0}} f.$$

The corresponding divisor $D + \mathrm{div}(f_x)$, denoted by D_x, is the (unique) *x-reduced divisor* linearly equivalent to D. This statement is a consequence of the maximum principle, see e.g. [BS13, Lemma 4.11].

Definition 2.1 (Slope sets and minimum slopes). Let D be an effective divisor on Γ. Given a point $x \in \Gamma$ and a tangent direction $\nu \in \mathrm{T}_x(\Gamma)$, let $\mathfrak{S}^\nu(D)$ denote the *slope set*

$$\mathfrak{S}^\nu(D) \coloneqq \{\mathrm{sl}_\nu f(x) : f \in \mathrm{Rat}(D)\}.$$

Let $s_0^\nu(D)$ denote the *minimum slope* in the slope set $\mathfrak{S}^\nu(D)$, i.e.,

$$s_0^\nu(D) := \min\{\mathrm{sl}_\nu f(x) : f \in \mathrm{Rat}(D)\}.$$

When the divisor D is clear from context, we will simply use s_0^ν to denote $s_0^\nu(D)$. ⋄

Lemma 2.2. *Suppose D is a divisor of rank r. Then, for every $x \in \Gamma$ and every $\nu \in \mathrm{T}_x(\Gamma)$, there are at least $r+1$ integers in the set of slopes $\{\mathrm{sl}_\nu f(x) : f \in \mathrm{Rat}(D)\}$.*

Proof. Let $x_1, \dots, x_r$ be a set of distinct points in the branch incident to x in the direction of ν sufficiently close to x. There exists a function $f \in \mathrm{Rat}(D)$ such that

$$D + \mathrm{div}(f) \geq (x_1) + \cdots + (x_r).$$

The function f changes slope at the points $x_1, \dots, x_r$. Each of the slopes taken at x_j in the direction of ν can be obtained as the slope of a function in $\mathrm{Rat}(D)$ at x along ν. □

The minimum slope $s_0^\nu(D)$ is related to the reduced divisor D_x at x.

Lemma 2.3. *Let D be an effective divisor on Γ, and x a point of Γ. Let D_x be the x-reduced divisor linearly equivalent to D.*

(a) Let f_x be the above defined rational function satisfying $\mathrm{div}(f_x) + D = D_x$, then, for any outgoing tangent vector $\nu \in \mathrm{T}_x(\Gamma)$,

$$s_0^\nu(D) = \mathrm{sl}_\nu f_x(x).$$

(b) The coefficient of D_x at x satisfies

$$D_x(x) = D(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} s_0^\nu(D).$$

Proof. The first result is obtained by observing that $f_x = \min h$ for $h \in \mathrm{Rat}(D)$ verifying $h(x) = 0$. The second result is a direct consequence of (a) and the definition of the principal divisor $\mathrm{div}(f_x)$. □

We now turn to the proof of Theorem 1.9. Let D be a divisor of rank r on Γ. Recall (Definition 1.3) that the Weierstrass locus of D, denoted by $L_W(D)$, is the subset of Γ formed by the points x such that there exists an effective divisor $E \sim D$ with $E(x) \geq r+1$. Equivalently, $L_W(D)$ is defined in terms of reduced divisors as

$$L_W(D) = \{x \in \Gamma : D_x(x) > r\},$$

where D_x denotes the x-reduced divisor linearly equivalent to D.

Proof of Theorem 1.9. We first assume that the open interval $(x, x+\varepsilon\nu)$ is disjoint from $L_W(D)$. Along the branch incident to x in the direction of ν, there is a small segment on which s_0^ν is the slope of a function in $\mathrm{Rat}(D)$, and it is the smallest slope taken by a function of $\mathrm{Rat}(D)$ on this segment. If a slope of $s_0^\nu + r + 1$ or larger is achieved at x, then, again on a small segment, it will be achieved at any point of that segment. This means that on the interior of this segment, the two minimum outgoing slopes at every point are s_0^ν and $-s_0^\nu - s'$ with $s' \geq r+1$. Therefore, by Lemma 2.3, we infer that on the interior of this segment, the reduced divisor at each point has coefficient at least $r+1$. This contradicts the assumption that a neighborhood $(x, x+\varepsilon\nu)$ is disjoint from $L_W(D)$, and shows that the highest possible slope is $s_0^\nu + r$. Combining this with Lemma 2.2, the slopes achieved at x along ν must be precisely $s_0^\nu, s_0^\nu + 1, \dots, s_0^\nu + r$. This proves (a).

We now assume that $(x, x+\varepsilon\nu) \subset L_W(D)$ and that ε is small enough so that the set of slopes of functions of $\mathrm{Rat}(D)$ along ν is constant on this interval. By Lemma 2.3, this means that on the interior of a small segment starting at x, the two minimum outgoing slopes at every point are s_0^ν and $-s_0^\nu - s'$ with $s' \geq r+1$. Therefore, close to x, a slope of at least $s_0^\nu + r + 1$ is achieved by a function in $\mathrm{Rat}(D)$. To prove (b), it is thus sufficient to show that the set of slopes $\mathrm{sl}_\nu f(x)$ of functions $f \in \mathrm{Rat}(D)$ is always made up of consecutive integers. Take $s_1 < s_2 < s_3$ to be three integers, and suppose that for $i \in \{1, 3\}$ there exists a function $f_i \in \mathrm{Rat}(D)$ such that $\mathrm{sl}_\nu f_i(x) = s_i$. Using f_1, f_3 and tropical operations, it is easy to construct a function f taking slopes s_3 and then s_1 away from x, changing slope at a point we denote by y (see Figure 1). We can then "chop up" the graph of f to construct a function h equal to f everywhere except on a small interval around y where it takes slope s_2. Since $(x, x+\varepsilon\nu)$ is disjoint from the support of D, h still belongs to $\mathrm{Rat}(D)$. The assumption made on ε at the beginning ensures that in fact there exists a function $f_2 \in \mathrm{Rat}(D)$ taking slope s_2 at x along ν, which concludes the argument. □

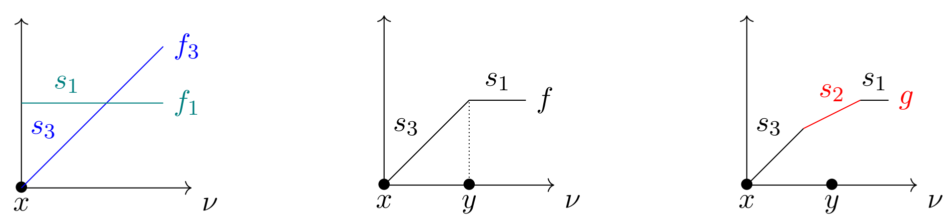

FIGURE 1. Construction of the functions f and g using functions f_1 and f_3 taking slopes $s_1 < s_3$.

Remark 2.4. In particular, note that along a given unit tangent vector ν attached to a point x, the slopes $\mathrm{sl}_\nu f(x)$ for $x \in \mathrm{Rat}(D)$ always form a set of consecutive integers. Moreover, if t is a positive integer such that for every $x \in e$, for e an edge of some model of Γ, the x-reduced divisor D_x satisfies $D_x \geq t\,(x)$, then for any $x \in \mathring{e}$, the set of slopes $\{\mathrm{sl}_\nu f(x) : f \in \mathrm{Rat}(D)\}$ contains at least $t+1$ consecutive integers. This claim is analogous to [Ami13, Theorem 14] and is proved using Theorem 3 of the same paper, which gives a concrete description of the variations of the reduced divisor D_x with respect to x. See also [AG22, Section 6.6]. ◇

For future use, we note the following generalization of part (b) of Lemma 2.3.

Proposition 2.5. *Suppose D is a divisor of rank r. Then, for any closed, connected subset $A \subset \Gamma$, we have*

$$\deg\left(D|_A\right) - \sum_{\nu \in \partial^{\mathrm{out}} A} s_0^\nu(D) \geq r.$$

Proof. Let E be an effective divisor of degree r, with support contained in A. Since D has rank r, there exists a function $f \in \mathrm{Rat}(D)$ such that $D + \mathrm{div}(f) \geq E$. Evaluating the respective degrees restricted to A yields

$$\deg\left(D|_A\right) - \sum_{\nu \in \partial^{\mathrm{out}} A} \mathrm{sl}_\nu f(x_\nu) \quad \geq \quad \deg\left(E|_A\right) = r,$$

where, we recall, x_ν is the point x of Γ with $\nu \in \mathrm{T}_x(\Gamma)$. By definition of the minimum slope $s_0^\nu(D)$, we have $s_0^\nu(D) \leq \mathrm{sl}_\nu f(x_\nu)$ for each $\nu \in \partial^{\mathrm{out}} A$, so the result follows. □

3. Weierstrass weights

Using the structure of slope sets in $\mathrm{Rat}(D)$, we prove Theorem 1.7, which will follow from the more general Theorem 3.9.

3.1. **Definition of weights and basic properties of the Weierstrass locus.** We start by establishing basic properties of Weierstrass loci. (Definition 1.3) The Weierstrass locus $L_W(D)$ is defined as the set of points x in Γ such that there exists an effective divisor E in the linear system of D whose coefficient at x is at least $r+1$. This is equivalent to requiring that $D_x(x) \geq r+1$. Let us now recall Definition 1.6 from the introduction. Given a connected component C of the Weierstrass locus $L_W(D)$, the tropical Weierstrass weight of C is defined as

$$\mu_W(C) = \mu_W(C; D) \coloneqq \deg\left(D|_C\right) + (g(C) - 1)\, r - \sum_{\nu \in \partial^{\mathrm{out}} C} s_0^{\nu}(D)$$

where $\deg\left(D|_C\right) = \sum_{x \in C} D(x)$ is the degree of D in C, $g(C) = \dim H_1(C, \mathbb{R})$ is the genus of C, $\partial^{\mathrm{out}} C$ is the set of outgoing unit tangent directions from C, and $s_0^{\nu}(D)$ is the minimum slope at x along a tangent direction ν, as defined in Definition 2.1. The following proposition shows that $L_W(D)$ is topologically nice.

Proposition 3.1. *The Weierstrass locus $L_W(D)$ is closed and has finitely many components. Each connected component is a metric graph.*

Proof. By the continuity of variation of reduced divisors proved in [Ami13, Theorem 3], the function $x \mapsto D_x(x)$ is upper semicontinuous, which implies that the subset $L_W(D)$ is closed. Then, by Theorem 1.9, the number of connected components of $L_W(D)$ is finite. The last statement follows as any connected component of a closed subset in a metric graph is itself a metric graph. □

Remark 3.2. We have the following geometric construction of $L_W(D)$, which gives another proof for Proposition 3.1. Let $\mathrm{Pic}^d(\Gamma)$ denote the space of divisor classes of degree d on Γ, and let $\mathrm{Eff}^d(\Gamma)$ denote the space of effective divisor classes of degree d. Let $\varphi : \Gamma \to \mathrm{Pic}^{d-r-1}(\Gamma)$ be the map defined by $\varphi(x) = [D - (r+1)\,(x)]$.

The condition that $D_x(x) > r$ is equivalent to $D_x \geq (r+1)\,(x)$. This is in turn equivalent to the condition that the divisor class $[D - (r+1)\,(x)]$ has an effective representative. Using this observation and the above terminology, $L_W(D) = \varphi^{-1}\left(\varphi(\Gamma) \cap \mathrm{Eff}^{d-r-1}(\Gamma)\right)$. In other words, $L_W(D)$ is described by the following pullback diagram.

$$\begin{array}{ccc} L_W(D) & \longrightarrow & \mathrm{Eff}^{d-r-1}(\Gamma) \\ \big\downarrow & \lrcorner & \big\downarrow \\ \Gamma & \xrightarrow{\ \varphi\ } & \mathrm{Pic}^{d-r-1}(\Gamma) \end{array}$$

Both $\mathrm{Eff}^{d-r-1}(\Gamma)$ and $\varphi(\Gamma)$ are polyhedral subsets of $\mathrm{Pic}^{d-r-1}(\Gamma)$ with finitely many facets. Thus their intersection has finitely many components, and each component is a union of finitely many closed intervals. ◇

Remark 3.3. As before, let D_x denote the x-reduced divisor linearly equivalent to D. Since the Weierstrass locus $L_W(D)$ is defined as $\{x \in \Gamma : D_x(x) - r > 0\}$, the expression $D_x(x) - r$ is a natural "naive" candidate for defining a tropical Weierstrass weight. In fact, this ends up being the correct definition when x is an isolated component of $L_W(D)$. When x is not an isolated component, our more technical definition of weight is required.

If the singleton $\{x\}$ is a connected component of $L_W(D)$, then we verify that the weight of x is simply given by $D_x(x) - r$. Since the genus of the component $\{x\}$ is zero, Definition 1.6 states that

$$\mu_W(x) = D(x) - r - \sum_{\nu \in \mathrm{T}_x(\Gamma)} s_0^\nu(D),$$

and Lemma 2.3 states that $D_x(x) = D(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} s_0^\nu(D)$. This verifies the claim.

Note that this applies for every connected component of $L_W(D)$ if D is W-finite. ◇

We now give two examples of metric graphs and their Weierstrass loci. The first Weierstrass locus is finite whereas the second one is infinite.

Example 3.4. Suppose Γ is the complete graph on four vertices with unit edge lengths; see Figure 2. This graph has genus three, and the rank of the canonical divisor K is $r = g - 1 = 2$. The Weierstrass locus $L_W(K)$ is finite and consists of the four vertex points. At a vertex v, the reduced divisor at v is $K_v = 4\,(v)$. Thus, $\mu_W(v) = K_v(v) - r = 4 - 2 = 2$.

FIGURE 2. Complete graph on four vertices, and its Weierstrass locus $L_W(K)$.

We will treat the example of the complete graph on five or more vertices in Section 6.5. ◇

Example 3.5. Suppose Γ is the "barbell graph" consisting of two cycles joined by a bridge edge; see Figure 3. (The edge lengths may be arbitrary.) This graph has genus two, and the canonical divisor K has rank $r = g - 1 = 1$.

The Weierstrass locus $L_W(K)$ consists of the middle edge and the outer midpoint on each cycle. The latter have weight one. If we divide each cycle into two equal parts according to its two distinguished points, then the slopes on each half-circle are $\{0, 1\}$ starting on the middle edge. This implies that the weight of the middle edge is also one.

FIGURE 3. The barbell graph and its Weierstrass locus $L_W(K)$.

We will show in greater generality in Section 6.4 that if e is a bridge edge of Γ such that each component of $\Gamma \smallsetminus \mathring{e}$ has positive genus, then e is contained in the canonical Weierstrass locus. ◇

3.2. **Positivity of Weierstrass weights.** We now prove the following theorem.

Theorem 3.6. *Let D be a divisor on Γ with non-negative rank r, and let C be a connected component of the Weierstrass locus $L_W(D)$. Then, the weight $\mu_W(C)$ given in Definition 1.6 is positive.*

Proof. We use the notations introduced previously. Let x be a point in the connected component C, and let D_x be the x-reduced divisor equivalent to D. By definition of the Weierstrass locus $L_W(D)$, we have $D_x(x)-r>0$. Let f_x be a rational function such that $D_x = D+\mathrm{div}(f_x)$. We have

$$D_x(x) = D(x) - \sum_{\nu\in \mathrm{T}_x(\Gamma)} \mathrm{sl}_\nu f_x(x).$$

Let A be any connected subgraph of Γ, and recall that $\deg\left(D_x|_A\right)$ denotes the sum $\sum_{y\in A} D_x(y)$. For a tangent vector $\nu \in \partial^{\mathrm{out}} A$, let, as before, x_ν denote the associated boundary point. We have

$$\deg\left(D_x|_A\right) = \deg\left(D|_A\right) - \sum_{\nu\in\partial^{\mathrm{out}} A} \mathrm{sl}_\nu f_x(x_\nu)$$

by applying Stokes theorem to the derivative of f_x on the region A.

Because $x\in C$ and D_x is effective, we have $\deg\left(D_x|_C\right) \geq D_x(x) > r$. For each tangent direction $\nu \in \partial^{\mathrm{out}} C$, the minimum slope $s_0^\nu(D)$ satisfies $s_0^\nu(D) \leq \mathrm{sl}_\nu f_x(x_\nu)$ by definition (Definition 2.1). Therefore,

$$\begin{aligned}\mu_W(C) = \deg\left(D|_C\right) + (g(C)-1)\,r - \sum_{\nu\in\partial^{\mathrm{out}} C} s_0^\nu(D) &\geq \deg\left(D|_C\right) - r - \sum_{\nu\in\partial^{\mathrm{out}} C} s_0^\nu(D) \\ \geq \deg\left(D|_C\right) - r - \sum_{\nu\in\partial^{\mathrm{out}} C} \mathrm{sl}_\nu f_x(x_\nu) \quad &= \quad \deg\left(D_x|_C\right) - r > 0\end{aligned}$$

as claimed. □

The proof of Theorem 3.6 shows the stronger bound $\mu_W(C) > g(C)r$. This is addressed later, in greater generality, in Corollary 3.13.

3.3. **Weierstrass measure.**

We prove Theorem 3.9 below, which will imply Theorem 1.7.

Definition 3.7. Fix a divisor D on a metric graph Γ, with Weierstrass locus $L_W(D)$. A subset $A\subset\Gamma$ is *$L_W(D)$-measurable* if A is a Borel set and, for every component C of the Weierstrass locus $L_W(D)$, we have either

$$C\subset A \qquad \text{or} \qquad C\subset \Gamma\setminus A.$$

Let $\mathcal{A}=\mathcal{A}(D)$ denote the σ-algebra of $L_W(D)$-measurable subsets of Γ. ◇

In other words, given a Weierstrass locus $L_W(D)\subset\Gamma$, we can construct the quotient map $\pi:\Gamma\to\Gamma_0$ in which each component $C_i\subset L_W(D)$ is contracted to a single point. Then, the $L_W(D)$-measurable sets of Γ are the preimages of Borel sets of Γ_0. If the divisor D is W-finite, then all Borel sets in Γ are $L_W(D)$-measurable.

Definition 3.8 (Weierstrass measure)**.** Notations as above, let D be an effective divisor of rank r on Γ, and let $\mathcal{A}$ denote the σ-algebra of $L_W(D)$-measurable subsets of Γ. We define the *Weierstrass measure* $\hat{\mu}_W$ as the "weighted counting measure" on Γ whose atoms are the connected components in the Weierstrass locus $L_W(D)$. More precisely, $\hat{\mu}_W$ is the measure on $(\Gamma,\mathcal{A})$ defined by

$$\hat{\mu}_W(A) \coloneqq \sum_{C\subset A} \mu_W(C),$$

where the sum is taken over components of $L_W(D)$ contained in A, and $\mu_W(C)$ is given by (1). ◇

We have the following description of the Weierstrass measure.

Theorem 3.9. *Notations as above, for any closed connected $A \in \mathcal{A}$, we have*

$$\hat{\mu}_W(A) = \deg\left(D|_A\right) + (g(A)-1)\,r - \sum_{\nu \in \partial^{\text{out}} A} s_0^\nu(D). \tag{2}$$

Proof. Let $\mathfrak{A} = \{C_1, \dots, C_n\}$ denote the set of components of $L_W(D)$ contained in A. Let $G = (V, E)$ be a model for Γ whose vertex set V contains the support of D, and let $V \cap (A \smallsetminus L_W(D)) = \{v_1, \dots, v_m\}$ denote the set of non-Weierstrass vertices in A. For each such vertex v_i, let $C_{n+i} = \{v_i\}$ denote the corresponding singleton, and let $\widetilde{\mathfrak{A}}$ denote the union

$$\widetilde{\mathfrak{A}} = \mathfrak{A} \cup \{\{v_1\}, \dots, \{v_m\}\} = \{C_1, \dots, C_n, C_{n+1}, \dots, C_{\tilde{n}}\} \qquad \text{where} \qquad \tilde{n} = n + m.$$

Finally, let $|\widetilde{\mathfrak{A}}| = \bigcup_{i=1}^{\tilde{n}} C_i$ be the underlying subset of Γ. Note that $|\widetilde{\mathfrak{A}}| \subset A$, and $A \smallsetminus |\widetilde{\mathfrak{A}}|$ consists of a union of finitely many open intervals; let k denote their number.

Let $V' \coloneqq V \smallsetminus L_W(D)$, as in Figure 4. For each $v \in V'$, we have $D_v(v) = r$, so

$$\mu_W(\{v\}) = D(v) - r - \sum_{\nu \in \mathrm{T}_v(\Gamma)} s_0^\nu(D) = D_v(v) - r = 0.$$

Thus, the "components" $C_{n+i} = \{v_i\}$ inside $\widetilde{\mathfrak{A}} \smallsetminus \mathfrak{A}$ do not contribute to the total weight, so it suffices to show that $\sum_{C_i \in \widetilde{\mathfrak{A}}} \mu_W(C_i)$ satisfies (2).

From Definition 1.6, we have

$$\sum_{i=1}^{\tilde{n}} \mu_W(C_i) = \sum_{i=1}^{\tilde{n}} \deg\left(D|_{C_i}\right) + r \sum_{i=1}^{\tilde{n}} (g(C_i) - 1) - \sum_{i=1}^{\tilde{n}} \left(\sum_{\nu \in \partial^{\text{out}} C_i} s_0^\nu(D) \right). \tag{3}$$

We treat separately the three terms appearing on the right-hand side of (3). The first term $\sum_i \deg\left(D|_{C_i}\right)$ is equal to $\deg\left(D|_A\right)$, since the vertex set V was chosen to contain the support of D.

For the second term, we apply the identity

$$\deg\left(K|_B\right) = 2g(B) - 2 + \mathrm{outval}(B) \qquad \text{for} \quad B \subset \Gamma \quad \text{closed and connected}$$

(see Lemma 3.16) twice to obtain

$$\begin{aligned} r \sum_{i=1}^{\tilde{n}} (g(C_i) - 1) &= \frac{r}{2} \sum_{i=1}^{\tilde{n}} \left(\deg\left(K|_{C_i}\right) - \mathrm{outval}(C_i) \right) \\ &= \frac{r}{2} \left(\deg\left(K|_A\right) - \mathrm{outval}(A) - 2k \right) \\ &= r\,(g(A) - 1) - rk, \end{aligned}$$

where k, we recall, denotes the numbers of edges of $\Gamma \smallsetminus |\widetilde{\mathfrak{A}}|$ whose endpoints are both in $|\widetilde{\mathfrak{A}}|$.

For the third term, the collection of all tangent directions $\bigcup_{C_i \in \widetilde{\mathfrak{A}}} \left\{ \nu \in \partial^{\text{out}} C_i \right\}$ can be partitioned into "paired" directions, if following ν leads to another component in $\widetilde{\mathfrak{A}}$, and "unpaired" directions, if following ν leads out of A. For any paired tangent direction $\nu \in \partial^{\text{out}} C_i$, there is a matching opposite direction $\overline{\nu} \in \partial^{\text{out}} C_j$ (see Section 1.6) and their minimum slopes satisfy $s_0^{\nu}(D) + s_0^{\overline{\nu}}(D) = -r$. For any unpaired tangent direction $\nu \in \partial^{\text{out}} C_i$, the minimum slope $s_0^{\nu}(D)$ is equal to $s_0^{\nu'}(D)$ for some parallel tangent direction $\nu' \in \partial^{\text{out}} A$. Moreover, this gives a bijection between $\partial^{\text{out}} A$ and the unpaired tangent directions. Using this, we have

$$\begin{aligned}\sum_{i=1}^{\tilde{n}} \left(\sum_{\nu \in \partial^{\text{out}} C_i} s_0^{\nu}(D) \right) &= \sum_{\text{unpaired } \nu} s_0^{\nu}(D) + \sum_{\text{paired } \nu} s_0^{\nu}(D) \\ &= \sum_{\nu \in \partial^{\text{out}} A} s_0^{\nu}(D) + \sum_{\ell=1}^{k} \left(s_0^{\nu_\ell}(D) + s_0^{\overline{\nu}_\ell}(D) \right) \\ &= \sum_{\nu \in \partial^{\text{out}} A} s_0^{\nu}(D) - rk.\end{aligned}$$

Combining the above identities shows that $\hat{\mu}_W(A)$ satisfies (2). □

Remark 3.10. For a closed subset $A \in \mathcal{A}$ with a finite number of connected components, the weight $\hat{\mu}_W(A)$ can be expressed equivalently as

$$\hat{\mu}_W(A) = \deg\left(D|_A\right) + (g(A) - c(A))\, r - \sum_{\nu \in \partial^{\text{out}} A} s_0^{\nu}(D)$$

where $c(A) = h_0(A)$ denotes the number of connected components of A. Note that $g(A) = h_1(A)$, so that in terms of the Euler characteristic χ, the middle term is $-r \cdot \chi(A)$. ◇

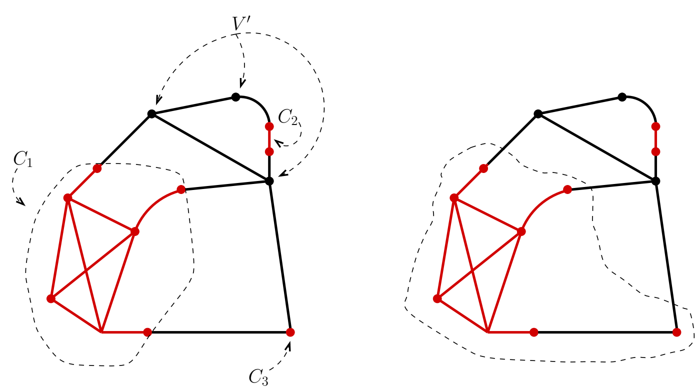

FIGURE 4. The part in red in the left figure is the (hypothetical) locus of Weierstrass points, and consists of three connected components. Red thickened points are on the boundary of the Weierstrass locus. Black vertices are those belonging to V', that is, outside the Weierstrass locus. They are three in number. The right figure is an example of a set A appearing in $\mathcal{A}$. There is no vertex in A outside the Weierstrass locus, so $m = 0$. There are two connected components of the Weierstrass locus in A, so $n = 2$. The subset $A \smallsetminus |\widetilde{\mathfrak{A}}|$ consists of four intervals. This means $k = 4$.

The following result can be obtained by the same method. Let U be a connected open subset of Γ which is $L_W(D)$-measurable.

Theorem 3.11. *Notations as above, the Weierstrass weight $\hat{\mu}_W(U)$ can be recovered from the slopes around the incoming branches as the sum*

$$\hat{\mu}_W(U) = \deg\left(D|_U\right) + (g(U)-1)\,r + \sum_{\nu\in\partial^{\mathrm{in}}U} s^{\nu}_{\max}(D)$$

where $\partial^{\mathrm{in}}U$ denotes the set of incoming unit tangent vectors from the boundary of U, and $s^{\nu}_{\max}(D)$ the maximum slope along the incoming tangent vector ν of any rational function in $\mathrm{Rat}(D)$.

Note that since U is open and $L_W(D)$ is closed, every $\nu \in \partial^{\mathrm{in}}U$ is tangent to an open interval on Γ which is outside $L_W(D)$ and thus $s^{\nu}_{\max}(D) = s^{\nu}_0(D)+r$ (see Theorem 1.9). Thus we have

$$\hat{\mu}_W(U) = \deg\left(D|_U\right) + (g(U)-1+\mathrm{inval}(U))\,r + \sum_{\nu\in\partial^{\mathrm{in}}U} s^{\nu}_0(D).$$

Proof of Theorem 1.7. We apply Theorem 3.9 with $A = \Gamma$. The statement about W-finite divisors follows from the first statement and Remark 3.3. □

3.4. **Consequences.** We now provide some direct consequences of the above results, starting with the following remark.

Remark 3.12. Theorem 1.7 and [Ric24, Theorem A] together imply that a generic divisor D of degree $d \geq g$ has a finite Weierstrass locus made up of $g\,(d-g+1)$ points, all of weight one. Indeed, the cardinality of $L_W(D)$ given by [Ric24, Theorem A] is $g\,(d-g+1)$, whereas the total weight given by Theorem 1.7 is $d-r+rg$. But $r = d-g$ generically and in this case we have $g\,(d-g+1) = g\,(r+1) = d-r+rg$. ◇

Corollary 3.13. *Suppose D is a divisor of rank r. For any closed, connected, $L_W(D)$-measurable subset $A \subset \Gamma$, we have*

$$\hat{\mu}_W(A) \geq g(A)\,r.$$

Proof. This follows from Theorem 3.9 and Lemma 2.5. □

Corollary 3.14 (Theorem 1.8)**.** *Suppose that the rank r of D is at least one. Then, the complement of the Weierstrass locus $L_W(D)$ is a disjoint union of (open) metric trees. In other words, every cycle in Γ intersects the tropical Weierstrass locus.*

Proof. For the sake of a contradiction, suppose that A is a cycle in Γ disjoint from the Weierstrass locus $L_W(D)$. Then, A is $L_W(D)$-measureable, and by definition (Definition 3.8), $\hat{\mu}_W(A) = 0$. However, Corollary 3.13 states that $\hat{\mu}_W(A) \geq g(A)\,r > 0$, which gives a contradiction. □

3.5. **Special cases of weights.** Here, we point out some special cases of the weight formula.

(i) If a divisor D has rank $r = 0$, then $\hat{\mu}_W(\Gamma) = d$. Suppose D is effective in its linear equivalence class. For any tangent direction ν outside the Weierstrass locus, the slope set $\mathfrak{S}^{\nu}(D)$ contains a single slope, and this slope must be zero since D is effective. Thus, a component C of the Weierstrass locus has weight $\mu_W(C) = \deg\left(D|_C\right)$.

(ii) If the genus $g = 0$, then for any divisor $\hat{\mu}_W(\Gamma) = d-r = 0$. (In general $0 \leq d-r \leq g$.) In particular, this implies that the Weierstrass locus $L_W(D)$ is empty.

(iii) If the genus $g = 1$, then for a divisor of degree d, the total Weierstrass weight is $\hat{\mu}_W(\Gamma) = d$. Every component C of the Weierstrass locus has weight $\mu_W(C) = 1$.

(iv) If the rank satisfies $r = d - g$, then $\hat{\mu}_W(\Gamma) = d - r + rg = g\,(r+1)$. In particular, this holds for a generic divisor class with degree $d \geq g$, and for every divisor with degree $d \geq 2g - 1$.
(v) If $D = K$ is the canonical divisor, then $d = 2g - 2$ and $r = g - 1$, so $\hat{\mu}_W(\Gamma) = g^2 - 1$. See Section 3.7 below for more discussion of this case.

3.6. **Combinatorial graphs.** In this section we assume Γ is a combinatorial graph. By this we mean Γ admits a model $(G = (V, E), \ell)$ which has unit edge lengths. We assume the divisor D is supported on the vertex set V.

Theorem 3.15. *Suppose $e = uv$ is an edge in G whose interior $\mathring{e}$ is $L_W(D)$-measurable. Let f_{uv} be a rational function that satisfies $\mathrm{div}(f_{uv}) = D_u - D_v$. Let ν be the unit tangent vector at v along e, towards u. Then, the Weierstrass weight of the interior of e is*

$$\hat{\mu}_W(\mathring{e}) = r - \mathrm{sl}_\nu(f_{uv}).$$

Proof. Let $U \coloneqq \mathring{e}$. Since $L_W(D)$ is closed, we can take the open interval U a little bit smaller so that its extremities are distinct from u and v and U still contains the same components of $L_W(D)$. Theorem 3.11 states that the sum of Weierstrass weights on $U = \mathring{e}$ is equal to

$$\hat{\mu}_W(U) = \deg\left(D|_U\right) + (g(U) - 1)\,r + \sum_{\nu \in \partial^{\mathrm{in}} U} s^\nu_{\max}(D).$$

Since D is supported on the vertex set, we have $\deg\left(D|_U\right) = 0$, and we also have $g(U) = 0$. Thus, the expression simplifies to

$$\hat{\mu}_W(U) = -r + \left(s^{v,\nu}_{\max}(D) + s^{u,\overline{\nu}}_{\max}(D)\right)$$

where ν and $\overline{\nu}$ are tangent directions towards u and v, respectively. If f_u and f_v satisfy

$$\mathrm{div}(f_u) = D_u - D \qquad \text{and} \qquad \mathrm{div}(f_v) = D_v - D,$$

then we have

$$\mathrm{sl}_{\overline{\nu}} f_u(u) = s^{u,\overline{\nu}}_0(D) = s^{u,\overline{\nu}}_{\max}(D) - r \qquad \text{and} \qquad \mathrm{sl}_\nu f_v(v) = s^{v,\nu}_0(D) = s^{v,\nu}_{\max}(D) - r,$$

and the relation $f_{uv} = f_u - f_v$ implies

$$\begin{aligned}\mathrm{sl}_\nu f_{uv}(u) &= \mathrm{sl}_\nu(f_u - f_v)(u) = -\left(s^{u,\overline{\nu}}_{\max}(D) - r\right) - \left(s^{v,\nu}_{\max}(D) - r\right)\\ &= 2r - \left(s^{v,\nu}_{\max}(D) + s^{u,\overline{\nu}}_{\max}(D)\right).\end{aligned}$$

Note that the slope of f_{uv} is constant along the interior of e, since the reduced divisors D_u and D_v are supported on vertices. □

3.7. **Canonical Weierstrass locus.** In this section we discuss the case of the canonical divisor on a metric graph.

3.7.1. *Weierstrass weight.* The weight formula (1) for $\mu_W(C; D)$ may be specialized to the case of the canonical divisor $D = K$. We need the following lemma.

Lemma 3.16. *Let $K = \sum_{x \in \Gamma} (\mathrm{val}(x) - 2)\,(x)$ denote the canonical divisor of Γ, and let $A \subset \Gamma$ be a closed connected subset. Then*

$$\deg\left(K|_A\right) = 2g(A) - 2 + \mathrm{outval}(A).$$

Proof. The proof can be obtained by direct calculation using an adapted graph model. The details are omitted. □

By direct summation, this result generalizes to closed subsets with finitely many connected components.

Theorem 3.17. *Suppose Γ is a metric graph of genus g, and let K be its canonical divisor. The weight of any component C of the Weierstrass locus $L_W(K)$ is*

$$\mu_W(C;K) = (g+1)(g(C)-1) - \sum_{\nu \in \partial^{\text{out}} C} (s_0^\nu(K) - 1).$$

More generally, for any closed, connected subset $A \subset \Gamma$ that is $L_W(K)$-measurable,

$$\hat{\mu}_W(A;K) = (g+1)(g(A)-1) - \sum_{\nu \in \partial^{\text{out}} A} (s_0^\nu(K) - 1).$$

Proof. Let $\partial^{\text{out}} C$ denote the set of outgoing tangent directions from C in Γ, and let $\text{outval}(C)$ denote its cardinality. From (1) we have

$$\mu_W(C;K) = \deg\left(K|_C\right) + r\,(g(C)-1) - \sum_{\nu \in \partial^{\text{out}} C} s_0^\nu(K).$$

The canonical divisor K has rank $r = g-1$. By Lemma 3.16, on a closed connected set $B \subset \Gamma$, the degree $\deg\left(K|_B\right)$ satisfies $\deg\left(K|_B\right) = 2g(B) - 2 + \text{outval}(B)$. Therefore,

$$\begin{aligned}
\mu_W(C;K) &= \deg\left(K|_C\right) + (g-1)(g(C)-1) - \sum_{\nu \in \partial^{\text{out}} C} s_0^\nu(K) \\
&= 2(g(C)-1) + \text{outval}(C) + (g-1)(g(C)-1) - \sum_{\nu \in \partial^{\text{out}} C} s_0^\nu(K) \\
&= (g+1)(g(C)-1) - \sum_{\nu \in \partial^{\text{out}} C} (s_0^\nu(K) - 1),
\end{aligned}$$

which concludes. ☐

If we repeat the same computation for the pluricanonical divisor nK, where $n \geq 2$, we find that

$$\hat{\mu}_W(A;nK) = (2n-1)g(g(A)-1) - \sum_{\nu \in \partial^{\text{out}} A} (s_0^\nu(nK) - n).$$

This next corollary to Theorem 3.17 is also a direct consequence of Theorem 1.7.

Corollary 3.18. *Suppose Γ is a genus g metric graph.*

(a) The sum of Weierstrass weights over all components of $L_W(K)$ is equal to $g^2 - 1$.
(b) For any integer $n \geq 2$, the sum of Weierstrass weights over all components of $L_W(nK)$ is equal to $(2n-1)g(g-1)$.

The next result is a special case of Corollary 3.13.

Corollary 3.19. *Suppose Γ is a metric graph of genus g. For any closed, connected, $L_W(K)$-measurable subset $A \subset \Gamma$, we have*

$$\hat{\mu}_W(A) \geq g(A)\,(g-1).$$

We end this section by providing a geometric interpretation of the tropical canonical Weierstrass locus. For the general description for any divisor D, see Remark 3.2.

The tropical canonical Weierstrass locus $L_W(K)$ can be described as an intersection as follows. Suppose $f : \Gamma \to \mathrm{Pic}^{g-2}(\Gamma)$ sends x to the divisor class $[K - g\,(x)]$, and let $h : \mathrm{Eff}^{g-2}(\Gamma) \to \mathrm{Pic}^{g-2}(\Gamma)$ be the inclusion of effective divisor classes in the space of all divisor classes of fixed degree $g-2$. The points in $L_W(K)$ are those such that $[K - g\,(x)] \geq 0$, or equivalently $[K - g\,(x)] \in \mathrm{Eff}^{g-2}(\Gamma)$. This description is summarized by the following pullback diagram.

$$\begin{array}{ccc} L_W(K) & \longrightarrow & \mathrm{Eff}^{g-2}(\Gamma) \\ \big\downarrow & \lrcorner & \big\downarrow h \\ \Gamma & \xrightarrow{\;f\;} & \mathrm{Pic}^{g-2}(\Gamma) \end{array}$$

The bottom horizontal map f sends x to the divisor class $[K - g\,(x)]$. The right vertical map h is the inclusion of effective divisor classes in the space of all divisor classes of fixed degree $g-2$. The points in $L_W(K)$ are those such that $[K - g\,(x)] \geq 0$, or equivalently $[K - g\,(x)] \in \mathrm{Eff}^{g-2}(\Gamma)$.

This description, which makes $L_W(K)$ sit inside the polyhedral complex $\mathrm{Eff}^{g-2}(\Gamma)$, brings forward the following open question.

Question 3.20. *It is possible to express the Weierstrass weights using this geometric description in a meaningful way?*

3.7.2. *Edge symmetry.* We now discuss properties of some specific Weierstrass points under some symmetry condition; see as well Section 6.5.

Definition 3.21. An edge e of a metric graph Γ is *reflexive* if there is an automorphism $\sigma : \Gamma \to \Gamma$ such that $\sigma(e) = \bar{e}$, i.e., σ reverses the direction of e. ◇

We show that the midpoint of a reflexive edge is either a Weierstrass point of K, or a Weierstrass point of nK for all $n \geq 2$.

Theorem 3.22. *Suppose Γ is a metric graph of genus $g \geq 2$, and let K denote the canonical divisor of Γ. Suppose e is a reflexive edge in Γ.*

(a) If g is even, then the midpoint of e is in the Weierstrass locus $L_W(K)$.

(b) If g is odd, then the midpoint of e is in the Weierstrass locus $L_W(nK)$ for any integer $n \geq 2$.

Proof. Let x denote the midpoint of the reflexive edge e. The tangent space $\mathrm{T}_x(\Gamma)$ contains two directions $\{\nu_1, \nu_2\}$, and the reflexive assumption implies that the minimum slopes are equal in both directions, i.e., $s_0^{\nu_1}(K) = s_0^{\nu_2}(K)$. If x is outside the Weierstrass locus, then the singleton $\{x\}$ is $L_W(K)$-measurable and we may apply the weight formula from Theorem 3.17,

$$\hat{\mu}_W(x; K) = (g+1)(-1) - 2(s_0^{\nu_1}(K) - 1) \equiv g + 1 \mod 2.$$

Hence if g is even, then $\hat{\mu}_W(x)$ is nonzero, which contradicts our assumption that x is outside the Weierstrass locus. This proves part (a).

Now consider $D = nK$ for $n \geq 2$. By a similar argument, if x is outside the Weierstrass locus $L_W(nK)$, then its Weierstrass weight is

$$\hat{\mu}_W(x; nK) = (2n-1)g(-1) - 2(s_0^{\nu_1}(nK) - n) \equiv g \mod 2.$$

If g is odd, then the weight $\hat{\mu}_W(x)$ is nonzero, which again gives a contradiction. This proves part (b). □

4. Generalizations

In this section, we generalize the setting of the previous sections to the case of augmented metric graphs, that is, in the presence of genera associated to the vertices.

Since the genus of a given vertex hides information about the geometry of the component, it turns out that there will be an ambiguity when talking about the Weierstrass locus of a divisor D. In fact, the right setup in this context is a divisor D endowed with the data of a closed sub-semimodule M of $\mathrm{Rat}(D)$, which plays the role of a (not necessarily complete) linear series on the augmented metric graph. In what follows, we will explain how the preceding definitions and results extend from divisors to semimodules in the more general setting of augmented metric graphs. We then introduce two special classes of semimodules, the *generic* semimodule associated to any divisor, and the *canonical* semimodule associated to the canonical divisor. We properly justify both of them using the framework of metrized complexes.

In the following, we assume all semimodules are nonempty unless specified otherwise.

4.1. **Weierstrass loci of semimodules and augmented metric graphs.**

4.1.1. *Semimodules.* Let Γ be a metric graph, and D a divisor of degree d on Γ. The set of functions $\mathrm{Rat}(D)$ naturally has the structure of a semimodule on the tropical semifield; we refer to [HMY12, AG22] for a discussion on this semimodule structure. Let M be a sub-semimodule of $\mathrm{Rat}(D)$. We endow $\mathrm{Rat}(D)$ with the topology induced by $\|\cdot\|_\infty$, and say $M \subset \mathrm{Rat}(D)$ is *closed* if it is closed with respect to this topology. The following is a direct extension to semimodules of the rank of divisors on graphs introduced by Baker and Norine [BN07].

Definition 4.1 (Divisorial rank)**.** The *divisorial rank* or simply *rank* of $M \subset \mathrm{Rat}(D)$ (also called the *rank of D with respect to M*) is the greatest integer r such that for any effective divisor E on Γ of degree r, there exists a function $f \in M$ verifying $D + \mathrm{div}(f) \geq E$. It is denoted by $r(M, D)$. ◇

In fact, as the following statement shows, the divisorial rank will only depend on the semimodule M, if we additionally assume that M is closed. Therefore, we will work only with closed semimodules in the following, and will denote their rank simply by $r(M)$. Note that any (nonempty) semimodule has rank $r(M) \geq 0$. Also note that by definition, we have the immediate inequality $r(M) \leq r(D)$.

Proposition 4.2. *The divisorial rank $r(M, D)$ of a closed semimodule $M \subset \mathrm{Rat}(D)$ depends only on M.*

Proof. First note that there is a unique minimal divisor D_0 such that $M \subset \mathrm{Rat}(D_0)$, which is obtained by taking the (point-wise) minimum of all such divisors.

Then, we denote $r(M, D)$ by r and $r(M, D_0)$ by r_0. It is clear from the inequality $D_0 \leq D$ that the inequality $r_0 \leq r$ holds. We thus prove that $r_0 \geq r$. We choose a model $G = (V, E)$ such that the vertex set contains the support of D.

First, we suppose that E is an effective divisor of degree r on Γ whose support is disjoint from the support of D. By definition of r, there exists $f \in M$ such that $D + \mathrm{div}(f) \geq E$. Since $M \subset \mathrm{Rat}(D_0)$ and D coincides with D_0 outside V, it follows that the divisor $D_0 + \mathrm{div}(f) \geq E$.

Now, let E be an effective divisor of degree r on Γ whose support may intersect that of D. Let $(E_n)_n$ be a sequence of divisors of degree r converging to E, such that for each n, the support of E_n is disjoint from V. By what precedes, for each n, there exists a function $f_n \in M$ such that $D_0 + \mathrm{div}(f_n) \geq E_n$. Without loss of generality, assume that $f_n(x_0) = 0$ for some $x_0 \in \Gamma$. Thanks to the boundedness of the slopes of functions in $\mathrm{Rat}(D_0)$ (see [GK08, Lemma 1.8]), we can assume that $(f_n)_n$ converges uniformly to a function f, which satisfies $D_0 + \mathrm{div}(f) \geq E$ at the limit. The limit function f is in M by assumption that M is closed, which concludes the argument. □

Remark 4.3. In essence, the above proof shows that the complement of the support of D is a "rank-determining set" for the semimodule M in the sense of [Luo11]. ⋄

The notion of minimum slopes naturally extends to closed semimodules.

Definition 4.4 (Slope sets and minimum slopes)**.** Let $M \subset \mathrm{Rat}(D)$ be a closed sub-semimodule. Given a point $x \in \Gamma$ and a tangent direction $\nu \in \mathrm{T}_x(\Gamma)$, let $\mathfrak{S}^\nu(M)$ denote the *slope set*

$$\mathfrak{S}^\nu(M) \coloneqq \{\mathrm{sl}_\nu f(x) : f \in M\}.$$

Let $s_0^\nu(M)$ denote the *minimum slope* along ν of functions in M. More generally, let $s_j^\nu(M)$ denote the $(j+1)$-smallest slope along ν of functions in M, i.e.,

$$s_0^\nu(M) = \min\{\mathfrak{S}^\nu(M)\}, \qquad s_j^\nu(M) = \min\{s \in \mathfrak{S}^\nu(M),\, s > s_{j-1}^\nu\}.$$

When the semimodule M is clear from context, we will simply use s_j^ν to denote $s_j^\nu(M)$. ⋄

The following result is obtained similarly to Proposition 2.5; we omit the details.

Proposition 4.5. *Suppose $M \subset \mathrm{Rat}(D)$ is a closed semimodule of divisorial rank r. Then for any closed, connected subset $A \subset \Gamma$, we have*

$$\deg\left(D|_A\right) - \sum_{\nu \in \partial^{\mathrm{out}} A} s_0^\nu(M) \geq r.$$

4.1.2. *Reduced divisors.* For closed $M \subseteq \mathrm{Rat}(D)$, there is a well-defined and well-behaved notion of x-reduced divisor denoted D_x^M linearly equivalent to D with respect to M for every $x \in \Gamma$. Simply, we define $f_x \colon \Gamma \to \mathbb{R}$ by setting

$$f_x(p) \coloneqq \inf_{\substack{f \in M \\ f(x)=0}} f(p) \quad \forall p \in \Gamma.$$

Using the boundedness of slopes [GK08, Lemma 1.8], the infimum in the definition above turns out to be a minimum, and f_x is the uniform limit of a sequence of elements in M. Therefore, $f_x \in M$. We set $D_x^M \coloneqq D + \mathrm{div}(f_x)$. It follows from the definition that $\mathrm{sl}_\nu f_x(x) = s_0^\nu$ for all $\nu \in \mathrm{T}_x(\Gamma)$, and $D_x^M(x) = D(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} s_0^\nu$. Therefore, the analogue of Lemma 2.3 holds.

4.1.3. *Augmented metric graphs.* An *augmented metric graph* is a metric graph Γ endowed with a model $(G = (V, E), \ell)$ and a genus function $\mathfrak{g} : V \to \mathbb{Z}_{\geq 0}$. The *genus* of $(\Gamma, \mathfrak{g})$, denoted by $g(\Gamma, \mathfrak{g})$ or simply g, is defined by

$$g(\Gamma, \mathfrak{g}) \coloneqq g(\Gamma) + \sum_{v \in V} \mathfrak{g}(v).$$

This terminology follows [ABBR15]; "vertex-weighted graph" is used in other places. Augmented metric graphs arise from the semistable reduction of smooth proper curves over a valued field, when remembering the genera $\mathfrak{g}(v) = g(X_v)$ of the components X_v, for $v \in V$.

Note that any metric graph is naturally an augmented metric graph, by declaring the genus function to be the zero function. This means that what we will discuss below applies equally to the setting of non-augmented metric graphs.

4.1.4. *Weierstrass locus.* We now extend the notion of tropical Weierstrass locus to semimodules in the general setting of augmented metric graphs. Let $(\Gamma, \mathfrak{g})$ be an augmented metric graph. Let D be a divisor on Γ and M be a closed sub-semimodule of $\mathrm{Rat}(D)$ of divisorial rank $r \leq r(D)$.

Definition 4.6 (Tropical Weierstrass locus of a closed semimodule)**.** The *tropical Weierstrass locus* of M, denoted by $L_W(M, D, \mathfrak{g})$ (or $L_W(M, \mathfrak{g})$ if D is clear from the context), is the set of all points $x \in \Gamma$ which verify $D^M_x(x) + (\mathfrak{g}(x) - 1)\, r > 0$.

In the case the genus function $\mathfrak{g}$ is zero, we lighten the notations and simply write $L_W(M, D)$, instead of $L_W(M, D, 0)$. We abbreviate $L_W(M, D)$ as $L_W(M)$ if D is clear from context. ⋄

The set $L_W(M, \mathfrak{g})$ is a closed subset of Γ that can in general be infinite. Note that for every $x \in \Gamma$, we have $D^M_x(x) \geq r$ and therefore $D^M_x(x) + (\mathfrak{g}(x) - 1)\, r \geq \mathfrak{g}(x)\, r \geq 0$. In particular, if $\mathfrak{g}(x) > 0$ and $r > 0$, then x belongs to the tropical Weierstrass locus.

We now associate an intrinsic weight to each connected component of the Weierstrass locus. The definition is analogous to Definition 1.6; here it is adapted to semimodules and depends on the genus function.

Let D be a divisor of degree d on Γ, and let $M \subset \mathrm{Rat}(D)$ be a closed sub-semimodule of divisorial rank r. We use the notations of Definition 1.6 for $\deg\big(D|_C\big)$, $g(C)$, and $\partial^{\mathrm{out}} C$; $s^\nu_0(M)$ is introduced in Definition 4.4.

Definition 4.7 (Intrinsic Weierstrass weight of a connected component)**.** Let C be a connected component of the tropical Weierstrass locus $L_W(M, \mathfrak{g})$. The *Weierstrass weight* of C, denoted by $\mu_W(C; M, D, \mathfrak{g})$, is defined by

$$\mu_W(C; M, D, \mathfrak{g}) \coloneqq \deg\big(D|_C\big) + \left(g(C) + \sum_{x \in C} \mathfrak{g}(x) - 1\right) r - \sum_{\nu \in \partial^{\mathrm{out}} C} s^\nu_0(M). \tag{4}$$

It is also denoted simply by $\mu_W(C; M, \mathfrak{g})$ or $\mu_W(C; \mathfrak{g})$ if M and D are understood from the context.

In the case the genus function is zero, we use $\mu_W(C; M, D)$, $\mu_W(C; M)$ or $\mu_W(C)$ for $\mu_W(C; M, D, 0)$. ⋄

This quantity is well-defined because any connected component of $L_W(M, \mathfrak{g})$ is a metric graph, a result that adapts directly from Proposition 3.1. As in the case of divisors (Proposition 3.1), $L_W(M, \mathfrak{g})$ has a finite number of connected components. And since Theorem 3.6 extends directly, we get $\mu_W(C; M, \mathfrak{g}) > 0$. We denote by $g(C, \mathfrak{g})$ the sum $g(C) + \sum_{x \in C} \mathfrak{g}(x)$, that is, the genus of C in the augmented metric graph $(\Gamma, \mathfrak{g})$.

Definition 4.8 (Tropical Weierstrass divisor)**.** We say that $(M, D, \mathfrak{g})$ is *Weierstrass finite* or simply *W-finite* if the tropical Weierstrass locus $L_W(M, D, \mathfrak{g})$ is finite. In this case, we define the *tropical Weierstrass divisor* $W(M, D, \mathfrak{g})$ as the effective divisor

$$W(M, D, \mathfrak{g}) \coloneqq \sum_{x \in L_W(M, \mathfrak{g})} \mu_W(x; M, D, \mathfrak{g})\, (x).$$

The tropical weight of x verifies $\mu_W(x; M, D, \mathfrak{g}) = D_x^M(x) + (\mathfrak{g}(x) - 1)\, r$. We abbreviate $W(M, D, \mathfrak{g})$ as $W(M, \mathfrak{g})$ if D is clear from the context. Note that the support $|(W(M, \mathfrak{g}))|$ of the tropical Weierstrass divisor is exactly the tropical Weierstrass locus $L_W(M, \mathfrak{g})$.

In the case the genus function is zero, we simply use $W(M, D)$ or $W(M)$ for $W(M, D, 0)$. $\diamond$

Remark 4.9. If we set $M = \mathrm{Rat}(D)$, and if the genus function is $\mathfrak{g} = 0$, then we recover the definitions given in Section 3 for a complete linear series on a non-augmented metric graph. Namely,

(i) For every $x \in \Gamma$, we have $D_x^{\mathrm{Rat}(D)} = D_x$.
(ii) We have $L_W(\mathrm{Rat}(D), 0) = L_W(D)$.
(iii) For every connected component C of $L_W(\mathrm{Rat}(D), 0)$, we have

$$\mu_W(C; \mathrm{Rat}(D), 0) = \mu_W(C; D).$$

(iv) D is W-finite if, and only if, $\mathrm{Rat}(D)$ is so. In this case, $W(\mathrm{Rat}(D), 0) = W(D)$. $\diamond$

The following proposition, a direct consequence of the definitions, states how the Weierstrass locus and Weierstrass weights on an augmented graph are related to the non-augmented definition.

Proposition 4.10. *If $M \subset \mathrm{Rat}(D)$ is a closed semimodule of rank r, then the following equalities hold.*
(a) $L_W(M, \mathfrak{g}) = L_W(M) \cup |\mathfrak{g}|$.
(b) For every connected component C of $L_W(M, \mathfrak{g})$, we have

$$\mu_W(C; M, D, \mathfrak{g}) = \mu_W(C; M, D) + r \sum_{x \in C} \mathfrak{g}(x).$$

4.1.5. *Total sum of Weierstrass weights.* The following theorem is an analogue of Theorem 1.9 for closed sub-semimodules of $\mathrm{Rat}(D)$, and is proved using a natural analogue of Lemma 2.3, given in Section 4.1.2. The only difference is that in the case of semimodules, sets of slopes are no longer necessarily made up of consecutive integers.

Theorem 4.11. *Let D be a divisor on Γ and M be a closed sub-semimodule of* $\mathrm{Rat}(D)$ *of divisorial rank r. We take a model for $(\Gamma, \mathfrak{g})$ such that the support of D is made up of vertices. Let $x \in \Gamma$ be a point and $\nu \in \mathrm{T}_x(\Gamma)$ be a tangent direction.*
(a) If the open interval $(x, x+\varepsilon\nu)$ is disjoint from the Weierstrass locus $L_W(M, \mathfrak{g})$, for $\varepsilon > 0$, then the set of slopes $\{\mathrm{sl}_\nu f(x) : f \in M\}$ consists of $r+1$ consecutive integers $\{s_0^\nu, s_0^\nu + 1, \dots, s_0^\nu + r\}$.
(b) If the open interval $(x, x+\varepsilon\nu)$ is contained in the Weierstrass locus $L_W(M, \mathfrak{g})$, then the set of slopes $\{\mathrm{sl}_\nu f(x) : f \in M\}$ consists of integers $\{s_0^\nu < s_1^\nu < \dots < s_t^\nu\}$ with $t \geq r$ and $s_t^\nu - s_0^\nu \geq r + 1$.

Part (a) implies in particular that for any edge e outside the Weierstrass locus of M, the number of slopes of functions on e is $r + 1$ and these slopes are consecutive.

As a corollary, following the same computation as in the case of a divisor, we get an analogue of Theorem 3.9.

Theorem 4.12 (Sum of Weierstrass weights for an incomplete series on an augmented metric graph)**.** *Suppose $(\Gamma, \mathfrak{g})$ is a genus $g = g(\Gamma, \mathfrak{g})$ augmented metric graph, D is a degree d divisor, and $M \subset \mathrm{Rat}(D)$ is a closed semimodule of divisorial rank $r \geq 0$.*

Then, the total sum of weights associated to connected components of $L_W(M,\mathfrak{g})$ *is equal to* $d-r+rg$. *In particular, if* M *is W-finite, then we have* $\deg(W(M,\mathfrak{g})) = d-r+rg$.

More generally, let $\mathcal{A}$ *denote the* σ*-algebra of* $L_W(M,\mathfrak{g})$*-measurable subsets of* Γ *and* $\hat{\mu}_W$ *the counting measure on* $(\Gamma,\mathcal{A})$ *associated to the weights* $\mu_W(C;M,\mathfrak{g})$ *given as above. Then, for any closed, connected* $A\in\mathcal{A}$, *we have*

$$\hat{\mu}_W(A;M,\mathfrak{g}) = \deg\left(D|_A\right) + (g(A,\mathfrak{g})-1)\,r - \sum_{\nu\in\partial^{\mathrm{out}}A} s_0^\nu(M), \tag{5}$$

where $g(A,\mathfrak{g})$ *denotes* $g(A)+\sum_{x\in A}\mathfrak{g}(x)$.

Theorem 4.12 implies the following analogue of Theorem 1.8.

Theorem 4.13. *If the divisorial rank* r *of* M *is at least one, then every closed connected subset* A *of* Γ *with* $g(A,\mathfrak{g})\geq 1$ *contains a point of* $L_W(M,\mathfrak{g})$.

Proof. Theorem 4.12 and Proposition 4.5 imply that for any closed, connected, $L_W(M,\mathfrak{g})$-measurable subset $A\subset\Gamma$, we have

$$\hat{\mu}_W(A;M,\mathfrak{g}) \geq g(A,\mathfrak{g})\,r,$$

an analogue of Corollary 3.13 for closed semimodules. Then, the argument used in the proof of Theorem 1.8 yields the result. □

4.1.6. *Coherence under inclusion of semimodules.* We have the following coherence property for the Weierstrass loci and weights associated to semimodules.

Proposition 4.14. *Let* $M\subset M'\subset \mathrm{Rat}(D)$ *be two closed semimodules of rank* r. *Then,* $L_W(M,\mathfrak{g})\subset L_W(M',\mathfrak{g})$ *and any* $L_W(M',\mathfrak{g})$*-measurable subset* A *of* Γ *is* $L_W(M,\mathfrak{g})$*-measurable. Moreover, the equality* $\hat{\mu}_W(A;M,\mathfrak{g}) = \hat{\mu}_W(A;M',\mathfrak{g})$ *holds.*

Proof. Note that the inclusion $M\subset M'$ implies that we have $D_y^M(y)\leq D_y^{M'}(y)$ for every $y\in\Gamma$. This, in turn, implies that $L_W(M,\mathfrak{g})\subset L_W(M',\mathfrak{g})$. The claim that A is $L_W(M,\mathfrak{g})$-measurable follows then, since A is assumed to be $L_W(M',\mathfrak{g})$-measurable.

To see that $\hat{\mu}_W(A;M,\mathfrak{g}) = \hat{\mu}_W(A;M',\mathfrak{g})$, it suffices to show that $s_0^\nu(M) = s_0^\nu(M')$ for each $\nu\in\partial^{\mathrm{out}}A$. Suppose ν is such a tangent direction pointing out of A. By part (a) of Theorem 4.11, there are exactly $r+1$ consecutive slopes of functions $F\in M'$ along ν. The same statement holds for M. Since $M\subset M'$, we infer that these slopes are the same. In particular, $s_0^\nu(M)=s_0^\nu(M')$, as desired. □

In the following two sections, we specialize the above constructions to two special families of closed semimodules M: the generic semimodule associated to any divisor D, and the canonical semimodule.

4.2. **The generic semimodule associated to a divisor.** Let $(\Gamma,\mathfrak{g})$ be an augmented metric graph. Denote by $|\mathfrak{g}|$ the support of $\mathfrak{g}$. For any divisor D on Γ, we define a closed semimodule $\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g})\subset\mathrm{Rat}(D)$.

Definition 4.15. The *generic linear series* or *generic semimodule* $\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g})$ consists of all rational functions f on Γ such that for every $x\in\Gamma$, we have the inequality

$$D(x)+\mathrm{div}(f)(x)\geq\mathfrak{g}(x). \qquad \diamond$$

Equivalently, we have the equality $\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g}) = \mathrm{Rat}(D_0)$ for the divisor D_0 defined by $D_0(x) \coloneqq D(x) - \mathfrak{g}(x)$, for every $x \in \Gamma$. (The claimed containment $\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g}) \subset \mathrm{Rat}(D)$ is clear.)

It follows that $\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g})$ is closed in the $\|\cdot\|_\infty$ topology of $\mathrm{Rat}(D)$.

Remark 4.16. The superscript "gen" stands for "generic" because, from the viewpoint of the degeneration of smooth projective curves, augmented metric graphs can be obtained from intermediate geometric objects called metrized complexes of curves. If this is the case, the above definition gives, precisely, the tropical part of the linear series of a divisor on the metrized complex in the case where the restriction of the divisor on every curve component of the metrized complex is generic. See Section 4.4 for more details. ⋄

The following statement computes the divisorial rank of the generic semimodule associated to a divisor.

Proposition 4.17. *Denote by r the divisorial rank of the generic semimodule $\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g})$, and let $r(D)$ and $r(D_0)$ denote the respective ranks of the two divisors D and D_0 in Γ without the genus function. We have the following (in)equalities.*

(a) $r \leq r(D)$;

(b) $r = r(D_0)$.

Proof. (a) The inequality follows from the containment $\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g}) \subset \mathrm{Rat}(D)$.

(b) This follows from Proposition 4.2 applied to $M \coloneqq \mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g}) = \mathrm{Rat}(D_0)$. □

Now that we have a closed sub-semimodule $\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g})$ of $\mathrm{Rat}(D)$ with a well-known divisorial rank, we can apply the machinery developed above.

Definition 4.18 (Generic tropical Weierstrass weights and locus of a divisor)**.** Notations as above, let D be a divisor on an augmented metric graph $(\Gamma,\mathfrak{g})$. The tropical Weierstrass locus, the Weierstrass weights, and the Weierstrass divisor (if it exists) are defined by plugging the semimodule $M \coloneqq \mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g})$ into Definitions 4.6, 4.7 and 4.8.

To lighten the notations while stressing the choice of the generic semimodule and the dependence on D and $\mathfrak{g}$, we write:

(i) $L_W^{\mathrm{gen}}(D,\mathfrak{g})$ for $L_W\left(\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g}),\mathfrak{g}\right)$;

(ii) $\mu_W^{\mathrm{gen}}(C;D,\mathfrak{g})$ for $\mu_W\left(C;\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g}),\mathfrak{g}\right)$; and

(iii) $W^{\mathrm{gen}}(D,\mathfrak{g})$ for $W\left(\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g}),\mathfrak{g}\right)$.

When D is clear from context, we simply use $\mu_W^{\mathrm{gen}}(C;\mathfrak{g})$ for $\mu_W^{\mathrm{gen}}(C;D,\mathfrak{g})$. ⋄

Note that when $\mathfrak{g}$ is the zero function, we have the equality $\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g}) = \mathrm{Rat}(D)$, and so the above definition recovers the one given in the previous sections for the Weierstrass divisor associated to a divisor.

Proposition 4.10 and a straightforward computation gives the following description of the generic Weierstrass locus.

Proposition 4.19. *The following equalities hold:*

(a) $L_W^{\mathrm{gen}}(D,\mathfrak{g}) = L_W(D_0) \cup |\mathfrak{g}|$;

(b) $\mu_W^{\mathrm{gen}}(C;D,\mathfrak{g}) = \mu_W\left(C;D_0\right) + (r+1)\sum_{x\in C}\mathfrak{g}(x)$.

In the remainder of this section, we discuss the generic semimodule associated to the canonical divisor. We first recall the definition of the canonical divisor in the augmented setting.

Definition 4.20 (Canonical divisor on an augmented metric graph)**.** Given an augmented metric graph $(\Gamma, \mathfrak{g})$, the *canonical divisor* K on $(\Gamma, \mathfrak{g})$ is defined by

$$K(x) \coloneqq \mathrm{val}(x) - 2 + 2\,\mathfrak{g}(x) \tag{6}$$

for each $x \in \Gamma$. ◇

Remark 4.21. In the context of augmented metric graphs, Lemma 3.16 becomes the following statement: for every closed connected subset $A \subset \Gamma$,

$$\deg\left(K|_A\right) = 2g(A) - 2 + 2\sum_{x\in A}\mathfrak{g}(x) + \mathrm{outval}(A). \quad ◇$$

The following statement gives the rank of the semimodule $\mathrm{Rat}^{\mathrm{gen}}(K, \mathfrak{g})$, which is not $g-1$ as one might expect.

Proposition 4.22 (Rank of the generic semimodule $\mathrm{Rat}^{\mathrm{gen}}(K, \mathfrak{g})$)**.** *If the genus function $\mathfrak{g}$ is nontrivial, the semimodule $\mathrm{Rat}^{\mathrm{gen}}(K, \mathfrak{g})$ has rank $g-2$.*

Proof. The rank of $\mathrm{Rat}^{\mathrm{gen}}(K, \mathfrak{g})$ coincides with the rank of $K_0 \coloneqq K - \sum \mathfrak{g}(x)\,(x)$ within the non-augmented metric graph Γ. Since the genus function is nontrivial, we have $\deg(K_0) = 2\,g(\Gamma) - 2 + \sum_x \mathfrak{g}(x) > 2\,g(\Gamma) - 2$ with $g(\Gamma)$ the genus of the non-augmented metric graph, and so, by Riemann–Roch on Γ, we have $r(K_0) = \deg(K_0) - g(\Gamma) = g(\Gamma, \mathfrak{g}) - 2$. □

In the next section, we define the canonical linear series for an augmented metric graph, and show it has the correct rank $g-1$.

Example 4.23. We compute the Weierstrass locus of the generic semimodule $\mathrm{Rat}^{\mathrm{gen}}(K, \mathfrak{g})$ on a cycle with one point of positive genus equal to two.

Let $(\Gamma, \mathfrak{g})$ be the augmented metric graph where Γ is the cycle of length one, parametrized by the interval $[0,\,1]$, the single vertex v coincides with the endpoints $v = 0 = 1$, and $\mathfrak{g}(v) = 2$. The genus of this augmented metric graph is $g = 3$.

We consider the canonical divisor K and the associated generic semimodule $\mathrm{Rat}^{\mathrm{gen}}(K, \mathfrak{g})$, as defined in the present section (see Definition 4.15). The rank is $r = g - 2 = 1$ according to Proposition 4.22, and the total weight of the Weierstrass locus is 6. The Weierstrass locus consists of the vertex v and the point of coordinate $\frac{1}{2}$. It is easy to compute that the weights are $\mu_W^{\mathrm{gen}}(v; K, \mathfrak{g}) = 5$ and $\mu_W^{\mathrm{gen}}\left(\frac{1}{2}; K, \mathfrak{g}\right) = 1$. Figure 5 shows the augmented metric graph and its Weierstrass locus. A generalization for any value of $\mathfrak{g}(v)$ is presented in Section 6.6.2. ◇

FIGURE 5. An augmented cycle graph with one point of genus two, the canonical divisor and its Weierstrass locus $L_W^{\mathrm{gen}}(D, \mathfrak{g})$.

4.3. **The canonical linear series on an augmented metric graph.** Consider the augmented metric graph $(\Gamma, \mathfrak{g})$ and its canonical divisor K, defined by $K(x) = \mathrm{val}(x) - 2 + 2\,\mathfrak{g}(x)$ for each $x \in \Gamma$. In this section, we define the linear series $\mathrm{KRat}(\mathfrak{g})$ associated to K, that we call the *canonical linear series* or *canonical semimodule*.

Definition 4.24. We define the *canonical semimodule* $\mathrm{KRat}(\mathfrak{g})$ as the set of all functions $f \in \mathrm{Rat}(\Gamma)$ which verify the following conditions:

(1) For every $x \in \Gamma$, we have $K(x) + \mathrm{div}(f)(x) \geq \mathfrak{g}(x) - 1$.
(2) If x has a tangent direction $\nu \in \mathrm{T}_x(\Gamma)$ such that $\mathrm{sl}_\nu f(x) \leq 0$, then $K(x)+\mathrm{div}(f)(x) \geq \mathfrak{g}(x)$. ⋄

The following set of conditions is equivalent to that of Definition 4.24.

(1) (local-minimum condition) If $x \in \Gamma$ is an *isolated local minimum* of f, i.e., $\mathrm{sl}_\nu f(x) \geq 1$ for every $\nu \in \mathrm{T}_x(\Gamma)$, then we impose $K(x) + \mathrm{div}(f)(x) \geq \mathfrak{g}(x) - 1$.
(2) (generic condition) For all other points $x \in \Gamma$, we impose the stricter condition $K(x)+\mathrm{div}(f)(x) \geq \mathfrak{g}(x)$.

Note that according to the above definition, if a point x has $\mathfrak{g}(x) = 0$, then x cannot be an isolated local minimum of $f \in \mathrm{KRat}(\mathfrak{g})$. Indeed, an isolated local minimum of f satisfies $\mathrm{div}(f)(x) \leq -\mathrm{val}(x)$, and so $K(x) + \mathrm{div}(f)(x) \leq -2$ assuming $\mathfrak{g}(x) = 0$, which would violate both conditions. This means that, for any $x \in \Gamma$ and $f \in \mathrm{KRat}(\mathfrak{g})$, we have $K(x) + \mathrm{div}(f)(x) \geq 0$, which implies that $\mathrm{KRat}(\mathfrak{g})$ is a subset of $\mathrm{Rat}(K)$. (It is easy to see that it is in fact a semimodule.) This shows, moreover, that the above definition is equivalent to Definition 4.15 outside of the support of $\mathfrak{g}$. Also note that we have the inclusion of semimodules $\mathrm{Rat}^{\mathrm{gen}}(K, \mathfrak{g}) \subset \mathrm{KRat}(\mathfrak{g})$.

Remark 4.25. The definition of the canonical semimodule differs from the generic semimodule $\mathrm{Rat}^{\mathrm{gen}}(K, \mathfrak{g})$ given by Definition 4.15. This is because the earlier definition, suitable for every divisor D on Γ, assumed D has "generic support" in the vertices with "hidden genus." The canonical divisor, however, is not generic. Its specific properties suggest a distinct definition for the complete linear series of K. The relevance of the above modification compared to Definition 4.15 will be further clarified in Section 4.4. ⋄

We have the following theorem which justifies the name given to the linear series $\mathrm{KRat}(\mathfrak{g})$. Recall that $g = g(\Gamma, \mathfrak{g})$.

Theorem 4.26. *The divisorial rank of the semimodule* $\mathrm{KRat}(\mathfrak{g})$ *is* $g - 1$.

Proof. The proof of this theorem will be given in Section 4.4.3. □

We have a closed sub-semimodule $\mathrm{KRat}(\mathfrak{g})$ of $\mathrm{Rat}(K)$ of divisorial rank $r = g - 1$, and we can apply the machinery developed for semimodules on augmented metric graphs.

Definition 4.27 (Canonical tropical Weierstrass weights and locus)**.** Notations as above, the canonical tropical Weierstrass locus, the Weierstrass weights, and the Weierstrass divisor on an augmented metric graph are defined by plugging the semimodule $M := \mathrm{KRat}(\mathfrak{g})$ into Definitions 4.6, 4.7 and 4.8.

To lighten the notations while stressing the choice of the canonical semimodule and the dependence on $\mathfrak{g}$, we write:

(i) $L_W(K, \mathfrak{g})$ for $L_W(\mathrm{KRat}(\mathfrak{g}), \mathfrak{g})$;
(ii) $\mu_W(C; K, \mathfrak{g})$ for $\mu_W(C; \mathrm{KRat}(\mathfrak{g}), \mathfrak{g})$; and
(iii) $W(K, \mathfrak{g})$ for $W(\mathrm{KRat}(\mathfrak{g}), \mathfrak{g})$. ⋄

Example 4.28. In this example, we compute the canonical Weierstrass locus on an augmented cycle with a point of genus two. For the case of the generic Weierstrass locus associated to the same divisor K, see Example 4.23.

Let $(\Gamma, \mathfrak{g})$ be the augmented metric graph where Γ is the cycle of length one, parametrized by the interval $[0, 1]$, the single vertex v coincides with the endpoints $v = 0 = 1$, and $\mathfrak{g}(v) = 2$. The genus of this augmented metric graph is $g = 3$.

We consider the canonical divisor K and the associated canonical semimodule $\mathrm{KRat}(\mathfrak{g})$, as defined in the present section (see Definition 4.24). The rank is $r = g - 1 = 2$ according to Theorem 4.26, and the total weight of the Weierstrass locus is $g^2 - 1 = 8$. The Weierstrass locus consists of the vertex v and the points of coordinates $\frac{1}{3}$ and $\frac{2}{3}$. The Weierstrass weights are $\mu_W(v; K, \mathfrak{g}) = 6$ and $\mu_W\left(\frac{1}{3}; K, \mathfrak{g}\right) = \mu_W\left(\frac{2}{3}; K, \mathfrak{g}\right) = 1$. Figure 6 shows the locus of Weierstrass points. A generalization for any value of $\mathfrak{g}(v)$ is presented in Section 6.6.1. ◇

FIGURE 6. An augmented cycle graph, the canonical divisor and its Weierstrass locus $L_W(K, \mathfrak{g})$.

In the rest of the paper, when handling the canonical divisor K on an augmented metric graph, the semimodule $\mathrm{KRat}(\mathfrak{g})$ will be preferred over $\mathrm{Rat}^{\mathrm{gen}}(K, \mathfrak{g})$, unless explicitly specified otherwise.

4.4. **Justification of the definition of Weierstrass loci for augmented metric graphs, in the generic and canonical case.** In this section, we provide a justification for the definitions we gave in Sections 4.2 and 4.3. This will be through divisor theory on metrized complexes, that we recall first. A purely metric graph justification, using metric graphs with shrinking parts, is sketched in Remark 4.34.

4.4.1. *Divisor theory on a metrized complex of curves.* We fix κ an algebraically closed field. A metrized complex of curves is, roughly speaking, the (metric realization of the) data of an augmented metric graph $(\Gamma, \mathfrak{g})$ endowed with a model $G = (V, E)$ and, for every $v \in V$, of a smooth, proper, connected, marked κ-curve $\mathcal{C}_v$ of genus $\mathfrak{g}(v)$ with marked points A_v in bijection with the branches of Γ incident to v. That is, a metrized complex of curves is a hybrid refinement of an augmented metric graph. For a full definition, see [AB15, Definition 2.17].

Let $\mathfrak{C}$ be a metrized complex of curves. A *divisor* $\mathfrak{D}$ on $\mathfrak{C}$ is a formal sum with integer coefficients of a finite number of points in $\mathfrak{C}$. We denote its hybrid rank on $\mathfrak{C}$ by $r_{\mathfrak{C}}(\mathfrak{D})$. By the forgetful projection map from $\mathfrak{C}$ to Γ, this gives rise to a divisor D on Γ of the same degree. Moreover, by restriction to each curve $\mathcal{C}_v$, for $v \in V$, we get a divisor $\mathcal{D}_v$ on $\mathcal{C}_v$. A rational function $\mathfrak{f}$ on $\mathfrak{C}$ consists of a rational function f on Γ and, for every $v \in V$, a nonzero rational function f_v on $\mathcal{C}_v$. The space of such functions $\mathfrak{f} = (f, f_v : v \in V)$ is denoted by $\mathrm{Rat}(\mathfrak{C})$.

Let now $\mathfrak{C}$ be a metrized complex of curves, with underlying metric graph Γ. Let $\mathfrak{D}$ be a divisor on $\mathfrak{C}$. We follow [AB15] and consider the linear series $\mathrm{Rat}(\mathfrak{D}, \mathfrak{C})$ defined as the subset of $\mathrm{Rat}(\mathfrak{C})$ consisting of all rational functions $\mathfrak{f} = (f \in \mathrm{Rat}(\Gamma); f_v \in \kappa(\mathcal{C}_v), v \in V)$ on $\mathfrak{C}$ that verify $\mathfrak{D} + \mathrm{div}(\mathfrak{f}) \geq 0$. This means that $D + \mathrm{div}(f)$ is effective on Γ, and for each $v \in V$, the divisor $\mathcal{D}_v - \sum_{\nu \in \mathrm{T}_v(\Gamma)} \mathrm{sl}_\nu f(v)\,(x_v^\nu) + \mathrm{div}(f_v)$ is effective on $\mathcal{C}_v$. Here, x_v^ν is the marked point on $\mathcal{C}_v$ that corresponds to ν.

Definition 4.29. We define $\mathrm{Rat}^{\mathrm{trop}}(D, \mathfrak{C})$ to be the subset of $\mathrm{Rat}(D)$ consisting of the tropical parts of all functions $\mathfrak{f} \in \mathrm{Rat}(\mathfrak{D}, \mathfrak{C})$. ◇

We omit the proof of the following result.

Proposition 4.30. $\mathrm{Rat}^{\mathrm{trop}}(D, \mathfrak{C})$ *is a closed sub-semimodule of* $\mathrm{Rat}(D)$.

We have the following comparison result, whose proof is direct from the definition of rank of divisors.

Proposition 4.31. *Let* $r(D, \mathfrak{C})$ *be the divisorial rank of the semimodule* $\mathrm{Rat}^{\mathrm{trop}}(D, \mathfrak{C})$. *Then we have the inequality*

$$r_{\mathfrak{C}}(\mathfrak{D}) \leq r(D, \mathfrak{C}).$$

The inequality in the above proposition can be strict in general. However, in some situations, e.g., for generic divisors on $\mathfrak{C}$ and for the canonical divisor, when the marked curves (C_v, A_v), for all $v \in V$, are generic in their moduli, we have the equality, as we explain now.

4.4.2. *The case of augmented metric graphs with generic divisors.* Condition (2) in the definition of $\mathrm{Rat}(\mathfrak{D}, \mathfrak{C})$ in the previous section justifies Definition 4.15. Indeed, take a rational function f on Γ such that for every $x \in \Gamma$, we have $D(x) + \mathrm{div}(f)(x) \geq \mathfrak{g}(x)$. Assume that the augmented metric graph $(\Gamma, \mathfrak{g})$ comes from a metrized curve complex $\mathfrak{C}$. Let $v \in \Gamma$ be a point underlying a curve $\mathcal{C}_v$. On the curve $\mathcal{C}_v$, the divisor $\mathcal{D}_v - \sum_{\nu \in \mathrm{T}_x(\Gamma)} \mathrm{sl}_\nu f(v)\,(x_v^\nu)$ has degree $\geq \mathfrak{g}(v)$ by assumption. Therefore, by the Riemann–Roch theorem, its rank is non-negative, which is precisely Condition (2) in the definition of $\mathrm{Rat}(\mathfrak{D}, \mathfrak{C})$. Now, in the other direction, if $\mathcal{D}_v$ is generic in the Picard group of $\mathcal{C}_v$ of relevant degree, then the divisor $\mathcal{D}_v - \sum_{\nu \in \mathrm{T}_v(\Gamma)} \mathrm{sl}_\nu f(v)\,(x_v^\nu)$ on $\mathcal{C}_v$ appearing in the second condition has non-negative rank only if it has degree at least $\mathfrak{g}(v)$. This means that Definition 4.15 is equivalent to the definition given for metrized complexes with a generic choice of divisors on components.

4.4.3. *The case of canonical divisor in augmented metric graphs.* We now justify Definition 4.24 using the terminology of Section 4.4.1, and also prove Theorem 4.26.

Let $G = (V, E)$ be a model of Γ whose vertex set contains all the points of valence different from two, and the support of $\mathfrak{g}$. Let $\mathfrak{C}$ be a metrized complex with underlying augmented metric graph $(\Gamma, \mathfrak{g})$. Denote by $\mathfrak{K}$ a canonical divisor for $\mathfrak{C}$ given by the collection of divisors $\mathcal{K}_{\mathcal{C}_v} + A_v = \mathcal{K}_{\mathcal{C}_v} + \sum_{\nu \in \mathrm{T}_v(\Gamma)} (x_v^\nu)$ on $\mathcal{C}_v$, where $\mathcal{K}_{\mathcal{C}_v}$ denotes a canonical divisor on $\mathcal{C}_v$, i.e., $\mathcal{O}(\mathcal{K}_{\mathcal{C}_v}) = \omega_{\mathcal{C}_v}$. The following claim justifies our definition of the canonical semimodule. We denote by $\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$ the tropical part of $\mathrm{Rat}(\mathfrak{K})$.

Proposition 4.32. *Notations as above, we have* $\mathrm{KRat}(\mathfrak{g}) \subseteq \mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$. *Moreover, if the markings* A_v *on the curves* $\mathcal{C}_v$ *are in general position, for all* $v \in V$, *then we have the equality* $\mathrm{KRat}(\mathfrak{g}) = \mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$.

Remark 4.33. This general position condition is the same as the one imposed in the work by Esteves and coauthors [EM02, ES07] in the special case of stable curves with two irreducible components, and stable curves in which any pair of components intersect. We will treat examples of augmented dipole graphs in Section 6. ◇

Proof of Proposition 4.32. We first prove the inclusion $\mathrm{KRat}(\mathfrak{g}) \subseteq \mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$. Consider an element $f \in \mathrm{KRat}(\mathfrak{g})$. We claim the existence of rational functions f_v on $\mathcal{C}_v$, for each $v \in V$, such that the collection $\{f, f_v, v \in V\}$ forms a rational function in $\mathrm{Rat}(\mathfrak{K})$. This proves the claim. Let $v \in V$, and consider the divisor $\mathcal{D}$ on $\mathcal{C}_v$ defined by f as follows:

$$\mathcal{D}_v \coloneqq \mathcal{K}_{\mathcal{C}_v} + \sum_{\nu \in \mathrm{T}_v(\Gamma)} (x_v^\nu) - \sum_{\nu \in \mathrm{T}_v(\Gamma)} \mathrm{sl}_\nu f(v)\,(x_v^\nu).$$

Note that the degree of $\mathcal{D}_v$ is precisely $K(v)+\mathrm{div}(f)(v)$. If the genus of v is zero, then by the condition $K(v)+\mathrm{div}(f)\geq 0$, the degree of $\mathcal{D}_v$ is non-negative and so there exists a rational function f_v on $\mathcal{C}_v$ such that $\mathcal{D}_v+\mathrm{div}(f_v)\geq 0$. If $\mathfrak{g}(v)\geq 1$ and v is not an isolated local minimum, then by the definition of $\mathrm{KRat}(\mathfrak{g})$, we have $\deg(\mathcal{D}_v)\geq \mathfrak{g}(v)$. By Riemann–Roch, this implies the existence of a function f_v such that $\mathcal{D}_v+\mathrm{div}(f_v)\geq 0$. Let $v\in\Gamma$ be a vertex of Γ such that $\mathfrak{g}(v)>0$ and which is an isolated local minimum of f. In this case, by the definition of $\mathrm{KRat}(\mathfrak{g})$, we have $\deg(\mathcal{D}_v)\geq \mathfrak{g}(v)-1$. The divisor $\mathcal{D}_v$ can be rewritten as $\mathcal{K}_{\mathcal{C}_v}-E$, where

$$E \coloneqq \sum_{\nu\in \mathrm{T}_v(\Gamma)} (\mathrm{sl}_\nu f(v)-1)\;(x_v^\nu)$$

is effective because v is an isolated local minimum of f. The Riemann–Roch theorem on $\mathcal{C}_v$, combined with the inequality $r(E)\geq 0$, thus yields

$$r(\mathcal{D}_v)=r\left(\mathcal{K}_{\mathcal{C}_v}-E\right)=r(E)+\deg(\mathcal{D}_v)-\mathfrak{g}(v)+1\geq 0.$$

That is, there exists a function f_v such that $\mathcal{D}_v+\mathrm{div}(f_v)\geq 0$. The rational function $\mathfrak{f}=(f,f_v:v\in V)$ on $\mathfrak{C}$ verifies $\mathfrak{K}+\mathrm{div}(\mathfrak{f})\geq 0$, as desired.

We now prove the inclusion $\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}\subseteq \mathrm{KRat}(\mathfrak{g})$ provided that the markings A_v on the curves $\mathcal{C}_v$, for all $v\in V$, are generic. First, we observe that $\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}\subseteq \mathrm{Rat}(K)$. Combining this with the results we proved in Section 2, it follows that the slopes taken by functions in $\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$ are bounded. Let f be an element of $\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$. We claim that under the general position assumption, we have $f\in\mathrm{KRat}(\mathfrak{g})$. Let v be a vertex of Γ. Resuming the notations introduced above, we write $\mathcal{D}_v$ for the divisor on $\mathcal{C}_v$ induced by f, and write it in the form $\mathcal{D}_v=\mathcal{K}_{\mathcal{C}_v}-E$.

First consider the case where v is an isolated local minimum of f. In this case, E is an effective divisor. We need to show that $\deg(E)\leq\mathfrak{g}(v)-1$. Indeed, otherwise, if $\deg(E)\geq\mathfrak{g}(v)$, then if the points x_v^ν, for $\nu\in\mathrm{T}_v(\Gamma)$, are in general position on $\mathcal{C}_v$, we will get $r(\mathcal{D}_v)\leq r(\mathcal{K}_{\mathcal{C}_v})-\mathfrak{g}(v)=-1$, which contradicts the assumption that $f\in\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$.

Consider the other case, where v is not an isolated minimum. In this case, the divisor E is not effective. We write $E=E_+-E_-$ where E_+ and E_- are the positive and negative parts of E, respectively. Note that E_+ and E_- are effective and they have disjoint support. Since E is not effective, E_- is non-zero, and so by Riemann–Roch, we have

$$r(\mathcal{K}_{\mathcal{C}_v}+E_-)=2\mathfrak{g}(v)-2+\deg(E_-)-\mathfrak{g}(v)=\mathfrak{g}(v)-2+\deg(E_-).$$

Now, we write

$$\mathcal{D}_v=\mathcal{K}_{\mathcal{C}_v}-E=\mathcal{K}_{\mathcal{C}_v}+E_--E_+$$

and observe, by the general position assumption on the points of A_v, that

$$r(\mathcal{D}_v)=\max\left\{-1,r(\mathcal{K}_{\mathcal{C}_v}+E_-)-\deg(E_+)\right\}.$$

Combining the two observations, we get

$$\begin{aligned} r(\mathcal{D}_v)=\max\left\{-1,\mathfrak{g}(v)-2+\deg(E_-)-\deg(E_+)\right\}&=\max\left\{-1,\deg(\mathcal{K}_{\mathcal{C}_v}-E)-\mathfrak{g}(v)\right\}\\ &=\max\left\{-1,\deg(\mathcal{D}_v)-\mathfrak{g}(v)\right\}.\end{aligned}$$

If $\deg(\mathcal{D}_v)<\mathfrak{g}(v)$, we get $r(\mathcal{D}_v)<0$, which would be a contradiction to the assumption that $f\in\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$. We conclude that $\deg(\mathcal{D}_v)\geq\mathfrak{g}(v)$, which leads to the inclusion $\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}\subseteq \mathrm{KRat}(\mathfrak{g})$. □

We now show that $\mathrm{KRat}(\mathfrak{g})$ has the expected rank $g-1$.

Proof of Theorem 4.26. We keep the notations as above. We denote by r the divisorial rank of $\mathrm{KRat}(\mathfrak{g})$.

It will be enough to show that if the markings A_v on the curves $\mathcal{C}_v$, for all $v \in V$, are in general position, then we have $r_{\mathfrak{C}}(\mathfrak{K}) = r$. By Riemann–Roch for metrized complexes proved in [AB15], we then obtain the equality $r = g-1$, as desired.

The inequality $r \geq r_{\mathfrak{C}}(\mathfrak{K})$ follows from the case of equality $\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}} = \mathrm{KRat}(\mathfrak{g})$ proved in the previous proposition, and by the definition of the rank in the metrized complex.

It remains to show the inequality $r_{\mathfrak{C}}(\mathfrak{K}) \geq r$. Let $\mathcal{E}$ be an effective divisor of degree r on $\mathfrak{C}$, and let E be the corresponding divisor on Γ. There exists a function $f \in \mathrm{KRat}(\mathfrak{g}) = \mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$ such that $E + \mathrm{div}(f) \geq 0$. If E has support outside the vertices of Γ, that is, $\mathcal{E}$ is entirely supported at the interior of edges of Γ, then using the arguments we used in the first part of Proposition 4.32, we deduce the existence of rational functions f_v on $\mathcal{C}_v$, for all $v \in V$, such that the rational function $\mathfrak{f} = (f, f_v : v \in V)$ on $\mathfrak{C}$ gives $\mathfrak{K} - \mathcal{E} + \mathrm{div}(\mathfrak{f}) \geq 0$, as desired.

Otherwise, if $\mathcal{E}$ has support in some of the curves $\mathcal{C}_v$, for $v \in V$, we write E as a limit of effective divisors E_n, for $n \geq 0$, of the same degree with support outside the vertices of Γ, and find elements f_n in $\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}} = \mathrm{KRat}(\mathfrak{g})$ which verify $K - E_n + \mathrm{div}(f_n) \geq 0$. Going to a subsequence, and using the boundedness of the slopes in $\mathrm{KRat}(\mathfrak{g})$, we can suppose that all the f_n have the same slopes along tangent directions at v, for each vertex $v \in V$. Moreover, changing the function $f \in \mathrm{KRat}(\mathfrak{g}) = \mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$ under the constraint that $E + \mathrm{div}(f) \geq 0$ if necessary, we can assume furthermore that f_n converges to f as n tends to infinity.

Denote by s_v^ν the slope of the f_n along the tangential direction $\nu \in \mathrm{T}_v(\Gamma)$. Let

$$\mathcal{D}_v \coloneqq \mathcal{K}_{\mathcal{C}_v} + \sum_{\nu \in \mathrm{T}_v(\Gamma)} (x_v^\nu) - \sum_{\nu \in \mathrm{T}_v(\Gamma)} \mathrm{sl}_\nu f(v)\,(x_v^\nu)$$

that we rewrite in the form

$$\mathcal{D}_v = \mathcal{K}_{\mathcal{C}_v} + \sum_{\nu \in \mathrm{T}_v(\Gamma)} (1 - s_v^\nu)\,(x_v^\nu) + \sum_{\nu} m_\nu\,(x_v^\nu)$$

with m_ν denoting the (weighted) number of points in the support of E_n tending to v through the tangential direction ν. Note that we have $\sum_{\nu \in \mathrm{T}_v(\Gamma)} m_\nu = E(v)$. Let

$$\mathcal{D}'_v \coloneqq \mathcal{K}_{\mathcal{C}_v} + \sum_{\nu \in \mathrm{T}_v(\Gamma)} (1 - s_v^\nu)\,(x_v^\nu).$$

Two cases can happen. Either, some of the slopes s_v^ν, for $\nu \in \mathrm{T}_v(\Gamma)$, are not positive, that is, v is not an isolated local minimum of f_n. In this case, the divisor $\mathcal{D}'_v$ has degree at least $\mathfrak{g}(v)$, which implies that it has non-negative rank. Or, all the slopes s_v^ν, for $\nu \in \mathrm{T}_v(\Gamma)$, are positive, that is, v is an isolated local minimum of f_n for all n. In this case, the divisor $\mathcal{D}'_v$ has degree at least $\mathfrak{g}(v) - 1$, and is the difference of $\mathcal{K}_{\mathcal{C}_v}$ and an effective divisor on $\mathcal{C}_v$. So again, it has non-negative rank.

In either case, we conclude that the divisor $\mathcal{D}'_v = \mathcal{D}_v - \sum_{\nu \in \mathrm{T}_v(\Gamma)} m_\nu\,(x_v^\nu)$ has non-negative rank. Since the points x_v^ν are assumed to be in general position on $\mathcal{C}_v$, it follows that the divisor $\mathcal{D}_v$ has rank at least $E(v) = \sum_{\nu \in \mathrm{T}_v(\Gamma)} m_\nu$. This shows the existence of a rational function f_v on $\mathcal{C}_v$ such that $\mathcal{D}_v - \mathcal{E}_v + \mathrm{div}(f_v) \geq 0$, with $\mathcal{E}_v$ being the part of $\mathcal{E}$ supported in $\mathcal{C}_v$. We conclude with the existence of a rational function $\mathfrak{f} = (f, f_v : v \in V)$ which verifies

$\mathfrak{K} - \mathcal{E} + \mathrm{div}(\mathfrak{f}) \geq 0$. This implies the inequality $r_{\mathfrak{C}}(\mathfrak{K}) \geq r$, and finishes the proof of our theorem. □

Remark 4.34. Definition 4.24 can be also justified using only the formalism of metric graphs and their limits. We briefly discuss this.

Suppose that the augmented metric graph $(\Gamma_0, \mathfrak{g})$ comes from a "limit family" of non-augmented metric graphs in the following sense. Let Γ be a (non-augmented) metric graph and $\Sigma \subset \Gamma$ a closed subset, where Σ has connected components $\Sigma_1, \dots, \Sigma_n$. For each $\varepsilon > 0$, consider the graph Γ_ε defined by shrinking every edge in Σ by the factor ε. As $\varepsilon \to 0$, the family Γ_ε converges to a metric graph Γ_0 (in the sense of Gromov–Hausdorff convergence). The limit metric graph Γ_0 is naturally equipped with a genus function $\mathfrak{g}$ where $\mathfrak{g}(v_i) = g(\Sigma_i)$ for each $v_i \in \Gamma_0$ that is the limit of a component Σ_i, and $\mathfrak{g}(x) = 0$ for all other $x \in \Gamma_0$. In this situation, we say that the augmented metric graph $(\Gamma_0, \mathfrak{g})$ is the limit of the shrinking family of the pair $\Sigma \subset \Gamma$.

Now consider the corresponding family of canonical series $\mathrm{Rat}(K_\varepsilon)$ on Γ_ε. For each $\varepsilon > 0$, the linear series $\mathrm{Rat}(K_\varepsilon)$ has rank $g-1$ on Γ_ε. The limit as $\varepsilon \to 0$ produces a semimodule of rational functions on Γ_0. We claim that this limit semimodule always contains $\mathrm{KRat}(\Gamma_0, \mathfrak{g})$, as described in Definition 4.24, and that if Σ is "generic" in an appropriate sense, then this limit is equal to $\mathrm{KRat}(\Gamma_0, \mathfrak{g})$.

We omit a proof of these claims here. The details can be verified using the theory of higher rank tropical curves and their algebro-geometric properties developed in [AN22, AN24]. ⋄

5. Tropical vs. algebraic Weierstrass loci

In the first sections of this paper, we associated a Weierstrass locus to a fixed divisor D on a metric graph, and then generalized this to a closed sub-semimodule of M the space $\mathrm{Rat}(D)$ on an augmented metric graph. However, those semimodules which come from tropicalization verify an extra set of properties, in particular, the following important one (see Section A.3):

($\star$) for each point x in Γ and any unit tangent direction $\nu \in \mathrm{T}_x(\Gamma)$, the set of slopes $\mathfrak{S}^\nu(M)$ taken by functions in M has size $r+1$.

In this section, we associate to any pair (M, D) consisting of a divisor D and a closed sub-semimodule $M \subset \mathrm{Rat}(D)$ that verifies ($\star$) a refined notion of Weierstrass divisor. It is inspired from the formula given in Theorem A.5, with the slopes being directly retrieved from M using property ($\star$). We then provide a comparison of this definition with that of Section 4. Using this link, we prove the main result of this section, Theorem 5.5, which relates tropical Weierstrass loci studied in the previous sections to tropicalization of Weierstrass divisors. We deduce then Theorem 1.10 as a special case of this result.

In the following, by an abuse of terminology, we refer to any pair (M, D) as above as a combinatorial limit linear series (clls). The terminology is borrowed from [AG22], however, the precise definition of combinatorial linear series requires more properties for the semimodule M. In our setting, we only need property ($\star$). The results can be thus applied more generally, in particular to the setting of tropical linear series developed in [JP22]. (No knowledge of [AG22, JP22] is required in this paper.)

5.1. **Weierstrass divisor of a combinatorial limit linear series.** Let D be a divisor on an augmented metric graph $(\Gamma, \mathfrak{g})$ and $M \subset \mathrm{Rat}(D)$ a closed sub-semimodule that verifies ($\star$). Since $M \subset \mathrm{Rat}(D)$ is closed, we can apply the machinery of Section 4. This point of view on Weierstrass loci however results in a loss in information provided by the slopes of M, unless the Weierstrass locus is finite. The following definition relies on the knowledge of the slopes along edges of G prescribed by M.

Definition 5.1. Suppose D is a divisor of degree d and M is a closed sub-semimodule in $\mathrm{Rat}(D)$ such that M verifies $(\star)$. The *clls Weierstrass divisor* of (M, D) is the divisor $W^{\mathrm{clls}}(M, D, \mathfrak{g})$ defined as

$$W^{\mathrm{clls}}(M, D, \mathfrak{g}) \coloneqq \sum_{x\in\Gamma} \mu_W^{\mathrm{clls}}(x)\,(x)$$

where *the clls Weierstrass weight* $\mu_W^{\mathrm{clls}}(x)$ of x is defined by

$$\mu_W^{\mathrm{clls}}(x) \coloneqq (r+1)\,D(x) + \frac{r(r+1)}{2}\,(\mathrm{val}(x) + 2\mathfrak{g}(x) - 2) - \sum_{\nu\in \mathrm{T}_x(\Gamma)} \sum_{j=0}^{r} s_j^{\nu}(M). \tag{7}$$

We write $W^{\mathrm{clls}}(M, D, \mathfrak{g})$ simply as $W^{\mathrm{clls}}(M, \mathfrak{g})$, the *clls Weierstrass divisor* of M, if D is understood from the context. If the genus function is trivial, $\mathfrak{g} = 0$, then we abbreviate $W^{\mathrm{clls}}(M, \mathfrak{g})$ to $W^{\mathrm{clls}}(M)$. ◇

Note that $W^{\mathrm{clls}}(M, \mathfrak{g})$ has finite support. Indeed, since elements in M are piecewise affine linear, we can find a model $G = (V, E)$ of Γ such that $s_0^\nu < s_1^\nu < \cdots < s_r^\nu$ is constant in the interior of any edge of $G = (V, E)$ for parallel unit tangent vectors ν based at points of the edge and pointing in the same direction. It follows that if $x \notin V$ and x is outside the supports of D and $\mathfrak{g}$, then $\mu_W^{\mathrm{clls}}(x) = 0$. Also note that the central term in the expression of $\mu_W^{\mathrm{clls}}(x)$ above is equal to $\frac{1}{2}r(r+1)K(x)$, where K is the canonical divisor on $(\Gamma, \mathfrak{g})$.

Example 5.2. Consider the non-augmented barbell graph Γ with edges of arbitrary length, see Figure 7. This metric graph has genus two and the canonical divisor has rank one. We define a sub-semimodule $M \subset \mathrm{Rat}(K)$ of rank one on Γ by prescribing the slopes $-1 < 1$ on the middle edge and, for $i = 1, 2$, slopes $0 < 1$ on both oriented edges $u_i v_i$. Then, we define M as the set of all functions in $\mathrm{Rat}(K)$ that, along any unit tangent vector at a given point of Γ, take one of the two prescribed slopes. It is easy to see that M is closed and verifies $(\star)$.

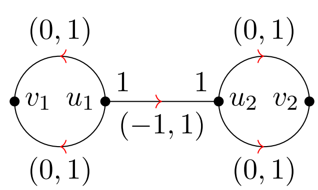

FIGURE 7. The barbell graph, the canonical divisor and the slope structure $\mathfrak{S}$.

The clls Weierstrass divisor is

$$W^{\mathrm{clls}}(M) = (u_1) + (u_2) + 2\,(v_1) + 2\,(v_2)$$

(see Figure 8, right). For comparison, the tropical Weierstrass locus $L_W(M, \mathfrak{g})$ of the semi-module M with trivial genus function $\mathfrak{g} = 0$ (as defined in Section 4), is shown on the same figure (left). Here, $L_W(M)$ turns out to be identical to the tropical Weierstrass locus of the complete linear series $L_W(K)$ (see Example 3.5). ◇

FIGURE 8. The tropical Weierstrass locus $L_W(M)$ (left) and the clls Weierstrass divisor $W^{\text{clls}}(M)$ (right) on the barbell graph.

5.2. **Comparison with the tropical Weierstrass locus.** The following proposition shows that the notion of clls Weierstrass divisor can be viewed as a refinement of the tropical Weierstrass locus defined in Section 4.

Proposition 5.3 (Comparison of the tropical and clls Weierstrass loci)**.** *Suppose $M \subset \mathrm{Rat}(D)$ is a combinatorial limit linear series of rank r with clls Weierstrass divisor $W^{\text{clls}}(M,\mathfrak{g})$. Let $L_W(M,\mathfrak{g})$ denote its Weierstrass locus, defined as in Section 4.1. If $A \subset \Gamma$ is closed, connected, and $L_W(M,\mathfrak{g})$-measurable, then we have the equality*

$$\deg\left(W^{\text{clls}}(M,\mathfrak{g})|_A\right) = (r+1)\,\hat{\mu}_W(A; M,\mathfrak{g}).$$

In particular, if M is W-finite as a semimodule, then the following equality holds:

$$W^{\text{clls}}(M,\mathfrak{g}) = (r+1)\,W(M,\mathfrak{g}).$$

Proof. We have

$$\deg\left(W^{\text{clls}}(M,\mathfrak{g})|_A\right) = (r+1)\sum_{x\in A} D(x) + \frac{r(r+1)}{2}\sum_{x\in A} K(x) - \sum_{x\in A}\left(\sum_{\nu\in \mathrm{T}_x(\Gamma)}\sum_{j=0}^{r} s_j^\nu\right)$$

where K denotes the canonical divisor on $(\Gamma,\mathfrak{g})$ (see Definition 4.20) and $s_j^\nu = s_j^\nu(M)$. The terms $(r+1)\,D(x)$ add up to the term $(r+1)\deg\left(D|_A\right)$. Remark 4.21 yields that the terms $K(x)$ add up to $2g(A)-2+2\sum_{x\in A}\mathfrak{g}(x)+\mathrm{outval}(A)$, where $\mathrm{outval}(A) \coloneqq \left|\partial^{\text{out}} A\right|$ is the number of outgoing branches from A.

The terms in the third part can be rearranged as a sum over directed edges of A, using some compatible model. Each edge has two in-going tangent directions, and the slope sums cancel out for this pair $(\nu,\overline{\nu})$ of opposing in-going directions since $s_j^\nu + s_{r-j}^{\overline{\nu}} = 0$. The only terms that do not cancel are the tangent directions which point out of A, i.e.,

$$\sum_{x\in A}\left(\sum_{\nu\in \mathrm{T}_x(\Gamma)}\sum_{j=0}^{r} s_j^\nu\right) = \sum_{\nu\in\partial^{\text{out}} A}\left(\sum_{j=0}^{r} s_j^\nu\right).$$

Combining these terms, we have

$$\begin{aligned}\deg\left(W^{\text{clls}}(M,\mathfrak{g})|_A\right) &= (r+1)\deg\left(D|_A\right) + \frac{r(r+1)}{2}\left(2g(A,\mathfrak{g})-2+\mathrm{outval}(A)\right)\\ &\qquad - \sum_{\nu\in\partial^{\text{out}} A}\sum_{j=0}^{r} s_j^\nu\\ &= (r+1)\deg\left(D|_A\right) + r(r+1)(g(A,\mathfrak{g})-1) - \sum_{\nu\in\partial^{\text{out}} A}\sum_{j=0}^{r}(s_j^\nu - j).\end{aligned}$$

Finally, we use the fact that $s_j^\nu = j + s_0^\nu$ for every j and for tangent directions ν outside the Weierstrass locus $L_W(M, \mathfrak{g})$, by Theorem 4.11. Thus,

$$\deg\left(W^{\mathrm{clls}}(M,\mathfrak{g})|_A\right) = (r+1)\left(\deg\left(D|_A\right) + (g(A,\mathfrak{g})-1)\,r - \sum_{\nu\in\partial^{\mathrm{out}}A} s_0^\nu\right),$$

which, using the same technique as in the proof of Theorem 3.9, gives the first statement. The second statement follows from the first by the expression of the Weierstrass weight of a connected component of the tropical Weierstrass locus which is reduced to a point. □

We have the following extension of the above proposition, using the notion of tangential ramifications introduced later in Section 5.5. In particular, the statement holds even if A is not $L_W(M, \mathfrak{g})$-measurable.

Proposition 5.4. *Notations as in Proposition 5.3, for any closed, connected $A \subset \Gamma$, the following equality holds*

$$\deg\left(W^{\mathrm{clls}}(M,\mathfrak{g})|_A\right) = (r+1)\left(\deg\left(D|_A\right) + (g(A,\mathfrak{g})-1)\,r - \sum_{\nu\in\partial^{\mathrm{out}}A} s_0^\nu(M)\right) - \sum_{\nu\in\partial^{\mathrm{out}}A}\sum_{j=0}^{r}\alpha_j^\nu(M),$$

where $\alpha_j^\nu(M) \coloneqq s_j^\nu(M) - j - s_0^\nu(M)$ are the tangential ramifications along ν.

5.3. **Tropicalization of Weierstrass loci.** The goal of this section is to prove Theorem 5.5, using the machinery developed for semimodules on augmented metric graphs (see Section 4.1). This provides a precise link between tropical Weierstrass loci and the tropicalization of Weierstrass divisors on algebraic curves. Using this result, we will deduce Theorem 1.11.

Let X be a smooth proper curve of genus g over an algebraically closed non-Archimedean field $\mathbb{K}$ of arbitrary characteristic with a non-trivial valuation. Let $\mathcal{L} = \mathcal{O}(\mathcal{D})$ be a line bundle of positive degree d on X. Let H be a vector subspace of global sections of $\mathcal{L}$ of rank r (i.e., $\dim H = r+1$), that we naturally view in the function field of X. When $\mathbb{K}$ has positive characteristic, we suppose that $\mathcal{L}$ is classical [Lak81, Nee84], that is, the gap sequence of H is the standard sequence $0 < 1 < \cdots < r$. We denote by $\mathcal{W} = \mathcal{W}(H)$ the corresponding Weierstrass divisor on X. Recall that $\mathcal{W}$ is the zero divisor of a global section, called the Wronskian, of the line bundle $\omega_X^{\otimes r(r+1)/2} \otimes \mathcal{L}^{\otimes(r+1)}$, see [Lak81] and Section A.4. In particular, we have

$$\deg(\mathcal{W}) = \frac{r(r+1)}{2}\,(2g-2) + (r+1)\,d = (r+1)\,(d - r + rg).$$

Let $(\Gamma, \mathfrak{g})$ be a skeleton of X^{an}, and let $\tau\colon X^{\mathrm{an}} \to \Gamma$ denote the specialization map. Let $W \coloneqq \tau_*(\mathcal{W})$ be the specialization of $\mathcal{W}$ to Γ. Note that $(\Gamma, \mathfrak{g})$ is an augmented metric graph. We let $D \coloneqq \tau_*(\mathcal{D})$ be the specialization of $\mathcal{D}$ to Γ, and let $M \subset \mathrm{Rat}(D)$ be the sub-semimodule consisting of the tropicalizations of non-zero rational functions in H. It follows from the slope formula that the divisorial rank of M is equal to the rank of H, see [AG22, Theorem 8.3] and [JP22, Proposition 4.1].

The following theorem compares the algebraic Weierstrass divisor of H on the curve X with the tropical Weierstrass divisor of M on the augmented metric graph $(\Gamma, \mathfrak{g})$.

Theorem 5.5 (Algebraic versus tropical weights: general case)**.** *Notations as above, let A be a closed, connected, $L_W(M,\mathfrak{g})$-measurable subset of Γ. Then, the total weight of Weierstrass points of $\mathcal{W}$ which tropicalize to A is given by*

$$\deg\left(\mathcal{W}|_{\tau^{-1}(A)}\right) = (r+1)\,\hat{\mu}_W(A;M,\mathfrak{g})$$

where

$$\hat{\mu}_W(A;M,\mathfrak{g}) = \deg\left(D|_A\right) + \left(g(A) + \sum_{x\in A}\mathfrak{g}(x) - 1\right)r - \sum_{\nu\in\partial^{\mathrm{out}}A} s_0^\nu(M).$$

In particular, if M is W-finite, we have the following equality of divisors on Γ:

$$\tau_*(\mathcal{W}) = (r+1)\,W(M,\mathfrak{g}).$$

Before proceeding to the proof, a remark is in order.

Remark 5.6. By Proposition 4.14, Theorem 5.5 holds in a slightly more general setting. Let M' be any closed sub-semimodule of $\mathrm{Rat}(D)$ of divisorial rank r containing M. Then, we have

$$\deg\left(\mathcal{W}|_{\tau^{-1}(A)}\right) = (r+1)\,\hat{\mu}_W(A;M',\mathfrak{g})$$

for every $L_W(M',\mathfrak{g})$-measurable subset A of Γ. ◇

Proof of Theorem 5.5. In the case the residue field of $\mathbb{K}$ has characteristic zero, we use Theorem A.5 which provides a description of the divisor $W = \tau_*(\mathcal{W})$ in terms of slope structures. As explained in Section A.3, the slopes at any point x and any unit tangent vector $\nu\in \mathrm{T}_x(\Gamma)$ of elements of the tropicalization M of H form a set of $r+1$ integers $s_0^\nu, s_1^\nu, \dots, s_r^\nu$. The definition of the Weierstrass divisor associated to a tropical linear series is chosen to ensure the equality $W = W^{\mathrm{clls}}(M,\mathfrak{g})$, which implies

$$\deg\left(\mathcal{W}|_{\tau^{-1}(A)}\right) = \deg\left(W^{\mathrm{clls}}(M,\mathfrak{g})|_A\right).$$

Proposition 5.3 states that if A is $L_W(M,\mathfrak{g})$-measurable, then

$$\deg\left(W^{\mathrm{clls}}(M,\mathfrak{g})|_A\right) = (r+1)\,\hat{\mu}_W(A;M,\mathfrak{g}),$$

from which the result follows.

In the general case, when the characteristic of $\mathbb{K}$ is arbitrary and the gap sequence of H is standard, we use the description of the reduction of the Weierstrass divisor to the skeleton given in Theorem A.2. Using the notations of Appendix A, letting $W = \tau_*(\mathcal{W})$, we have

$$W(x) = (r+1)D(x) + \frac{r(r+1)}{2}K(x) - \sum_{\nu\in\mathrm{T}_x(\Gamma)} \mathrm{sl}_\nu F,$$

with $F = \mathrm{trop}(\mathrm{Wr}_{\mathcal{F}})$. Furthermore, $\mathrm{sl}_\nu F = \frac{r(r+1)}{2} + \mathrm{ord}_{p_x^\nu}\widetilde{\mathrm{Wr}_{\mathcal{F}_x}}$.

Since the slopes along the unit tangent vectors $\nu\in\mathrm{T}_x(\Gamma)$ which are outgoing from A form a consecutive sequence of integers, by Proposition A.4 we infer that the quantity $\mathrm{ord}_{p_x^\nu}\widetilde{\mathrm{Wr}_{\mathcal{F}_x}}$ is equal to $s_0^\nu + \cdots + s_r^\nu$. Using Theorem A.2, we get $\mathrm{sl}_\nu F = s_0^\nu + \cdots + s_r^\nu$.

Moreover, since F belongs to $\mathrm{Rat}(\Gamma)$, the total sum of the slopes of F for the edges which appear in the interior of A vanish. We infer that

$$\begin{aligned}\deg\left(\mathcal{W}|_{\tau^{-1}(A)}\right) &= (r+1)\sum_{x\in A} D(x) + \frac{r(r+1)}{2}\sum_{x\in A} K(x) - \sum_{x\in A}\sum_{\nu\in \mathrm{T}_x(\Gamma)} \mathrm{sl}_\nu F \\ &= (r+1)\sum_{x\in A} D(x) + \frac{r(r+1)}{2}\sum_{x\in A} K(x) - \sum_{x\in A}\left(\sum_{\nu\in \mathrm{T}_x(\Gamma)}\sum_{j=0}^{r} s_j^\nu\right) \\ &= \deg\left(W^{\mathrm{clls}}(M,\mathfrak{g})|_A\right) = (r+1)\,\hat{\mu}_W(A; M, \mathfrak{g}),\end{aligned}$$

as required. □

5.4. **Proofs of Theorems 1.10 and 1.11.** We deduce Theorem 1.10 from Theorem 5.5.

Proof of Theorem 1.10. Since D and $\mathcal{D}$ have the same rank r, we can plug $H \coloneqq \mathrm{Rat}(\mathcal{D})$ and $M' \coloneqq \mathrm{Rat}(D)$ into Remark 5.6, following Theorem 5.5, to get

$$\deg\left(\mathcal{W}(\mathcal{D})|_{\tau^{-1}(A)}\right) = (r+1)\left(\deg\left(D|_A\right) + r\left(g(A,\mathfrak{g})-1\right) - \sum_{\nu\in\partial^{\mathrm{out}} A} s_0^\nu(D)\right). \tag{8}$$

In the context of Theorem 1.10, $\mathfrak{g} = 0$. The result follows. □

Using this result, we can prove Theorem 1.11.

Proof of Theorem 1.11. This follows from the combination of Theorem 4.12, Proposition 4.5 and Theorem 5.5. □

5.5. **Tangential ramification sequence and effectivity.** Unlike the tropical Weierstrass divisors defined earlier in this paper, the Weierstrass divisor defined in Definition 5.1 is not automatically effective. We can rewrite the Weierstrass weight as

$$\begin{aligned}\mu_W^{\mathrm{clls}}(x) &= (r+1)\left(D(x) + \frac{r}{2}\mathrm{val}(x) + (\mathfrak{g}(x)-1)\,r\right) \\ &\qquad - \sum_{\nu\in \mathrm{T}_x(\Gamma)}\sum_{j=0}^{r}\left(s_0^\nu + j + \left(s_j^\nu - s_0^\nu - j\right)\right) \\ &= (r+1)\left(D_x^M(x) + (\mathfrak{g}(x)-1)\,r\right) - \sum_{\nu\in \mathrm{T}_x(\Gamma)}\sum_{j=0}^{r}\left(s_j^\nu - s_0^\nu - j\right) \\ &= r(r+1)\mathfrak{g}(x) + \underbrace{(r+1)\left(D_x^M(x) - r\right)}_{\geq 0} - \underbrace{\sum_{\nu\in \mathrm{T}_x(\Gamma)}\sum_{j=0}^{r}\left(s_j^\nu - s_0^\nu - j\right)}_{\geq 0}.\end{aligned}$$

Definition 5.7 (Tangential ramification sequence)**.** We call the sequence

$$\{\alpha_j^\nu(M) \coloneqq s_j^\nu(M) - s_0^\nu(M) - j : j = 0, 1, \dots, r\}$$

the *ramification sequence* of M at x along the tangential direction ν. This sequence is non-decreasing. ⋄

This motivates the following definition.

Definition 5.8 ($\mathfrak{g}$-effective linear series)**.** Let $\mathfrak{g}$ be a genus function on Γ. The combinatorial limit linear series M is called $\mathfrak{g}$-*effective* if $W^{\text{clls}}(M, \mathfrak{g})$ is effective. That is, for all $x \in \Gamma$,

$$r(r+1)\mathfrak{g}(x) + (r+1)\left(D_x^M(x) - r\right) \geq \sum_{\nu \in \mathrm{T}_x(\Gamma)} \sum_{j=0}^{r} \alpha_j^\nu(M). \tag{9}$$

◇

We say that $M \subset \mathrm{Rat}(\Gamma, \mathfrak{g})$ is *realizable* if there exists a smooth proper curve X of genus g over $\mathbb{K}$, a line bundle $\mathcal{L} = \mathcal{O}(\mathcal{D})$ of degree d and a subspace $H \subset H^0(X, \mathcal{L})$ of rank r such that $(\Gamma, \mathfrak{g})$ is a skeleton of X^{an}, and $M = \{\mathrm{trop}\,(f) : f \in H \smallsetminus \{0\}\}$. If this happens over $\mathbb{K}$ of equicharacteristic zero, we say M is realizable in equicharacteristic zero.

Proposition 5.9. *If M is realizable in equicharacteristic zero, then the following hold:*

(i) $W^{\text{clls}}(M, \mathfrak{g})$ is effective, i.e., M is $\mathfrak{g}$-effective.
(ii) the divisor of degree zero

$$W^{\text{clls}}(M, \mathfrak{g}) - (r+1)\,D - \frac{r(r+1)}{2}\,K = \sum_{x \in \Gamma} \left(\sum_{\nu \in \mathrm{T}_x(\Gamma)} \sum_{j=0}^{r} s_j^\nu \right)(x)$$

is principal.

Proof. Both statements follow from Theorem A.5. □

Example 5.10. Consider the non-augmented metric graph Γ below and its canonical divisor K. We consider the following combinatorial limit linear series $M \subset \mathrm{Rat}(K)$. For each bridge edge oriented outwards (towards the adjacent circle), allow slopes $-1 < 1 < 3$. Divide each circle into three equal parts, in a way compatible with the position of the attachment points. On the two edges adjacent to the attachment points, allow slopes $0 < 1 < 2$ away from the attachment points, and on the remaining edges, allow slopes $-1 < 0 < 1$ (see Figure 9).

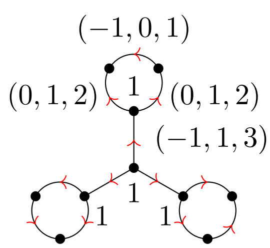

FIGURE 9. Three-cycle graph with a specified slope structure on $\mathrm{Rat}(K)$, defining a combinatorial limit linear series $M \subset \mathrm{Rat}(K)$.

We can define a suitable closed sub-semimodule $M \subset \mathrm{Rat}(K)$ of rank two of functions compatible with this choice of slopes. The tropical Weierstrass locus $L_W(M)$ of the semimodule M, in the sense of Section 4.1 (with $\mathfrak{g} = 0$), contains the bridge edges and the points of coordinates $\frac{1}{3}$ and $\frac{2}{3}$ on the circles (see Figure 10, left). In particular, M is not W-finite. The clls Weierstrass divisor $W^{\text{clls}}(M)$ is also shown in the figure (right). In particular, M is not $\mathfrak{g}$-effective. ◇

FIGURE 10. The tropical Weierstrass locus $L_W(M)$ (left) and the clls Weierstrass divisor $W^{\text{clls}}(M)$ (right).

6. EXAMPLES

We here discuss several examples in order to illustrate the results of the previous sections.

6.1. **Dipole graph.** Suppose Γ is a dipole graph of genus $g \geq 2$ (also known as a "banana" graph), consisting of two vertices joined by $g+1$ edges, possibly of different lengths. The canonical divisor K has coefficient $g-1$ on each vertex. The Weierstrass locus $L_W(K)$ consists of the interval $[\ell/g, \ell - \ell/g]$ on every edge, with ℓ the length of that edge (see Figure 11 for $g=3$). Each component $C \subset L_W(K)$ has two outgoing directions, and in each outgoing direction, the minimum slope is $s_0^\nu = -(g-1)$. By Theorem 3.17, the Weierstrass weight of each component is

$$\mu_W(C) = (g+1)(g(C)-1) - \sum_{\nu \in \partial^{\text{out}} C} (s_0^\nu - 1) = (g+1)(-1) - (-g-g) = g-1.$$

The total Weierstrass weight of $L_W(K)$ is g^2-1, as expected (Corollary 3.18 (a)).

FIGURE 11. Dipole graph of genus $g=3$ and its Weierstrass locus $L_W(K)$.

6.2. **Tent graph.** Consider the tent graph G, consisting of three vertices and five edges, as shown in Figures 12, 13 and 14. We first consider the case $D = K$, a divisor of rank $r = g-1 = 2$. We have, for each of the Weierstrass points located at the endpoints of the bottom edge in Figure 12, $\mu_W(v) = (3+1)(-1) - (-2-2-2) = 2$. The other four Weierstrass points are located on either of the four other edges respectively, one third of the distance from the top vertex to the other endpoints. Their weight is 1.

Now consider the case $D = K + (v)$ for v the vertex of degree four, a divisor also of rank $r = 2$. $L_W(D)$ has a unique component, see Figure 13, and by Theorem 3.17, its weight is $\mu_W(C) = \deg\left(D|_C\right) + (g(C)-1)\,r - \sum_{\nu \in \partial^{\text{out}} C} s_0^\nu = 5 + (2-1)\cdot 2 - (-1-1) = 9$.

Finally, consider $D = K + (u)$ for u one of the vertices of degree three, a divisor still of rank $r = 2$. See Figure 14. The two singleton components of $L_W(D)$ each have weight one. Suppose C is the non-singleton component of $L_W(D)$, whose boundary points on both left-hand edges are located one third of the distance from the top vertex. Theorem 3.17 gives $\mu_W(C) = 7$.

FIGURE 12. Tent graph and its Weierstrass locus $L_W(K)$.

FIGURE 13. A divisor on the tent graph and its Weierstrass locus.

FIGURE 14. A divisor on the tent graph and its Weierstrass locus.

6.3. **Cube graph.** The cube graph is shown in Figure 15, with all edges of length one. It has genus 5 and the canonical divisor K has rank 4. The Weierstrass locus $L_W(K)$ consists of the closed segment $[2/5, 3/5]$ on each edge, and excludes the vertices. Each component C of $L_W(K)$ has out-valence 2, with minimum slopes in $\mathrm{Rat}(K)$ in each outgoing direction equal to -3. Theorem 3.17 gives $\mu_W(C) = 2$. There are 12 components, so the total weight is 24.

FIGURE 15. Cube graph with its Weierstrass locus $L_W(K)$.

6.4. **Bridge edges.** We expand on the barbell graph (Example 3.5).

Theorem 6.1 (Weierstrass loci and bridge edges)**.** *Let Γ be a metric graph which has a bridge edge e such that each component of $\Gamma \smallsetminus \mathring{e}$ has positive genus. Then, the edge e is contained in the canonical Weierstrass locus $L_W(K)$.*

Proof. To show this, let u_1 and u_2 denote the endpoints of e, and Γ_1 and Γ_2 be the components of $\Gamma \smallsetminus \mathring{e}$ containing u_1 and u_2, respectively. If g, g_1 and g_2 are the genera of Γ, Γ_1 and Γ_2 respectively, then $g = g_1 + g_2$. Let $r = g - 1$ be the rank of the canonical divisor on Γ. We want to show that we can move $r + 1 = g_1 + g_2$ chips to every point $x \in e$. For $i = 1, 2$, denoting by K_i the canonical divisor of Γ_i, we have $r_{\Gamma_i}\Big(K|_{\Gamma_i} - (u_i)\Big) = r_{\Gamma_i}(K_i) = g_i - 1 \geq 0$, which implies that, using only functions on Γ that are constant outside Γ_i, we can move g_i chips to u_i. It is easy to see that we can move chips along e to put $g_1 + g_2$ chips at x. $\square$

Figure 16 shows an example where the Weierstrass locus strictly contains the bridge edges. Γ has two bridge edges and is of genus 5. All the edges of Γ are taken of unit length. The boundary points of the Weierstrass locus on the left and right circle are the points of coordinates $\frac{1}{5}$, $\frac{3}{5}$ and $\frac{4}{5}$ on each of the six corresponding edges, 1 being the outermost point. The sum of all Weierstrass weights is $12 + 6 \cdot 2 = 24 = g^2 - 1$, as expected.

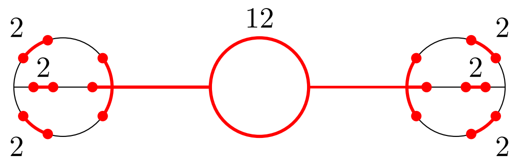

FIGURE 16. A graph with two bridge edges and its Weierstrass locus $L_W(K)$.

6.5. **Cases where the whole graph is Weierstrass.** We provide two infinite families of examples for which the Weierstrass locus is the whole graph.

Example 3.4 treated the complete graph on four vertices with unit edge lengths, with the Weierstrass locus consisting of the four vertices. Now consider the case Γ is the complete graph on $n \geq 5$ vertices with unit edge lengths. This graph has genus $g = \frac{n^2-3n+2}{2}$, and the canonical divisor K has rank $\frac{n^2-3n}{2}$. The Weierstrass locus of K is the whole graph as $K_x(x) \geq g$. Indeed, $K(v) = n - 3$ on each vertex v, and the reduced divisor at v is $K_v = (n^2 - 3n)\,(v)$ (move all chips to v in a single firing). For x in the interior of an edge, the reduced divisor K_x leaves $(n-1)$ chips away from x, i.e., $K_x(x) = n^2 - 4n + 1$. Note that $n^2 - 4n + 1 \geq \frac{n^2-3n+2}{2}$ for $n \geq 5$. This provides a first infinite family with the whole metric graph Weierstrass.

We now give a second such family. In this family, the choice of the length function is free and there are infinitely many possible choices of divisors with this property on the same metric graph. See also [Ric24, Example 4.6]. Let Γ be the metric graph generalizing the barbell graph (Example 3.5) to any number of cycles. More precisely, take $g \geq 2$ cycles of arbitrary length and join them all to a central vertex v with a bridge edge of positive length, as in Figure 17. Consider the divisor $D = d\,(v)$, with $d \geq 3$. By Clifford's theorem, the rank of D satisfies the bound $r \leq d - 2$. Since a divisor of positive degree on a cycle has rank one less than the degree, and since chips can move freely on bridge edges, it is easy to show $D_x(x) \geq d - 1 \geq r + 1$ for every $x \in \Gamma$. Therefore, the Weierstrass locus is the whole graph.

FIGURE 17. The generalized barbell graph, the divisor D and its Weierstrass locus $L_W(D)$.

Note that in the second family the quantity $\min_{x\in\Gamma}(D_x(x) - r)$ can be arbitrarily large. The existence of these two families of examples, with very different combinatorial properties (for example, the first is made up of graphs with high connectivity, whereas the graphs of the second have many bridge edges), suggests the following.

Question 6.2. *Provide a classification of all graphs G that admit a length function and a divisor with Weierstrass locus the whole metric graph. Among them, what are the ones for which this property holds for every choice of edge lengths?*

6.6. **Augmented cycle with one point of positive genus.** We compute Weierstrass loci for the canonical divisor with respect to the canonical and generic linear systems on an augmented cycle on which one point has positive genus, generalizing Examples 4.28 and 4.23. The canonical case recovers a result of Diaz [Dia85, Theorem A2.1]: the generic non-separating node on a uninodal stable curve is a limit of exactly $g(g-1)$ Weierstrass points on nearby smooth curves.

Let a be a positive integer, and consider the augmented metric graph $(\Gamma, \mathfrak{g})$ where Γ is the cycle of length one, parametrized by the interval $[0, 1]$, the single vertex v coincides with the endpoints $v = 0 = 1$, and $\mathfrak{g}(v) = a$. The genus of this augmented metric graph is $g = a + 1$.

6.6.1. *The case of the canonical linear system.* We expand on Example 4.28 for which $a = 2$ was fixed. Consider the canonical divisor K and the associated canonical semimodule $\mathrm{KRat}(\mathfrak{g})$, as defined in Section 4.3. The rank is $r = g-1 = a$ according to Theorem 4.26, and the total weight of the Weierstrass locus is $g^2 - 1 = a^2 + 2a$. The Weierstrass locus consists of the vertex v and all the points of the form $\frac{k}{a+1}$ for $k = 1, \dots, a$. The Weierstrass weights are $\mu_W(v; K, \mathfrak{g}) = a^2 + a$ and $\mu_W\left(\frac{k}{a+1}; K, \mathfrak{g}\right) = 1$. Figure 18 shows the canonical divisor and its (canonical) Weierstrass locus depicted in the middle.

6.6.2. *The case of the generic linear system.* In the second case, we generalize Example 4.23 and consider the same divisor K as above, but take the generic semimodule $\mathrm{Rat}^{\mathrm{gen}}(K, \mathfrak{g})$ as defined in Section 4.2. In this case, the rank is $r = g - 2 = a - 1$ (see Proposition 4.22) and the total weight of the Weierstrass locus is $a^2 + a$. The Weierstrass points are v, and all the points $\frac{k}{a}$ with $k = 1, \dots, a-1$. The weights are $\mu_W(v; K, \mathfrak{g}) = a^2 + 1$ and $\mu_W\left(\frac{k}{a}; K, \mathfrak{g}\right) = 1$. Figure 18 shows the canonical divisor on the left, and its generic Weierstrass locus depicted on the right.

We note that the Weierstrass loci are different even though they are both finite. The total weights are also different, as the underlying semimodules have different ranks.

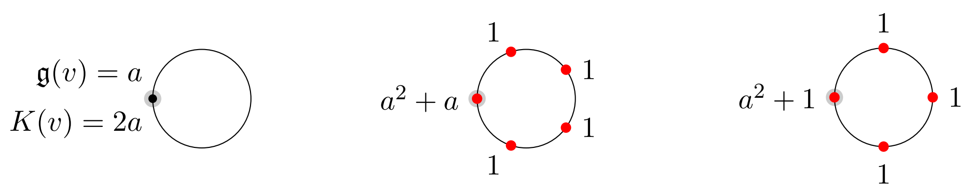

FIGURE 18. An augmented cycle graph, with its canonical Weierstrass locus $L_W(K, \mathfrak{g})$ in the middle, and its Weierstrass locus $L_W^{\mathrm{gen}}(K, \mathfrak{g})$ on the right. The drawing is made for $a = 4$.

6.7. **Augmented cycle with two points of positive genus.** Now consider an augmented cycle with exactly two points of positive genus. We describe the Weierstrass locus $L_W(K, \mathfrak{g})$.

Suppose the augmented metric graph $(\Gamma, \mathfrak{g})$ consists of two vertices u and v connected by two edges of length α and β, and the vertices have genus $\mathfrak{g}(u) = g_1$ and $\mathfrak{g}(v) = g_2$. The genus of $(\Gamma, \mathfrak{g})$ is $g = g_1 + g_2 + 1$, and the rank of the canonical linear system $\mathrm{KRat}(\mathfrak{g})$ is $g_1 + g_2$. We parametrize Γ by the interval $[0, \alpha + \beta]$ with 0 and $\alpha + \beta$ identified, $u = 0$ and $v = \alpha$ (see Figure 19).

The canonical system is W-finite, and $W(K, \mathfrak{g}) = W_{\mathrm{aug}} + W_{\mathrm{met}}$ where

- $W_{\mathrm{aug}} = g_1 g\,(u) + g_2 g\,(v) = g_1(g_1 + g_2 + 1)\,(u) + g_2(g_1 + g_2 + 1)\,(v)$, see Figure 20; and
- $W_{\mathrm{met}} = \sum_{i=1}^{g_1}(x_i) + \sum_{j=1}^{g_2}(y_j)$ where

$$x_i \coloneqq \alpha + \frac{i}{g_1 + g_2 + 1}\beta - \frac{g_1 + 1 - i}{g_1 + g_2 + 1}\alpha, \quad \text{and} \quad y_j \coloneqq \frac{j}{g_1 + g_2 + 1}\alpha - \frac{g_2 + 1 - j}{g_1 + g_2 + 1}\beta$$

modulo $\alpha + \beta$, for every $1 \le i \le g_1$, and $1 \le j \le g_2$, see Figure 20. It turns out that these $g_1 + g_2$ points are all distinct.

If we additionally assume that the edge lengths α and β are generic, then all x_i's and y_j's are also distinct from u and v. The total weight is $g^2 - 1$. If $g_2 = 0$, we recover the example in Section 6.6.

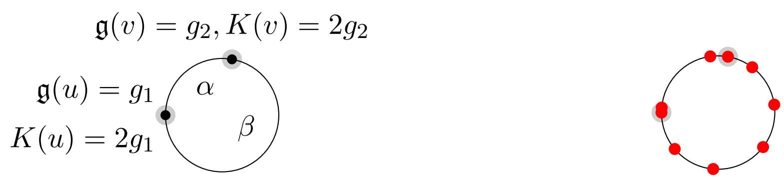

FIGURE 19. An augmented cycle with two points of positive genus and its Weierstrass locus $L_W(K, \mathfrak{g})$ (here, $g_1 = 4$, $g_2 = 3$). Weights are given in Figure 20.

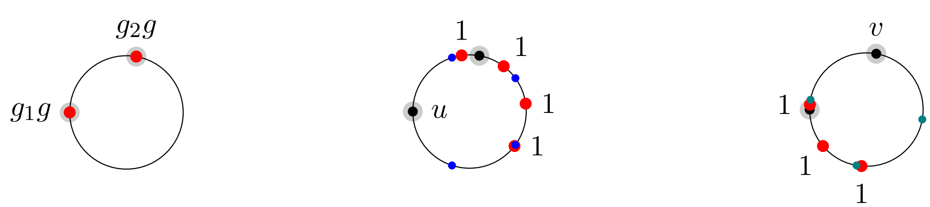

FIGURE 20. The three different types of Weierstrass points with $g_1 = 4$, $g_2 = 3$. The points in blue are the (g_1+1)-torsion points with respect to u, and the points in teal are the $(g_2 + 1)$-torsion points with respect to v.

6.8. **Augmented dipole graph.** We now consider an augmented dipole graph made up of two vertices u and v joined by $n = h + 1$ edges of arbitrary lengths, where h is the genus of the corresponding metric graph. We assume $\mathfrak{g}$ has support in $\{u, v\}$, and denote by a and b the genus of u and v, respectively, with $a \le b$. This metric graph is the one that appears in the work by Esteves and Medeiros [EM02]. As we explained previously, the canonical linear series reflects the genericity of the points of intersection on each of the two components.

The canonical divisor has coefficients $K(u) = h - 1 + 2a$ and $K(v) = h - 1 + 2b$. The total genus is $g = g(\Gamma, \mathfrak{g}) = h + a + b$, and the rank r of the canonical linear series is equal to $g - 1 = h + a + b - 1$ according to Theorem 4.26. We compute $L_W(K, \mathfrak{g})$.

In the case $h = 0$, if a and b are both positive, then $L_W(K, \mathfrak{g}) = \Gamma$. Otherwise, if $a = 0$, and b is at least two, then $L_W(K, \mathfrak{g}) = \{v\}$, and the Weierstrass weight is $b^2 - 1$. If $b = 1$ or 0, then the Weierstrass locus is empty. The case $h = 1$ was treated separately in Section 6.7.

We now suppose $h \geq 2$. The determination of the Weierstrass locus turns out to be complicated in general, and its shape depends on the values of a, b, h and the edge lengths. We illustrate the computation in two concrete cases.

6.8.1. *First particular case.* Suppose $a = b = 1$, all the edges have unit length, and the genus of the metric graph is $h \geq 2$. We have $r = h + 1$ and $g = h + 2$.

Then the Weierstrass locus is made up of both vertices u and v, along with the segment $\left[\frac{2}{h+2}, \frac{h}{h+2}\right]$ on each edge (see Figure 21). The vertices u and v and have weight $2h + 2$ and each segment in the interior of an edge has weight $h - 1$. The total weight is $g^2 - 1$.

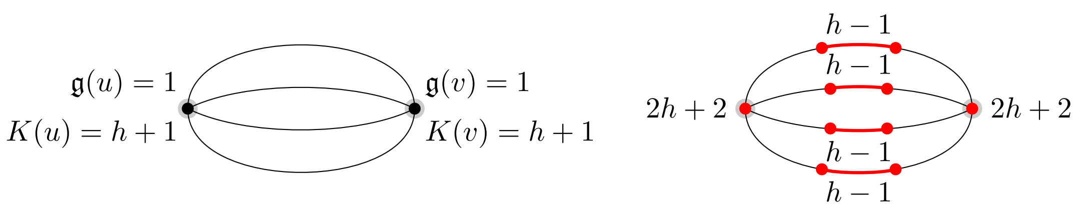

FIGURE 21. Augmented dipole graph with combinatorial genus $h = 3$, genera $a = b = 1$, all edges of unit length, and its Weierstrass locus $L_W(K, \mathfrak{g})$.

6.8.2. *Second particular case.* Suppose $a = 3$, $b = 5$, $h = 2$, and all the edges have unit length. We have $r = 9$ and $g = 10$. The Weierstrass locus is made up of the vertex v (weight 50), the union of the three segments $[0,\ 1/10]$ lying on each edge (weight 34), the point of coordinate $6/10$ on each edge (weight 1), and the segments $[3/10,\ 4/10]$ and $[8/10,\ 9/10]$ on each edge (each of weight 2). See Figure 22. The total weight is $50 + 34 + 3 \cdot (2 + 1 + 2) = 99 = g^2 - 1$.

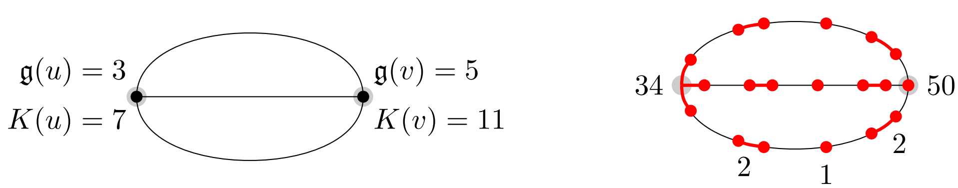

FIGURE 22. Augmented dipole graph with $h = 2$, $a = 3$ and $b = 5$, edges all of unit length, and its Weierstrass locus $L_W(K, \mathfrak{g})$.

6.9. **Weierstrass divisor of a combinatorial limit linear series.** We go back to the non-augmented dipole graph with four edges (of unit length to simplify the notations), a particular case of the class of examples presented in Section 6.1. The genus is $g = 3$ and the rank of the canonical divisor K is $r = 2$. Denote by u and v the two vertices and by e_1, e_2, e_3 and e_4 the four edges of Γ (see Figure 23, left). For $i = 1, 2, 3, 4$, let $t_i \in \left[0,\ \frac{1}{6}\right]$. For each choice of the t_i's, we will construct a semimodule M of rank two that verifies condition $(\star)$ from Section 5, and compute its clls Weierstrass divisor $W^{\mathrm{clls}}(M)$ (here, $\mathfrak{g} = 0$).

Assume t_i's are fixed. We prescribe the set of slopes taken by functions in M as in Figure 23. For each i, we endow the edge e_i with the slope sets $0 < 1 < 2$ on the interval $\left[0, \frac{1}{2} - t_i\right]$, slopes $-1 < 0 < 1$ on the interval $\left[\frac{1}{2} - t_i, \frac{1}{2} + t_i\right]$, and slopes $-2 < -1 < 0$ on the interval $\left[\frac{1}{2} + t_i, 1\right]$.

We define M as the sub-semimodule of $\mathrm{Rat}(K)$ consisting of all the functions that take one of the prescribed slopes above at any point of Γ along any unite tangent vector. We thus get a 4-parameter family of pairs (M, D) of rank two, $M \subset \mathrm{Rat}(K)$, that verify $(\star)$.

The clls Weierstrass divisor is given by (7), and yields $W^{\mathrm{clls}}(M) = 3\sum_{i=1}^{4}\left((x_i) + (y_i)\right)$, where x_i and y_i denote the points of coordinates $1/2 - t_i$ and $1/2 + t_i$ on the edge e_i, respectively. Figure 23 gives a visual rendering of $W^{\mathrm{clls}}(M)$ for the choice $(t_1, t_2, t_3, t_4) = \left(\frac{1}{6}, 0, \frac{1}{12}, \frac{1}{8}\right)$.

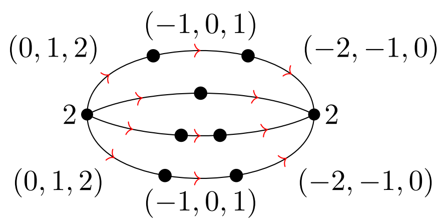

FIGURE 23. Dipole graph and its clls Weierstrass divisor $W^{\mathrm{clls}}(M)$.

6.10. **Combinatorial graphs without Weierstrass points.** There exist combinatorial graphs which do not have any Weierstrass point. Using [HKN13], this is equivalent to saying that in the metric graph obtained by assigning uniform edge lengths equal to one to all edges of G, the connected components of the Weierstrass locus $L_W(K)$ of the canonical divisor live in the interior of the edges of G. Such graphs are interesting from the point of view of arithmetic geometry, see [Bak08, Section 4] and [Ogg78, LN64, Atk67, AP03].

The dipole graph is an example of such a graph, see Figure 11. So is the cube graph, see Figure 15. Figure 24 shows another example. We refer to Section 7.5 for further discussion.

FIGURE 24. The canonical divisor of a combinatorial graph and the distribution of the Weierstrass weights on the edges of the corresponding metric graph with unit lengths. A black edge has total weight zero, and the interior of a light-red edge has total weight one. This indicates that the Weierstrass locus is concentrated in the interior of certain edges and does not contain any vertex.

7. FURTHER DISCUSSIONS

We discuss other interesting questions and results related to the content of the paper.

7.1. **Total locus of Weierstrass points.** Let $G = (V, E, \mathfrak{g})$ be a stable augmented graph of genus g, that is, a combinatorial graph of genus h endowed with a genus function $\mathfrak{g}\colon V \to \mathbb{N} \cup \{0\}$ such that any vertex of genus zero has valence at least three. Its total genus is $g = h + \sum_{v \in V} \mathfrak{g}(v)$. We view G as the dual graph of a stable curve X of total genus g with components X_v, for $v \in V$. Any one-parameter family of curves $\mathfrak{X}_t$ with fiber $\mathfrak{X}_0 = X$ and smooth fibers away from 0 gives rise to an edge length function $\ell\colon E \to (0, +\infty)$. Reparametrization of the family leads to another length function which is a homothety of ℓ. Every length function ℓ arises in this way from a one-parameter family of curves $\mathfrak{X}_t$, see e.g. [ABBR15, Theorem 3.24].

Given a fixed edge $e \in E$, consider the set of all the edge length functions ℓ which give $\ell(e) = 1$. Any family of curves with a stable curve X as fiber at zero whose dual graph is G gives rise, after a possible reparametrization, to such a length function. We will refer to such a family as being (G, e)-*admissible*.

Denote by Γ_ℓ the augmented metric graph associated to the pair (G, ℓ) augmented with the genus function $\mathfrak{g}$. The metric graphs Γ_ℓ all share an interval of length one corresponding to the edge e. We denote by $L_W^{\mathrm{can}}(\Gamma_\ell)$ the Weierstrass locus of the canonical divisor in $(\Gamma_\ell, \mathfrak{g})$, using the semimodule $\mathrm{KRat}(\mathfrak{g})$ of functions on Γ_ℓ as in Section 4.3.

We define the *total Weierstrass locus of the canonical divisor*, denoted $L_W^{\mathrm{tot}}(e)$, as the portion of the edge e covered by Weierstrass points of all the augmented metric graphs Γ_ℓ, for those verifying $\ell(e) = 1$, that is,

$$L_W^{\mathrm{tot}}(e) \coloneqq \bigcup_{\ell \text{ with } \ell(e)=1} L_W^{\mathrm{can}}(\Gamma_\ell) \cap e.$$

Question 7.1.

(i) What is the shape of $L_W^{\mathrm{tot}}(e)$, that is, how many components can it have on the edge e?

(ii) What is the size of $L_W^{\mathrm{tot}}(e)$? That is, what proportion of e is covered by Weierstrass points of metric graphs of combinatorial type $(G, \mathfrak{g})$?

(iii) How is $L_W^{\mathrm{tot}}(e)$ placed on e? That is, characterize the boundary of $L_W^{\mathrm{tot}}(e)$.

(iv) Characterize all the points in $L_W^{\mathrm{tot}}(e)$ which can arise as a limit of Weierstrass points on nearby smooth curves. More precisely, characterize those points p for which there exists a (G, e)-admissible family of curves $\mathfrak{X}_t$ and a Weierstrass point p_t on $\mathfrak{X}_t$ such that p is the tropical limit of p_t.

(v) What is the quantity $\sup_\ell |L_W^{\mathrm{can}}(\Gamma_\ell) \cap e|$, where $|L_W^{\mathrm{can}}(\Gamma_\ell) \cap e|$ refers to the Lebesgue measure of $L_W^{\mathrm{can}}(\Gamma_\ell) \cap e$ and the supremum is taken over all length functions ℓ such that $\ell(e) = 1$?

Inspired by Baker [Bak08, Lemma 4.7], we can prove Theorem 7.5 below which shows that the total Weierstrass locus $L_W^{\mathrm{tot}}(e)$ on the edge e is not always connected. This provides a partial answer to Question (i) above. We do not know of any example with a number of connected components larger than two.

We can define a refined version of $L_W^{\mathrm{tot}}(e)$ by requiring the stable curve in the admissible family to be a fixed stable curve X, as follows. We define $L_W^{\mathrm{tot}}(e, X)$ as the locus of all the points in e that are limits of Weierstrass points in a one-parameter family of smooth curves converging to X.

Question 7.2. *What is the quantity $\sup_X \left|L_W^{\mathrm{tot}}(e, X)\right|$, where $\left|L_W^{\mathrm{tot}}(e, X)\right|$ refers to the Lebesgue measure of $L_W^{\mathrm{tot}}(e, X)$?*

Here, the supremum is taken over all stable curves X with the same stable dual graph G.

The discussion above is related to the work of Diaz [Dia85] and Gendron [Gen21]. Translated into the above language, Diaz and Gendron show in *loc. cit.* that the set $L_W^{\mathrm{tot}}(e)$ is nonempty. In fact, they prove that for any X with dual graph G, the set $L_W^{\mathrm{tot}}(e, X)$ is nonempty provided that e is not a bridge edge in G. If e is a bridge, then Gendron has a characterization of the situations where $L_W^{\mathrm{tot}}(e, X)$ is nonempty. The statement on non-bridge edges can be proved by using tropical arguments, by reducing to the example of the augmented cycle 6.6.

In a similar vein, we cite the following theorem of Eisenbud and Harris.

Theorem 7.3 (Eisenbud–Harris [EH87a])**.** *Suppose X is a smooth curve of genus g, and E is an elliptic curve with identity $e_0 \in E$. Let $X' = X \cup_x E$ denote the nodal curve obtained by joining $e_0 \in E$ to $x \in X$ by a node. If x is not a Weierstrass points of X, then the limit Weierstrass points of X' contained in E are exactly the torsion points of order g on E.*

Remark 7.4. Let G be a simple graph of genus g. Assume that G is 2-connected, that is, it does not have bridge edges. Then, we believe the following should be true. Given an edge e, there should exist a choice of edge lengths for which the Weierstrass locus contains a connected component in the interior of e. ◇

The above questions and the results we proved in this paper provide a tropical refinement of the problem raised by Eisenbud and Harris on the determination of the limit Weierstrass loci on stable curves.

An example with a disconnected locus $L_W^{\mathrm{tot}}(e)$. We first prove the following result.

Theorem 7.5. *Let $G = (V, E)$ be a graph containing an edge $e = uv$ such that deleting e along with a small open neighborhood of its endpoints creates a tree. Assume e is parametrized by the interval $[0, 1]$, and suppose that its endpoints have valence $\mathrm{val}(u) = a + 2$ and $\mathrm{val}(v) = b + 2$. Then, $L_W^{\mathrm{tot}}(e)$ is disjoint from the interval $\left[\frac{b}{a+b+1}, \frac{b+1}{a+b+1}\right]$ in e.*

Note that a graph satisfying the conditions in Theorem 7.5 has genus $g = a + b + 1$.

Proof. Let Γ_ℓ be metric graph of model G with $\ell(e) = 1$. Consider a point x in the interval $\left[\frac{b}{a+b+1}, \frac{b+1}{a+b+1}\right]$ and let $D \coloneqq K - g\,(x)$. In order to prove that x is not a Weierstrass point in Γ_ℓ, we will prove that the rank of D is negative. We proceed as follows.

Let $T \coloneqq G - u - v$ be the tree obtained by removing u, v, and all the incident edges to them from G. Let y be a point in the interior of e in Γ_ℓ, that will be determined later as a function of x. The set $V' = V \cup \{y\}$ is the vertex set of another model of Γ_ℓ. We enumerate the vertices of the tree T as $v_0, \dots, v_n$ such that each vertex v_j for $j \in \{0, 1, \dots, n\}$ is connected to exactly one vertex among $v_0, \dots, v_{j-1}$. Consider the total order $\mathcal{O}$ on V' given by the enumeration $v_0, \dots, v_n, u, v, y$. The corresponding divisor $D_{\mathcal{O}}$ is explicitly given as $D_{\mathcal{O}} = a\,(u) + b\,(v) + (y) - (v_0)$. Denote by $\overline{\mathcal{O}}$ the total order on V' opposite to $\mathcal{O}$, and $D_{\overline{\mathcal{O}}}$ the corresponding divisor. The divisors $D_{\mathcal{O}}$ and $D_{\overline{\mathcal{O}}}$ have degree $g - 1 = a + b$, they are of negative rank, and moreover, $D_{\mathcal{O}} + D_{\overline{\mathcal{O}}} = K$, see [BN07, BJ16].

We now write

$$D = K - g(x) = D_{\mathcal{O}} + D_{\overline{\mathcal{O}}} - g\,(x) = D_{\overline{\mathcal{O}}} - E - (v_0)$$

where $E = g\,(x) - D_{\mathcal{O}} - (v_0) = (a+b+1)\,(x) - a\,(u) - b\,(v) - (y)$. The claim $r(D) = -1$ now follows by observing that for x in the above interval, there exists y in e such that the divisor E is principal, that is, $E = \mathrm{div}(f)$ for a function $f \in \mathrm{Rat}(\Gamma_\ell)$. Explicitly, using the parametrization of e by the interval $[0,1]$ for a given x, we take $y = (a+b+1)\,x - b$. We have $y \in [0,1]$ because of the assumption that $x \in \left[\frac{b}{a+b+1}, \frac{b+1}{a+b+1}\right]$. The desired function f on Γ_ℓ is constant outside e, has slopes $\mathrm{sl}_e f(u) = a$ and $\mathrm{sl}_e f(v) = b$, and has orders of vanishing at x and y given by $a+b+1$ and -1, respectively. □

Now consider a graph G verifying conditions of Theorem 7.5. Note that this implies there is a single edge between u and v. Assume that the leaves in the tree T are connected to both u and v. In this case, if $\mathrm{val}(u) > 2$ (which is equivalent, according to the previous assumption, to the fact that T has at least two leaves, i.e., T is not made up of a single vertex), then $v \in L_W^{\mathrm{tot}}(e)$, and similarly, if $\mathrm{val}(v) > 2$, then $u \in L_W^{\mathrm{tot}}(e)$.

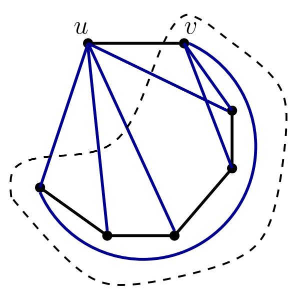

FIGURE 25. The cut in G used to prove that u is in the total Weierstrass locus $L_W^{\mathrm{tot}}(e)$ if $\mathrm{val}(v) > 2$.

To prove this, using symmetry and keeping the notations of Theorem 7.5, we assume $b > 0$. Take the union of T and v, as in Figure 25. Set the length of edges between u and T equal to b, that of uv equal to one, and the others arbitrary. Let f be the function defined to be affine linear on edges and which takes value b at u and zero at other vertices. Then, $b > 0$ implies that $E = K + \mathrm{div}(f)$ is effective and has coefficient at least $a+b+1$ at u. So u belongs to the total Weierstrass locus, as required.

In the case a and b are both positive, this implies that u and v are both in the total Weierstrass locus $L_W^{\mathrm{tot}}(e)$. Therefore, $L_W^{\mathrm{tot}}(e)$ will be disconnected.

7.2. **Variation of Weierstrass loci over the moduli space of metric graphs.** Let $G = (V,E)$ be a stable graph of genus g. Consider the cone $\eta_G := \mathbb{R}_+^E$ of positive metrics on G, and let $\bar{\eta}_G$ be its closure. The (coarse) moduli space of metric graphs of genus g, denoted by $\mathscr{M}_g^{gr}$, is obtained by glueing of the cones $\bar{\eta}_G$, for every stable graph G of genus g. More precisely, it is the direct limit of the diagram of inclusions $\bar{\eta}_H \hookrightarrow \bar{\eta}_G$ for pairs H and G of stable graphs of genus g such that H is obtained by contraction of some edges in G; see [ACP15] for more details. We endow $\mathscr{M}_g^{gr}$ with the topology induced by those on η_G as the corresponding quotient topology on the limit. For each stable graph G of genus g, we get a canonical map $\eta_G \to \mathscr{M}_G^{gr}$. The universal metric graph $\mathscr{G}_g$ is defined over these charts. That is, over the cone η_G, we have the universal metric graph $\mathscr{G}_G$.

Let $\mathcal{D} = (D_t)_{t \in \eta_G}$ be a continuous family of effective divisors of degree d and rank r. At each point $t \in \eta_G$, we consider the Weierstrass locus $L_W(D_t)$ which lives in the metric graph

$\mathscr{G}_{G,t}$. We denote by $L_W(\mathcal{D})$ the Weierstrass locus of the family defined as the union of all $L_W(D_t)$, for $t \in \eta_G$. We have the following theorem.

Theorem 7.6. *The Weierstrass locus $L_W(\mathcal{D})$ is a closed subset of $\mathscr{G}_G$.*

Sketch of the proof. We need to show that any point x_{t_0} in a fiber $\mathscr{G}_{G,t_0}$ which is a limit of Weierstrass points x_t in $\mathscr{G}_{G,t}$, as t tends to t_0, is Weierstrass. This amounts to showing the existence of a function f in $\mathrm{Rat}(D_{t_0})$ such that $D_{t_0} - (r+1)\,(t_0) + \mathrm{div}(f) \geq 0$. By assumption, there exists $f_t \in \mathrm{Rat}(D_{t_0})$ such that $D_t - (r+1)\,(t) + \mathrm{div}(f_t) \geq 0$, and such that moreover $f_t(x_{t_0}) = 0$. A compactness argument then shows the existence of a subsequence of f_t's converging to a function f on $\mathscr{G}_{G,t_0}$. This limit function is in $\mathrm{Rat}(D_{t_0})$, from which the theorem follows. □

More generally, we can define the Weierstrass locus over the full moduli space $\mathscr{M}_g^{gr}$. Let $\mathcal{D} = (D_t)$, for $t \in \mathscr{M}_g^{gr}$, be a continuous family of effective divisors of degree d and rank r over the moduli space of metric graphs of genus g.

Theorem 7.7. *The Weierstrass locus $L_W(\mathcal{D})$ is closed.*

Proof. The proof is similar to that of Theorem 7.6. □

7.3. **Effective determination of minimum slopes.** We discuss a concrete way of determining the Weierstrass locus and weights in a given metric graph.

Let D be an effective divisor on Γ. There is an algorithmic way for determining all the minimum slopes of functions in $\mathrm{Rat}(D)$ along unit tangent vectors in Γ. This is based on chip-firing on metric graphs. More precisely, [Luo11] gives a generalization of Dhar's burning algorithm for metric graphs, which allows us to test whether a divisor is x-reduced for any point $x \in \Gamma$ and eventually to compute reduced divisors. See Definition 2.10, Algorithm 2.13 and Theorem 2.15 in [Luo11].

We can extract the minimum slopes from this procedure. Let x be a point of Γ and $\nu \in \mathrm{T}_x(\Gamma)$ be a tangent direction at x. At step i of the algorithm, following the notations of [Luo11, Definition 2.10], we count the number n_i of indices $1 \leq j \leq J$ such that $Q_j^{(1)}$ contains a segment of Γ starting at x and supporting the direction ν. The number n_i is either zero or one and represents the number of chips that go through this segment toward the point x at step i. We denote by n the sum of the n_i's. It is the total number of chips that are brought to x by Dhar's algorithm via the branch supporting ν. This means that $s_0^\nu = -n$, which shows that the minimum slope on ν can be computed using Dhar's algorithm.

7.4. **Tropical Weierstrass points in positive characteristic.** The treatment of Weierstrass points for curves over positive characteristic fields suggests the following possible modification of the theory of tropical Weierstrass points in the isolated cases where the whole graph is Weierstrass. We replace the rank r with the integer

$$b = b(\Gamma, D) := \min_{x \in \Gamma} D_x(x),$$

and define the Weierstrass locus as the subset of points $x \in \Gamma$ verifying $D_x(x) \geq b+1$. The weight of a connected component C of this modified Weierstrass locus is modified by setting

$$\mu_W(C; D) := \deg\left(D|_C\right) + (g(C) - 1)\, b - \sum_{\nu \in \partial^{\mathrm{out}} C} s_0^\nu(D).$$

This leads to a consistent theory on the tropical side, with the weights of components of the Weierstrass locus adding up to $d - b + bg$ (instead of $d - r + rg$). This is reminiscent of the setting of curves in the situation where the standard sequence of vanishing orders differs from the sequence $0, 1, \dots, r$, cf. [Lak81]. However, at this point, we are not aware of any geometric meaning to this tropical count.

7.5. **Weierstrass points of random combinatorial graphs.** There exist combinatorial graphs without any Weierstrass points among their vertices (see Section 6.10). This seems, however, to be a rare phenomenon, as a computer verification of examples indicates. Examples of random graphs were created and visualized using Python, Matplotlib [Hun07], and NetworkX [HSS08].

Question 7.8. *What is the proportion of combinatorial graphs which do not have any Weierstrass point among their vertices? That is, what is the probability that a combinatorial graph on n vertices has no Weierstrass point?*

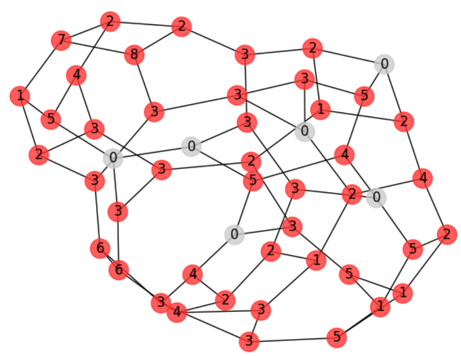

FIGURE 26. Random trivalent graph and its Weierstrass locus $L_W(K)$. The graph has genus 26, and the vertex labels indicate the coefficients $K_v(v) - 25$.

Randomness is understood within a class of graphs, for example regular graphs of given degree, or Erdös–Rényi random graphs. This is related to the following question of Baker.

Question 7.9 (Baker [Bak08])**.** *Provide a classification of combinatorial graphs without Weierstrass points among their vertices.*

APPENDIX A. TROPICALIZATION OF WEIERSTRASS POINTS
BY OMID AMINI

In this appendix we describe the tropicalization of the Weierstrass divisor of a line bundle on a smooth curve over a non-Archimedean field.

Let $\mathbb{K}$ be an algebraically closed complete non-Archimedean field with a non-trivial valuation by $\mathfrak{v}$. Let R, $\mathfrak{m}$, and $\kappa = \mathrm{R}/\mathfrak{m}$ be the valuation ring, the maximal ideal of R, and the residue field, respectively. We also denote by $|\cdot|$ the corresponding norm on $\mathbb{K}$, so that $\mathfrak{v}(\cdot) = -\log|\cdot|$.

Let X be a smooth proper curve defined over $\mathbb{K}$. Let $\mathcal{D}$ be a divisor of degree d on X and let $\mathcal{L} = \mathcal{O}(\mathcal{D})$ be the corresponding line bundle, with $\mathcal{O} = \mathcal{O}_X$, the structure sheaf of X. Denote by ω_X the canonical line bundle on X.

Let $H \subset H^0(X, \mathcal{L})$ be a space of sections of dimension $r+1$ and denote by $\mathcal{W} = \mathcal{W}(\mathcal{D}, H)$ the corresponding Weierstrass divisor. We assume that the gap sequence of H is the standard sequence $0, 1, \dots, r$, that is, for a general point $x \in X(\mathbb{K})$, the orders of vanishing of sections of $\mathcal{L}$ in H are $0, 1, \dots, r$. In particular, if $\mathbb{K}$ is of characteristic zero, this is automatic.

A.1. **Tropicalization.** We denote by X^{an} the Berkovich analytification of X. We assume familiarity with the theory of Berkovich analytic curves, and refer to [AB15, Section 4] and [BPR16, Section 5] that contain what we need.

A semistable vertex set for X^{an} is a finite set of type 2 points V in X^{an} such that the complement $X^{\mathrm{an}} \smallsetminus V$ is a disjoint union of finitely many open annuli and infinitely many open disks. A semistable model for X is an integral proper relative curve $\mathfrak{X}$ over R with generic fiber $\mathfrak{X}_\eta = X$ and special fiber $\mathfrak{X}_0$ that is reduced and has nodal singularities. Any irreducible component of the special fiber $\mathfrak{X}_0$ of a semistable model $\mathfrak{X}$ gives a valuation on $\mathbb{K}(X)$ and defines a point of type 2 in X^{an}. The set V of points in X^{an} associated to the irreducible components of $\mathfrak{X}_0$ is a semistable vertex set for X^{an}. This process provides, in fact, a bijection between semistable vertex sets of X^{an} and semistable models of $\mathfrak{X}$ (see [BPR16, Thm. 5.38]). Moreover, each point of type 2 appears in a semistable vertex set.

A semistable vertex set V gives rise to a skeleton Γ for X^{an}, defined as the union in X^{an} of V and the skeletons of the open annuli in $X^{\mathrm{an}} \smallsetminus V$. The canonical metric on the skeletons of the open annuli gives the skeleton a metric graph structure, naturally embedded in X^{an}.

The underlying graph $G = (V, E)$ has vertex set V and edge set E in bijection with the set of open annuli in $X^{\mathrm{an}} \smallsetminus V$. There is an edge between a pair of vertices v and u in V for each open annulus whose closure contains the points v and u. Moreover, the edge length function $\ell \colon E \to (0, +\infty)$ associates to each edge of G the modulus of the corresponding annulus. Using the correspondence between semistable models and semistable vertex sets, the graph G is identified with the dual graph of $\mathfrak{X}_0$, the special fiber of $\mathfrak{X}$, with vertices in bijection with the irreducible components of $\mathfrak{X}_0$, and edges in bijection with the nodes of $\mathfrak{X}_0$. There is an edge $e = uv$ in G for each node that lies on the irreducible components associated to u and v. The length of an edge corresponds to the singularity degree in $\mathfrak{X}$ of the corresponding node.

For each point x of type 2 in X^{an}, the extension $\kappa(x)/\kappa$ is of transcendence degree one. We denote by C_x the corresponding smooth proper curve over κ. In a semistable model $\mathfrak{X}$ in which x is in the vertex set, C_x is the normalization of the irreducible component in $\mathfrak{X}_0$ associated to x, and $\kappa(x)$ is the function field of this component.

We denote by $\mathbb{B}_+$ the standard open ball in the Berkovich affine line $\mathbb{A}^{1,\mathrm{an}}$. The complement of Γ in X^{an} is a disjoint union of open balls B_ν in bijection with $\nu \in \mathrm{T}_x(X^{\mathrm{an}}) \smallsetminus \mathrm{T}_x(\Gamma)$ for all points x of type 2 in Γ, each isomorphic to $\mathbb{B}_+$. For a given ball B_ν in $X^{\mathrm{an}} \smallsetminus \Gamma$, the corresponding point x is the unique point in Γ that lies in the closure of B_ν. Denote by p_x^ν the point of $\mathrm{C}_x(\kappa)$ corresponding to $\nu \in \mathrm{T}_x(X^{\mathrm{an}}) \smallsetminus \mathrm{T}_x(\Gamma)$.

Let Γ be a metric graph skeleton of X^{an} with underlying graph $G = (V, E)$ and denote by $\tau \colon X^{\mathrm{an}} \to \Gamma$ the canonical retraction map. We call τ the tropicalization map. In the notations of the previous paragraph, the tropicalization map sends all the points in B_ν to the point x. The restriction of τ to $X(\mathbb{K}) \subset X^{\mathrm{an}}$ is compatible with the specialization map from the generic fiber $\mathfrak{X}_\eta$ to $\mathfrak{X}_0$, that is, a point specialized to a node is sent by τ to a point in the corresponding edge, and a point specialized to a smooth point of $\mathfrak{X}_0$ is sent by τ to the vertex of G corresponding to this component. We get a tropicalization map $\tau_* \colon \mathrm{Div}(X) \to \mathrm{Div}(\Gamma)$ that sends a divisor $\mathcal{D} = \sum_{x \in X(\mathbb{K})} a_x(x)$ on X to the divisor $\tau_*(\mathcal{D}) = \sum_{x \in X(\mathbb{K})} a_x(\tau(x))$.

We denote by $\mathfrak{v}_x\colon \mathbb{K}(X) \to \mathbb{R} \cup \{+\infty\}$ the valuation of a point $x \in X^{\mathrm{an}} \smallsetminus X(\mathbb{K})$ with $\mathfrak{v}_x(f) = +\infty$ only if $f = 0$. The residue field of this valuation is denoted by $\kappa(x)$. We also denote by $|\cdot|_x = \exp(-\mathfrak{v}_x)$ the corresponding norm.

For each non-zero $f \in \mathbb{K}(X)$, we define the tropicalization of f, denoted $\mathrm{trop}(f)\colon \Gamma \to \mathbb{R}$, as the map that sends each $x \in \Gamma \subset X^{\mathrm{an}} \smallsetminus X(\mathbb{K})$ to $\mathfrak{v}_x(f)$. This induces a tropicalization map $\mathrm{trop}\colon \mathbb{K}(X) \smallsetminus \{0\} \to \mathrm{Rat}(\Gamma)$.

For a vector subspace $H \subset \mathbb{K}(X)$, we call $M = \mathrm{trop}(H \smallsetminus \{0\})$ the tropicalization of H, and denote it, by a slight abuse of notation, by $\mathrm{trop}(H)$.

We define the genus function $\mathfrak{g}$ on X^{an} to be the genus of C_x for a point of type 2, extended by zero everywhere else. The restriction of $\mathfrak{g}$ to Γ gives an augmented metric graph of genus g equal to that of X. We denote by K the canonical divisor of the augmented metric graph $(\Gamma, \mathfrak{g})$, with $K(x) = 2\mathfrak{g}(x) - 2 + \mathrm{val}(x)$ for all $x \in \Gamma$.

A.2. **Reduction.** For a point of type 2, the valuation $\mathfrak{v}_x$ has the same value group as $\mathfrak{v}$. For each nonzero $f \in \mathbb{K}(X)$, choosing $a \in \mathbb{K}$ with $\mathfrak{v}(a) = \mathfrak{v}_x(f)$, we get that $a^{-1}f$ has valuation $\mathfrak{v}_x(a^{-1}f) = 0$, and therefore gives an element in the residue field $\kappa(x)$ that we denote by $\tilde{f}_x$. We call this the reduction of f at x, which is nonzero and defined only up to multiplication by a non-zero scalar in κ. For a vector subspace $H \subset \mathbb{K}(X)$ of dimension $r+1$, denote by $\widetilde{H}_x \subset \kappa(x)$ the κ-vector subspace spanned by the reductions $\tilde{f}_x$ of elements $f \in H$ [AB15, Section 4.4]. By [AB15, Lemma 4.3], $\widetilde{H}_x$ has dimension $r+1$ over κ.

A.3. **Slopes.** For point x in Γ of type 2 in X^{an}, each unit tangent direction $\nu \in \mathrm{T}_x(\Gamma)$ gives a point $p_x^\nu \in \mathrm{C}_x(\kappa)$. By the slope formula [BPR16], for any non-zero $f \in \mathbb{K}(X)$, we have $\mathrm{sl}_\nu(\mathrm{trop}(f)) = \mathrm{ord}_{p_x^\nu}\left(\tilde{f}_x\right)$. Moreover,

$$\tau_*(\mathrm{div}(f)) = \mathrm{div}(\mathrm{trop}(f)).$$

If $H \subset \mathbb{K}(X)$ is a $\mathbb{K}$-vector subspace of dimension $r+1$, for any point $x \in \Gamma$ and unit tangent vector $\nu \in \mathrm{T}_x(\Gamma)$, we get a collection of integers $\mathrm{sl}_\nu(\mathrm{trop}(f)) = \mathrm{ord}_{p_x^\nu}(\tilde{f}_x)$, $f \in H$. Since $\widetilde{H}_x$ has dimension $r+1$, this collection has size $r+1$. This means that the collection of slopes $\mathrm{sl}_\nu(h)$, for $h \in M = \mathrm{trop}(H)$, has size $r+1$. In particular, Property $(\star)$ in Section 5 is satsified by $M = \mathrm{trop}(H)$.

For each unit tangent vector ν, we order the slopes $\mathrm{sl}_\nu(h)$, for $h \in M$, in the form $s_0^\nu < s_1^\nu < \cdots < s_r^\nu$. Since elements of $\mathrm{trop}(H)$ are piecewise affine linear, adding more points of Γ to the semistable vertex set, we can suppose that the set of slopes

$$s_0^\nu < s_1^\nu < \cdots < s_r^\nu$$

is constant in the interior of any edge of $G = (V, E)$ for parallel tangent directions ν at the point of the edge that point in the same direction.

A.4. **Weierstrass divisor and Wronskian.** Let X be a smooth proper curve defined over $\mathbb{K}$. Let $\mathcal{D}$ be a divisor of degree d on X and let $\mathcal{L} = \mathcal{O}(\mathcal{D})$ be the corresponding line bundle, with $\mathcal{O} = \mathcal{O}_X$, the structure sheaf of X. Denote by ω_X the canonical line bundle on X.

Let $H \subset H^0(X, \mathcal{L})$ be a space of sections of dimension $r+1$ and denote by $\mathcal{W} = \mathcal{W}(\mathcal{D}, H)$ the corresponding Weierstrass divisor. The Weierstrass divisor $\mathcal{W}$ is the divisor of a global section of the line bundle $\omega_X^{\otimes\frac{r(r+1)}{2}} \otimes \mathcal{L}^{\otimes(r+1)}$ called the *Wronskian*. It is described as follows.

In local coordinates, for any point $p \in X(\mathbb{K})$, the local ring $\mathcal{O}_p$ is a discrete valuation ring. We choose a uniformizer that we denote by $\mathfrak{t}_p$. We have $\mathcal{L}_p \simeq \mathcal{O}_p$ as an $\mathcal{O}_p$-module. Taking the generator $g_p = \mathfrak{t}_p^{\mathcal{D}(p)}$ of $\mathcal{L}_p$, each global section f of $\mathcal{L}$ can be written in the form $f = f_p g_p$ with $f_p \in \mathcal{O}_p$.

We define the Hasse derivative of order j, for $j \in \mathbb{Z}_{\geq 0}$, on $\mathbb{K}[\mathfrak{t}_p]$ by

$$\mathrm{D}^{(j)}\mathfrak{t}^m = \binom{m}{j}\mathfrak{t}^{m-j} \qquad \text{for } m > 0,$$

and extend it by linearity to all $\mathbb{K}[\mathfrak{t}]$, and then to all $\mathbb{K}(\mathfrak{t})$. Since the extension $\mathbb{K}(X)/\mathbb{K}(\mathfrak{t})$ is separable, $\mathrm{D}^{(j)}$ is extended to $\mathbb{K}(X)$. Note that if $\mathbb{K}$ has characteristic zero, we can recursively define for any $j \geq 0$, the j-th derivative $f_p^{(j)}$ by

$$f_p^{(j)} = \frac{\mathrm{d}}{\mathrm{d}\mathfrak{t}_p} f_p^{(j-1)}$$

with $f_p^{(0)} = f_p$. In this case, we have $j!\,\mathrm{D}^{(j)} f_p = f_p^{(j)}$.

Let $\mathcal{F} := \{f_0, \dots, f_r\}$ be a basis for $H \subset H^0(X, \mathcal{L})$, and for each i, write $f_i = f_{i,p} g_p$. Viewing $\mathrm{Wr}_{\mathcal{F}}$ as a meromorphic section of $\omega_X^{\otimes \frac{r(r+1)}{2}}$, the stalk of the Wronskian $\mathrm{Wr}_{\mathcal{F}}$ at p is given by

$$\mathrm{Wr}_{\mathcal{F},p} = \det\left(\mathrm{D}^{(j)} f_p\right)_{0 \leq i,j \leq r} (\mathrm{d}\mathfrak{t}_p)^{\frac{r(r+1)}{2}} \in \omega_p^{\otimes \frac{r(r+1)}{2}}.$$

We have

$$\mathcal{W} = (r+1)\mathcal{D} + \mathrm{div}(\mathrm{Wr}_{\mathcal{F}}).$$

We note, without going into details, that the Wronskian $\mathrm{Wr}_{\mathcal{F}}$ can also be defined without local coordinates, in terms of a filtration of the jet bundle and the diagonal embedding of X.

A.5. **Slope formula for meromorphic differentials.** We denote by $\|\cdot\|$ the Kähler norm introduced by Temkin in [Tem16] on the sheaf of differentials ω_X that at any point $x \in X^{\mathrm{an}}$ associates to any section α of ω_X the real number $\|\alpha\|_x$. For each positive integer n, the Kähler norm $\|\cdot\|$ induces a metric on $\omega_X^{\otimes n}$ which, by an abuse of notation, we still denote by $\|\cdot\|$. Given a meromorphic section α of $\omega_X^{\otimes n}$, the tropicalization of α denoted by $\mathrm{trop}(\alpha)$ is the map

$$\mathrm{trop}(\alpha)\colon \Gamma \to \mathbb{R}, \qquad x \mapsto -\log \|\alpha\|_x.$$

The tropicalization of any meromorphic n-form on X is a rational function on Γ, that is, $\mathrm{trop}(\alpha) \in \mathrm{Rat}(\Gamma)$. Moreover, the following slope formula holds.

Lemma A.1 (Slope formula for meromorphic differentials)**.** *For any meromorphic section α of $\omega_X^{\otimes n}$, we have*

$$\tau_*(\mathrm{div}(\alpha)) = nK + \mathrm{div}(\mathrm{trop}(\alpha)).$$

Moreover, for any point $x \in \Gamma$ of type 2 and for any $\nu \in \mathrm{T}_x(\Gamma)$, we have $\mathrm{sl}_\nu(\mathrm{trop}(\alpha)) = \mathrm{ord}_{p_x^\nu}(\tilde{\alpha}_x) + n$.

Here, $\tilde{\alpha}_x$ is the reduction of α at x, and is a meromorphic form on C_x, see [TT22, Section 2].

Proof. The case $n = 1$ is proved in [TT22], see also [KRZ16, Theorem 2.6]. The proof in general is similar and we only sketch it. The second statement, that $\mathrm{sl}_\nu(\mathrm{trop}(\alpha)) = \mathrm{ord}_{p_x^\nu}(\tilde{\alpha}_x) + n$ for $\nu \in \mathrm{T}_x(\Gamma)$, is the analogue of [TT22, Proposition 2.3.3], and both are special cases of [BT20, Lemma 3.3.2].

Notations as in Section A.1, for each ball $B_\nu \simeq \mathbb{B}_+$, for $\nu \in \mathrm{T}_x(X^{\mathrm{an}}) \setminus \mathrm{T}_x(\Gamma)$, α can be written in the form $f\mathrm{d}T^n$ for a meromorphic function f on B_ν, with T a parameter for the ball $B_\nu \simeq \mathbb{B}_+$, and the slope formula applied to f implies

$$\sum_{a \in B_\nu(\mathbb{K})} \mathrm{ord}_a(\alpha) = \mathrm{ord}_{p_x^\nu} \tilde{\alpha}_x.$$

Write $\tau_*(\mathrm{div}(\alpha)) = \sum_{x\in\Gamma} a_x(x)$, for $a_x \in \mathbb{Z}$. For x not of type 2, $a_x = 0$, and for x of type 2, we have

$$\begin{aligned} a_x &= \sum_{\nu \in \mathrm{T}_x(X^{\mathrm{an}}) \setminus \mathrm{T}_x(\Gamma)} \left(\sum_{a \in B_\nu(\mathbb{K})} \mathrm{ord}_a(\alpha) \right) = \sum_{\nu \in \mathrm{T}_x(X^{\mathrm{an}}) \setminus \mathrm{T}_x(\Gamma)} \mathrm{ord}_{p_x^\nu} \tilde{\alpha}_x \\ &= n(2\mathfrak{g}(x) - 2) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} \mathrm{ord}_{p_x^\nu} \tilde{\alpha}_x \qquad (\text{since } \tilde{\alpha}_x \text{ is a meromorphic } n\text{-form on } \mathrm{C}_x) \\ &= n(2\mathfrak{g}(x) - 2) + n\,\mathrm{val}(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} \mathrm{sl}_\nu(\mathrm{trop}(\alpha)) = nK(x) + \mathrm{ord}_x(\mathrm{trop}(\alpha)), \end{aligned}$$

as required. □

A.6. **Tropicalization of the Wronskian.** Notations as in Section A.4, let $F \coloneqq \mathrm{trop}(\mathrm{Wr}_{\mathcal{F}})$ be the tropicalization of the Wronskian $\mathrm{Wr}_{\mathcal{F}}$, which is a meromorphic $(r(r+1)/2)$-form. Let $W = \tau_*(\mathcal{W})$ be the tropicalization of $\mathcal{W}$ to Γ. Let $D = \tau_*(\mathcal{D})$. The following result is a direct consequence of Lemma A.1, with $\alpha = \mathrm{Wr}_{\mathcal{F}}$.

Theorem A.2. *Notations as above, we have*

$$W(x) = (r+1)D(x) + \frac{r(r+1)}{2} K(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} \mathrm{sl}_\nu F,$$

where $F = \mathrm{trop}(\mathrm{Wr}_{\mathcal{F}})$. *Furthermore,* $\mathrm{sl}_\nu F = \frac{r(r+1)}{2} + \mathrm{ord}_{p_x^\nu} \widetilde{\mathrm{Wr}_{\mathcal{F}}}_x$.

Here, $\widetilde{\mathrm{Wr}_{\mathcal{F}}}_x$ denotes the reduction of $\mathrm{Wr}_{\mathcal{F}}$ at x.

A.7. **Wronskian of analytic functions on annuli.** Let $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{K}[T])$ and $\mathbb{A}^{1,\mathrm{an}}$ its Berkovich analytification. Let $A(\rho)$ be the closed annulus in $\mathbb{A}^{1,\mathrm{an}}$ of center 0 with outer radius one and inner radius $\rho \in (0,1)$,

$$A(\rho) = \left\{ x \in \mathbb{A}^{1,\mathrm{an}} \,\middle|\, \rho \le |T|_x \le 1 \right\}.$$

Let $R(\rho)$ be the ring of analytic functions on $A(\rho)$. An analytic function f on $A(\rho)$ admits a formal power series expansion

$$f = \sum_{n \in \mathbb{Z}} a_n T^n$$

with $\lim_{n\to\pm\infty} |a_n| s^n = 0$ for all $s \in [\rho, 1]$. The skeleton of $A(\rho)$ is a closed interval, which can be identified with $I \coloneqq [0, -\log\rho]$: each point q in this interval corresponds to the norm $|\cdot|_{\zeta_q}(f) = \sup_{n\in\mathbb{Z}} |a_n| \exp(-qn) = \max_{n\in\mathbb{Z}} |a_n| \exp(-qn)$ on any analytic function f as above. The tropicalization of an analytic function f is the function $\mathrm{trop}(f)$ on the interval I given by

$$\mathrm{trop}(f)(q) = \min\{\mathfrak{v}(a_n) + nq \,|\, n \in \mathbb{Z}\} \qquad \forall q \in I.$$

Let $\zeta = \zeta_0 \in A(\rho)$ be the boundary point corresponding to the extremity 0 of I, that is, $|f|_\zeta = \max_{n\in\mathbb{Z}} |a_n|$ on any analytic function f as above. The reduction at ζ of an analytic function f with $|f|_\zeta = 1$ is a Laurent polynomial

$$\tilde{f}_\zeta = \sum_{\substack{n\in\mathbb{Z}\\ |a_n|=1}} \tilde{a}_n t^n$$

where t is the reduction of T at ζ and $\tilde{a}_n \in \kappa$ is a_n modulo $\mathfrak{m}$. The slope of $\mathrm{trop}(f)$ at 0 along the unit tangent direction $\nu \in \mathrm{T}_0(I)$ is the minimum exponent that appears in $\tilde{f}_\zeta$.

Let $f_0, \dots, f_r$ be $r+1$ $\mathbb{K}$-linearly independent analytic functions on $A(\rho)$ with

$$f_i = \sum_{n\in\mathbb{Z}} a_{i,n} T^n, \quad a_{i,n} \in \mathbb{K}$$

the analytic expansion of f_i. Suppose that $|f_i|_\zeta = 1$ for all $i = 0, 1, \dots, r$ and that $\mathrm{trop}(f_0), \dots, \mathrm{trop}(f_r)$ have slopes $s_0 < \dots < s_r$ at 0 along the unit tangent direction $\nu \in \mathrm{T}_0(I)$. This means the reduction $\tilde{f}_{i,\zeta}$ of f_i at ζ has initial term t^{s_i}.

Consider the analytic function on $A(\rho)$ defined by

$$h := \det\left(\mathrm{D}^{(j)} f_i\right)_{i,j=0}^{r},$$

where $\mathrm{D}^{(j)} f_i$ denotes the Hasse derivative of f_i. The following proposition describes the slope of $\mathrm{trop}(h)$ at the point 0 along the direction ν.

Proposition A.3. *Notations as above, assume*

- *either, the residue field κ is of characteristic zero,*
- *or, the sequence $s_0, \dots, s_r$ forms an interval, that is, $s_j = s_0 + j$ for all $j = 0, \dots, r$.*

Then, we have

$$\mathrm{sl}_\nu(\mathrm{trop}(h)) = s_0 + \dots + s_r - \frac{r(r+1)}{2}.$$

Proof. The coefficient $a_{s_i,i}$ of f_i has norm 1, so replacing f_i with $a_{s_i,i}^{-1} f_i$, we can assume that $a_{i,s_i} = 1$. We write $f_i = T^{s_i} + f_i'$, so that all the coefficients of T^n for $n \le s_i$ that show up in the power series expansion of f_i' have norm strictly smaller than one. We have

$$h = \det\left(\mathrm{D}^{(j)} T^{s_i^\nu} + \mathrm{D}^{(j)} f_i'\right)_{i,j=0}^{r} = \det\left(\binom{s_i}{j} T^{s_i^\nu - j} + \mathrm{D}^{(j)} f_i'\right)_{i,j=0}^{r}.$$

Developing the determinant, we observe that

$$h = \det\left(\binom{s_i}{j} T^{s_i^\nu - j}\right)_{i,j=0}^{r} + h' = C\, T^{s_0 + \dots + s_r - \frac{r(r+1)}{2}} + h'$$

with h' an analytic function on $A(\rho)$ whose power series expansion in T has the property that the coefficient of T^n for $n \le s_0 + \dots + s_r - \frac{r(r+1)}{2}$ has norm strictly smaller than one, and

$$C = \det\left(\binom{s_i}{j}\right)_{i,j=0}^{r}.$$

Note that C is a non-zero integer. If the residue field κ has characteristic zero, then C has norm one in $\mathbb{K}$, and the reduction of h has minimum exponent $s_0 + \dots + s_r - \frac{r(r+1)}{2}$.

On the other hand, if κ has positive characteristic, then by assumption $s_j = s_0 + j$ for all $j = 0, \dots, r$, and we use the identity for any $m \in \mathbb{Z}$

$$\det \begin{pmatrix} \binom{m}{0} & \binom{m}{1} & \dots & \binom{m}{r} \\ \binom{m+1}{0} & \binom{m+1}{1} & \dots & \binom{m+1}{r} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{m+r}{0} & \binom{m+r}{1} & \dots & \binom{m+r}{r} \end{pmatrix} = 1$$

to infer that $C = 1$ so again the reduction of h at ζ has minimum exponent $s_0+\cdots+s_r-\frac{r(r+1)}{2}$.

We have shown that in either case, the slope of $\mathrm{trop}(h)$ along $\nu \in \mathrm{T}_0(I)$ is $s_0 + \cdots + s_r - \frac{r(r+1)}{2}$. □

A.8. **Order of vanishing of the reduction of Wronskian.** A consequence of Proposition A.3 is the following description of the slopes appearing in Theorem A.2.

Proposition A.4. *Let x be a point of type 2 in Γ and $\nu \in \mathrm{T}_x(\Gamma)$. Denote by $s_0^\nu, \dots, s_r^\nu$ the sequence of slopes associated by tropicalization of H to ν. Assume*

- *either, the residue field κ is of characteristic zero,*
- *or, the sequence $s_0^\nu, \dots, s_r^\nu$ forms an interval, that is, $s_j^\nu = s_0^\nu + j$.*

Then, we have

$$\mathrm{ord}_{p_x^\nu}\widetilde{\mathrm{Wr}_{\mathcal{F}_x}} = s_0^\nu + \cdots + s_r^\nu - \frac{r(r+1)}{2}.$$

Proof. Consider a segment I in Γ that contains x with $\nu \in \mathrm{T}_x(I)$. I is the skeleton of a closed annulus A in X^{an} isomorphic to $A(\rho)$ for $0 < \rho < 1$. The elements of $\mathcal{F}$ give a collection of analytic functions $f_0, \dots, f_r$ on $A \simeq A(\rho)$ which are $\mathbb{K}$-linearly independent, and whose reductions have the initial term t^{s_i}. The statement now follows by applying Proposition A.3 to the analytic function $h \coloneqq \det \left(\mathrm{D}^{(i)} f_i\right)_{i,j=0}^{r}$, and observing that the restriction of $\mathrm{Wr}_{\mathcal{F}}$ to A coincides with $h(\mathrm{d}T)^{\frac{r(r+1)}{2}}$ and that $\mathrm{ord}_{p_x^\nu}\widetilde{\mathrm{Wr}_{\mathcal{F}_x}} = \mathrm{sl}_\nu(\mathrm{trop}(h))$. □

A.9. **Reduction of Weierstrass points in equal characteristic zero.** Assume the residue field κ has characteristic zero. As in Section A.3, denote by s_i^ν, $i = 0, \dots, r$, the slopes of functions of the form $\mathrm{trop}(f) \in \mathrm{trop}(H)$, for $f \in H$.

Theorem A.5. *Let $W = \tau_*(\mathcal{W})$. We have*

$$W(x) = (r+1)D(x) + \frac{r(r+1)}{2}K(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} \sum_{i=0}^{r} s_i^\nu.$$

Proof. This follows by combining Proposition A.4 with Theorem A.2. □

References

[AB15] Omid Amini and Matthew Baker. Linear series on metrized complexes of algebraic curves. *Math. Ann.*, 362(1-2):55–106, 2015.

[ABBR15] Omid Amini, Matthew Baker, Erwan Brugallé, and Joseph Rabinoff. Lifting harmonic morphisms I: metrized complexes and Berkovich skeleta. *Res. Math. Sci.*, 2(1):1–67, 2015.

[ACP15] Dan Abramovich, Lucia Caporaso, and Sam Payne. The tropicalization of the moduli space of curves. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(4):765–809, 2015.

[AG22] Omid Amini and Lucas Gierczak. Limit linear series: combinatorial theory. Preprint available at arXiv:2209.15613, 2022.

[Ami13] Omid Amini. Reduced divisors and embeddings of tropical curves. *Trans. Amer. Math. Soc.*, 365(9):4851–4880, 2013.

[Ami14] Omid Amini. Equidistribution of Weierstrass points on curves over non-Archimedean fields. Preprint available at arXiv:1412.0926, 2014.

[AN22] Omid Amini and Noema Nicolussi. Moduli of hybrid curves II: Tropical and hybrid Laplacians. Preprint available at arXiv:2203.12785, 2022.

[AN24] Omid Amini and Noema Nicolussi. Moduli of hybrid curves III: Intrinsic geometry of hybrid curves and metric degeneration. Preprint, 2024.

[AP03] Scott Ahlgren and Matthew Papanikolas. Higher Weierstrass points on $X_0(p)$. *Trans. Amer. Math. Soc.*, 355(4):1521–1535, 2003.

[Arb74] Enrico Arbarello. Weierstrass points and moduli of curves. *Compos. Math.*, 29(3):325–342, 1974.

[Atk67] A.O.L. Atkin. Weierstrass points at cusps of $\Gamma_0(n)$. *Ann. of Math.*, pages 42–45, 1967.

[Bak08] Matthew Baker. Specialization of linear systems from curves to graphs. *Algebra Number Theory*, 2(6):613–653, 2008.

[BG95] Edoardo Ballico and Letterio Gatto. On the monodromy of Weierstrass points on Gorenstein curves. *J. Algebra*, 175(2):633–643, 1995.

[BJ16] Matthew Baker and David Jensen. Degeneration of linear series from the tropical point of view and applications. In *Nonarchimedean and Tropical Geometry*, pages 365–433. Springer, 2016.

[BN07] Matthew Baker and Serguei Norine. Riemann–Roch and Abel–Jacobi theory on a finite graph. *Adv. Math.*, 215(2):766–788, 2007.

[BPR16] Matthew Baker, Sam Payne, and Joseph Rabinoff. Nonarchimedean geometry, tropicalization, and metrics on curves. *Algebr. Geom*, 3(1):63–105, 2016.

[BS13] Matthew Baker and Farbod Shokrieh. Chip-firing games, potential theory on graphs, and spanning trees. *J. Combin. Theory Ser. A*, 120(1):164–182, 2013.

[BT20] Uri Brezner and Michael Temkin. Lifting problem for minimally wild covers of Berkovich curves. *J. Algebraic Geom.*, 29(1):123–166, 2020.

[Cap15] Lucia Caporaso. Algebraic and tropical curves: comparing their moduli spaces. In *Handbook of Moduli. Volume I*, pages 119–160. Somerville, MA: International Press, 2015.

[CEG08] Caterina Cumino, Eduardo Esteves, and Letterio Gatto. Limits of special Weierstrass points. *Int. Math. Res. Pap. IMRP*, 2, 2008.

[CF91] Fernando Cukierman and Lung-Ying Fong. On higher Weierstrass points. *Duke Math. J.*, 62(1):179–203, 1991.

[Cuk89] Fernando Cukierman. Families of Weierstrass points. *Duke Math. J.*, 58(2):317–346, 1989.

[DC08] Andrea Del Centina. Weierstrass points and their impact in the study of algebraic curves: a historical account from the "Lückensatz" to the 1970s. *Annali dell'Universita di Ferrara*, 54(1):37–59, 2008.

[dCS94] Cicero Fernandes de Carvalho and Karl-Otto Stöhr. Higher order differentials and Weierstrass points on Gorenstein curves. *Manuscripta Math.*, 85:361–380, 1994.

[Dia85] Steven Diaz. Exceptional Weierstrass points and the divisor on moduli space that they define. *Mem. Amer. Math. Soc.*, 56(327), 1985.

[EH86] David Eisenbud and Joe Harris. Limit linear series: basic theory. *Invent. Math.*, 85(2):337–371, 1986.

[EH87a] David Eisenbud and Joe Harris. Existence, decomposition, and limits of certain Weierstrass points. *Invent. Math.*, 87(3):495–515, 1987.

[EH87b] David Eisenbud and Joe Harris. The Kodaira dimension of the moduli space of curves of genus ≥ 23. *Invent. Math.*, 90(2):359–387, 1987.

[EM02] Eduardo Esteves and Nivaldo Medeiros. Limit canonical systems on curves with two components. *Invent. Math.*, 149(2):267–338, 2002.

[ES07] Eduardo Esteves and Parham Salehyan. Limit Weierstrass points on nodal reducible curves. *Trans. Amer. Math. Soc.*, 359(10):5035–5056, 2007.

[Est98] Eduardo Esteves. Linear systems and ramification points on reducible nodal curves. *Mat. Contemp.*, 14:21–35, 1998.

[Gen21] Quentin Gendron. Sur les nœuds de Weierstraß. *Annales Henri Lebesgue*, 4:571–589, 2021.

[GK08] Andreas Gathmann and Michael Kerber. A Riemann–Roch theorem in tropical geometry. *Math. Z.*, 259(1):217–230, 2008.

[GL95] Arnaldo García and Robert F. Lax. On canonical ideals, intersection numbers, and Weierstrass points on Gorenstein curves. *J. Algebra*, 178(3):807–832, 1995.

[HKN13] Jan Hladkỳ, Daniel Král, and Serguei Norine. Rank of divisors on tropical curves. *J. Combin. Theory Ser. A*, 120(7):1521–1538, 2013.

[HMY12] Christian Haase, Gregg Musiker, and Josephine Yu. Linear systems on tropical curves. *Math. Z.*, 270(3):1111–1140, 2012.

[HSS08] Aric A. Hagberg, Daniel A. Schult, and Pieter J. Swart. Exploring network structure, dynamics, and function using NetworkX. In Gaël Varoquaux, Travis Vaught, and Jarrod Millman, editors, *Proceedings of the 7th Python in Science Conference*, pages 11 – 15, Pasadena, CA USA, 2008.

[Hun07] J. D. Hunter. Matplotlib: A 2D graphics environment. *Computing in Science & Engineering*, 9(3):90–95, 2007.

[Hur92] Adolf Hurwitz. Über algebraische Gebilde mit eindeutigen Transformationen in sich. *Math. Ann.*, 41(3):403–442, 1892.

[JP22] Dave Jensen and Sam Payne. Tropical linear series and tropical independence, 2022. Preprint available at arXiv:2209.15478.

[KRZ16] Eric Katz, Joseph Rabinoff, and David Zureick-Brown. Uniform bounds for the number of rational points on curves of small Mordell–Weil rank. *Duke Math. J.*, 165(16):3189–3240, 2016.

[Lak81] Dan Laksov. Weierstrass points on curves. *Astérisque*, 87(88):221–247, 1981.

[Lax75] Robert F. Lax. Weierstrass points of the universal curve. *Math. Ann.*, 216:35–42, 1975.

[Lax87] Robert F. Lax. Weierstrass points on rational nodal curves. *Glasgow Math. J.*, 29(1):131–140, 1987.

[LN64] Joseph Lehner and Morris Newman. Weierstrass points of $\Gamma_0(n)$. *Ann. of Math.*, 85:360–368, 1964.

[Luo11] Ye Luo. Rank-determining sets of metric graphs. *J. Combin. Theory Ser. A*, 118(6):1775–1793, 2011.

[LW90] Robert F. Lax and Carl Widland. Weierstrass points on Gorenstein curves. *Pacific J. Math.*, 142(1):197–208, 1990.

[Nee84] Amnon Neeman. Weierstrass points in characteristic p. *Invent. Math.*, 75(2):359–376, 1984.

[Ogg78] Andrew P. Ogg. On the Weierstrass points of $X_0(n)$. *Illinois J. Math.*, 22(1):31–35, 1978.

[Ric24] David Harry Richman. The distribution of Weierstrass points on a tropical curve. Preprint available at arXiv:1809.07920v2, to appear in *Selecta Math.*, 2024.

[Tem16] Michael Temkin. Metrization of differential pluriforms on Berkovich analytic spaces. In *Nonarchimedean and tropical geometry*, pages 195–285. Springer, 2016.

[TT22] Michael Temkin and Ilya Tyomkin. Reduction and lifting problem for differential forms on Berkovich curves. *Adv. Math.*, 397:108208, 2022.

[Wei67] Karl Weierstrass. *Mathematische Werke, 7 vols (1894–1927) Berlin, reprint by G.* Olms, Hildesheim, 1967.

CNRS - CMLS, École polytechnique, Institut polytechnique de Paris
Email address: omid.amini@polytechnique.edu

CMLS, École polytechnique, Institut polytechnique de Paris
Email address: lucas.gierczak-galle@polytechnique.edu

Department of Mathematics, University of Washington
Email address: hrichman@uw.edu