MATCUBOIDS AND FLAG ARRANGEMENTS

OMID AMINI AND LUCAS GIERCZAK

ABSTRACT. We introduce combinatorial objects named *matcuboids* that provide a generalization of the theory of matroids. As matroids provide a combinatorial axiomatization of hyperplane arrangements, matcuboids provide a combinatorial axiomatization of arrangements of flags in a vector space. We give cryptomorphic axiomatic systems in terms of rank function, flats, independent sets, and circuits, and formulate a duality concept. We also provide precise links between matcuboids, permutation arrays and matroids.

CONTENTS

1. INTRODUCTION

Consider a vector space H of finite dimension over a ground field κ and a collection $\mathcal{A}$ of m initial flags $\mathrm{F}_1^\bullet, \dots, \mathrm{F}_m^\bullet$. For $j = 1, \dots, m$, the flag $\mathrm{F}_j^\bullet$ consists of a chain of $\varrho_j + 1$ vector subspaces, $0 \leqslant \varrho_j \leqslant \dim_\kappa(\mathrm{H})$,

$$\mathrm{H} = \mathrm{F}_j^0 \supseteq \mathrm{F}_j^1 \supseteq \dots \supseteq \mathrm{F}_j^{\varrho_j - 1} \supseteq \mathrm{F}_j^{\varrho_j} \supsetneq (0)$$

where, for every $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, \varrho_j\}$, F_j^i is a vector subspace of codimension 0 or 1 in F_j^{i-1}. We call this collection of flags a *flag arrangement*. In the case $\varrho_j = 1$ for all j, and codimension of F_j^1 in H equal to one, we obtain a hyperplane arrangement.

The aim of this paper is to introduce *matcuboids* that give a combinatorial axiomatization for the intersection patterns of a finite collection of flags in a vector space, as the one above. The case $\varrho_1, \dots, \varrho_m = 1$ recovers the theory of matroids. Like matroids which in the representable case come from matrices, representable matcuboids come from cubical matrices (i.e., three-dimensional matrices).

Let us start with a few notations. For n a non-negative integer, we set $[n] := \{0, \dots, n\}$. Let m be a positive integer, and $\varrho_1, \dots, \varrho_m$ be non-negative integers. Let $\underline{\varrho} := (\varrho_1, \dots, \varrho_m)$. The hypercuboid $\square_{\underline{\varrho}}$ of width $\underline{\varrho}$ is the product $\prod_{j=1}^m [\varrho_j]$.

The hypercuboid $\square_{\underline{\varrho}}$ is endowed with a natural partial order $\preceq$ defined by declaring $\underline{a} \preceq \underline{b}$ for elements $\underline{a} = (a_1, \dots, a_m)$ and $\underline{b} = (b_1, \dots, b_m)$ in $\square_{\underline{\varrho}}$, if $a_j \leqslant b_j$ for all j. We define two

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operations $\vee$ and $\wedge$ of join and meet by taking the maximum and the minimum coordinatewise, respectively:

$$\underline{a} \vee \underline{b} := (\max(a_1, b_1), \dots, \max(a_m, b_m)), \qquad \underline{a} \wedge \underline{b} := (\min(a_1, b_1), \dots, \min(a_m, b_m)).$$

The minimum and maximum elements of $\square_{\underline{\varrho}}$ are the points $\underline{0} := (0, \dots, 0)$ and $\underline{\varrho} := (\varrho_1, \dots, \varrho_m)$. For $i \in \{1, \dots, m\}$ and $t \in [\varrho_i]$, we denote by $t\underline{e}_i$ the point of $\square_{\underline{\varrho}}$ whose i-th coordinate is t and whose other coordinates are zero.

We first give the definition of matcuboids in terms of their rank functions, and then provide cryptomorphic axiomatic systems in terms of their flats, circuits and independent sets.

1.1. **Definition in terms of rank function.** A function $f \colon \square_{\underline{\varrho}} \to \mathbb{Z}$ is called *submodular* if for every two elements $\underline{a}$ and $\underline{b}$, we have

$$f(\underline{a}) + f(\underline{b}) \geqslant f(\underline{a} \vee \underline{b}) + f(\underline{a} \wedge \underline{b}).$$

A *matcuboid* $\mathscr{M}$ with ground set $\square_{\underline{\varrho}}$ is defined in terms of a function $\mathbf{r} \colon \square_{\underline{\varrho}} \to \mathbb{N} \cup \{0\}$ called *the rank function of* $\mathscr{M}$ that satisfies the following conditions:

(R1) $\mathbf{r}(\underline{0}) = 0$, and for every $1 \leqslant i \leqslant m$ and $1 \leqslant t \leqslant \varrho_i$, we have $\mathbf{r}(t\underline{e}_i) - \mathbf{r}((t-1)\underline{e}_i) \leqslant 1$.

(R2) $\mathbf{r}$ is non-decreasing, that is, if $\underline{a} \preceq \underline{b}$, then $\mathbf{r}(\underline{a}) \leqslant \mathbf{r}(\underline{b})$.

(R3) $\mathbf{r}$ is submodular.

We call $r = r(\mathscr{M}) := \mathbf{r}\left(\underline{\varrho}\right)$, the maximum value taken by the function $\mathbf{r}$, the *rank* of $\mathscr{M}$. In the case $\varrho_j = 1$ for all j, $\mathscr{M}$ will be a matroid with ground set $E = \{1, \dots, m\}$ (with loops those elements $i \in E$ with $\mathbf{r}(\underline{e}_i) = 0$).

Note that it follows from (R1) and (R2) that $\mathbf{r}(t\underline{e}_i) \leqslant t$. We say that $\mathscr{M}$ is *simple* if the following stronger version of (R1) holds:

(R1*) $\varrho_i > 0$ and $\mathbf{r}(t\underline{e}_i) = t$ for all $i = 1, \dots, m$ and $t \in [\varrho_i]$.

For a collection $\mathcal{A}$ of initial flags $\mathrm{F}_1^{\bullet}, \dots, \mathrm{F}_m^{\bullet}$ in a vector space of dimension n, as above, the codimensions of intersection patterns of their elements define a rank function. That is, the function $\mathbf{r} \colon \square_{\underline{\varrho}} \to \mathbb{Z}$ defined, for every $\underline{a} = (a_1, \dots, a_m)$, by

$$\mathbf{r}(\underline{a}) := \operatorname{codim}_\kappa \left(\mathrm{F}_1^{a_1} \cap \dots \cap \mathrm{F}_m^{a_m}\right) = n - \dim_\kappa \left(\mathrm{F}_1^{a_1} \cap \dots \cap \mathrm{F}_m^{a_m}\right)$$

is a rank function. We denote by $\mathscr{M}_{\mathcal{A}}$ the corresponding matcuboid. Note that $\mathscr{M}_{\mathcal{A}}$ is simple if all the inclusions in each flag are strict. As for matroids, a matcuboid appearing in this way will be called *representable* over the field κ. Note that by duality of vector spaces, a representable matcuboid can be described equivalently by a collection of initial increasing flags in the dual space. This point of view allows to associate a matcuboid to any three-dimensional matrix with entries in a given field. We refer to Section 2.3 for more details.

Abstracting the example given above of an arrangement of initial flags in a vector space, we show in Section 2.4 that a finite collection of initial flag matroids, all defined on the same ground set, defines a matcuboid.

In the next three sections, we present alternative axiomatic systems for matcuboids, that will be discussed more thoroughly in the paper.

1.2. **Flats of matcuboids.** Let $\mathscr{M}$ be a matcuboid of rank r with ground set $\square_{\underline{\varrho}}$ and rank function $\mathbf{r}$. A point $\underline{a}$ in $\square_{\underline{\varrho}}$ is called a *flat* of $\mathscr{M}$ if

($*$) for every $1 \leqslant i \leqslant m$ such that $\underline{a} + \underline{e}_i$ belongs to $\square_{\underline{\varrho}}$, we have $\mathbf{r}(\underline{a} + \underline{e}_i) = \mathbf{r}(\underline{a}) + 1$.

Note that in particular, $\underline{\varrho}$ is a flat of $\mathscr{M}$. We denote by $\mathscr{F}(\mathscr{M})$ the set of flats of $\mathscr{M}$. In the case $\mathscr{M}$ is a matroid, $\mathscr{F}(\mathscr{M})$ is the set of flats of the matroid.

As in the case of matroids, a matcuboid can be defined in terms of its flats. The axiomatic system of flats of a matcuboid are (F1), (F2), (F3) given as follows.

Given a poset $(P, \preceq)$ and two elements $x, y \in P$, we say that that y *covers* x and write $y \gtrdot x$ if $y \succ x$ in P and there is no element $z \in P$ such that $y \succ z \succ x$. Let $\mathscr{F}$ be a subset of $\square_{\underline{\varrho}}$. Endowed with the partial order $\preceq$ of the hypercuboid $\square_{\underline{\varrho}}$, $\mathscr{F}$ is a poset. We prove in Section 3 that $\mathscr{F}$ is the set of flats of a matcuboid with underling ground set $\square_{\underline{\varrho}}$ if, and only if, the following properties hold. In other words, the two axiomatic systems (F1)-(F2)-(F3), given below, and (R1)-(R2)-(R3) are equivalent.

(F1) $\underline{\varrho}$ is in $\mathscr{F}$.
(F2) $\mathscr{F}$ is closed under meet.
(F3) If $\underline{a}$ is an element of $\mathscr{F}$ and i is such that $\underline{a} + \underline{e}_i \in \square_{\underline{\varrho}}$, then there exists an element $\underline{b} \succeq \underline{a} + \underline{e}_i$ in $\mathscr{F}$ such that $\underline{b} \gtrdot \underline{a}$ in $\mathscr{F}$.

1.3. **Duality, and circuits of matcuboids.** Again, let $\mathscr{M}$ be a matcuboid on the ground set $\square_{\underline{\varrho}}$. In Section 2.6 we define the dual matcuboid $\mathscr{M}^*$ on the same ground set $\square_{\underline{\varrho}}$. In terms of rank functions, the rank function $\mathbf{r}^*$ of $\mathscr{M}^*$ is given by

$$\mathbf{r}^*(\underline{x}) := |\underline{x}|_{\ell_1} + \mathbf{r}(\underline{x}^c) - \mathbf{r}(\mathscr{M}) \qquad \forall\, \underline{x} \in \square_{\underline{\varrho}}$$

where $\underline{x}^c := \underline{\varrho} - \underline{x}$ is the *complement* of $\underline{x}$ in $\square_{\underline{\varrho}}$, and $\mathbf{r}$ denotes the rank function of $\mathscr{M}$.

Denote by $\mathscr{F}(\mathscr{M}^*)$ the set of flats of the dual matcuboid, and consider

$$\check{\mathscr{C}} := \left\{\underline{a}^c \;\middle|\; \underline{a} \in \mathscr{F}(\mathscr{M}^*)\right\} \subseteq \square_{\underline{\varrho}}.$$

Then, a point $\underline{c}$ in $\square_{\underline{\varrho}}$ is called a *circuit* of $\mathscr{M}$ if

($*$) $\underline{c}$ is an element of $\check{\mathscr{C}}$ which is not the join of any set of elements of $\check{\mathscr{C}} \smallsetminus \{\underline{c}\}$.

We denote by $\mathscr{C}(\mathscr{M})$ the set of circuits of $\mathscr{M}$. This definition extends that of circuits in matroids. Moreover, as is the case of matroids, a matcuboid can be defined in terms of its circuits, via the following axiomatic system. We prove in Section 4 that $\mathscr{C} \subseteq \square_{\underline{\varrho}}$ is the set of circuits of a matcuboid with underlying ground set $\square_{\underline{\varrho}}$ if, and only if, the following properties hold.

(C1) $\underline{0}$ is *not* in $\mathscr{F}$.
(C2) All elements of $\mathscr{C}$ are join-irreducible in $\mathscr{C}$.
(C3) If $\underline{a} \in \check{\mathscr{C}}$ and $i \in \{1, \dots, m\}$ is such that $\underline{a} - \underline{e}_i \in \square_{\underline{\varrho}}$, then there exists an element $\underline{b} \preceq \underline{a} - \underline{e}_i$ in $\check{\mathscr{C}} \cup \{\underline{0}\}$ such that $\underline{b} \lessdot \underline{a}$ in $\check{\mathscr{C}} \cup \{\underline{0}\}$.

In other words, the axiomatic systems (C1)-(C2)-(C3) and (R1)-(R2)-(R3) are equivalent.

1.4. **Independents of matcuboids.** Let $\mathscr{M}$ be a matcuboid on the ground set $\square_{\underline{\varrho}}$. We say that an element $\underline{p}$ of $\square_{\underline{\varrho}}$ is an *independent of* $\mathscr{M}$ if

($*$) for each $i = 1, \dots, m$ such that $\underline{p} - \underline{e}_i \in \square_{\underline{\varrho}}$, we have $\mathbf{r}\left(\underline{p} - \underline{e}_i\right) = \mathbf{r}\left(\underline{p}\right) - 1$.

We denote by $\mathscr{I}(\mathscr{M}) \subseteq \square_{\underline{\varrho}}$ the set of independents of $\mathscr{M}$.

The set of independents of a matcuboid is nonempty and closed under meet $\wedge$. Moreover, deletion reduces the rank in the following sense: for every independent $\underline{a} \in \mathscr{I}(\mathscr{M})$, and every distinct elements $i_1, \dots, i_k \in \{1, \dots, m\}$ with $a_{i_j} \neq 0$, $1 \leqslant j \leqslant k$, we have $\mathbf{r}(\underline{a} - \underline{e}_{i_1} - \cdots - \underline{e}_{i_k}) = \mathbf{r}(\underline{a}) - k$.

We provide an axiomatic system for independent sets of a matcuboid. In order to do this, we need to define an operation of deletion in independents.

Let $\mathscr{J}$ be a subset of $\square_{\underline{\varrho}}$. Let $\underline{a}$ be an element of $\mathscr{J}$ and $i \in \{1, \dots, m\}$ so that $a_i \neq 0$. If there is at least one element $\underline{b} \prec \underline{a}$ in $\mathscr{J}$ which differs from $\underline{a}$ only in the i-th component, we define $\underline{a} \smallsetminus i$ to be such an element in $\mathscr{J}$ with the largest i-th coordinate. In this case, we say that $\underline{a} \smallsetminus i$ is the *deletion of i in $\underline{a}$*.

Definition 1.1. Let $\mathcal{J}$ be a subset of $\boxcube_{\underline{\varrho}}$.

(a) We say that *deletions exist in* $\mathcal{J}$ if for every $i \in \{1,\dots,m\}$, if $a_i \geqslant 1$, the deletion $\underline{a} \smallsetminus i$ exists in $\mathcal{J}$.

If deletions exist in $\mathcal{J}$, then $\underline{0} \in \mathcal{J}$ and for every element $\underline{a}$, there exists a sequence of deletions that reduce $\underline{a}$ to $\underline{0}$.

(b) We say that $\mathcal{J}$ is *orderable* if deletions exist in $\mathcal{J}$ and for every $\underline{a} \in \mathcal{J}$, all the sequences of deletions that bring $\underline{a}$ to $\underline{0}$ have the same length.

If $\mathcal{J}$ is orderable, we define the *size* of $\underline{a}$, denoted by $|\underline{a}|$, as the number of deletions that reduce $\underline{a}$ to $\underline{0}$. $\diamond$

The axiomatic system of independents can be formulated as follows. Let $\mathscr{I}$ be a subset of $\boxcube_{\underline{\varrho}}$ that verifies the following property:

(I1) Deletions exist in $\mathscr{I}$ and the following holds. For all $\underline{p} \in \mathscr{I}$ and deletions $\underline{p} \smallsetminus i$ and $\underline{p} \smallsetminus j$, with $i, j \in \{1,\dots,m\}$, the meet $\underline{q} := \big(\underline{p} \smallsetminus i\big) \wedge \big(\underline{p} \smallsetminus j\big)$ belongs to $\mathscr{I}$ and, moreover, the two intervals $\big[\underline{q}, \underline{p} \smallsetminus i\big]$ and $\big[\underline{q}, \underline{p} \smallsetminus j\big]$ in $\mathscr{I}$ have the same size.

Here, the interval $[\underline{a}, \underline{b}]$ in $\mathscr{I}$ means the set of all $\underline{c} \in \mathscr{I}$ which verify $\underline{a} \preceq \underline{c} \preceq \underline{b}$.

We prove in Section 5 that (I1) is equivalent to the orderability of the set $\mathscr{I}$. In particular, if $\mathscr{I}$ verifies (I1), we can define the size of $\underline{a}$ as the number of deletions needed to reduce $\underline{a}$ to $\underline{0}$. This enables to formulate the second property of interest. For $\underline{a}, \underline{b} \in \boxcube_{\underline{\varrho}}$, denote by $D(\underline{a}, \underline{b})$ the set of elements in $\{1,\dots,m\}$ such that $a_i < b_i$. The following is understood as a matcuboid analogue of the augmentation property for independents of matroids.

(I2) $|\cdot|$ is increasing on independents, i.e., for all $\underline{a}, \underline{b} \in \mathscr{I}$ such that $\underline{a} \prec \underline{b}$, we have $|\underline{a}| < |\underline{b}|$. Moreover, let $\underline{a}, \underline{b}$ be two elements of $\mathscr{I}$ such that $|\underline{a}| < |\underline{b}|$ and $D(\underline{a}, \underline{b})$ contains at least two elements. Then, there exists $\underline{c} \in \mathscr{I}$ that verifies:
- $\underline{c} \preceq \underline{a} \vee \underline{b}$,
- $|\underline{c}| > |\underline{a}|$.
- There exists $i \in D(\underline{a}, \underline{b})$ such that $c_i < b_i$.

We prove in Theorem 5.5 that (I1)-(I2) are equivalent to (R1)-(R2)-(R3).

1.5. **Permutation arrays.** A combinatorial approach to the study of intersection patterns of a configuration of complete flags was introduced by Eriksson–Linusson in the notion of permutation arrays [EL00a, EL00b]. Our Theorem 7.1 shows that permutation arrays are in one-to-one correspondence with matcuboids of rank r or $r+1$ on the ground set $\boxcube_r^d := \boxcube_{\underline{\varrho}}$ with $\underline{\varrho} = (r, r, \dots, r)$, that is, with all $\varrho_j = r$.

1.6. **Matcuboids as coherent complexes of matroids.** As we explain in Section 8, a matcuboid locally gives rise to a collection of matroids. Local obstructions for the representability of a matcuboid can then be formulated in terms of matroid representability. In the case of permutation arrays, via our Theorem 7.1, this gives obstructions for representability that generalize the examples found in the work of Billey and Vakil [BV08].

In Section 8.3, we provide a matroidal characterization of matcuboids by establishing an equivalence between matcuboids and coherent complexes of matroids labeled by the elements of a hypercuboid satisfying Properties (CC1) and (CC2) below. Namely, let $(\mathrm{M}_{\underline{a}})_{\underline{a} \in \boxcube_{\underline{\varrho}}}$ be a family of matroids indexed by $\boxcube_{\underline{\varrho}}$, with $\mathrm{M}_{\underline{a}}$ a matroid on the set $I_{\underline{a}}$ consisting of all $j \in \{1,\dots,m\}$ with $a_j < \varrho_j$. Denote the rank function of $\mathrm{M}_{\underline{a}}$ by $\mathbf{r}_{\underline{a}}$. We say that $(\mathrm{M}_{\underline{a}})$ forms a *coherent complex of matroids* if the following two conditions are met:

(CC1) For all $i \in \{1,\dots,m\}$ and $0 \leqslant t < \varrho_i$, we have $\mathbf{r}_{t\underline{e}_i}(\underline{e}_i) \leqslant 1$.

(CC2) The matroids satisfy the following relation:

$$\mathrm{M}_{\underline{a}+\underline{e}_i} = \begin{cases} \mathrm{M}_{\underline{a}} \,/\, i & \text{if } a_i = \varrho_i - 1 \\ \mathrm{M}_{\underline{a}} \,/\, i \sqcup \{i\} & \text{else} \end{cases}.$$

Theorem 8.6 provides an equivalence between (CC1)-(CC2) and (R1)-(R2)-(R3).

1.7. **Combinatorial limit linear series.** Our original motivation for developing the theory of matcuboids exposed here comes from the problem of describing tropical degenerations of linear series on algebraic curves. In companion work [AG22], matcuboids are used as the combinatorial structure underlying a combinatorial theory of limit linear series on metric graphs (the geometric situation behind this theory is briefly discussed in Section 9.6). While working on the degeneration problem for linear series, we gradually realized how similarly matcuboids and matroids behave. Apart from the concept of *bases*, that we do not cover in this paper, the other relevant constructions in the theory of matroids have their analogues in the setting of matcuboids.

1.8. **Related work.** The work directly related to the content of this paper is the one by Eriksson–Linusson [EL00a, EL00b] on permutation arrays, that we will discuss in Section 7, through Theorem 7.1.

Remembering only the data of the subspaces in a flag arrangement results in a subspace arrangement, whose combinatorics is encoded in an integer polymatroid. In the same way, any matcuboid on the ground set $⧉_{\underline{\varrho}}$, $\underline{\varrho} = (\varrho_1, \dots, \varrho_m)$, gives rise to an integer polymatroid on the ground set the disjoint union $[\varrho_1] \sqcup \cdots \sqcup [\varrho_m]$. Bases and exchange properties for integer polymatroids have been studied by Herzog and Hibi [HH02]. Csirmaz [Csi20] gives axiomatics of cyclic flats. Recent work by Bonin, Chun and Fife [BCF23] studies bases, circuits, and cyclic flats in integer polymatroids, connecting it to a classical construction going back to McDiarmid [McD73], Lovász [Lov77], and Helgason [Hel06] that shows that the data of an integer polymatroid on a ground set E is equivalent to the data of a matroid on a larger set $\widehat{E}$ obtained from E by replacing each element e of E with $\rho(e)$ distinct copies of that element. (This has gained recent interest in the work [OSW19], in developing a decomposition theorem for 2-polymatroids, as well as in the works [CHL$^+$22, EL23] related to combinatorial Hodge theory.) In our setting, starting from a matcuboid, we can thus associate to it first an integer polymatroid and then use this construction to replace the integer polymatroid by a matroid in a larger ground set. We will review this construction in Section 9.2. As we explain there, this leads to a story different from the theory presented in this paper.

In [BDP18], Bollen, Draisma and Pendavingh show that each representation of an algebraic matroid M over a field of positive characteristic comes naturally with a valuation, that they name the *Lindström valuation* of that representation. To this end, using the Frobenius map of the base field, they associate to any such representation what they call a *matroid flock*, an infinite family of linear matroids of the same rank as M, indexed by $\mathbb{Z}^E$, where E is the ground set of M. It is interesting to note that, although these notions arise in totally different contexts, the axiomatic systems of coherent complexes and matroid flocks are reminiscent of each other. There are however some major differences. Namely, the matroids appearing in a matroid flock all have the same rank, and there is an invariance property with respect to the direction $(1, \dots, 1)$. Besides, the boundary condition imposed in the coherent complexes does not appear in matroid flocks.

Submodular functions on distributive lattices are a central topic in the study of some combinatorial optimization problems. We refer to the books by Schrijver [Sch03] and Fuji [Fuj05] for a discussion of these aspects.

The work of Murota [Mur98] investigates a convex analysis theory in the discrete setting, involving functions $f\colon \mathbb{Z}^n \to \mathbb{Z}$ with (global) convexity properties. Discrete convexity in that setting is similar to the submodularity property we study in the present paper, based on matroid theory. Classical duality theorems about real convex functions are proved in the discrete setting.

1.9. **Organization of the text.** In Section 2, we define matcuboids using rank functions and give basic examples, including uniform and representable matcuboids. We define operations of deletion and contraction on matcuboids, and formulate a duality concept.

In Sections 3, 4 and 5, we explore alternative axiomatic systems for matcuboids, relying on flats, circuits and independents, respectively. They give different approaches on matcuboids but are all equivalent.

In Section 6, we prove elementary combinatorial results useful throughout the paper, which give a simpler way of checking whether a function on a hypercube is a rank function.

In Section 7, we show that particular kinds of matcuboids are in natural one-to-one correspondence with permutation arrays, previously studied in the literature.

In Section 8, we provide the equivalence of matcuboids with coherent complexes of matroids, and provide obstructions for the representability.

Finally, in Section 9, we discuss some further properties of matcuboids and a few open questions.

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2. Basic properties

Let n be a non-negative integer and $[n] = \{0, 1, \dots, n\}$. For elements $\varrho_1, \dots, \varrho_m \in [n]$, the *hypercuboid* $\square_{\underline{\varrho}}$ of *width* $\underline{\varrho}$ is the product $\prod_{j=1}^{m}[\varrho_j]$. When $\varrho_1 = \cdots = \varrho_d = r$, we simply denote the hypercuboid by $\square_r^m$. We denote the elements of $\square_{\underline{\varrho}}$ by vectors $\underline{a} = (a_1, \dots, a_m)$, for $a_1 \in [\varrho_1], \dots, a_m \in [\varrho_m]$. In the hypercuboid, we define, for every $i = 1, \dots, r$ and $t \in [\varrho_i]$, the t-th layer in the direction i as $L_t^i := \Big\{\underline{x} \in \square_{\underline{\varrho}}, x_i = t\Big\}$.

There is a partial order $\preceq$ on $\square_{\underline{\varrho}}$ where for a pair of elements $\underline{a}, \underline{b} \in \square_{\underline{\varrho}}$, we write $\underline{a} \preceq \underline{b}$ whenever for all $j = 1, \dots, m$, $a_j \leqslant b_j$. The smallest and largest elements of $\square_{\underline{\varrho}}$ with respect to this partial order are $\underline{0} := (0, \dots, 0)$ and $\underline{\varrho} = (\varrho_1, \dots, \varrho_m)$, respectively. Moreover, there is a lattice structure on $\square_{\underline{\varrho}}$, the two operations of join $\vee$ and meet $\wedge$ correspond to taking the maximum and the minimum coordinate-wise, respectively. Namely, for $\underline{a}, \underline{b} \in \square_{\underline{\varrho}}$, we have

$$\underline{a} \vee \underline{b} = (\max(a_1, b_1), \dots, \max(a_m, b_m)), \qquad \underline{a} \wedge \underline{b} = (\min(a_1, b_1), \dots, \min(a_m, b_m)).$$

A function $f\colon \square_{\underline{\varrho}} \to \mathbb{Z}$ is called *submodular* if for every pair of elements $\underline{a}$ and $\underline{b}$, we have

$$f(\underline{a}) + f(\underline{b}) \geqslant f(\underline{a} \vee \underline{b}) + f(\underline{a} \wedge \underline{b}).$$

We will be interested in a special kind of submodular function on $\square_{\underline{\varrho}}$. For each integer $1 \leqslant i \leqslant m$, we denote by $\underline{e}_i$ the vector whose coordinates are all zero except the i-th coordinate, which is equal to one. For $0 \leqslant t \leqslant \varrho_i$, the vector $t\underline{e}_i$ lies in $\square_{\underline{\varrho}}$.

Definition 2.1 (Matcuboid)**.** A *matcuboid* $\mathscr{M}$ with ground set $\square_{\underline{\varrho}}$ is defined in terms of a function $\mathbf{r}\colon \square_{\underline{\varrho}} \to \mathbb{Z}$ called *the rank function of* $\mathscr{M}$ that satisfies the following conditions:

(R1) $\mathbf{r}(\underline{0}) = 0$, and for every $1 \leqslant i \leqslant m$ and $1 \leqslant t \leqslant \varrho_i$, we have $\mathbf{r}(t\underline{e}_i) - \mathbf{r}((t-1)\underline{e}_i) \leqslant 1$.

(R2) $\mathbf{r}$ is non-decreasing with respect to the partial order of $\square_{\underline{\varrho}}$, that is, if $\underline{a} \preceq \underline{b}$, then $\mathbf{r}(\underline{a}) \leqslant \mathbf{r}(\underline{b})$.

(R3) The rank function $\mathbf{r}$ is submodular.

We call $r = r(\mathscr{M}) := \mathbf{r}\left(\underline{\varrho}\right)$, the maximum value taken by the function $\mathbf{r}$, the *rank* of $\mathscr{M}$. $\diamond$

Note that (R1) implies that $\mathbf{r}(t\underline{e}_i) \leqslant t$ for all $i = 1, \ldots, m$ and $t \in [\varrho_i]$.

We say that $\mathscr{M}$ is *simple* if the following alternate form of (R1) holds:

(R1*) $\varrho_i > 0$ and $\mathbf{r}(t\underline{e}_i) = t$ for all $i = 1, \ldots, m$ and $t \in [\varrho_i]$.

Remark 2.2. In $\mathscr{M}$ is simple, then the above properties imply that if $\underline{a} \in \square_{\underline{\varrho}}$ has rank j, then $a_i \leqslant j$ for all $i = 1, \ldots, m$. In particular, $\underline{0}$ is the only element of rank 0 in $\square_{\underline{\varrho}}$. $\diamond$

To be able to give examples of rank functions easily, we adopt the following notation.

Convention 2.3 (Cases $m = 1, 2, 3$)**.** In this article, for $m = 1$, a function on $\square_r$ is described by a tuple with $r + 1$ entries $(x_0, \ldots, x_r)$, which means that the value of the function on the i-th entry of $\square_r$ is x_i.

In the same way, for $m = 2$, a function on $\square_{(\varrho_1,\varrho_2)}$ will be often described by a array of size $(\varrho_1 + 1) \times (\varrho_2 + 1)$, $(x_{ij})_{0\leqslant i\leqslant \varrho_1, 0\leqslant j\leqslant \varrho_2}$, which means that the function takes value x_{ij} on $(i, j) \in \square_{(\varrho_1,\varrho_2)}$. We take as a convention that the first direction is horizontal, the second direction is vertical, and the origin is the bottom left-hand corner.

When $m = 3$ and $\underline{\varrho} = (\varrho_1, \varrho_2, \varrho_3)$, a function defined $\square_{\underline{\varrho}}$ will be specified by $\varrho_3 + 1$ arrays $R_0, \ldots, R_{\varrho_3}$ of size $(\varrho_1 + 1) \times (\varrho_2 + 1)$, where R_k describes the values of the function on $\square_{(\varrho_1,\varrho_2)} \times \{k\} \subseteq \square_{\underline{\varrho}}$. $\diamond$

Here are two examples of matcuboids with $\underline{\varrho} = (4, 3)$. The one on the left is simple, the one on the right is not.

$$\begin{pmatrix} 3 & 3 & 3 & 4 & 5 \\ 2 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \qquad\qquad \begin{pmatrix} 3 & 3 & 3 & 3 & 4 \\ 2 & 2 & 2 & 2 & 3 \\ 1 & 2 & 2 & 2 & 3 \\ 0 & 1 & 2 & 2 & 3 \end{pmatrix}$$

The following proposition implies that the set of values of a matcuboid of rank r is the interval $[r]$.

Proposition 2.4. *Let $\mathbf{r}$ be a rank function on $\square_{\underline{\varrho}}$. Let $i \in \{1, \ldots, m\}$. For an element $\underline{a} \in \square_{\underline{\varrho}}$ such that $\underline{a} + \underline{e}_i \in \square_{\underline{\varrho}}$, we have*

$$\mathbf{r}(\underline{a}) \leqslant \mathbf{r}(\underline{a} + \underline{e}_i) \leqslant \mathbf{r}(\underline{a}) + 1.$$

Proof. Let $\underline{b} = (a_i + 1)\,\underline{e}_i$, and note that $\underline{a} \vee \underline{b} = \underline{a} + \underline{e}_i$ and $\underline{a} \wedge \underline{b} = a_i\,\underline{e}_i$. Applying the submodularity of $\mathbf{r}$ to the vectors $\underline{a}$ and $\underline{b}$, and using (R1) in Definition 2.1, we get $\mathbf{r}(\underline{a}) + 1 \geqslant \mathbf{r}(\underline{a} + \underline{e}_i)$. The first inequality follows from the non-decreasing property of $\mathbf{r}$. $\square$

2.1. **Uniform matcuboids.** Notations as in the previous section, let $\underline{\varrho} = (\varrho_1, \ldots, \varrho_m)$, and consider the corresponding hypercuboid $\square_{\underline{\varrho}}$. Let $r \in [\varrho_1 + \cdots + \varrho_m]$ be a non-negative integer. We define the uniform matcuboid $\mathscr{U}_{\underline{\varrho},r}$ of width $\underline{\varrho}$ and rank r as the matcuboid defined by the *standard rank function*

$$\mathbf{r}^{\mathrm{st}}(\underline{a}) := \min(r, a_1 + \cdots + a_m) \qquad \text{for } \underline{a} = (a_1, \ldots, a_m) \in \square_{\underline{\varrho}}.$$

Notice that the uniform matcuboid $\mathscr{U}_{\underline{\varrho},r}$ is simple if, and only if, $r \geqslant \max_i \varrho_i$.

Below are the uniform matcuboids $\mathscr{U}_{(4,3),3}$ and $\mathscr{U}_{(4,3),5}$.

$$\begin{pmatrix} 3 & 3 & 3 & 3 & 3 \\ 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 & 3 \end{pmatrix} \qquad \begin{pmatrix} 3 & 4 & 5 & 5 & 5 \\ 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

Proposition 2.5. *The rank function* $\mathbf{r}$ *of any matcuboid* $\mathscr{M}$ *of rank* r *on the ground set* $\square_{\underline{\varrho}}$ *is dominated by the rank function* $\mathbf{r}^{\mathrm{st}}$ *of the uniform matcuboid* $\mathscr{U}_{\underline{\varrho},r}$*. That is, for every* $\underline{a} \in \square_{\underline{\varrho}}$*, we have* $\mathbf{r}(\underline{a}) \leqslant \mathbf{r}^{\mathrm{st}}(\underline{a})$*.*

Proof. It follows directly from Proposition 2.4 that we have $\mathbf{r}(\underline{a}) \leqslant a_1 + \cdots + a_m$. Combined with $\mathbf{r}(\underline{a}) \leqslant r$, we deduce the result. $\square$

2.2. **Representable matcuboids.** Let n be a non-negative integer, and let H be a vector space of dimension n over some field κ. An *initial (decreasing) flag* of H of length r consists of a chain of vector subspaces

$$\mathrm{H} = \mathrm{F}^0 \supseteq \mathrm{F}^1 \supseteq \cdots \supseteq \mathrm{F}^{r-1} \supseteq \mathrm{F}^r \supsetneq (0),$$

where for each positive $i \in [r]$, F^i is a vector subspace of codimension 0 or 1 in F^{i-1}. We say that $\mathrm{F}^\bullet$ is *simple* if each F^i has codimension i in H. A *complete flag* is a simple initial flag of length $n-1$.

Let m be a positive integer, and let $\mathcal{A}$ be a collection of m initial flags $\mathrm{F}^\bullet_1, \dots, \mathrm{F}^\bullet_m$ of H of lengths $\varrho_1, \dots, \varrho_m$, respectively. Define the function $\mathbf{r} \colon \square_{\underline{\varrho}} \to \mathbb{Z}$ by

$$\mathbf{r}(\underline{a}) := \mathrm{codim}_\kappa \left(\mathrm{F}_1^{a_1} \cap \cdots \cap \mathrm{F}_m^{a_m} \right) \qquad \forall\, \underline{a} = (a_1, \dots, a_m) \in \square_{\underline{\varrho}}. \tag{1}$$

Proposition 2.6. *The hypercuboid* $\square_{\underline{\varrho}}$ *endowed with the function* $\mathbf{r}$ *defined in* (1) *is a matcuboid. This matcuboid is simple if, and only if, all the initial flags are simple.*

Proof. Let $\underline{a}$ and $\underline{b}$ be two points $\square_{\underline{\varrho}}$, and let $\underline{x} := \underline{a} \wedge \underline{b}$ and $\underline{y} := \underline{a} \vee \underline{b}$. We have an injection

$$\mathrm{F}_1^{a_1} \cap \cdots \cap \mathrm{F}_m^{a_m} \big/ \mathrm{F}_1^{y_1} \cap \cdots \cap \mathrm{F}_m^{y_m} \hookrightarrow \mathrm{F}_1^{x_1} \cap \cdots \cap \mathrm{F}_m^{x_m} \big/ \mathrm{F}_1^{b_1} \cap \cdots \cap \mathrm{F}_m^{b_m},$$

from which, comparing the dimensions, we get $\mathbf{r}(\underline{b}) - \mathbf{r}(\underline{x}) \geqslant \mathbf{r}(\underline{y}) - \mathbf{r}(\underline{a})$. This proves the submodularity of $\mathbf{r}$. Properties (R1) and (R2) in Definition 2.1 are trivially verified. This proves the first assertion. The matcuboid is simple if, and only if, each F_j^i has codimension i in H, that is if, and only if, $\mathrm{F}_j^\bullet$ is simple, for $j = 1, \dots, m$. $\square$

We denote by $\mathscr{M}_{\mathcal{A}}$ the matcuboid associated to $\mathcal{A}$.

Definition 2.7 (Representable matcuboid)**.** A matcuboid $\mathscr{M}$ on ground set $\square_{\underline{\varrho}}$ is called *representable over a field* κ if it is the matcuboid associated to an arrangement of m initial flags $\mathrm{F}^\bullet_1, \dots, \mathrm{F}^\bullet_m$ of lengths $\varrho_1, \dots, \varrho_m$, respectively, in a κ-vector space H. $\diamond$

Example 2.8. The matcuboid $\mathscr{U}_{\underline{\varrho},r}$ is representable. Indeed, it is the matcuboid associated to an arrangement of m initial flags of lengths $\varrho_1, \dots, \varrho_m$ in H of dimension r which are in *general relative position*, that is, whose intersection patterns have the smallest possible dimensions. $\diamond$

Let $\mathscr{M}$ be a matcuboid associated to an arrangement of m initial (decreasing) flags $\mathrm{F}^\bullet_1, \dots, \mathrm{F}^\bullet_m$ inside H. For every $i \in \{1, \dots, m\}$, duality transforms the initial (decreasing) flag in H

$$\mathrm{H} = \mathrm{F}_i^0 \supseteq \mathrm{F}_i^1 \supseteq \cdots \supseteq \mathrm{F}_i^{\varrho_i - 1} \supseteq \mathrm{F}_i^{\varrho_i} \supsetneq (0)$$

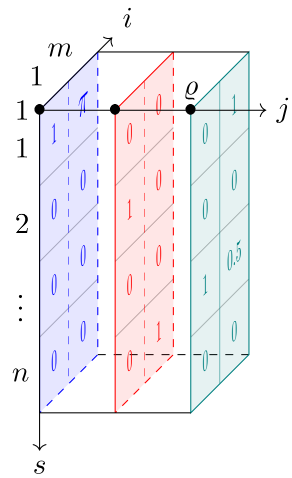
into an *initial (increasing) flag* in H^*

$$(0) = \mathrm{G}_0^i \subseteq \mathrm{G}_1^i \subseteq \cdots \subseteq \mathrm{G}_{\varrho_i - 1}^i \subseteq \mathrm{G}_{\varrho_i}^i \subsetneq \mathrm{H},$$

where G^i_j is the orthgonal to F^j_i for the duality pairing $\langle \cdot\,,\cdot \rangle\colon \mathrm{H} \times \mathrm{H}^* \to \kappa$, that is, $\mathrm{G}^i_j := \bigcap_{v \in \mathrm{F}^j_i} \ker(v\colon \mathrm{H}^* \to \kappa)$ and $\mathrm{F}^j_i := \bigcap_{\ell \in \mathrm{G}^i_j} \ker(\ell\colon \mathrm{H} \to \kappa)$, so that one filtration can be recovered from the other. Note that in the case of matroids, this duality corresponds to the one between arrangements of hyperplanes in H and arrangements of vectors in the dual H^*.

The rank function $\mathbf{r}$ of $\mathscr{M}$, defined in Equation (1) using intersections of the F^j_i, can be alternatively described using the flags G^j_i in the following way:

$$\mathbf{r}(\underline{a}) = \dim_\kappa \left(\mathrm{G}^1_{a_1} + \cdots + \mathrm{G}^m_{a_m}\right) \qquad \forall\ \underline{a} = (a_1, \ldots, a_m) \in \square_{\underline{\varrho}}. \tag{2}$$



$$\begin{pmatrix} 3 & 3 & 4 & 4 \\ 2 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

FIGURE 1. The left figure represents a three-dimensional matrix $A = (A^i_{js})$ of size $m \times \varrho \times n$ with $m = 2$, $\varrho = 3$ and $n = 4$. The blue (resp. red, resp. green) layer contains vertically the coordinates of the vectors v^1_1 and v^2_1 (resp. v^1_2 and v^2_2, resp. v^1_3 and v^2_3). The associated matcuboid is given on the right.

2.3. **Matcuboids induced by three-dimensional matrices.** Using the duality between initial (decreasing) flags in H and initial (increasing) flags in H^*, we explain a procedure that associates a representable matcuboid to any three-dimensional matrix with coordinates in a field κ. This construction extends the representation of representable matroids by matrices. As in the case of matroids, this justifies the terminology *matcuboid*, which encompasses both the idea of *cubical matrix*, i.e., three-dimensional matrix (like the terminology "matroid", with the same root as "matrix") and *hypercuboid* (a matcuboid is described by a hypercuboid of numbers, given by the rank function).

Notation as in the previous section, assume first that $\varrho_1 = \cdots = \varrho_m = \varrho$ and let $i \in \{1, \ldots, m\}$. We choose vectors $v^i_1, \ldots, v^i_\varrho \in \mathrm{H}^*$ such that for every $j \in \{1, \ldots, \varrho\}$, $\mathrm{G}^i_j = \langle v^i_1, \ldots, v^i_j \rangle$. This shows that a representable matcuboid $\mathscr{M}$ can be determined by the collection of vectors v^i_j, for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, \varrho\}$. Said otherwise, choosing a basis of H^*, $\mathscr{M}$ is determined by a three-dimensional matrix $A = (A^i_{js})$ where $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, \varrho\}$ and $s \in \{1, \ldots, n\}$, n being the dimension of H^*. Inversely, using the definition of rank function given in (2), this procedure gives a way to associate to every three-dimensional matrix $A = (A^i_{js})$ of size $m \times \varrho \times n$ with entries in a field κ a matcuboid $\mathscr{M}_A$ on the hypercube $\square^m_\varrho$.

In the general case, if not all ϱ_i are equal, we set $\varrho := \max_i \varrho_i$ and choose, for every $i \in \{1, \ldots, m\}$, a family of vectors $v^i_1, \ldots, v^i_{\varrho_i}, 0, \ldots, 0$, with v^i_j as before, completed now with $\varrho - \varrho_i$ copies of the zero vector. This gives a matrix A of size $m \times \varrho \times n$. In the matcuboid $\mathscr{M}_A$ associated to A, we delete, for every $i \in \{1, \ldots, m\}$, $\varrho - \varrho_i$ times the element i (we refer to Section 2.5.1 below for the definition of deletion operation in matcuboids). This gives the matcuboid associated to the original family of flags.

It follows from the construction above that every representable matcuboid is associated to some three-dimensional matrix, possibly after a few deletions corresponding to zero vectors.

An example of a matcuboid associated to a three-dimensional real matrix with $m = 2$, $\varrho_1 = \varrho_2 = 3$ and $n = 4$, is depicted in Figure 1. The ground set of the corresponding matcuboid is $[3] \times [3]$.

2.4. **Matcuboids induced by a collection of flag matroids.** We show that a finite collection of initial flag matroids on the same ground set E gives rise to a (simple) matcuboid. We refer to [BGW03, Chapter 1] and [CDMS17] for an introduction to the theory of flag matroids.

Let E be a finite set and ϱ a positive integer. An *initial flag* of size s is an increasing chain of subsets $F_0 = \varnothing \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_s \subseteq E$ with $|F_j| = j$ for $j = 1, \ldots, s$. Note that the data of an initial flag is equivalent to an ordered sequence $\varepsilon_1, \ldots, \varepsilon_s$ of distinct elements of E, F_j consisting of the first j elements $\varepsilon_1, \ldots, \varepsilon_j$.

A total order $<_{\mathcal{O}}$ on E induces a partial, element-wise order on E^s. Through the bijection between initial flags of size s and ordered sequences of size s in E, $<_{\mathcal{O}}$ induces a partial order on initial flags of size s.

An *initial flag matroid* of rank s is a collection $\mathscr{F}$ of initial flags of size s as above such that for any total order $<_{\mathcal{O}}$ on E, there exists a unique flag in $\mathscr{F}$ maximal with respect to the induced order on initial flags of size s. In this case, the following properties hold:

- the collection consisting of the terms F_j of flags $F_\bullet$ appearing in $\mathscr{F}$ forms the set of bases of a matroid M_j of rank j on the ground set E, for $j = 0, \ldots, s$;
- the matroid M_j is a quotient of the matroid M_{j+1};
- any sequence $F_0 = \varnothing \subset F_1 \subset F_2 \subset \cdots \subset F_s$ with F_j a basis of M_j is an element of $\mathscr{F}$.

(These properties are equivalent to $\mathscr{F}$ defining a flag matroid, see [BGW03, Theorem 1.7.1].) The matroid M_j is called the *j-th constituent* of M.

Let now $\underline{\varrho} = (\varrho_1, \ldots, \varrho_m)$ be a vector with positive integer entries. Consider a collection $M^1, \ldots, M^m$ of initial flag matroids of respective ranks $\varrho_1, \ldots, \varrho_m$ on the ground set E. For each $i = 1, \ldots, m$, and each $j = 0, \ldots, \varrho_i$, denote by M^i_j the j-th constituent of M^i. This is a matroid of rank j.

For each $\underline{a} \in \square_{\underline{\varrho}}$, denote by $M_{\underline{a}}$ the matroid union $M^1_{a_1} \cup \cdots \cup M^m_{a_m}$ of $M^1_{a_1}, \ldots, M^m_{a_m}$. Recall that the independent sets of the matroid union $M_{\underline{a}}$ are subsets of E of the form $I_1 \cup \cdots \cup I_m$ where each I_i is an independent of the matroid $M^i_{a_i}$ for $i = 1, \ldots, m$.

Consider the function $\mathbf{r} \colon \square_{\underline{\varrho}} \to \mathbb{Z}_{\geqslant 0}$ on the hypercuboid defined by

$$\mathbf{r}(\underline{a}) := r(M_{\underline{a}}) \qquad \forall \underline{a} \in \square_{\underline{\varrho}},$$

where $r(M_{\underline{a}})$ is the rank of the matroid $M_{\underline{a}}$.

Theorem 2.9. *Notations as above, $\mathbf{r}$ is the rank function of a simple matcuboid $\mathscr{M} = \mathscr{M}(M^1, \ldots, M^m)$ with ground set the hypercuboid $\square_{\underline{\varrho}}$.*

Proof. We first note that $\mathbf{r}(t\underline{e}_i) = r(M^i_t) = t$ for all $t = 0, \ldots, \varrho_i$. This shows that (R1*) is verified. The axiom (R2) is obviously verified by the definition of the matroid union. It thus remains to show (R3), i.e., that $\mathbf{r}$ is submodular. By Theorem 6.2, it will be enough to show that $\mathbf{r}$ verifies the diamond property, that is, for all $\underline{x} \in \square_{\underline{\varrho}}$ and distinct $1 \leqslant i, j \leqslant m$ with $\underline{x} + \underline{e}_i, \underline{x} + \underline{e}_j \in \square_{\underline{\varrho}}$, we have

$$\mathbf{r}(\underline{x} + \underline{e}_i) + \mathbf{r}(\underline{x} + \underline{e}_j) \geqslant \mathbf{r}(\underline{x} + \underline{e}_i + \underline{e}_j) + \mathbf{r}(\underline{x}). \tag{3}$$

Removing an element from each independent set of $M^i_{x_i+1}$ results in an independent set of $M^i_{x_i}$. This implies that $\mathbf{r}(\underline{x}+\underline{e}_i) \leqslant \mathbf{r}(\underline{x})+1$. We thus have $\mathbf{r}(\underline{x}) \leqslant \mathbf{r}(\underline{x}+\underline{e}_i+\underline{e}_j) \leqslant \mathbf{r}(\underline{x})+2$.

Let $\underline{a} = \underline{x}+\underline{e}_i$, $\underline{b} = \underline{x}+\underline{e}_j$, and $\underline{c} = \underline{x}+\underline{e}_i+\underline{e}_j$. Three cases can occur, depending on whether $\mathbf{r}(\underline{c}) = \mathbf{r}(\underline{x})$, $\mathbf{r}(\underline{x})+1$, or $\mathbf{r}(\underline{x})+2$.

• In the first case, $\mathbf{r}(\underline{c}) = \mathbf{r}(\underline{x})$, inequality (3) holds trivially.

• Consider the third case $\mathbf{r}(\underline{c}) = \mathbf{r}(\underline{x})+2$. In this case, using the inequality $\mathbf{r}(\underline{y}+\underline{e}_k) \leqslant \mathbf{r}(\underline{y})+1$ for all $\underline{y}, \underline{y}+\underline{e}_k \in \cube_{\underline{\varrho}}$, we infer that $\mathbf{r}(\underline{a}) = \mathbf{r}(\underline{b}) = \mathbf{r}(\underline{x})+1$, and inequality (3) holds again trivially.

• It remains to treat the case $\mathbf{r}(\underline{c}) = \mathbf{r}(\underline{x})+1$. Let $I = I_1 \cup \cdots \cup I_m$ be a basis of $M_{\underline{x}}$ with I_k an independent of $M^k_{x_k}$ for $k = 1, \dots, m$. There exists a basis J of $M_{\underline{c}}$ which contains I and an extra element ε of E. Write $J = J_1 \cup \cdots \cup J_m$ with J_k an independent of $M^k_{c_k}$, for $k = 1, \dots, m$. Since J is not an independent of $M_{\underline{x}}$, ε appears in either J_i or J_j. Removing it if necessary from one of the two, we can suppose that ε appears in exactly one of the two sets J_i or J_j, say, without loss of generality, in J_i. Then, J will be an independent set of $M_{\underline{a}}$, and so $\mathbf{r}(\underline{a}) = \mathbf{r}(\underline{x})+1$. This shows that inequality (3) holds. The theorem follows. $\square$

Note that if, in the definition of an initial flag, we relax strict inclusions to large inclusions, the same construction as above gives rise as well to matcuboids, which are not necessarily simple.

2.5. **Operations on matcuboids.** Let $\mathscr{M}$ be a matcuboid with underlying ground set $\cube_{\underline{\varrho}}$, $\underline{\varrho} = (\varrho_1, \dots, \varrho_m)$.

2.5.1. *Deletion.* Let $i \in \{1, \dots, m\}$. We define the *deletion* of i in $\mathscr{M}$, denoted by $\mathscr{M} \setminus i$, as the matcuboid with ground set $\cube_{\underline{\varrho}'}$, $\underline{\varrho}' = (\varrho_1, \dots, \varrho_i - 1, \dots, \varrho_m)$, defined as follows. We view $\cube_{\underline{\varrho}'}$ as the subset of $\cube_{\underline{\varrho}}$ consisting of all the points $\underline{x}$ with $x_i < \varrho_i$ and define the rank function $\mathbf{r}'$ of $\mathscr{M} \setminus i$ to be the restriction of $\mathbf{r}$ on $\cube_{\underline{\varrho}'}$. Obviously, $\mathbf{r}'$ verifies the axiomatic system (R1)-(R2)-(R3) of matcuboid rank functions. Furthermore, note that if $\mathscr{M}$ is simple, then so is $\mathscr{M} \setminus i$.

As an example, here is a (simple) matcuboid $\mathscr{M}$ with $\underline{\varrho} = (4,3)$ (left) and its deletion $\mathscr{M} \setminus 2$ in the vertical direction (right).

$$\begin{pmatrix} 3 & 3 & 3 & 4 & 5 \\ 2 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

2.5.2. *Contraction.* Let $i \in \{1, \dots, m\}$. We define the *contraction* of i in $\mathscr{M}$, denoted by $\mathscr{M} \,/\, i$, as the matcuboid with ground set $\cube_{\underline{\varrho}'}$, $\underline{\varrho}' = (\varrho_1, \dots, \varrho_i - 1, \dots, \varrho_m)$, defined as follows. We define an embedding of $\cube_{\underline{\varrho}'}$ in $\cube_{\underline{\varrho}}$ by sending each point $\underline{x}$ to $\underline{x}+\underline{e}_i$. We then define the rank function $\mathbf{r}'$ of $\mathscr{M} \,/\, i$ by setting $\mathbf{r}'(\underline{x}) := \mathbf{r}(\underline{x}+\underline{e}_i) - \mathbf{r}(\underline{e}_i)$. The embedding of $\cube_{\underline{\varrho}'}$ in $\cube_{\underline{\varrho}}$ respects the two operations of $\wedge$ and $\vee$. It is easy to see that $\mathbf{r}'$ verifies the axiomatic system (R1)-(R2)-(R3) of matcuboid rank functions. Note that $\mathscr{M} \,/\, i$ is not necessarily simple, even if $\mathscr{M}$ is so.

As an example, here is a (simple) matcuboid $\mathscr{M}$ with $\underline{\varrho} = (4,3)$ (left) and its (non-simple) contraction $\mathscr{M} \,/\, 2$ in the vertical direction (right).

$$\begin{pmatrix} 3 & 3 & 3 & 4 & 5 \\ 2 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 2 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 & 3 \end{pmatrix}$$

2.5.3. *Minors.* A matcuboid $\mathscr{M}'$ is a *minor* of another matcuboid $\mathscr{M}$ if it can be obtained by a sequence of deletions and contractions from $\mathscr{M}$.

Both operations of contraction and deletion respect the representability over a given field κ. It follows that if $\mathscr{M}$ is representable over κ, then all of its minors $\mathscr{M}'$ are also representable over κ. We will discuss the connection between representability and minors in Section 9.3.

2.6. **Duality.** Let $\mathscr{M}$ be a matcuboid on the ground set $\square_{\underline{\varrho}}$ with rank function $\mathbf{r}$. The *dual matcuboid* $\mathscr{M}^*$ is the matcuboid on $\square_{\underline{\varrho}}$ with rank function $\mathbf{r}^*$ defined by

$$\mathbf{r}^*(\underline{x}) := |\underline{x}|_{\ell_1} + \mathbf{r}(\underline{x}^c) - \mathbf{r}(\mathscr{M}) \qquad \forall\, \underline{x} \in \square_{\underline{\varrho}}.$$

Here,

$$\underline{x}^c := \underline{\varrho} - \underline{x} = (\varrho_1 - x_1, \dots, \varrho_m - x_m)$$

is the *complement* of $\underline{x}$ and $|\underline{x}|_{\ell_1} := x_1 + \dots + x_m$ is the ℓ_1-norm of $\underline{x} = (x_1, \dots, x_m)$. A direct verification shows that $\mathbf{r}^*$ verifies the axiomatic system (R1)-(R2)-(R3) of matcuboid rank functions. Moreover, $\mathscr{M}^*$ has rank $\mathbf{r}^*(\mathscr{M}^*) = \left|\underline{\varrho}\right|_{\ell_1} - \mathbf{r}(\mathscr{M})$, and we have $(\mathscr{M}^*)^* = \mathscr{M}$. Note however that $\mathscr{M}$ can be simple without $\mathscr{M}^*$ being so, and vice-versa.

Here is a (simple) matcuboid $\mathscr{M}$ with $\underline{\varrho} = (4,3)$ (left) and its (non-simple) dual $\mathscr{M}^*$ (right).

$$\begin{pmatrix} 3 & 3 & 3 & 4 & 5 \\ 2 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

3. Flats

In this section, we define flats of a matcuboid and provide an axiomatic system for the set of flats of a matcuboid. This extends the axioms of flats in matroid theory.

Definition 3.1 (Flats of a matcuboid)**.** Let $\mathscr{M} = (\square_{\underline{\varrho}}, \mathbf{r})$ be a matcuboid of rank r. A point $\underline{a}$ is called a *flat* for $\mathbf{r}$ if for every $1 \leqslant i \leqslant m$ such that $\underline{a} + \underline{e}_i$ belongs to $\square_{\underline{\varrho}}$, we have $\mathbf{r}(\underline{a} + \underline{e}_i) = \mathbf{r}(\underline{a}) + 1$. We denote by $\mathscr{F} = \mathscr{F}(\mathscr{M}) \subseteq \square_{\underline{\varrho}}$ the set of flats of $\mathscr{M}$. $\diamond$

Here are two (simple) matcuboids with $\underline{\varrho} = (4,3)$ for the first, and $\underline{\varrho} = (5,4)$ for the second. The flats in each case are depicted in blue.

$$\begin{pmatrix} 3 & 3 & \color{blue}{3} & \color{blue}{4} & \color{blue}{5} \\ 2 & 2 & \color{blue}{2} & \color{blue}{3} & \color{blue}{4} \\ \color{blue}{1} & 2 & 2 & 3 & 4 \\ \color{blue}{0} & \color{blue}{1} & 2 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 4 & \color{blue}{4} & 5 & 5 & \color{blue}{5} & \color{blue}{6} \\ \color{blue}{3} & 4 & 5 & 5 & 5 & 6 \\ \color{blue}{2} & \color{blue}{3} & 4 & 4 & \color{blue}{4} & \color{blue}{5} \\ \color{blue}{1} & \color{blue}{2} & 3 & \color{blue}{3} & 4 & 5 \\ \color{blue}{0} & \color{blue}{1} & \color{blue}{2} & 3 & 4 & 5 \end{pmatrix}.$$

Proposition 3.2 (Stability of flats under meet)**.** *The set of flats $\mathscr{F}(\mathscr{M})$ of a matcuboid $\mathscr{M}$ is stable under $\wedge$.*

We need the following lemma.

Lemma 3.3. *Let $i \in \{1, \dots, m\}$ and $\underline{x}, \underline{y}, \underline{x} + \underline{e}_i, \underline{y} + \underline{e}_i$ elements of $\square_{\underline{\varrho}}$ with $\underline{x} \preceq \underline{y}$ and $x_i = y_i$. If $\mathbf{r}\left(\underline{y} + \underline{e}_i\right) = \mathbf{r}\left(\underline{y}\right) + 1$, then we have $\mathbf{r}(\underline{x} + \underline{e}_i) = \mathbf{r}(\underline{x}) + 1$.*

Proof. This follows directly from the submodularity of $\mathbf{r}$ to $\underline{x} + \underline{e}_i$ and $\underline{y}$, and Proposition 2.4. $\square$

Proof of Proposition 3.2. Let $\underline{a}$ and $\underline{b}$ be elements of $\cube_{\underline{\varrho}}$ and let $\underline{c} = \underline{a} \wedge \underline{b}$. Let $1 \leqslant i \leqslant m$ be such that $\underline{c} + \underline{e}_i$ belongs to $\cube_{\underline{\varrho}}$. We have to show that $\mathbf{r}(\underline{c} + \underline{e}_i) = \mathbf{r}(\underline{c}) + 1$. By symmetry, we can suppose that $a_i \leqslant b_i$, that is, $c_i = a_i$. Since $\underline{a}$ is a flat, we have $\mathbf{r}(\underline{a} + \underline{e}_i) = \mathbf{r}(\underline{a}) + 1$. Applying Lemma 3.3 to $\underline{x} = \underline{c}$ and $\underline{y} = \underline{a}$, we conclude. □

The above result implies the following.

Theorem 3.4. *The set $\mathscr{F}(\mathscr{M})$ of flats of a matcuboid endowed with the partial order $\preceq$ is a graded lattice. The grading is induced by the rank function.*

Proof. $\mathscr{F}(\mathscr{M})$ has a minimum and a maximum element, and is stable under meet. It follows that it is a lattice, with the operation $\vee$ between two elements $\underline{a}$ and $\underline{b}$ in $\mathscr{F}(matcub)$ defined as the meet of all the upper bounds $\underline{c}$ for $\underline{a}$ and $\underline{b}$.

Note that for $\underline{a} \prec \underline{b}$ two different flats of $\mathscr{M}$, we have $\mathbf{r}(\underline{a}) < \mathbf{r}(\underline{b})$. The statement that $\mathscr{F}(\mathscr{M})$ is graded is a consequence of Proposition 3.6 below. □

Lemma 3.5. *Let $\mathscr{M}$ be a matcuboid on the ground set $\cube_{\underline{\varrho}}$. Let $\underline{x}$ be an element of $\cube_{\underline{\varrho}}$, and let $\underline{c}$ be the minimum flat with $\underline{c} \succeq \underline{x}$. Then, we have*

1. $\mathbf{r}(\underline{c}) = \mathbf{r}(\underline{x})$.
2. *Let $i \in \{1, \dots, m\}$ be such that $\underline{x} + \underline{e}_i \in \cube_{\underline{\varrho}}$. Then,*
   - *if $c_i > x_i$, then we have $\mathbf{r}(\underline{x} + \underline{e}_i) = \mathbf{r}(\underline{x})$.*
   - *if $c_i = x_i$, then we have $\mathbf{r}(\underline{x} + \underline{e}_i) = \mathbf{r}(\underline{x}) + 1$.*

Proof. To prove (1), it will be enough to show there exists a flat $\underline{b} \succeq \underline{x}$ with $\mathbf{r}(\underline{b}) = \mathbf{r}(\underline{x})$. Then, since flats are closed under meet, $\underline{b}$ will coincide with $\underline{c}$ and (1) follows. We proceed by a reverse induction on the ℓ_1-norm of $\underline{x}$. If $\underline{x}$ is a flat, in particular, if $\underline{x} = \underline{\varrho}$, there is nothing to prove. Otherwise, there exists $1 \leqslant i \leqslant m$ with $\underline{y} := \underline{x} + \underline{e}_i \in \cube_{\underline{\varrho}}$ and $\mathbf{r}(\underline{x} + \underline{e}_i) = \mathbf{r}(\underline{x})$. By the induction hypothesis, there is a flat $\underline{b} \succeq \underline{y}$ with $\mathbf{r}(\underline{b}) = \mathbf{r}(\underline{y}) = \mathbf{r}(\underline{x})$, and we conclude.

We prove (2). If $c_i \geqslant x_i + 1$, then $\underline{x} \preceq \underline{x} + \underline{e}_i \preceq \underline{c}$, and so $\mathbf{r}(\underline{c}) \leqslant \mathbf{r}(\underline{x} + \underline{e}_i) \leqslant \mathbf{r}(\underline{c}) = \mathbf{r}(\underline{x})$. We infer that $\mathbf{r}(\underline{x} + \underline{e}_i) = \mathbf{r}(\underline{x})$.

If $c_i = x_i$, then using $\mathbf{r}(\underline{c} + \underline{e}_i) = \mathbf{r}(\underline{c}) + 1$, we apply Lemma 3.3 and deduce that $\mathbf{r}(\underline{x} + \underline{e}_i) = \mathbf{r}(\underline{x}) + 1$, as required. □

We get the following corollary.

Proposition 3.6. *Let $\underline{a} \prec \underline{b}$ be two different flats of $\mathscr{M}$ with $\mathbf{r}(\underline{b}) \geqslant \mathbf{r}(\underline{a}) + 2$. There exists a flat $\underline{a} \prec \underline{c} \prec \underline{b}$ with $\mathbf{r}(\underline{c}) = \mathbf{r}(\underline{a}) + 1$.*

Proof. Since $\underline{a} \prec \underline{b}$, there is an index i with $a_i < b_i$. Let $\underline{x} := \underline{a} + \underline{e}_i \preceq \underline{b}$. Since $\underline{a}$ is a flat, $\mathbf{r}(\underline{x}) = \mathbf{r}(\underline{a}) + 1$. Let $\underline{c}$ be the minimum flat with $\underline{c} \succeq \underline{x}$. Obviously, $\underline{a} \prec \underline{c}$. Also $\underline{c} \preceq \underline{b}$, as $\underline{b}$ is a flat and $\underline{c}$ is minimum. By the previous lemma, $\mathbf{r}(\underline{c}) = \mathbf{r}(\underline{x}) = \mathbf{r}(\underline{a}) + 1$. We thus have strict inequality $\underline{c} \prec \underline{b}$, and the result follows. □

3.1. Axiomatic system of flats.

- (F1) $\underline{\varrho}$ is in $\mathscr{F}$.
- (F2) $\mathscr{F}$ is closed under meet.
- (F3) If $\underline{a}$ is an element of $\mathscr{F}$ and $i \in \{1, \dots, m\}$ is such that $\underline{a} + \underline{e}_i \in \cube_{\underline{\varrho}}$, then there exists an element $\underline{b} \succeq \underline{a} + \underline{e}_i$ in $\mathscr{F}$ such that $\underline{b} \gtrdot \underline{a}$ in $\mathscr{F}$.

We also introduce the following non-degeneracy property.

- (F*) Each layer L^i_t, $i = 1, \dots, m$, $t \in [\varrho_i]$, contains an element of $\mathscr{F}$.

We prove the following result.

Theorem 3.7. *The set of flats $\mathscr{F}$ of a matcuboid $\mathscr{M}$ with ground set $\boxplus_{\underline{\varrho}}$ verifies (F1)-(F2)-(F3). Conversely, let $\mathscr{F} \subseteq \boxplus_{\underline{\varrho}}$ be a subset verifying (F1)-(F2)-(F3). Then, $\mathscr{F}$ is the set of flats of a matcuboid $\mathscr{M}$ with underling ground set $\boxplus_{\underline{\varrho}}$.*

Moreover, the matcuboid $\mathscr{M}$ is simple if, and only if, Property (F) holds.*

Remark 3.8. The first three conditions (F1)-(F2)-(F3) are the matcuboid analogs of the three axioms that define the set of flats of a matroid. Property (F*) requires that the matcuboid does not contain any *loop*: if a layer does not verify this condition, deleting it, we get a smaller hypercuboid with the same collection Fl that verifies the same axioms. $\diamond$

Remark 3.9. The axioms (F2) and (F*) together imply that $\underline{0}$ is automatically in $\mathscr{F}$. Indeed, (F*) yields a flat in L_0^i for each $i = 1, \dots, m$. The meet of all those elements is $\underline{0}$, and by (F2) belongs to $\mathscr{F}$. $\diamond$

Remark 3.10. If $\mathscr{F}$ verifies the axiom (F2), then the element $\underline{b}$ in (F3) is automatically unique. Indeed, let $\underline{b}, \underline{c} \in \mathscr{F}$ such that $\underline{b}, \underline{c} \succeq \underline{a} + \underline{e}_i$ and $\underline{b}, \underline{c} \gtrdot \underline{a}$ in $\mathscr{F}$. Then, $\underline{b} \wedge \underline{c} \succeq \underline{a} + \underline{e}_i$ so $\underline{b} \wedge \underline{c} \neq \underline{a}$. Since $\underline{a} \preceq \underline{b} \wedge \underline{c} \preceq \underline{b}$ and $\underline{b} \gtrdot \underline{a}$, we must have $\underline{b} \wedge \underline{c} = \underline{b}$. By symmetry, $\underline{b} \wedge \underline{c} = \underline{c}$ and so $\underline{b} = \underline{c}$. $\diamond$

3.2. **Flats of a matcuboid verify the axioms.** Let $\mathscr{M}$ be a matcuboid $\boxplus_{\underline{\varrho}}$ and let $\mathscr{F}$ be the set of flats of $\mathscr{M}$. We prove that $\mathscr{F} = \mathscr{F}(\mathscr{M})$ verifies properties (F1)-(F2)-(F3). Moreover, if $\mathscr{M}$ is simple, then (F*) holds.

Proof of the first part of Theorem 3.7. Property (F1) follows from the definition of flats of a matcuboid. We already proved property (F2) in Proposition 3.2. It remains to show (F3). If $\mathbf{r}\left(\underline{\varrho}\right) = \mathbf{r}(\underline{a}) + 1$, then $\underline{b} = \underline{\varrho}$ satisfies (F3). Otherwise, we have $\mathbf{r}\left(\underline{\varrho}\right) \geqslant \mathbf{r}(\underline{a}) + 2$, and by Proposition 3.6, we have an element $\underline{b}$ in $\mathscr{F}$ of rank $\mathbf{r}(\underline{a}) + 1$ with $\underline{b} \succeq \underline{a} + \underline{e}_i$. Again, $\underline{b} \gtrdot \underline{a}$, as required.

Now suppose that $\mathscr{M}$ is simple. Property (F*) is a consequence of Lemma 3.5 above applied to $t\underline{e}_i$. Let $\underline{c}$ be the minimum flat with $\underline{c} \succeq t\underline{e}_i$. If Property (F*) does not hold for the layer L_t^i, then necessary $c_i > t$ and so $\mathbf{r}((t+1)\underline{e}_i) = \mathbf{r}(t\underline{e}_i)$ contradicting the simpleness of $\mathscr{M}$. $\square$

In the rest of the section, we prove the second part of the theorem.

3.3. **Diamond property.** We recall the following definition.

Definition 3.11 (Diamond property for lattices)**.** Let $(L, \preceq)$ be a lattice with the meet and join operations $\wedge$ and $\vee$, respectively. We say that L has the *diamond property* if for every triple of elements $a, b, c \in L$ such that $b \neq c$ and b and c both cover a, the join $b \vee c$ covers both b and c. $\diamond$

Lemma 3.12. *Let L be a lattice that satisfies the diamond property. Then, it admits a grading, i.e., all its maximal chains have the same length.*

Proof. This is well-known. We give a rather informal proof. We apply the diamond property multiple times to show that two maximal chains C and C' in L have the same length. This is done by induction on the elements of C and C'; the "diamonds" drawn in the Hasse diagram by application of the diamond property provide a finite sequence of chains of constant length between C and C', ultimately proving that C and C' have the same length. $\square$

3.4. **The axiomatic system of flats implies the diamond property.**

Lemma 3.13. *Let $\mathscr{F} \subseteq \boxplus_{\underline{\varrho}}$ be a subset verifying axioms (F1) and (F2). Then, $\mathscr{F}$ is a lattice. If additionally $\mathscr{F}$ verifies axioms (F3), then it verifies the diamond property.*

Proof. Since $\mathscr{F}$ is closed under meet and it has a maximum element, it is a lattice. The join operation $\vee$ is defined by

$$\underline{a} \vee \underline{b} := \bigwedge_{\substack{\underline{c} \in \mathscr{F} \\ \underline{c} \succeq \underline{a}, \underline{c} \succeq \underline{b}}} \underline{c}.$$

Now let $\underline{a}, \underline{b}, \underline{c} \in \mathscr{F}$ such that $\underline{b} \neq \underline{c}$, and $\underline{b}$ and $\underline{c}$ both cover $\underline{a}$. By assumption, $\underline{b}$ and $\underline{c}$ are not comparable, so there exist $j \neq k \in \{1, \dots, m\}$ such that $b_j > c_j$ and $c_k > b_k$. In fact, $c_j = a_j$, because otherwise, we would have $\underline{a} \prec \underline{b} \wedge \underline{c} \prec \underline{b}$, which is impossible because $\underline{b} \succ\!\!\succ \underline{a}$.

Now, applying (F3) yields $\underline{x} \in \mathscr{F}$ such that $\underline{x} \succeq \underline{c} + \underline{e}_j$ and $\underline{x} \succ\!\!\succ \underline{c}$. We show that $\underline{x} \succ \underline{b}$. Indeed, first, if $\underline{x} \not\succeq \underline{b}$, then we would have $\underline{a} \prec \underline{x} \wedge \underline{b} \prec \underline{b}$, the strictness of the first inequality coming from the inequalities $x_j \geqslant a_j + 1$ (because $\underline{x} \succeq \underline{c} + \underline{e}_j$ and $c_j = a_j$) and $b_j > a_j$. This would be in contradiction with $\underline{b} \succ\!\!\succ \underline{a}$. Second, $\underline{x} = \underline{b}$ is not possible because $\underline{x} \succeq \underline{c} + \underline{e}_j \succeq \underline{c}$ and $\underline{b}$ and $\underline{c}$ are not comparable. Therefore, $\underline{x} \succ \underline{b}$.

Symmetrically, (F3) provides an element $\underline{y} \in \mathscr{F}$ such that $\underline{y} \succeq \underline{b} + \underline{e}_k$, $\underline{y} \succ\!\!\succ \underline{b}$ and $\underline{y} \succ \underline{c}$.

We now define $\underline{u} := \underline{x} \wedge \underline{y}$ and notice that $\underline{u}$ verifies the chains of inequalities $\underline{b} \preceq \underline{u} \preceq \underline{y}$ and $\underline{c} \preceq \underline{u} \preceq \underline{x}$. In other words, $\underline{u}$ belongs to the interval $\big[\underline{b}, \underline{y}\big]$, which is equal to $\big\{\underline{b}, \underline{y}\big\}$ since $\underline{y} \succ\!\!\succ \underline{b}$, and, likewise, $\underline{u} \in [\underline{c}, \underline{x}] = \{\underline{c}, \underline{x}\}$. Since $\underline{b}$ and $\underline{c}$ are not comparable, $\underline{y} \succ\!\!\succ \underline{b}$ and $\underline{x} \succ\!\!\succ \underline{c}$, the only possibility for the sets $\big\{\underline{b}, \underline{y}\big\}$ and $\{\underline{c}, \underline{x}\}$ to have the element $\underline{u}$ in common is that $\underline{u} = \underline{x} = \underline{y}$. This shows that $\underline{u}$ covers $\underline{b}$ and $\underline{c}$. Then $\underline{u}$ is necessarily equal to $\underline{b} \vee \underline{c}$, therefore $\mathscr{F}$ verifies the diamond property. □

Applying Lemma 3.12, we infer the following.

Proposition 3.14. *Let $\mathscr{F} \subseteq \square_{\underline{\varrho}}$ be a subset verifying axioms (F1)-(F2)-(F3). Then, $\mathscr{F}$ is a graded lattice.*

We denote by $\mathbf{r}\colon \mathscr{F} \to \mathbb{N} \cup \{0\}$ the corresponding grading. The following properties hold.

(a) The function $\mathbf{r}$ is increasing on $\mathscr{F}$, in the following sense. If $\underline{a} \prec \underline{b} \in \mathscr{F}$, then $\mathbf{r}(\underline{a}) < \mathbf{r}(\underline{b})$.
(b) If $\underline{a}, \underline{b} \in \mathscr{F}$ are such that $\underline{b} \succ\!\!\succ \underline{a}$, then $\mathbf{r}(\underline{b}) = \mathbf{r}(\underline{a}) + 1$.

3.5. **Proof of the second part of Theorem 3.7.** Let $\mathscr{F} \subseteq \square_{\underline{\varrho}}$ be a subset verifying the axioms (F1)-(F2)-(F3). We define a map

$$\varphi\colon \square_{\underline{\varrho}} \to \mathscr{F} \subseteq \square_{\underline{\varrho}}$$

as follows. For every point $\underline{x} \in \square_{\underline{\varrho}}$, we define $\varphi(\underline{x})$ to be the minimum flat $\underline{b} \in \mathscr{F}$ such that $\underline{b} \succeq \underline{x}$.

Lemma 3.15. *Notations as above, the map φ is well-defined, and φ and $\mathscr{F}$ have the following properties.*

(i) The map $\varphi\colon \square_{\underline{\varrho}} \to \mathscr{F} \subseteq \square_{\underline{\varrho}}$ is non-decreasing.
(ii) Let $\underline{x} \in \square_{\underline{\varrho}}$ and $i \in \{1, \dots, m\}$ such that $\underline{x} + \underline{e}_i \in \square_{\underline{\varrho}}$. Then, either $\varphi(\underline{x} + \underline{e}_i) = \varphi(\underline{x})$, or $\varphi(\underline{x} + \underline{e}_i) \succ\!\!\succ \varphi(\underline{x})$.

Proof. The first part is immediate by definition. We prove the second one. If we assume that $\varphi(\underline{x} + \underline{e}_i) \neq \varphi(\underline{x})$, then we must have $\varphi(\underline{x} + \underline{e}_i) \succeq \varphi(\underline{x}) + \underline{e}_i$. Indeed, otherwise, $\varphi(\underline{x} + \underline{e}_i) \succ \varphi(\underline{x}) \succeq \underline{x} + \underline{e}_i$, contradicting the minimality of $\varphi(\underline{x} + \underline{e}_i)$.

Now, (F3) yields the existence of $\underline{b} \succeq \varphi(\underline{x}) + \underline{e}_i$ in $\mathscr{F}$ such that $\underline{b} \succ\!\!\succ \varphi(\underline{x})$. Then, using Property (i) above and the definition of φ, we get $\underline{b} \succeq \varphi(\varphi(\underline{x}) + \underline{e}_i) \succeq \varphi(\underline{x} + \underline{e}_i)$. The chain of inequalities $\underline{b} \succeq \varphi(\underline{x} + \underline{e}_i) \succeq \varphi(\underline{x})$ and the fact that $\underline{b} \succ\!\!\succ \underline{x}$ imply that $\underline{b} = \varphi(\underline{x} + \underline{e}_i)$ and therefore that $\varphi(\underline{x} + \underline{e}_i) \succ\!\!\succ \varphi(\underline{x})$, as required. □

We can now complete the proof of Theorem 3.7.

Proof of the second part of Theorem 3.7. Let $\mathscr{F}$ be a subset of $\boxcube_{\underline{\varrho}}$ verifying axioms (F1)-(F2)-(F3). As we have shown already in Proposition 3.14, $\mathscr{F}$ is a graded lattice.

We first extend the function $\mathbf{r}\colon \mathscr{F} \to \mathbb{N} \cup \{0\}$ to a function $\mathbf{r}\colon \boxcube_{\underline{\varrho}} \to \mathbb{N} \cup \{0\}$ by setting, for each $\underline{x} \in \boxcube_{\underline{\varrho}}$, $\mathbf{r}(\underline{x}) := \mathbf{r}(\varphi(\underline{x}))$. We claim that $\mathbf{r}$ is the rank function of a matcuboid.

We will use the properties proven in Lemma 3.15.

Part (ii) of the lemma implies directly that $\mathbf{r}(\underline{x}+\underline{e}_i) \leqslant \mathbf{r}(\underline{x})+1$ for each $\underline{x} \in \boxcube_{\underline{\varrho}}$, proving (R1).

The fact that $\mathbf{r}$ is non-decreasing on $\boxcube_{\underline{\varrho}}$ is a consequence of Property (i) in the lemma and Fact (a) stipulating that $\mathbf{r}$ is increasing on elements of $\mathscr{F}$.

We show that $\mathbf{r}$ is submodular on $\boxcube_{\underline{\varrho}}$. By Theorem 6.2 that we will prove in Section 6, it is sufficient to prove the diamond property for functions on hypercuboids (see Section 6). Let $\underline{x} \in \boxcube_{\underline{\varrho}}$ and $i \neq j \in \{1, \dots, m\}$ be such that $\underline{x} + \underline{e}_i + \underline{e}_j \in \boxcube_{\underline{\varrho}}$. Using Property (ii), we may assume that $\mathbf{r}(\underline{x}+\underline{e}_i+\underline{e}_j) = \mathbf{r}(\underline{x}+\underline{e}_j)+1$, and then need to prove that $\mathbf{r}(\underline{x}+\underline{e}_i) = \mathbf{r}(\underline{x})+1$. The equality $\mathbf{r}(\underline{x}+\underline{e}_i+\underline{e}_j) = \mathbf{r}(\underline{x}+\underline{e}_j)+1$ means that $\varphi(\underline{x}+\underline{e}_i+\underline{e}_j) \succ \varphi(\underline{x}+\underline{e}_j)$. This implies that $\varphi(\underline{x}+\underline{e}_j) \in L^i_{x_i}$ (as otherwise, we would get $\varphi(\underline{x}+\underline{e}_j)_i \geqslant x_i+1$, that is, $\varphi(\underline{x}+\underline{e}_j) \geqslant \underline{x}+\underline{e}_i+\underline{e}_j$, and so we would have $\varphi(\underline{x}+\underline{e}_i+\underline{e}_j) = \varphi(\underline{x}+\underline{e}_j)$). Since $\varphi(\underline{x}+\underline{e}_j) \geq \varphi(\underline{x})$, this in turn implies that $\varphi(\underline{x}) \in L^i_{x_i}$. However, $\varphi(\underline{x}+\underline{e}_i)_i \geqslant x_i+1$. We infer that $\varphi(\underline{x}+\underline{e}_i) \succ \varphi(\underline{x})$, and so $\mathbf{r}(\underline{x}+\underline{e}_i) = \mathbf{r}(\underline{x})+1$, as desired. We have shown that $\mathbf{r}$ is a the rank of a matcuboid with ground set $\boxcube_{\underline{\varrho}}$. The fact that the set of flats of $\mathbf{r}$ is exactly $\mathscr{F}$ is immediate by the definition of $\mathbf{r}$ and map φ, and the fact that $\mathbf{r}$ is increasing on $\mathscr{F}$, Fact (a).

It remains to show that if (F*) holds, then $\mathscr{M}$ is simple. Let $i \in \{1, \dots, m\}$. We show by induction that for every $0 \leqslant t \leqslant \varrho_i$, we have $\mathbf{r}(t\underline{e}_i) = t$. The base case $t=0$ holds by definition. We now suppose that $\mathbf{r}(t\underline{e}_i) = t$ with $0 \leqslant t < \varrho_i$ and show that $\mathbf{r}((t+1)\underline{e}_i) = t+1$. (F*) implies that $\varphi(t\underline{e}_i) \in L^i_t$ and $\varphi((t+1)\underline{e}_i) \in L^i_{t+1}$. In particular, $\varphi((t+1)\underline{e}_i) \neq \varphi(t\underline{e}_i)$. Again, Lemma 3.15 implies that $\varphi((t+1)\underline{e}_i) \succ \varphi(t\underline{e}_i)$ which, using that $\mathbf{r}$ is increasing on $\mathscr{F}$ implies $\mathbf{r}((t+1)\underline{e}_i) = \mathbf{r}(\varphi((t+1)\underline{e}_i)) = \mathbf{r}(\varphi(t\underline{e}_i))+1 = \mathbf{r}(t\underline{e}_i)+1 = t+1$, as desired. □

4. Circuits

In this section, we define circuits in matcuboids, using the duality of matcuboids. We provide an intrinsic axiomatic system for them which extends the axioms of circuits in matroid theory.

4.1. Circuits and duality.

Let $\mathscr{M}$ be a matcuboid on the ground set $\boxcube_{\underline{\varrho}}$, and denote by $\mathscr{M}^*$ its dual. Denote by $\mathscr{F}(\mathscr{M}^*)$ the set of flats of the dual matcuboid, and consider

$$\check{\mathscr{C}} := \left\{\underline{a}^c \;\middle|\; \underline{a} \in \mathscr{F}(\mathscr{M}^*)\right\} \subseteq \boxcube_{\underline{\varrho}},$$

where, we recall, $\underline{a}^c = \underline{\varrho} - \underline{a}$. Since $\mathscr{F}(\mathscr{M})$ is closed under meet, $\check{\mathscr{C}}$ will be closed under the join operation.

Given a subset $\mathcal{A} \subset \boxcube_{\underline{\varrho}}$, we say that an element $\underline{a}$ of $\mathcal{A}$ is *join-irreducible in* $\mathcal{A}$ if it is not the join of any set of elements of $\mathcal{A} \smallsetminus \{\underline{a}\}$.

Definition 4.1 (Circuits)**.** The collection of circuits of $\mathscr{M}$, denoted by $\mathscr{C}$, is defined as the set of nonzero join-irreducible elements of $\check{\mathscr{C}}$. ⋄

Here is a (simple) matcuboid $\mathscr{M}$ with $\underline{\varrho} = (5,4)$. On the left, $\mathscr{M}$ is represented by its rank function $\mathbf{r}$, with its circuits in red and the join-reducible elements of $\check{\mathscr{C}}$ in blue. On the right, the dual $\mathscr{M}^*$ of $\mathscr{M}$, which is not simple, is represented by its rank function $\mathbf{r}^*$, with its flats in teal.

$$(4)\qquad \begin{pmatrix} 4 & 4 & 4 & 4 & 5 & 6 \\ 3 & 3 & 4 & 4 & 5 & 6 \\ 2 & 2 & 3 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} \qquad \begin{pmatrix} 3 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix}.$$

Obviously, by definition, $\mathscr{C}$ determines $\check{\mathscr{C}}$, and therefore, gives the set of flats of the dual matroid $\mathscr{M}^*$. By Theorem 3.7, this implies that $\mathscr{C}$ determines $\mathscr{M}$.

4.2. **Axiomatic system of circuits.** For a subset $\mathscr{C}$ of $\cube_{\underline{\varrho}}$, denote by $\check{\mathscr{C}}$ the join-closure of $\mathscr{C}$, obtained by taking the join of any set of elements of $\mathscr{C}$. Consider the following set of properties:

(C1) $\underline{0}$ is *not* in $\mathscr{F}$.
(C2) All elements of $\mathscr{C}$ are join-irreducible in $\mathscr{C}$.
(C3) If $\underline{a} \in \check{\mathscr{C}}$ and $i \in \{1, \dots, m\}$ is such that $\underline{a} - \underline{e}_i \in \cube_{\underline{\varrho}}$, then there exists an element $\underline{b} \leq \underline{a} - \underline{e}_i$ in $\check{\mathscr{C}} \cup \{\underline{0}\}$ such that $\underline{b} \lessdot \underline{a}$ in $\check{\mathscr{C}} \cup \{\underline{0}\}$.

We also introduce the following simpleness property.

(C*) For every $i \in \{1, \dots, m\}$ and $t \in [\varrho_i]$, $t\underline{e}_i$ is *not* in $\mathscr{C}$.

We prove the following result.

Theorem 4.2. *The set of circuits $\mathscr{C}$ of a matcuboid $\mathscr{M}$ with ground set $\cube_{\underline{\varrho}}$ verifies (C1)-(C2)-(C3). Conversely, let $\mathscr{C} \subseteq \cube_{\underline{\varrho}}$ be a subset verifying (C1)-(C2)-(C3). Then, $\mathscr{C}$ is the set of circuits of a matcuboid $\mathscr{M}$ with underlying ground set $\cube_{\underline{\varrho}}$.*

Moreover, the matcuboid $\mathscr{M}$ is simple if, and only if, Property (C) holds.*

Proof. ($\Longrightarrow$) Let $\mathscr{C}$ be the set of circuits of a matcuboid $\mathscr{M}$ with ground set $\cube_{\underline{\varrho}}$. Properties (C1) and (C2) hold by definition of $\mathscr{C}$ (see Definition 4.1). As for Property (C3), it is a translation through duality of Property (F3) which holds for the set of flats of the dual matcuboid $\mathscr{M}^*$.

Assume moreover that $\mathscr{M}$ is simple. Let $i \in \{1, \dots, m\}$ and $t \in [\varrho_i]$. We have $\mathbf{r}(t\underline{e}_i) = t$ and therefore, denoting by $\mathbf{r}^*$ the rank function on $\mathscr{M}^*$, we have $\mathbf{r}^*\left(\underline{\varrho} - t\underline{e}_i\right) = \mathbf{r}^*\left(\underline{\varrho}\right)$. This implies that for every i and t, $\underline{\varrho} - t\underline{e}_i$ is not a flat in $\mathscr{M}^*$, which means that $t\underline{e}_i$ is not in $\check{\mathscr{C}}$. As a consequence, $t\underline{e}_i \notin \mathscr{C}$.

($\Longleftarrow$) Let $\mathscr{C} \subseteq \cube_{\underline{\varrho}}$ be a subset verifying (C1)-(C2)-(C3), and $\check{\mathscr{C}}$ the join-closure of $\mathscr{C}$. Define

$$\mathscr{F} := \left\{\underline{a}^c \mid \underline{a} \in \check{\mathscr{C}}\right\} \cup \left\{\underline{\varrho}\right\}.$$

We claim that $\mathscr{F}$ is the set of flats of a matcuboid. We need to show that it satisfies (F1)-(F2)-(F3). By construction, (F1) and (F2) hold. Property (F3) is a translation through duality of Property (C3) which holds for $\mathscr{C}$. Therefore, $\mathscr{F}$ is the set of flats of a matcuboid. We denote the dual of this matcuboid by $\mathscr{M}$, so that $\mathscr{F} = \mathscr{F}(\mathscr{M}^*)$. It is immediate by construction that $\mathscr{C}$ is its set of circuits of $\mathscr{M}$.

Let $\mathbf{r}$ be the rank function of $\mathscr{M}$ and $\mathbf{r}^*$ that of $\mathscr{M}^*$.

To prove the last assertion, assume that for every $i \in \{1, \dots, m\}$ and $t \in [\varrho_i]$, $t\underline{e}_i$ is not in $\mathscr{C}$. Then, for every i and t, $\underline{\varrho} - t\underline{e}_i$ is not a flat of $\mathscr{M}^*$. By a simple induction, this implies that for every i and t, $\mathbf{r}^*\left(\underline{\varrho} - t\underline{e}_i\right) = \mathbf{r}^*\left(\underline{\varrho}\right)$, and consequently, $\mathbf{r}(t\underline{e}_i) = t$. We infer that $\mathscr{M}$ is simple. $\square$

5. Independents

In this section, we define the independents of a matcuboid and study their properties. As in the case of flats, we give the axiomatic system of independents of a matcuboid.

5.1. Definition and basic properties.

Definition 5.1 (Independents of a matcuboid)**.** Let $\mathscr{M}$ be a matcuboid on the ground set $\boxcube_{\underline{\varrho}}$. We say that an element $\underline{p}$ of $\boxcube_{\underline{\varrho}}$ is an *independent of* $\mathscr{M}$ if for each $i = 1, \dots, m$ such that $\underline{p} - \underline{e}_i \in \boxcube_{\underline{\varrho}}$, we have $\mathbf{r}\left(\underline{p} - \underline{e}_i\right) = \mathbf{r}\left(\underline{p}\right) - 1$. We denote by $\mathscr{I}(\mathscr{M}) \subseteq \boxcube_{\underline{\varrho}}$ the set of independents of $\mathscr{M}$. ⋄

Here are two matcuboids with $\underline{\varrho} = (4, 3)$ for the first, and $\underline{\varrho} = (5, 4)$ for the second. The independents in each case are depicted in blue.

$$\begin{pmatrix} 3 & 3 & 3 & 4 & 5 \\ 2 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 4 & 4 & 5 & 5 & 5 & 6 \\ 3 & 4 & 5 & 5 & 5 & 6 \\ 2 & 3 & 4 & 4 & 4 & 5 \\ 1 & 2 & 3 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

The following proposition provides a list of properties of independent sets in a matcuboid.

Proposition 5.2. *Let $\mathscr{I}(\mathscr{M})$ be the set of independents of a matcuboid $\mathscr{M}$. The following properties hold.*

- *$\mathscr{I}(\mathscr{M})$ is non-empty and closed under meet.*
- *For every independent $\underline{p} \in \mathscr{I}(\mathscr{M})$ and every distinct elements $i_1, \dots, i_k \in \{1, \dots, m\}$ with $p_{i_j} \neq 0$, $j = 1, \dots, k$, we have $\mathbf{r}\left(\underline{p} - \underline{e}_{i_1} - \dots - \underline{e}_{i_k}\right) = \mathbf{r}(\underline{a}) - k$.*

Proof. Both statements follow from Lemma 3.3, as in the proof of Proposition 3.2. □

The set of independents in general does not contain a maximum, and lacks the existence of a join. In order to study more refined properties of independents, we will associate a notion of *size* to each independent element in $\mathscr{M}$ by defining a deletion operation on elements of $\mathcal{J}(\mathscr{M})$.

5.2. **Deletion and size.** Let $\mathcal{J}$ be a subset of $\boxcube_{\underline{\varrho}}$. Let $\underline{a}$ be an element of $\mathcal{J}$ and $i \in \{1, \dots, m\}$ such that $a_i \neq 0$. If there is at least one element $\underline{b} \prec \underline{a}$ in $\mathcal{J}$ that differs from $\underline{a}$ only in the i-th component, we define $\underline{a} \smallsetminus i$ to be such an element in $\mathcal{J}$ with the largest i-th coordinate. In this case, we say that $\underline{a} \smallsetminus i$ is the *deletion of i in $\underline{a}$ in $\mathcal{J}$.*

Definition 5.3. Let $\mathcal{J}$ be a subset of $\boxcube_{\underline{\varrho}}$.

(a) We say that *deletions exist in* $\mathcal{J}$ if for every $i \in \{1, \dots, m\}$, if $a_i \geqslant 1$, the deletion $\underline{a} \smallsetminus i$ exists in $\mathcal{J}$.

If deletions exist in $\mathcal{J}$, then necessarily, we have $\underline{0} \in \mathcal{J}$. Moreover, for every element $\underline{a} \in \mathcal{J}$, there exists a sequence of deletions in $\mathcal{J}$, that reduce $\underline{a}$ to $\underline{0}$.

(b) We say that $\mathcal{J}$ is *orderable* if deletions exist in $\mathcal{J}$ and for every $\underline{a} \in \mathcal{J}$, all the sequences of deletions in $\mathcal{J}$ that bring $\underline{a}$ to $\underline{0}$ have the same length.

If $\mathcal{J}$ is orderable, we define the *size* of each element $\underline{a} \in \mathcal{J}$ denoted by $|\underline{a}|$ as the number of deletions that reduce $\underline{a}$ to $\underline{0}$. ⋄

In Lemma 5.7 below, we formulate a simple orderability criterion.

5.3. **Axiomatic system of independents.** We first define the following that turn out to be useful in the proof of the main theorem of this section.

Definition 5.4. Let $\underline{a}, \underline{b} \in \boxplus_{\underline{\varrho}}$ be two elements. We define:

$$\Delta(\underline{a}, \underline{b}) := \left\{k = 1, \dots, m \;\middle|\; a_k < b_k\right\} \qquad \text{and} \qquad E(\underline{a}, \underline{b}) := \sum_{k \in D(\underline{a}, \underline{b})} (b_k - a_k). \qquad \diamond$$

Now, consider, for $\mathscr{I}$ a subset of $\boxplus_{\underline{\varrho}}$, the following property:

(I1) Deletions exist in $\mathscr{I}$ and the following holds. For all $\underline{p} \in \mathscr{I}$ and deletions $\underline{p} \setminus i$ and $\underline{p} \setminus j$, with $i, j \in \{1, \dots, m\}$, the meet $\underline{q} := \left(\underline{p} \setminus i\right) \wedge \left(\underline{p} \setminus j\right)$ belongs to $\mathscr{I}$ and, moreover, the two intervals $\left[\underline{q}, \underline{p} \setminus i\right]$ and $\left[\underline{q}, \underline{p} \setminus j\right]$ in $\mathscr{I}$ have the same size.

It follows from Lemma 5.7 proved in Section 5.5 that a subset $\mathscr{I} \subseteq \boxplus_{\underline{\varrho}}$ that verifies (I1) is orderable. We can thus define the size $|\underline{a}|$ of each element $\underline{a} \in \mathscr{I}$. This enables to formulate the second property of interest:

(I2) $|\cdot|$ is increasing on independents, i.e., for all $\underline{a}, \underline{b} \in \mathscr{I}$ such that $\underline{a} \prec \underline{b}$, we have $|\underline{a}| < |\underline{b}|$. Moreover, let $\underline{a}, \underline{b}$ be two elements of $\mathscr{I}$ such that $|\underline{a}| < |\underline{b}|$ and $D(\underline{a}, \underline{b})$ contains at least two elements. Then, there exists $\underline{c} \in \mathscr{I}$ that verifies:
- $\underline{c} \preceq \underline{a} \vee \underline{b}$,
- $|\underline{c}| > |\underline{a}|$.
- There exists $i \in D(\underline{a}, \underline{b})$ such that $c_i < b_i$.

We also introduce the following notion of simpleness.

(I*) For $i = 1, \dots, m$, the points $t\underline{e}_i$ for $t \in [\varrho_i]$ are all in $\mathscr{I}$.

This is the main result of this section.

Theorem 5.5. *The set of independents $\mathscr{I}(\mathscr{M})$ of a matcuboid $\mathscr{M}$ with ground set $\boxplus_{\underline{\varrho}}$ verifies (I1)-(I2). Conversely, let $\mathscr{I} \subseteq \boxplus_{\underline{\varrho}}$ be a subset that verifies (I1)-(I2). Then, $\mathscr{I}$ is the set of independents of a matcuboid $\mathscr{M}$ with underling ground set $\boxplus_{\underline{\varrho}}$.*

Moreover, the matcuboid $\mathscr{M}$ is simple if, and only if, $\mathscr{I}$ verifies (I).*

Remark 5.6. Axiom (I1) is an analog of the hereditary property for independents of matroids. It also implies that $\underline{0} \in \mathscr{I}$, analog of the first axiom of independents in matroids. Axiom (I2) plays the role of the augmentation property for independents. These axioms take into account the more singular nature of independents in the context of matcuboids: for example, in matroids, all maximal (for inclusion) independents have the same cardinality, whereas in matcuboids, maximal independents (for the partial order $\preceq$) can have different sizes as the examples above show (see Section 9.1 for further discussion). Axiom (I*) requires that there is no "loop". $\diamond$

5.4. **Independents of a matcuboid verify the axioms.** Let $\mathscr{M}$ be a matcuboid $\boxplus_{\underline{\varrho}}$. We prove that $\mathscr{I}(\mathscr{M})$ verifies properties (I1)-(I2). Moreover, if $\mathscr{M}$ is simple, then (I*) holds.

The proof shows that the size function on $\mathscr{I}(\mathscr{M})$ coincides with the rank function.

Proof of the first part of Theorem 5.5. We start by proving (I1). Let $\underline{p} \in \mathscr{I}(\mathscr{M})$ and let $i \in \{1, \dots, m\}$ be such that $p_i \neq 0$. We claim that $\underline{a} := \underline{p} - p_i\underline{e}_i \in \mathscr{I}(\mathscr{M})$. By design, $a_i = 0$. Now, let $j \in \{1, \dots, m\}$ be different from i such that $a_j \neq 0$. We have $p_j = a_j \neq 0$. We have $\mathbf{r}\left(\underline{p} - \underline{e}_j\right) = \mathbf{r}\left(\underline{p}\right) - 1$. Applying Lemma 3.3 with $\underline{x} = \underline{a} - \underline{e}_j$ and $\underline{y} = \underline{p} - \underline{e}_j$, we get $\mathbf{r}(\underline{a} - \underline{e}_j) = \mathbf{r}(\underline{a}) - 1$. This shows that $\underline{a} \in \mathscr{I}(\mathscr{M})$. Therefore, deletions exist in $\mathscr{I}(\mathscr{M})$.

We next show that $\mathbf{r}\left(\underline{p} \setminus i\right) = \mathbf{r}\left(\underline{p}\right) - 1$. For the sake of a contradiction, suppose this not being the case, that is, $\mathbf{r}\left(\underline{p} \setminus i\right) \leqslant \mathbf{r}\left(\underline{p}\right) - 2$. Then, there would exist $\underline{p} \setminus i \prec \underline{b} \prec \underline{p}$ such that $\mathbf{r}(\underline{b}) = \mathbf{r}(\underline{p}) - 1$ and $\mathbf{r}(\underline{b} - \underline{e}_i) = \mathbf{r}(\underline{b}) - 1$. Note that $b_j = p_j$ for all $j \neq i$. Applying again

Lemma 3.3 as above, we infer that $\underline{b}$ belongs to $\mathscr{I}(\mathscr{M})$. This would be a contradiction to the definition of the deletion.

This implies that every sequence of deletions bringing $\underline{p} \in \mathscr{I}(\mathscr{M})$ to $\underline{0}$ has size precisely $\mathbf{r}(\underline{p})$.

Now, for distinct $i, j \in \{1, \dots, m\}$, we consider the deletions $\underline{p} \smallsetminus i$ and $\underline{p} \smallsetminus j$ in $\mathscr{I}(\mathscr{M})$, as well as $\underline{q} := \left(\underline{p} \smallsetminus i\right) \wedge \left(\underline{p} \smallsetminus j\right)$. By Proposition 5.2, $\underline{q} \in \mathscr{I}(\mathscr{M})$. Note that we have $\mathbf{r}\left(\underline{p} \smallsetminus j\right) = \mathbf{r}\left(\underline{p}\right) - 1 = \mathbf{r}\left(\underline{p} \smallsetminus i\right)$.

Note that $\underline{q}$ differs from $\underline{p} \smallsetminus i$ only in the j-th component, and therefore can be obtained from it by a sequence of deletions of j. It follows that $\left[\underline{q}, \underline{p} \smallsetminus i\right]$ has cardinality $\mathbf{r}\left(\underline{p} \smallsetminus i\right) - \mathbf{r}\left(\underline{q}\right) + 1$. Similarly, $\left[\underline{q}, \underline{p} \smallsetminus j\right]$ has cardinaltiy $\mathbf{r}\left(\underline{p} \smallsetminus j\right) - \mathbf{r}\left(\underline{q}\right) + 1$.

We conclude that $\left[\underline{q}, \underline{p} \smallsetminus i\right]$ and $\left[\underline{q}, \underline{p} \smallsetminus j\right]$ in $\mathscr{I}(\mathscr{M})$ have the same cardinality, and (I1) follows. We thus get a well-defined size function $|\cdot|$ on $\mathscr{I}(\mathscr{M})$. As the proof shows, we have $|\underline{a}| = \mathbf{r}(\underline{a})$ for every $\underline{a} \in \mathscr{I}$.

The first half of Property (I2) results from the fact that if $\underline{a} \prec \underline{b}$ are two independents, then $\Delta(\underline{a}, \underline{b}) \neq \varnothing$. Then, taking $k \in \Delta(\underline{a}, \underline{b})$, we get $\mathbf{r}(\underline{a}) \leqslant \mathbf{r}(\underline{b} - \underline{e}_k) = \mathbf{r}(\underline{b}) - 1$.

For the second half of Property (I2), let $\underline{a}, \underline{b}$ be two independents such that $|\underline{a}| < |\underline{b}|$ and $|\Delta(\underline{a}, \underline{b})| \geqslant 2$. We consider two cases depending on whether $|\underline{a}| \leqslant |\underline{b}| - 2$ or $|\underline{a}| = |\underline{b}| - 1$.

First consider the case $|\underline{a}| \leqslant |\underline{b}| - 2$. Let $i \in \Delta(\underline{a}, \underline{b})$. Since $b_i > \underline{a}_i \geqslant 0$, we can define $\underline{c} := \underline{b} \smallsetminus i$. Note that $|\underline{c}| = |\underline{b}| - 1 > |\underline{a}|$. Furthermore, by construction, $\underline{c} \preceq \underline{a} \vee \underline{b}$ and $c_i < b_i$. This shows that $\underline{c}$ is suitable.

We now consider the case $|\underline{a}| = |\underline{b}| - 1$. Let $\underline{d} := \underline{a} \vee \underline{b}$. Let $\underline{y} \preceq \underline{d}$ be an element of $\boxed{\varrho}$ minimal for $\preceq$ under the constraints that $\mathbf{r}\left(\underline{y}\right) = \mathbf{r}(\underline{d})$ and for all $k \notin \Delta(\underline{a}, \underline{b})$, $y_k = d_k$ (note that for all those k, we have $d_k = a_k$). For all $k \in \Delta(\underline{a}, \underline{b})$ with $\underline{y} - \underline{e}_k \in \boxed{\varrho}$, we thus have $\mathbf{r}\left(\underline{y} - \underline{e}_k\right) = \mathbf{r}\left(\underline{y}\right) - 1$.

Next, let $\underline{x} \preceq \underline{y}$ be an element of $\boxed{\varrho}$ minimal for $\preceq$ under the constraint that $\mathbf{r}(\underline{x}) = \mathbf{r}\left(\underline{y}\right) = \mathbf{r}(\underline{d})$, and for all $k \in \Delta(\underline{a}, \underline{b})$, $x_k = y_k$. Since for all $k \in \Delta(\underline{a}, \underline{b})$ with $x_k = y_k > 0$, we have $\mathbf{r}\left(\underline{y} - \underline{e}_k\right) = \mathbf{r}\left(\underline{y}\right) - 1$, Lemma 3.3 implies that $\mathbf{r}(\underline{x} - \underline{e}_k) = \mathbf{r}(\underline{x}) - 1$. Moreover, by the choice of $\underline{x}$, we also have for all $i \notin \Delta(\underline{a}, \underline{b})$, $\mathbf{r}(\underline{x} - \underline{e}_i) = \mathbf{r}(\underline{x}) - 1$ provided that $\underline{x} - \underline{e}_i$ belongs to $\boxed{\varrho}$. Therefore, combining all this, we conclude that $\underline{x} \in \mathscr{I}(\mathscr{M})$.

If $\underline{y} \prec \underline{d}$, let $\underline{c} := \underline{x} \in \mathscr{I}(\mathscr{M})$. There exists then $k \in \Delta(\underline{a}, \underline{b})$ such that $y_k < d_k = b_k$. As a consequence, $c_k < b_k$. Moreover, $\mathbf{r}(\underline{c}) = \mathbf{r}(\underline{d}) \geqslant \mathbf{r}(\underline{b}) = \mathbf{r}(\underline{a}) + 1$ and therefore $|\underline{c}| > |\underline{a}|$. Since by construction $\underline{c} \prec \underline{d}$, $\underline{c}$ is suitable.

It remains to consider the case $\underline{y} = \underline{d}$. This means that for every $k \in \Delta(\underline{a}, \underline{b})$, $\mathbf{r}(\underline{d} - \underline{e}_k) < \mathbf{r}(\underline{d})$. We claim that in this case, the strict inequality $\mathbf{r}(\underline{d}) > \mathbf{r}(\underline{b})$ holds. Indeed, for the sake of a contradiction, suppose $\mathbf{r}(\underline{d}) = \mathbf{r}(\underline{b})$. Since $|\Delta(\underline{a}, \underline{b})| \geqslant 2$, there are two distinct elements $i, j \in \Delta(\underline{a}, \underline{b})$, and for these i, j, we would have $\mathbf{r}(\underline{d} - \underline{e}_i) = \mathbf{r}(\underline{d} - \underline{e}_j) = \mathbf{r}(\underline{d}) - 1$. By submodularity (see Lemma 3.3), we would get $\mathbf{r}(\underline{d} - \underline{e}_i - \underline{e}_j) = \mathbf{r}(\underline{d}) - 2$. Since $i, j \in \Delta(\underline{a}, \underline{b})$, we have $\underline{a} \preceq \underline{d} - \underline{e}_i - \underline{e}_j$, and therefore, we would have $\mathbf{r}(\underline{a}) \leqslant \mathbf{r}(\underline{d}) - 2 = \mathbf{r}(\underline{b}) - 2$, contradicting the assumption that $|\underline{a}| = |\underline{b}| - 1$. This proves the claim that $\mathbf{r}(\underline{d}) > \mathbf{r}(\underline{b})$.

Let now $\underline{c} := \underline{x} \smallsetminus k$ for an element $k \in \Delta(\underline{a}, \underline{b})$. We have $|\underline{c}| = \mathbf{r}(\underline{c}) = \mathbf{r}(\underline{x}) - 1 = \mathbf{r}(\underline{d}) - 1 \geqslant \mathbf{r}(\underline{b}) = |\underline{b}| > |\underline{a}|$, and therefore $|\underline{c}| > |\underline{a}|$. Besides, $c_k < x_k = d_k = b_k$, and obviously $\underline{c} \preceq \underline{d}$. This shows that $\underline{c}$ is suitable in this last case. This ends the proof of (I2). We have proved that $\mathscr{I}(\mathscr{M})$ verifies (I1) and (I2).

To finish the proof, note that if $\mathscr{M}$ is simple, then (R1*) immediately implies (I*). $\square$

5.5. **Orderability lemma.** Before going to the proof of the second part of Theorem 5.5, we show the following criterion for orderability.

Lemma 5.7. *Let $\mathscr{J}$ be a subset of $\boxed{\varrho}$. The following are equivalent:*

(1) $\mathcal{J}$ *satisfies (I1).*
(2) $\mathcal{J}$ *is orderable in the sense of Definition 5.3.*

Proof. We first prove (1) $\Rightarrow$ (2). Assume (I1) holds. Proceeding by induction under the partial order $\preceq$, we show that for every $\underline{a} \in \mathcal{J}$, the following property holds:

$$P(\underline{a}): \quad \text{All the sequences of deletions that bring } \underline{a} \text{ to } \underline{0} \text{ have the same length.}$$

Obviously, $P(\underline{0})$ holds. Let $\underline{a} \in \mathcal{J}$ be an element such that for every $\underline{b} \in \mathcal{J}$ with $\underline{b} \prec \underline{a}$, $P(\underline{b})$ holds. We prove that $P(\underline{a})$ is true. Let

$$\underline{a} = \underline{b}_0 \succ \underline{b}_1 \succ \cdots \succ \underline{b}_k = \underline{0} \quad \text{and} \quad \underline{a} = \underline{c}_0 \succ \underline{c}_1 \succ \cdots \succ \underline{c}_\ell = \underline{0}$$

be two sequences of deletions bringing $\underline{a}$ to $\underline{0}$. We need to prove that $k = \ell$. Let $i, j \in \{1, \dots, m\}$ be such that $\underline{b}_1 = \underline{a} \smallsetminus i$ and $\underline{c}_1 = \underline{a} \smallsetminus j$, and let $\underline{q} := \underline{b}_1 \wedge \underline{c}_1 \in \mathcal{J}$. Let

$$\underline{q} = \underline{x}_0 \succ \underline{x}_1 \succ \cdots \succ \underline{x}_m = \underline{0}$$

be every sequence of deletions bringing $\underline{q}$ to $\underline{0}$.

By (I1), the linear intervals $[\underline{q}, \underline{a}_1]$ and $[\underline{q}, \underline{b}_1]$ have the same size, that we denote by $s+1$, with $s \geqslant 0$. Let

$$[\underline{q}, \underline{b}_1] = \left\{\underline{b}_1 = \underline{y}_0 \succ \underline{y}_1 \succ \cdots \succ \underline{y}_s = \underline{q}\right\} \text{ and } [\underline{q}, \underline{c}_1] = \left\{\underline{c}_1 = \underline{z}_0 \succ \underline{z}_1 \succ \cdots \succ \underline{z}_s = \underline{q}\right\}.$$

Then, $\underline{b}_1 = \underline{y}_0 \succ \underline{y}_1 \succ \cdots \succ \underline{y}_s = \underline{q} = \underline{x}_0 \succ \underline{x}_1 \succ \cdots \succ \underline{x}_m = \underline{0}$ is a sequence of deletions that brings $\underline{b}_1$ to $\underline{0}$. Property $P(\underline{b}_1)$ therefore implies that the length of this sequence is equal to the length of the sequence $\underline{b}_1 \succ \cdots \succ \underline{b}_k = \underline{0}$, that is, $k - 1 = s + m$. The same argument applied to $\underline{c}_1$ yields $\ell - 1 = s + m$. We conclude that $k = \ell$.

The implication (2) $\Rightarrow$ (1) follows from the identities

$$|\underline{q}| + \left|[\underline{q}, \underline{p} \smallsetminus i]\right| = |\underline{p}| = |\underline{q}| + \left|[\underline{q}, \underline{p} \smallsetminus j]\right|,$$

using a sequence of deletions of j (resp. i) that brings $p \smallsetminus i$ (resp. $p \smallsetminus j$) to q. $\square$

5.6. **Proof of the second part of Theorem 5.5.** We need the following lemma.

Lemma 5.8. *Let* $\mathcal{J}$ *be a subset of* $\boxplus_{\underline{\varrho}}$ *that verifies (I2). Then, for two elements* $\underline{a} \prec \underline{b}$ *of* $\mathcal{J}$ *such that* $D(\underline{a}, \underline{b})$ *has at least two elements, we have* $|\underline{a}| \leqslant |\underline{b}| - 2$.

Proof. The first part of Property (I2) ensures that $|\underline{a}| < |\underline{b}|$. The second part of Property (I2) now implies that there exists an element $\underline{c} \in \mathcal{J}$ and $i \in D(a, b)$ such that $\underline{c} \preceq \underline{a} \vee \underline{b} = \underline{b}$, $|\underline{c}| > |\underline{a}|$, and $c_i < b_i$. Combining the latter with $\underline{c} \preceq \underline{b}$ yields that $\underline{c} \prec \underline{b}$. Applying (I2) again, we get $|\underline{c}| < |\underline{b}|$. All in all, we get $\underline{a} \prec \underline{c} \prec \underline{b}$, and the inequality $|\underline{a}| \leqslant |\underline{b}| - 2$ follows. $\square$

We now prove the second part of the main theorem.

Proof of the second part of Theorem 5.5. Notations as in the statement of the theorem, by Lemma 5.7, we have a well-defined size function $|\cdot|$ on $\mathscr{I}$. We define a function $\mathbf{r}$ on $\boxplus_{\underline{\varrho}}$ by setting

$$\mathbf{r}(\underline{x}) := \max_{\underline{a} \in \mathscr{I} \text{ with } \underline{a} \preceq \underline{x}} |\underline{a}| \qquad \forall\, \underline{x} \in \boxplus_{\underline{\varrho}},$$

and show that $\mathbf{r}$ is the rank function of a matcuboid. Note that by (I2), $\mathbf{r}(\underline{a}) = |\underline{a}|$ for $\underline{a} \in \mathscr{I}$.

Obviously, $\mathbf{r}(\underline{0}) = 0$. Moreover, for $1 \leqslant i \leqslant m$ and $1 \leqslant t \leqslant \varrho_i$, we have either $\mathbf{r}(t\underline{e}_i) - \mathbf{r}((t-1)\underline{e}_i) = 0$ or $\mathbf{r}(t\underline{e}_i) - \mathbf{r}((t-1)\underline{e}_i) = 1$, by orderability of $\mathscr{I}$. Therefore (R1) holds.

Since $|\cdot|$ is increasing by (I2), $\mathbf{r}$ is non-decreasing on $\boxplus_{\underline{\varrho}}$. That is, Property (R2) holds.

We show $\mathbf{r}$ is submodular. Using Theorem 6.2 proved in Section 6, it will be enough to show that $\mathbf{r}$ verifies the diamond property for functions on hypercuboid.

We first observe that, by orderability of $\mathscr{I}$, for every $i \in \{1, \dots, m\}$, we have $\mathbf{r}(\underline{x} + \underline{e}_i) - \mathbf{r}(\underline{x}) \leqslant 1$, provided that $\underline{x} + \underline{e}_i \in \boxplus_{\underline{\varrho}}$.

Now let $\underline{x} \in \boxdot_{\underline{\varrho}}$ and let $i \neq j$ be elements of $\in \{1, \ldots, m\}$ such that $\underline{x} + \underline{e}_i + \underline{e}_j \in \boxdot_{\underline{\varrho}}$. Let $\underline{y} := \underline{x} + \underline{e}_i$, $\underline{z} := \underline{x} + \underline{e}_j$ and $\underline{w} := \underline{x} + \underline{e}_i + \underline{e}_j$. Proving the diamond property for $\underline{x}, \underline{y}, \underline{z}, \underline{w}$ reduces to showing that the situation where $\mathbf{r}(\underline{x}) = \mathbf{r}(\underline{y}) = \mathbf{r}(\underline{z})$ and $\mathbf{r}(\underline{w}) = \mathbf{r}(\underline{x}) + 1$ never happens. For the sake of a contradiction, assume we are in the situation where the above equalities hold. This implies in particular that $\underline{y}, \underline{z} \notin \mathscr{I}$. The rest of the argument is a case-by-case analysis. We treat first the case $\underline{w} \in \mathscr{I}$, then, $\underline{w} \notin \mathscr{I}$ but $\underline{x} \in \mathscr{I}$, and then generalize the argument to treat the remaining case $\underline{w}, \underline{x} \notin \mathscr{I}$.

First consider the case where $\underline{w} \in \mathscr{I}$. Let $\underline{a} \preceq \underline{x}$ be an element of $\mathscr{I}$ such that $\mathbf{r}(\underline{a}) = \mathbf{r}(\underline{x})$. Since $\underline{a} \prec \underline{w}$ and $|D(\underline{a}, \underline{w})| \geqslant |D(\underline{x}, \underline{w})| = 2$, applying Lemma 5.8, we get the inequality $\mathbf{r}(\underline{x}) \leqslant \mathbf{r}(\underline{w}) - 2$, which is a contradiction. This implies $\underline{w} \notin \mathscr{I}$.

At this point, we have deduced $\underline{y}, \underline{z}, \underline{w} \notin \mathscr{I}$. Now consider the case where $\underline{x} \in \mathscr{I}$. Let $\underline{b} \prec \underline{w}$ be an element of $\mathscr{I}$ such that $\mathbf{r}(\underline{w}) = \mathbf{r}(\underline{b})$. Notice that $|\underline{b}| = \mathbf{r}(\underline{w}) > \mathbf{r}(\underline{x}) = |\underline{x}|$. Moreover, $b_i = w_i = x_i + 1$ because otherwise we would have $\underline{b} \preceq \underline{y}$ and $\mathbf{r}(\underline{b}) > \mathbf{r}(\underline{y})$, which would be impossible since $\mathbf{r}$ is non-decreasing. Likewise, we have $b_j = w_j = x_j + 1$. Since $\underline{b} \prec \underline{w} = \underline{x} + \underline{e}_i + \underline{e}_j$, this shows that $D(\underline{x}, \underline{b}) = \{i, j\}$. By (I2), there exists an independent $\underline{c} \in \mathscr{I}$ such that $\underline{c} \preceq \underline{x} \vee \underline{b} = \underline{w}$, $|\underline{c}| > |\underline{x}|$ and $c_k < b_k$ for some $k \in \{i, j\}$. But if $k = i$, then $\underline{c} \preceq \underline{y}$ and therefore $|\underline{c}| \leqslant \mathbf{r}(\underline{y}) = \mathbf{r}(\underline{x}) = |\underline{x}|$, a contradiction; we conclude similarly if $k = j$. We have shown that $\underline{x} \notin \mathscr{I}$.

We now treat the remaining case. We define a finite procedure by applying repeatedly an analogue of the preceding construction, as follows. Let $\underline{a} \prec \underline{x}$, $\underline{b} \prec \underline{w}$ be elements of $\mathscr{I}$ such that $\mathbf{r}(\underline{x}) = |\underline{a}|$ and $\mathbf{r}(\underline{w}) = |\underline{b}|$. We have $|\underline{b}| > |\underline{a}|$. We claim $b_i = w_i > x_i \geqslant a_i$. Indeed, otherwise, we would have $\underline{b} \preceq \underline{y}$, impossible by the inequality $|\underline{b}| = \mathbf{r}(\underline{w}) > \mathbf{r}(\underline{y})$. Likewise, we have $b_j = w_j > a_j$. Consequently, we have $D(\underline{a}, \underline{b}) \supseteq \{i, j\}$. By (I2), there exists an independent $\underline{c}^1 \in \mathscr{I}$ such that

$$\underline{c}^1 \preceq \underline{a} \vee \underline{b} \preceq \underline{w}, \qquad \left|\underline{c}^1\right| > |\underline{a}|, \qquad \text{and} \qquad c^1_{k_1} < b_{k_1} \quad \text{for some } k_1 \in D(\underline{a}, \underline{b}).$$

Next, if $D(\underline{a}, \underline{c}^1)$ contains itself at least two elements, since we have $|\underline{c}^1| > |\underline{a}|$, we can apply (I2), and the same procedure as above, replacing the pair $\underline{a}, \underline{b}$ by the pair $\underline{a}, \underline{c}^1$, yields an element $\underline{c}^2 \in \mathscr{I}$ such that

$$\underline{c}^2 \preceq \underline{a} \vee \underline{c}^1 \preceq \underline{w}, \qquad \left|\underline{c}^2\right| > |\underline{a}|, \qquad \text{and} \qquad c^2_{k_2} < c^1_{k_2} \quad \text{for some } k_2 \in D(\underline{a}, \underline{c}^1).$$

Repeating the procedure while it is possible, we get a sequence $\underline{c}^1, \underline{c}^2, \ldots, \underline{c}^j, \ldots$ of elements of $\mathscr{I}$, satisfying for every $j \geqslant 1$,

$$\underline{c}^j \preceq \underline{a} \vee \underline{c}^{j-1} \preceq \underline{w}, \qquad \left|\underline{c}^j\right| > |\underline{a}|, \qquad \text{and} \qquad c^j_{k_j} < c^{j-1}_{k_j} \quad \text{for some } k_j \in D(\underline{a}, \underline{c}^{j-1}),$$

with $\underline{c}^0 = \underline{b}$. We claim that this sequence is necessarily finite. Indeed, we observe that for every $j \geqslant 1$, we have by construction:

$$D(\underline{a}, \underline{c}^{j-1}) \supseteq D(\underline{a}, \underline{c}^j) \quad \text{and} \quad E(\underline{e}, \underline{c}^{j-1}) > E(\underline{e}, \underline{c}^j).$$

Since the integers $E(\underline{a}, \underline{c}^j)$ are all non-negative, we infer that the sequence $\underline{c}^\bullet$ ends at some integer $j > 0$. This means that the condition $|D(\underline{a}, \underline{c}^j)| \geqslant 2$ fails, and so it is impossible to have both i and j included in $D(\underline{a}, \underline{c}^j)$.

Without loss of generality, assume $i \notin D(\underline{a}, \underline{c}^j)$. This implies that $\underline{c}^j_i \leqslant a_i$, and so, we have $\underline{a}^j \preceq \underline{y}$. We infer that $|\underline{a}| < |\underline{c}^j| \leqslant \mathbf{r}(\underline{y}) = \mathbf{r}(\underline{x}) = |\underline{a}|$, which is a contradiction.

At this point, we have shown the diamond property, and therefore we condlude that $\mathbf{r}$ is submodular, and (R3) follows.

It follows that $\mathbf{r}$ is the rank function of a matcuboid $\mathscr{M}$. Moreover, by definition of the rank function, $\mathscr{I}$ coincides with the set of independents of $\mathscr{M}$.

Finally, by definition of $\mathbf{r}$, Property (R1*) is seen to be equivalent to (I*), and so $\mathscr{M}$ is simple if, and only if, (I*) holds. $\square$

6. Diamond property for functions

The aim of this section is to generalize to the setting of matcuboids the well-known result in matroid theory that the submodularity of the rank function of a matroid is equivalent to the *diamond property* for its graded lattice of flats [Sta11, Proposition 3.3.2].

To this end, we here introduce a weaker version of submodularity.

Definition 6.1 (Diamond property for functions on hypercuboids)**.** We say an integer-valued function $\mathbf{r}$ on $\boxplus_{\underline{\varrho}}$ satisfies the *diamond property* if the following holds. For every point $\underline{x} \in \boxplus_{\underline{\varrho}}$ and distinct integers $i \neq j \in \{1, \dots, m\}$ such that $\underline{x} + \underline{e}_i, \underline{x} + \underline{e}_j \in \boxplus_{\underline{\varrho}}$, we have

$$\mathbf{r}(\underline{x} + \underline{e}_i) - \mathbf{r}(\underline{x}) \geqslant \mathbf{r}(\underline{x} + \underline{e}_i + \underline{e}_j) - \mathbf{r}(\underline{x} + \underline{e}_j). \tag{5}$$

The following theorem shows that the above property is equivalent to submodularity.

Theorem 6.2 (Equivalence of submodularity and the diamond property)**.** *Let* $\mathbf{r}$ *be an integer-valued function on* $\boxplus_{\underline{\varrho}}$*. The following properties are equivalent:*

(i) $\mathbf{r}$ *is submodular.*
(ii) $\mathbf{r}$ *verifies the diamond property.*

In preparation for the proof, we provide generalizations of the diamond property, that allow to proceed by induction. We say an integer-valued function $\mathbf{r}$ on $\boxplus_{\underline{\varrho}}$ satisfies the *unidirectional submodularity at distance one* if the following holds. For all $i \in \{1, \dots, m\}$ and for all points $\underline{x} \preceq \underline{y} \in \boxplus_{\underline{\varrho}}$ such that $x_i = y_i$ and $\underline{x} + \underline{e}_i \in \boxplus_{\underline{\varrho}}$, the inequality

$$\mathbf{r}(\underline{x} + \underline{e}_i) - \mathbf{r}(\underline{x}) \geqslant \mathbf{r}\left(\underline{y} + \underline{e}_i\right) - \mathbf{r}\left(\underline{y}\right) \tag{6}$$

holds. More generally, we have the following generalization of Property (6) in *several directions* and at *higher distance*.

Definition 6.3 (Mutidirectional submodularity at a given distance)**.** For positive integers k and n, we define the *k-directional submodularity at distance up to n*, denoted $(*)^n_k$, as follows.
$(*)^n_k$: Pick any integer $1 \leqslant s \leqslant k$, any integers $1 \leqslant i_1 < \dots < i_s \leqslant m$ and $0 \leqslant n_{i_1}, \dots, n_{i_s} \leqslant n$. Then, for every pair of elements $\underline{x} \preceq \underline{y} \in \boxplus_{\underline{\varrho}}$ such that $x_{i_j} = y_{i_j}$ for all $1 \leqslant j \leqslant s$, and $\underline{x} + \sum_{1 \leqslant j \leqslant s} n_j \underline{e}_{i_j} \in \boxplus_{\underline{\varrho}}$, we have

$$\mathbf{r}\Big(\underline{x} + \sum_{j=1}^{s} n_{i_j} \underline{e}_{i_j}\Big) - \mathbf{r}(\underline{x}) \geqslant \mathbf{r}\Big(\underline{y} + \sum_{j=1}^{s} n_{i_j} \underline{e}_{i_j}\Big) - \mathbf{r}\left(\underline{y}\right). \qquad \diamond$$

Notice that the property stated in (6) is exactly $(*)^1_1$ as defined above, and the terminologies are consistent. Moreover, any $(*)^n_k$ with $k, n \geqslant 1$ implies $(*)^1_1$.

Remark 6.4 (Alternative description of $(*)^n_k$)**.** Using the notations of Definition 6.3, after the change of variables $\underline{a} := \underline{x}$, $\quad \underline{b} := \underline{y} - \sum_{j=1}^{s} n_{i_j} \underline{e}_{i_j}$, property $(*)^n_k$ can be rewritten as follows.

For all elements $\underline{a}$ and $\underline{b} \in \boxplus_{\underline{\varrho}}$, we have the submodularity inequality

$$\mathbf{r}(\underline{a}) + \mathbf{r}(\underline{b}) \geqslant \mathbf{r}(\underline{a} \vee \underline{b}) + \mathbf{r}(\underline{a} \wedge \underline{b})$$

as long as there exist an integer $1 \leqslant s \leqslant k$ and integers $1 \leqslant i_1 < \dots < i_s \leqslant m$ such that $\underline{b} + \sum_{1 \leqslant j \leqslant s} (a_{i_j} - b_{i_j}) \underline{e}_{i_j}$ is an element of $\boxplus_{\underline{\varrho}}$ greater than or equal to $\underline{a}$ and, that, for all $1 \leqslant j \leqslant s$, we have $0 \leqslant a_{i_j} - b_{i_j} \leqslant n$.

This parametrization using $\underline{a}$ and $\underline{b}$ enables to see instantaneously that the submodularity property of $\mathbf{r}$ in the hypercuboid implies all the properties $(*)^n_k$. The other parametrization, using $\underline{x}$ and $\underline{y}$, will be useful to prove Theorem 6.2 below, in that it behaves linearly (contrary to formulas involving the symbols $\wedge$ and $\vee$). $\diamond$

Proof of Theorem 6.2. Obviously, (i) implies (ii).

We explain how to deduce (6), that is $(*)_1^1$, from (ii). Let $\underline{x}$ and $\underline{y}$ be as in Definition 6.1. The fact that $x_i = y_i$ implies that $\underline{y}$ can be written as $\underline{y} = \underline{x} + \sum_{j \neq i} n_j \underline{e}_j$ with $n_j \geqslant 0$, and we can sum inequalities of the form (5) to get the inequality (6).

We explain how to deduce (i) from $(*)_1^1$. Proceeding by induction, we show that the property $(*)_1^1$ implies $(*)_k^n$ for all $k, n \geqslant 1$. We first show that $(*)_1^1$ implies $(*)_1^n$ for all $n \geqslant 1$. Let $i \in \{1, \dots, m\}$ and $0 \leqslant n_i \leqslant n$, and let $\underline{x} \preceq \underline{y}$ be elements of $\square_{\underline{\varrho}}$ such that $\underline{x} + n_i\, \underline{e}_i \in \square_{\underline{\varrho}}$ and $x_i = y_i$. For all $0 \leqslant t < n_i$, the pair $\left(\underline{x} + t\underline{e}_i, \underline{y} + t\underline{e}_i\right)$ satisfies the hypotheses needed to apply $(*)_1^1$ in direction i, so we know that

$$\mathbf{r}(\underline{x} + (t+1)\underline{e}_i) - \mathbf{r}(\underline{x} + t\underline{e}_i) \geqslant \mathbf{r}\left(\underline{y} + (t+1)\underline{e}_i\right) - \mathbf{r}\left(\underline{y} + t\underline{e}_i\right).$$

Summing all these inequalities for $0 \leqslant t < n_i$, and cancelling out the terms which appear on both sides, yields

$$\mathbf{r}(\underline{x} + n_i\underline{e}_i) - \mathbf{r}(\underline{x}) \geqslant \mathbf{r}\left(\underline{y} + n_i\underline{e}_i\right) - \mathbf{r}\left(\underline{y}\right),$$

which gives $(*)_1^n$.

We now show that properties $(*)_1^n$ for $n \geqslant 1$ imply properties $(*)_2^n$. Let $i, j \in \{1, \dots, m\}$ and $0 \leqslant n_i, n_j \leqslant n$, and let $\underline{x} \preceq \underline{y}$ be elements of $\square_{\underline{\varrho}}$ such that $\underline{x} + n_i\, \underline{e}_i + n_j\, \underline{e}_j \in \square_{\underline{\varrho}}$, $x_i = y_i$ and $x_j = y_j$. We apply $(*)_1^n$ to the pair $\left(\underline{x}, \underline{y}\right)$ in direction i and get

$$\mathbf{r}(\underline{x} + n_i\underline{e}_i) - \mathbf{r}(\underline{x}) \geqslant \mathbf{r}\left(\underline{y} + n_i\underline{e}_i\right) - \mathbf{r}\left(\underline{y}\right).$$

The pair $\left(\underline{x} + n_i\, \underline{e}_i, \underline{y} + n_i\, \underline{e}_i\right)$ satisfies the hypotheses required for applying $(*)_1^n$ again, but this time in direction j. This yields

$$\mathbf{r}(\underline{x} + n_i\underline{e}_i + n_j\underline{e}_j) - \mathbf{r}(\underline{x} + n_i\underline{e}_i) \geqslant \mathbf{r}\left(\underline{y} + n_i\underline{e}_i + n_j\underline{e}_j\right) - \mathbf{r}\left(\underline{y} + n_i\underline{e}_i\right).$$

Summing up these two inequalities shows that $\mathbf{r}$ satisfies $(*)_2^n$. The same procedure inductively proves that $\mathbf{r}$ satisfies all $(*)_k^n$, i.e., $\mathbf{r}$ is submodular. □

Remark 6.5 (Discrete partial derivatives and transverse local convexity)**.** For $i \in \{1, \dots, m\}$, we can define the *discrete partial derivative of* $\mathbf{r}$ *in the direction* i as the function $\partial_i\mathbf{r}$ defined by

$$\partial_i\mathbf{r}(\underline{x}) := \mathbf{r}(\underline{x} + \underline{e}_i) - \mathbf{r}(\underline{x}), \qquad \underline{x} \in \square_{\underline{\varrho}} \text{ such that } x_i < \varrho_i.$$

We notice that property $(*)_1^1$ is equivalent to the fact that for all $i \in \{1, \dots, m\}$ and for all $0 \leqslant t < \varrho_i$, $\partial_i\mathbf{r}|_{L_t^i}$ is non-increasing. This is why $(*)_1^1$ may be alternatively called *transverse local concavity*. Submodularity is thus equivalent to transverse local concavity.

In other contexts, submodularity is sometimes referred to as the discrete analogue of concavity: see, for example, [Sch03, Theorem 44.1]. While this is fully relevant for a function $\mathbf{r}$ defined on the collection $\mathcal{P}(S)$ of all subsets of a given set S, that is on the hypercube $\square_1^m$, it is not exactly true for supermodular functions on $\square_{\underline{\varrho}}$ for larger values of $\varrho_1, \dots, \varrho_m$. This is because the functions $\partial_i\mathbf{r}$ are non-decreasing only in directions different from i. For the function $\mathbf{r}$ defined on $\square_{(2,2)}$ by $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}$, we have $\mathbf{r}((2,1)) - \mathbf{r}((2,0)) \ngeqslant \mathbf{r}((2,2)) - \mathbf{r}((2,1))$, i.e., $\partial_2\mathbf{r}$ is not non-increasing in the direction $\underline{e}_2$. ⋄

7. Permutation arrays

The aim of this section is to study simple matcuboids with ground set an actual hypercube ($\varrho_1 = \varrho_2 = \dots = \varrho_d = r$) and with minimum possible rank r or $r+1$. Our Theorem 7.1 shows that they correspond precisely to permutation arrays introduced by Eriksson-Linusson [EL00a, EL00b].

7.1. **Permutation arrays.** First, we recall some terminology from [EL00a]. Our presentation differs slightly from the original setting as our indexing of flags is by codimension while in their work, Eriksson and Linusson use an indexing by dimension. (Concretely, this amounts to having lower blocks in [EL00a, EL00b] replaced here by upper blocks.)

Let $\varrho_1, \dots, \varrho_m$ be m non-negative integers. An *m-dimensional dot array* P is a m-dimensional array of type $[\varrho_1] \times \cdots \times [\varrho_m]$ where some of the entries are dotted.

For a dot array P, and $\underline{x} \in \cube_{\underline{\varrho}}$, we denote by $P[\underline{x}]$ the *upper principal subarray* of P that consists of all $\underline{y}$ with $\underline{y} \succeq \underline{x}$. It is naturally a dot array itself.

To be precise, for $P[\underline{x}]$ to become a dot array, we must coordinate-wise substract the point $(x_1, \dots, x_m)$ to all its elements. In the following, we will use both parametrization conventions freely for the sake of convenience.

For a dot array P and $j \in \{1, \dots, m\}$, the *rank along the j-axis*, denoted by $\mathrm{rank}_j(P)$, is the total number of $0 \leqslant t \leqslant \varrho_j$ such that there is at least one dot in some position whose j-th index is equal to t, i.e., there is at least one dot in the layer L_t^j of P. A dot array P is called *rankable* if we have $\mathrm{rank}_j(P) = \mathrm{rank}_i(P)$ for all $i, j \in \{1, \dots, m\}$. If P is rankable, then we call $\mathrm{rank}_j(P)$ the *rank* of P for any $j \in \{1, \dots, m\}$.

A dot array P is called *totally rankable* if every upper principal subarray of P is rankable.

We recall that in the terminology of [EL00a] and [EL00b], a position $\underline{x}$ is *redundant* if there exist dot positions $\underline{y}_1, \dots, \underline{y}_m \neq \underline{x}$ for some $m \geqslant 2$, such that each $\underline{y}_i$ has at least one coordinate in common with $\underline{x}$, and such that $\underline{x} = \bigwedge_{i=1}^m \underline{y}_i$. (In the language of lattices, non-dotted positions would rather be called *meet-irreducible*.) The set of redundant positions of P is denoted by $R(P)$. A *redundant dot* is a redundant position that is dotted. The reason for the term "redundant" is that placing or removing a redundant dot does not change the rank of any upper principal subarray of P.

If A is a subset of $\cube_{\underline{\varrho}}$, then $P \cup A$ (resp. $P \smallsetminus A$) denotes the dot array based on P where, for every $\underline{x} \in A$, we dot (resp. undot) the position $\underline{x}$ in P, if necessary.

A *permutation array* of width r and dimension m is a totally rankable dot array P of shape $\cube_r^m = \cube_{\underline{\varrho}} = [r]^m$, $\underline{\varrho} = (r, \dots, r)$, of rank $r+1$, and with no redundant dots.

7.2. **Equivalence of permutation arrays with simple matcuboids of rank r or $r+1$ on $\cube_r^m$.** Our next theorem establishes an equivalence between permutation arrays and simple matcuboids of rank r or $r+1$ on the hypercube $\cube_r^m$.

Theorem 7.1. *Let P be a permutation array of width r and dimension m. The function $\mathbf{r}_P$ defined by $\mathbf{r}_P(\underline{a}) := r + 1 - \mathrm{rank}(P[\underline{a}])$ for every $\underline{a} \in \cube_r^m$ is the rank function of a simple matcuboid $\mathscr{M}_P$ with ground set $\cube_r^m$. This matcuboid is of rank r or $r+1$ depending on whether the position $\underline{\varrho}$ in $\cube_r^m$ is dotted or not. The set of flats of $\mathbf{r}_P$ is precisely the union of the set of dot positions in P with $R(P)$, and $\underline{\varrho}$.*

Conversely, the rank function $\mathbf{r}$ of every simple matcuboid $\mathscr{M}$ of rank r or $r+1$ on the hypercube $\cube_r^m$ defines a dot array $P_{\mathscr{M}}$ on $\cube_r^m = [r]^m$ with dots positioned on the set of flats $\underline{a} \neq \underline{\varrho}$ of $\mathscr{M}$, and also a dot positioned on $\underline{\varrho}$ if $\mathbf{r}(\mathscr{M}) = r$. Then, $P := P_{\mathscr{M}} \smallsetminus R(P_{\mathscr{M}})$ is a permutation array.

The proof of this theorem is given in the next section.

7.3. **Proof of Theorem 7.1.** We start by proving the first part of the theorem. Let P be a permutation array on $\cube_r^m = [r]^m$. We claim that the function

$$\mathbf{r}_P(\underline{a}) := r + 1 - \mathrm{rank}(P[\underline{a}]), \qquad \forall \underline{a} \in \cube_r^m,$$

is the rank function of a simple matcuboid on the ground set $\cube_r^m$. We need to show properties (R1*)-(R2)-(R3).

Since $\underline{a} \preceq \underline{b}$ implies $P[\underline{a}] \supseteq P[\underline{b}]$, we deduce that $\mathbf{r}_P$ is non-decreasing, which shows (R2).

We now prove (R1). Let $i \in \{1, \dots, m\}$. We have to show that $\mathbf{r}_P(t\,\underline{e}_i) = t$ for $t \in [r]$. By definition, $\mathrm{rank}(P[t\,\underline{e}_i]) \leqslant r+1-t$, which implies $\mathbf{r}_P(t\underline{e}_i) \geqslant t$. The reverse inequality is shown by induction on t. The case $t = 0$ is true by the definition of permutation arrays which requires $\mathrm{rank}(P) = r+1$. Assume that $\mathrm{rank}(P[t\,\underline{e}_i]) \geqslant r+1-t$, we show that $\mathrm{rank}(P[(t+1)\,\underline{e}_i]) \geqslant r-t$. This follows from the inequality $\mathrm{rank}(P[\underline{a} + \underline{e}_i]) = \mathrm{rank}_i(P[\underline{a} + \underline{e}_i]) \geqslant \mathrm{rank}_i(P[\underline{a}]) - 1 = \mathrm{rank}(P[\underline{a}]) - 1$, valid for every $\underline{a} \in \cube^m_r$ such that $\underline{a} + \underline{e}_i \in \cube^m_r$.

It remains to show that $\mathbf{r}_P$ is submodular. Thanks to Theorem 6.2, it is sufficient to show that $\mathbf{r}_P$ satisfies the diamond property for functions. We thus take two distinct integers $i \neq j \in \{1, \dots, m\}$ and an element $\underline{x} \in \cube^m_r$ such that $\underline{x} + \underline{e}_i, \underline{x} + \underline{e}_j \in \cube^m_r$. We assume that $\mathbf{r}(\underline{x} + \underline{e}_i + \underline{e}_j) - \mathbf{r}(\underline{x} + \underline{e}_j) = 1$ and show that $\mathbf{r}(\underline{x} + \underline{e}_i) - \mathbf{r}(\underline{x}) = 1$. The hypothesis implies that the layer $L^i_{x_i}$ in the dot array $P[\underline{x} + \underline{e}_j]$ contains a dotted point. This point will be counted in the difference $\mathbf{r}(\underline{x} + \underline{e}_i) - \mathbf{r}(\underline{x})$, which proves the result.

This matcuboid $\mathscr{M}_P$ is of rank r or $r+1$ depending on whether $\mathrm{rank}(P[\underline{\varrho}]) = 1$ or 0, that is, whether $\underline{\varrho}$ is dotted or not.

Finally, to see the statement about the flats, consider $\underline{x} \neq \underline{\varrho}$ and assume first that $\underline{x}$ is dotted. Then, for each direction $\underline{e}_i$ with $\underline{x}+\underline{e}_i \in \cube^m_r$, we get $\mathrm{rank}_i(P[\underline{x}]) - \mathrm{rank}_i(P[\underline{x}+\underline{e}_i]) = 1$. This shows that $\underline{x}$ is a flat.

Next, assume that $\underline{x}$ is not dotted. Since flats are closed under meet, if $\underline{x}$ is a redundant position, then it is a flat. It remains to consider the case where $\underline{x}$ is neither dotted nor a redundant point. This means there is an $i \in \{1, \dots, m\}$ such that the layer $L^i_{x_i}$ in $P[\underline{x}]$ does not contain any dot. Two cases can happen:

- If $\underline{x} + \underline{e}_i \in \cube^m_r$, then
$$\mathrm{rank}(P[\underline{x}]) = \mathrm{rank}_i(P[\underline{x}]) = \mathrm{rank}_i(P[\underline{x} + \underline{e}_i]) = \mathrm{rank}(P[\underline{x} + \underline{e}_i]),$$
and so $\mathbf{r}(\underline{x}) = \mathbf{r}(\underline{x} + \underline{e}_i)$, and $\underline{x}$ is not a flat of $\mathscr{M}_P$.
- Otherwise, $x_i = r$, and so $\mathrm{rank}_i(P[\underline{x}]) = 0$, that is, $\mathbf{r}(\underline{x}) = r+1$. This implies that $\mathscr{M}$ is of rank $r+1$, and since $\underline{x} \neq \underline{\varrho}$, then, again $\underline{x}$ is not a flat.

This finishes the proof of the first direction.

We now show the other direction. Suppose that $\mathscr{M}$ is a simple matcuboid of rank $r+1$ or r on $\cube^m_r$ with rank function $\mathbf{r}$. Let $P_{\mathscr{M}}$ be the corresponding dot array where a dot is positioned on every flat $\underline{a}$ of $\mathscr{M}$ different from $\underline{\varrho}$, and if the rank of $\mathscr{M}$ is r, then a dot is also positioned on $\underline{\varrho}$. Let $P = P_{\mathscr{M}} \smallsetminus R(P_{\mathscr{M}})$. We show that P is a permutation array.

By construction, P has no redundant dots. We thus need to show that P is totally rankable and has rank $r+1$. We have to prove that for every $\underline{x} \in P_{\mathscr{M}}$ and $i, j \in \{1, \dots, m\}$, $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = \mathrm{rank}_j(P_{\mathscr{M}}[\underline{x}])$. This is a direct consequence of Proposition 7.2 below, which also shows that the rank of $P_{\mathscr{M}}$ is $r+1$. We conclude that P is a permutation array. $\square$

It remains to prove the following proposition.

Proposition 7.2. *Suppose that $\mathscr{M}$ is a simple matcuboid of rank r or $r+1$ on the ground set $\cube^m_r$ and denote by $\mathbf{r}$ its rank function. Let $P_{\mathscr{M}}$ be the corresponding dot array with a dot positioned at each flat $\underline{a} \neq \underline{\varrho}$, and also a dot positioned at $\underline{\varrho}$ in the case $\mathbf{r}(\mathscr{M}) = r$.*

Let $\underline{x}$ be an element of the dot array $P_{\mathscr{M}}$ and $1 \leqslant i \leqslant m$. Then, $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = r+1-\mathbf{r}(\underline{x})$.

Proof of Proposition 7.2. We proceed by reverse induction in the lattice $\cube^m_r$, starting from $\underline{\varrho}$. For $\underline{\varrho}$, we have $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{\varrho}]) = 0$ or 1 depending on whether $\mathbf{r}(\mathscr{M}) = r+1$ or r, respectively, for each $i \in \{1, \dots, m\}$, as required. Assume $\underline{x} \neq \underline{\varrho}$. We suppose the following equalities hold:
$$\mathrm{rank}_i(P_{\mathscr{M}}[\underline{y}]) = r+1-\mathbf{r}(\underline{y}) \quad \forall\ \underline{y} \succ \underline{x} \qquad \text{and} \qquad \forall\ i \in \{1, \dots, m\}.$$
We prove that the equalities hold as well for $\underline{x}$.

Suppose first that $\underline{x}$ is a flat. Two cases can occur.

(I.1) If $\underline{x}$ has rank r, then for each $i \in \{1,\dots,m\}$ with $\underline{x}+\underline{e}_i \in \square^m_r$, we have $\mathbf{r}(\underline{x}+\underline{e}_i) = r+1$. By induction, $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}+\underline{e}_i]) = r+1-\mathbf{r}(\underline{x}+\underline{e}_i) = 0$. It follows that $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = \mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}+\underline{e}_i])+1 = 1$, as required. For the other values of i, we have $\underline{x}+\underline{e}_i \notin \square^m_r$, that is, $x_i = r$, and, in this case, we have as well $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = 1$.

(I.2) If $\mathbf{r}(\underline{x}) < r$, then using the inequality $x_i = \mathbf{r}(x_i\underline{e}_i) \leqslant \mathbf{r}(\underline{x})$, we get $x_i < r$. This implies $\underline{x}+\underline{e}_i \in \square^m_r$ for all i. We get

$$\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = \mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}+e_i])+1 = r+1-\mathbf{r}(\underline{x}+e_i)+1 = r+1-\mathbf{r}(\underline{x}),$$

as required.

Suppose now that $\underline{x}$ is not a flat of $\mathscr{M}$. Again, two cases can occur.

(II.1) If $x_i < r$ for all $i \in \{1,\dots,m\}$, then let $\underline{a}$ be the minimum flat with $\underline{a} \succeq \underline{x}$ and let $i \in \{1,\dots,m\}$. Applying the hypothesis of the induction to $\underline{y} = \underline{x}+\underline{e}_i$, we get $\mathrm{rank}_i\left(P_{\mathscr{M}}\left[\underline{y}\right]\right) = r+1-\mathbf{r}\left(\underline{y}\right)$. By Lemma 3.5, we deduce that

$$\mathbf{r}\left(\underline{y}\right) = \begin{cases}\mathbf{r}(\underline{x})+1 & \text{if } a_i = x_i,\\ \mathbf{r}(\underline{x}) & \text{if } a_i > x_i.\end{cases}$$

In the first case, when $a_i = x_i$, there is a dot in the layer $L^i_{x_i}$ of $P_{\mathscr{M}}[\underline{x}]$, and so $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = \mathrm{rank}\left(P_{\mathscr{M}}\left[\underline{y}\right]\right)+1$. In the second case, when $a_i > x_i$, there is no dot in the layer $L^i_{x_i}$ of $P_{\mathscr{M}}[\underline{x}]$, and so $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = \mathrm{rank}\left(P_{\mathscr{M}}\left[\underline{y}\right]\right)$. We infer that $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = r+1-\mathbf{r}(\underline{x})$, as required.

(II.2) We now treat the remaining case where $x_i = r$ for some indices i among $1,\dots,m$. In this case, $\mathbf{r}(\underline{x})$ is either r or $r+1$. We treat each of these possibilities separately.

(1) Suppose first $\mathbf{r}(\underline{x}) = r$. Consider an index i with $x_i = r$. Then, if $\mathbf{r}(\mathscr{M}) = r$, there is a dot positioned at $\underline{\varrho}$, and so $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = 1 = r+1-\mathbf{r}(\underline{x})$. If $\mathbf{r}(\mathscr{M}) = r+1$, the minimum flat $\underline{a} \succeq \underline{x}$ has rank r, and lives in the layer L^i_r of $P_{\mathscr{M}}[\underline{x}]$. Again, we get $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = 1$.
Consider now an index j with $x_j < r$, so that $\underline{x}+\underline{e}_j \in \square^m_r$. Let $\underline{a}$ be the minimum flat dominating $\underline{x}$. A reasoning similar to (II.1), based on the use of Lemma 3.5, shows that $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = r+1-\mathbf{r}(\underline{x})$, as required.

(2) Suppose now that $\mathbf{r}(\underline{x}) = r+1$. The unique flat $\underline{a}$ that dominates $\underline{x}$ is $\underline{\varrho}$, which is not dotted. We infer that $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = 0 = r+1-\mathbf{r}(\underline{x})$, as required. □

8. Local matroids

In this section, we define local matroids of matcuboids, and formulate a local obstruction to their representability. We then turn this into an equivalent characterization of matcuboids.

8.1. **Local matroids of a matcuboid.** In the following, for all $\underline{a} \in \square_{\underline{\varrho}}$, we define $I_{\underline{a}}$ as the set of all $i \in \{1,\dots,m\}$ such that $\underline{a}+\underline{e}_i \in \square_{\underline{\varrho}}$.

To motivate the definition, first consider a representable matcuboid, given by m initial flags $\mathrm{F}^\bullet_1,\dots,\mathrm{F}^\bullet_m$ of length $\varrho_1,\dots,\varrho_m$, respectively, in a κ-vector space H. Let

$$\mathrm{F}_{\underline{a}} := \bigcap_{1\leqslant i\leqslant m} \mathrm{F}^{a_i}_i.$$

Let $J_{\underline{a}} \subseteq I_{\underline{a}}$ be the set of $i \in I_{\underline{a}}$ such that $\mathrm{F}_{\underline{a}+\underline{e}_i}$ is a proper vector subspace of $\mathrm{F}_{\underline{a}}$, and so, is a hyperplane in $\mathrm{F}_{\underline{a}}$. The arrangement of hyperplanes in $\mathrm{F}_{\underline{a}}$ given by $J_{\underline{a}}$ defines a matroid $\mathrm{M}_{\underline{a}}$ on the ground set $I_{\underline{a}}$. The rank one elements of this matroid are precisely given by $J_{\underline{a}}$ and all the elements in the complement $I_{\underline{a}} \setminus J_{\underline{a}}$ are loops. For the terminology regarding matroids, we refer to [Oxl06].

We now explain how to generalize the above picture to any matcuboid, and define local matroids associated to elements of the hypercuboid $\square_{\underline{\varrho}}$. Let $\mathbf{r}$ be a rank function on $\square_{\underline{\varrho}}$. Let $\underline{a} \in \square_{\underline{\varrho}}$, and define a function $\mathbf{r}_{\underline{a}} \colon 2^{I_{\underline{a}}} \to \mathbb{Z}_{\geqslant 0}$ as follows. For every subset $X \subseteq I_{\underline{a}}$, set

$$\mathbf{r}_{\underline{a}}(X) := \mathbf{r}\Big(\underline{a} + \sum_{i \in X} \underline{e}_i\Big) - \mathbf{r}(\underline{a}).$$

Proposition 8.1. *With notations as above, the pair $(I_{\underline{a}}, \mathbf{r}_{\underline{a}})$ defines a matroid $\mathrm{M}_{\underline{a}}$ on the set of elements $I_{\underline{a}}$.*

Proof. By Proposition 2.4, $\mathbf{r}_{\underline{a}}$ takes values in the set $\{0, \dots, |I_{\underline{a}}|\}$, and $\mathbf{r}_{\underline{a}}(X) \leqslant |X|$. Since $\mathbf{r}$ is non-decreasing, we also have $\mathbf{r}_{\underline{a}}(Y) \leqslant \mathbf{r}_{\underline{a}}(X)$ for $Y \subseteq X$. Therefore, it is enough to show that $\mathbf{r}_a$ is submodular, that is,

$$\forall\, X, Y \subseteq I_{\underline{a}} \quad \mathbf{r}_{\underline{a}}(X) + \mathbf{r}_{\underline{a}}(Y) \geqslant \mathbf{r}_{\underline{a}}(X \cup Y) + \mathbf{r}_{\underline{a}}(X \cap Y).$$

This follows directly from the submodularity of $\mathbf{r}$ applied to the elements $\underline{b} = \underline{a} + \sum_{i \in X} \underline{e}_i$ and $\underline{c} = \underline{a} + \sum_{i \in Y} \underline{e}_i$ of the hypercuboid $\square_{\underline{\varrho}}$. □

Proposition 8.2. *A necessary condition for the representability of a matcuboid $\mathscr{M}$ with ground set $\square_{\underline{\varrho}}$ on a field κ is the representability of all the matroids $M_{\underline{a}}$, $\underline{a} \in \square_{\underline{\varrho}}$, on κ.*

Proof. This follows directly from the above discussions. □

8.2. **The case of permutation arrays.** Let κ be a field. By Theorem 7.1, the representability of a simple matcuboid $\mathscr{M}$ of rank r or $r+1$ with the ground set the hypercube $\square_r^m$ is equivalent to the representability of the corresponding permutation array in the terminology of [EL00b]. Billey and Vakil [BV08] provide several examples of permutation arrays which are non-representable. Proposition 8.2 above provides a conceptual explanation of the examples treated in [BV08].

8.3. **A matroidal characterization of matcuboids.** The aim of this section is to show that the data of a matcuboid on the ground set $\square_{\underline{\varrho}}$ is equivalent to the data of a set of matroids indexed by $\square_{\underline{\varrho}}$ satisfying certain compatibility properties.

Notations as in the previous section, for the sake of convenience, if $i \in I_{\underline{a}}$, we sometimes write $\mathbf{r}_{\underline{a}}(i)$ or $\mathbf{r}_{\underline{a}}(\underline{e}_i)$ instead of $\mathbf{r}_{\underline{a}}(\{i\})$. We start with the definitions below.

Definition 8.3 (Increasing path)**.** Let $\underline{a}$ and $\underline{b}$ be points of $\square_{\underline{\varrho}}$ with $\underline{a} \preceq \underline{b}$. We define an *increasing path* from $\underline{a}$ to $\underline{b}$ to be any finite sequence

$$\underline{a} = \underline{c}_0, \underline{c}_1, \dots, \underline{c}_k = \underline{b}$$

such that for every $0 \leqslant j < k$, we have $\underline{c}_{j+1} = \underline{c}_j + \underline{e}_\ell$ for some $\ell \in \{1, \dots, m\}$. ◇

Note that the integer k is equal to $\sum_{i=1}^m (b_i - a_i)$.

Definition 8.4 (Coherent complex of matroids)**.** Let $(\mathrm{M}_{\underline{a}})_{\underline{a} \in \square_{\underline{\varrho}}}$ be a set of matroids indexed by $\square_{\underline{\varrho}}$, with $\mathrm{M}_{\underline{a}}$ a matroid on the set $I_{\underline{a}}$ and with rank function $\mathbf{r}_{\underline{a}}$. We say $(\mathrm{M}_{\underline{a}})$ form a *coherent complex of matroids* if the following two properties are satisfied:

(CC1) For all $i \in \{1, \dots, m\}$ and $0 \leqslant t < \varrho_i$, we have $\mathbf{r}_{t\underline{e}_i}(\underline{e}_i) \leqslant 1$.

(CC2) The matroids satisfy the following relation.

$$\mathrm{M}_{\underline{a}+\underline{e}_i} = \begin{cases} \mathrm{M}_{\underline{a}} \,/\, i & \text{if } a_i = \varrho_i - 1 \\ \mathrm{M}_{\underline{a}} \,/\, i \sqcup \{i\} & \text{else} \end{cases}.$$

Here, M / e denotes the contraction of a matroid M by its element e, and i is the element of the matroid set corresponding to the direction i. Moreover, $\mathrm{M}_{\underline{a}} / i \sqcup \{i\}$ denotes an extension of $\mathrm{M}_{\underline{a}} / i$ by a single element denoted i. ◇

In the following, we denote by $\mathbf{r}_{\underline{a}/i}$ the rank function on $I_{\underline{a}} \smallsetminus \{i\}$ that defines the matroid $\mathrm{M}_{\underline{a}} \,/\, i$. We recall that this is given by the following equation in terms of the rank function $\mathbf{r}_{\underline{a}}$:

$$\mathbf{r}_{\underline{a}/i}(X) = \mathbf{r}_{\underline{a}}(X \cup \{i\}) - \mathbf{r}_{\underline{a}}(i) \qquad \text{for all } X \subseteq I_{\underline{a}} \smallsetminus \{i\}.$$

Remark 8.5. Property (CC2) above implies the following: let $\underline{x} \preceq \underline{y}$ be two points of $\square_{\underline{\varrho}}$ and $i \in \{1, \dots, m\}$ such that $x_i = y_i$ and $i \in I_{\underline{x}}$. Then, i being a loop in $\mathrm{M}_{\underline{x}}$ implies that i is a loop in $\mathrm{M}_{\underline{y}}$. Indeed, $\mathrm{M}_{\underline{y}}$ is obtained from $\mathrm{M}_{\underline{x}}$ through a sequence of operations consisting of either the contraction of an element different from i or an extension. These operations do not change the property of i being a loop. $\diamond$

Theorem 8.6 (Matroidal characterization of matcuboids)**.** *There is a one-to-one correspondence between coherent complexes of matroids indexed by the hypercuboid $\square_{\underline{\varrho}}$ and matcuboids with ground set $\square_{\underline{\varrho}}$.*

Proof. ($\Longleftarrow$) If we start with a matcuboid $\mathscr{M}$ with rank function $\mathbf{r}$ on $\square_{\underline{\varrho}}$, the collection of matroids $\mathrm{M}_{\underline{a}}$ defined above forms a coherent complex of matroids. Indeed, Condition (CC1) is of course satisfied because of (R1) in Definition 2.1. We now check Condition (CC2). Let $\underline{a} \in \square_{\underline{\varrho}}$ and $i \in I_{\underline{a}}$. If $a_i = \varrho_i - 1$, then $I_{\underline{a}+\underline{e}_i} = I_{\underline{a}} \smallsetminus \{i\}$, which is the ground set of the matroid $\mathrm{M}_{\underline{a}} \,/\, i$. If $a_i < \varrho_i - 1$, then $I_{\underline{a}+\underline{e}_i} = I_{\underline{a}}$, which is the ground set of the matroid $\mathrm{M}_{\underline{a}} \,/\, i \sqcup \{i\}$. To finish, we have to check equality of the rank functions of the matroids on subsets of $I_{\underline{a}} \smallsetminus \{i\}$. It is sufficient to consider $X \subseteq I_{\underline{a}+\underline{e}_i}$ not containing the element i, and show that

$$\mathbf{r}_{\underline{a}+\underline{e}_i}(X) = \mathbf{r}_{\underline{a}}(X \cup \{i\}) - \mathbf{r}_{\underline{a}}(i).$$

The left-hand side is by definition $\mathbf{r}\big(\underline{a} + \underline{e}_i + \sum_{j \in X} \underline{e}_j\big) - \mathbf{r}(\underline{a} + \underline{e}_i)$, and the right-hand side is

$$\mathbf{r}\Big(\underline{a} + \underline{e}_i + \sum_{j \in X} \underline{e}_j\Big) - \mathbf{r}(\underline{a}) - \mathbf{r}(\underline{a} + \underline{e}_i) + \mathbf{r}(\underline{a}).$$

Both sides are therefore equal.

($\Longrightarrow$) The other way around, we consider a coherent complex of matroids $(\mathrm{M}_{\underline{a}})_{\underline{a}}$ and associate a matcuboid $\mathscr{M}$ on $\square_{\underline{\varrho}}$ by specifying its rank function $\mathbf{r}$. Let $\underline{a} \in \square_{\underline{\varrho}}$. We take any increasing path $\underline{0} = \underline{b}_0, \underline{b}_1, \dots, \underline{b}_k = \underline{a}$ from $\underline{0}$ to $\underline{a}$, and define

$$\mathbf{r}(\underline{a}) := \sum_{j=0}^{k-1} \mathbf{r}_{\underline{b}_j}\left(\underline{b}_{j+1} - \underline{b}_j\right).$$

We first prove that $\mathbf{r}$ is well-defined, which amounts to showing that $\mathbf{r}(\underline{a})$ does not depend on the choice of the increasing path $\big(\underline{b}_j\big)$. Two different such paths can be linked by a finite sequence of increasing paths such that between two consecutive increasing paths in the sequence, the only change is an inversion between two consecutive elementary moves $\underline{e}_i$ and $\underline{e}_j$, $i \neq j$. We thus have to check that, for every $\underline{a} \in \square_{\underline{\varrho}}$ and $i, j \in I_{\underline{a}}$ with $i \neq j$, we have

$$\mathbf{r}_{\underline{a}}(\underline{e}_i) + \mathbf{r}_{\underline{a}+\underline{e}_i}(\underline{e}_j) = \mathbf{r}_{\underline{a}}(\underline{e}_j) + \mathbf{r}_{\underline{a}+\underline{e}_j}(\underline{e}_i).$$

But by (CC2), we have $\mathbf{r}_{\underline{a}+\underline{e}_i}(\underline{e}_j) = \mathbf{r}_{\underline{a}/i}(\underline{e}_j) = \mathbf{r}_{\underline{a}}(\underline{e}_j + \underline{e}_i) - \mathbf{r}_{\underline{a}}(\underline{e}_i)$ in the left-hand part and $\mathbf{r}_{\underline{a}+\underline{e}_j}(\underline{e}_i) = \mathbf{r}_{\underline{a}/j}(\underline{e}_i) = \mathbf{r}_{\underline{a}}(\underline{e}_i + \underline{e}_j) - \mathbf{r}_{\underline{a}}(\underline{e}_j)$ in the right-hand part, so the desired equality holds.

We now check (R1)-(R2)-(R3). It is obvious by construction that $\mathbf{r}$ takes integer values, is non-increasing and that $\mathbf{r}(t\underline{e}_i) - \mathbf{r}((t-1)\underline{e}_i) = 0$ or 1 for all $i \in \{1, \dots, m\}$ and $0 \leqslant t \leqslant \varrho_i$. These imply (R1) and (R2).

It remains to show that $\mathbf{r}$ is submodular. By Theorem 6.2, it is sufficient to check the diamond property. We show property $(*)^1_1$. Let thus $i \in \{1, \dots, m\}$ and $\underline{x}, \underline{y} \in \square_{\underline{\varrho}}$ such that $\underline{x} \preceq \underline{y}$, $i \in I_{\underline{x}}$ and $x_i = y_i$. We assume that $\mathbf{r}_{\underline{x}}(\underline{e}_i) = 0$ and show that $\mathbf{r}_{\underline{y}}(\underline{e}_i) = 0$. But this has been shown to be true in Remark 8.5.

We have defined two maps linking coherent complexes of matroids and rank functions. It is straightforward to check that they are inverse of each other. $\square$

9. Further discussions

In this final section, we discuss some related questions.

9.1. **Bases of matcuboids and special features of independents.** We do not know how to define a good notion of bases for matcuboids. We review some natural attemps in this section. For each definition, we show with an example that the data of the set of bases according to that definition does not determine the matcuboid in a unique way. Below, $\mathscr{M}$ refer to a matcuboid of rank r on the ground set $\square_{\underline{\varrho}}$ and $\underline{a}$ is an element of $\square_{\underline{\varrho}}$.

Consider first the idea closest to that of matroids.

(a) *A basis of $\mathscr{M}$ is an independent $\underline{a} \in \mathscr{I}$ which is maximal for the partial order $\preceq$.*

This does not carry enough information. The two matcuboids below, with $\underline{\varrho} = (2,2)$, have the same set of bases according to Definition (a), but not the same sets of independents. The elements verifying (a) are highlighted in red, the other independents in blue.

$$\begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \tag{7}$$

We note that, unlike matroids, a maximal independent of a matcuboid is not necessarily of maximal rank. This is not visible in Example (7) but the following matcuboid provides such an example, two maximal independents (in red) have distinct ranks.

$$\begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \tag{8}$$

Consider thus the following alternative to (a).

(b) *A basis of $\mathscr{M}$ is an independent $\underline{a} \in \mathscr{I}$ of maximal rank r.*

Example (7) shows that this does not work neither.

Definitions (a) and (b) are global. Seeking for local counterparts, similar to flats, circuits, independents, treated in the previous sections, leads to the following candidates.

(c) *A basis of $\mathscr{M}$ is an independent $\underline{a} \in \mathscr{I}$ which is locally maximal, in the sense that for every $i \in \{1, \dots, m\}$ with $\underline{a} + \underline{e}_i \in \square_{\underline{\varrho}}$, we have $\underline{a} + \underline{e}_i \notin \mathscr{I}$.*

(d) *A basis of $\mathscr{M}$ is an independent $\underline{a} \in \mathscr{I}$ which is locally of maximal rank, in the sense that for every $i \in \{1, \dots, m\}$ with $\underline{a} + \underline{e}_i \in \square_{\underline{\varrho}}$, we have $\mathbf{r}(\underline{a} + \underline{e}_i) = \mathbf{r}(\underline{a})$.*

It is easily shown that in fact Definitions (c) and (d) are equivalent. The two matcuboids below, with $\underline{\varrho} = (5,4)$, have the same set of bases according to Definition (c), but not the same sets of independents. The elements verifying (c) are depicted in red, and the other independents are in blue.

$$\begin{pmatrix} 4 & 4 & 4 & 4 & 5 & 6 \\ 3 & 3 & 4 & 4 & 5 & 6 \\ 2 & 2 & 3 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} \qquad \begin{pmatrix} 4 & 4 & 5 & 5 & 5 & 6 \\ 3 & 4 & 5 & 5 & 5 & 6 \\ 2 & 3 & 4 & 4 & 4 & 5 \\ 1 & 2 & 3 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} \tag{9}$$

Considering the deletion operation defined in Section 5.2, produces smaller independents. Consider the following proposition.

(e) *A basis of $\mathscr{M}$ is an independent $\underline{a} \in \mathscr{I}$ which is not the deletion $\underline{b} \setminus i$ for some $\underline{b} \in \mathscr{I}$ and $i \in \{1, \dots, m\}$.*

This is genuinely new, but, if we consider the matcuboids in Example (7) or in Example (9), the bases given by (e) and (a) are the same.

In a matroid with ground set E and rank function $\mathbf{r}$, every subset $S \subseteq E$ satisfies $\mathbf{r}(S) + \mathbf{r}^*(S^c) \leqslant |E|$, where $\mathbf{r}^*$ is the dual rank function and S^c is the complement of S, with equality if, and only if, S is a basis. This leads to the following candidate.

(f) *A basis of $\mathscr{M}$ is an independent $\underline{a} \in \mathscr{I}$ such that $\mathbf{r}(\underline{a}) + \mathbf{r}^*(\underline{a}^c) = \left|\underline{\varrho}\right|_{\ell_1}$, where $\underline{a}^c = \underline{\varrho} - \underline{a}$ is the complement of $\underline{a}$ and $\left|\underline{\varrho}\right|_{\ell_1} = \sum_i \varrho_i$.*

The inequality $\mathbf{r}(\underline{x}) + \mathbf{r}^*(\underline{x}^c) \leqslant \left|\underline{\varrho}\right|_{\ell_1}$ holds for every element $\underline{x} \in \square_{\underline{\varrho}}$, however, some matcuboids have no bases at all according to this definition. This is for instance the case for both matcuboids in Example (9), or in the following matcuboid (the independents are highlighted in blue).

$$\begin{pmatrix} 3 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 \\ 1 & 2 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \tag{10}$$

The question of finding a good notion of bases in matcuboids therefore remains open.

9.2. **The polymatroid and the matroid associated to a matcuboid.** We refer to [HH02] for the definition and basic properties of polymatroids.

Let P be an integer polymatroid on the ground set E with rank function $\rho\colon 2^E \to \mathbb{Z}_{\geqslant 0}$. Replace each element e of E with $\rho(e)$ elements, and let $\widehat{E}$ be the resulting set. For each subset $S \subseteq E$, let $\widehat{S} \subseteq \widehat{E}$ be the union of all the elements associated to each $e \in S$. Define $\widehat{\rho}\colon 2^{\widehat{E}} \to \mathbb{Z}_{\geqslant 0}$ by the formula

$$\widehat{\rho}(Y) \coloneqq \min_{S \subseteq E} \Big(\rho(S) + |Y \smallsetminus \widehat{S}|\Big).$$

This defines a matroid on the ground set $\widehat{E}$ which is symmetric with respect to the permutation of the $\rho(e)$ elements associated to each e.

To a given matcuboid $\mathscr{M}$ on the ground set $\square_{\underline{\varrho}}$, we can associate an integer polymatroid P on the ground set $E \coloneqq [\varrho_1] \sqcup \cdots \sqcup [\varrho_m]$. The rank function ρ of P is given by associating to any subset $S \subseteq E$ the integer value $\rho(S) \coloneqq \mathbf{r}\big(\bigvee_{\underline{a} \in S} \underline{a}\big)$, where the join is taken in the hypercuboid $\square_{\underline{\varrho}}$ and $\mathbf{r}$ is the rank function of $\mathscr{M}$.

There is, moreover, a natural way to send elements of $\mathscr{M}$ to subsets of E, by mapping every $\underline{x} = (x_1, \dots, x_m) \in \square_{\underline{\varrho}}$ to the subset $[0, x_1] \sqcup \cdots \sqcup [0, x_m] \subset E$, where each interval $[0, x_i]$ is taken inside $[0, \varrho_i]$. The polymatroid P can be seen as a polytope in the vector space $\mathbb{R}^E$, but there is no natural way to associate to elements of the matcuboid vectors in this polytope.

Proceeding as above, we can thus associate to a given matcuboid $\mathscr{M}$ a matroid M on the ground set $\widehat{E}$. Note that M has $\sum_{i=1}^{m} \sum_{t=1}^{\varrho_i} \mathbf{r}(t\varrho_i)$ elements.

Using this construction, it seems natural to transfer the notions of flats, independents, bases, and circuits from the matroid M to the matcuboid, in the spirit of the work [BCF23] on integer polymatroids. This seems however to give a story complementary to the theory exposed in this paper.

In the case of flats, for example, our definition coincides with the definition of flats in the corresponding polymatroid, in the sense that the map described above establishes a bijection between the flats of $\mathscr{M}$ and the flats of P. However, we are not aware of any intrinsic axiomatic system for flats in polymatroids, and the one for flats in matcuboids given in the present paper does not seem to be directly related to the one for cyclic flats in polymatroids, due to Csirmaz [Csi20], see as well [BCF23, Section 5].

When it comes to independents of matcuboids, our definition differs entirely from that of independents in a polymatroid [BCF23, Section 3]. As we observed in the previuos section, maximal independents in matcuboids can be of various ranks, whereas maximal independents in polymatroids all have the same rank. The independents in polymatroids are only defined as vectors in the corresponding polytope, not set-theoretically. Besides, as we mentioned previously, we do not have yet a good notion of bases for matcuboids.

The same situation holds for circuits: the definition and axiomatic system we give in the present paper differ from the ones given for polymatroids in [BCF23, Section 4]. As for independents, circuits in polymatroids can only be defined as vectors. Our definition does not rely on independents, and yields to a different story. Note in particular that in matcuboids, we can have comparable circuits, for the partial order $\preceq$ (see Example (4) in Section 4.1), whereas two distinct circuits in a polymatroid are never comparable.

9.3. **Representability and minors.** In analogy with Rota's conjecture on the characterization of the representability of matroids over finite fields using a finite set of forbidden minors, we formulate the following question.

Question 9.1 (Rota's conjecture for matcuboids)**.** *Let κ be a finite field. Does there exist a finite collection of matcuboids such that a matcuboid is representable over κ if and only if it does not contain any of the matcuboids in the collection as a minor?*

Carrying on the discussion of the previous sections, we note that as it was recently shown by Oxley, Semple, and Whittle [OSW16], the analogue of Rota's conjecture for 2-polymatroids fails in general.

We do not know whether the local obstructions given by Proposition 8.2 are the only obstructions for the representability of a hypercube rank function. On the other hand, the representability of a matcuboid over a field is equivalent to the representability of the corresponding matroid. This follows from the observation that the representablity of a matcuboid over a field is equivalent to the representability of the associated polymatroid by a subspace arrangement over the same field. And, the representability of an integer polymatroid by a subspace arrangement is equivalent to the representability of the matroid associated to the polymatroid.

Theorem 9.2. *A matcuboid $\mathscr{M}$ is representable over a field κ if and only if the corresponding matroid M is representable over κ.*

We call a matcuboid $\mathscr{M}$ *regular* if it is representable over every field.

Question 9.3. *Provide a characterization of regular matcuboids.*

For matroids, there are several such characterizations. Seymour's theorem describes regular matroids in terms of sums of graphic, cographic, and an exceptional regular matroid on 10 elements [Sey80]. A theorem by Tutte characterizes regular matroids as those representable over the fields $\mathbb{F}_2$ and $\mathbb{F}_3$ with two and three elements, respectively. Another result by Tutte characterizes regular matroids as those that contain as a minor neither the Fano matroid F_7 nor its dual F_7^*. We refer to [Tru92, Chap. 9] for a presentation of these results.

9.4. **Polymatcuboids.** We can define a notion of polymatcuboids. A polymatcuboid is a function $f\colon \boxdot_{\underline{\varrho}} \to \mathbb{R}$ with $f(\underline{0}) = 0$ which is non-decreasing and submodular, that is, it verifies (R2) and (R3). Examples of polymatcuboids are those associated to a collection of arbitrary (instead of initial) flags in a vector space. A polymatcuboid can be viewed in this way as a special type of polymatroid.

9.5. **Tutte polynomial.** An important algebraic invariant associated to a matroid is its Tutte polynomial. This is a two-variable polynomial which specializes to the characteristic polynomial of the matroid (which is itself a generalization to matroid of the chromatic polynomial of graphs).

The Tutte polynomial of a matroid M on the ground set E is the unique polynomial $T_M(X,Y)$ that verifies the following recursive equation for every $e \in E$:

$$T_M(X,Y) = \begin{cases} X\,T_{M\setminus e}(X,Y) & \text{if } e \text{ is a coloop} \\ Y\,T_{M/e}(X,Y) & \text{if } e \text{ is a loop} \\ T_{M\setminus e}(X,Y) + T_{M/e}(X,Y) & \text{if } e \text{ is neither a loop nor a coloop,} \end{cases}$$

and is defined for the matroid $\varnothing$ with empty ground set by $T_\varnothing \equiv 1$.

We can define the notion of loop and coloop in matcuboids. We say $i \in [m]$ is a *loop* of $\mathscr{M}$ if $\varrho_i > 0$ and $\mathbf{r}(\underline{e}_i) = 0$. We say i is a *coloop* of $\mathscr{M}$ if i is a loop of the dual matcuboid $\mathscr{M}^*$. This is equivalent to having $\mathbf{r}(\mathscr{M} \setminus i) = \mathbf{r}(\mathscr{M}) - 1$. The recursive equation above however does not lead to an invariant of matcuboids.

The Tutte polynomial of a matroid M on the ground set E can be defined directly by the following formula:

$$T_M(X,Y) = \sum_{S \subseteq E} (X-1)^{r-\mathbf{r}(S)}(Y-1)^{|S|-\mathbf{r}(S)}.$$

This definition naturally extends to any matcuboid. Namely, we define the Tutte polynomial of a matcuboid $\mathscr{M}$ as follows.

Definition 9.4. Let $\mathscr{M}$ be a matcuboid of rank r on the ground set $\square_{\underline{\varrho}}$. The *Tutte polynomial* of $\mathscr{M}$ is the two-variable polynomial

$$T_{\mathscr{M}}(X,Y) := \sum_{\underline{x} \in \square_{\underline{\varrho}}} (X-1)^{r-\mathbf{r}(\underline{x})}(Y-1)^{|\underline{x}|_{\ell_1}-\mathbf{r}(\underline{x})},$$

where $|\underline{x}|_{\ell_1}$ is the ℓ_1-norm of $\underline{x}$. $\diamond$

Obviously, in the case $\mathscr{M}$ is a matroid, we find the Tutte polynomial of that matroid.

Question 9.5. *Does there exist a recursive identity which defines $T_{\mathscr{M}}$, in terms of deletions and contractions?*

Remark 9.6. As explained in Section 9.2, a matcuboid naturally gives rise to a polymatroid on the same ground set, see. There is a version of the Tutte polynomial for polymatroids defined by Cameron and Fink [CF22]. This polynomial sastifies a relation involving elementary operations reminiscent of deletion and contraction in polymatroids, and specializes to the Tutte polynomial for matroids. We do not know if there is any relation between the Tutte polynomial of a matcuboid defined above and the Tutte polynoial of the corresponding polymatroid. $\diamond$

We define the direct sum $\mathscr{M} \oplus \mathscr{M}'$ of two matcuboids $\mathscr{M}$ and $\mathscr{M}'$ on hypercuboids $\square_{\underline{\varrho}}$ and $\square_{\underline{\varrho}}$, respectively, by setting

$$\mathbf{r}_{\mathscr{M}\oplus\mathscr{M}'}(\underline{x} \oplus \underline{x}') = \mathbf{r}_{\mathscr{M}}(\underline{x}) + \mathbf{r}_{\mathscr{M}'}(\underline{x}').$$

Proposition 9.7. *Tutte polynomials of matcuboids verify the following properties.*

- *Let $\mathscr{M}^*$ denote the dual of the matcuboid $\mathscr{M}$. Then, we have $T_{\mathscr{M}^*}(X,Y) = T_{\mathscr{M}}(Y,X)$.*
- *For two matcuboids $\mathscr{M}$ and $\mathscr{M}'$, we have $T_{\mathscr{M}\oplus\mathscr{M}'} = T_{\mathscr{M}} \cdot T_{\mathscr{M}'}$.*

Proof. The first statement can be verifed by using the definition of the rank function of $\mathscr{M}^*$. The second statement is straightforward. $\square$

9.6. **Matcuboids arising from linear series on curves.** As we pointed out in Section 1.7, matcuboids naturally arise in our work on tropical degenerations of linear series on algebraic curves. We provide hereafter a brief hint to this by explaining how a finite collection of points and a finite dimensional space of rational functions on an algebraic curve gives rises to a matcuboid.

Let κ be an algebraically closed field, and let C be a smooth proper curve over κ. Let r be a non-negative integer, and let p be a κ-point on C. Let $\kappa(C)$ be the function field of C, and let $\mathrm{H} \subseteq \kappa(C)$ be a vector subspace of rational functions of dimension $r+1$ over κ. The point p leads to a complete flag $\mathrm{F}_p^\bullet$ of H by looking at the orders of vanishing at p of functions in H, as follows. Define the set $S_p := \bigl\{\mathrm{ord}_p(f) \bigm| f \in \mathrm{H} \smallsetminus \{0\}\bigr\}$. We have the following basic result.

Proposition 9.8. *The set S_p is finite of cardinality $r+1$.*

Denote by $s_0^p < \cdots < s_r^p$ the elements of S_p, enumerated in increasing order. The flag $\mathrm{F}_p^\bullet$ is defined by setting, for $j = 0, \dots, r$,

$$\mathrm{F}_p^j := \bigl\{f \in \mathrm{H} \smallsetminus \{0\} \bigm| \mathrm{ord}_p(f) \geqslant s_j^p\bigr\} \cup \{0\}.$$

The fact that F_p^j has codimension j in H follows from the previous proposition.

Let now m be a natural number, and let $A = \{p_1, \dots, p_m\}$ be a collection of m distinct κ-points on C. By the construction above, each point p_i leads to a complete flag $\mathrm{F}_i^\bullet$. Denoting $S_i := \{\mathrm{ord}_{p_i}(f) \bigm| f \in H \smallsetminus \{0\}\}$, and enumerating the elements of S_i in increasing order $s_0^i < \cdots < s_r^i$, the flag $\mathrm{F}_i^\bullet$ is defined by setting

$$\mathrm{F}_i^j := \bigl\{f \in H \smallsetminus \{0\} \bigm| \mathrm{ord}_{p_i}(f) \geqslant s_j^i\bigr\} \cup \{0\}.$$

The data of $C, \mathrm{H}, p_1, \dots, p_m$ defines a matcuboid on the ground set $\square_r^m = [r]^m$.

We call *geometric* a matcuboid with ground set $\square_r^m$ that arises from the above construction for a curve C over an algebraically closed field κ. By construction, geometric matcuboids are all representable over the field κ over which the curve C is defined. The following question is interesting.

Question 9.9. *Is it true that all representable matcuboids on $\square_r^m$ of rank r or $r+1$ are geometric? What is the smallest possible genus of a curve realizing* $\mathbf{r}$*?*

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CNRS - CMLS, École polytechnique, Institut polytechnique de Paris
Email address: omid.amini@polytechnique.edu

CMLS, École polytechnique, Institut polytechnique de Paris
Email address: lucas.gierczak-galle@polytechnique.edu