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Analyse mathématique de quelques systèmes d'équations de type fluide-cinétique

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[1] et même d'apprendre un peu le Plumfoot !

[2] liste non exhaustive...

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RÉSUMÉ

Cette thèse est consacrée à l'analyse mathématique de systèmes de type fluide-cinétique, qui décrivent l'évolution d'une suspension de particules au sein d'un fluide ambiant. Le point de vue adopté est celui de la théorie cinétique pour la phase dispersée et celui de la mécanique des fluides pour la phase continue.

Les Chapitres 2 et 3 sont dédiés à l'étude du comportement en temps long pour les équations de Vlasov-Navier-Stokes dans un domaine, avec condition d'absorption au bord pour les particules. Au Chapitre 2, nous analysons la compétition entre concentration en vitesse et absorption dans un domaine borné. Nous démontrons que la fonction de distribution des particules possède un comportement monocinétique en vitesse, et exhibons une grande variété de scénarios pour le profil asymptotique spatial. Au Chapitre 3, nous nous plaçons dans le cas du demi-espace en prenant en compte l'action de la force de gravité sur les particules. Nous montrons que les effets d'absorption au bord, combinés à la gravité, permettent d'obtenir la stabilité de la solution triviale pour ce système. Notre obtenons une famille d'estimations de décroissance en temps pour tous les moments en vitesse de la fonction de distribution, grâce à l'introduction d'une condition de contrôle géométrique appropriée.

Dans le Chapitre 4, nous utilisons les idées précédentes pour étudier une limite hydrodynamique des équations de Vlasov-Navier-Stokes avec gravité, dans un régime haute-friction. Nous obtenons ainsi la dérivation globale en temps d'un système de type Boussinesq-Navier-Stokes.

Finalement, le Chapitre 5 est consacré à l'étude mathématique d'un système des sprays épais, qui est un couplage singulier entre une équation cinétique et les équations des fluides compressibles. Dans le cas d'un fluide visqueux, nous démontrons l'existence et l'unicité de solutions à régularité Sobolev, localement en temps, pour des données initiales satisfaisant un critère de stabilité à la Penrose. Il s'agit de la première construction rigoureuse de solution pour ce type de système, inspirée de travaux récents sur les équations de Vlasov singulières.

ABSTRACT

This thesis delves into the mathematical analysis of fluid-kinetic systems, which describe the evolution of a suspension of particles in an ambient fluid. In this framework, a kinetic equation is coupled with the standard equations of fluid mechanics.

The focus of Chapters 2 and 3 is the long-time behaviour of the Vlasov-Navier-Stokes equations in a domain, with absorption boundary condition for the particles. In Chapter 2, we analyse the competition between concentration in velocity and absorption in a bounded domain. We show that the particle distribution function has a monokinetic behaviour in velocity, and exhibit a wide variety of scenarios for the spatial asymptotic profile. In Chapter 3, we consider the half-space case, taking into account the action of the gravity force on the particles. The stability of the trivial solution for the system is explored and proven by combining both the absorption at the boundary and the gravity effects. Our approach is based on time-decay estimates for all moments in velocity of the distribution function, obtained by introducing an appropriate geometric control condition.

Chapter 4 builds on the previous ideas to study a hydrodynamic limit of the Vlasov-Navier-Stokes equations with gravity, in a high-friction regime. We obtain the global in time derivation of a Boussinesq-Navier-Stokes type system.

Chapter 5 is dedicated to the mathematical study of a thick spray system, which is a singular coupling between a kinetic equation and the compressible fluid equations. In the case of a viscous fluid, we prove the local in time strong well-posedness of the equations with Sobolev regularity, for initial data satisfying a Penrose stability condition. This is the first rigorous construction of solutions to this type of system, in the spirit of some recent works on singular Vlasov equations.

LISTE DES TRAVAUX

Cette thèse a donné lieu aux (pré)publications suivantes.

- [EHK23] Lucas Ertzbischoff and Daniel Han-Kwan, On well-posedness for thick spray equations, *soumis pour publication.*
- [Ert22] Lucas Ertzbischoff, Global derivation of a Boussinesq-Navier-Stokes type system from fluid-kinetic equations, *soumis pour publication.*
- [Ert21] Lucas Ertzbischoff, Decay and absorption for the Vlasov-Navier-Stokes system with gravity in a half-space, *soumis pour publication.*
- [EHKM21] Lucas Ertzbischoff, Daniel Han-Kwan, and Ayman Moussa, Concentration versus absorption for the Vlasov–Navier–Stokes system on bounded domains, *Nonlinearity*, 34(10):6843, 2021.

Note de lecture. Les chapitres qui composent ce manuscrit peuvent être lus de façon indépendante. Le premier chapitre, en français, est une introduction générale à cette thèse. Les autres chapitres, en anglais, réunissent les contributions tirées des travaux ci-dessus.

Table of contents

Chapter 1

Introduction et synthèse

Cette thèse porte sur l'étude et l'analyse mathématique d'équations aux dérivées partielles décrivant l'évolution d'un nuage de particules en suspension dans un fluide. Ce chapitre introductif a pour but de poser les bases mathématiques de la théorie des systèmes d'équations dits fluide-cinétique, qui constituent le cœur du manuscrit. On présente d'abord les systèmes qui vont nous intéresser (équations de Vlasov-Navier-Stokes et des sprays épais) ainsi que le cadre mathématique jusqu'ici développé pour les analyser. On s'achemine ensuite petit à petit vers les problématiques abordées dans cette thèse, rappelant les résultats récents qui l'ont précédée, et développant les contributions et résultats obtenus.

1.1 Couplages fluide-cinétique et modélisation

Cette première section, très générale et introductive, a pour but de présenter les équations et modèles sous-jacents à la description des *sprays*. Elle contient en particulier quelques rappels succincts de théorie cinétique et du transport, et de dynamique des fluides.

1.1.1 Suspensions de particules dans un fluide : vers l'étude des sprays

Derrière la dénomination de *spray* (ou encore *aérosol*) issue des applications physiques, se cache un terme très général désignant une phase constituée de petites particules se déplaçant au sein d'une phase fluide. On peut par exemple penser à l'évolution d'un nuage de fines gouttelettes ou de petites particules solides immergées dans un gaz ambiant. De manière plus générale, il s'agit d'un sous-domaine des systèmes dits *multiphasiques*.

La complexité, mais aussi l'intérêt, de ces systèmes provient de l'écoulement du fluide en lui-même, du couplage entre la phase dispersée et le fluide où les particules se déplacent (ces deux phases ne sont pas indépendantes mais vont interagir entre elles), mais aussi de l'interaction entre les particules elles-mêmes. L'étude de tels modèles a conduit à une importante littérature du point de vue physique [Wil85, O'R81, Koc90, DE88, GM11, Sir10].

De par leur généralité, ce type de systèmes permet de décrire de nombreux phénomènes naturels ou issus des applications. Mentionnons, entre autres, l'évolution de particules dans l'atmosphère et leur lien avec la formation des nuages, les phénomènes de combustion dans les moteurs [AOB89, Lau02], l'évaporation de brouillards de gouttes [MV01, DC09], l'utilisation d'aérosols à visée médicamenteuse dans les voies respiratoires [Mou09, BGG$^+$20, BGLM15, BM21], la description d'aérosols marins [VM20] ou d'aérosols volcaniques [MPO03] et leur impact sur l'atmosphère, la capture du dioxyde de carbone dans les océans [BDC20], ou encore les aérosols dans l'atmosphère de géantes gazeuses ou d'exoplanètes [WST86, GWMP21]. Évoquons également la description des phénomènes de sédimentation de particules dans un fluide, avec l'influence de la gravité sur chacune d'elles, qui est également un objet d'étude majeur du point de vue physique [GM11].

Grosso modo, le comportement d'une particule isolée dans un fluide visqueux (voir les travaux de Stokes [Sto50]) est très différent de celui d'un système de $N > 1$ particules immergées dans celui-ci : même si elles ne sont pas directement en interaction (particules non chargées et avec un mélange initialement bien dilué), chaque particule va influencer le fluide environnant qui va à son tour modifier la dynamique de chaque particule, *etc*... Les interactions entre particules dans un fluide visqueux se font donc "à longue portée"[1]. Lorsque le nombre de particules devient très important ($N \gg 1$), il peut être compliqué d'utiliser des simulations numériques pour suivre chaque particule une par une.

[1]En comparaison, on pourra penser à la dynamique d'un système de particules chargées en interaction (de type Coulomb), à ceci près qu'une suspension de particules dans un fluide ne fait pas nécessairement intervenir des interactions répulsives. De plus, c'est le fluide ambiant qui permet indirectement les interactions.

Un enjeu important est donc l'étude d'équations et modèles mathématiques permettant de comprendre de tels systèmes physiques. Ces équations doivent être suffisamment simples pour que l'on puisse les analyser précisément, tout en s'assurant qu'elles sont pertinentes du point de vue de la modélisation. En retour, une telle complexité est source de problèmes mathématiques non-triviaux.

D'un point de vue mathématique, il existe plusieurs moyens permettant de décrire un spray. Suivant Desvillettes [Des10], on peut par exemple

- étudier un mélange liquide-gaz en considérant les équations de la mécanique des fluides à l'intérieur des particules et au sein du gaz, ces deux phases étant séparées par une frontière libre. Ce type de modélisation est difficile à gérer d'un point de vue mathématique (et numérique). Considérer un système de $N \gg 1$ boules rigides de rayon $R_N > 0$ suivant leur propre dynamique au sein d'un fluide lui-même décrit de façon macroscopique apparaît déjà comme un modèle intéressant.

- voir le système particules-fluide comme un mélange de deux phases macroscopiques (description Eulérienne-Eulérienne), en particulier lorsque le nuage de particules n'est pas très dilué au sein du mélange. On obtient des systèmes dits bi-fluides [IH10] où la frontière libre entre les deux phases disparaît mais où l'on considère la fraction volumique de chacune d'entre-elles. L'analyse mathématique de tels systèmes est assez peu développée, même s'il semble plus courante du point de vue numérique.

La description retenue dans cette thèse, intermédiaire entre les deux précédentes, est celle des **équations fluide-cinétique**. Elle se situe à l'intersection de la **théorie cinétique** et de la **dynamique des fluides**, et apparaît parfois sous la dénomination de description Eulérienne-Lagrangienne. Introduite par Williams [Wil85] et O'Rourke [O'R81] (voir aussi Caflisch et Papanicolaou [CP83]), elle est basée sur le principe général suivant :

- on adopte un **point de vue macroscopique** sur le fluide ambiant (phase continue) à l'aide des équations classiques de la **mécanique des fluides** (équations de Navier-Stokes ou d'Euler) qui portent sur la vitesse du fluide $u(t,x)$, sa densité $\varrho(t,x)$ ou sa pression $p(t,x)$ - voir la présentation faite en Section 1.1.2. Plusieurs choix de modélisation sont alors possibles pour le fluide; visqueux ou parfait, compressible ou incompressible, homogène ou non, *etc*....

- on décrit le nuage de particules (phase dispersée) du **point de vue mésoscopique** en s'inspirant de la **théorie cinétique** (équations de Vlasov ou Boltzmann) qui s'intéresse à l'évolution de la fonction de distribution f des particules. On obtient des informations intéressantes sur le nuage de particules en faisant par exemple des moyennes selon les variables de position, vitesse, rayon ou énergie interne de cette fonction de distribution. Ce type de description, et plus généralement les équations mises en jeu, est détaillé en Section 1.1.3.

Du point de vue physique, les couplages fluide-cinétique permettent de pouvoir facilement tenir compte de la variabilité en taille des différentes particules au sein de la phase dispersée. Ils autorisent également l'écriture de nombreuses interactions physiques au niveau des particules (collisions, coalescence, abrasion, *etc*...) et entre la phase dispersée et le fluide (force de traînée, échanges thermiques, *etc*...).

Les système fluide-cinétique ont fait l'objet de très nombreux travaux depuis une vingtaine d'années, tant du point de vue théorique que numérique. Dans cette thèse, on aborde leur étude *via* l'analyse théorique des équations aux dérivées partielles. Précisons que ce point de vue, très en amont des applications, n'est pas dénué d'intérêt au vu de la complexité des modèles en jeu.

Dans ce manuscrit, on s'intéressera plus précisément au comportement quantitatif et qualitatif des solutions à ces équations à travers leur **dynamique en temps long**, ainsi qu'à leur **caractère bien posé au sens de Hadamard** (c'est-à-dire existence, unicité et stabilité par rapport aux donnée initiales des solutions). Dans l'esprit du 6ème problème de Hilbert concernant l'axiomatisation mathématique de la physique, on étudiera également le lien entre les différentes descriptions des sprays évoquées auparavant, grâce aux **limites hydrodynamiques** des équations fluide-cinétique.

La saveur et la difficulté mathématique des systèmes fluide-cinétique sont essentiellement contenues dans le couplage choisi entre le fluide et les particules. On présentera les différentes interactions retenues dans cette thèse, et donc les équations étudiées (à savoir le système de Vlasov-Navier-Stokes et des sprays épais), dans les Sections 1.1.4 et 1.1.5.

Pour commencer, on propose quelques rappels généraux sur les équations de la mécanique des fluides (Section 1.1.2) et sur les équations cinétiques (Section 1.1.3). Comme les couplages fluide-cinétique combinent ces deux aspects, cela nous permettra de poser les équations et de fixer les notations.

1.1.2 Mécanique des fluides : les équations de Navier-Stokes

On s'intéresse donc d'abord à la description d'un fluide visqueux, que l'on voit comme un milieu continu régi par des quantités macroscopiques comme sa densité $\rho(t,x) \in \mathbb{R}^+$ ou sa vitesse $u(t,x) \in \mathbb{R}^d$. On choisit une formulation Eulérienne, par opposition à une formulation Lagrangienne qui consisterait à suivre la position de chaque particule de fluide au cours du temps.

Dans cette thèse, on n'abordera pas (ou seulement très peu) les équations d'Euler pour les fluides parfaits[2], c'est-à-dire non visqueux. On décidera plutôt d'inclure les effets de dissipation interne liés à la viscosité du fluide (à la suite des travaux de Navier et Stokes dans la première moitié du 19ème siècle). On obtient alors le **système de Navier-Stokes** pour un fluide visqueux, sur lequel nous nous concentrerons dans la suite. On renvoie par exemple à [BF12] pour une présentation détaillée et complète des équations présentées ici.

Dans cette section, le terme F_{ext} représente une force volumique extérieure donnée agissant sur le fluide, et pouvant dépendre du temps et de l'espace. On travaille en dimension $d = 2$ ou $d = 3$.

1.1.2.1 Fluide visqueux incompressible

On considère tout d'abord le cas des fluides dits *incompressibles*. Cela signifie que l'espace occupé par une certaine quantité de fluide peut changer de forme au cours de l'évolution, mais pas de volume. Cette condition est équivalente au fait que la densité du fluide reste constante le long des trajectoires associées au champ de vitesse, et correspond *grosso modo* à un régime à faible nombre de Mach.

Le système de Navier-Stokes pour un fluide incompressible homogène prend la forme suivante :

$$\text{(NS)} \begin{cases} \partial_t u + (u \cdot \nabla_x) u + \nabla_x p - \nu \Delta_x u = F_{\text{ext}}, \\ \operatorname{div}_x u = 0, \\ u_{|t=0} = u^{\text{in}}. \end{cases} \tag{1.1.1}$$

L'inconnue est le champ de vitesse $u(t,x) \in \mathbb{R}^d$, à l'instant $t \in \mathbb{R}^+$ et à la position $x \in \Omega$, où le domaine spatial est $\Omega \subset \mathbb{R}^d$. Son évolution est prescrite par la première équation. Si le domaine Ω possède un bord, on prescrit une condition de Dirichlet homogène pour la vitesse du

[2]Celles-ci ont été écrites pour la première fois en 1757 par Euler.

fluide, c'est-à-dire

$$u(t)_{|\partial\Omega} = 0.$$

L'opérateur différentiel $(u \cdot \nabla_x)$ est un opérateur vectoriel d'auto-advection défini par

$$\forall j \in [\![1; d]\!], \quad [(u \cdot \nabla_x)u]_j = \sum_{i=1}^{d} u_i \partial_{x_i} u_j,$$

tandis que Δ_x désigne le Laplacien vectoriel. Le coefficient $\nu > 0$ est quant à lui le coefficient de viscosité dynamique du fluide. La première équation traduit ainsi la balance des forces au sein du fluide. La seconde équation du système (1.1.1) assure l'incompressibilité du fluide. Enfin, la fonction scalaire $p(t,x) \in \mathbb{R}^d$ peut être vue comme une inconnue annexe du problème. En effet, en prenant la divergence dans la première équation de (1.1.1), on obtient formellement

$$-\Delta_x p = \mathrm{div}_x \left[(u \cdot \nabla_x u)\right] - \mathrm{div}_x F_{\mathrm{ext}}.$$

On peut donc obtenir la pression $p(t,x)$ à l'instant t en résolvant une équation de Poisson faisant intervenir la vitesse $u(t,x)$ à l'instant t. C'est donc l'unique fonction scalaire (à une fonction du temps près) permettant d'assurer la condition de divergence nulle au cours du temps. Autrement dit, la pression est un multiplicateur de Lagrange lié à la contrainte d'incompressibilité.

1.1.2.2 Fluide visqueux compressible

Passons maintenant au système de Navier-Stokes pour un fluide *compressible*. Dans le régime dit barotropique (et où il n'y a pas d'équation sur l'énergie interne ou la température du fluide), les équations ont alors la forme suivante :

$$(\mathrm{NSc}) \begin{cases} \partial_t \varrho + \mathrm{div}_x(\varrho u) = 0, \\ \partial_t(\varrho u) + \mathrm{div}_x(\varrho u \otimes u) + \nabla_x p - \mu \Delta_x u - (\lambda + \mu)\nabla_x \mathrm{div}_x\, u = \varrho F_{\mathrm{ext}}, \\ (\varrho_{|t=0}, u_{|t=0}) = (\varrho^{\mathrm{in}}, u^{\mathrm{in}}). \end{cases} \tag{1.1.2}$$

Ici, les inconnues sont donc la densité du fluide $\varrho(t,x)$ et sa vitesse $u(t,x) \in \mathbb{R}^d$. La première équation s'obtient par la conservation de la masse tandis que la seconde provient de l'évolution de la quantité de mouvement pour le fluide.

La pression $p : \mathbb{R}^+ \to \mathbb{R}$ est une fonction donnée et n'est pas une inconnue. Par simplicité, on considère ici le régime barotropique où la pression dépend uniquement de la densité du fluide, *via* une loi d'état qui est prescrite, c'est-à-dire $p = p(\varrho)$. Des hypothèses (mathématiquement) raisonnables sur la loi de pression sont par exemple $p(0) = 0$ et la croissance de cette fonction. Le cas des fluides polytropiques donne par exemple $p(\varrho) = a\rho^\gamma$ où $a > 0$ et $\gamma > 1$. Les coefficients μ and λ sont appelés coefficients de Lamé, et peuvent dépendre de ϱ. On suppose généralement que $\mu > 0$ et $\lambda + 2\mu > 0$.

1.1.3 Principes de la théorie cinétique

On propose ici de brefs rappels sur les équations standards de la théorie cinétique. Rappelons que ce type d'équations permettra dans la suite de décrire la phase dispersée de particules au sein du couplage avec le fluide ambiant.

1.1.3.1 Une histoire d'échelles

Il existe plusieurs niveaux de description pour étudier l'évolution d'un nuage de particules, suivant l'échelle à laquelle on se place.

• Échelle microscopique : il s'agit du point de vue Newtonien sur un système de $N \in \mathbb{N}$ particules, déjà évoqué auparavant. On décrit chaque particule à l'aide de sa position et de sa vitesse qui satisfont les équations de Newton. Cette approche est extrêmement précise, mais demande d'une part de connaître toutes les interactions à l'œuvre dans le système, et fournit d'autre part beaucoup trop d'informations sur le système.

En dimension 3, cette approche requiert la résolution d'un système couplé de $6N$ équations différentielles ordinaires. Typiquement, pour un litre d'air à température ambiante, on aura $N \sim 10^{23}$ avec N le nombre d'atomes, tandis que la Voie Lactée se comporte comme un gaz d'étoiles avec $N \sim 10^{11}$. Par ailleurs, le problème à 3 corps ($N = 3$) en dimension 3 constitue déjà un système dynamique extrêmement complexe à étudier.

• Échelle macroscopique : on décrit l'ensemble des particules à l'aide de quantités macroscopiques (pression, vitesse ou densité). Ceci s'effectue par exemple par un bilan de masse, de quantité de mouvement et d'énergie dans un petit cube qui contient néanmoins un grand nombre de particules. Cela suppose que celles-ci soient à l'équilibre thermodynamique. On a perdu de l'information par rapport à l'échelle précédente mais on espère que cette description hydrodynamique, bien moins coûteuse en terme de nombre d'équations et de calculs, soit encore assez précise.

• Échelle mésoscopique et théorie cinétique : cette échelle est une sorte de compromis entre les deux précédentes. Ici, on ne s'intéresse pas à l'évolution de chaque particule prise séparément, mais plutôt au nombre de particules ayant tel ou tel comportement. Il s'agit donc d'un point de vue statistique sur le nuage de particules. La théorie cinétique décrit ainsi la *distribution* de particules dans l'espace des phases, c'est-à-dire selon leur position x (dans un domaine spatial $\Omega \subset \mathbb{R}^d$) et leur vitesse admissible v (dans $\mathbb{R}^d$), et éventuellement selon d'autres variables. L'objet principal de la théorie cinétique est une fonction de distribution

$$\begin{array}{rccc} f & : & \mathbb{R}^+ \times \Omega \times \mathbb{R}^d & \longrightarrow & \mathbb{R}^+ \\ & & (t,x,v) & \longmapsto & f(t,x,v) \end{array}.$$

On voit $f(t)$, à chaque instant t, comme la densité d'une mesure sur l'espace des phases $\Omega \times \mathbb{R}^d$.

Ainsi, si $\mathcal{O}_1 \subset \Omega$ et $\mathcal{O}_2 \subset \mathbb{R}^d$ désignent respectivement deux ouverts du domaine spatial et de l'ensemble des vitesses admissibles, la quantité

$$\int_{\mathcal{O}_1 \times \mathcal{O}_2} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v$$

représentera le nombre total de particules à l'instant t ayant une position dans $\mathcal{O}_1$ et animées d'une vitesse dans $\mathcal{O}_2$. Pour des systèmes ayant un nombre fini de particules, il est légitime de supposer que $f(t)$ soit intégrable sur l'espace des phases $\Omega \times \mathbb{R}^d$. Après normalisation de l'intégrale, on peut donc voir $f(t)$ comme étant à chaque instant une densité de probabilité dans l'espace des phases, expliquant l'aspect statistique de la description cinétique.

Notons qu'en général, on ne s'intéresse pas vraiment aux valeurs ponctuelles de la fonction de distribution $f(t)$, qui n'est d'ailleurs pas réellement observable. On s'intéresse plutôt à des moyennes de f, qui représentent naturellement des quantités physiques. Par exemple, les intégrales

$$\rho_f(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \qquad \text{(densité locale)}, \tag{1.1.3}$$

$$j_f(t,x) := \int_{\mathbb{R}^d} f(t,x,v)v\,\mathrm{d}v, \qquad \text{(moment local)}, \tag{1.1.4}$$

$$E_f(t) := \int_{\Omega\times\mathbb{R}^d} \frac{|v|^2}{2} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v, \qquad \text{(énergie cinétique totale)}, \tag{1.1.5}$$

sont des quantités d'intérêt dans l'étude du nuage de particules, que l'on retrouvera d'ailleurs dans toute cette thèse.

Historiquement, la théorie cinétique a rigoureusement été introduite par Maxwell et Boltzmann à la fin du 19ème siècle. Le développement d'une telle théorie repose d'une part sur l'étude préliminaire de la thermodynamique des gaz tout au long de ce siècle, et d'autre part sur les progrès de la statistique, notamment dans le domaine des sciences sociales.

Notons que le point de vue cinétique, fondamentalement basé sur une description atomiste de la nature, a été formalisé alors que l'existence même des atomes était encore largement controversée. Il est notable que cette théorie se retrouve aujourd'hui en physique des plasmas, en astrophysique, en géophysique ou encore en biologie (swarming, flocking, *etc*...) et en théorie de la turbulence faible. En fait, cette approche apparaît pertinente dès que l'on étudie le comportement d'une grande collection d'objets où les informations statistiques sur leur dynamique sont plus intéressantes que leur évolution individuelle. On renvoie à [MV15] pour une description et des références détaillées.

1.1.3.2 Équations cinétiques

Présentons maintenant les équations typiques satisfaites par la fonction de distribution f. Il en existe de très nombreuses variantes, selon les systèmes décrits et les effets pris en compte. Ces **équations cinétiques** sont des équations de transport dans l'espace des phases, c'est-à-dire en (x,v). On considère $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$ pour simplifier et éviter tout problème au bord. On prescrit une condition initiale $(x,v)\mapsto f^{\mathrm{in}}(x,v)$ au temps $t=0$.

La brique de base de toutes les équations cinétiques classiques est celle du **transport libre**, qui s'écrit

$$\partial_t f + v\cdot\nabla_x f = 0. \tag{1.1.6}$$

Elle décrit l'évolution de particules se déplaçant en ligne droite à vitesse constante $v\in\mathbb{R}^d$, sans être accélérées par une force extérieure et sans interagir entre elles. L'équation (1.1.6) est si simple qu'elle se résout de façon explicite : sa solution est donnée par la formule

$$f_{\mathrm{lib}}(t,x,v) = f^{\mathrm{in}}(x-tv,v).$$

Cependant, il ne faut pas se fier à la simplicité trompeuse de l'équation de transport. Celle-ci recèle de nombreuses propriétés qui sont souvent à la base d'effets importants dans des modèles cinétiques plus complets.

Si l'on considère l'influence d'une force extérieure $F_{\mathrm{ext}} = F_{\mathrm{ext}}(t,x)\in\mathbb{R}^d$ agissant sur le nuage de particules (et en normalisant la masse des particules), on obtient une **équation de Vlasov linéaire** de la forme

$$\partial_t f + v\cdot\nabla_x f + F_{\mathrm{ext}}\cdot\nabla_v f = 0. \tag{1.1.7}$$

Ici, les particules sont donc accélérées sous l'effet de la force F_{ext}.

Dans de nombreux cas, on souhaite également tenir compte des interactions entre les particules. Le type de modèle à considérer dépend fortement de l'interaction et de sa portée, et de nombreuses variantes intermédiaires sont possibles. On présente ici deux principales classes d'interactions.

- lorsque l'interaction s'effectue à longue portée (*i.e.* sur des échelles spatiales non négligeables par rapport à l'échelle du système), on considère plutôt une **équation de champ moyen**;
- lorsque l'interaction s'effectue à courte portée et est localisée en espace, on considère plutôt une **équation collisionnelle**.

Équations de champ moyen. Traditionnellement, une équation de type champ moyen se présente sous la forme d'une **équation de Vlasov non-linéaire**

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v\,[F[f]f] = 0, \tag{1.1.8}$$

où la force $F[f] = F[f](t,x,v)$ résulte ici de l'interaction des particules entre-elles. La dynamique de l'équation (1.1.8) est essentiellement encodée dans cette force. Par ailleurs, dans les modèles physiques, elle ne dépend en général de f qu'au travers de ses moments en vitesse ou espace-vitesse, comme ρ_f et j_f (voir (1.1.3) et (1.1.4)). Citons ici deux exemples classiques (voir [Gla96]) :

- le **système de Vlasov-Poisson**

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ E = -\nabla_x U, \\ -\gamma \Delta_x U = \rho_f, \end{cases} \tag{1.1.9}$$

avec $\gamma = 1$ pour l'interaction Coulombienne (pour des électrons dans les plasmas - d'après les travaux de Vlasov [Vla38]) et $\gamma = -1$ pour l'interaction Newtonienne (pour les galaxies - d'après les travaux de Jeans [Jea15]). Il s'agit d'un couplage entre une équation de transport dans l'espace des phases et une équation de Poisson. Ici, on a $F[f](t,x) = E(t,x)$ et on observe que le champ E s'obtient à partir du moment ρ_f puisqu'on a formellement $E = \gamma \nabla_x (\Delta_x)^{-1} \rho_f$. On reverra apparaître le système de Vlasov-Poisson (pour des ions) avec la problématique de la limite quasi-neutre au moment de la Section 1.6.2.

- le **système de Vlasov-Maxwell** (dont (1.1.9) est une approximation électrostatique lorsque la vitesse de la lumière $c \to +\infty$) décrivant la distribution d'électrons dans un plasma relativiste

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \left(E + \dfrac{1}{c}\hat{v} \wedge B\right) \cdot \nabla_v f = 0, \\ \dfrac{1}{c}\partial_t B + \mathrm{rot}_x\, E = 0, \quad \mathrm{div}_x\, E = \rho_f, \\ -\dfrac{1}{c}\partial_t E + \mathrm{rot}_x\, B = \dfrac{1}{c}\hat{j}_f, \quad \mathrm{div}_x\, B = 0, \end{cases} \tag{1.1.10}$$

où

$$\widehat{v} := \frac{v}{\sqrt{1+|v|^2/c^2}}, \quad \widehat{j}_f := \int_{\mathbb{R}^d} \widehat{v} f \,\mathrm{d}v.$$

Il s'agit là d'un couplage entre une équation de transport dans l'espace des phases et les équations de Maxwell sur les champs électrique E et magnétique B. Ici, on a $F[f](t,x,v) = E(t,x) + \frac{1}{c}\widehat{v} \wedge B(t,x)$ et on observe que les champs E et B sont obtenus grâce aux moments ρ_f et $\widehat{j}_f$.

Les équations cinétiques considérées dans cette thèse seront toutes de type Vlasov. La force $F[f]$ sera dans notre cas associée à des équations de la mécanique des fluides (voir la Section 1.1.4).

Équations avec collisions. Lorsque l'interaction entre les particules est localisée, on peut considérer

$$\partial_t f + v \cdot \nabla_x f = \mathcal{C}(f), \tag{1.1.11}$$

où $\mathcal{C}(f)$ est un opérateur intégro-différentiel, dit **opérateur de collisions**. Voici quelques exemples très importants.

- Opérateur de Boltzmann : Pour un nuage de particules suffisamment dilué, on considère des collisions binaires entre celles-ci et $\mathcal{C}(f)(v)$ représente la variation du nombre de particules se déplaçant à la vitesse v due aux collisions. Traditionnellement, on considère des collisions élastiques (conservation de la masse, du moment, et de l'énergie cinétique après collision) et réversibles à l'échelle microscopique (par exemple pour les gaz neutres raréfiés, dits *moléculaires*). Cependant, on peut aussi vouloir tenir compte d'éventuels effets dissipatifs au cours des collisions. Pour décrire cette potentielle perte, on s'appuie sur la théorie des *milieux dits granulaires* [Vil06]. Celle-ci décrit un nuage de particules dont la taille est plus grande que le cas standard avec collisions élastiques. Dans ce manuscrit, on considérera donc également des collisions inélastiques.

  On considère ainsi le ,paramètre $\beta \in (0,1]$ fixé, appelé *coefficient de restitution.* Le cas $\beta = 1$ correspondra au cas élastique tandis que le cas $\beta \in (0,1)$ correspondra au cas purement inélastique. On écrit $\mathcal{C}(f) = \mathcal{Q}_\beta(f,f)$ où $\mathcal{Q}_\beta$ est une différence entre un terme de gain et un terme de perte

  $$\mathcal{Q}_\beta(f,f)(v) = \mathcal{Q}_\beta^+(f,f)(v) - \mathcal{Q}_\beta^-(f,f)(v).$$

  Le terme de gain (resp. de perte) $\mathcal{Q}_\beta^+(f,f)(v)$ (resp. $\mathcal{Q}_\beta^-(f,f)(v)$) représente le nombre de particules ayant acquis (resp. perdu) la vitesse v après collision. Ces deux termes peuvent être calculés de la façon suivante: si $'v$ et $'v_\star$ sont les vitesses de deux particules avant collision, leurs vitesses respectives v et $v_\star$ après collision sont données par

  $$v = {}'v - \frac{1+\beta}{2}({}'u \cdot n)n, \quad v_\star = {}'v_\star + \frac{1+\beta}{2}({}'u \cdot n)n, \quad {}'u := {}'v - {}'v_\star, \tag{1.1.12}$$

  où $n \in \mathbb{S}^{d-1}$ est un vecteur unité pointant du centre de la particule à vitesse v vers le centre de la particule à vitesse $v_\star$ au moment de l'impact. On observe que dans le régime inélastique $\beta \in (0,1)$, on a

  $$|v|^2 + |v_\star|^2 = |{}'v|^2 + |{}'v_\star|^2 - \frac{1-\beta^2}{2}({}'u \cdot n)^2,$$

  ce qui induit bien une perte d'énergie cinétique après chaque collision, tandis que la masse et la quantité de mouvement sont conservées.

  Deux fonctions de distribution $f = f(v)$ and $g = g(v)$ étant données, les relations (1.1.12) permettent de considérer (voir [ALT20]) le régime (in)élastique dit des sphères dures avec

  $$\begin{aligned} \mathcal{Q}_\beta^+(f,g)(v) &:= \frac{1}{\beta^2}\int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |u \cdot n| b(\widehat{u} \cdot n) f({}'v) g({}'v_\star)\, \mathrm{d}v_\star \mathrm{d}n, \\ \mathcal{Q}_\beta^-(f,g)(v) &:= f(v) \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |u \cdot n| b(\widehat{u} \cdot n) g(v_\star)\, \mathrm{d}v_\star \mathrm{d}n, \\ u &:= v - v_\star, \quad \widehat{u} = u/|u|. \end{aligned} \tag{1.1.13}$$

  Ici, la fonction b est un noyau de collision angulaire donné. L'opérateur $\mathcal{Q}_\beta(f,f)$ est ainsi un opérateur quadratique intégral, agissant uniquement en la variable de vitesse (les collisions étant localisées en espace-temps). L'équation (1.1.11) devient alors la célèbre équation de Boltzmann (pour la première fois écrite dans sa forme forte dans [Bol72]). Citons également les opérateurs de Landau et de Lénard-Balescu, utilisés par exemple pour les collisions dans les plasmas (voir [Vil02b]).

- Opérateur de type Fokker-Planck : une simplification diffusive (en vitesse) du cas élastique précédent et qui permet de s'affranchir de la géométrie des collisions (voir [MV15]) conduit à

$$\mathcal{C}(f) = \mathrm{div}_v\,[a(v)f] + \Delta_v f,$$

où $a(v) \in \mathbb{R}$ est un coefficient relié à la friction. Le Laplacien en vitesse, réminiscent du mouvement Brownien, modélise l'agitation thermique des particules.

On renvoie à l'article de synthèse de Villani [Vil02b] pour plus de détails sur ce type de modèle[3]. La présence de collisions sera notamment prise en compte au cours du Chapitre 5 (voir également les Sections 1.5 et 1.6.1.2).

1.1.3.3 Conditions au bord

Les équations cinétiques précédentes peuvent aussi être posées dans un domaine $\Omega \times \mathbb{R}^d$ de l'espace des phases (avec $\Omega \subset \mathbb{R}^d$). Dans ce cas, on doit essentiellement prescrire la valeur de la fonction de distribution f sur un sous-ensemble $\Sigma^- \subset \partial\Omega \times \mathbb{R}^d$ correspondant aux trajectoires rentrantes de l'espace des phases (voir plus précisément la Section 1.2.2). Il existe de nombreuses conditions aux limites possibles en théorie cinétique (voir par exemple [CIP13, Ber20]). Elles sont de type absorption/réflexion: on peut choisir

- Condition d'absorption/émission au bord: pour un certaine source g, on prescrit

$$\forall t > 0, \quad f(t)_{|\Sigma^-} = g. \tag{1.1.14}$$

La conservation de la masse n'est plus nécessairement assurée.

- Condition de réflexion spéculaire pure: suivant les lois de Snell-Descartes, on prescrit

$$\forall t > 0,\ \forall (x,v) \in \Sigma^-, \quad f(t,x,v) = f(t,x,R_x v),$$

où $R_x v = v - 2(v\cdot n(x))n(x)$ est la vitesse avant la collision avec le bord. On a en particulier conservation de la masse, du moment et de l'énergie. On note $f_{|\Sigma^-} = Rf_{|\Sigma^+}$.

- Condition de réflexion diffuse: pour tenir compte d'une perte d'information quand les particules sont ré-émises après contact avec le bord, on peut considérer

$$\forall t > 0,\ \forall (x,v) \in \Sigma^-, \quad f(t,x,v) = \int_{v'\cdot n(x)>0} \mathbf{K}(x,v,v') f(t,x,v')\,\mathrm{d}v',$$

pour un certain noyau positif $\mathbf{K}$ qui doit satisfaire $\int_{v'\cdot n(x)<0} \mathbf{K}(x,v,v')\,\mathrm{d}v' = 1$. (pour assurer la conservation de la masse). On peut par exemple considérer un noyau Gaussien. On note $f_{|\Sigma^-} = K f_{|\Sigma^+}$.

[3] On ne s'attarde pas ici sur la dérivation des équations collisionnelles de type Boltzmann, dans la limite dite de Boltzmann-Grad (voir [CIP13]) et on renvoie aux travaux et articles de synthèse récents de Bodineau, Gallagher, Saint-Raymond, Simonella et Texier [GSRT13, BGSRS22].

- Condition de Maxwell: il s'agit d'une sorte d'interpolation entre les deux conditions de réflexion précédentes: pour un certain $\alpha \in [0,1]$, on considère

$$f_{|\Sigma^-} = \alpha R f_{|\Sigma^+} + (1-\alpha) K f_{|\Sigma^+}.$$

Dans le cadre de ce manuscrit, on se focalisera[4] sur la **condition d'absorption au bord** (correspondant à $g = 0$ dans l'égalité ci-dessus). On renvoie à nouveau à la Section 1.2.2 pour des définitions plus précises.

1.1.4 Interaction entre fluide et particules

Dans cette section, on présente les deux principaux modèles fluide-cinétique qui sont au cœur de cette thèse. Afin de comparer et d'expliquer brièvement (et formellement) leur dérivation et le sens des termes qui apparaissent, on commence par considérer des quantités non adimensionnées et on se place également en dimension $d = 3$.

Ici, on suppose que le spray est monodispersé (une seule taille de particule), constitué de particules sphériques de rayon $r_p > 0$ et de masse volumique $\rho_p > 0$ donnés. La fonction de distribution $f(t,x,v)$ ne dépend donc pas d'une variable de rayon. On suppose également qu'elle ne dépend pas de la température ou de l'énergie interne des particules. On pose enfin $m_p = \frac{4}{3}\pi r_p^3 \rho_p$.

Les deux modèles qui suivent sont classifiés selon une quantité cruciale qui est la fraction volumique de la phase dispersée $1 - \alpha(t,x)$ où

$$\alpha(t,x) := 1 - \frac{4}{3}\pi r_p^3 \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v,$$

désigne la fraction volumique de la phase fluide. Formellement, et suivant la classification introduite par O'Rourke [O'R81] (voir aussi [Rei96]) on considère deux régimes :

- dans le cas des **sprays fins**, la fraction volumique de la phase dispersée est négligeable devant celle du fluide ambiant, correspondant à $\alpha \sim 1$. Cette quantité n'apparaît donc pas dans les équations. Cependant, on ne néglige pas la masse de chaque particule si bien qu'un couplage entre la phase dispersée et le fluide apparaît dans les équations.

  On désigne par F_{fin} la force exercée par le fluide sur le nuage de particules et par F_{ext} une force extérieure donnée agissant sur les particules (typiquement, la force de gravité). Un modèle des sprays fins s'écrit alors

$$\begin{cases} \partial_t u + (u\cdot\nabla_x)u + \nabla_x p - \nu\Delta_x u = -\displaystyle\int_{\mathbb{R}^d} F_{\text{fin}} f\,\mathrm{d}v, \\ \operatorname{div}_x u = 0, \\ \partial_t f + v\cdot\nabla_x f + \operatorname{div}_v\left[f\dfrac{F_{\text{fin}} + F_{\text{ext}}}{m_p}\right] = 0. \end{cases} \tag{1.1.15}$$

  Il s'agit donc d'un couplage entre les équations de Navier-Stokes pour un fluide homogène incompressible (voir en Section 1.1.2.1) et une équation de Vlasov (voir en Section 1.1.3.2) avec une force $F_{\text{fin}} + F_{\text{ext}}$. De plus, on considère une *rétroaction* de la phase dispersée sur le fluide *via* la force F_{fin} (en vertu de la 3ème loi de Newton) qui s'exprime comme un terme source dans les équations de Navier-Stokes, en intégrant sur toutes les vitesses admissibles pour les particules.

[4] Dans le cadre du transport de particules, la déposition des particules au bord du domaine semble pertinente, par exemple, pour l'évolution d'un spray dans les voies respiratoires tapissées de mucus, expliquant ce choix de condition.

- dans le cas des **sprays épais**, la fraction volumique de la phase de particules n'est plus négligeable devant celle du fluide ambiant, et on considère typiquement des cas où $\alpha \sim 0,8$ ou $\alpha \sim 0,9$. La fraction volumique α pour le fluide apparaît donc dans les équations et induit un couplage supplémentaire entre les deux phases. De plus, les interactions binaires (de type collision) entre chaque particule ne sont plus négligées. Ce régime est traditionnellement considéré dans le cas d'un fluide compressible.

  En notant $F_{\text{épais}}$ la force exercée par le fluide sur le nuage de particules, un modèle des sprays épais s'écrit alors

$$\begin{cases} \partial_t(\alpha\varrho) + \mathrm{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u \otimes u) + \nabla_x p - \mu\Delta_x u - (\lambda+\mu)\nabla_x \mathrm{div}_x\, u = -\displaystyle\int_{\mathbb{R}^d} F_{\text{épais}} f \,\mathrm{d}v, \\ \partial_t f + v\cdot\nabla_x f + \mathrm{div}_v\left[f\dfrac{F_{\text{épais}}+F_{\text{ext}}}{m_p}\right] = \mathcal{Q}(f,f), \\ \alpha = 1 - \dfrac{4}{3}\pi r_p^3 \displaystyle\int_{\mathbb{R}^d} f\,\mathrm{d}v. \end{cases} \tag{1.1.16}$$

  Il s'agit ici d'un couplage entre les équations de Navier-Stokes pour un fluide compressible barotropique (voir en Section 1.1.2.2) et une équation de Vlasov-Boltzmann (voir en Section 1.1.3.2) avec force $F_{\text{épais}} + F_{\text{ext}}$ et un opérateur de collision $\mathcal{Q}(f,f)$ de type Boltzmann. Il y a toujours une rétroaction des particules sur le fluide, et la fraction volumique $\alpha(t,x)$ est présente dans les équations de masse et de moment pour le fluide.

Détaillons à présent les forces de couplage F_{fin} et $F_{\text{épais}}$ présentes dans les couplages (1.1.15) et (1.1.16), et qui traduisent l'action du fluide sur la phase dispersée. Traditionnellement, on décompose ces forces en différentes contributions. Celles retenues ici sont les suivantes :

- celle qui est commune à tous les couplages fluide-cinétique est la **force de traînée** : elle traduit le mouvement des particules au passage du fluide ambiant. On peut par exemple imaginer le lâcher d'une sphère dans un fluide en mouvement. L'expression d'une telle force est très complexe en toute généralité mais l'effet principal attendu est que la vitesse des particules tende à s'aligner sur celle du fluide.

  Une expression de la force de traînée est

$$F_{\text{tra}}(t,x,v) = \pi r_p^2 \varrho(t,x)\mathrm{C}_D|u(t,x)-v|(u(t,x)-v), \tag{1.1.17}$$

  où C_D est un paramètre adimensionnel appelé coefficient de traînée. Il dépend *a priori* de ϱ et $|u-v|$ et son évaluation se fait souvent de manière semi-empirique, en fonction du système étudié. Pour un nombre de Reynolds particulaire $\mathrm{Re}_p = 2r\varrho|u-v|/\nu$ (avec ν la viscosité du fluide) peu élevé, une simplification souvent utilisée (voir [O'R81, Duf05, DC09, Sir10]) consiste à poser

$$F_{\text{tra}}(t,x,v) = 6\pi\mu r_p(u(t,x)-v), \tag{1.1.18}$$

  réminiscente de la loi de Stokes en mécanique des fluides (voir aussi l'article fondateur de Brinkman [Bri49]). On observe que cette force est proportionnelle à l'opposée de la vitesse relative des particules par rapport au fluide. C'est cette expression (linéaire) de la force de traînée que l'on utilisera principalement dans cette thèse. La terme en $-v$ est appelé terme de friction.

- dans le cas des sprays épais, une force additionnelle (de type force d'Archimède) due aux gradients de pression dans le fluide est prise en compte. Elle s'exprime comme

$$F_{\text{press}}(t,x) = -\frac{m_p}{\rho_p}\nabla_x p(t,x), \tag{1.1.19}$$

  et provient du fait que le nuage de particules occupe un volume qui n'est plus négligeable devant celui du fluide. On pourra par exemple penser au cas d'une onde de choc traversant une surface solide et emportant une phase de particules dans son passage. Ce terme est déjà pris en compte dans l'analyse de Dukowicz [Duk80]. Dans [O'R81], O'Rourke considère (formellement) le régime des sprays épais dans des cas où cette force n'est plus négligeable devant la force de traînée, et doit donc être prise en compte.

Insistons également sur la rétroaction de la phase dispersée sur le fluide, présent au travers des termes source intégraux dans l'équation fluide de (1.1.15) et (1.1.16).

L'effet de rétroaction provenant de la force de traînée F_{tra}, et qui agit comme un forçage dans l'équation fluide, s'appelle la **force de Brinkman**. Si $F_{\text{tra}}(t,x,v) = 6\pi\mu r_p(u(t,x) - v)$, ce terme intégral s'écrit

$$\begin{aligned} -\int_{\mathbb{R}^d} F_{\text{tra}}(t,x,v)f(t,x,v)\,\mathrm{d}v &= 6\pi\mu r_p \int_{\mathbb{R}^d}(v - u(t,x)f(t,x,v)\,\mathrm{d}v \\ &= 6\pi\mu r_p(j_f(t,x) - \rho_f(t,x)u(t,x)), \end{aligned} \tag{1.1.20}$$

où ρ_f et j_f sont les densités et moments locaux des particules (voir (1.1.3) et (1.1.4)). On observe ainsi un échange de quantité mouvement entre la phase dispersée et le fluide.

Remarque 1.1.1. Terminons par quelques commentaires sur la spécificité des modèles de sprays épais

1. Contrairement au système (1.1.15), l'équation cinétique dans (1.1.16) fait intervenir l'opérateur de collisions $\mathcal{Q}(f,f)$. La présence d'un opérateur de collision est liée au fait que si la fraction volumique de la phase dispersée n'est pas négligeable dans le système, des collisions vont intervenir au sein du nuage de particules. Plus précisément, on sait que le régime généralement utilisé dans l'asymptotique de Boltzmann-Grad (pour dériver l'équation de Boltzmann) donne un volume final nul avec tout de même des collisions (voir [CIP13]). Ici, les collisions doivent donc en quelque sorte être dominantes dans le système. Ce point de vue permet de connecter les équations des sprays épais aux équations des fluides multiphasiques (voir [DM10] et la Section 1.6.1).

   Par ailleurs, l'idée générale est que dans le régime épais, on ne s'attend pas à la conservation de l'énergie cinétique de la phase dispersée au cours du temps : de l'énergie est dissipée lors du processus de collision. On considère ainsi un opérateur de collisions inélastiques de type Boltzmann (voir Section 1.1.3.2), défini par $\mathcal{Q}(f,f) = \mathcal{Q}_\beta(f,f)$ avec (voir (1.1.13))

$$\mathcal{Q}_\beta(f,g) = \mathcal{Q}^+_{B,\beta}(f,g) - \mathcal{Q}^-_{B,\beta}(f,g).$$

2. La présence du gradient de pression fluide dans l'équation cinétique est en fait réminiscente des systèmes de fluide multiphasiques, où l'on aurait traité les particules et le gaz ambiant comme deux phases macroscopiques. L'hypothèse de pression commune entre les deux phases est standard dans ce type d'équations (voir à nouveau la Section 1.6.1).

Pour finir, insistons sur le fait qu'il existe une constellation de couplages fluide-cinétique, prenant en compte de nombreux phénomènes physiques que l'on n'a pas retenus ici. Citons par exemple les phénomènes de coalescence, de fragmentation ou d'évaporation des particules ainsi que les échanges thermiques et chimiques qui peuvent avoir lieu. On renvoie par exemple à [Bar04, Duf05, DC09, Mic21]. On peut néanmoins motiver l'importance de tous les problèmes de cette thèse par le fait que les systèmes (certes idéalisés) que l'on va étudier sont des prototypes au cœur de tout couplage fluide-cinétique.

1.1.5 Systèmes considérés dans le manuscrit

Dans la suite, on normalise toutes les constantes apparaissant dans les équations (on reviendra néanmoins sur l'adimensionnement des équations au moment du Chapitre 4).

1.1.5.1 Sprays fins : le système de Vlasov-Navier-Stokes

Suivant la discussion qui précède (voir (1.1.18)), on considèrera dans cette thèse

$$F_{\text{fin}}(t,x,v) = u(t,x) - v, \quad F_{\text{ext}} = \delta G, \quad G \in \mathbb{R}^d, \quad \delta \in \{0,1\}.$$

Ce dernier terme, lorsque $\delta = 1$, traduit l'action d'un champ de force constant représentant la force de gravité subie par les particules. Cela s'inscrit dans la modélisation des phénomènes de sédimentation de particules (voir l'introduction très éclairante de la thèse de Höfer [Höf20]) On aura dans ce cas une force totale

$$u(t,x) - v + \delta G$$

dans l'équation cinétique. Partant de (1.1.15) pour les sprays fins, on obtient le **système de Vlasov-Navier-Stokes** (VNS)

$$\text{(VNS)} \begin{cases} \partial_t u + (u \cdot \nabla_x) u + \nabla_x p - \Delta_x u = \displaystyle\int_{\mathbb{R}^d} (v-u) f \, \mathrm{d}v, \\ \operatorname{div}_x u = 0, \\ \partial_t f + v \cdot \nabla_x f + \operatorname{div}_v \left[f(u-v) + \delta f G \right] = 0, \end{cases} \tag{1.1.21}$$

d'inconnues

$$f = f(t,x,v) \in \mathbb{R}^+, \quad u(t,x) \in \mathbb{R}^d.$$

Le couplage entre les deux équations s'effectue par la force de traînée et la force de Brinkman (1.1.20). On notera que contrairement au système de Vlasov-Poisson (1.1.9), cette force dépend ici de la variable de vitesse v. Par ailleurs, la force de Brinkman (terme source intégral dans les équations de Navier-Stokes) s'écrivant

$$\int_{\mathbb{R}^d} (v-u) f \, \mathrm{d}v = j_f - \rho_f u, \tag{1.1.22}$$

on observe que la dépendance de u par rapport à f ne se fait qu'au travers des moments ρ_f et j_f (définis en (1.1.3) et (1.1.4)).

Remarque 1.1.2. Notons que l'on peut aussi tenir compte de l'action de la gravité sur le fluide. Le terme de gravité, s'il est présent, peut toujours être absorbé dans le gradient de pression $\nabla_x p$ en écrivant $G = (g_1, g_2, g_3) = \nabla_x (g_1 x_1, g_2 x_2, g_3 x_3)$ en dimension 3.

1.1.5.2 Sprays épais

On considère ici, dans le régime barotropique où $p = p(\varrho)$, la force (voir (1.1.18) et (1.1.19))

$$F_{\text{épais}}(t, x, v) = u(t, x) - v - \nabla_x p[\varrho](t, x).$$

Comme pour les sprays fins, le terme $u(t, x) - v$ a un effet rétroactif dans l'équation *via* la force de Brinkman $j_f - \rho_f u$ au membre de droite de l'équation sur le moment du fluide (voir (1.1.20)). Pour le terme de pression $-\nabla_x p[\varrho](t, x)$, la rétroaction produite est

$$-\int_{\mathbb{R}^d} (-\nabla_x p[\varrho](t, x)) f(t, x, v) \, \mathrm{d}v = (1 - \alpha(t, x)) \nabla_x p[\varrho](t, x),$$

si bien que la second équation de (1.1.16) s'écrit

$$\partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u \otimes u) + \nabla_x p(\varrho) - \Delta_x u - \nabla_x \mathrm{div}_x \, u = j_f - \rho_f u + (1 - \alpha)\nabla_x p[\varrho].$$

On peut donc éliminer le dernier terme du membre de droite de l'équation, en obtenant uniquement un terme $\alpha\nabla_x p[\varrho]$ au membre de gauche. Enfin, on négligera dans cette thèse l'action d'une force extérieure sur les particules en prenant $F_{\text{ext}} = 0$ (même s'il aurait été possible d'inclure un champ de force constant comme pour le cas des sprays fins).

Partant de (1.1.16), on obtient les équations des **sprays épais** (parfois appelées couplages "Eulériens-Lagrangiens")

$$\begin{cases} \partial_t(\alpha\varrho) + \mathrm{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u \otimes u) + \alpha\nabla_x p(\varrho) - \Delta_x u - \nabla_x \mathrm{div}_x \, u = \displaystyle\int_{\mathbb{R}^d} (v - u) f \, \mathrm{d}v, \\ \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v \left[f(u - v) - f \nabla_x p(\varrho) \right] = \mathcal{Q}(f, f), \end{cases} \tag{1.1.23}$$

avec $\alpha = 1 - \int_{\mathbb{R}^d} f \, \mathrm{d}v$, d'inconnues

$$f = f(t, x, v) \in \mathbb{R}^+, \quad \varrho(t, x) \in \mathbb{R}^+, \quad u(t, x) \in \mathbb{R}^d.$$

Le couplage entre les équations a donc lieu grâce à la force de traînée en $u - v$, le gradient de pression $-\nabla_x p[\varrho]$, et la fraction volumique α.

Généralisations possibles pour (1.1.23).

1. Terme de traînée dépendant de la densité du fluide. Prenant en compte la discussion faite auparavant sur la force de traînée (voir la formule (1.1.17)), il semble naturel de vouloir considérer le cas d'une force dépendant de la densité ϱ du fluide. On souhaitera donc traiter le cas simplifié (car linéaire en ϱ) suivant

$$F_{\text{épais}}(t, x, v) = \varrho(t, x)(u(t, x) - v) - \nabla_x[p(\varrho)](t, x).$$

2. Fluide non-barotrope. On peut également vouloir s'affranchir de l'hypothèse d'un fluide barotrope (voir [O'R81]). Dans ce cas, on tient compte de l'énergie interne $\mathfrak{e}(t, x) \in \mathbb{R}^+$ du fluide (qui devient une nouvelle inconnue des équations) et la pression $p = p(\varrho, \mathfrak{e})$ dépend maintenant de la densité et de l'énergie interne du fluide. Par exemple, $p(\varrho, \mathfrak{e}) = \varrho\mathfrak{e}$ correspond au régime d'un gaz parfait. Les équations des sprays épais prennent alors la forme suivante :

$$\begin{cases} \partial_t(\alpha\varrho) + \mathrm{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u \otimes u) + \alpha\nabla_x p(\varrho, \mathfrak{e}) - \Delta_x u - \nabla_x \mathrm{div}_x u = \displaystyle\int_{\mathbb{R}^d} (v - u) f \, \mathrm{d}v, \\ \partial_t(\alpha\varrho\mathfrak{e}) + \mathrm{div}_x(\alpha\varrho\mathfrak{e}u) + p(\varrho, \mathfrak{e}) \left(\partial_t\alpha + \mathrm{div}_x(\alpha u) \right) = \displaystyle\int_{\mathbb{R}^d} |u - v|^2 f \, \mathrm{d}v, \\ \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v \Big[f(u - v) - f \nabla_x p(\varrho, \mathfrak{e}) \Big] = 0. \end{cases} \tag{1.1.24}$$

Remarque 1.1.3. Traditionnellement (et suivant la littérature physique - voir [O'R81]), le système des sprays épais (1.1.23) est écrit avec les équations d'Euler compressible plutôt qu'avec les équations de Navier-Stokes compressible : cela veut dire que le terme $-\Delta_x u - \nabla_x \mathrm{div}_x\, u$ est négligé dans l'équation sur le moment. On reviendra sur ce point au moment de l'analyse mathématique du système (voir la Section 1.6).

1.2 Cadre mathématique des couplages fluide-cinétique

Comme expliqué en Sections 1.1.4 et 1.1.5, les modèles fluide-cinétique considérés dans ce manuscrit seront donc principalement composés d'un couplage entre

– un système de Navier-Stokes, compressible ou incompressible, pour la partie fluide;

– une équation de transport-cinétique de type Vlasov.

On présente ici quelques outils et cadres généraux dont nous nous servirons dans la suite, en particulier concernant le problème de Cauchy.

1.2.1 Théorie mathématique des équations de Navier-Stokes

On commence par le cas des équations de Navier-Stokes pour le fluide[5]. Dans le cadre de ce manuscrit, on rencontrera deux cadres principaux concernant le problème de Cauchy pour équations de Navier Stokes (1.1.1) et (1.1.2). Insistons sur le fait que la liste des références données ci-dessous est bien loin d'être exhaustive.

- La première approche concerne la construction de **solutions faibles**, vérifiant les équations uniquement au sens des distributions. C'est le point de vue principal utilisé pour le système de Navier-Stokes incompressible (1.1.1) dans ce manuscrit.

  L'idée est d'utiliser le fait que le système (1.1.1) vient de la physique et possède des quantités conservées ou dissipées, comme l'énergie. On cherchera donc des solutions qui vérifient une identité d'énergie naturelle, ce qui dessine un cadre fonctionnel adéquat indiquant où chercher ces solutions. Pour simplifier, on se place dans le cas du tore plat $\Omega = \mathbb{T}^d$ (avec $d = 2, 3$).

  Voici le célèbre résultat obtenu en 1934 par Leray.

  Théorème 1.2.1 (Leray, [Ler34])**.** *Soit $u^{\mathrm{in}} \in \mathrm{L}^2(\mathbb{T}^d)$ à divergence nulle. Il existe une solution faible globale u des équations de Navier-Stokes incompressible* (1.1.1) *telle que*

$$u \in \mathrm{L}^\infty(\mathbb{R}^+; \mathrm{L}^2(\mathbb{T}^d)) \cap \mathrm{L}^2(\mathbb{R}^+; \mathrm{H}^1(\Omega)),$$

  avec condition initiale u^{in}, et vérifiant pour tout $t > 0$ et presque tout $s \in (0, t)$ ($s = 0$ inclus)

$$\|u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^d)} + 2\nu \int_0^t \|\nabla_x u(\tau)\|^2_{\mathrm{L}^2(\mathbb{T}^d)} \mathrm{d}\tau \leq \|u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^d)}.$$

  En réalité, le théorème de Leray se place dans le cadre de l'espace entier. Le cas d'un domaine borné avec condition au bord de Dirichlet a par exemple été obtenu par Hopf [Hop51] (voir plus généralement [RRS16]). Ces théorèmes s'adaptent par ailleurs au cas d'une force extérieure

[5] Si l'on considère le cas $f = 0$ (pas de particules) dans le systèmes (1.1.21) et (1.1.23), on retrouve naturellement les équations de Navier-Stokes (1.1.1) et (1.1.2). Il paraît dès lors légitime d'examiner le cas des équations fluide en tant que telles, sans couplage. En un sens, toute théorie mathématique des couplages fluide-cinétique se doit de considérer d'abord le cas du fluide seul.

$F_{\text{ext}} \in \mathrm{L}^2_{\text{loc}}(\mathbb{R}^+; \mathrm{H}^{-1}(\Omega))$. Ces solutions faibles vérifiant l'inégalité d'énergie précédente sont généralement appelées *solutions à la Leray* (ou de Leray-Hopf). Elles sont essentiellement construites par une procédure, maintenant standard, de régularisation-compacité.

Si ce type d'approche a l'avantage de produire des solutions (faibles) globales, il ne dit rien au sujet de l'unicité de telles solutions. De plus, si l'on démarre avec une donnée initiale très régulière (disons $\mathscr{C}^\infty$), on ne sait pas dire s'il existe une solution qui conserve cette régularité pour tout temps. On peut néanmoins démontrer que ces solutions sont uniques et régulières en dimension 2 (voir [Lad59]).

- Il existe également une seconde approche pour construire une solution à (1.1.1), qui exploite le fait que ce système ressemble *grosso modo* à un système d'équations de la chaleur, auquel on ajoute la non-linéarité $(u \cdot \nabla_x)u$. Celle-ci est basée sur les **invariances d'échelles** de l'équation. Suivant une méthode initiée par Kato, on tente de chercher une solution vue comme un point fixe d'une formulation de type intégral (à la Duhamel) de l'équation. Pour pouvoir espérer appliquer un théorème de point fixe, l'idée générale est alors de chercher des solutions dans un espace fonctionnel où l'influence de tous les termes de l'équation, *a priori* en compétition, est la même (on parle parfois d'espaces critiques).

  Cela permet généralement d'obtenir existence et unicité de solutions fortes au moins localement au temps. On espère pouvoir par contre prouver existence globale, unicité et dépendance continue par rapport aux données dans des espaces critiques, pour des données initiales petites. Insistons sur le fait que des données initiales quelconques ne donnent en général que l'existence locale en temps.

  On n'utilisera pas directement ce type d'approche pour (1.1.2) dans ce manuscrit et on renvoie donc directement aux références suivantes (voir de façon plus générale [BCD11]) : mentionnons les travaux de Fujita et Kato [FK64], Kato [Kat84], Chemin [Che92] Cannonne, Meyer et Planchon [CMP94] ou encore Koch et Tataru [KT01].

Le problème de Cauchy dans le cas compressible (1.1.2) a également été très étudié. Contrairement à (1.1.1), on a affaire à un couplage entre une équation de type hyperbolique sur ϱ et une équation de type parabolique sur u. Une nouvelle difficulté liée à la présence de vide, c'est-à-dire de régions où $\varrho = 0$, fait également son apparition. Dans ce manuscrit, on s'intéressera à la construction de solutions à très forte régularité et en temps court, mais les deux approches standards présentées auparavant, pour le cas incompressible, existent cependant.

- Pour la construction de solutions faibles dans des espaces d'énergie, le pendant du Théorème 1.2.1 de Leray a d'abord été obtenu par Lions [Lio98] (pour des pressions $p(\varrho) = \rho^\gamma$ avec $\gamma > 3d/(d+2)$) dans les années 1990 (voir aussi les travaux de Hoff [Hof87] considérant des solutions ayant des discontinuités). Mentionnons l'important élargissement à des exposants $\gamma > d/2$ permis par [FNP01]. De nombreuses extensions de ces résultats ont été données récemment : elles concernent la prise en compte de lois de pression non monotones, de viscosités anisotropes et de coefficients λ, μ dépendant de la densité. On renvoie par exemple aux travaux [BD03, BD06b, MV07, BJ18].

- L'existence de solutions régulières (locales en temps ou globales avec des hypothèses sur les données) a par exemple été considérée dans [Ser59, Nas62] dans un cadre classique, dans [Sol80] avec des solutions mild, et dans [MN80, MN83] pour des solutions à hautes régularités Sobolev près d'un d'équilibre. Concernant la construction de solutions dans des espaces invariants par changement d'échelles (dans le régime barotropique), on renvoie aux résultats de Danchin [Dan00, Dan01b, Dan01a, Dan05b], Danchin et Charve [CD10], Danchin, Fanelli et Paicu [DFP20], et Danchin et Tolksdorff [DT22].

1.2.2 Théorie générale du transport (cadre cinétique)

Dans cette section, on s'intéresse aux équations cinétiques du type Vlasov (1.1.8) avec le point de vue général des équations de transport. On considère le cas d'un domaine à bord Ω, puisqu'il en sera directement question dans les Chapitres 2-3-4. Ceci étant, les résultats demeurent vrais dans les cas sans bord $\Omega = \mathbb{R}^d$ ou $\mathbb{T}^d$.

Soit $T > 0$ et $\Omega \subset \mathbb{R}^d$ un ouvert Lipschitz (borné ou non) tel que sa normale extérieure $n(x)$ est bien définie en tout point $x \in \partial\Omega$. On considère également[6] un champ de vecteurs

$$F : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d.$$

Pour une donnée initiale f^{in} définie sur l'espace des phases $\Omega \times \mathbb{R}^d$ et prescrite au temps $t = 0$, on s'intéresse à l'équation de Vlasov

$$\partial_t f + v \cdot \nabla_x f + \operatorname{div}_v [\mathrm{F} f] = 0, \quad (t, x, v) \in (0, T) \times \Omega \times \mathbb{R}^d. \tag{1.2.1}$$

Il s'agit bien d'un cas particulier d'une équation de transport de type conservatif (ou encore équation de continuité) d'inconnue w, sous la forme

$$\partial_t w + \operatorname{div}_z(wb) = 0, \quad \text{où} \quad z = (x, v), \quad b(t, x, v) = (v, \mathrm{F}(t, x, v)).$$

Par exemple, le cas du système de Vlasov-Navier-Stokes (1.1.21) (resp. le système des sprays épais (1.1.23)) correspond à $\mathrm{F}(t, x, v) = u(t, x) - v$ (resp. à $\mathrm{F}(t, x, v) = u(t, x) - v - \nabla_x[p(\varrho)](t, x)$) avec u (resp. (ϱ, u)) solution des équations de Navier-Stokes associées. Dans les cas qui nous intéresseront, le champ de vecteurs F sera en fait toujours solution d'une autre équation couplée à l'équation (1.2.1).

Par analogie avec une équation de transport classique, où l'on doit imposer une condition au bord pour des trajectoires rentrantes dans le domaine, on définit ici

$$\begin{aligned} \Sigma^{\pm} &:= \Big\{(x, v) \in \partial\Omega \times \mathbb{R}^d \mid \pm v \cdot n(x) > 0\Big\}, \\ \Sigma_0 &:= \Big\{(x, v) \in \partial\Omega \times \mathbb{R}^d \mid v \cdot n(x) = 0\Big\}, \\ \Sigma &:= \Sigma^+ \sqcup \Sigma^- \sqcup \Sigma_0 = \partial\Omega \times \mathbb{R}^d. \end{aligned} \tag{1.2.2}$$

L'ensemble Σ correspond au bord de l'espace des phases, et le sous-ensemble Σ^- au bord rentrant de celui-ci. C'est sur la partie Σ^- que l'on peut prescrire une condition pour la fonction de distribution f solution à l'équation (1.2.1). Pour un certaine source g, on impose

$$\forall t > 0, \quad f(t)_{|\Sigma^-} = g. \tag{1.2.3}$$

Dans le cas $g = 0$, qui nous intéressera par la suite, cela signifie qu'une particule rencontrant le bord (de façon non rasante) est absorbée et on parle de **condition d'absorption au bord**. Dit autrement, aucune trajectoire ne peut rentrer dans le système une fois sortie. En particulier, on n'a plus nécessairement conservation de la masse totale au cours de la dynamique. Cette condition d'absorption sera à l'origine de plusieurs mécanismes intéressants pour l'analyse du système de Vlasov-Navier-Stokes. On renvoie à la Section 1.1.3.3 pour la définition d'autres conditions au bord.

[6]Dans la pratique, il s'agira souvent de l'extension appropriée d'un champ de vecteurs défini sur $[0, T] \times \Omega \times \mathbb{R}^d$.

Jusqu'à la fin de cette section, on considère donc principalement le problème

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v\,[Ff] = 0, \quad (t,x,v) \in (0,T) \times \Omega \times \mathbb{R}^d, \\ f_{|t=0} = f^{\mathrm{in}}, \\ f_{|\Sigma^-} = 0, \end{cases} \tag{1.2.4}$$

1.2.2.1 Le cas d'une force régulière et de données régulières

Un point de vue clé sur les équations de transport cinétique, que l'on utilisera en permanence dans ce manuscrit, est le point de vue Lagrangien : on peut relier l'étude de l'équation (1.2.1) au flot du champ de vecteurs F, *via* un système d'équations différentielles.

Si F est lisse et globalement Lipschitzien en la variable (x, v), on peut définir, pour tout $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ et $t \in [0, T]$, l'unique solution $s \mapsto (\mathrm{X}^{s;t}(x, v), \mathrm{V}^{s;t}(x, v))$ sur $[0, T]$ du système d'équations différentielles suivant

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{X}^{s;t}(x,v) = \mathrm{V}^{s;t}(x,v), \quad \mathrm{X}^{t;t}(x,v) = x, \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}^{s;t}(x,v) = \mathrm{F}(s, \mathrm{X}^{s;t}(x,v), \mathrm{V}^{s;t}(x,v)), \quad \mathrm{V}^{t;t}(x,v) = v. \end{cases} \tag{1.2.5}$$

Dans la suite, on notera pour $s, t \in [0, T]$ et $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$

$$\mathrm{Z}^{s;t}(x,v) =: (\mathrm{X}^{s;t}(x,v), \mathrm{V}^{s;t}(x,v)),$$

et on rappelle la propriété de semi-groupe

$$\forall t_1, t_2, t_3 \in [0,T], \quad \mathrm{Z}^{t_3;t_1} := \mathrm{Z}^{t_3;t_2} \circ \mathrm{Z}^{t_2;t_1}.$$

De plus, pour tout $s, t \in [0, T]$, l'application $(x, v) \mapsto \mathrm{Z}^{s;t}(x, v)$ est un $\mathscr{C}^1$ difféomorphisme de $\mathbb{R}^d \times \mathbb{R}^d$ dans $\mathbb{R}^d \times \mathbb{R}^d$, d'inverse $(\mathrm{Z}^{s,t})^{-1} = \mathrm{Z}^{t,s}$ et de déterminant Jacobien valant $J^{s,t}(x, v) = \exp\left(\int_t^s \mathrm{div}_v \mathrm{F}(\tau, \mathrm{Z}^{\tau;t}(x, v))\,\mathrm{d}\tau\right)$.

La méthode dite des caractéristiques, énoncée ici dans le cas sans bord $\Omega = \mathbb{R}^d$, est la suivante.

Théorème 1.2.2 (Cas $\Omega = \mathbb{R}^d$). *On suppose que* $\mathrm{F} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ *est un champ de vecteurs satisfaisant*

$$\mathrm{F} \in \mathscr{C}^1([0,T] \times \mathbb{R}^d \times \mathbb{R}^d) \cap \mathrm{L}^1(0,T; \mathrm{W}^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)).$$

Soit $f^{\mathrm{in}} \in \mathscr{C}^1(\mathbb{R}^d \times \mathbb{R}^d)$. *Alors il existe une unique solution* $f \in \mathscr{C}^1([0, T] \times \mathbb{R}^d \times \mathbb{R}^d)$ *de l'équation de Vlasov* (1.2.1) *avec donnée initiale* f^{in}. *Elle est donnée par la formule :*

$$\forall t \geq 0, \forall (x,v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad f(t,x,v) = f^{\mathrm{in}}(\mathrm{Z}^{0;t}(x,v)) \exp\left(-\int_0^t \mathrm{div}_v \mathrm{F}(\tau, \mathrm{Z}^{\tau;t}(z))\,\mathrm{d}\tau\right). \tag{1.2.6}$$

Ainsi, *modulo* un terme quantifiant la variation du volume due au champ F, la solution f est constante le long des courbes caractéristiques. En particulier, la solution reste positive au court du temps si elle l'est initialement.

Remarque 1.2.3. La formule de représentation (1.2.6) est en fait équivalente à

$$\forall t \geq 0, \quad f(t,\cdot)\,\mathrm{d}x\,\mathrm{d}x = \mathrm{Z}^{t;0}\sharp\left(f^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v\right),$$

où $\mathrm{Z}^{t;0}\sharp\mu$ est la mesure-image (push-forward) de la mesure μ par l'application $\mathrm{Z}^{t;0}$.

Dans le cas d'un domaine à bord, il est aussi possible d'obtenir une formule de représentation pour les solutions de (1.2.4), en suivant précisément les trajectoires et leur temps de vie à l'intérieur du domaine (on précisera ce cas au cours des sections suivantes si nécessaire). Parmi les premiers résultats sur les équations de transport dans un domaine, mentionnons les travaux de Bardos [Bar70] pour les équations de transport générales avec champ de vecteurs Lipschitz et à l'aide des caractéristiques (voir aussi Ukai [Uka67]).

1.2.2.2 Théorie de DiPerna et Lions dans un domaine à bord

Lorsque le champ de vecteurs F n'est plus régulier (disons moins que Lipschitz en la variable d'espace), on sort du cadre standard fourni par le Théorème de Cauchy-Lipschitz et il peut être difficile de résoudre les EDO du système (1.2.5). En particulier, on ne peut plus définir les courbes caractéristiques de façon classique et le Théorème 1.2.2 ne s'applique donc pas. Dans le cadre de cette thèse, il ne s'agit pas d'une question anecdotique : les champs de vecteurs considérés seront typiquement des solutions faibles des équations de Navier-Stokes et ont juste une régularité de type Sobolev en espace (voir le Théorème 1.2.1 de Leray).

On cherche donc à obtenir une bonne théorie de Cauchy pour l'équation (1.2.1) lorsque le champ F est peu régulier, et pour des données initiales peu régulières elles-aussi. On s'intéresse à des solutions faibles de (1.2.1) au sens des distributions. Pour le problème au bord (1.2.4) avec condition d'absorption, celles-ci sont définies de la façon suivante.

Définition 1.2.4. *Supposons* $\mathrm{F} \in \mathrm{L}^1(0,T;\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^d))$*. Soit* $f^{\mathrm{in}} \in \mathrm{L}^1 \cap \mathrm{L}^\infty(\Omega \times \mathbb{R}^d)$*. On dit que* $f \in \mathrm{L}^\infty((0,T) \times \Omega \times \mathbb{R}^d)$ *est une solution faible de* (1.2.4) *si pour toute fonction test* $\varphi \in \mathscr{C}^\infty_c([0,T) \times \overline{\Omega} \times \mathbb{R}^d)$ *nulle sur* $[0,T) \times (\Sigma^+ \sqcup \Sigma_0)$*, on a*

$$\int_0^T \int_{\Omega \times \mathbb{R}^d} f\left[\partial_t \varphi + v \cdot \nabla_x \varphi + \mathrm{F} \cdot \nabla_x \varphi\right] \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v = -\int_{\Omega \times \mathbb{R}^d} f^{\mathrm{in}} \varphi(0)\,\mathrm{d}x\,\mathrm{d}v.$$

La condition d'absorption est prise en compte dans le choix de la fonction test. On notera que cette formulation est en particulier satisfaite dans $\mathscr{D}'((0,T) \times \Omega \times \mathbb{R}^d)$.

Pour un champ F peu régulier (disons localement intégrable en temps-espace, et à divergence bornée) et une donnée f^{in} intégrable et bornée, l'existence d'une solution faible globale au sens précédent peut être obtenue par régularisation-compacité, en se servant de la linéarité de l'équation.

Le cœur de l'affaire réside dans l'unicité et la stabilité (par rapport au champ F et aux conditions initiales) de telles solutions. Rappelons que l'on n'a plus accès à la formule de représentation (1.2.6). Identifier un cadre minimal (pour la régularité du champ de vecteurs) assurant l'unicité à une équation de transport est en fait un enjeu complexe (voir [Aiz78, Dep02]).

La théorie de DiPerna et Lions [DL89c], initiée dans les années 80, s'attache ainsi à identifier des conditions suffisantes sur le champ F pour obtenir unicité et stabilité des solutions faibles. Ces conditions s'énoncent sous la forme suivante :

- **Hypothèse A1** : $\mathrm{F} \in \mathrm{L}^1(0,T;\mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^d))$,
- **Hypothèse A2** : $\mathrm{div}_v\,\mathrm{F} \in \mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{R}^d \times \mathbb{R}^d))$,
- **Hypothèse A3** : $(t,x,v) \mapsto (1+|(x,v)|)^{-1}\mathrm{F}(t,x,v) \in \mathrm{L}^1(0,T;\mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^d \times \mathbb{R}^d))$.

Une idée essentielle est d'introduire le concept de **solutions renormalisées** pour (1.2.1) : on dit qu'une solution faible de (1.2.1) est renormalisée si pour tout $\beta \in \mathscr{C}^1(\mathbb{R};\mathbb{R})$, la fonction $\beta(f)$ est encore une solution faible de (1.2.1), c'est-à-dire

$$\partial_t \beta(f) + v \cdot \nabla_x \beta(f) + \mathrm{div}_v(\mathrm{F}\beta(f)) = (\beta(f) - f\beta'(f))\mathrm{div}_v \mathrm{F}. \tag{1.2.7}$$

On dit que le champ de vecteur F possède la propriété de renormalisation si toute solution faible de l'équation de continuité associée est renormalisée. Cette propriété mène formellement à l'unicité pour (1.2.1) en choisissant $\beta(x) = x^2$ et est satisfaite pour des solutions classiques et un champ F régulier.

Définir une notion appropriée de **trace au bord** pour les solutions faibles d'une équation de transport (cinétique), et donc de type hyperbolique, n'est pas direct. En effet, dans un cadre à régularité faible et pour des données dans des espaces de Lebesgue, on ne s'attend pas à obtenir des solutions à régularité de type Sobolev en espace (qui permettrait de définir leur trace au bord de façon usuelle).

Dans le cas du transport libre cinétique (1.1.6), Cessenat [Ces84] construit cependant une trace au bord en un sens faible grâce à l'annulation de l'opérateur de transport libre (voir aussi les travaux [Ham92, CC91, AC93, Hei99] pour le cas de l'équation de Boltzmann dans un domaine). Ce type de résultat a été revisité et étendu par Mischler [Mis00b] et Boyer dans [Boy05] (voir aussi le livre de [BF12]) dans le cadre de la théorie de DiPerna et Lions.

Le point essentiel est que l'on peut définir une trace faible pour une solution d'une équation de transport-cinétique, précisément *grâce* à l'équation qui est satisfaite au sens des distributions. On peut démontrer (voir [Mis00b]) l'existence d'une trace $\gamma f \in \mathrm{L}^\infty_{\mathrm{loc}}((0,T)\times\Sigma, \mathrm{d}t\mathrm{d}\sigma\mathrm{d}v)$ (la mesure σ étant la mesure superficielle sur $\partial\Omega$), et l'équation (1.2.1) admet alors une formulation dans $\mathscr{D}'([0,T]\times\overline{\Omega}\times\mathbb{R}^d)$ faisant apparaître ce terme. Plus précisément, pour toute fonction test $\varphi \in \mathscr{C}^\infty_c([0,T]\times\overline{\Omega}\times\mathbb{R}^d)$ et pour tout $t\in[0,T]$

$$\int_0^t \int_{\Omega\times\mathbb{R}^d} f\left[\partial_t\varphi + v\cdot\nabla_x\varphi + \mathrm{F}\cdot\nabla_x\varphi\right]\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v$$
$$= \int_{\Omega\times\mathbb{R}^d} f(t)\varphi(t)\,\mathrm{d}x\,\mathrm{d}v - \int_{\Omega\times\mathbb{R}^d} f^{\mathrm{in}}\varphi(0)\,\mathrm{d}x\,\mathrm{d}v + \int_0^t\int_{\partial\Omega\times\mathbb{R}^d}(\gamma f)\varphi v\cdot n(x)\,\mathrm{d}s\,\mathrm{d}\sigma(x)\,\mathrm{d}v.$$

On peut alors obtenir une théorie de Cauchy pour les solutions renormalisées de (1.2.4) dans un domaine à bord. Un énoncé informel est le suivant.

Théorème 1.2.5 (DiPerna et Lions [DL89c], Mischler [Mis00b], Boyer [Boy05])**.** *Soit* F *un champ de vecteurs satisfaisant les hypothèses **A1-A2-A3**. Soit* $f^{\mathrm{in}}\in\mathrm{L}^1\cap\mathrm{L}^\infty(\Omega\times\mathbb{R}^d)$. *Alors*

- ***Unicité** : il existe une unique solution faible f de l'équation* (1.2.4) *sur* $[0,T]$ *au sens de la Définition 1.2.4. De plus, on a* $f\in\mathscr{C}([0,T];\mathrm{L}^p(\Omega\times\mathbb{R}^d))$ *pour tout* $p\in[1,\infty)$.
- ***Renormalisation** : pour tout* $\beta\in\mathscr{C}^1(\mathbb{R};\mathbb{R})$, $\beta(f)$ *satisfait l'équation* (1.2.7) *au sens des distributions.*
- ***Stabilité** : Il y a stabilité (séquentielle) forte de l'unique solution faible f, c'est-à-dire continuité*[7] *de l'application* $(\mathrm{F}, f^{\mathrm{in}})\mapsto f$.

Cette théorie a été utilisée dans différents contextes : citons par exemple (dans un domaine sans bord) la construction de solutions faibles renormalisées pour le système de Vlasov-Poisson [DL88] ou le système de Vlasov-Maxwell [DL89a], ou la construction de solutions à l'équation de transport pour le système de Navier-Stokes compressible [Lio98]. On renvoie aussi à son utilisation par Mischler dans [Mis00a, Mis10] (dans des domaines à bord).

La stabilité (séquentielle) des solutions renormalisées fournit un outil très puissant pour manipuler les solutions faibles de (1.2.4) : étant donnée l'unique solution faible renormalisée de l'équation

[7] *Grosso modo* pour des topologies associées à des espaces L^p $(1\le p<\infty)$ en position-vitesse.

associée à des données $(\mathrm{F}, f^{\mathrm{in}})$, on peut régulariser ces dernières pour obtenir une suite $(\mathrm{F}_k, f_k^{\mathrm{in}}) \longrightarrow (\mathrm{F}, f^{\mathrm{in}})$ pour lesquelles on a des solutions classiques (f_k) (données par les caractéristiques). On peut facilement mener des calculs sur l'équation de continuité satisfaite par les f_k et obtenir des estimations sur la solution faible de départ grâce à la propriété de stabilité forte. Il s'agit d'une procédure standard dont on se servira très souvent (implicitement) dans ce manuscrit.

Remarque 1.2.6. La théorie initiée par DiPerna et Lions ne s'arrête pas là. On peut chercher à abaisser l'hypothèse de régularité sur le champ F de sorte que la propriété de renormalisation ait toujours lieu, et donc unicité et stabilité pour l'équation de transport. Dans [Amb04], Ambrosio montre que la propriété de renormalisation est satisfaite pour les équations de transport si le champ de vecteurs a une *régularité BV en espace* (voir [CDS14, ACM05] pour l'extension de ce cadre aux domaines à bord). L'étude de la notion de flot dans ce contexte à régularité faible a quant à elle été initiée par Crippa et De Lellis (voir [DL06, CDL08, Amb17]).

1.3 Le système de Vlasov-Navier-Stokes

Dans cette section, nous abordons l'étude du système de Vlasov-Navier-Stokes (1.1.21) (pour le cas des sprays fins), que l'on rappelle ici :

$$\begin{cases} \partial_t u + (u \cdot \nabla_x) u + \nabla_x p - \Delta_x u = \displaystyle\int_{\mathbb{R}^d} (v-u) f \, \mathrm{d}v, \\ \mathrm{div}_x \, u = 0, \\ \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v \, [f(u-v)] = 0, \\ f_{|t=0} = f^{\mathrm{in}}, \; u_{|t=0} = u^{\mathrm{in}}. \end{cases} \tag{1.3.1}$$

Comme expliqué en Section 1.1.5, le couplage entre fluide et particules a lieu *via* la force de traînée $u - v$, produisant un terme source (force de Brinkman)

$$\int_{\mathbb{R}^d} (v-u) f \, \mathrm{d}v = j_f - \rho_f u,$$

en source de l'équation fluide. Les équations sont non-linéaires et, à cause du point de vue fluide-cinétique, les inconnues $f(t,x,v)$ et $u(t,x)$ ne dépendent pas du même jeu de variables.

Le système (1.3.1) est un couplage entre une système de type parabolique pour le fluide et une équation de transport cinétique. En comparaison, mentionnons le système de Vlasov-Poisson (1.1.9) qui couple une équation de transport dans l'espace des phases avec une équation elliptique, tandis que le système de Vlasov-Maxwell (1.1.10) est un couplage avec des équations de nature hyperbolique, de type ondes (voir Section 1.1.3).

Pour simplifier, on se place dans le cas d'un domaine spatial sans bord, comme le tore $\mathbb{T}^d$.

1.3.1 Estimations *a priori*

On présente tout d'abord les principales estimations *a priori* disponibles pour (1.3.1). Celles-ci fournissent des informations cruciales pour construire des solutions au système, mais aussi pour étudier leur comportement (voir en Section 1.4).

Estimations d'énergie. Supposons que (1.3.1) possède une solution lisse (f,u). En prenant le produit scalaire de l'équation de Navier-Stokes par $u(t,x)$ et en intégrant sur $\mathbb{T}^d$, on obtient par intégration par parties

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^d} |u|^2 \, \mathrm{d}x + \int_{\mathbb{T}^d} |\nabla_x u|^2 \, \mathrm{d}x = \int_{\mathbb{T}^d \times \mathbb{R}^d} u \cdot (v-u) f \, \mathrm{d}x \, \mathrm{d}v. \tag{1.3.2}$$

Ici, on a utilisé le fait que

$$\int_{\mathbb{T}^d} (u \cdot \nabla_x) u \cdot u \,\mathrm{d}x = -\frac{1}{2} \int_{\mathbb{T}^d} |u|^2 \mathrm{div}_x\, u \,\mathrm{d}x = 0, \quad \int_{\mathbb{T}^d} \nabla_x p \cdot u \,\mathrm{d}x = -\int_{\mathbb{T}^d} p \mathrm{div}_x\, u \,\mathrm{d}x = 0,$$

par intégration par parties et grâce à la condition de divergence nulle. D'autre part, en multipliant l'équation de Vlasov par $|v|^2/2$ et en intégrant sur $\mathbb{T}^d \times \mathbb{R}^d$, on obtient après intégration par parties

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^d \times \mathbb{R}^d} f |v|^2 \,\mathrm{d}x \,\mathrm{d}v = \int_{\mathbb{T}^d \times \mathbb{R}^d} v \cdot (u - v) f \,\mathrm{d}x \,\mathrm{d}v. \tag{1.3.3}$$

En sommant les deux identités (1.3.2) et (1.3.3) précédentes, on aboutit à

$$\mathrm{E}(f, u)(t) + \int_0^t \mathrm{D}(f, u)(\tau) \,\mathrm{d}\tau = \mathrm{E}(f^{\mathrm{in}}, u^{\mathrm{in}}), \tag{1.3.4}$$

où l'on a défini l'**énergie cinétique totale** par

$$\mathrm{E}(f, u)(t) := \frac{1}{2} \int_{\mathbb{T}^d} |u(t)|^2 \,\mathrm{d}x + \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} f(t) |v|^2 \,\mathrm{d}x \,\mathrm{d}v, \tag{1.3.5}$$

et la **dissipation totale** par

$$\mathrm{D}(f, u)(t) := \int_{\mathbb{T}^d} |\nabla_x u(t)|^2 \,\mathrm{d}x + \int_{\mathbb{T}^d \times \mathbb{R}^d} f(t) |v - u(t)|^2 \,\mathrm{d}x \,\mathrm{d}v. \tag{1.3.6}$$

On observe que $\mathrm{E}(f, u)$ est exactement la somme de l'énergie cinétique du fluide et du nuage de particules, tandis que $\mathrm{D}(f, u)$ est la somme de la dissipation interne au fluide (visqueux) et de la dissipation liée au couplage entre les deux phases. La remarquable identité (1.3.4), dite **d'énergie-dissipation**, est une propriété fondamentale du couplage (1.3.1). Elle sera au cœur des Chapitres 2–3–4. Dans la suite, ces deux fonctionnelles seront souvent notées E et D pour alléger l'écriture.

Si $\mathrm{E}(f^{\mathrm{in}}, u^{\mathrm{in}}) < \infty$, l'inégalité d'énergie précédente dessine alors un premier cadre fonctionnel minimal pour chercher de solutions, à savoir :

$$u \in \mathrm{L}^\infty(\mathbb{R}^+; \mathrm{L}^2(\mathbb{T}^d)) \cap \mathrm{L}^2(\mathbb{R}^+; \mathrm{H}^1(\mathbb{T}^d)), \quad |v|^2 f \in \mathrm{L}^\infty(\mathbb{R}^+; \mathrm{L}^1(\mathbb{T}^d \times \mathbb{R}^d)).$$

Principes du maximum *via* les caractéristiques. Comme expliqué en Section 1.2.2, il est possible de voir l'équation de Vlasov présente dans (1.3.1) comme une équation de continuité de la forme (1.2.1) sur $\mathbb{T}^d \times \mathbb{R}^d$, pour le champ de vecteurs

$$\mathrm{F}(t, x, v) = u(t, x) - v, \quad u \text{ solution des équations de Navier-Stokes dans (1.3.1).}$$

Considérons alors les courbes caractéristiques pour le système de Vlasov-Navier-Stokes (1.3.1), définies par

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s} \mathrm{X}^{s;t}(x, v) = \mathrm{V}^{s;t}(x, v), \quad \mathrm{X}^{t;t}(x, v) = x, \\
&\frac{\mathrm{d}}{\mathrm{d}s} \mathrm{V}^{s;t}(x, v) = u(s, \mathrm{X}^{s;t}(x, v)) - \mathrm{V}^{s;t}(x, v), \quad \mathrm{V}^{t;t}(x, v) = v.
\end{aligned}$$

Pour une donnée initiale f^{in}, la solution f de l'équation de Vlasov s'écrit alors

$$f(t, x, v) = e^{dt} f^{\mathrm{in}}(\mathrm{X}^{0;t}(x, v), \mathrm{V}^{0;t}(x, v)),$$

puisque $\mathrm{div}_v\, [u(t, x) - v] = -d$. La positivité de la donnée initiale est donc propagée au cours de l'évolution. On obtient également les bornes *a priori* suivantes sur f :

$$\forall t > 0, \quad \|f(t)\|_{\mathrm{L}^\infty_{x,v}} = e^{dt} \|f^{\mathrm{in}}\|_{\mathrm{L}^\infty_{x,v}}, \quad \|f(t)\|_{\mathrm{L}^1_{x,v}} = \|f^{\mathrm{in}}\|_{\mathrm{L}^1_{x,v}}. \tag{1.3.7}$$

1.3.2 Problème de Cauchy pour le système de Vlasov-Navier-Stokes

1.3.2.1 Solutions faibles globales

Dans cette section, on explique comment construire une solution faible globale (f, u) pour (1.3.1).

L'idée générale est la suivante : f étant donnée, on peut résoudre les équations de Navier-Stokes par la méthode de Leray; tandis que pour un champ de vecteurs (Sobolev) u donné, on sait construire une solution de l'équation de Vlasov par la théorie de DiPerna et Lions. Cependant, comme les équations sont fortement couplées (par le terme de traînée et la force de Brinkman), on ne peut pas résoudre l'une des équations et se servir du résultat pour résoudre l'autre. On va donc utiliser une méthode de point fixe pour en quelque sorte découpler les équations.

A cause des termes $(u-v)f$ et $(u \cdot \nabla_x)u$ dans les équations, on est également obligé d'appliquer cette méthode sur un système où l'on a régularisé et tronqué ces termes non-linéaires. Il s'agit ensuite de montrer que l'on peut remonter jusqu'au système initial dont on cherche une solution. Ce processus de troncature/régularisation doit permettre de contrôler les termes non-linéaires du type $(u-v)f$, tout en préservant l'inégalité d'énergie-dissipation (1.3.4) et les bornes (1.3.7) auparavant obtenues. Le choix de troncature/régularisation est donc l'étape cruciale pour de tels couplages.

Il existe plusieurs travaux construisant des solutions faibles par des méthodes plus ou moins proches : citons Anoshchenko et Boutet de Monvel [ABdMB97] dans un domaine borné (avec réflection spéculaire), Hamdache [Ham98] dans le cas Stokes instationnaire pour le fluide dans un domaine borné, Boudin Desvillettes, Grandmont et Moussa [BDGM09] pour le cas du tore, ou encore Boudin, Grandmont et Moussa [BGM17] dans le cas général d'un domaine borné mobile avec condition d'absorption au bord (voir aussi Boudin, Michel et Moussa [BMM20]).

Jusqu'à la fin de cette sous-section, on se place dans le cas du tore $\mathbb{T}^d$ (avec $d = 2$ ou $d = 3$) pour simplifier la présentation. La méthode est cependant assez robuste pour traiter de domaines à bord, bornés ou non, avec diverses conditions au bords[8]. On présente ici une technique utilisée dans les travaux [BGM17, BMM20].

Sans perte de généralité, on travaille sur un intervalle de temps $[0, T]$ avec $T > 0$ fixé. Pour tout $n \in \mathbb{N}$, on introduit le projecteur orthogonal suivant :

$$P_n : \mathrm{L}^2(\mathbb{T}^d) \longrightarrow F_n := \{u \in \mathrm{L}^2(\mathbb{T}^d) \mid \forall |k| \leq n, \quad \widehat{u}_k = 0\},$$

qui permet de tronquer les fréquences d'une fonction en éliminant les modes supérieurs à n. On considère alors le système tronqué et régularisé[9] suivant :

$$\partial_t u + P_n \mathbb{P}(u \cdot \nabla_x)u - \Delta_x u = P_n \mathbb{P} \int_{\mathbb{R}^d} \chi(v-u) f \, \mathrm{d}v, \tag{1.3.8}$$

$$\mathrm{div}_x\, u = 0, \tag{1.3.9}$$

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v \left[f \chi(u-v) \right] = 0, \tag{1.3.10}$$

$$f_{|t=0} = \eta f^{\mathrm{in}}, \quad u_{n|t=0} = P_n u^{\mathrm{in}}. \tag{1.3.11}$$

Ici, $\chi \in \mathscr{C}^\infty(\mathbb{R})$ est une fonction impaire bornée satisfaisant $0 \leq z \cdot \chi(z)$ et $|\chi(z)| \leq |z|$, et pour tout $w \in \mathbb{R}^d$, on note $\chi(w)$ le vecteur où l'on a appliqué χ sur chaque coordonnée. La fonction η

[8] On verra apparaître une variante de cette stratégie dans l'Appendice 3.B du Chapitre 3, prenant en compte le terme gravité dans l'équation cinétique.

[9] Notons que la régularisation peut aussi s'effectuer par une convolution avec une fonction lisse, une projection sur les modes propres de l'opérateur de Stokes $-\mathbb{P}(\Delta_x)$, ou encore une régularisation de Yosida.

est quant à elle une troncature (lisse et bornée) en vitesse. Enfin, $\mathbb{P}$ est le projecteur de Leray sur les champs de vecteurs à divergence nulle.

La méthode générale est alors la suivante :

- Pour tout $n \in \mathbb{N}$, on construit une solution (f, u) aux équations (1.3.8)–(1.3.9)–(1.3.10)–(1.3.11). On utilise pour cela un théorème de point fixe (par exemple de Schauder). Cela nécessite d'invoquer les estimations d'énergie du système régularisé, ainsi que des propriétés de stabilité forte fournies par la théorie de DiPerna et Lions.

- On considère une suite (χ_n) et (η_n) satisfaisant les mêmes propriétés qu'auparavant et telles que $\chi_n \to Id$ et $\eta_n \to 1$ lorsque $n \to +\infty$. On considère alors la suite $(f_n, u_n)_{n\in\mathbb{N}}$ de solutions de (1.3.8)–(1.3.9)–(1.3.10)–(1.3.11), fournie par la première étape. Le but est de montrer que, à sous-suite près, cette suite converge vers une solution des équations de Vlasov-Navier-Stokes.

  Les estimations fournies par la première étape permettent d'obtenir de la compacité faible sur la suite. Par le lemme d'Aubin-Lions, on peut récupérer de la compacité forte sur la partie fluide (u_n). Cette convergence forte est ensuite transférée à (f_n) par stabilité des solutions renormalisées. On peut enfin passer à la limite dans tous les termes de l'équation et obtenir une solution (f, u).

On est alors en mesure d'obtenir un résultat du type :

Théorème 1.3.1 (Boudin, Desvillettes, Grandmont et Moussa [BDGM09])**.** *Soit $u^{\text{in}} \in \mathrm{L}^2(\mathbb{T}^d)$ à divergence nulle. Soit $f^{\text{in}} \in \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ telle que $f^{\text{in}} \geq 0$ et $\int_{\mathbb{T}^d\times\mathbb{R}^d} f^{\text{in}}|v|^2 \,\mathrm{d}x\,\mathrm{d}v < \infty$. Il existe une solution faible globale (f, u) du système* (1.3.1) *avec condition initiale $(f^{\text{in}}, u^{\text{in}})$, où*

- *u est une solution de Leray des équations de Navier-Stokes;*
- *f est une solution renormalisée positive de l'équation de Vlasov, au sens de DiPerna et Lions.*

De plus, pour tout $t > 0$ et presque tout $s \in (0, t)$ ($s = 0$ inclus), on a

$$\mathrm{E}(f, u)(t) + \int_s^t \mathrm{D}(f, u)(\tau)\,\mathrm{d}\tau \leq \mathrm{E}(f, u)(s).$$

Dans la suite, on appellera *condition initiale admissible* toute paire $(f^{\text{in}}, u^{\text{in}})$ qui satisfait les conditions du Théorème 1.3.1. Le théorème précédent sera le point de départ pour toute notre analyse du système de Vlasov-Navier-Stokes.

Remarque 1.3.2. Le cadre présenté ici (voir [BGM17]) permet d'énoncer le même type de théorème sur un domaine à bord, avec condition d'absorption au bord pour la partie cinétique (voir Section 1.2.2) et une condition de Dirichlet homogène pour la vitesse du fluide.

Remarque 1.3.3. On peut comparer le Théorème 1.3.1 avec la théorie des solutions faibles globales pour le système de Vlasov-Poisson (1.1.9). Dans [Ars75], Arsenev démontre l'existence globale d'une solution faible bornée, de masse et d'énergie finies si la donnée initiale f^{in} l'est aussi. Comme pour le Théorème 1.3.1, la preuve est basée sur une procédure de régularisation/point fixe/compacité, sur la régularité elliptique fournie par l'équation de Poisson, et sur l'inégalité d'énergie

$$\forall t \geq 0, \;\; \mathfrak{E}(t) := \frac{1}{2}\int_{x,v} f(t)|v|^2 \,\mathrm{d}x\,\mathrm{d}v + \frac{1}{2}\int_x |\nabla_x U(t)|^2 \,\mathrm{d}x \leq \mathfrak{E}(0),$$

où U désigne le potentiel électrique. On peut en fait relaxer l'hypothèse initiale en supposant seulement $f^{\text{in}} \in \mathrm{L}^1_{x,v} \cap \mathrm{L}^2_{x,v}$, grâce au travail [HH84] de Horst et Hünze[10]. Dans [DL88], DiPerna et Lions obtiennent quant à eux l'existence d'une solution faible globale renormalisée, de masse et d'entropie finie. On renvoie enfin à certains travaux récents [BBC16, CCM18].

[10] Ce qui, à notre connaissance, n'est pas connu pour les équations de Vlasov-Navier-Stokes.

1.3.2.2 Unicité pour le système de Vlasov-Navier-Stokes en dimension 2

Concluons brièvement cette section sur le problème de Cauchy pour les équations de Vlasov-Navier-Stokes (1.3.1) : les solutions faibles données par le Théorème 1.3.1 sont-elles-uniques ? En effet, la méthode esquissée dans la Section 1.3.2.1 pour construire de telles solutions ne dit rien sur leur éventuelle unicité.

Si l'on s'intéresse à ce type de régularité faible, il semble déraisonnable de vouloir aborder ce problème en dimension 3 : en effet, en prenant une condition initiale nulle pour f, on observe que ce problème est au moins aussi difficile que l'unicité pour les équations de Navier-Stokes en dimension 3. La question semble plus abordable pour (1.3.1) en dimension 2 : chacune des équations découplées (Navier-Stokes sur u en posant $f = 0$ et une équation de Vlasov amortie sur f en posant $u = 0$) admet une unique solution faible globale (voir [CDGG06] pour les équations de Navier-Stokes en dimension 2). Ici, la difficulté concernant l'unicité des solutions (f, u) vient bien sûr du couplage (non-linéaire) entre les deux équations.

Le résultat que nous présentons ici est un résultat d'unicité des solutions faibles de (1.3.1) (à la Leray pour u, renormalisée pour f) en dimension $d = 2$: il s'applique pour le même type de conditions initiales $(u^{\text{in}}, f^{\text{in}})$ qui donnent naissance à des solutions faibles globales (Théorème 1.3.1), en supposant une certaine décroissance en vitesse de f^{in}. Il a été obtenu par Han-Kwan, Miot, Moussa et Moyano dans [HKMMM20]. À notre connaissance, il s'agit du seul résultat d'unicité de solutions faibles pour (1.3.1).

Théorème 1.3.4 (Han-Kwan, Miot, Moyano et Moussa, [HKMMM20])**.** *Soit $\Omega = \mathbb{T}^2$ ou $\mathbb{R}^2$, et soit $q > 4$. Soit $u^{\text{in}} \in \mathrm{L}^2(\Omega)$ à divergence nulle. Soit $f^{\text{in}} \in \mathrm{L}^1 \cap \mathrm{L}^\infty(\Omega \times \mathbb{R}^2)$ telle que $f^{\text{in}} \geq 0$ et $\int_{\Omega\times\mathbb{R}^2} f^{\text{in}}|v|^2 \,\mathrm{d}x\,\mathrm{d}v < \infty$. On suppose que*

$$(x, v) \mapsto (1 + |v|^q)f^{\text{in}}(x, v) \in \mathrm{L}^\infty(\Omega \times \mathbb{R}^2). \tag{1.3.12}$$

Il existe une unique solution faible globale (f, u) du système (1.3.1) sur $\Omega\times\mathbb{R}^2$ avec condition initiale $(f^{\text{in}}, u^{\text{in}})$.

La condition de décroissance (1.3.12) est réminiscente de celle introduite par Lions et Perthame [LP91] pour obtenir l'unicité du système de Vlasov-Poisson. Le cadre des solutions faibles (f, u) de (1.3.1) en dimension 2 est en fait plus favorable qu'il n'y paraît : grâce à l'effet régularisant des équations de Navier-Stokes, on peut montrer de l'intégrabilité supplémentaire pour u et obtenir par la même occasion que les moments en vitesse d'ordre $0, 1$ et 2 de f sont uniformément bornés localement en temps. La preuve se déroule ensuite dans l'esprit des travaux de Loeper [Loe06] pour l'unicité du système de Vlasov-Poisson[11].

1.3.3 Interlude : dérivation du système de Vlasov-Navier-Stokes

Dans cette section, on explore le problème de la dérivation rigoureuse du système de Vlasov-Navier-Stokes. Cette question a été l'objet d'une recherche récente, encore à ses balbutiements, et on en donne donc un aperçu par souci de complétude. Comme cette thèse n'aborde pas cette problématique, cette section contient peu de détails et on renvoie donc directement aux références mentionnées.

Une question cruciale de la physique mathématique concerne le passage rigoureux de la description microscopique (grâce aux lois de Newton) à la description mésoscopique ou macroscopique (voir Section 1.1.3). Ce problème s'inscrit plus généralement dans le cadre du 6ème problème de Hilbert

[11] En s'appuyant sur des arguments de type fonction maximale de Hardy-Littlewood plutôt que sur une distance de Wasserstein quadratique.

concernant l'axiomatisation mathématique des modèles physiques au niveau continu. Lorsque l'on cherche à dériver une équation de type champ moyen ou de type Vlasov à partir d'un système de N particules, on appelle cette dérivation une **limite de champ moyen**.

Cette section offre aussi l'occasion d'introduire un type de modèle de sédimentation au niveau macroscopique : dans la Section 1.5, on abordera un autre type de dérivation partant des modèles cinétiques et allant vers ces systèmes *via* des limites hydrodynamiques.

On considère un système de N particules en interaction dans un champ de force dérivant d'un potentiel Φ (symétrique), les positions et vitesses $(\mathrm{X}_i, \mathrm{V}_i)_{1\leq i\leq N}$ satisfaisant le système d'équations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{X_i} = \mathrm{V}_i, \quad \mathrm{X}_{i|t=0} = \mathrm{X}_i^{\mathrm{in}}, \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}_i = -\dfrac{1}{N}\displaystyle\sum_{\substack{j=1\\ j\neq i}}^{N} \nabla_x \Phi(\mathrm{X}_j - \mathrm{X}_i), \quad \mathrm{V}_{i|t=0} = \mathrm{V}_i^{\mathrm{in}}. \end{cases} \tag{1.3.13}$$

En définissant la mesure empirique

$$\mu_N(t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\mathrm{X}_i(t)} \otimes \delta_{\mathrm{V}_i(t)},$$

et en supposant que $(\mu_N(t))_N$ converge en un certain sens vers une fonction $f(t)$ dans la limite $N \to +\infty$, on s'attend formellement à ce que f soit solution de l'équation de Vlasov (1.1.8) avec une force donnée par $F = -\nabla_x \Phi \star_x \rho_f$.

On dit que la limite de champ moyen est valide si pour toute configuration initiale $(\mathrm{X}_i^{\mathrm{in}}, \mathrm{V}_i^{\mathrm{in}})_{1\leq i\leq N}$ telle que $(\mu_{N|t=0})_N$ converge (faiblement) vers une fonction régulière $f^{\mathrm{in}} = f^{\mathrm{in}}(x, v)$, la convergence de $(\mu_N(t))_N$ vers $f(t) = f(t, x, v)$ (avec donnée initiale f^{in}) a lieu sur un intervalle de temps $[0, T]$ pour un certain $T > 0$ indépendant de N.

En général, une observation clé est que la mesure empirique μ_N satisfait elle aussi une équation de Vlasov (*modulo* un terme diagonal non principal), si bien que des estimations de stabilité pour cette équation se révèlent cruciales dans ce genre de problème. Le cas des potentiels réguliers ($\Phi \in \mathscr{C}^2$) est bien compris depuis les travaux de Braun et Hepp [BH77], Neunzert et Wick [NW74], ou Dobrushin [Dob79]. Dans les cas des potentiels singuliers (comme le potentiel Coulombien $\Phi(y) = 1/|y|$ en dimension 3), le problème est encore largement ouvert. On renvoie aux notes de cours de Golse [Gol16] et à l'article de synthèse de Jabin [Jab14] pour plus de détails. Mentionnons, entre autres, les travaux récents de Hauray et Jabin [HJ07, HJ15], de Lazarovici et Pickl [LP17], de Serfaty et Duerinckx [DS20], et de Bresch, Jabin et Wang [BJW19].

Dans le cas de particules dans un fluide, la situation de départ est déjà plus complexe : l'interaction entre les particules n'est pas donnée explicitement par des équations du type (1.3.13) qui font intervenir directement la distance entre les particules. Au contraire, les particules vont interagir les unes avec les autres de manière implicite en modifiant la vitesse du fluide. Dans un fluide visqueux, ce type de couplage fait également apparaître des interactions comportant des singularités.

On considère ainsi un nuage de N particules $\{B_i\}_{1\leq i\leq N}$ (dans $\mathbb{R}^3$) de rayon $R_N > 0$ décrites par leur centre de masse X_i et leur vitesse V_i, et immergées dans un fluide visqueux ayant une vitesse $u(t, x)$ et une pression $p(t, x)$. La dynamique de chaque boule $B_i = B(\mathrm{X}_i, R_N)$ est décrite par les lois de Newton :

$$\forall i = 1, \cdots, N, \quad \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{X_i} = \mathrm{V}_i, \quad \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{V_i} = \int_{\partial B_i} \left(\nabla_x u + \nabla_x u^t - p\mathrm{I}\right) n \mathrm{d}\sigma. \tag{1.3.14}$$

Le couple (u, p) pour le fluide satisfait alors les équations de Navier-Stokes dans un domaine $\Omega^N := \mathbb{R}^3 \setminus \left(\bigcup_{1 \le i \le N} \overline{B_i}\right)$

$$\begin{cases} \partial_t u + (u \cdot \nabla_x) u + \nabla_x p - \Delta_x u = 0, \quad \text{dans} \quad \Omega^N, \\ \operatorname{div}_x u = 0, \quad \text{dans} \quad \Omega^N, \\ u(x) = \mathrm{V}_i \quad \text{sur} \quad \partial B_i \ (1 \le i \le N), \\ u(x) \longrightarrow 0, \quad |x| \to +\infty. \end{cases} \tag{1.3.15}$$

Une dérivation envisageable des équations de Vlasov-Navier-Stokes consisterait à montrer que, lorsque $N \to +\infty$, les équations couplées (1.3.14)–(1.3.15) conduisent au système (1.3.1). Notons que l'analyse de la force de Brinkman montre qu'il faut considérer un scaling tel que $NR_N = 1$ pour voir apparaître ce terme à la limite dans l'équation fluide. Ce programme ambitieux comporte deux aspects inévitables, et fortement liés :

- la dérivation de l'équation de Vlasov dans (1.3.1), *via* la convergence de la mesure empirique associée μ_N au système de N particules, est un **problème de champ moyen**;
- la dérivation de l'équation macroscopique pour le fluide, avec en particulier l'obtention de la force de Brinkman en terme source : il s'agit d'un **problème d'homogénéisation** pour une équation fluide dans un domaine perforé. Cela correspond au fait que les particules peuvent se déplacer avec une vitesse qui ne correspond pas à celle du fluide, introduisant un effet de frottement *via* un terme d'ordre 0.

Une approche rigoureuse prenant en compte ces deux problématiques est, à notre connaissance, totalement inconnue à l'heure actuelle. Pour résumer, les travaux existants traitent plutôt de l'une ou l'autre de ces directions de recherche.

• Concernant la partie homogénéisation, il est possible de justifier l'apparition de la force de Brinkman dans l'équation de (Navier-)Stokes stationnaire lorsque $N \to +\infty$ et lorsque la dynamique des particules est absente. *Grosso modo*, on se donne les positions et vitesse des particules en supposant la convergence de la mesure empirique et on procède à l'homogénéisation pour le fluide. Ce type de résultat, dans la continuation du travail d'Allaire dans [All91], a été obtenu par Desvillettes, Golse et Ricci [DGR08], puis ensuite amélioré par Fereisl, Namlyeyeva et Nečasová [FNN16], Hillairet [Hil18, Hil21], Hillairet, Moussa et Sueur [HMS19], Giunti et Höfer [GH19], Carrapatoso et Hillairet [CH20], et Höfer et Jansen [HJ20]. On obtient ainsi l'influence asymptotique des particules sur le fluide lorsque $N \to +\infty$. Tous ces travaux incluent essentiellement une hypothèse de dilution pour le nuage de particules (faisant intervenir une distance minimale), qui ne serait *a priori* pas propagée par la dynamique.

• Il n'existe pas, à notre connaissance et hormis[12] [FLR19, FLR21], de résultats portant sur la dérivation de l'équation cinétique *via* la limite de champ moyen. Les travaux précédents considèrent en fait un système fluide déjà modifié initialement et avec un terme de diffusion en vitesse pour les particules.

Une autre ligne de recherche récente, concernant les problèmes de sédimentation, considère le cas où les particules sont sans inertie. Dans ce cas, les interactions ne déterminent pas l'accélération des particules mais leur vitesse. On peut espérer que pour cette dynamique mettant en jeu des équations d'ordre 1, des particules proches auront quasiment la même vitesse.

Si on considère l'action de la gravité $G \in \mathbb{R}^3$ sur le nuage de particules sans inertie (toujours entourées d'un fluide visqueux), une limite de champ moyen sur $\rho^N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{\mathrm{X}_i(t)}$ mène au

[12] Pour un modèle de fluide intermédiaire où l'interaction particules-fluide est donnée par un noyau explicite avec une singularité tronquée, au lieu de conditions aux limites entre les particules et le fluide.

système macroscopique suivant, posé dans tout le domaine :

$$\begin{cases} \partial_t \rho + \mathrm{div}_x[\rho(u+G)] = 0, \\ -\Delta_x u + \nabla_x p = \rho G, \\ \mathrm{div}_x\, u = 0. \end{cases} \tag{1.3.16}$$

Ce couplage entre la densité limite ρ et la vitesse du fluide u est appelé système de Stokes-Transport. Après les travaux préliminaires de Jabin et Otto [JO04], ce système a été dérivé par Höfer [Höf18] et Mecherbet [Mec19]. Dans le cas sans inertie, une hypothèse initiale de distance minimale entre les particules peut être propagée au cours de l'évolution. Un outil important permettant l'analyse est la *méthode des réflections*. On renvoie à (l'introduction de) la thèse de Höfer [Höf20] pour une présentation et des explications détaillées. Mentionnons également les travaux récents de Höfer et Schubert [HS21, HS22] qui incluent le phénomène de viscosité effective dans la dérivation ou bien un régime de très faible inertie.

Analogue du système de Boussinesq (voir (1.5.5) plus bas) où l'on aurait négligé l'auto-convection pour le fluide, le système de Stokes-Transport (1.3.16) fait intervenir un transport de la densité par un champ de vitesses non local. Il est également intéressant en lui-même et on renvoie aux travaux récents portant sur ces équations [Mec20, Leb22, MS88, GI22, GGBS22, Cob23, DGL23].

Pour résumer, la dérivation rigoureuse du système de Vlasov-Navier-Stokes est encore très partielle : inclure à la fois la dynamique complète des particules tout en ayant de bonnes informations sur la distance entre chacune d'elles est un défi encore totalement ouvert. Il est raisonnable de penser que savoir traiter le cas sans inertie pour des hypothèses de dilution affaiblies est une première étape importante.

Remarque 1.3.5. Mentionnons enfin une stratégie alternative proposée par Bernard, Desvillettes, Golse et Ricci pour dériver les équations de Vlasov-Navier-Stokes. Partant d'un système bi-phasique avec deux équations de Boltzmann couplées (une pour le fluide et une pour les particules), les travaux [BDGR17, BDGR18, Ric17] identifient un scaling précis menant au couplage fluide-cinétique. Ce programme, encore formel à l'heure actuel, est dans l'esprit de celui de Bardos, Golse et Levermore [BG91, BGL93] sur les limites hydrodynamiques pour l'équation de Boltzmann (vers celles de Navier-Stokes). Justifier rigoureusement cette dérivation est cependant un défi encore ouvert.

1.4 Comportement en temps long pour le système de Vlasov-Navier-Stokes

Dans cette section, on aborde la question de la dynamique en temps long du système de Vlasov-Navier-Stokes. On a en effet montré en Section 1.3.2.1 que l'on savait construire des solutions faibles globales pour (1.3.1), partant de conditions initiales d'énergie finie. Or, cela ne nous dit rien de leur comportement en temps long.

La situation est assez similaire à celle du système de Vlasov-Poisson (1.1.9) (en dimension $d \leq 3$) ou des équations d'Euler incompressible bidimensionnelles (en formulation vorticité), pour lesquels on sait aussi construire des solutions faibles globales. Du fait de la nature Hamiltonienne de ces équations, l'étude de leur stabilité, de leur comportement asymptotique ou de leur instabilité est cependant source de questions très complexes : citons par exemple le problème de l'amortissement Landau (voir l'article de revue [Bed22]) ou de l'amortissement non-visqueux (voir la présentation [GV14]).

1.4.1 Approche heuristique

Les premières questions naturelles à se poser sont les suivantes. Existe-il des états stationnaires du système (1.3.1) ? Attirent-ils les solutions du système ou sont-ils au contraire instables ?

On sait par exemple que toute fonction de distribution $\mu = \mu(|v|^2)$ (et même $\mu = \mu(v)$) ne dépendant que de v (dit profil homogène en espace) est une solution stationnaire des équations de Vlasov-Poisson (1.1.9), générant un champ électrique nul. De même, tout profil de cisaillement $U(x_1, x_2) = (V(x_2), 0)$ satisfait les équations d'Euler 2d. Il existe cependant une première différence notable entre ces deux exemples et le système de Vlasov-Navier-Stokes.

Plaçons-nous dans le cas d'un domaine sans bord, disons $\mathbb{R}^d$ ou $\mathbb{T}^d$. Cherchons par exemple une solution stationnaire sous la forme $\mu = F(|v|^2)$ (avec $F : \mathbb{R}^+ \to \mathbb{R}^+$ lisse et $\int_{\mathbb{R}^d} v\mu(v)\,\mathrm{d}v = 0$) pour (1.3.1). Le couple $(f, u) = (\mu, 0)$ est solution (1.3.1) si seulement si $\mathrm{div}_v(v\mu) = 0$. Cela donne $2|v|^2F'(|v|^2) = -dF(|v|^2)$. L'étude de l'équation différentielle $y'(r) = -dy(r)/2r$ donne alors $\mu(v) = C|v|^{-d}$ et celle-ci n'est localement intégrable que si $C = 0$.

En conclusion, tout solution stationnaire de la forme précédente (pour la partie cinétique) est nécessairement nulle. Ceci est dû au terme de friction dans l'équation de Vlasov.

Allons légèrement plus loin dans l'analyse et regardons le système de Vlasov-Navier-Stokes sur $\mathbb{R}^d \times \mathbb{R}^d$ linéarisé autour de l'équilibre trivial $(\overline{f}, \overline{u}) = (0, 0)$, qui s'écrit

$$\begin{cases} \partial_t u - \Delta_x u + \nabla_x p = \displaystyle\int_{\mathbb{R}^d} vf\,\mathrm{d}v, \\ \partial_t f + v\cdot\nabla_x f - \mathrm{div}_v(vf) = 0. \end{cases} \tag{1.4.1}$$

La seconde équation, qui est une équation de Vlasov avec friction autour d'une vitesse fluide nulle, se résout explicitement par

$$f(t, x, v) = e^{dt} f_{|t=0}(x + (1 - e^t)v, v).$$

Le comportement en temps long de f (lorsque $t \to +\infty$) fait donc apparaître une compétition entre le transport et la friction. Cependant, un calcul explicite permet d'obtenir la convergence faible sur $\mathbb{R}^d \times \mathbb{R}^d$

$$f(t, x, v) \underset{t\to+\infty}{\rightharpoonup} \left(\int_{\mathbb{R}^d} f_{|t=0}(x - v, v)\right) \underset{x,v}{\otimes} \delta_{v=0}.$$

On a également $j_f(t) \to 0$ d'où $u(t) \to 0$ quand $t \to +\infty$, l'équation de Stokes instationnaire avec source tendant vers 0 étant dissipative.

Comme expliqué dans [GHKM18], la conclusion des observations précédentes est la suivante, dans un domaine sans bord :

- la friction tend à l'emporter, si bien que la distribution des particules se singularise en temps long et fait apparaître un **comportement monocinétique** (présence d'une masse de Dirac en vitesse). Asymptotiquement, la phase fluide et la phase dispersée alignent leurs vitesses. On parle aussi de phénomène de concentration en vitesse;
- en dehors de l'équilibre trivial 0, les seules solutions stationnaires de (1.4.1) sont des mesures;
- la solution triviale $(0, 0)$ de (1.4.1) est **instable au sens de Lyapunov** car toute distribution initiale non triviale donne naissance à une solution qui converge faiblement vers une masse de Dirac lorsque $t \to +\infty$. Ceci contraste fortement avec le système de Vlasov-Poisson (voir par exemple [BD85]) ou de Vlasov-Maxwell (voir par exemple [GS87]).

Dans un régime proche de l'équilibre, on s'attend génériquement à retrouver ce comportement singulier pour le système de Vlasov-Navier-Stokes complet (dans un cas sans bord).

Remarque 1.4.1. On peut aussi considérer, comme dans [BDD23], un profil homogène en espace mais dépendant du temps

$$\mu(t, v) = e^{dt} F(|e^t v|^2), \quad t > 0, \quad v \in \mathbb{R}^d,$$

qui, associé à $u = 0$, est une solution particulière du système de Vlasov-Navier-Stokes puisque

$$\partial_t \mu - \mathrm{div}_v(v\mu) = 0.$$

On a donc une famille de profils homogènes en espace et dépendants du temps qui sont solutions. L'observation importante, à nouveau liée à la friction, est que la partie cinétique $\mu(t, v)$ converge vers une masse de Dirac en vitesse, centrée en $v = 0$, lorsque $t \to +\infty$ (*modulo* renormalisation de F).

Remarque 1.4.2. La situation est très différente si l'on rajoute un opérateur de type Fokker-Planck dans l'équation cinétique. Dans ce cas, la présence d'un terme dissipatif du type $-\Delta_v f$ modifie totalement la dynamique : pour des données initiales bien préparées, on s'attend à obtenir convergence vers un équilibre Maxwellien (comme prouvé dans [GHMZ10]). Le comportement monocinétique évoqué plus haut correspond au cas limite d'une Maxwellienne à température nulle.

On peut encore pousser l'heuristique un peu plus loin pour tenter de comprendre le comportement en temps long de (1.3.1). L'identité d'énergie-dissipation (1.3.4)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) + \mathrm{D}(t) = 0,$$

inhérente à (1.3.1) (voir les définitions (1.3.5) et (1.3.6)), implique, en intégrant en temps,

$$\int_0^\infty \int_{x,v} |u(t) - v|^2 f(t) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \leq \mathrm{E}(0) < \infty, \tag{1.4.2}$$

si l'on considère des données initiales d'énergie finie.

Supposons alors que $u(t) \to u^\infty \in \mathbb{R}^d$ et $f(t) \to \mu^\infty$ quand $t \to +\infty$, où $\mu^\infty \, \mathrm{d}x \, \mathrm{d}v$ est une mesure sur $\mathbb{R}^d \times \mathbb{R}^d$ telle que $\int_{\mathbb{R}^d} \mu^\infty(\cdot, v) \, \mathrm{d}v = \rho^\infty$ (densité locale asymptotique). Au vu de (1.4.2), on s'attend de façon très formelle à avoir

$$\int_{x,v} |u(t) - v|^2 f(t) \, \mathrm{d}x \, \mathrm{d}v \underset{t \to +\infty}{\longrightarrow} 0.$$

Cette convergence de $f(t)$ contre la fonction test $|u(t) - v|^2$, si l'on pouvait la remplacer par une convergence contre la fonction test $|u^\infty - v|^2$, signifie que l'action de la mesure limite μ^∞ est nulle contre la fonction test $|u^\infty - v|^2$. Cela implique que son support en vitesse est réduit au singleton $\{u^\infty\}$. On obtient donc le profil monocinétique

$$\mu^\infty = \rho^\infty \otimes \delta_{v = u^\infty}.$$

En un certain sens, cette dynamique devrait avoir lieu dès lors que l'énergie cinétique initiale est finie.

Mais comment connaître le profil limite ? Comme dans l'heuristique faite dans [HKMM20], l'analyse précédente nous permet de supposer que $f(t, x, v) \sim \rho_f(t, x) \otimes \delta_{v = u(t,x)}$ quand $t \to +\infty$. Cette convergence en temps annule alors la force de Brinkman $j_f - \rho_f u$ dans le terme source de l'équation sur u. Sur tout l'espace $\mathbb{R}^d$, on sait alors que la vitesse $u(t)$ devrait tendre vers 0 quand $t \to +\infty$ et on peut espérer obtenir

$$f(t,x,v) \sim \rho_f(t,x) \otimes \delta_{v=0}.$$

Sur le tore, en notant $\langle \cdot \rangle$ la moyenne spatiale sur $\mathbb{T}^d$, on sait que $u(t)$ va se rapprocher de $\langle u(t) \rangle$, si bien que $f(t,x,v) \sim \rho_f(t,x) \otimes \delta_{v=\langle u(t) \rangle}$. En particulier, $\langle j_f(t) \rangle \sim \langle \rho_f(t) \rangle \langle u(t) \rangle$. Après renormalisation de la masse totale, on s'attend à avoir $\langle j_f(t) \rangle \sim \langle u(t) \rangle$. En utilisant la conservation du moment total au cours de la dynamique

$$\langle j_f(t) + u(t) \rangle = \langle j_{f^{\mathrm{in}}} + u^{\mathrm{in}} \rangle,$$

on obtiendrait finalement

$$f(t,x,v) \sim \rho_f(t,x) \otimes \delta_{v=\langle j_{f^{\mathrm{in}}} + u^{\mathrm{in}} \rangle /2}.$$

Cette dernière heuristique a également été proposée dans [HKMM20].

1.4.2 Premiers résultats rigoureux et convergence conditionnelle

La question de la dynamique en temps long pour le système Vlasov-Navier-Stokes a connu plusieurs développements récents.

Dans un travail préliminaire [Jab00b], Jabin aborde un couplage plus simple que (1.3.1) (sorte de modèle-jouet pour une suspension de particules dans un fluide) et prouve rigoureusement la convergence en temps long de la distribution des particules vers un profil monocinétique. La preuve est basée sur l'étude de l'énergie cinétique totale des particules.

Dans [CK15], Choi et Kwon étudient le couplage complet (1.3.1) dans le cas du tore spatial tridimensionnel $\mathbb{T}^3$ et apportent une réponse partielle au problème de la dynamique en temps long. Sous des hypothèses *a priori* sur les solutions globales elle-mêmes, ils obtiennent le comportement monocinétique asymptotique. Plus précisément, les auteurs demandent certaines bornes sur la densité locale $\rho_f(t)$ qui doivent être satisfaites pour tout temps $t > 0$.

L'idée, initiée dans [CK15], est de moduler l'énergie cinétique totale $\mathrm{E}(t)$ en introduisant la fonctionnelle

$$\mathscr{E}_{\mathbb{T}^3}(t) = \frac{1}{2} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f(t,x,v) |v - \langle j_f(t) \rangle|^2 \, \mathrm{d}v \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{T}^3} |u(t,x) - \langle u(t) \rangle|^2 \, \mathrm{d}x + \frac{1}{4} |\langle j_f(t) \rangle - \langle u(t) \rangle|^2. \quad (1.4.3)$$

Ici, $\langle \cdot \rangle$ désigne à nouveau la moyenne spatiale sur $\mathbb{T}^3$. On peut ainsi voir apparaître les déviations par rapport aux moyennes $\langle j_f(t) \rangle$ et $\langle u(t) \rangle$ mises en évidence dans l'heuristique précédente. Il s'agit *grosso modo* d'une modulation autour d'un équilibre local de type masse de Dirac. Le dernier terme vient équilibrer les deux premiers, et permet de prouver que l'identité d'énergie-dissipation (1.3.4) persiste au niveau de l'énergie modulée, c'est-à-dire :

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{E}_{\mathbb{T}^3}(t) + \mathrm{D}(t) = 0.$$

Ceci repose sur la conservation du moment total et l'identité d'énergie-dissipation classique (1.3.4).

L'intérêt d'une telle fonctionnelle provient du fait suivant : le contrôle de la dynamique de l'énergie modulée $\mathscr{E}_{\mathbb{T}^3}(t)$ permet essentiellement d'encoder le comportement en temps long des solutions sur le tore. En définissant $\mathrm{U}^{\mathrm{in}} := \langle j_{f^{\mathrm{in}}} + u^{\mathrm{in}} \rangle / 2$, on peut en effet montrer, après normalisation de la masse, que

$$\forall t \geq 0, \quad \mathrm{W}_1\left(f(t), \rho_f(t) \otimes \delta_{v=\mathrm{U}^{\mathrm{in}}}\right) + \left\| u(t) - \mathrm{U}^{\mathrm{in}} \right\|_{\mathrm{L}^2(\mathbb{T}^3)} \lesssim \mathcal{E}_{\mathbb{T}^3}(t)^{1/2}.$$

Ici, la distance W_1 est la distance de Wasserstein sur $\mathbb{T}^3 \times \mathbb{R}^3$ provenant du transport optimal. Elle métrise la convergence faible des mesures et peut être vue comme la distance duale pour la semi-norme Lipschitz (voir par exemple [Vil21]). Ainsi, le comportement monocinétique attendu se ramène à la convergence $\mathscr{E}_{\mathbb{T}^3}(t) \to 0$ lorsque $t \to +\infty$.

Le résultat conditionnel suivant dit alors que l'on peut contrôler la dissipation par le bas *via* l'énergie modulée et obtenir la décroissance vers 0 de cette dernière.

Lemme 1.4.3 (Choi et Kwon, [CH20])**.** *Soit (f, u) une solution de (1.3.1) d'énergie finie telle que*

$$\sup_{t \in [0,T]} \|\rho_f(t)\|_{\mathrm{L}^{3/2}(\mathbb{T}^3)} < \infty. \tag{1.4.4}$$

pour un $T > 0$. Alors pour tout $t \in [0, T]$, on a

$$\mathscr{E}_{\mathbb{T}^3}(t) \lesssim \mathrm{D}(t),$$

et on a en particulier décroissance exponentielle de $\mathscr{E}_{\mathbb{T}^3}(t)$ sur $[0, T]$.

Ce résultat est cependant partiel au sens où rien ne permet d'affirmer *a priori* que toute solution non-triviale de (1.3.1) satisfait l'hypothèse (1.4.4) pour tout temps.

1.4.3 Le cas du tore et de l'espace entier

La première réponse complète concernant la dynamique en temps long a été apportée par Han-Kwan, Moussa et Moyano dans [HKMM20]. Dans le cas du tore $\mathbb{T}^3$, leur travail vise à montrer que, sous des hypothèses où les données sont bien préparées, le contrôle (1.4.4) est vrai avec $T = +\infty$. Leur stratégie permet en fait d'obtenir

$$\sup_{t \in \mathbb{R}^+} \|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} < \infty. \tag{1.4.5}$$

En définissant à nouveau $\mathrm{U}^{\mathrm{in}} = \langle j_{f^{\mathrm{in}}} + u^{\mathrm{in}} \rangle / 2$, on peut énoncer une version du théorème principal de [HKMM20] de la façon suivante.

Théorème 1.4.4 (Han-Kwan, Moussa et Moyano [HKMM20])**.** *Soit $(u^{\mathrm{in}}, f^{\mathrm{in}})$ une condition initiale admissible pour (1.3.1) avec f^{in} suffisament décroissante en vitesse (uniformément en x) et telle que*

$$\mathscr{E}_{\mathbb{T}^3}(0) + \|u^{\mathrm{in}}\|_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)} \ll 1.$$

Alors pour toute solution faible globale (f, u) de (1.3.1) partant de $(f^{\mathrm{in}}, u^{\mathrm{in}})$, la conclusion du Lemme 1.4.3 est satisfaite et il existe un profil $\rho^\infty \in \mathrm{L}^\infty(\mathbb{T}^3)$ et $\lambda > 0$ tels que pour tout $t \geq 0$

$$\mathrm{W}_1\left(f(t), \rho^\infty\left(x - t\mathrm{U}^{\mathrm{in}}\right) \otimes \delta_{v=\mathrm{U}^{\mathrm{in}}}\right) + \left\| u(t) - \mathrm{U}^{\mathrm{in}} \right\|_{\mathrm{L}^2(\mathbb{T}^3)} \lesssim \mathcal{E}_{\mathbb{T}^3}(0)^{1/2} e^{-\lambda t}.$$

La condition de petitesse $\mathscr{E}_{\mathbb{T}^3}(0) + \|u^{\mathrm{in}}\|_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)} \ll 1$ peut-être vue comme une condition de type “données proches de l'équilibre”. La semi-norme $\|\cdot\|_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)}$ (impliquée dans l'invariance par scaling des équations de Navier-Stokes en dimension 3) intervient aussi pour propager de la régularité supplémentaire sur u en temps long.

L'existence du profil asymptotique spatial ρ^∞, dans l'esprit de [Jab00b], n'est pas constructive et utilise la décroissance exponentielle de l'énergie modulée. Sa description contenue dans [HKMM20] repose sur l'histoire entière des solutions (f, u).

La première partie de la preuve du Théorème 1.4.4 consiste à identifier une condition suffisante assurant (1.4.5). Grâce au changement de variable en vitesse $\Gamma^{t,x} : v \mapsto \mathrm{V}^{0,t}(x,v)$ basé sur les courbes caractéristiques

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{X}^{s;t}(x,v) = \mathrm{V}^{s;t}(x,v), \quad \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}^{s;t}(x,v) = u(s, \mathrm{X}^{s;t}(x,v)) - \mathrm{V}^{s;t}(x,v) \tag{1.4.6}$$

pour (1.3.1) (avec $(\mathrm{X}^{t;t}(x,v), \mathrm{V}^{t;t}(x,v)) = (x,v)$), on peut utiliser une formule de représentation pour f et écrire

$$\begin{aligned}\rho_f(t,x) &= \int_{\mathbb{T}^3} e^{3t} f^{\mathrm{in}}(\mathrm{X}^{0;t}(x,v), \mathrm{V}^{0;t}(x,v))\, \mathrm{d}v \\ &= \int_{\mathbb{T}^3} e^{3t} f^{\mathrm{in}}(\mathrm{X}^{0;t}(x, [\Gamma^{t,x}]^{-1}(w)), w) |\det \mathrm{D}_w\, [\Gamma^{t,x}]^{-1}(w)|\, \mathrm{d}w.\end{aligned}$$

Cette procédure est inspirée du travail de Bardos et Degond pour le système de Vlasov-Poisson [BD85]. On peut montrer que ce changement de variable, assurant la borne $\|\rho_f\|_{\mathrm{L}^\infty(\mathbb{R}^+;\mathrm{L}^\infty(\mathbb{T}^3))} < \infty$, est valide si

$$\int_0^\infty \|\nabla_x u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\, \mathrm{d}s \ll 1. \tag{1.4.7}$$

La suite de la preuve est alors basée sur un argument de bootstrap visant a propager cette condition sur $\mathbb{R}^+$ tout entier. Un argument d'interpolation permet de contrôler la semi-norme Lipschitz précédente par l'énergie modulée $\mathscr{E}_{\mathbb{T}^3}(t)$, en payant un peu de régularité supplémentaire sur u (*via* $\|u^{\mathrm{in}}\|_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)} \ll 1$). Notons que sur le tore, la décroissance exponentielle de l'énergie modulée fournit virtuellement n'importe quelle intégrabilité en temps. La condition $\mathscr{E}_{\mathbb{T}^3}(0) \ll 1$ donne alors la petitesse désirée dans (1.4.7).

Le cas de l'espace entier $\mathbb{R}^3$ a quant à lui été traité par Han-Kwan dans [HK22]. Dans ce cas, la bonne fonctionnelle encodant le comportement en temps long est l'énergie cinétique E elle-même (définie en (1.3.5)).

Théorème 1.4.5 (Han-Kwan [HK22]). *Soit $(u^{\mathrm{in}}, f^{\mathrm{in}})$ une condition initiale admissible pour (1.3.1) avec f^{in} suffisament décroissante en vitesse (uniformément en x) et telle que*

$$\mathrm{E}(0) + \|u^{\mathrm{in}}\|_{\mathrm{L}^1\cap \mathrm{H}^1(\mathbb{R}^3)} + \|f^{\mathrm{in}}\|_{\mathrm{L}^1_v(\mathbb{R}^3);\mathrm{L}^\infty_x(\mathbb{R}^3)} \ll 1.$$

Alors toute solution faible globale (f,u) de (1.3.1) partant de $(f^{\mathrm{in}}, u^{\mathrm{in}})$ vérifie pour tout $t \geq 0$

$$\mathrm{W}_1\left(f(t), \rho_f(t) \otimes \delta_{v=0}\right) + \|u(t)\|_{\mathrm{L}^2(\mathbb{R}^3)} \lesssim \frac{\mathrm{E}(0)^{1/2} + \|u^{\mathrm{in}}\|_{\mathrm{L}^1(\mathbb{R}^3)}}{(1+t)^\alpha}, \quad \alpha \in (0, 3/4).$$

L'hypothèse ici faite de type “données petites” conduit alors à l'instabilité de la solution $(0,0)$. Notons que la décroissance de l'énergie cinétique n'est ici que polynomiale et provient de l'absence d'inégalité de Poincaré dans tout l'espace : c'est déjà le cas pour les équations de Navier-Stokes (sans source) sur tout l'espace, dont le comportement asymptotique en temps est moralement dicté par la décroissance de l'équation de la chaleur.

La première partie de la preuve consiste à montrer cette décroissance polynomiale de l'énergie cinétique totale si la borne $\|\rho_f\|_{\mathrm{L}^\infty(\mathbb{R}^+;\mathrm{L}^\infty(\mathbb{R}^3))} < \infty$ est vérifiée. Ceci est basé sur une étude fine du couplage entre fluide et particules, dans l'esprit du Fourier-splitting de Schonbek [Sch86] et Wiegner [Wie87] sur les équations de Navier-Stokes avec source.

La condition de petitesse (1.4.7) permet à nouveau d'assurer le contrôle sur ρ_f. À cause de la faible décroissance en temps de l'énergie, un argument d'interpolation ne permet pas de conclure directement. Ceci nécessite de mieux comprendre le comportement de la force de Brinkman agissant en source dans Navier-Stokes, et requiert l'introduction d'une famille de fonctionnelles de dissipation d'exposants plus élevés, à savoir $\mathrm{D}_p(t) = \int_{\mathbb{R}^3\times\mathbb{R}^3} f(t)|v-u(t)|^p \,\mathrm{d}x\,\mathrm{d}v$. Des estimations à poids permettent alors de conclure par un argument de bootstrap.

1.4.4 Le cas du rectangle: influence du bord et contrôle

Dans [GHKM18], Glass, Han-Kwan et Moussa ont réussi à exhiber un cadre géométrique particulier (et pertinent pour la modélisation) où il existe des solutions stationnaires non-triviales et régulières pour (1.3.1), étant, de plus, asymptotiquement stables. Le système est étudié sur le rectangle bidimensionnel

$$\Omega := \{(x_1,x_2) \in (-L,L)\times(-1,1)\}, \quad L>0.$$

Dans ce domaine, on considère une zone d'injection de particules sur le bord de gauche, tandis que les bords horizontaux et de droite sont associés à des conditions d'absorption des particules. Le fluide est lui associé à un écoulement de Poiseuille au bord, défini par

$$u_p(x) = u_{\max}(1-x_2^2)e_1, \quad u_{\max}>0,$$

et solution stationnaire des équations de Navier-Stokes, satisfaisant en particulier des conditions de Dirichlet homogènes sur les bord horizontaux.

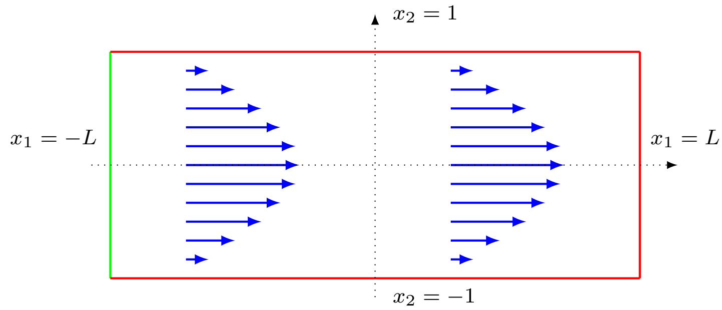

Figure 1.1: *En vert : injection des particules (condition au bord entrante) ; en rouge : absorption des particules ; en bleu : profil de Poiseuille* u_p.

On a donc

$$u(t,x) = u_p(x), \quad x \in \partial\Omega,$$
$$f(t,x,v) = \psi(x,v), \quad (x,v)\in\Sigma^-,$$

avec ψ à support en x dans la zone d'injection (en vert sur la Figure 1.1) et à support compact en vitesse, et où Σ^- désigne le 'bord rentrant' de l'espace des phases (défini en (1.2.2)). L'idée est que le mécanisme d'absorption au bord, combiné à l'injection, doit permettre d'éviter le scénario de concentration en vitesse pour la partie cinétique qui a été obtenu en Section 1.4.3. Pour profiter de l'absorption au bord, les auteurs de [GHKM18] introduisent des conditions de contrôle géométrique adaptées à l'injection/absorption dans le domaine Ω. Celles-ci doivent permettre d'assurer que les particules rejoignent les bords horizontaux ou de droite en un temps fixé à l'avance.

Pour cela, on regarde les trajectoires associées aux EDO (1.4.6) pour l'équation de Vlasov avec vitesse fluide U. Une définition informelle est la suivante.

Définition 1.4.6. *Soit $T > 0$ et K un compact de l'espace des phases. Un champ de vecteurs U satisfait la condition de sortie géométrique en temps T par rapport à K si toute trajectoire issue de K a un temps de vie dans Ω strictement inférieur à T (et sort de façon transversale).*

Cette condition clé est réminiscente de la Condition de Contrôle Géométrique (GCC) introduite par Bardos, Lebeau et Rauch [BLR92] dans le contexte du contrôle de l'équation de ondes. Elle est essentiellement utilisée pour $U = u_p$ et $K = \operatorname{supp} \psi$. En utilisant la stabilité de la condition de sortie géométrique par petite perturbation, on peut travailler dans un voisinage du flot de Poiseuille et construire des solutions stationnaires en remontant le long des trajectoires ayant été absorbées. Leur stabilité est ensuite obtenue en propageant la condition de sortie géométrique au cours du temps pour la solution issue d'une perturbation de l'état stationnaire.

Une version simplifiée du résultat est alors la suivante.

Théorème 1.4.7 (Glass, Han-Kwan et Moussa [GHKM18])**.** *Considérons le système* (1.3.1) *dans le rectangle Ω avec les conditions au bord précédentes. Alors:*

1. *pour un flot de Poiseuille u_p et une donnée d'injection ψ bien choisis, il existe une solution stationnaire régulière $(\overline{f}, \overline{u})$ non triviale (au sens où $\overline{f} \neq 0$);*

2. *sous les mêmes hypothèses, il y a la stabilité asymptotique de $(\overline{f}, \overline{u})$ dans l'espace L^2 pour des perturbations à support compact adéquat.*

Grosso modo, on peut dire que le phénomène d'absorption et d'injection au bord permet ici d'empêcher l'apparition de profil singulier pour la partie cinétique.

1.4.5 Bilan et questions sur le comportement en temps long

Dressons un rapide bilan sur la dynamique en temps long des équations de Vlasov-Navier-Stokes, ainsi que sur les questions qu'elle soulève.

Effet de bord. Les résultats énoncés dans les Sections 1.4.3 et 1.4.4 mènent aux conclusions suivantes:

- dans un domaine sans bord, des données initiales d'énergie finie (et petites ou proches de l'équilibre) conduisent à la concentration en vitesse de la partie cinétique, c'est-à-dire à la convergence vers un profil monocinétique;

- il existe des cas où, dans un domaine à bord avec des conditions mixtes d'absorption/injection au bord, le scénario de concentration en vitesse peut être évité.

Dans un domaine à bord $\Omega \subset \mathbb{R}^3$ quelconque, et avec la condition d'absorption au bord $f(t)_{|\Sigma^-} = 0$ pour la fonction de distribution il semble difficile de caractériser l'absence ou non de concentration en vitesse pour les particules.

Notons que l'inégalité d'énergie-dissipation (1.3.4) (à la base des stratégies mentionnées en Section 1.4.3) est toujours vérifiée, malgré les termes de bord. En effet, en multipliant l'équation de Vlasov du système (1.3.1) par $|v|^2$ et en intégrant sur $\Omega \times \mathbb{R}^3$, on obtient formellement par intégration par parties

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\times\mathbb{R}^3} f|v|^2 \,\mathrm{d}x\,\mathrm{d}v + \int_{\partial\Omega\times\mathbb{R}^3} |v|^2(\gamma f) v\cdot n_\Omega(x)\,\mathrm{d}\sigma(x)\,\mathrm{d}v = \int_{\Omega\times\mathbb{R}^3} v\cdot(u-v) f\,\mathrm{d}x\,\mathrm{d}v,$$

où γf désigne la trace de f au bord du domaine (voir Section 1.2.2). En sommant cette identité avec (1.3.2) pour les équations de Navier-Stokes (pour des conditions de Dirichlet homogène), on obtient

$$\mathrm{E}(f,u)(t) + \int_0^t \mathrm{D}(f,u)(\tau)\,\mathrm{d}\tau + \int_0^t \int_{\partial\Omega\times\mathbb{R}^3} |v|^2(\gamma f) v\cdot n_\Omega(x)\,\mathrm{d}\sigma(x)\,\mathrm{d}v\,\mathrm{d}\tau \leq \mathrm{E}(f^{\mathrm{in}}, u^{\mathrm{in}}).$$

Le dernier terme du membre de gauche est *a priori* non signé mais il est en fait positif puisque $(\gamma f)(t,x,v) = 0$ est nul lorsque $v\cdot n_\Omega(x) < 0$, en vertu de la condition d'absorption pour f.

Une question naturelle est alors la suivante : peut-on trouver des conditions nécessaires et suffisantes pour déterminer qui, de la concentration ou de l'absorption, l'emporte en temps long ? Peut-on quantifier le plus précisément possible cette compétition ?

Nous apporterons une réponse partielle à cette question au Chapitre 2, que nous présentons informellement dans la prochaine Section 1.4.6.

Influence de la gravité. Un autre problème, pertinent du point de vue physique (voir [Höf20]), est celui de la dynamique des équations de Vlasov-Navier-Stokes lorsque les effets de gravité sont pris en compte. On considère une force de gravité $G \in \mathbb{R}^3$ constante agissant sur les particules (voir Section 1.1.5) et on s'intéresse au système

$$\begin{cases} \partial_t u + (u\cdot\nabla_x)u + \nabla_x p - \Delta_x u = \displaystyle\int_{\mathbb{R}^d} (v-u) f\,\mathrm{d}v, \\ \mathrm{div}_x\, u = 0, \\ \partial_t f + v\cdot\nabla_x f + \mathrm{div}_v\,[f(u-v) + fG] = 0. \end{cases} \tag{1.4.8}$$

L'ajout d'une force de gravité dans l'équation cinétique paraît anodine mais est source de plusieurs difficultés dans l'analyse du couplage. Si l'on reprend l'identité d'énergie-dissipation, l'estimation d'énergie dans l'équation de Vlasov (1.4.8) devient

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{x,v} f(t)|v|^2\,\mathrm{d}x\,\mathrm{d}v = \int_{x,v} v\cdot(u-v) f\,\mathrm{d}x\,\mathrm{d}v + \int_{x,v} G\cdot v f\,\mathrm{d}x\,\mathrm{d}v,$$

de telle sorte que l'identité d'énergie (1.3.4), ici sous forme différentielle, doit être modifiée en

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) + \mathrm{D}(t) = \int_{x,v} G\cdot v f(t)\,\mathrm{d}x\,\mathrm{d}v.$$

Le terme additionnel au membre de droite n'a *a priori* pas de signe. Cependant, on peut penser que la vitesse des particules va finir par s'aligner avec le champ de gravité (par exemple dans un régime à vitesse fluide petite), si bien que ce terme deviendrait positif. Par conséquent, la décroissance en temps de l'énergie cinétique E ne semble pas assurée dans ce cadre (voir néanmoins la Remarque 1.4.16). Une image un peu vague est que la gravité fournit ici un réservoir d'énergie qui est continuellement injectée dans le système. Cette remarque est aussi faite par Höfer [Hö18] et Han-Kwan et Michel [HKMar].

Les travaux sur le comportement en temps long [HKMM20, HK22] étant basés de façon cruciale sur la décroissance de l'énergie, il apparaît difficile d'étudier la dynamique du système en suivant exactement la même stratégie. Sur tout l'espace (et abandonnant l'idée d'un champ de gravité avec des conditions au bord périodiques) et dans un régime à donnée petite, on attendrait un comportement de la forme

$$u(t) \underset{t\to+\infty}{\longrightarrow} 0, \quad f(t) \underset{t\to+\infty}{\sim} \rho_f(t)\otimes\delta_{v=G}.$$

L'approche de [HK22] ne permet cependant pas directement de traiter une telle dynamique, qui reste à notre connaissance un problème ouvert.

Pour étudier le comportement en temps long de (1.4.8), nous proposons au Chapitre 3 un cadre géométrique particulier, celui du demi-espace, enrichi de la condition d'absorption au bord pour les particules. Nous montrerons que ce cadre permet de prouver la convergence des solutions au système vers l'état trivial. Nous présentons les résultats obtenus dans la Section 1.4.7.

1.4.6 Contribution du Chapitre 2 : concentration et absorption pour le système de Vlasov-Navier-Stokes dans un domaine borné

Le Chapitre 2 reproduit le contenu de l'article [EHKM21], écrit en collaboration avec Daniel Han-Kwan et Ayman Moussa, et publié dans *Nonlinearity*. On s'intéresse à la dynamique en temps long du système de Vlasov-Navier-Stokes sur un domaine borné, avec condition d'absorption au bord pour les particules (voir en particulier la Section 1.2.2).

Soit $\Omega \subset \mathbb{R}^3$ un domaine borné régulier. On considère le système (1.3.1) enrichi des conditions au bord

$$f(t)_{|\Sigma^-} = 0, \quad u(t)_{|\partial\Omega} = 0. \tag{1.4.9}$$

À cause de la condition au bord pour la vitesse du fluide, on peut s'attendre classiquement à une relaxation de celle-ci vers 0. Pour la partie cinétique, le problème est d'étudier la compétition entre absorption et convergence vers un profil singulier en vitesse.

1.4.6.1 Résultat principal

Les résultats principaux du Chapitre 2 sont les suivants.

Théorème 1.4.8 (Chapitre 2 - basé sur [EHKM21])**.** *Soit $(f^{\mathrm{in}}, u^{\mathrm{in}})$ une condition initiale admissible pour le système* (1.3.1) *avec f^{in} suffisament décroissante en vitesse (uniformément en x) et telle que*

$$\mathrm{E}(0) + \|\nabla u^{\mathrm{in}}\|_{\mathrm{L}^2(\Omega)} \ll 1.$$

où E *est définie par* (1.3.5)*. Alors toute solution faible globale (f, u) de* (1.3.1) *partant de $(f^{\mathrm{in}}, u^{\mathrm{in}})$ et satisfaisant* (1.4.9) *vérifie pour tout $t \geq 0$*

$$\mathrm{W}_1\Big(f(t), \rho_f(t) \otimes \delta_{v=0}\Big) + \|u(t)\|_{\mathrm{L}^2(\Omega)} \lesssim \mathrm{E}(0)^{1/2} e^{-\lambda t}.$$

pour un certain $\lambda > 0$, et où W_1 désigne la distance de Wasserstein. De plus, il existe $\rho^\infty \in \mathrm{L}^\infty(\Omega)$ tel que

$$\rho_f(t) \overset{t \to +\infty}{\longrightarrow} \rho^\infty \quad \text{dans} \quad \mathrm{H}^{-1}(\Omega),$$

à vitesse exponentielle.

Il existe donc un phénomène de concentration en vitesse dans un domaine borné avec absorption des particules au bord. Cependant, il ne s'agit que d'une réponse partielle car le comportement asymptotique se cache dans le profil spatial ρ^∞. Comme dans [HKMM20], il est possible de décrire (implicitement) ce profil, qui dépend encore une fois de toute la dynamique des solutions.

Théorème 1.4.9 (Chapitre 2 - basé sur [EHKM21]). *Sous les hypothèses du Théorème 1.4.8, il existe un champ de vecteurs régulier* $\mathrm{X}_\infty : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *donné par*

$$\mathrm{X}_\infty(y,v) = y + v + \int_0^\infty u(\tau, \mathrm{X}_0^\tau(y,v))\,\mathrm{d}\tau,$$

avec

$$\mathrm{X}_0^t(y,v) = y + (1-e^{-t})v + \int_0^t (1-e^{\tau-t})u(\tau, \mathrm{X}_0^\tau(y,v))\,\mathrm{d}\tau,$$

tel que pour tout $v \in \mathbb{R}^3$*, l'application* $\mathrm{X}_{\infty,v} : y \mapsto \mathrm{X}_\infty(y,v)$ *est un difféomorphisme de* $\mathbb{R}^3$ *satisfaisant l'identité suivante: pour presque tout* $x \in \Omega$*, on a*

$$\rho^\infty(x) = \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{U}^\infty}(x,v) f^{\mathrm{in}}\left(\mathrm{X}_{\infty,v}^{-1}(x), v\right) |\det \mathrm{D}_x \mathrm{X}_{\infty,v}^{-1}(x)|\,\mathrm{d}v, \tag{1.4.10}$$

où

$$(x,v) \in \mathcal{U}^\infty \subset \Omega \times \mathbb{R}^3 \Longleftrightarrow \exists! y \in \Omega, \quad x = \mathrm{X}_\infty(y,v) \quad et \quad \forall t \geq 0, \quad \mathrm{X}_0^t(y,v) \in \Omega.$$

Remarque 1.4.10. Le théorème 1.4.8 est aussi un théorème d'existence globale de solutions fortes (pour la partie fluide uniquement) pour (1.3.1) puisque l'on prouve par la même occasion, et pour les besoins de la preuve, que

$$u \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}^1(\Omega)) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}^2(\Omega)).$$

On peut en fait essayer d'obtenir plus d'informations sur le profil asymptotique limite ρ^∞. Une information intéressante est par exemple encodée dans la masse finale $\int_\Omega \rho^\infty(x)\,\mathrm{d}x$: rappelons en effet que la masse totale $\int_\Omega \rho_f(t,x)\,\mathrm{d}x$ n'est pas conservée au cours de l'évolution à cause de la condition d'absorption au bord. Il existe en fait une large gamme de scénarios, allant de l'absorption totale des particules jusqu'à leur confinement 'loin' du bord. Le résultat suivant montre que l'on peut obtenir n'importe quelle masse finale (inférieure à la masse initiale), pour des données initiales bien choisies.

Proposition 1.4.11 (Chapitre 2 - basé sur [EHKM21]). *Soit* $m \in [0,1]$*. Il existe une condition initiale* $(u^{\mathrm{in}}, f^{\mathrm{in}})$ *admissible pour* (1.3.1) *avec* $\int_\Omega \rho_{f^{\mathrm{in}}}(x)\,\mathrm{d}x = 1$ *telle que pour toute solution faible globale* (f,u) *de* (1.3.1) *partant de* $(f^{\mathrm{in}}, u^{\mathrm{in}})$ *et satisfaisant* (1.4.9)*, il existe* $\rho^\infty \in \mathrm{L}^\infty(\Omega)$ *vérifiant*

$$\rho_f(t) \xrightarrow[t\to+\infty]{} \rho^\infty \quad \text{dans} \quad \mathscr{C}(\overline{\Omega})', \quad \int_\Omega \rho^\infty(x)\,\mathrm{d}x = m.$$

Eléments de preuve. Esquissons quelques idées de preuve. Comme dans [HKMM20], la ligne directrice est la décroissance d'une bonne fonctionnelle capturant le comportement en temps long. Ici, la fonctionnelle d'énergie à considérer est l'énergie cinétique elle-même, traduisant la convergence des vitesses fluide et des particules vers 0.

La décroissance exponentielle de l'énergie provient de l'existence d'une inégalité de Poincaré pour la vitesse du fluide dans l'ouvert Ω, sous réserve d'une borne $\mathrm{L}^\infty(\mathbb{R}^+; \mathrm{L}^\infty(\Omega))$ sur la densité locale ρ_f.

On adopte ensuite une approche formellement basée sur la formule de représentation de la solution f, prenant en compte les effets d'absorption au bord, et en écrivant :

$$\rho_f(t,x) = \int_{\mathbb{R}^3} e^{3t} \mathbf{1}_{\mathcal{O}^t}(x,v) f^{\mathrm{in}}(\mathrm{X}^{0;t}(x,v), \mathrm{V}^{0;t}(x,v))\,\mathrm{d}v.$$

où $s \mapsto (\mathrm{X}^{s;t}(x,v), \mathrm{V}^{s;t}(x,v))$ sont les courbes caractéristiques définies comme en Section 1.2.2.1 et où

$$\mathcal{O}^t = \left\{(x,v) \in \Omega \times \mathbb{R}^d \mid \forall \sigma \in [0,t],\ \mathrm{X}^{\sigma;t}(x,v) \in \Omega\right\}.$$

Une étude des courbes caractéristiques montre que la condition $\|\nabla_x u\|_{\mathrm{L}^1(\mathbb{R}^+;\mathrm{L}^\infty(\Omega))} \ll 1$ permet à nouveau d'assurer la borne sur ρ_f.

Un argument de bootstrap est alors mis en place pour assurer le contrôle précédent : comme pour [HKMM20], on interpole entre la décroissance exponentielle de l'énergie (assurant intégrabilité en temps et petitesse *via* les données initiales) et des estimations d'ordre plus élevé pour u. Des outils clés sont la théorie de la régularité maximale pour le système de Stokes sur un domaine et la régularisation parabolique due aux équations de Navier-Stokes. Plusieurs arguments basés sur des outils d'analyse harmonique, qui pouvaient être invoqués sur le tore, ne sont plus disponibles ici.

La caractérisation de la masse du profil asymptotique passe par une étude minutieuse des trajectoires en fonction du support en espace et en vitesse de la donnée initiale f^{in}. Nous utilisons le fait que la solution f possède un support dans l'espace des phases qui est l'image du support initial par le flot, c'est-à-dire

$$\forall t \geq 0, \quad \operatorname{supp} f(t) \subset \left(\mathrm{X}(t;0,\operatorname{supp} f^{\mathrm{in}}) \times \mathrm{V}(t;0,\operatorname{supp} f^{\mathrm{in}})\right) \cap \mathcal{O}^t \subset \mathbb{R}^3 \times \mathbb{R}^3.$$

Nous étudions alors séparément le cas d'un support initial ne contenant que des basses vitesses et qui est, soit localisé spatialement loin du bord du domaine (donnant des trajectoires piégées, pour lesquelles la masse est conservée), soit localisé spatialement dans une couronne près du bord et avec des vitesses sortantes (conduisant à des trajectoires totalement absorbées, et donc à une perte de masse). Ces deux situations sont ensuite combinées pour prouver la Proposition 1.4.11.

1.4.6.2 Perspectives

On peut envisager plusieurs directions de recherche pour prolonger l'étude de la dynamique du système de Vlasov-Navier-Stokes (1.3.1).

- **Question 1**. Comme annoncé auparavant, le résultat obtenu au Théorème 1.4.8 n'est que partiel et tout réside dans la possible description du profil asymptotique ρ^∞. Ce problème est déjà présent dans le cas du tore mais le phénomène supplémentaire d'absorption rajoute une grande variété de scénarios limites. Il pourrait être intéressant de réussir à caractériser d'autres propriétés pertinentes pour ce profil.

- **Question 2**. Dans une direction un peu différente, sait-on si une donnée initiale qui serait monocinétique donne naissance à des solutions qui le restent au cours de l'évolution ? Cette question a été étudiée par Jabin dans [Jab02], dans un cadre simplifié où il n'y a pas de couplage car la vitesse du fluide est donnée et très régulière. Le problème se ramène à un système d'Euler compressible sans pression, mais posé dans un domaine à frontière libre. Bien évidemment, on quitte le cadre des solutions autorisées par le Théorème 1.3.1. Il serait intéressant d'essayer d'étendre ce type de résultat au couplage total, d'abord en dimension 2.

- **Question 3**. Comme expliqué en Section 1.4.3, la stratégie de [HKMM20] vise à obtenir une borne $\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^\infty(\mathbb{T}^3))$ sur ρ_f pour obtenir la dynamique en temps long sur le tore, alors que le résultat conditionnel énoncé au Lemme 1.4.3 ne demande qu'un contrôle dans $\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^{3/2}(\mathbb{T}^3))$. Il pourrait être intéressant de savoir si une preuve différente permet d'accéder au comportement en temps long en n'assurant que cette condition plus faible.

- **Question 4**. Enfin, à la suite des travaux [HKMM20, HK22] et du résultat présenté ici, on peut se demander s'il existe un autre moyen qu'une méthode "par énergie" pour étudier les phénomènes de concentration en vitesse pour le système de Vlasov-Navier-Stokes. Une difficulté majeure provient du fait qu'il est difficile de linéariser les équations autour d'un profil singulier en vitesse. Dans un travail en cours d'écriture avec Aymeric Baradat et Daniel Han-Kwan, nous développons une approche par représentation multiphasique des solutions, qui semble particulièrement adaptée pour traiter ce type de problème.

1.4.7 Contribution du Chapitre 3 : dynamique en temps long du système de Vlasov-Navier-Stokes avec gravité dans le demi-espace

Le Chapitre 3 reproduit le contenu de l'article prépublié [Ert21]. On s'intéresse au comportement en temps long du système de Vlasov-Navier-Stokes avec gravité (1.4.8), dans un contexte géométrique particulier et détaillé ci-dessous.

Rappelons que la principale difficulté concernant la dynamique en temps pour (1.4.8) est la non décroissance *a priori* de l'énergie cinétique totale du système. La présence de la force de gravité vient donc briser l'inégalité d'énergie dissipation clé qui avait permis d'aboutir à la concentration en vitesse dans [HKMM20, HK22] (voir la Section 1.4.3 et le Chapitre 2). Pour contourner cet obstacle, nous étudions (1.4.8) dans le cadre simplifié suivant.

Cadre géométrique et hypothèse. Le domaine spatial retenu est le demi-espace en dimension 3, à savoir

$$\mathbb{R}^3_+ := \mathbb{R}^2 \times (0, +\infty).$$

Il s'agit donc d'un domaine non borné avec une frontière. Ce choix est combiné avec un champ de gravité constant pointant verticalement vers la frontière, à savoir

$$G = -ge_3, \quad g > 0, \; e_3 = (0, 0, 1).$$

On peut normaliser cette force en prenant $g = 1$.

Conditions au bord. Nous considérons les conditions au bord

$$f(t)_{|\Sigma^-} = 0, \quad u(t)_{|x_3=0} = 0, \tag{1.4.11}$$

c'est-à-dire à nouveau une condition d'absorption au bord pour la fonction de distribution et une condition de Dirichlet homogène pour le fluide. En ce sens, les travaux du Chapitre 3 prolongent ceux du Chapitre 2 concernant la dynamique du système de Vlasov-Navier-Stokes dans un domaine à bord et avec absorption.

L'idée majeure est que l'alliance de l'absorption *et* de la force de gravité va permettre d'obtenir suffisamment de décroissance des solutions, et *in fine* de prouver la stabilité asymptotique de la solution $(0, 0)$.

1.4.7.1 Résultat principal

Une version simplifiée du théorème principal est la suivante (on renvoie à l'énoncé précis du Théorème 3.2.1 au Chapitre 3).

Théorème 1.4.12 (Chapitre 3 - basé sur [Ert21])**.** *Soit* $(f^{\text{in}}, u^{\text{in}})$ *une condition initiale admissible pour le système* (1.4.8) *telle que*

$$\mathrm{E}(0) + \|u^{\text{in}}\|_{\mathrm{L}^1 \cap \mathrm{H}^1(\mathbb{R}^3_+)} \ll 1, \qquad \sup_{(x,v) \in \mathbb{R}^3_+ \times \mathbb{R}^3} (1 + x_3)^q (1 + |v|)^q f^{\text{in}}(x, v) \ll 1,$$

pour un $q > 0$ assez grand, et où E est définie par (1.3.5). Alors toute solution faible globale (f, u) de (1.4.8) partant de $(f^{\mathrm{in}}, u^{\mathrm{in}})$ vérifie pour tout $t \geq 0$

$$\|u(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \frac{1}{(1+t)^{3/4}}.$$

De plus, il existe $k = k(q) > 0$ tel que pour tout $r \in [1, +\infty]$ et tout $t \geq 0$

$$\|m_\ell f(t)\|_{\mathrm{L}^r(\mathbb{R}^3_+)} \lesssim \frac{1}{(1+t)^k}, \quad 0 \leq \ell < q - 3,$$

où l'on a noté

$$m_\ell f(t, x) := \int_{\mathbb{R}^3} f(t, x, v)|v|^\ell \, \mathrm{d}v.$$

Remarque 1.4.13. Le jeu d'exposants admissibles dans cet énoncé peut être quantifié explicitement - voir la Proposition 3.2.2 au Chapitre 3. Le point clé est le suivant : plus la décroissance initiale sur f^{in} est forte dans l'espace des phases, plus la décroissance en temps d'un nombre élevé de moments de la solution $f(t)$ est permise. En particulier, on obtient un résultat de propagation des moments pour les équations de Vlasov-Navier-Stokes.

Remarque 1.4.14. Comme au Chapitre 2, la preuve s'acccompagne également d'un résultat d'existence globale de solutions fortes (pour la partie fluide uniquement) pour le couplage. La régularité sur u^{in} permet en effet de propager des estimations H^1 sur u. Notons que l'on a en fait besoin d'un peu plus de régularité sur u^{in}, dans le but d'invoquer la théorie de la régularité maximale pour l'équation de Stokes au cours de la preuve.

Eléments de preuve. Essayons d'expliquer le mécanisme principal à l'œuvre dans la preuve. Afin de profiter du phénomène d'absorption au bord, nous adoptons un point de vue Lagrangien sur l'équation cinétique. Cette approche est basée sur la formule de représentation

$$f(t, x, v) = e^{3t} \mathbf{1}_{\mathcal{O}^t}(x, v) f^{\mathrm{in}}(\mathrm{X}^{0;t}(x, v), \mathrm{V}^{0;t}(x, v)),$$

où l'on a défini les courbes caractéristiques par

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{X}^{s;t}(x, v) = \mathrm{V}^{s;t}(x, v), \quad \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}^{s;t}(x, v) = u(s, \mathrm{X}^{s;t}(x, v)) - \mathrm{V}^{s;t}(x, v) + G,$$

avec $(\mathrm{X}^{t;t}(x, v), \mathrm{V}^{t;t}(x, v)) = (x, v)$ et où

$$\mathcal{O}^t = \left\{(x, v) \in \Omega \times \mathbb{R}^d \mid \forall \sigma \in [0, t], \ \mathrm{X}^{\sigma;t}(x, v) \in \mathbb{R}^3_+\right\}. \tag{1.4.12}$$

Dans l'esprit de [GHKM18] (voir la Définition 1.4.6), nous introduisons alors une condition de contrôle géométrique adaptée au demi-espace.

Définition 1.4.15. *Etant donnés $L, R > 0$ et un temps $T > 0$, on dit qu'un champ de vecteurs $u : \mathbb{R} \times \mathbb{R}^3_+ \to \mathbb{R}^3$ satisfait une condition de sortie géométrique $\mathrm{EGC}_T(L, R)$ si toute trajectoire issue du domaine $(\mathbb{R}^2 \times (0, L)) \times \mathrm{B}(0, R)$ de l'espace des phases quitte le demi-espace avant le temps T.*

L'intérêt principal de l'ECG est la suivante : si $\mathrm{EGC}_T(L, R)$ est satisfaite alors pour tout $t > T$, on a

$$\mathbf{1}_{|\mathrm{V}^{0;t}(x,v)|>R} \, \mathbf{1}_{\mathrm{X}^{0;t}(x,v)_3>L} f(t, x, v) = 0,$$

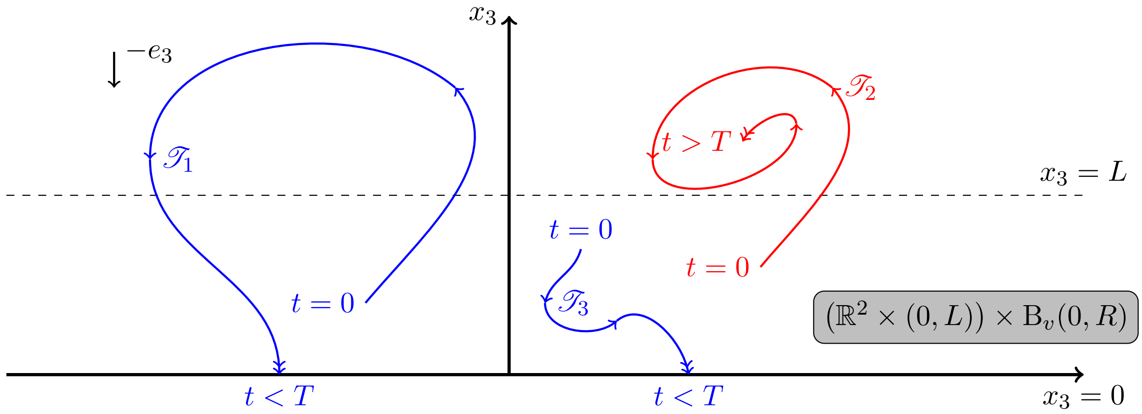

Figure 1.2: $\mathrm{EGC}_T(L, R)$ non satisfaite (trajectoire $\mathscr{T}_2$ non absorbée avant $t = T$)

puisque les trajectoires issues de ce domaine de l'espace des phases ont été absorbées avant le temps T. On en déduit essentiellement pour tout $k > 0$ et $\ell > 0$

$$f(t, x, v) \leq \frac{1}{R^k}\frac{1}{L^\ell}|\mathrm{V}^{0;t}(x, v)|^k|\mathrm{X}^{0;t}(x, v)|^\ell e^{3t}\mathbf{1}_{\mathcal{O}^t}(x, v)f^{\mathrm{in}}(\mathrm{X}^{0;t}(x, v), \mathrm{V}^{0;t}(x, v)). \tag{1.4.13}$$

Si la donnée initiale f^{in} est suffisamment décroissante en (x, v), l'inégalité précédente permet alors d'obtenir de la décroissance sur les moments en vitesse de la solution $f(t)$. Pour cela, nous utilisons le changement de variable en vitesse $v \mapsto \mathrm{V}^{0;t}(x, v)$ qui est valide si la condition de petitesse $\|\nabla_x u\|_{\mathrm{L}^1_t\mathrm{L}^\infty_x} \ll 1$ est vérifiée, comme dans la méthode de preuve présentée en Section 1.4.3 (et qui a l'avantage de compenser le facteur exponentiel en temps).

Afin de propager la condition de sortie géométrique, nous nous appuyons sur un résultat de stabilité de l'EGC qui autorise moralement à ne regarder que la dynamique du système "libre avec gravité", associée à

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{X}_g^{s;t}(x, v) = \mathrm{V}_g^{s;t}(x, v), \quad \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}_g^{s;t}(x, v) = -\mathrm{V}_g^{s;t}(x, v) + G,$$

pourvu que le champ u satisfasse la condition de petitesse $\|u\|_{\mathrm{L}^1_t\mathrm{L}^\infty_x} \ll 1$. L'avantage des trajectoires précédentes est qu'elles sont très faciles à analyser explicitement. On montre que pour $T > 0$, la condition $\mathrm{EGC}_T(L, R)$ est satisfaite pour $L, R \sim 1 + T$.

Revenant à (1.4.13), on obtient donc de la décroissance polynomiale en temps sur les moments de la fonction de distribution. Insistons sur le fait que la présence du terme de gravité est ici *nécessaire* pour mener cette analyse.

On souhaite donc propager la condition $\|u\|_{\mathrm{L}^1(\mathbb{R}^+;\mathrm{W}^{1,\infty}(\mathbb{R}^3_+))} \ll 1$ par un argument de bootstrap. Cette petitesse sera essentiellement obtenue si la vitesse du fluide décroît suffisamment en temps. Pour cela, on se souvient que le terme source dans les équations de Navier-Stokes, à savoir la force de Brinkman $j_f - \rho_f u$, est composée de deux premiers moments en vitesse de f, qui sont justement supposés décroître en temps à vitesse polynomiale.

Cela rentre dans le cadre de la théorie de Schonbek et Wiegner [Sch86, Wie87] pour les équations de Navier-Stokes en domaine non borné : si la données initiale u^{in} est intégrable et d'énergie finie, la solution u décroît polynomialement comme la solution de l'équation de la chaleur (nous utilisons une adaptation au cas du demi-espace dans l'esprit de [BM90] - voir l'Appendixe 3.E au Chapitre 3). Comme dans [HK22], cette décroissance est trop lente pour être directement utilisée mais des estimations à poids propagées au cours du bootstrap et la décroissance des moments en vitesse permettent de conclure.

Remarque 1.4.16. En comparaison, la preuve utilisée par Han-Kwan dans [HK22] (sur tout l'espace et sans gravité) n'utilise pas directement le résultat de Schonbek et Wiegner [Sch86, Wie87] : elle est en effet basée sur le couplage algébrique fin entre les équations de Navier-Stokes et de Vlasov, *via* l'inégalité d'énergie-dissipation. À cause de l'ajout de la force de gravité, nous ne pouvons pas utiliser cette approche et devons invoquer un mécanisme supplémentaire provenant de l'absorption des particules.

Richard Höfer nous a néanmoins mentionné la remarque suivante : en introduisant la fonctionnelle d'énergie potentielle

$$\mathrm{E}^{\mathrm{p}}(t) := -\int_{x,v} G \cdot x f(t)\,\mathrm{d}x\,\mathrm{d}v$$

pour (1.4.8), on peut montrer que l'on a formellement

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}^{\mathrm{p}}(t) = -\int_{x,v} G \cdot v f(t)\,\mathrm{d}x\,\mathrm{d}v$$

si bien que, avec les notations standards pour l'énergie cinétique et la dissipation, on obtient

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{E} + \mathrm{E}^{\mathrm{p}}(t)\right) + \mathrm{D}(t) = 0.$$

Il est donc tentant de considérer l'énergie totale $\mathrm{E}+\mathrm{E}^{\mathrm{p}}$, qui décroît bel et bien, et qui est positive dans le cas du demi-espace avec $G = -e_3$. Néanmoins, l'adaptation de la preuve de [HK22] nécessiterait de prouver que l'on peut contrôler par le bas la dissipation D par une partie de l'énergie potentielle E^{p}. Cette information est, à notre connaissance, difficile à obtenir. Le seul moyen d'intégrer le terme $\mathrm{E}^{\mathrm{p}}(t)$ à l'analyse serait de prouver qu'il décroît lui aussi à vitesse polynomiale. On retombe donc essentiellement sur la stratégie détaillée plus haut à propos du Chapitre 3.

1.4.7.2 Perspectives

On pourrait tenter de prolonger le Théorème 1.4.12 dans plusieurs directions.

- **Question 1**. Peut-on abaisser les différentes hypothèses de petitesse et de régularité sur les données initiales faites au Théorème 1.4.8, de même que les hypothèses de décroissance dans l'espace des phases sur la donnée cinétique initiale ? Cette dernière semble naturelle et adaptée à notre stratégie et il paraît difficile de contrôler d'une autre manière l'effet des "grandes" vitesses qui mèneraient à des trajectoires n'étant pas *in fine* absorbées.

- **Question 2**. Quelle est la situation dans un domaine extérieur (*i.e.* le complémentaire d'un compact) ? Cette variante pourrait plus généralement s'insérer dans le cadre de la correction d'Oberbeck-Boussinesq (au niveau macroscopique) : la force de gravitation constante serait remplacée par le gradient d'une fonction harmonique décroissant comme le potentiel Newtonien associé au complémentaire du domaine (voir par exemple [FS12]).

- **Question 3**. La question du comportement en temps long pour (1.4.8) reste ouverte dans le cas de l'espace entier. Aller plus loin que la remarque 1.4.16 demanderait de mieux comprendre la dynamique de l'énergie potentielle. Des calculs préliminaires basés sur la dérivée seconde de l'énergie ne semblent pour l'instant pas suffisants.

1.5 Limites hydrodynamiques pour le système de Vlasov-Navier-Stokes

Dans cette section, nous abordons le problème des limites hydrodynamiques pour les équations de Vlasov-Navier-Stokes.

Cette problématique est à nouveau dans l'esprit du 6ème problème de Hilbert à propos de l'axiomatisation de la mécanique et de la physique. Si la dérivation (partielle) des couplages fluide-cinétique depuis les lois de Newton a été évoquée en Section 1.3.3, la question concerne ici **le passage du fluide-cinétique vers les équations hydrodynamiques**.

Plus précisément, on aimerait remplacer la description à l'échelle mésoscopique des particules par un comportement effectif décrit à l'aide de grandeurs macroscopiques (basée sur l'espace physique et non pas sur l'espace des phases).

L'idée vague est la suivante : les équations au niveau cinétique apparaissent en réalité avec des coefficients physiques (positifs) devant chaque terme, que l'on a jusqu'à présent normalisés. En adimensionnant les équations, on peut faire apparaître plusieurs ratios sans dimension de quantités physiques, qui peuvent être d'ordre 1, ε, ε^2, ... avec $0 < \varepsilon \ll 1$. Cela traduit différents régimes de l'équation, où l'on suppose que certains termes ont une contribution dominante par rapport à d'autres. A l'inverse, on aimerait n'obtenir que des constantes d'ordre 1 dans le système macroscopique limite, ce qui est notamment plus facile à manipuler numériquement.

Un exemple célèbre de tel problème concerne la limite hydrodynamique de l'équation collisionnelle de Boltzmann (voir Section 1.1.3.2). Après adimensionnement, on considère

$$\tau_\varepsilon \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon} \mathcal{Q}(f_\varepsilon, f_\varepsilon),$$

où $\varepsilon > 0$ est un petit paramètre appelée *nombre de Knudsen* et $\tau_\varepsilon > 0$ est un autre paramètre à choisir en fonction du régime fluide.

De manière informelle, un régime physique où les collisions sont prédominantes ($\varepsilon \to 0$) comme ici doit conduire la distribution des particules f_ε à se rapprocher d'un équilibre Maxwellien local. Ce dernier est associé à des quantités hydrodynamiques vérifiant les équations d'Euler, de (Navier-) Stokes ou de l'acoustique. La littérature étant si développée sur ce sujet, on renvoie plutôt à l'article de revue de Villani [Vil02a], aux monographies de Golse [Gol05] et Saint-Raymond [SR09a] et aux quelques références ci-dessous. On reverra apparaître formellement une limite hydrodynamique de type collisionnel concernant les sprays épais en Section 1.6.1.2.

Il existe plusieurs techniques générales pour aborder ce problème. On peut par exemple utiliser des méthodes **de compacité faible**, dans l'esprit du programme de Bardos, Golse et Levermore [BG91, BGL93], permettant d'étudier la limite de l'équation de Boltzmann vers les équations de Navier-Stokes incompressible (voir [GSR04, GSR09, Ars12]). On peut aussi **moduler une fonctionnelle** bien choisie par les solutions du problème limite. Cette méthode permet par exemple de traiter la limite vers les équations d'Euler incompressible (voir [SR03, SR09b]). Enfin, on peut travailler dans un cadre à forte régularité et procéder à une **étude spectrale** de l'opérateur de Boltzmann linéarisé (voir [Bri15, BMAM19, GT20, Ger22]).

On renvoie par exemple à la discussion détaillée et référencée faite dans l'introduction de [HKMar, ALT20].

1.5.1 Richesse des scalings pour le système de Vlasov-Navier-Stokes

Il existe de nombreux régimes possibles pour les équations de Vlasov-Navier-Stokes, preuve de la richesse physique et mathématique derrière ce modèle. Afin de ne pas alourdir la présentation, on ne détaille pas ici le processus d'adimensionnement et présente directement le système obtenu. On renvoie au Chapitre 4 pour un exemple détaillé d'une telle procédure.

Afin de considérer une écriture unifiée, on s'intéresse aux solutions $(f_\varepsilon, u_\varepsilon)$ des équations

$$
(\mathrm{VNS}_\varepsilon)\begin{cases}
\partial_t f_\varepsilon + \dfrac{1}{\varepsilon^a} v\cdot\nabla_x f_\varepsilon + \dfrac{1}{\varepsilon}\mathrm{div}_v\left[f_\varepsilon(\varepsilon^a u_\varepsilon - v)\right] = 0,\\
\partial_t u_\varepsilon + (u_\varepsilon\cdot\nabla_x)u_\varepsilon - \Delta_x u_\varepsilon + \nabla_x p_\varepsilon = \dfrac{1}{\varepsilon^b}(j_\varepsilon - \rho_\varepsilon u_\varepsilon),\\
\mathrm{div}_x\, u_\varepsilon = 0,\\
\rho_\varepsilon(t,x) := \displaystyle\int_{\mathbb{R}^3} f_\varepsilon(t,x,v)\,\mathrm{d}v, \quad j_\varepsilon(t,x) := \dfrac{1}{\varepsilon^a}\int_{\mathbb{R}^3} v f_\varepsilon(t,x,v)\,\mathrm{d}v,
\end{cases} \tag{1.5.1}
$$

où $\varepsilon > 0$ est un petit paramètre et pour un couple $(a, b) \in \{0, 1/2\} \times \{0, 1\}$ à choisir (voir ci-dessous). Le problème de la limite hydrodynamique revient à savoir si l'on peut remplacer (1.5.1) par un système hydrodynamique qui restitue le comportement des équations lorsque $\varepsilon \to 0$. Il s'agit d'un problème de limite singulière, dans un **régime de haute-friction**.

Dans le cas des couplages fluide-cinétique comme (1.5.1), on s'intéresse à la convergence de $(f_\varepsilon, u_\varepsilon)$ quand $\varepsilon \to 0$. En réalité, c'est plutôt la limite des moments en vitesse ρ_ε ou j_ε qui apparaitra dans le système final. Notons par ailleurs que pour tout $\varepsilon > 0$, le système (1.5.1) possède une solution faible globale $(f_\varepsilon, u_\varepsilon)$ puisque le Théorème 1.3.1 s'applique *mutatis mutandis* aux régimes dépendant de ε.

Précisons les scalings retenus dans (1.5.1) et la littérature les concernant :

- Cas $(a = 0, b = 0)$: ce régime est celui des *particules légères* (suivant [HKMar]). Il a été étudié dans un travail préliminaire de Jabin [Jab00a] sur un modèle-jouet où le fluide est donné par une convolution avec des moments de f. Le cas de la dimension 1 a été étudiée par Goudon dans [Gou01], mais la preuve est spécifique au cadre unidimensionnel (voir aussi [CWY20] pour une variante compressible). Plus récemment, Han-Kwan et Michel [HKMar] ont réussi à justifier la limite hydrodynamique dans ce cas pour (1.5.1), sur le tore en dimension 3. Le système obtenu à la limite $\varepsilon \to 0$ est le système de Transport-Navier-Stokes incompressible, où le fluide ne subit plus l'influence des particules, et où la densité est seulement transportée par celui-ci.

- Cas $(a = 1/2, b = 0)$: ce régime est celui des *particules légères et rapides* (suivant [HKMar]). La limite hydrodynamique est étudiée par Han-Kwan et Michel [HKMar] sur le tore en dimension 3. Le système obtenu à la limite $\varepsilon \to 0$ est encore le système de Transport-Navier-Stokes incompressible.

- Cas $(a = 0, b = 1)$: ce régime est celui des *particules fines* (suivant [GJV04b] et [HKMar]). Ce scaling correspond au régime le plus singulier de cette liste. En dimension 1, il est étudié par Goudon [Gou01]. Le cas tridimensionnel sur le tore est obtenu par Han-Kwan et Michel [HKMar].

  Dans ce cas, le système obtenu dans l'asymptotique $\varepsilon \to 0$ est le système est de Navier-Stokes inhomogène incompressible :

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho u) = 0, \\ \partial_t((1+\rho)u) + \mathrm{div}_x((1+\rho)u \otimes u) - \Delta_x u + \nabla_x p = 0, \\ \mathrm{div}_x\, u = 0. \end{cases} \tag{1.5.2}$$

Le couplage est encore présent entre les deux phases, et la densité spatiale des particules vient s'ajouter à celle du fluide. Mentionnons également l'apparition de ce régime dans le travail de Benjelloun, Desvillettes et Moussa [BDM14] sur un système bidispersé en taille mais dont l'asymptotique n'est que formellement dérivée. Elle est ensuite rigoureusement justifiée dans [HKMar].

A la différence de l'équation de Boltzmann, l'asymptotique $\varepsilon \to 0$ dans (1.5.1) fait apparaître des objets singuliers en vitesse pour la partie cinétique. Dans les cas $a = 0$ ci-dessus, elle est reliée à la convergence

$$u_\varepsilon(t) \underset{\varepsilon \to 0}{\longrightarrow} u(t), \quad f_\varepsilon(t) \underset{\varepsilon \to 0}{\longrightarrow} \rho(t) \otimes \delta_{v=u(t)},$$

où (ρ, u) est solution du système hydrodynamique limite. Notons que l'on a déjà exhibé ce type de profil monocinétique à la Section 1.4 lors de notre étude du comportement en temps long. Cela peut se comprendre ici par la présence du facteur ε^{-1} devant le terme $\mathrm{div}_v\,[f_\varepsilon(u_\varepsilon - v)]$, qui agit comme pénalisation. Après intégration en vitesse, la limite $\varepsilon \to$ tend ainsi à rapprocher les vitesses des particules autour de la vitesse du fluide.

A notre connaissance, le travail de Han-Kwan et Michel est donc le plus complet obtenu jusqu'à présent pour (1.5.1) (sur le tore tridimensionnel). On n'insiste pas sur un énoncé précis et on renvoie plutôt directement à l'article [HKMar]. Nous ne donnons que quelques grandes idées de preuve, qui emprunte certaines techniques au travail [HKMM20] sur la dynamique en temps de long de (1.3.1) sur le tore.

Comme on s'attend à obtenir une masse de Dirac en vitesse à la limite, travailler dans un cadre à forte régularité ou directement par linéarisation s'avère délicat. Le but est plutôt d'obtenir des estimations uniformes, notamment sur ρ_ε dans $\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^3))$. Comme pour la dynamique en temps long, cela ne suit découle directement de la conservation de la masse. Suivant [HKMM20], un contrôle sur $\nabla_x u_\varepsilon$ dans $\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3)$ permet d'obtenir les bornes souhaitées, au moins en temps court. L'utilisation de la fonctionnelle d'énergie modulée $\mathscr{E}_{\mathbb{T}^3,\varepsilon}$, adaptée au scaling, permet de récupérer certaines estimations en temps long (pour des données "proches de l'équilibre") *via* un argument de bootstrap.

La difficulté supplémentaire des limites hydrodynamiques consiste à obtenir des bornes uniformes pour la force de Brinkman

$$\varepsilon^{-b}(j_\varepsilon - \rho_\varepsilon u_\varepsilon).$$

Pour ce faire, une délicate stratégie de désingularisation (en ε) est mise en place. Des taux de convergence (en ε) sont obtenus à l'aide d'une entropie relative adaptée au problème (1.5.1), à savoir la fonctionnelle

$$H_\varepsilon(t) = \frac{1}{2}\int_{x,v} f_\varepsilon(t)|v-u(t)|^2\,\mathrm{d}x\,\mathrm{d}v + \frac{1}{2}\int_x |u_\varepsilon(t)-u(t)|^2\,\mathrm{d}x. \tag{1.5.3}$$

Remarque 1.5.1. Mentionnons enfin que la situation est bien plus singulière que pour le cas du système de Vlasov-Fokker-Planck-Navier-Stokes (avec un Laplacien en vitesse dans l'équation cinétique). Pour ce dernier, on s'attend ici à ce que $f_\varepsilon(t,x,v)$ converge vers une Maxwellienne locale

$$M_{[\rho,u]}(t,x,v) = (2\pi)^{-3/2}\rho(t,x)\exp(-|v-u(t,x)|^2/2),$$

où (ρ, u) satisfait le système hydrodynamique limite (de type Smoluchowski-Navier-Stokes). Encore une fois, le cas dégénéré d'une Maxwellienne à température nulle est celui de (1.5.1). On renvoie aussi à l'analyse formelle de Carrillo et Goudon dans [CG06] où le cas de conditions au bord et de potentiels extérieurs est discuté. Dans ce cadre, le cas $(a = 1/2, b = 0)$ a été traité par Goudon, Jabin et Vasseur [GJV04a] en dimension 2. Les mêmes auteurs abordent le cas $(a = 0, b = 1)$ dans [GJV04b] (dans cette direction, mentionnons également les résultats de Mellet et Vasseur dans [MV08] pour le cas du système Navier-Stokes compressible, et de Su et Yao dans [SY20] pour le cas inhomogène incompressible).

La présence de diffusion en vitesse dans l'équation cinétique offre alors une borne supplémentaire sur l'entropie des solutions. Notons que le cas le plus singulier traité dans [GJV04b] se base sur une méthode d'entropie relative, à l'aide de la fonctionnelle

$$\mathcal{H}_\varepsilon(t) = \int_{x,v} \left(f_\varepsilon(t) \log \frac{f_\varepsilon(t)}{M_{[\rho(t),u(t)]}} + M_{[\rho(t),u(t)]} - f_\varepsilon(t) \right) \mathrm{d}x \, \mathrm{d}v + \frac{1}{2} \int_x |u_\varepsilon(t) - u(t)|^2 \, \mathrm{d}x,$$

dont (1.5.3) est une version dégénérée lorsque le paramètre de diffusion en vitesse tend vers 0.

1.5.2 Limite hydrodynamique avec force de gravité

Un autre problème intéressant concerne l'ajout des effets dus à la gravité sur les particules. On a déjà évoqué brièvement cette extension du système de Vlasov-Navier-Stokes qui inclut cette force supplémentaire en Section 1.4.5 (voir à nouveau [Höf20] sur la pertinence de ce modèle de sédimentation de particules dans un fluide).

Dans le reste de cette section, on considère un nombre de Reynolds $\mathrm{Re} \in \{0, 1\}$ pour le fluide, permettant de négliger ou non l'auto-convection du fluide. Pour un petit paramètre $0 < \varepsilon \ll 1$ et un vecteur $G \in \mathbb{R}^3$ donné, on s'intéresse au système[13]

$$(\mathrm{VNS}_{\varepsilon,\mathrm{G}}) \begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \dfrac{1}{\varepsilon} \mathrm{div}_v \left[f_\varepsilon (u_\varepsilon - v + G) \right] = 0, \\ \mathrm{Re} \left(\partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla_x) u_\varepsilon \right) - \Delta_x u_\varepsilon + \nabla_x p_\varepsilon = \displaystyle\int_{\mathbb{R}^3} (v - u_\varepsilon) f_\varepsilon, \\ \mathrm{div}_x \, u_\varepsilon = 0. \end{cases} \tag{1.5.4}$$

Sans rentrer dans les détails, cela correspond à un régime où le nombre de Stokes pour le nuage de particules tend vers 0. Ce nombre est le quotient du temps d'adaptation d'une particule au fluide ambiant par le temps caractéristique du fluide lui-même. Un régime à faible nombre de Stokes correspond à une réponse très rapide du nuage de particules qui va donc suivre le fluide, si bien que les effets inertiels ne sont pas très importants. On renvoie au Chapitre 4 pour la procédure d'adimensionnement. Notons que ceci est cohérent avec l'analyse dimensionnelle faite par Höfer dans [Höf20], où cette limite hydrodynamique est appelée *limite sans inertie.*

Considérons alors une famille de solutions $(f_\varepsilon, u_\varepsilon)$ de (1.5.4) et supposons que

$$u_\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} u, \quad \rho_\varepsilon := \int_{\mathbb{R}^3} f_\varepsilon \, \mathrm{d}v \overset{\varepsilon \to 0}{\longrightarrow} \rho, \quad j_\varepsilon := \int_{\mathbb{R}^3} v f_\varepsilon \, \mathrm{d}v \overset{\varepsilon \to 0}{\longrightarrow} j.$$

En intégrant l'équation de Vlasov en vitesse contre $\mathrm{d}v$ et contre $v\mathrm{d}v$, on obtient la conservation de la masse et du moment

$$\begin{cases} \partial_t \rho_\varepsilon + \mathrm{div}_x \, j_\varepsilon = 0, \\ \partial_t j_\varepsilon + \mathrm{div}_x \left(\displaystyle\int_{\mathbb{R}^3} v \otimes v f_\varepsilon \, \mathrm{d}v \right) = \dfrac{1}{\varepsilon} \left(\rho_\varepsilon (u_\varepsilon + G) - j_\varepsilon \right). \end{cases}$$

[13] On a pris $a = 0$ dans la définition donnée plus haut pour (1.5.1).

Formellement, on en déduit $\rho_\varepsilon(u_\varepsilon + G) - j_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} 0$, d'où $j = \rho(u+G)$. La limite de (1.5.4) quand $\varepsilon \to 0$ est donc le système

$$(\mathrm{BNS_G}) \begin{cases} \partial_t \rho + \mathrm{div}_x[\rho(u+G)] = 0, \\ \mathrm{Re}\,(\partial_t u + (u\cdot\nabla_x)u) - \Delta_x u + \nabla_x p = \rho G, \\ \mathrm{div}_x\, u = 0. \end{cases} \tag{1.5.5}$$

On reconnaît un couplage de type Boussinesq-Navier-Stokes (*modulo* le terme de gravité dans la première équation) : la densité des particules est transportée par le fluide et la gravité, tandis que l'action de particules apparaît comme un terme de forçage dans l'équation fluide, le long du champ de gravité. Les équations de Boussinesq-Navier-Stokes forment un modèle standard en géophysique (voir [Sal98, Val17, Maj03]). Du point de vue mathématique, elles ont récemment reçu une attention grandissante (voir entre autres [HL05, Cha06, AH07, HK07, DP08, HR10] pour certains résultats d'existence, et d'autres références au Chapitre 4).

A nouveau, l'asymptotique $\varepsilon \to 0$ vient pénaliser la force $u_\varepsilon(t) - v + G$ si bien que la convergence attendue ci-dessus est reliée au comportement monocinétique

$$u_\varepsilon(t) \underset{\varepsilon\to 0}{\longrightarrow} u(t), \quad f_\varepsilon(t) \underset{\varepsilon\to 0}{\longrightarrow} \rho(t)\otimes\delta_{v=u(t)+G}, \quad \text{où} \quad (\rho,u) \quad \text{est solution de (1.5.5).}$$

La justification de cette limite hydrodynamique (de (1.5.4) vers (1.5.5)) a été énoncée comme un problème ouvert par Han-Kwan et Michel dans [HKMar]. La stratégie qui a été mise en place dans leur travail permet en fait d'obtenir la dérivation **en temps court**, dans le cas du tore ou de l'espace entier. Une obstruction majeure concernant la validité de cette limite **globalement en temps** est l'ajout d'énergie dans le système due à la force de gravité (comme pour le comportement en temps long - voir la Section 1.4.5). La question posée par [HKMar] est donc la suivante.

Question 1.5.2 (Han-Kwan, Michel [HKMar])**.** *Dans le cas* $\mathrm{Re} = 1$*, peut-on justifier la limite de* (1.5.4) *vers* (1.5.5)*, lorsque* $\varepsilon \to 0$ *et sur tout intervalle de temps ?*

Ce problème sera partiellement résolu au Chapitre 4 (voir la présentation faite en Section 1.5.3).

Pour $\mathrm{Re} = 0$, on retrouve le système de Stokes-Transport (1.3.16) qui avait été dérivé en Section 1.3.3, précisément dans une limite de champ moyen où l'inertie des particules était négligée. Dans ce cas, la limite hydrodynamique $\varepsilon \to 0$ partant de Vlasov-Stokes a été obtenue par Höfer dans [Hö18]. Le domaine spatial considéré est ici $\mathbb{R}^3$.

Théorème 1.5.3 (Höfer [Hö18], cas $\mathrm{Re} = 0$)**.** *Soit* $T > 0$*. Pour tout* $f^{\mathrm{in}} \in \mathrm{W}^{1,\infty}(\mathbb{R}^3\times\mathbb{R}^3)$ *à support compact, considérons la solution* $(f_\varepsilon, u_\varepsilon)$ *de* (1.5.4) *telle que* $f_\varepsilon \in \mathrm{W}^{1,\infty}((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$ *soit à support compact et* $u_\varepsilon \in \mathrm{L}^\infty(0,T;\mathrm{W}^{2,\infty}(\mathbb{R}^3\times\mathbb{R}^3)) \cap \mathrm{W}^{1,\infty}((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$*. Alors pour tout* $\beta \in [0,1)$*, on a*

$$\rho_\varepsilon = \int_{\mathbb{R}^3} f_\varepsilon \,\mathrm{d}v \underset{\varepsilon\to 0}{\longrightarrow} \rho, \quad \text{dans} \quad \mathscr{C}^{0,\beta}((0,T)\times\mathbb{R}^3),$$
$$u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u, \quad \text{dans} \quad \mathrm{L}^1((0,T);\mathrm{W}^{1,\infty}(\mathbb{R}^3\times\mathbb{R}^3)),$$

où (ρ,u) *est solution de* (1.3.16)*.*

Esquissons quelques éléments de preuve, permettant de mettre en lumière pourquoi celle-ci ne s'applique pas au système de Vlasov-Navier-Stokes avec $\mathrm{Re} = 1$.

Insistons tout d'abord sur le fait que la stratégie mise en œuvre est vraiment spécifique au cas de l'équation de Stokes pour le fluide, à la fois par son caractère linéaire (en u) et son caractère quasi-stationnaire (et donc non directement applicable pour le cas Re = 1). Rappelons également que, comme pour le comportement en temps long du système de Vlasov-Navier-Stokes avec gravité (1.4.8) (voir en Section 1.4.5), la présence du terme de gravité induit formellement une injection d'énergie dans le système.

Une idée clef est que, dans ce couplage, la croissance de l'énergie cinétique des particules est linéaire en le terme de gravité mais moralement quadratique en la friction qui a le bon signe : cela permet d'empêcher une trop forte croissance de cette énergie si on dispose d'une borne uniforme pour ρ_ε dans $\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3))$.

Comme pour [HKMar], cela passe par un argument de bootstrap. La preuve utilise certains arguments basés sur les courbes caractéristiques, notamment le changement de variable en vitesse (ré-)utilisé dans [HKMM20, HK22, HKMar]. Un avantage du couplage avec l'équation de Stokes est que cette équation fournit plusieurs contrôles (ponctuels en temps) sur $u_\varepsilon(t)$ et $\nabla_x u_\varepsilon(t)$.

La stratégie consiste ensuite à introduire une solution intermédiaire $\widetilde{u}_\varepsilon$, solution de l'équation de Stokes avec source $\rho_\varepsilon G$. Comparer u_ε à $\widetilde{u}_\varepsilon$ et $\widetilde{u}_\varepsilon$ à u nécessite des estimations à très haute régularité pour f_ε, ρ_ε et u_ε. Cette étape utilise fortement la linéarité et l'ellipticité des différentes équations de Stokes mises en jeu, ainsi que la représentation explicite des solutions *via* une convolution avec le tenseur d'Oseen.

Il semble que la méthode ci-dessus ne permette donc pas de traiter le cas Re = 1 correspondant au système complet de Vlasov-Navier-Stokes (1.5.4), et donc de répondre à la Question 1.5.2 posée par Han-Kwan et Michel. Au Chapitre 4, nous proposons une voie différente (plus proche de [HKMar] - et dans un cadre géométrique adapté) afin d'établir la limite hydrodynamique vers les équations de Boussinesq-Navier-Stokes (1.5.5). Comme indiqué plus haut, la principale difficulté consiste à justifier cette dérivation pour tout intervalle de temps. Notre stratégie sera donc fortement liée aux travaux sur le comportement en temps long du système avec gravité effectué au Chapitre 3.

1.5.3 Contribution du Chapitre 4 : limite hydrodynamique vers un système de type Boussinesq-Navier-Stokes sur le demi-espace

Le Chapitre 4 reproduit le contenu de l'article prépublié [Ert22].

Le but est d'obtenir la limite hydrodynamique du couplage complet (1.5.4) avec gravité vers (1.5.5) lorsque $\varepsilon \to 0$, sans négliger l'inertie du fluide (*i.e.* Re = 1). On renvoie à [Höf20, Section 2.3.5] pour une discussion détaillée qui montre qu'il est physiquement pertinent de considérer des nombres de Reynolds d'ordre 1.

Nous allons ici tenter de suivre le cadre introduit dans [HKMar], dans un contexte géométrique adéquat. On considère ainsi les équations de Vlasov-Navier-Stokes (1.5.4) avec $G = -ge_3$ ($g > 0$) posées dans le demi-espace $\mathbb{R}^3_+ := \mathbb{R}^2 \times (0, +\infty)$ et à nouveau avec les conditions au bord

$$f_\varepsilon(t)_{|\Sigma^-} = 0, \quad u_\varepsilon(t)_{|x_3=0} = 0. \tag{1.5.6}$$

Le système de Boussinesq-Navier-Stokes (1.5.5) sur (ρ, u) est quant à lui complété par une condition de Dirichlet homogène pour la vitesse du fluide

$$u(t)_{|x_3=0} = 0.$$

Il n'y a pas de condition au bord pour la densité ρ : en effet, le champ de vecteurs $u + G$ qui la transporte vérifie $n_{\mathbb{R}^3_+}(x) \cdot (u(t,x) + G) > 0$.

1.5.3.1 Résultat principal

Une version informelle du théorème est le suivant (on renvoie aux énoncés détaillés de la Section 4.1.4 du Chapitre 4, notamment pour le jeu d'hypothèses complet).

Théorème 1.5.4 (Chapitre 4 - basé sur [Ert22])**.** *Soit (ρ, u) une solution forte globale du système (1.5.5) avec donnée initiale $(\rho^{\mathrm{in}}, u^{\mathrm{in}})$. Sous des hypothèses de bornes, décroissance et de petitesse uniformes pour une famille de données initiales admissibles $(f_\varepsilon^{\mathrm{in}}, u_\varepsilon^{\mathrm{in}})_\varepsilon$, il existe $\varepsilon_0 > 0$ et une fonction positive continue φ vérifiant $\varphi(\cdot, 0) = 0$ tels que si $T > 0$, alors pour $t \in [0, T]$ et $\varepsilon \in (0, \varepsilon_0)$*

$$\|u_\varepsilon(t) - u(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|\rho_\varepsilon(t) - \rho(t)\|_{\mathrm{H}^{-1}(\mathbb{R}^3_+)} \lesssim e^{1+T}\left(\|u_\varepsilon^{\mathrm{in}} - u^{\mathrm{in}}\|_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|\rho_\varepsilon^{\mathrm{in}} - \rho^{\mathrm{in}}\|_{\mathrm{H}^{-1}(\mathbb{R}^3_+)} + \varphi(T, \varepsilon)\right),$$

où $(f_\varepsilon, u_\varepsilon)$ est une solution faible globale de (1.5.4) avec condition d'absorption (1.5.6) partant de $(f_\varepsilon^{\mathrm{in}}, u_\varepsilon^{\mathrm{in}})$.

On peut en fait affaiblir ce résultat en ne supposant que la convergence faible des données initiales : celle-ci est transférée aux solutions, à savoir pour u_ε faiblement dans $\mathrm{L}^2(0, T; \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3_+))$ et pour ρ_ε faiblement$-\star$ dans $\mathrm{L}^\infty((0, T) \times \mathbb{R}^3_+)$. C'est la convergence forte qui est quantifiée (on renvoie à nouveau à la Section 4.1.4 du Chapitre 4).

Concernant les hypothèses sur la vitesse initiale du fluide $u_\varepsilon^{\mathrm{in}}$, on demande essentiellement de la petitesse dans $\mathrm{H}^1(\mathbb{R}^3_+)$, un peu d'intégrabilité et de la régularité dans un espace d'interpolation bien choisi.

Remarque 1.5.5. Contrairement à [Hö18] (pour le cas du système de Vlasov-Stokes), on ne suppose pas que la donnée initiale cinétique $f_\varepsilon^{\mathrm{in}}$ soit à support compact - elle a seulement des moments contrôlés. De plus, celle-ci est uniquement supposée bornée dans les espaces de Lebesgue, ce qui est naturel pour la théorie du transport et pour l'équation limite sur ρ.

Le théorème 1.5.4 apporte donc une réponse partielle à la Question 1.5.2 de Han-Kwan et Michel, dans un cadre géométrique bien choisi.

Eléments de preuve. Donnons quelques idées de la preuve. Comme indiqué plus haut, la stratégie mise en place par Han-Kwan et Michel dans [HKMar] peut *a priori* formellement s'adapter[14] au cas avec gravité, mais ne semble fournir qu'une convergence valide en temps court. Le but est donc d'obtenir une dérivation du système limite qui est globale en temps. Bien sûr, la stratégie présentée pour le comportement en temps long en Section 1.4.7 s'applique *stricto sensu* mais fournit des estimations qui ne sont pas uniformes en ε.

On observe premièrement que le couple $(\rho_\varepsilon, u_\varepsilon)$ vérifie

$$\begin{cases} \partial_t \rho_\varepsilon + \mathrm{div}_x\, j_\varepsilon = 0, \\ \partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla_x) u_\varepsilon - \Delta_x u_\varepsilon + \nabla_x p_\varepsilon = j_\varepsilon - \rho_\varepsilon u_\varepsilon, \\ \mathrm{div}_x\, u_\varepsilon = 0. \end{cases} \tag{1.5.7}$$

Le but est d'obtenir des estimations uniformes en ε sur $(\rho_\varepsilon, u_\varepsilon)$ (pour mettre en place un argument de compacité dans un premier temps) et la convergence de la force de Brinkman $F_\varepsilon = j_\varepsilon - \rho_\varepsilon u_\varepsilon$ (ou plutôt de $F_\varepsilon - \rho_\varepsilon G$).

[14]Mettons de côté l'aspect non borné du domaine.

L'idée principale est d'estimer les quantités intégrales

$$\rho_\varepsilon(t,x) = \int_{\mathbb{R}^3} f_\varepsilon(t,x,v)\,\mathrm{d}v, \quad F_\varepsilon(t,x) = \int_{\mathbb{R}^3} f_\varepsilon(t,x,v)\,(v - u_\varepsilon(t,x))\,\mathrm{d}v,$$

grâce aux trajectoires associées à l'équation de Vlasov, c'est-à-dire :

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{X}_\varepsilon^{s;t}(x,v) = \mathrm{V}^{s;t}(x,v), \quad \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}_\varepsilon^{s;t}(x,v) = \frac{1}{\varepsilon}\Big(u_\varepsilon(s,\mathrm{X}_\varepsilon^{s;t}(x,v)) - \mathrm{V}_\varepsilon^{s;t}(x,v) + G\Big).$$

Pour obtenir des bornes valides sur tout intervalle de temps, nous utilisons les méthodes et techniques du Chapitre 3 (voir la Section 1.4.7), à savoir des changements de variables en vitesse (et en espace), la condition de sortie géométrique (voir Définition 1.4.15) ou encore des estimations précises des trajectoires. On peut montrer qu'une condition suffisante assurant un passage à la limite est une condition de petitesse et d'uniformité sur u_ε (pour des normes intégrées en temps), et donc de la décroissance en temps.

Le point crucial consiste à contrôler la force de Brinkman F_ε, en source des équations de Navier-Stokes. Contrairement à la stratégie exposée en Section 1.4.7, il ne suffit pas d'utiliser la décroissance des moments ρ_ε et j_ε séparément car l'utilisation du changement de variable

$$\Gamma_\varepsilon^{t,x} : v \mapsto \mathrm{V}_\varepsilon^{0;t}(x,v),$$

fait apparaître (pour le moment d'ordre 1 en vitesse) un terme singulier en ε provenant de $[\Gamma_\varepsilon^{t,x}]^{-1}$. On a en effet

$$[\Gamma_\varepsilon^{t,x}]^{-1}(w) = \mathrm{Good}_\varepsilon + \frac{1}{\varepsilon}\int_0^t e^{\frac{\tau-t}{\varepsilon}}(Pu_\varepsilon)(\tau, \mathrm{X}_\varepsilon(\tau;t,x,[\Gamma_\varepsilon^{t,x}]^{-1}(w)))\,\mathrm{d}\tau. \tag{1.5.8}$$

où $\mathrm{Good}_\varepsilon$ est un terme non singulier en ε. Suivant [HKMar], nous ne séparons pas l'expression de la force de Brinkman en deux et nous servons pleinement de l'effet de friction en écrivant, *via* la représentation par les trajectoires (voir la notation (1.4.12))

$$F_\varepsilon(t,x) = e^{\frac{3t}{\varepsilon}}\int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,v) f_\varepsilon^{\mathrm{in}}(\mathrm{X}_\varepsilon^{0;t}(x,v), \mathrm{V}_\varepsilon^{0;t}(x,v))\,(v - u_\varepsilon(t,x))\,\mathrm{d}v. \tag{1.5.9}$$

Utilisant le changement de variable $w = \Gamma_\varepsilon^{t,x}(v)$, on effectue d'abord une intégration par parties en temps dans l'intégrale de (1.5.8) pour *désingulariser* ce terme en ε (grâce au facteur exponentiel). En l'injectant dans (1.5.9), on obtient une intégrande avec

$$[\Gamma_\varepsilon^{t,x}]^{-1}(w) - u_\varepsilon(t,x) = \mathrm{Good}_\varepsilon + \mathrm{Terme}[u_\varepsilon^{\mathrm{in}}].$$

Cette stratégie donne lieu à une décomposition de F_ε (et aussi de $F_\varepsilon - \rho_\varepsilon G$ suivant le même canevas). Nous poussons l'analyse plus loin en visant l'obtention d'estimations de décroissance polynomiale en temps, provenant de la condition d'absorption au bord et de la condition de contrôle géométrique. Nous suivons alors une stratégie de bootstrap permettant de propager la condition

$$\int_0^T \|u_\varepsilon(s)\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^3_+)}\,\mathrm{d}s \ll 1,$$

en temps long et pour ε assez petit.

La convergence forte (quantifiée) énoncée plus haut provient ensuite d'un argument de stabilité sur l'équation satisfaite par les différences $u_\varepsilon - u$ et $\rho_\varepsilon - \rho$: cela nécessite des bornes d'ordre supérieure, globales en temps et uniformes en ε, qui sont obtenues auparavant dans le cœur de la preuve.

1.5.3.2 Perspectives

De nombreuses questions restent encore ouvertes concernant les limites hydrodynamiques du système de Vlasov-Navier-Stokes.

- **Question 1**. Peut-on relaxer les hypothèses de petitesse et de régularité faites au Théorème 1.5.4 ? Cette question est naturelle car on dispose d'une part de solutions faibles globales pour le système de Vlasov-Navier-Stokes (1.5.4) et d'un cadre de solutions faibles à la Leray pour le système de Boussinesq-Navier-Stokes (1.5.5) (voir [DP08]).

  On pourrait s'attendre à une justification de la limite hydrodynamique faisant le pont entre ces deux théories. Rappelons en contraste que l'on sait passer des solutions faibles renormalisées pour l'équation de Boltzmann aux solutions faibles de Leray pour les équations de Navier-Stokes. La stratégie de preuve du Théorème 1.5.4 semble cependant difficilement pouvoir se passer d'estimations d'ordre supérieur et d'un gain de régularité sur la partie fluide.

- **Question 2**. Comme pour le comportement en temps long, et suivant la Question 1.5.2, la question de la limite hydrodynamique de (1.5.4) dans $\mathbb{R}^3$ (globalement en temps) se pose. Elle est cependant ouverte à notre connaissance.

  Notons qu'une stratégie dans l'esprit de [HKMar], basée sur de la décroissance en temps de l'énergie, semble difficile à imaginer : en effet, on sait que le système limite de Boussinesq-Navier-Stokes a l'inconvénient de présenter une large classe de solutions telles que $\|u(t)\|_{\mathrm{L}^2(\mathbb{R}^3)}$ diverge polynomialement. À l'inverse, ce n'est pas le cas pour le demi-espace (voir la discussion détaillée dans l'article de synthèse [BS18] et les articles [BS12, BM17, KW20]), ce qui est cohérent avec le résultat du Théorème 1.5.4 et l'approche présentée ci-dessus.

- **Question 3**. Il existe d'autres limites hydrodynamiques partant des équations de Vlasov-Navier-Stokes qui n'ont pas encore été explorées. Par exemple (voir [Höf20]), la limite de

$$\begin{cases} \partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \mathrm{div}_v\left[f_\varepsilon(u_\varepsilon - v) + \varepsilon f_\varepsilon G\right] = 0, \\ \partial_t u_\varepsilon + (u_\varepsilon\cdot\nabla_x)u_\varepsilon - \Delta_x u_\varepsilon + \nabla_x p_\varepsilon = \dfrac{1}{\varepsilon}\left(j_\varepsilon - \rho_\varepsilon u_\varepsilon\right), \\ \mathrm{div}_x\, u_\varepsilon = 0, \end{cases}$$

  devrait formellement conduire, quand $\varepsilon \to 0$, au système de type Vlasov-Darcy suivant

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + \mathrm{div}_v\left[f(u - v)\right] = 0, \\ \rho_f u + \nabla_x p = j_f, \\ \mathrm{div}_x\, u = 0. \end{cases}$$

  Mentionnons également la limite intéressante du système de Vlasov-Stokes suivant

$$\begin{cases} \partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \mathrm{div}_v\left[\dfrac{1}{\varepsilon}f_\varepsilon(u_\varepsilon - v) + f_\varepsilon G\right] = 0, \\ -\Delta_x u_\varepsilon + \nabla_x p_\varepsilon = \dfrac{1}{\varepsilon}(j_\varepsilon - \rho_\varepsilon u_\varepsilon), \\ \mathrm{div}_x\, u_\varepsilon = 0. \end{cases}$$

  Ce régime très particulier fait formellement apparaître le système limite

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho u) = 0, \\ \rho\left(\partial_t u + (u\cdot\nabla_x)u\right) - \Delta_x u + \nabla_x p = \rho G, \\ \mathrm{div}_x\, u = 0, \end{cases}$$

avec à présent de l'inertie pour le fluide, alors que l'on était parti d'une équation de Stokes quasi-stationnaire.

L'analyse de ces différentes limites hydrodynamiques fait l'objet d'un travail en cours avec Richard Höfer.

1.6 Analyse mathématique du système des sprays épais

Dans cette section, on aborde l'étude du système des sprays épais (voir Section 1.1.5). Rappelons qu'il s'agit du couplage fluide-cinétique d'inconnue (f, ϱ, u) qui s'écrit

$$\begin{cases} \partial_t(\alpha\varrho) + \mathrm{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u \otimes u) + \alpha\nabla_x p(\varrho) - \nu\,(\Delta_x u + \nabla_x \mathrm{div}_x\, u) = j_f - \rho_f u, \\ \partial_t f + v\cdot\nabla_x f + \mathrm{div}_v\,[f(u-v) - f\nabla_x p(\varrho)] = 0. \end{cases} \tag{1.6.1}$$

avec $\alpha = 1 - \rho_f$. Le cas $\nu = 0$ est celui des équations d'Euler pour un fluide parfait, tandis que le cas $\nu = 1$ est celui des équations de Navier-Stokes pour un fluide visqueux. On néglige ici l'effet des collisions provenant de l'opérateur de Boltzmann $\mathcal{Q}$. Pour simplifier la présentation, on se restreint au cas d'un fluide barotrope (la fonction $p : \mathbb{R}^+ \to \mathbb{R}^+$ est donc donnée) et avec un terme de traînée linéaire dans l'équation cinétique.

1.6.1 Difficultés et résultats connus

L'étude rigoureuse du système (1.6.1) du point de vue mathématique est très lacunaire. Bien que considéré dans des simulations numériques (voir [BDM14, BDGN12]), ce système a presque uniquement été analysé de manière formelle du point de vue mathématique. Ce constat contraste fortement avec la littérature existante concernant le système de Vlasov-Navier-Stokes (1.3.1) pour les sprays fins.

1.6.1.1 Obstruction formelle à l'existence de solutions

La problème concerne déjà l'existence même de solutions au système (1.6.1), qui est restée une question ouverte jusqu'à présent (en opposition au système de Vlasov-Navier-Stokes (1.3.1) - voir les Sections 1.3.2.1 et 1.3.2.2).

Lorsque $f = 0$ (et donc $\alpha = 1$), le système (1.6.1) se réduit au système d'Euler compressible ($\nu = 0$) ou de Navier-Stokes compressible ($\nu = 1$) standards, pour lequel plusieurs résultats sont connus. Citons, entre autres :

- dans le cas des équations d'Euler compressible : l'existence de solutions fortes en temps petit peut être obtenu en se ramenant d'une manière ou d'une autre à la théorie générale des systèmes de lois de conservation hyperboliques (voir entre autres [Maj12, Che90, BGS07]). La théorie de Glimm [Gli65] fournit quant à elle l'existence de solutions faibles globales mais uniquement en dimension 1 d'espace.
- dans le cas des équations de Navier-Stokes compressible (dans le cas barotropique), on renvoie aux références classiques données en Section 1.1.2.

Lorsque l'on considère le système complet (1.6.1), il faut tenir compte du couplage entre le fluide et le nuage de particules qui s'effectue *via* : le terme de traînée $u - v$ (comme pour le système de

Vlasov-Navier-Stokes), le gradient de pression dans l'équation cinétique, et la fraction volumique α. Plusieurs difficultés apparaissent alors.

• D'une part, il semble difficile de considérer la construction de solutions faibles globales pour le couplage (1.6.1), même dans le cas visqueux $\nu = 1$. Certes, des bornes *a priori* sont formellement données dans ce cas par l'identité d'énergie-dissipation

$$\frac{\mathrm{d}}{\mathrm{d}t}E(f,\alpha,\varrho,u)(t) + \int \left(|\nabla_x u(t)|^2 + |\mathrm{div}_x\, u(t)|^2\right) \mathrm{d}x + \int f(t)|u(t)-v|^2 \,\mathrm{d}x\,\mathrm{d}v = 0,$$

où

$$E(f,\alpha,\varrho,u)(t) := \int \left(\alpha(t)\varrho(t)\frac{|u(t)|^2}{2} + \alpha(t)\varrho(t)e(\varrho(t))\right) \mathrm{d}x + \frac{1}{2}\int f(t)|v|^2 \,\mathrm{d}x\,\mathrm{d}v,$$
$$e(\varrho) := \int_0^{\varrho} p(r)/r^2 \,\mathrm{d}r,$$

ce qui est comparable à ce que l'on obtient pour le système de Navier-Stokes compressible (1.1.2). Cependant, la théorie standard à la Lions [Lio98], basée sur cette approche, donne essentiellement de l'intégrabilité en espace et en temps pour la solution $\alpha\varrho$ de l'équation de conservation de la masse (par exemple $\alpha\varrho \in \mathrm{L}^\infty_t \mathrm{L}^\gamma_x$ pour une pression $p(r) = r^\gamma$), et il difficile d'espérer mieux pour la densité ϱ elle-même. Dès lors, le champ de force $\nabla_x p(\varrho)$ qui intervient dans l'équation cinétique sur f ne permet pas d'utiliser le cadre des solutions renormalisées à la DiPerna et Lions ou même à la Ambrosio, puisque l'on est *a priori* loin d'avoir $\nabla_x p(\varrho(t)) \in \mathrm{W}^{1,1}_x$ ou $\nabla_x p(\varrho(t)) \in \mathrm{BV}_x$.

• D'autre part, même dans le cas $\nu = 1$, des estimations d'énergie naïves sur la partie transport et cinétique de (1.6.1) semblent formellement révéler **plusieurs pertes de dérivées**. En effet, supposons que l'on dispose d'une solution (f,ϱ,u) aussi régulière que souhaitée et à support compact. En utilisant les relations $\alpha = 1-\rho_f$ et $\partial_t \rho_f + \mathrm{div}_x\, j_f = 0$, il est possible de réécrire formellement l'équation sur ϱ comme

$$\partial_t \varrho + u\cdot\nabla_x\varrho + \frac{\varrho}{1-\rho_f}\mathrm{div}_x\,[j_f - \rho_f u] = -\frac{\varrho}{1-\rho_f}\mathrm{div}_x\, u.$$

D'une part, une estimation d'énergie standard pour le transport donne pour tout $T > 0$

$$\forall t \in [0,T], \quad \|\varrho(t)\|_{\mathrm{H}^\ell_x} \lesssim \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{\ell+1}_x)} + \|f\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{\ell+1}_{x,v})}. \tag{1.6.2}$$

Un contrôle de ℓ dérivées de ϱ semble ainsi demander un contrôle de $\ell+1$ dérivées de f à première vue. Ceci est du au couplage avec α dans l'équation de conservation dans la masse. De même, une estimation d'énergie dans l'équation de transport cinétique sur f donne

$$\forall t \in [0,T], \quad \|\rho_f(t)\|_{\mathrm{H}^\ell_x} + \|j_f(t)\|_{\mathrm{H}^\ell_x} \lesssim \|f(t)\|_{\mathrm{H}^\ell_{x,v}} \lesssim \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^\ell_x)} + \|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{\ell+1}_x)}. \tag{1.6.3}$$

Cette estimation provient de la présence du gradient de pression dans la force agissant dans l'équation de Vlasov. Un contrôle de ℓ dérivées de f semble donc nécessiter un contrôle de $\ell+1$ dérivées de ϱ.

En conclusion, on obtient un contrôle de ℓ dérivées de f par $\ell+2$ dérivées de f (et de même pour ϱ), et ainsi de suite. Des estimations brutales de type transport semblent ainsi impliquer une perte formelle de 2 dérivées sur f ou ϱ. Il s'agit donc d'un **couplage singulier.** Cela empêche formellement l'utilisation de techniques standards pour construire des solutions. Le cas des équations d'Euler ($\nu = 0$) est encore plus défavorable à cause du manque de régularisation pour la vitesse du fluide u elle-même.

Dans [BD06a], Baranger et Desvillettes indiquent ainsi que le système des sprays épais (1.6.1) dans le cas non-visqueux $\nu = 0$ pourrait être mal posé. Cette remarque est faite par analogie avec les modèles de fluide diphasique (voir la section suivante). Dans le cas visqueux $\nu = 1$, ils conjecturent toutefois qu'il devrait être possible de construire des solutions à ce système[15].

Conjecture 1.6.1 (Baranger, Desvillettes [BD06a])**.** *Dans le cas $\nu = 1$, le système des sprays épais* (1.6.1) *est (localement) bien posé au sens de Hadamard.*

Au Chapitre 5, nous apporterons une réponse positive partielle à la conjecture de Baranger et Desvillettes : sous certaines conditions de stabilité des données initiales, on construira des solutions[16] à régularité Sobolev pour (1.6.1) et pour plusieurs variantes du système (dans le cas $\nu = 1$).

1.6.1.2 Quelques résultats connus jusqu'à présent

Limite formelle vers les systèmes bi-fluide. Comme évoqué dans la Section 1.1.4, il existe une connexion entre le système des sprays épais et les modèles de fluide biphasique. Ces systèmes bi-fluide ont été évoqués au début de l'introduction concernant les descriptions possibles des sprays. Pour faire ce lien, on réintroduit un opérateur de collisions $\mathcal{Q}$ de type Boltzmann dans l'équation cinétique de (1.6.1).

Ce passage a été formellement explicité par Desvillettes et Mathiaud dans [DM10] (voir aussi [Mat06]), à travers une procédure de limite hydrodynamique : ils considèrent un scaling à faible nombre de Knudsen dans le système des sprays épais (1.6.1), qui rend dominante la contribution des collisions entre particules. On considère ainsi pour tout $\varepsilon > 0$

$$\begin{cases} \partial_t(\alpha_\varepsilon \varrho_\varepsilon) + \mathrm{div}_x(\alpha_\varepsilon \varrho u_\varepsilon) = 0, \\ \partial_t(\alpha_\varepsilon \varrho_\varepsilon u_\varepsilon) + \mathrm{div}_x(\alpha_\varepsilon \varrho u_\varepsilon \otimes u_\varepsilon) + \alpha_\varepsilon \nabla_x p(\varrho_\varepsilon) - \nu\left(\Delta_x u_\varepsilon + \nabla_x \mathrm{div}_x\, u_\varepsilon\right) = \displaystyle\int_{\mathbb{R}^d} (v - u_\varepsilon) f_\varepsilon, \\ \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \mathrm{div}_v\left[f_\varepsilon \dfrac{1}{m}(u_\varepsilon - v) - f_\varepsilon \dfrac{1}{\widetilde{\rho}} \nabla_x p(\varrho_\varepsilon)\right] = \dfrac{1}{\varepsilon}\mathcal{Q}(f_\varepsilon, f_\varepsilon), \\ \alpha_\varepsilon = 1 - \dfrac{m}{\widetilde{\rho}} \displaystyle\int_{\mathbb{R}^d} f_\varepsilon \,\mathrm{d}v. \end{cases} \tag{1.6.4}$$

Ici, on a réintroduit des constantes $m > 0$ et $\widetilde{\rho} > 0$ (voir la Section 1.1.4) pour rendre l'écriture finale plus parlante. L'opérateur de collisions considéré ici est de type Boltzmann inélastique (voir en Section 1.1.3.2). On a donc conservation de la masse et du moment

$$\int_{\mathbb{R}^d} \mathcal{Q}(g, g)\,\mathrm{d}v = 0, \quad \int_{\mathbb{R}^d} v\mathcal{Q}(g, g)\,\mathrm{d}v = 0, \tag{1.6.5}$$

mais pas de l'énergie cinétique qui est dissipée au cours de la dynamique (voir la Section 1.1.3.2). Sans préciser explicitement la forme de $\mathcal{Q}$, on peut supposer (typiquement pour des collisions inélastiques) que les fonctions $g(t, x, v)$ annulant cet opérateur intégral sont de la forme

$$g(t, x, v) = G(t, x)\delta_{w=w(t,x)},$$

pour un certain profil de masse $G(t, x) \geq 0$ et de vitesse $w(t, x) \in \mathbb{R}^d$. Il s'agit du cas limite d'une Maxwelienne en vitesse à température nulle et contraste avec le cas élastique purement Maxwellien. Supposons alors formellement les convergences

$$(\varrho_\varepsilon, u_\varepsilon, f_\varepsilon, \alpha_\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} (\varrho, u, f, \alpha).$$

[15] Cette remarque est aussi faite par Desvillettes et Mathiaud à la toute fin de l'article [DM10].

[16] Notons que le résultat de Choi [Cho17] semble seulement confirmer que la partie fluide de l'équation peut induire explosion en temps fini du couplage, ce qui n'enlève pas la possibilité d'un résultat d'existence locale en temps.

La conservation de la masse au cours de l'évolution donne alors

$$f(t,x,v) = \widetilde{\rho}\frac{1-\alpha(t,x)}{m_p}\delta_{v=w(t,x)}.$$

Comme on le fait traditionnellement pour les limites hydrodynamiques de l'équation de Boltzmann, on peut alors intégrer l'équation cinétique en vitesse après multiplication par m et mv, puis utiliser les conservations (1.6.5) pour en déduire formellement le passage à la limite $\varepsilon \to 0$: le quadruplet (ϱ, u, α, w) satisfait

$$\begin{cases} \partial_t(\alpha\varrho) + \mathrm{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u \otimes u) + \alpha\nabla_x p - \nu\,(\Delta_x u + \nabla_x \mathrm{div}_x\, u) = \dfrac{\widetilde{\rho}}{m}(1-\alpha)(w-u), \\ \partial_t((1-\alpha)\widetilde{\rho}) + \mathrm{div}_x((1-\alpha)\widetilde{\rho} w) = 0, \\ \partial_t((1-\alpha)\widetilde{\rho} w) + \mathrm{div}_x((1-\alpha)\widetilde{\rho} w \otimes w) + (1-\alpha)\nabla_x p = -\dfrac{\widetilde{\rho}}{m}(1-\alpha)(w-u). \end{cases} \tag{1.6.6}$$

avec

$$\widetilde{\rho} = \mathrm{Cst}, \quad p = p(\varrho).$$

Le scaling de Desvillettes et Mathiaud permet ainsi de passer du couplage fluide-cinétique des sprays épais (1.6.4) à un système macroscopique (1.6.6) de fluide à deux phases. Cette dérivation, purement formelle, est valide dans le cas $\nu = 0$ ou $\nu = 1$.

Dans le cas $\nu = 0$, le système (1.6.6) correspond à un système d'Euler compressible multiphasique. La présence d'une pression commune entre les deux phases semble standard dans ce type de modèles (voir [IH10]). Il est aussi connu que ces équations, répandues dans le milieu de l'ingénierie, souffrent de plusieurs problèmes au niveau mathématique.

D'une part, ces équations sont écrites sous forme non-conservative (avec l'inconnue α "en-dehors" des gradients). En cas d'apparition de chocs (ce qui est bien sûr attendu), la théorie standard pour trouver les conditions de Rankine-Hugoniot au niveau du choc ne s'applique pas. Cela complique toute tentative de régularisation visqueuse de l'équation pour sélectionner les solutions[17]. D'autre part, et plus inquiétant encore, ce type d'équations présente un domaine d'ellipticité non trivial qui les rend *a priori* non hyperboliques [KSS03, NKDVLG05, HdM21] et permet à des instabilités de se développer.

Il n'est cependant pas clair de savoir si le modèle fluide-cinétique des sprays épais, qui donne naissance à ces modèles fluide-fluide, souffre des mêmes défauts. Mentionnons que l'ajout de diffusion en vitesse dans les modèles à deux fluides semble bénéfique du point de la stabilité du système (voir [Ram00]). On peut penser que cette remarque vague va dans le sens de la Conjecture 1.6.1 de Baranger et Desvillettes sur le caractère bien posé du système des sprays épais dans le cas $\nu = 1$.

Stabilité linéaire autour de profils homogènes. Très récemment [BDD23], Buet, Després et Desvillettes ont étudié la linéarisation de (1.6.1) autour de solutions explicites du type

$$\overline{f}(t) = \overline{n}e^{dt}F(e^{2t}|v|^2), \quad \overline{\varrho} = 1, \quad \overline{u} = 0, \tag{1.6.7}$$

avec $\overline{n} \in (0,1)$ et $F : \mathbb{R}^+ \to \mathbb{R}^+$. Le profil cinétique est homogène en espace, radialement symétrique en vitesse, et est corrigé par un facteur dépendant du temps à cause du terme de friction.

[17] Voir néanmoins la théorie de Dal Maso, LeFloch et Murat [DMLM95].

Sous la condition de décroissance du profil $F' < 0$, ils parviennent à obtenir une fonction de Lyapunov pour le linéarisé autour de (1.6.7). Cela donne essentiellement une estimation de stabilité L^2 pour le système des sprays épais, linéarisé autour d'un profil homogène radialement décroissant en vitesse. En comparaison des systèmes multiphasiques évoqués plus haut, la situation semble ainsi bien plus favorable.

Cependant, cette procédure ne suffit pas pour donner un résultat d'existence, même local en temps, pour le système non-linéaire complet (1.6.1). En effet, puisque les équations sont quasilinéaires, il faudrait prouver l'analogue de ces estimations de stabilité pour toutes les fonctions dans un voisinage de la solution (1.6.7).

Par ailleurs, la condition de stabilité $F' < 0$ demandée sur le profil définissant (1.6.7) est réminiscente du critère de stabilité de Penrose en physique des plasmas (voir [Pen60]). Il s'agit en fait d'un cas particulier de la condition de Penrose généralisée que nous employons au Chapitre 5 pour prouver le caractère bien posé de (1.6.1).

Existence pour un système moyenné. Très récemment enfin, Buet, Després et Fournet ont proposé dans [FBD22] une modification du système des sprays épais (1.6.1) (avec $\nu = 0$ et sans collisions) : le gradient de pression $-\nabla_x p(\varrho)$ et la fraction volumique α y sont régularisés au moyen d'une convolution par une fonction à variation bornée $\omega_r = \mathbf{1}_{B(0,r)}$ fixée, le rayon $r > 0$ étant donné. En notant $\langle g \rangle_r = \omega_r \star_x g$, ce nouveau système respectant les lois de conservation s'écrit

$$\begin{cases} \partial_t(\alpha\varrho) + \mathrm{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u \otimes u) + \alpha\langle \nabla_x p \rangle_r = j_f - \rho_f u, \\ \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v \left[f(u-v) - f\langle \nabla_x p \rangle_r \right] = 0, \\ \alpha = 1 - \langle \rho_f \rangle_r. \end{cases} \tag{1.6.8}$$

Cette régularisation du système originel permet de compenser les pertes formelles de dérivées mentionnées plus haut. Suivant les idées de Baranger et Desvillettes [BD06a] (voir aussi [Mat10]), il est alors possible d'invoquer la théorie générale des systèmes hyperboliques symétrisables de lois de conservation (en traitant les termes perdant originellement une dérivée comme des termes d'ordre 0) : cela permet de construire des solutions classiques localement en temps pour le système modifié (1.6.8) (avec un temps d'existence dépendant de $r > 0$).

À notre connaissance, la stratégie utilisée ne permet pas de passer à la limite lorsque $r \to 0$ (et donc $\omega_r \to \delta_{x=0}$) pour retrouver le système (1.6.1) de départ.

1.6.2 Parallèle avec les équations de Vlasov singulières

Dans le Chapitre 5, notre stratégie pour aborder l'étude du système des sprays épais (présentant des estimations (1.6.2)–(1.6.3) *a priori* à perte) sera basée sur l'étude récente de certaines équations de Vlasov singulières. Ces dernières proviennent de la physique des plasmas.

Plus précisément, considérons le système de Vlasov-Poisson décrivant la distribution d'une population d'ions:

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + E_\varepsilon \cdot \nabla_v f_\varepsilon = 0, \\ E_\varepsilon = -\nabla_x U_\varepsilon, \\ U_\varepsilon - \varepsilon^2 \Delta_x U_\varepsilon = \displaystyle\int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\,\mathrm{d}v - 1. \end{cases} \tag{1.6.9}$$

Ici, le paramètre sans dimension ε est le rapport de la longueur de Debye sur la longueur typique d'observation. Dans des conditions physiques réalistes, ce nombre est très petit et la limite $\varepsilon \to 0$, correspondant à un plasma localement quasiment neutre, est appelée **limite quasineutre**.

Formellement, si l'on considère $\varepsilon \to 0$ et si l'on suppose que $f_\varepsilon \to f$, on obtient

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \rho_f \cdot \nabla_v f = 0, \\ \rho_f(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v. \end{cases} \tag{1.6.10}$$

Cette équation a été baptisée *équation de Vlasov-Dirac-Benney* par Bardos dans [Bar13] : en effet, le potentiel de Coulomb de (1.6.9) a été remplacé par une masse de Dirac dans (1.6.10), et ce système est lié formellement (au moins en dimension 1) au système de Benney pour les Water-Waves. Dans la suite, on utilisera la dénomination **équation de Vlasov-Benney**.

L'observation principale sur l'équation de Vlasov-Benney est la suivante : la champ de force dans (1.6.10) perd une dérivée par rapport à f, *a contrario* des équations de Vlasov-Poisson (1.6.9) où le champ de force a un cran de régularité supplémentaire. Partant d'un système couplé entre une équation de transport cinétique et une équation elliptique, la limite $\varepsilon \to 0$ conduit ainsi à une équation de Vlasov singulière, où il n'y a plus de régularisation. On s'attend à ce que l'analyse mathématique de (1.6.10) soit très différente de celle de (1.6.9).

On comprend alors le parallèle qui se dessine entre le système de Vlasov-Benney (1.6.10) et le système des sprays épais: il s'agit également d'un **couplage cinétique singulier**. D'un point de vue purement mathématique, on peut voir (1.6.9) comme une régularisation de (1.6.10) *via* l'opérateur $(\mathrm{Id} - \varepsilon^2 \Delta_x)^{-1}$. L'enjeu de la limite quasineutre est de savoir si l'on peut ou non passer à la limite et se débarrasser de cette régularisation pour obtenir une solution au système limite. La stratégie que nous suivrons au Chapitre 5 dans le cas du système des sprays épais impliquera formellement la même problématique.

Dressons un bref panorama des résultats existants sur le caractère bien posé ou non de (1.6.10). Comme pour (1.6.9), tout profil homogène $\mu(v)$ est solution stationnaire de (1.6.10). Bardos et Nouri [BN12] ont alors observé qu'il existe des profils homogènes autour desquels les équations linéarisées possèdent un spectre instable non borné.

L'existence de tels profils instables, provenant de la perte de dérivée au niveau du champ de force, a la conséquence suivante : le système complet (1.6.10) est génériquement **mal posé au sens de Hadamard** dans des espaces de Sobolev, même avec une perte arbitraire de dérivées et sur un temps arbitrairement petit, comme prouvé par Han-Kwan et Nguyen dans [HKN16] (il s'agit d'un résultat quantitatif précisant un raisonnement par l'absurde de Bardos et Nouri dans [BN12]). Mentionnons aussi le résultat de Baradat [Bar20] qui permet de traiter des instabilités autour de profils très peu réguliers (de type mesure). Le caractère mal posé de l'équation de Vlasov-Benney est en opposition totale avec la théorie de Cauchy existant pour le système de Vlasov-Poisson (solutions faibles ou fortes), toujours basée sur la régularisation elliptique *via* le potentiel de Coulomb.

Il existe cependant des situations où le système (1.6.10) est bien posé. Vu la discussion précédente, il semble que des hypothèses de très forte régularité ou de structure sur les données ou les solutions soient nécessaires pour éviter des potentielles instabilités. Pour obtenir l'existence locale de solutions analytiques avec données analytiques, un théorème de type Cauchy-Kowaleskaya peut être appliqué directement sur (1.6.10), comme montré par exemple par Jabin et Nouri [JN11] en dimension 1 (voir aussi [BFJJ13]), ou par les arguments de [MV11, Section 9].

En dimension $d = 1$, des données initiales de Sobolev avec un profil de vitesse à une bosse (pour chaque x) conduisent à une solution locale en temps pour (1.6.10) (voir les travaux de Bardos et Besse [BB13, BB15]). En toute dimension (et sur le tore $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$), l'existence de solutions pour (1.6.10) dans un cadre Sobolev a été obtenue par Han-Kwan et Rousset dans [HKR16]. Ce travail s'appuie sur une **condition de stabilité** multidimensionnelle à la Penrose (introduite dans l'article pionnier [Pen60]) et définie comme suit :

Définition 1.6.2. *On dit qu'un profil homogène en vitesse $v \mapsto \mathrm{f}(v)$ satisfait la condition de stabilité de Penrose s'il existe $c > 0$ tel que*

$$\inf_{(\gamma,\tau,k)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\setminus\{0\}} \left| 1 - \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot (\mathcal{F}_v \nabla_v \mathrm{f})(ks)\,\mathrm{d}s \right| > c, \tag{1.6.11}$$

où $\mathcal{F}_v$ désigne la transformée de Fourier en vitesse sur $\mathbb{R}^d$.

Cette condition a été introduite sous cette forme générale par Mouhot et Villani dans leur importante contribution [MV11] sur l'amortissement Landau pour les équations de Vlasov-Poisson sur le tore. Mouhot et Villani prouvent la stabilité non-linéaire asymptotique de ce système, autour d'un équilibre $\mu(v)$ satisfaisant la condition de stabilité (1.6.11). On renvoie également à [BMM16, GNR21] pour des preuves plus récentes de l'amortissement Landau sur le tore.

La condition de Penrose est essentiellement une hypothèse sur la forme du profil en vitesse $\mathrm{f}(v)$. Voici quelque exemples (voir [MV11]) :

- tout profil suffisamment petit en taille satisfait (1.6.11);
- en dimension 1, tout profil à une bosse (d'abord croissant puis décroissant) est lui aussi stable. En dimension $d \geq 1$, tout profil radial décroissant satisfait la condition. Ceci inclut en particulier le cas important des Maxwelliennes (locales lisses) en vitesse;
- enfin, tout perturbation suffisamment petite et régulière d'un profil satisfaisant (1.6.11) reste dans cette classe (voir Figure 1.3).

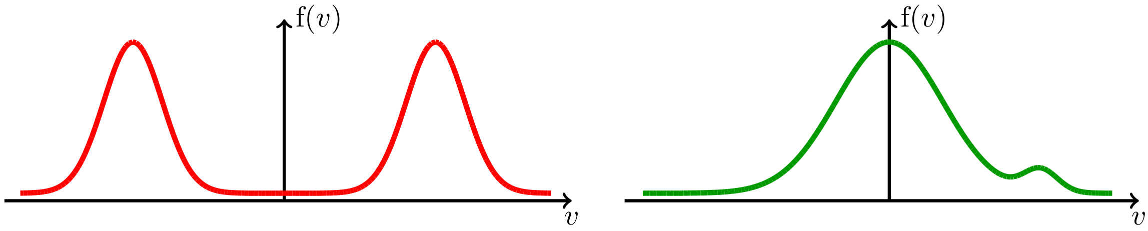

Figure 1.3: À gauche, un profil unidimensionnel à deux bosses (instable). À droite, un profil unidimensionnel à une bosse légèrement perturbé (stable).

Définition 1.6.3. *Pour $k \in \mathbb{N}$, $r \geq 0$ et $\mathrm{f} : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}^+$, on définit la norme de Sobolev avec poids en vitesse suivante:*

$$\|\mathrm{f}\|_{\mathcal{H}^k_r} := \left(\sum_{|\alpha|+|\beta|\leq k} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (1+|v|^2)^r |\partial_x^\alpha \partial_v^\beta \mathrm{f}(x,v)|^2 \,\mathrm{d}x\,\mathrm{d}v \right)^{\frac{1}{2}}.$$

Le résultat obtenu par Han-Kwan et Rousset est alors le suivant.

Théorème 1.6.4 (Han-Kwan et Rousset [HKR16])**.** *Soit $f^{\mathrm{in}} \in \mathcal{H}^m_r$ avec $m, r > 0$ assez grands et tel que pour tout $x \in \mathbb{T}^d$, $v \mapsto f^{\mathrm{in}}(x,v)$ satisfait une condition de Penrose (1.6.11). Alors il existe $T > 0$ et une unique solution $f \in \mathscr{C}([0,T];\mathcal{H}^{m-1}_r)$ de l'équation de Vlasov-Benney (1.6.10) avec $\rho_f \in \mathrm{L}^2(0,T;\mathrm{H}^m(\mathbb{T}^d))$ et donnée initiale f^{in} tels que $v \mapsto f(t,x,v)$ satisfait une condition de Penrose pour tout $t \in [0,T]$ et $x \in \mathbb{T}^d$.*

Il s'agit donc d'un résultat d'existence locale à régularité Sobolev pour des données initiales satisfaisant une condition généralisée de Penrose. On remarquera que la condition (1.6.11) doit être satisfaite initialement pour un profil en vitesse, ponctuellement en chaque $x \in \mathbb{T}^d$. Mentionnons également le récent travail de Chaub [Cha23] concernant les équations de Vlasov modérément singulières, pouvant être traitées sans condition de stabilité.

Concluons cette section en annonçant un peu plus précisément le parallèle avec le Chapitre 5 concernant le système des sprays épais : dans l'esprit du couplage singulier (1.6.10) et du Théorème 1.6.4, nous montrerons qu'il existe une condition de stabilité à la Penrose adaptée au système (1.6.1), ouvrant la voie à la construction de solutions pour celui-ci. Cette condition de stabilité sera en fait très proche de (1.6.11). Son identification et son utilité dans le domaine des couplages fluide-cinétique est à notre connaissance nouvelle, et fait l'objet de la section suivante.

1.6.3 Contribution du Chapitre 5 : caractère localement bien posé pour le système des sprays épais

Le Chapitre 5 reproduit le contenu de la prépublication à venir [EHK23], écrite en collaboration avec Daniel Han-Kwan. On s'intéresse à la construction en temps court de solutions pour le système des sprays épais (1.6.1) (et de ses variantes), dans le cas des équations de Navier-Stokes pour le fluide et sous une hypothèse de stabilité à la Penrose des données initiales.

Dans toute cette section, on travaille dans l'espace des phases $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$ ($d \in \mathbb{N}\backslash\{0\}$) où $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ désigne le tore plat de dimension d, muni de la mesure de Lebesgue normalisée. Pour alléger, on note H^m l'espace de Sobolev standard sur le tore.

Commençons par introduire la condition de stabilité adaptée aux système des sprays épais. Elle est réminiscente de la condition de Penrose (1.6.11) pour le système de Vlasov-Poisson.

Définition 1.6.5. *On dit qu'un profil $(x, v) \mapsto (\mathrm{f}(x, v), \rho(x))$ satisfait la condition de stabilité de Penrose pour les sprays épais (pour une pression p donnée) s'il existe $c > 0$ tel que pour tout $x \in \mathbb{T}^d$*

$$\inf_{(\gamma,\tau,k)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\}} \left| 1 - \frac{p'(\rho(x))\rho(x)}{1-\rho_{\mathrm{f}}(x)} \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot (\mathcal{F}_v \nabla_v \mathrm{f})(x, ks)\, \mathrm{d}s \right| > c. \quad (1.6.12)$$

En comparaison de (1.6.11), elle est satisfaite si pour tout $x \in \mathbb{T}^d$, le profil $v \mapsto \mathrm{f}(x, v)$ vérifie l'une des conditions évoquées après la condition (1.6.11) en Section 1.6.2. Rappelons que cela couvre une large gamme de données initiales, pas seulement de taille petite, et en particulier les Maxwelliennes en vitesse. On renvoie également à une reformulation équivalente de cette condition faite dans les Remarques 5.1.9-5.2.12 du Chapitre 5.

1.6.3.1 Résultat principal

Rappelons la Définition 1.6.3 de la norme Sobolev à poids $\mathcal{H}^m_r$ pour les fonctions définies sur l'espace des phases. Le résultat principal du Chapitre 5 est le suivant. Il concerne le cas $\nu = 1$ pour le système des sprays épais (1.6.1), pour des lois de pressions suffisamment régulières.

Théorème 1.6.6 (Chapitre 5 - basé sur [EHK23])**.** *Soit une donnée initiale*

$$f^{\mathrm{in}} \in \mathcal{H}^m_r, \quad \varrho^{\mathrm{in}} \in \mathrm{H}^{m+1}, \quad u^{\mathrm{in}} \in \mathrm{H}^m,$$

avec $m > 0$ et $r > 0$ assez grands, telle que $(f^{\mathrm{in}}, \varrho^{\mathrm{in}})$ satisfait la condition de Penrose (1.6.12) et

$$0 \leq f^{\mathrm{in}}, \quad \rho_{f^{\mathrm{in}}} < \Theta < 1, \quad 0 < \mu \leq \varrho^{\mathrm{in}}, \quad 0 < \underline{\theta} \leq (1 - \rho_{f^{\mathrm{in}}})\varrho^{\mathrm{in}} \leq \overline{\theta},$$

pour des constantes $\Theta, \mu, \underline{\theta}, \overline{\theta}$. Alors il existe $T > 0$ et une unique solution (f, ϱ, u) au système des sprays épais (1.6.1) *(pour $\nu = 1$) satisfaisant*

$$f \in \mathscr{C}\left([0,T]; \mathcal{H}^{m-1}_r\right), \quad \varrho \in \mathrm{L}^2(0,T;\mathrm{H}^m), \quad u \in \mathscr{C}([0,T];\mathrm{H}^m) \cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1}),$$

avec donnée initiale $(f^{\mathrm{in}}, \varrho^{\mathrm{in}}, u^{\mathrm{in}})$ et tels que $t \mapsto (f(t), \varrho(t))$ satisfait une condition de Penrose (1.6.12) *pour tout $t \in [0,T]$.*

On obtient donc, dans l'esprit de [HKR16], le caractère bien posé localement en temps et à régularité Sobolev du système des sprays épais (pour un fluide visqueux), pour des données stables au sens de Penrose. Les bornes initiales, propagées en temps court, sont là pour assurer la non-annulation des densités et de la fraction volumique.

Il s'agit d'une réponse partielle à la Conjecture 1.6.1 de Baranger-Desvillettes et constitue, à notre connaissance, le premier résultat d'existence de solutions pour un modèle de sprays épais (hormis le cas modifié et régularisé de [FBD22]).

Si la condition de Penrose (1.6.12) n'est pas vérifiée, le système des sprays épais (1.6.1) est en fait[18] mal posé au sens de Hadamard, ce qui signifie *grosso modo* que le système présente bel et bien des pertes de dérivées. Ce fait est réminiscent du mécanisme à l'œuvre pour l'équation de Vlasov-Benney (1.6.10) (voir [HKN16, Bar20]). La condition de Penrose (1.6.12) est donc une condition nécessaire et suffisante assurant le caractère localement bien posé de (1.6.1).

Eléments de preuve. La preuve est détaillée en Section 5.1.5 du Chapitre 5. Essayons d'en dégager les étapes et idées clefs.

Dans l'esprit du problème de la limite quasineutre présentée en Section 1.6.2, on commence par introduire un système des sprays épais régularisé où l'on lisse le gradient de pression dans l'équation cinétique :

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v[f(u-v)] - p'(\varrho)\nabla_x [\mathrm{J}_\varepsilon \varrho] \cdot \nabla_v f = 0, \quad \mathrm{J}_\varepsilon := (\mathrm{Id} - \varepsilon^2 \Delta_x)^{-1},$$

pour un paramètre $\varepsilon \in (0,1)$ (les autres équations sont laissées intactes). Par des méthodes standards, on obtient l'existence d'une solution $(f_\varepsilon, \rho_\varepsilon, u_\varepsilon)$ avec un temps de vie $T_\varepsilon > 0$ pour ce couplage non singulier. Malheureusement, ce temps de vie peut dégénérer vers 0 lorsque $\varepsilon \to 0$. On cherche ensuite à borner uniformément (en ε) la quantité

$$\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) := \|f_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)} + \|\varrho_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} + \|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1})},$$

sur un certain intervalle de temps (court) indépendant de ε. La perte de dérivée provenant de l'équation cinétique est ici prise en compte dans cette quantité *via* le décalage d'une dérivée entre les deux premiers termes. Une borne uniforme sur $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T)$ permettra de passer à la limite quand $\varepsilon \to 0$ grâce un argument de compacité.

Le reste de la preuve repose sur un argument de bootstrap. Par des estimations d'énergie, il vient que la quantité clé à estimer est le terme $\|\varrho_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}$. L'observation essentielle est que la densité ϱ_ε satisfait, *modulo* un terme S d'ordre plus bas,

$$\partial_t \varrho_\varepsilon + u_\varepsilon \cdot \nabla_x \varrho_\varepsilon + \frac{\varrho_\varepsilon}{1-\rho_{f_\varepsilon}} \mathrm{div}_x \left[j_{f_\varepsilon} - \rho_{f_\varepsilon} u_\varepsilon\right] = S, \tag{1.6.13}$$

et ne dépend donc de f_ε que par ses moments en vitesse j_{f_ε} et ρ_{f_ε}. La condition de Penrose (1.6.12) permettra d'établir une estimation sans perte sur ϱ_ε.

[18] Voir en Section 1.6.3.2.

On travaille donc d'abord sur les moments cinétique eux-mêmes. En prenant m dérivées spatiales dans l'équation cinétique, des commutateurs impliquant des termes de la forme

$$\partial_x \nabla_x \varrho_\varepsilon \cdot \partial_x^{m-1} \nabla_v f_\varepsilon \;\text{ ou }\; \partial_x^2 \nabla_x \varrho_\varepsilon \cdot \partial_x^{m-2} \nabla_v f_\varepsilon$$

apparaissent. Comme on ne contrôle que $m-1$ dérivées de f_ε, ces expressions sont mauvaises au vu du terme en divergence sur les moments dans (1.6.13). Dans l'esprit de [HKR16], l'idée est d'introduire une inconnue augmentée $\mathcal{F} = (\partial_{x,v}^I f_\varepsilon)_{|I|=m-1,m}$, qui satisfait un système d'équations de Vlasov couplées où le terme dominant est du type $\partial_x^I (p'(\varrho_\varepsilon) \nabla_x [\mathrm{J}_\varepsilon \varrho_\varepsilon]) \cdot \nabla_v f_\varepsilon$ pour $|I| = m-1, m$.

L'étape suivante consiste à simplifier la structure de ce système en adoptant une vision Lagrangienne. Partant de la dynamique totale

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} \mathrm{X}^{s;t}(x,v) = \mathrm{V}^{s;t}(x,v), \\ \dfrac{\mathrm{d}}{\mathrm{d}s} \mathrm{V}^{s;t}(x,v) = -\mathrm{V}^{s;t}(x,v) + u(s, \mathrm{X}^{s;t}(x,v)) - p'(\varrho(s, \mathrm{X}^{s;t}(x,v))) \nabla_x \mathrm{J}_\varepsilon \varrho(s, \mathrm{X}^{s;t}(x,v)), \end{cases}$$

nous utilisons un changement de variable en vitesse qui redresse les trajectoires en espace : en temps petit, il existe un difféomorphisme $\psi_x^{s,t} : \mathbb{R}^d \to \mathbb{R}^d$ tel que

$$\mathrm{X}^{s;t}(x, \psi_x^{s,t}(v)) = x + (1 - e^{t-s})v.$$

Cela permet de ramener la dynamique de l'équation de Vlasov à celle du transport libre avec friction

$$\partial_t g + v \cdot \nabla_x g - v \cdot \nabla_v g = 0.$$

Avec une formule de Duhamel le long des trajectoires pour $\mathcal{F}$, on obtient

$$\partial_x^m \rho_{f_\varepsilon} = \mathrm{K}_G^{\text{free}}[\mathrm{J}_\varepsilon \partial_x^m \varrho_\varepsilon] + R, \quad \partial_x^m j_{f_\varepsilon} = \mathrm{K}_{vG}^{\text{free}}[\mathrm{J}_\varepsilon \partial_x^m \varrho_\varepsilon] + R, \tag{1.6.14}$$

où $G = p'(\varrho_\varepsilon) \nabla_v f_\varepsilon$ et R est un reste contrôlé dans $\mathrm{L}_t^2 \mathrm{H}_x^1$. Ici, $\mathrm{K}_G^{\text{free}}$ est un opérateur intégral à noyau agissant sur $F(t,x)$ par

$$\mathrm{K}_G^{\text{free}}[F](t,x) = \int_0^t \int_{\mathbb{R}^d} [\nabla_x F](s, x - (t-s)v) \cdot G(t,s,x,v) \, \mathrm{d}v \, \mathrm{d}s.$$

Malgré une perte apparente de dérivée, cet opérateur est en fait borné sur $\mathrm{L}_{t,x}^2$ si le noyau G est suffisamment lisse et décroissant en vitesse. Il s'agit d'un résultat obtenu dans [HKR16].

Pour aboutir à (1.6.14), on a en fait besoin de plusieurs extensions de ce résultat pour des opérateurs de moyenne associés à la dynamique avec friction, à savoir

$$\mathrm{K}_G^{\text{fric}}[F](t,x) = \int_0^t \int_{\mathbb{R}^d} [\nabla_x F](s, x + (1 - e^{t-s})v) \cdot G(t,s,x,v) \, \mathrm{d}v \, \mathrm{d}s.$$

Un argument crucial est que $\mathrm{K}_G^{\text{fric}}$ est aussi borné sur $\mathrm{L}_{t,x}^2$ (pour un bon noyau G). De plus, comme observé dans [HKR23], l'annulation du noyau G sur la diagonale $s = t$ implique que les deux opérateurs précédents deviennent bornés de $\mathrm{L}_{t,x}^2$ dans $\mathrm{L}_t^2 \mathrm{H}_x^1$, donnant une régularisation additionnelle. Enfin, nous prouvons que la *différence* entre ces deux opérateurs fait gagner une dérivée supplémentaire : l'opérateur $\mathrm{K}_G^{\text{free}} - \mathrm{K}_G^{\text{fric}}$ est lui aussi borné de $\mathrm{L}_{t,x}^2$ dans $\mathrm{L}_t^2 \mathrm{H}_x^1$.

Ces résultats sont réminiscents des célèbres lemmes de moyenne en théorie cinétique [GLPS88]. Notons qu'ils permettent de regagner (au moins) une dérivée, ce qui n'est pas fourni directement par cette théorie (voir la discussion détaillée par Han-Kwan dans [HK19, Section 6.2]).

La dernière étape de la preuve consiste à insérer (1.6.14) dans l'équation (1.6.13), après avoir pris m dérivées dans celle-ci. Le but est d'obtenir un contrôle sur $h := \partial_x^m \varrho_\varepsilon$ dans $\mathrm{L}^2_{t,x}$. Grâce à une commutation entre div_v et les opérateurs intégraux de (1.6.14), on peut montrer que h satisfait

$$\left(\mathrm{Id} - \frac{\varrho_\varepsilon}{1-\rho_{f_\varepsilon}} \mathrm{K}_G^{\mathrm{free}} \circ \mathrm{J}_\varepsilon\right) \Big[\partial_t h + u_\varepsilon \cdot \nabla_x h\Big] = \text{Termes d'ordre inférieur}, \tag{1.6.15}$$

où $G(t,x,v) = p'(\varrho_\varepsilon(t,x))\nabla_v f_\varepsilon(t,x,v)$. Il s'agit donc d'une factorisation explicite entre un opérateur de transport suivant u_ε et un opérateur integro-différentiel dépendant de ϱ_ε et f_ε. La condition de Penrose (1.6.12), va permettre d'assurer des estimations $\mathrm{L}^2_{t,x}$ sur les solutions $H = \partial_t h + u_\varepsilon \cdot \nabla_x h$ de l'équation précédente.

Suivant [HKR16], on peut relier (*modulo* conjugaison en temps) l'opérateur intégro-différentiel précédent à un opérateur pseudodifférentiel semi-classique en temps-espace, de symbole

$$\mathscr{P}_{f,\varrho}(t,x,\gamma,\tau,k) := \frac{p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)} \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot (\mathcal{F}_v \nabla_v f)(t,x,ks)\, \mathrm{d}s.$$

On observe alors que la condition de Penrose (1.6.12), prolongée en temps court pour les solutions elle-mêmes, s'écrit

$$\inf_{(t,x,\gamma,\tau,k)} |1 - \mathscr{P}_{f,\varrho}(t,x,\gamma,\tau,k)| > c,$$

et signifie que l'opérateur intégro-différentiel dans (1.6.15) est *elliptique*. Un calcul pseudodifférentiel semi-classique adapté et avec grand paramètre permet alors d'en déduire des estimations $\mathrm{L}^2_{t,x}$ sur H. On obtient de même de telles estimations sur h *via* l'opérateur de transport. Cela permet essentiellement de conclure l'argument de bootstrap.

La condition de stabilité (1.6.12) permet donc principalement d'assurer que l'équation sur la densité du fluide se factorise en une partie *elliptique* et une partie *hyperbolique*, fournissant des estimées sans perte.

Remarque 1.6.7. L'approche utilisée permet de traiter le caractère bien posé en temps court de plusieurs variantes du système des sprays épais (comme présentées dans la Section 1.1.5.2), le cœur du problème étant le système (1.6.1). Cela comprend (voir les Sections 5.8–5.9–5.10 du Chapitre 5) :

1. le cas d'un opérateur de collisions dans l'équation cinétique de (1.6.1), de type élastique ou inélastique pour les sphères dures (voir Section 1.1.5.2);
2. le cas d'un fluide barotrope avec une équation supplémentaire sur la variable d'énergie interne $\mathfrak{e}(t,x) \in \mathbb{R}^+$ du fluide (voir le système (1.1.24)) : il existe une condition de stabilité de Penrose adaptée à ce contexte, permettant de traiter des lois de pression du type $p = p(\varrho\mathfrak{e})$;
3. le cas d'une force de traînée du type $\varrho(u-v)$ dans l'équation cinétique de (1.6.1) : notre approche nécessite ici de supposer que la donnée initiale f^{in} est à support compact en vitesse.

1.6.3.2 Perspectives

Les perspectives sont nombreuses pour le système des sprays épais, dont l'étude rigoureuse vient seulement d'être initiée.

- **Question 1**. Que se passe-t-il si la condition de stabilité de Penrose (1.6.12) n'est pas satisfaite par les données initiales ? Le système des sprays épais (1.6.1) est en fait mal posé au sens de Hadamard, au sens où le flot associé aux solutions, quand il existe, ne peut être Höldérien d'exposant $\alpha \in (0, 1]$, et ce sur n'importe quel espace de Sobolev d'indice positif quelconque et en temps arbitrairement petit. Ce fait, dans l'esprit des travaux sur le caractère mal posé de l'équation de Vlasov-Benney [HKN16, Bar20], fait l'objet d'un travail en cours d'écriture avec Aymeric Baradat et Daniel Han-Kwan. Il s'agit, en combinaison avec le Théorème 1.6.6, d'une réponse à la Conjecture 1.6.1 de Baranger et Desvillettes, dans le cas d'un fluide visqueux.

- **Question 2**. Peut-on espérer une théorie de Cauchy globale en temps pour le système des sprays épais (1.6.1) (dans le cas d'un fluide visqueux) ? Quelle est la dynamique du système en temps long ?

  Des calculs préliminaires semblent indiquer qu'il existe une fonctionnelle de type énergie modulée (comme présentée en Section 1.4.3) dont la décroissance encoderait également une convergence de la partie cinétique vers un profil singulier en vitesse. Notons par ailleurs que l'ajout d'un opérateur de collisions inélastique va dans le même sens car celui-ci entraîner une relaxation vers le même type de profil asymptotique. Une masse de Dirac en vitesse est par ailleurs un cas extrême d'une Maxwellienne en vitesse, qui satisfait la condition de Penrose (1.6.12). Cependant, il semble extrêmement difficile d'inclure la théorie de Cauchy locale (à très haute régularité) du Théorème 1.6.6 dans une analyse visant à capturer de tels comportements singuliers.

- **Question 3**. *Quid* du cas Euler pour le fluide ? A ce jour, les travaux [BDD23, FBD22] et notre stratégie mise en place au Chapitre 5 ne permettent pas encore de traiter ce cas et de construire une solution. Notons que notre méthode utilise pleinement l'effet de régularisation des équations de Navier-Stokes pour la partie fluide et ne s'applique pas directement si l'équation fluide fait elle aussi perdre une dérivée dans les estimations. On constate par ailleurs que la condition de Penrose (1.6.12) ne fait pas intervenir la vitesse initiale du fluide. On peut cependant espérer pouvoir adapter notre approche au cas d'un fluide non-visqueux, même si cela demanderait un travail important. De même, il serait intéressant d'étudier le caractère mal posé de ce système dans ce cas.

Chapter 2

Concentration and absorption for the Vlasov-Navier-Stokes system on a bounded domain

Based on the article [EHKM21] published in *Nonlinearity* (2021), in collaboration with Daniel Han-Kwan and Ayman Moussa.

2.1 Introduction

Let Ω be a smooth connected and bounded open set of $\mathbb{R}^3$. We consider the Vlasov-Navier-Stokes system in $\Omega \times \mathbb{R}^3$

$$\partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p = j_f - \rho_f u, \quad (t,x) \in \mathbb{R}_+^* \times \Omega, \tag{2.1.1}$$

$$\operatorname{div} u = 0, \quad (t,x) \in \mathbb{R}_+^* \times \Omega, \tag{2.1.2}$$

$$\partial_t f + v \cdot \nabla_x f + \operatorname{div}_v((u-v)f) = 0, \quad (t,x,v) \in \mathbb{R}_+^* \times \Omega \times \mathbb{R}^3, \tag{2.1.3}$$

where we define

$$\rho_f(t,x) := \int_{\mathbb{R}^3} f(t,x,v)\,\mathrm{d}v,$$

$$j_f(t,x) := \int_{\mathbb{R}^3} v f(t,x,v)\,\mathrm{d}v.$$

This system of nonlinear PDEs aims at describing the transport of small particles (the *dispersed phase*) immersed within a Newtonian viscous and incompressible fluid (the *continous phase*). Here, we describe the particles thanks to a kinetic distribution function $f(t,\cdot,\cdot)$ on the phase space $\Omega \times \mathbb{R}^3$ while the fluid is described thanks to its velocity $u(t,\cdot)$ and pressure $p(t,\cdot)$. The function f satisfies absorption boundary condition and we impose no-slip boundary condition for the vector-field u. We refer to Subsection 2.1.1 for more details.

In the large gallery of such *fluid-kinetic systems*, which stem from the works of O'Rourke [O'R81] and Williams [Wil85], the dispersed phase and the fluid are related through a particular coupling (see [Des10]). Here, we neglect the effect of collisions, coalescence and fragmentation between particles and we work under the assumption of *thin sprays*, considering that the volume occupied by the droplets is negligeable compared to that occupied by the fluid. This modelling is for instance a prototype for the description of an aerosol in the air. The coupling is thus made of a drag term in the Vlasov equation (2.1.3), and of a source term in the Navier-Stokes equations (2.1.1), called the *Brinkman force*, which describes the exchange of momentum between the particles and the fluid. In this system called the Vlasov-Navier-Stokes system, where physical constants are all normalized, both unknowns f and u depend on each other.

The Vlasov-Navier-Stokes system and its mathematical analysis have received a lot of attention in the past twenty years. The question of the global existence of weak solutions (or local existence of strong solutions) to the Cauchy problem is now well-understood on a large class of spatial tridimensional domains, like the flat torus $\mathbb{T}^3$ in [BDGM09, CK15], a fixed bounded domain in [ABdMB97] and even a time-dependent bounded domain in [BGM17, BMM20].

Concerning the rigorous derivation of theses equations from "first principles", little is known about the whole system. We refer to Section 1.3.3 of the introduction for more details.

The large time behavior for global solutions to the Vlasov-Navier-Stokes system appears as one of the next important steps in the understanding of such fluid kinetic model. A general setting has been highlighted by Jabin in [Jab00b] where an asymptotic scenario with concentration in velocity for the particles (namely, a convergence of the distribution function towards a Dirac mass for the velocity part) has been described for some class of kinetic equations (but with a different coupling between the fluid and the particles).

The asymptotic dynamics of the Vlasov-Navier-Stokes has been studied in a recent series of papers [GHKM18, HKMM20, HK22]: a concentration phenomenon in velocity, leading to monokinetic large time behavior for the particles, is proven in [HKMM20] in the case of the torus $\mathbb{T}^3$ and in [HK22] in the case of the whole space $\mathbb{R}^3$. This asymptotics is obtained for data that are in some sense close to equilibrium (we mention that a first contribution of Choi and Kwon has been made in the same direction on the torus in [CK15], but with an *a priori* assumption on the solutions which is not made in the two previous articles). We refer to Section 2.1.2 of this chapter where we give more details about the strategy which is used. When a Fokker-Planck dissipation term (namely, a term of the form $-\Delta_v f$) is added in the kinetic equation (2.1.3), global classical solutions can be constructed for data close to Maxwellian equilibria and this non-singular steady states locally attract these solutions (see [GHMZ10]).

Fore more general and physical domains, very few articles deal with the question of the large time behavior of the Vlasov-Navier-Stokes system. In [GHKM18], the authors consider the particular case of the Vlasov-Navier-Stokes system on a rectangle in $\mathbb{R}^2$, where the particular geometry of the domain allows to construct weak solutions around non-singular stationary equilibrium and where a geometric control condition helps to avoid concentration in velocity scenario for the particles (we also refer to Section 2.1.2 below for more details).

Main contribution of this chapter. We study the Vlasov-Navier-Stokes system on a bounded domain of $\mathbb{R}^3$ with absorption boundary condition for the distribution function and homogeneous Dirichlet boundary condition for the fluid velocity. We aim at giving a proof of the large time monokinetic behavior to solutions to the system. This singular behavior is adressed for global weak solutions satisfying a natural energy-dissipation inequality and starting at initial data close to equilibrium. Furthermore, there is a competition between concentration and absorption which determine the final dynamics. A broad range of outcomes is possible for the asymptotic spatial profile as we illustrate by exhibiting examples of initial data, leading to a variety of behaviors, from total absorption of the particles to no absorption at all.

2.1.1 Notations and definitions

We denote by $\mathscr{D}_{\mathrm{div}}(\Omega)$ the set of smooth $\mathbb{R}^3$ valued divergence free vector-fields having compact support in Ω. The closures of $\mathscr{D}_{\mathrm{div}}(\Omega)$ in $\mathrm{L}^2(\Omega)$ and in $\mathrm{H}^1(\Omega)$ are respectively denoted by $\mathrm{L}^2_{\mathrm{div}}(\Omega)$ and by $\mathrm{H}^1_{\mathrm{div}}(\Omega)$. We write $\mathrm{H}^{-1}_{\mathrm{div}}(\Omega)$ for the dual of the later. In the following, the outer-pointing normal to the boundary at a point $x \in \partial\Omega$ will be denoted by $n(x)$.

We first define the class of admissible initial data for (2.1.1)-(2.1.3).

Definition 2.1.1 (Initial condition)**.** *We shall say that a couple* (u_0, f_0) *is an admissible initial condition if*

$$\begin{aligned} u_0 &\in \mathrm{L}^2_{\mathrm{div}}(\Omega), \\ f_0 &\in \mathrm{L}^1 \cap \mathrm{L}^\infty(\Omega \times \mathbb{R}^3), \\ f_0 &\geq 0, \quad \int_{\Omega\times\mathbb{R}^3} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v = 1, \\ (x,v) &\mapsto f_0(x,v)|v|^2 \in \mathrm{L}^1(\Omega\times\mathbb{R}^3). \end{aligned}$$

The system (2.1.1)-(2.1.3) is supplemented with the following initial conditions for the fluid velocity u and the distribution function f

$$\begin{aligned} u_{|t=0} &= u_0 \text{ in } \Omega, \\ f_{|t=0} &= f_0 \text{ in } \Omega\times\mathbb{R}^3. \end{aligned}$$

We prescribe the following homogeneous Dirichlet boundary condition for the fluid velocity

$$u(t,\cdot) = 0, \text{ on } \partial\Omega. \tag{2.1.4}$$

We also need to introduce the following outgoing/incoming phase-space boundary for the dispersed phase:

$$\begin{aligned}
\Sigma^{\pm} &:= \Big\{(x,v) \in \partial\Omega \times \mathbb{R}^3 \mid \pm v \cdot n(x) > 0\Big\}, \\
\Sigma_0 &:= \Big\{(x,v) \in \partial\Omega \times \mathbb{R}^3 \mid v \cdot n(x) = 0\Big\}, \\
\Sigma &:= \Sigma^+ \sqcup \Sigma^- \sqcup \Sigma_0 = \partial\Omega \times \mathbb{R}^3.
\end{aligned}$$

Then, we prescribe the following absorption boundary condition for the distribution function f:

$$f(t,\cdot,\cdot) = 0, \text{ on } \Sigma^-, \tag{2.1.5}$$

meaning that all the particles reaching transversally the physical boundary are deposited.

We then define the energy and the dissipation of the whole system.

Definition 2.1.2. *1. The* ***kinetic energy*** *of the Vlasov-Navier-Stokes system is defined for all $t \geq 0$ as*

$$\mathrm{E}(t) := \frac{1}{2}\int_{\Omega} |u(t,x)|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\Omega\times\mathbb{R}^3} f(t,x,v)|v|^2 \,\mathrm{d}x\,\mathrm{d}v. \tag{2.1.6}$$

2. The ***dissipation*** *of the Vlasov-Navier-Stokes system is defined for all $t \geq 0$ as*

$$\mathrm{D}(t) := \int_{\Omega} |\nabla u(t,x)|^2 \,\mathrm{d}x + \int_{\Omega\times\mathbb{R}^3} f(t,x,v)|u(t,x) - v|^2 \,\mathrm{d}x\,\mathrm{d}v. \tag{2.1.7}$$

These two functionals naturally appear when looking for *a priori* estimates satisfied by solutions to the Vlasov-Navier-Stokes system. One can indeed check that the following energy-dissipation identity formally holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) + \mathrm{D}(t) = 0. \tag{2.1.8}$$

We then introduce the notion of weak solution to the system.

Definition 2.1.3 (Weak solution)**.** *Consider an admissible initial condition (u_0, f_0) in the sense of Definition 2.1.1. A global weak solution to the Vlasov-Navier-Stokes system with initial condition (u_0, f_0) on Ω is a pair (u, f) such that*

$$\begin{aligned}
&u \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2_{\mathrm{div}}(\Omega)) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}^1_{\mathrm{div}}(\Omega)), \\
&f \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^1 \cap \mathrm{L}^{\infty}(\Omega \times \mathbb{R}^d)), \\
&j_f - \rho_f u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}^{-1}_{\mathrm{div}}(\Omega)), \\
&f(t,x,v) \geq 0 \ \ \textit{for almost all } (t,x,v) \in \mathbb{R}^+ \times \Omega \times \mathbb{R}^3,
\end{aligned}$$

with u being a Leray solution to the Navier-Stokes equations (2.1.1)-(2.1.2) with strong energy inequality (with initial condition u_0) and f being a renormalized solution in the sense of DiPerna-Lions (see Appendix 2.A) to the Vlasov equation (2.1.3) (with initial condition f_0). Furthermore, we require that the following energy estimate holds for almost all $s \geq 0$ (including $s = 0$) and all $t \geq s$

$$\mathrm{E}(t) + \int_s^t \mathrm{D}(\sigma)\mathrm{d}\sigma \leq \mathrm{E}(s). \tag{2.1.9}$$

Remark 2.1.4. A weak solution (u, f) to the Vlasov-Navier-Stokes system satisfies the following weak formulations.

- For all $T > 0$, for all $\phi \in \mathscr{D}([0,T] \times \Omega)$ such that $\phi(T) = 0$ and $\operatorname{div}_x \phi = 0$

$$\int_0^T \int_\Omega [u \cdot \partial_t \phi + (u \otimes u) : \nabla_x \phi - \nabla_x u : \nabla_x \phi]\,(t,x)\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_0^T \langle j_f - \rho_f u, \phi \rangle(t)\,\mathrm{d}t - \int_\Omega u_0(x) \cdot \phi(0,x)\,\mathrm{d}x.$$

- For all $T > 0$, for all $\psi \in \mathscr{D}([0,T] \times \overline{\Omega} \times \mathbb{R}^3)$ such that $\psi(T) = 0$ and vanishing on $[0,T] \times (\Sigma^+ \sqcup \Sigma^0)$

$$\int_0^T \iint_{\Omega \times \mathbb{R}^3} f\,[\partial_t \psi + v \cdot \nabla_x \psi + (u - v) \cdot \nabla_v \psi]\,(t,x,v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t$$
$$= -\iint_{\Omega \times \mathbb{R}^3} f_0(x,v)\psi(0,x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

Furthermore, for such a weak solution to the Vlasov equation, we can define a trace on the phase space boundary Σ in the DiPerna-Lions framework for transport equations $\Omega \times \mathbb{R}^3$: we refer to Section 2.A of the Appendix for further properties of the solution to this initial boundary value problem (see [BF12, Section 1 - Chap 6] and [Mis00b]).

Finally, we introduce the following definitions that will be useful later.

Definition 2.1.5. *For any $\alpha > 0$, and any measurable function $f : \mathbb{R}^+ \times \Omega \times \mathbb{R}^3 \to \mathbb{R}^+$, we set*

$$m_\alpha f(t,x) := \int_{\mathbb{R}^3} |v|^\alpha f(t,x,v)\,\mathrm{d}v,$$
$$M_\alpha f(t) := \int_{\Omega \times \mathbb{R}^3} |v|^\alpha f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_\Omega m_\alpha f(t,x)\,\mathrm{d}x.$$

Definition 2.1.6. *We say that an initial kinetic condition f_0 satisfies the pointwise decay assumption of order $q > 0$ if*

$$N_q(f_0) := \sup_{\substack{x \in \Omega \\ v \in \mathbb{R}^3}} (1 + |v|^q) f_0(x,v) < \infty. \tag{2.1.10}$$

2.1.2 The torus, the whole space and the rectangle

As already said, the large-time behavior of the Vlasov-Navier-Stokes has been tackled in the case of the torus $\mathbb{T}^3$ in [HKMM20] or the whole space $\mathbb{R}^3$ in [HK22], and for which concentration in velocity happens (that is to say, a convergence towards a Dirac mass in velocity) in some regime close to equilibrium . We refer to Section 1.2 of the introduction of [HKMM20] for heuristics about this monokinetic asymptotic phenomenon when there are no boundaries, relying on an explicit formula for the solutions to the linearized equations around states of the form $(\overline{U}, \overline{f} = 0)$ with $\overline{U} \in \mathbb{R}^3$.

On the contrary, non-singular equilibra have been constructed in the case of a bidimensional rectangle in [GHKM18]. Taking advantage of the specific geometry of the domain and of absorbing boundary conditions for the particles, it has been proven that such stationary solutions are locally asymptotically stable relatively to compact perturbations

Let us describe more specifically the main strategy that has been employed in each of these situations.

- On the torus, Choi and Kwon have introduced in [CK15] a version of the following so-called *modulated energy*

$$\mathcal{E}_{\mathbb{T}^3}(t) := \frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} f(t,x,v)|v-\langle j_f(t)\rangle|^2\,\mathrm{d}v\,\mathrm{d}x \\ +\frac{1}{2}\int_{\mathbb{T}^3}|u(t,x)-\langle u(t)\rangle|^2\,\mathrm{d}x+\frac{1}{4}|\langle j_f(t)\rangle-\langle u(t)\rangle|^2, \quad (2.1.11)$$

where $\langle\cdot\rangle$ stands for the average on $\mathbb{T}^3$. They have used this functional to describe the large-time dynamics of the system but under an *a priori* assumption on the solutions. In [HKMM20], the authors have worked in a regime close to equilibrium (in the sense of having $\mathcal{E}_{\mathbb{T}^3}(0)$ small enough) and have provided the first complete description of the asymptotic behavior of the system. Loosely speaking, under a condition of the type

$$\mathcal{E}_{\mathbb{T}^3}(0)+\|u_0\|_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)}\ll 1, \quad (2.1.12)$$

the authors have shown that the fluid velocity $u(t)$ homogenizes when $t\to+\infty$ to the constant velocity $U_0 := \langle u_0 + j_{f_0}\rangle/2$, while the kinetic distribution function $f(t)$ converges in velocity to the Dirac mass supported at U_0. Moreover, this convergence is exponentially fast and can be measured thanks to the 1-Wasserstein distance on $\mathbb{T}^3\times\mathbb{R}^3$.

The previous modulated energy is at the heart of the proof of this monokinetic large time behavior for the system on the torus and is linked to the dissipation thanks to the formal identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\mathbb{T}^3}(t)+\mathrm{D}(t)=0.$$

As a matter of fact, this modulated energy essentially captures concentration phenomena so that controlling such a quantity is the main key to understand the large-time dynamics of the system. In short, under the assumption that $\rho_f\in\mathrm{L}^\infty(\mathbb{R}^+;\mathrm{L}^{3/2}(\mathbb{T}^3))$, Choi and Kwon proved in [CK15] that

$$\forall t\geq 0,\quad \mathcal{E}_{\mathbb{T}^3}(t)\lesssim e^{-\lambda t}\mathcal{E}_{\mathbb{T}^3}(0), \quad (2.1.13)$$

for some $\lambda>0$. Thanks to this exponential decay, they deduced that the asymptotics we have mentioned above hold for the fluid velocity and for the kinetic distribution.

The main strategy of [HKMM20] is based on a bootstrap analysis whose aim is to ensure that $\|\rho_f\|_{\mathrm{L}^\infty(\mathbb{R}^+;\mathrm{L}^\infty(\mathbb{T}^3))}<\infty$. Thanks to a straigthening change of variable in velocity, it is shown in [HKMM20] that this condition is actually implied by an estimate of the type

$$\int_0^\infty\|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\,\mathrm{d}s\ll 1. \quad (2.1.14)$$

Then, the framework of the Fujita-Kato solutions to the Navier-Stokes system has been leveraged in [HKMM20] to ensure that such a control holds under the condition (2.1.12).

- In the case of the euclidean space $\mathbb{R}^3$, the large-time behavior of the system has been investigated in [HK22] (for small data) where it has been shown that concentration in velocity towards a Dirac mass supported at 0 occurs also for the kinetic distribution, while the fluid velocity converges to 0. Because of the unbounded nature of the spatial domain, this convergence is only at a polynomial rate. More precisely, the good functional to look at to understand the concentration phenomenon turns out to be the kinetic energy E. One of the the main results of [HK22] is a conditional decay of the form

$$\forall t\geq 0,\quad \mathrm{E}(t)\leq\frac{\varphi_\alpha(\mathrm{E}(0))}{(1+t)^\alpha},\quad \text{for all}\quad \alpha\in]0,3/2[,$$

for some function φ_α, up to an *a priori* control on the moment ρ_f. As in the torus case, a bootstrap analysis is used to obtain such a control for small data solutions. But in this unbounded context where the decay of the energy is only polynomial, the Brinkman force requires a more careful treatment which has led to the derivation of a new family of identities for higher order dissipation functionals.

- In [GHKM18], the authors have dealt with the particular case of a bidimensional rectangle $\Omega := (-L, L) \times (-1, 1)$. Here, the boundary condition for the fluid part is a Dirichlet boundary condition matching a Poiseuille flow u_p (which is a stationary solution to the Navier-Stokes equations). For the particles, partly absorbing boundary conditions are used with absorption boundary conditions on the horizontal parts and an injection boundary condition on the vertical left part. Thanks to a geometric control condition (referred to as the *exit geometric condition*), compelling the particles to be absorbed by the boundary before a fixed finite time, one can construct non-trivial smooth equilibria $(\overline{u}, \overline{f})$ for the system, so that the concentration in velocity scenario does not occur. Furthermore, if $(\overline{u}, \overline{f})$ is a smooth stationary solution close to $(u_p, 0)$ (for some small injection term and a small Poiseuille flow u_p), any (u_0, f_0) which is a small perturbation of $(\overline{u}, \overline{f})$ gives birth to a weak solution (u, f) to the system satisfying

$$\forall t \in \mathbb{R}^+, \quad \|f(t) - \overline{f}\|_{\mathrm{L}^2_{x,v}} + \|u(t) - \overline{u}\|_{\mathrm{L}^2_x} \lesssim e^{-t}.$$

In short, the previous equilibria are asymptotically stable in L^2, under small localized perturbations.

In the following, we will consider solutions starting at initial data close to the equilibrium $(0, 0)$ and prove the existence of an asymptotic profile $\rho^\infty \in \mathrm{L}^\infty(\Omega)$ such that the following weak convergence holds

$$f(t) \xrightharpoonup[t\to+\infty]{} \rho^\infty \otimes \delta_{v=0},$$

where the tensor product is in (x, v). As said before, this kind of singular limit was already present in the case of the torus and of the whole space while geometric control was preventing such monokinetic behavior in the case of the rectangle. Once this monokinetic large time behavior is established, we will be interested in a possible further study of the spatial profile ρ^∞: because of the boundary effects, it may indeed exist some particle trajectories which are leaving the domain during the evolution. Therefore, we would like to study different possible scenarios, where absorption of the particles can prevail or not, in order to describe the asymptotic local density.

2.1.3 Main results

The main result of this chapter is stated in the following theorem and corollary.

Theorem 2.1.7. *There exists a universal constant $\varepsilon_0 > 0$ and a nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that the following holds. Let (u_0, f_0) be an admissible initial condition in the sense of Definition 2.1.1 satisfying*

$$\begin{aligned} &u_0 \in \mathrm{H}^1_{\mathrm{div}}(\Omega), \\ &M_6 f_0 + N_q f_0 < \infty, \quad \textit{for some } q > 4. \end{aligned} \tag{2.1.15}$$

If

$$\varphi\Big(1 + N_q f_0\Big)\mathrm{E}(0) < \varepsilon_0, \quad \|\nabla u_0\|_{\mathrm{L}^2(\Omega)} < \varepsilon_0, \tag{2.1.16}$$

where E *is defined in (2.1.6), then for any weak solution (u, f) to the Vlasov-Navier-Stokes system with initial data (u_0, f_0), there exist constants $\lambda, C_\lambda > 0$ such that*

$$\rho_f \in \mathrm{L}^\infty(\mathbb{R}^+; \mathrm{L}^\infty(\Omega)), \tag{2.1.17}$$

$$\mathrm{E}(t) \leq \mathrm{E}(0) C_\lambda \exp(-\lambda t), \quad t \geq 0. \tag{2.1.18}$$

Remark 2.1.8. In the case of the torus [HKMM20], the required assumption on the initial fluid velocity u_0 was $u_0 \in \dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)$ and allowed to rely on some parabolic smoothing for the solution u. Here, we have preferred to state the result with the assumption (2.1.15) in order to avoid unnecessary technical developments.

Corollary 2.1.9. *Under the same assumptions of Theorem (2.1.7), for any weak solution (u, f) to the Vlasov-Navier-Stokes system with admissible initial data (u_0, f_0), there exist constants $\lambda, C_\lambda > 0$ and $\rho^\infty \in \mathrm{L}^\infty(\Omega)$ such that for all $t \geq 0$*

$$\mathrm{W}_1\Big(f(t), \rho_f(t) \otimes \delta_{v=0}\Big) + \|u(t)\|_{\mathrm{L}^2(\Omega)} \leq \mathrm{E}(0)^{1/2} C_\lambda \exp(-\lambda t), \tag{2.1.19}$$

$$\rho_f(t) \overset{t \to +\infty}{\longrightarrow} \rho^\infty \quad \text{in } \mathrm{H}^{-1}(\Omega), \tag{2.1.20}$$

where W_1 stands for the Wasserstein distance on $\overline{\Omega} \times \mathbb{R}^3$. Moreover, the last convergence also occurs with an exponential rate.

As in [HKMM20], it is possible to provide a further description of the limit profile ρ^∞, using a notion of asymptotic characteristics.

Theorem 2.1.10. *For δ small enough, under the assumptions of Theorem 2.1.7, and if*

$$\int_0^{+\infty} \|\nabla u(\tau)\|_{\mathrm{L}^\infty(\Omega)} \, \mathrm{d}\tau \leq \delta, \tag{2.1.21}$$

then there exists a vector field

$$\begin{aligned} \mathbb{R}^3 \times \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, v) &\longmapsto \mathrm{X}_\infty(x, v), \end{aligned}$$

belonging to $\mathscr{C}^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and such that the following holds. For all $v \in \mathbb{R}^3$, the mapping $\mathrm{X}_{\infty,v} : x \mapsto \mathrm{X}_\infty(x, v)$ is a $\mathscr{C}^1$-diffeomorphism from $\mathbb{R}^3$ to itself and we have for almost every $x \in \Omega$

$$\rho^\infty(x) = \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{U}^\infty}(x, v) f_0\left(\mathrm{X}_{\infty,v}^{-1}(x), v\right) |\det \mathrm{D}_x \mathrm{X}_{\infty,v}^{-1}(x)| \, \mathrm{d}v, \tag{2.1.22}$$

where the set $\mathcal{U}^\infty$ is defined as follows: for $(x, v) \in \Omega \times \mathbb{R}^3$,

$$(x, v) \in \mathcal{U}^\infty \iff \exists ! y \in \Omega, \quad x = \mathrm{X}_\infty(y, v) \quad \text{and} \quad \forall t \geq 0, \quad \mathrm{X}_0^t(y, v) \in \Omega,$$

where

$$\mathrm{X}_0^t(y, v) = x + (1 - e^{-t})v + \int_0^t (1 - e^{\tau - t}) u(\tau, \mathrm{X}_0^\tau(x, v)) \, \mathrm{d}\tau, \tag{2.1.23}$$

$$\mathrm{X}_\infty(y, v) = x + v + \int_0^\infty u(\tau, \mathrm{X}_0^\tau(y, v)) \, \mathrm{d}\tau. \tag{2.1.24}$$

Remark 2.1.11. In the previous statement, we can actually get rid of the assumption (2.1.21) by only considering the evolution of the system from time $t = 1$. Indeed, the proof of Theorem 2.1.7 and Corollary 2.1.9 will ensure (see the bootstrap procedure in Section 2.6) that for all $\epsilon > 0$, the quantity $\|\nabla u\|_{\mathrm{L}^1(\varepsilon, +\infty; \mathrm{L}^\infty(\Omega))}$ can be ensured as small as required up to imposing a relevant smallness assumption (2.1.16). Nevertheless, for the sake of clarity, we have decided to state the result from time $t = 0$. If one wants to get the result without further assumption near the time 0, one has to replace f_0 by $f_{|t=1}$ together with integrals starting at $t = 1$ and for a set $\mathcal{U}^\infty$ defined with the function

$$\mathrm{X}_1^t(y, v) = x + (1 - e^{1-t})v + \int_1^t (1 - e^{\tau - t}) u(\tau, \mathrm{X}_1^\tau(x, v)) \, \mathrm{d}\tau.$$

As mentioned before, an important difference with the case of the torus is that the particle trajectory may possibly escape the domain Ω because we prescribe absorption boundary conditions for the Vlasov equation. This means that, in Corollary 2.1.9 and Theorem 2.1.10, we obtain an asymptotic spatial profile ρ^∞ whose total mass is unknown: indeed, part of the initial mass of the system may have disappeared throughout the evolution.

In the following result, we show that any fixed mass which is less than or equal to the initial mass can be reached by the system for some well-chosen data. We recall that we consider initial distribution functions f_0 such that $\int_{\Omega\times\mathbb{R}^3} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v = 1$.

Proposition 2.1.12. *Let $\alpha \in [0,1]$. There exists (u_0, f_0) an admissible initial condition in the sense of Definition 2.1.1 such that for any weak solution (u,f) to the Vlasov-Navier-Stokes system starting at (u_0,f_0), there exists $\rho^\infty \in \mathrm{L}^\infty(\Omega)$ which satisfies*

$$\rho_f(t) \xrightarrow[t\to+\infty]{} \rho^\infty \quad \textit{in} \quad \mathscr{C}(\overline{\Omega})', \tag{2.1.25}$$

$$\int_\Omega \rho^\infty(x)\,\mathrm{d}x = \alpha. \tag{2.1.26}$$

Strategy and outline of the chapter. Let us describe the main strategy that we use in this chapter and how it is organized. Our method is reminiscent of the work of [HKMM20].

In **Section 2.2**, we study the conditional decay of the kinetic energy and its consequences. As explained before, such a decay will essentially be enough for the concentration in velocity to happen. Here, we strongly rely on the Poincaré inequality which holds on the domain Ω for the fluid velocity. In short, we can hope for an exponential decay of the energy provided that the local density ρ_f is controlled in $\mathrm{L}^\infty\mathrm{L}^\infty$. This result is stated in Proposition 2.2.4. We then show in Proposition 2.2.5 that it also provides the existence of an asymptotic spatial profile ρ^∞.

In **Section 2.3**, we investigate a way to get the desired control on this moment ρ_f. We mainly rely on trajectorial estimates, by introducing the characteristic curves (X,V) for the Vlasov equation as the solution of the differential system

$$\begin{cases} \dot{\mathrm{X}}(s;t,x,v) = \mathrm{V}(s;t,x,v), \\ \dot{\mathrm{V}}(s;t,x,v) = u(s,\mathrm{X}(s;t,x,v)) - \mathrm{V}(s;t,x,v), \end{cases}$$

with $(\mathrm{X}(t;t,x,v),\mathrm{V}(t;t,x,v)) = (x,v)$ and where we extend u by 0 outside of Ω. In Proposition 2.3.2, we then derive a representation formula for weak solutions to the Vlasov equation that is valid on a bounded domain and that takes into account the boundary effects. This leads to

$$\rho_f(t,x) = e^{3t}\int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t}(x,v)\, f_0(\mathrm{X}(0;t,x,v),\mathrm{V}(0;t,x,v))\,\mathrm{d}v, \tag{2.1.27}$$

where

$$\mathcal{O}^t = \left\{(x,v)\in\Omega\times\mathbb{R}^3 \mid \forall\sigma\in[0,t],\ \mathrm{X}^{\sigma;t}(\sigma;t,x,v)\in\Omega\right\}.$$

Thanks to the change of variable in velocity $v\mapsto \mathrm{V}(0;t,x,v)$, obtaining a uniform control on ρ_f until time t actually reduces to an inequality of the type $\|\nabla u\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\Omega))} \ll 1$. This fact is stated in Proposition 2.3.4.

A third step is made of a bootstrap analysis and aims at obtaining the previous control on $\|\nabla u\|_{\mathrm{L}^\infty(\Omega)}$. To do so, we interpolate this quantity by second order derivatives of u together with the kinetic energy of the system, in order to get integrability in time thanks to the exponential decay of this energy. This procedure requires some integrability and higher regularity estimates for the solutions to the Navier-Stokes equations. **Section 2.4** yields preliminaries for this subsequent bootstrap argument. In Proposition 2.4.8, we obtain local in time controls for the velocity field u, as well as for the moments ρ_f and j_f. We also provide H^1 energy estimates for the velocity field u, using the framework of strong solutions to the Navier-Stokes equations. This one is allowed if the initial condition and the Brinkman force are small enough, as seen in Proposition 2.4.9. In order to propagate this condition, we introduce the notion of strong existence time.

In **Section 2.5**, we start to estimate the second order derivatives $\mathrm{D}^2 u$ in some $\mathrm{L}^q\mathrm{L}^p$ space. To do so, we rely on maximal regularity for the Stokes system. This requires to control the Brinkman force $j_f - \rho_f u$ and the convection term $(u \cdot \nabla)u$ in some $\mathrm{L}^a\mathrm{L}^b$ space, which is possible thanks to the higher order energy estimates satisfied by the fluid velocity.

The bootstrap analysis then takes place in **Section 2.6** where the smallness condition (2.1.16) we have used in our statements plays a key role. We essentially show that this one can ensure the global control $\|\nabla u\|_{\mathrm{L}^1(1,+\infty;\mathrm{L}^\infty(\Omega))} \ll 1$, ending the proof of Theorem 2.1.7 and Corollary 2.1.9.

In **Section 2.7**, we give a further description of the asymptotic spatial density profile ρ^∞ of the particles. In order to prove Theorem 2.1.10, we rely on the Lagrangian structure of the Vlasov equation. The main guiding line is to pass to the limit in the representation formula (2.1.27) when $t \to +\infty$, by considering asymptotic characteristic curves. Here, our analysis is based upon a fine analysis of the flow (X, V) where we have to deal again carefully with the possible exit of these trajectories from the domain Ω.

As the structure of the asymptotic profile remains quite implicit, the question of its total mass is discussed in **Section 2.8**. We first study sufficient conditions on the support of the initial distribution f_0 and on the initial velocity u_0 to ensure the possible following scenarios: the total mass of the system can be preserved throughout the evolution or, on the contrary, can vanish after a certain finite time. These two opposite situations are stated in Propositions 2.8.1-2.8.5 and strongly depend on the support in velocity of f_0. Their analysis are strongly based on the consequences of the previous bootstrap. Combining these two examples eventually leads to Proposition 2.1.12.

2.2 Energy dissipation: towards concentration

We start with a Lemma which states that the total mass of the system is nonincreasing along the evolution. The proof of this result is postponed to Subsection 2.3.2 because it requires the use of a representation formula for the distribution function f.

Lemma 2.2.1. *For any weak solution (u, f) to the Vlasov-Navier-Stokes system and for all $t \geq 0$, we have*

$$\int_{\Omega\times\mathbb{R}^3} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \leq \int_{\Omega\times\mathbb{R}^3} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

We can now state the following inequality which highlights the role of the kinetic energy for the study of the asymptotics.

Lemma 2.2.2. *For any weak solution (u, f) to the Vlasov-Navier-Stokes system and for all $t \geq 0$, we have the following inequality*

$$\mathrm{W}_1\left(f(t), \rho_f(t) \otimes \delta_{v=0}\right) + \|u(t)\|_{\mathrm{L}^2(\Omega)} \lesssim \mathrm{E}(t)^{1/2}, \tag{2.2.1}$$

where W_1 stands for the Wasserstein distance on $\overline{\Omega} \times \mathbb{R}^3$.

Proof. First, we observe that for all times $t \geq 0$, the measures $f(t)\,\mathrm{d}x\,\mathrm{d}v$ and $(\rho_f(t)\mathrm{d}x) \otimes \delta_{v=0}$ have the same mass on $\Omega \times \mathbb{R}^3$. Therefore we use the Monge-Kantorovich duality for W_1 on $\overline{\Omega} \times \mathbb{R}^3$ (see

Appendix 2.C) to get

$$\begin{aligned}\mathrm{W}_1\left(f(t), \rho_f(t) \otimes \delta_{v=0}\right) &= \sup_{\|\nabla_{x,v}\phi\|_\infty \leq 1} \left\{ \left| \int_{\Omega \times \mathbb{R}^3} f(t,x,v)(\phi(x,v) - \phi(x,0)) \, \mathrm{d}x \, \mathrm{d}v \right| \right\} \\ &\leq \int_{\Omega \times \mathbb{R}^3} f(t,x,v)|v| \, \mathrm{d}x \, \mathrm{d}v.\end{aligned}$$

The Cauchy-Schwarz inequality and the previous Lemma 2.2.1 together with the normalisation for the density f_0 give us

$$\begin{aligned}&\mathrm{W}_1\left(f(t), \rho_f(t) \otimes \delta_{v=0}\right) \\ &\qquad \leq \left(\int_{\Omega \times \mathbb{R}^3} f(t,x,v)|v|^2 \, \mathrm{d}x \, \mathrm{d}v \right)^{1/2} \left(\int_{\Omega \times \mathbb{R}^3} f(t,x,v) \, \mathrm{d}x \, \mathrm{d}v \right)^{1/2} \leq \sqrt{2}\mathrm{E}(t)^{1/2},\end{aligned}$$

by definition of the kinetic energy. The inequality $\|u(t)\|_{\mathrm{L}^2(\Omega)} \lesssim \mathrm{E}(t)^{1/2}$ is also straigthforward. □

Before going further, we state a Grönwall's lemma under integral form in which exponential decay is can be obtained (a proof can be found in [HKMM20, Appendix]).

Lemma 2.2.3. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be an integrable and decaying function such that $g(0)$ is well defined and which satisfies, for some $\lambda > 0$, for almost every $t \geq 0$*

$$\lambda \int_t^{+\infty} g(\tau) \, \mathrm{d}\tau \leq g(t).$$

Then, for almost every $t \geq 0$, we have

$$g(t) \lesssim_\lambda g(0) e^{-\lambda t},$$

where $\lesssim_\lambda$ refers to a constant only depending on λ.

We now state the following proposition relating the dissipation and the kinetic energy.

Proposition 2.2.4. *There exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that the following holds. Let (u, f) be a weak solution to the Vlasov-Navier-Stokes system such that $\rho_f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\Omega))$. Fix $T \in (0, +\infty]$ and set*

$$\lambda := \psi\left(\sup_{[0,T]} \|\rho_f(t)\|_{\mathrm{L}^\infty(\Omega)} \right),$$

then

$$\forall t \in [0,T], \quad \lambda \mathrm{E}(t) \leq \mathrm{D}(t), \tag{2.2.2}$$

and we have the following exponential decay

$$\forall t \in [0,T], \quad \mathrm{E}(t) \lesssim_\lambda \mathrm{E}(0) e^{-\lambda t}, \tag{2.2.3}$$

where $\lesssim_\lambda$ refers to a constant only depending on λ.

Proof. The proof follows the same arguments as in [HKMM20, Lemma 3.4] and was originally obtained in [CK15, Theorem 1.2] in the case of the torus. It mainly relies on the fact that the Poincaré inequality is valid for the fluid velocity on Ω.

- First, we note that (2.2.2) implies (2.2.3). Indeed, if (2.2.2) holds, the energy-dissipation inequality (2.1.9) shows that for almost all $0 \leq s \leq t \leq T$,

$$\mathrm{E}(t) + \lambda \int_s^t \mathrm{E}(\tau)\,\mathrm{d}\tau \leq \mathrm{E}(s),$$

from which we get

$$\lambda \int_t^T \mathrm{E}(\tau)\,\mathrm{d}\tau \leq \mathrm{E}(t), \qquad \mathrm{E}(t) \leq \mathrm{E}(s), \qquad s \leq t.$$

We get the desired exponential decay (2.2.3) thanks to Lemma 2.2.3.

- It remains to get a $\lambda > 0$ such that $\lambda \mathrm{E} \leq \mathrm{D}$ on $[0, T]$. Thanks to the Poincaré inequality in $\mathrm{H}^1_0(\Omega)$, there exists a constant $c_P > 0$ such that

$$\mathrm{D}(t) \geq \frac{1}{2} \int_{\Omega\times\mathbb{R}^3} f(t)|u(t) - v|^2\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2} c_P \|u(t)\|^2_{\mathrm{L}^2(\Omega)}.$$

Let us denote

$$\widetilde{\mathrm{E}}(t) := \mathrm{E}(t) - \frac{1}{2}\|u(t)\|^2_{\mathrm{L}^2(\Omega)} = \frac{1}{2}\int_{\Omega\times\mathbb{R}^3} f(t,x,v)|v|^2\,\mathrm{d}v\,\mathrm{d}x.$$

We see that it is sufficient to find some $\gamma > 0$ and $\beta \in (0, c_P)$ such that

$$\int_{\Omega\times\mathbb{R}^3} f(t)|u(t) - v|^2\,\mathrm{d}v\,\mathrm{d}x \geq \gamma \widetilde{\mathrm{E}}(t) - \beta \|u(t)\|^2_{\mathrm{L}^2(\Omega)}. \tag{2.2.4}$$

Indeed, if this holds, we can define $\lambda := \min(\gamma/2, c_P - \beta) > 0$ and the previous inequalities yield

$$\mathrm{D}(t) \geq \lambda \widetilde{\mathrm{E}}(t) + \frac{1}{2}\left(c_P - \beta - \frac{\lambda}{2}\right)\|u(t)\|^2_{\mathrm{L}^2(\Omega)} \geq \lambda \mathrm{E}(t).$$

In order to get the inequality (2.2.4), we first write

$$\int_{\Omega\times\mathbb{R}^3} f(t)|u(t) - v|^2\,\mathrm{d}v\,\mathrm{d}x = \int_{\Omega\times\mathbb{R}^3} f|v|^2\,\mathrm{d}v\,\mathrm{d}x + \int_\Omega \rho_f(t)|u(t)|^2\,\mathrm{d}x - 2\int_{\Omega\times\mathbb{R}^3} f(t)\,v\cdot u(t)\,\mathrm{d}v\,\mathrm{d}x.$$

For the last term, we use Cauchy-Schwarz and Young inequalities to get

$$-2\int_{\Omega\times\mathbb{R}^3} f(t)\,v\cdot u(t)\,\mathrm{d}v\,\mathrm{d}x \geq -\alpha \int_{\Omega\times\mathbb{R}^3} f(t)|v|^2\,\mathrm{d}v\,\mathrm{d}x - \alpha^{-1}\int_\Omega \rho_f(t)|u(t)|^2\,\mathrm{d}x,$$

for some $\alpha \in (0, 1)$. We infer that

$$\int_{\Omega\times\mathbb{R}^3} f(t)|v - u(t)|^2\,\mathrm{d}v\,\mathrm{d}x \geq (1-\alpha)\int_{\Omega\times\mathbb{R}^3} f(t)|v|^2\,\mathrm{d}v\,\mathrm{d}x - (\alpha^{-1} - 1)\int_\Omega \rho_f(t)|u(t)|^2\,\mathrm{d}x.$$

From the definition of $\widetilde{\mathrm{E}}(t)$, we can deduce that

$$\begin{aligned}\int_{\Omega\times\mathbb{R}^3} f(t)|u(t) - v|^2\,\mathrm{d}v\,\mathrm{d}x &\geq (1-\alpha)\widetilde{\mathrm{E}}(t) - (\alpha^{-1} - 1)\int_\Omega \rho_f|u(t)|^2\,\mathrm{d}x \\ &\geq (1-\alpha)\widetilde{\mathrm{E}}(t) - (\alpha^{-1} - 1)\sup_{s\in[0,T]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\Omega)}\|u(t)\|^2_{\mathrm{L}^2(\Omega)}.\end{aligned}$$

By our assumptions, the quantity $\sup_{s\in[0,T]} \|\rho_f(s)\|_{L^\infty(\Omega)}$ is finite and we can also assume that it is not equal to 0. In order to get the inequality (2.2.4), we eventually choose

$$\alpha := \left(1 + \frac{c_P}{2 \sup_{s\in[0,T]} \|\rho_f(s)\|_{L^\infty(\Omega)}}\right)^{-1} \in (0,1),$$

and then $\beta := (\alpha^{-1} - 1) \sup_{s\in[0,T]} \|\rho_f(s)\|_{L^\infty(\Omega)} = c_P/2$ and $\gamma := 1 - \alpha$. Such an α is a continuous nondecreasing function of the variable $\sup_{s\in[0,T]} \|\rho_f(s)\|_{L^\infty(\Omega)}$ and does not vanish. Thus, we get

$$\lambda := \min((1-\alpha)/2, c_P/2),$$

which is of the desired form. The proof is therefore complete. □

The following proposition highlights the fact that a global bound on the local density ρ_f is enough to obtain the convergence towards an asymptotic profile.

Proposition 2.2.5. *For any weak solution (u, f) to the Vlasov-Navier-Stokes system for which*

$$\sup_{t\geq 0} \|\rho_f(t)\|_{L^\infty(\Omega)} < \infty, \tag{2.2.5}$$

there exists a profile $\rho^\infty \in L^\infty(\Omega)$ such that

$$\rho_f(t) \overset{t\to+\infty}{\longrightarrow} \rho^\infty \quad \text{in } H^{-1}(\Omega), \tag{2.2.6}$$

exponentially fast.

Proof. It relies strongly on the exponential decay of the kinetic energy: indeed, as a consequence of Proposition 2.2.4, we have $E(t) \longrightarrow 0$ exponentially fast when $t \to +\infty$. We will use Cauchy's criterion in $H^{-1}(\Omega)$ in order to obtain the existence of an asymptotic profile.

To derive an estimate in H^{-1} norm, we fix $\psi \in \mathscr{C}_c^\infty(\Omega)$ and we use the weak formulation for the Vlasov equation (see Appendix 2.A) to write

$$\int_\Omega \psi\rho_f(t) - \int_\Omega \psi\rho_f(s) = \int_s^t \int_\Omega \nabla\psi \cdot j_f(\tau)\, d\tau - \int_s^t \int_{\partial\Omega\times\mathbb{R}^3} (\gamma f)\psi(x) v\cdot n(x)\, d\sigma(x)\, dv\, d\tau,$$

for all $0 \leq s \leq t$. Since ψ is compactly supported in Ω, the last term vanishes. For the other term, we then use the Cauchy-Schwarz inequality to get, omitting the variable to simplify the formula

$$\left|\int_\Omega \psi\rho_f(t)\, dx - \int_\Omega \psi\rho_f(s)\, dx\right| \leq \int_s^t \left(\int_\Omega \rho_f |\nabla\psi|^2\, dx\right)^{1/2} \left(\int_{\Omega\times\mathbb{R}^3} f|v|^2\, dx\, dv\right)^{1/2} d\tau,$$

so that we eventually obtain

$$\left|\int_\Omega \psi\rho_f(t)\, dx - \int_\Omega \psi\rho_f(s)\, dx\right| \leq \|\rho_f\|^{1/2}_{L^\infty(\mathbb{R}^+;L^\infty(\Omega))} \|\nabla\psi\|_{L^2(\Omega)} \int_s^t E(\tau)^{1/2}\, d\tau.$$

Thanks to the assumption (2.2.5), we infer that

$$\|\rho_f(t) - \rho(s)\|_{H^{-1}(\Omega)} \lesssim \|\rho_f\|^{1/2}_{L^\infty(\mathbb{R}^+;L^\infty(\Omega))} \int_s^t E(\tau)^{1/2}\, d\tau. \tag{2.2.7}$$

By the exponential decay of the energy on $\mathbb{R}^+$, the function $t \mapsto \mathrm{E}^{1/2}(t)$ in integrable on $\mathbb{R}^+$. Therefore, the Cauchy criterion for the $\|\cdot\|_{\mathrm{H}^{-1}(\Omega)}$ norm applied to the function $\rho_f(t)$ when $t \to +\infty$ gives us the convergence $\rho_f(t) \to \rho^\infty$ in $\mathrm{H}^{-1}(\Omega)$ when $t \to +\infty$, for some $\rho^\infty \in \mathrm{H}^{-1}(\Omega)$.

But again thanks to (2.2.5), any sequence $(t_n)_n \to +\infty$ produces a sequence of functions $(\rho_f(t_n))_n$ bounded in $\mathrm{L}^\infty(\Omega)$. We deduce that $\rho^\infty \in \mathrm{L}^\infty(\Omega)$ and it is easy to see that the convergence towards this asymptotic profile is exponentially fast by looking at the estimate (2.2.7) and letting $t \to +\infty$. □

2.3 The particle trajectory

In this section, we first derive a representation formula for the kinetic distribution thanks to the method of characteristics for the Vlasov equation, taking into account the absorption boundary condition. Then, we explain how to use a straightening change of variable in velocity in the integral formula giving ρ_f to get a bound of the form

$$\sup_{0 \le s \le t} \|\rho_f(s)\|_{\mathrm{L}^\infty(\Omega)} \lesssim 1.$$

This idea has already been used in [HKMM20, HK22] and can be applied if the change of variable is valid, which is the case if the semi norm $\|\nabla u\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\Omega))}$ is small enough.

2.3.1 Characteristic curves for the system

Given a time-dependent vector field u on $\mathbb{R}^+ \times \Omega$, a time $t \in \mathbb{R}^+$ and a point $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$, we define the characteristic curves $s \in \mathbb{R}^+ \mapsto (\mathrm{X}(s;t,x,v), \mathrm{V}(s;t,x,v)) \in \mathbb{R}^3 \times \mathbb{R}^3$ for the Vlasov equation (associated to u) as the solution of the following system of ordinary differential equations

$$\begin{cases} \dot{\mathrm{X}}(s;t,x,v) = \mathrm{V}(s;t,x,v), \\ \dot{\mathrm{V}}(s;t,x,v) = (Pu)(s, \mathrm{X}(s;t,x,v)) - \mathrm{V}(s;t,x,v), \\ \mathrm{X}(t;t,x,v) = x, \\ \mathrm{V}(t;t,x,v) = v, \end{cases} \tag{2.3.1}$$

where the dot means derivative along the first variable. Here, P is the linear extension operator continuous from $\mathrm{L}^\infty(\Omega)$ to $\mathrm{L}^\infty(\mathbb{R}^3)$ and from $\mathrm{W}^{1,\infty}_0(\Omega)$ to $\mathrm{W}^{1,\infty}(\mathbb{R}^3)$ defined by

$$\forall x \in \mathbb{R}^d, \quad (Pw)(x) := \begin{cases} w(x), & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^3 \setminus \Omega, \end{cases} \tag{2.3.2}$$

and which satisfies

$$\forall w \in \mathrm{L}^\infty(\Omega), \ \|Pw\|_{\mathrm{L}^\infty(\mathbb{R}^3)} = \|w\|_{\mathrm{L}^\infty(\Omega)}, \tag{2.3.3}$$

$$\forall w \in \mathrm{W}^{1,\infty}_0(\Omega), \quad \|\nabla(Pw)\|_{\mathrm{L}^\infty(\mathbb{R}^3)} \le \|\nabla w\|_{\mathrm{L}^\infty(\Omega)}. \tag{2.3.4}$$

Indeed, for $w \in \mathrm{W}^{1,\infty}_0(\Omega)$, let us show that the extension Pw is Lipschitz on $\mathbb{R}^d$. To do so, take $x, y \in \mathbb{R}^d$.

- If $x \in \overline{\Omega}$ and $y \in \overline{\Omega}$, then we have

$$|(Pw)(x) - (Pw)(y)| = |w(x) - w(y)| \le \|\nabla w\|_{\mathrm{L}^\infty(\Omega)}|x - y|.$$

- If $x \notin \overline{\Omega}$ and $y \notin \overline{\Omega}$, then $|(Pw)(x) - (Pw)(y)| = 0 \le \|\nabla w\|_{\mathrm{L}^\infty(\Omega)}|x - y|$.

- If $x \in \overline{\Omega}$ and $y \notin \overline{\Omega}$, then we use the connectedness of the segment $[x, y]$ to find a point $z \in [x, y] \cap \partial\Omega$. We thus write

$$|(Pw)(x) - (Pw)(y)| = |w(x)| = |w(x) - w(z)| \leq \|\nabla w\|_{\mathrm{L}^\infty(\Omega)}|x - z| \leq \|\nabla w\|_{\mathrm{L}^\infty(\Omega)}|x - y|.$$

Here, we have used the fact that the Lipschitz semi-norm of w on Ω is smaller than $\|\nabla w\|_{\mathrm{L}^\infty(\Omega)}$ (see e.g: [BF12, Proposition III.2.9]). All in all, we end up with

$$|(Pw)(x) - (Pw)(y)| \leq \|\nabla w\|_{\mathrm{L}^\infty(\Omega)}|x - y|,$$

so that $Pw \in \mathrm{W}^{1,\infty}(\mathbb{R}^3)$ with the desired inequalities, because of the identification of $\mathrm{W}^{1,\infty}(\mathbb{R}^3)$ with the space of bounded Lipschitz functions on $\mathbb{R}^3$.

Also, we will use the convention

$$(Pu)(t, \cdot) = P(u(t, \cdot)).$$

Now, let $T > 0$ be fixed and take

$$u \in \mathrm{L}^2(0, T; \mathrm{H}^1_0(\Omega)) \cap \mathrm{L}^1(0, T; \mathrm{W}^{1,\infty}(\Omega)).$$

We can apply the Cauchy-Lipschitz theorem to show the following proposition.

Proposition 2.3.1. *Given $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ and a time $t \in [0, T]$, the system (2.3.1) admits a unique solution $s \mapsto \mathrm{Z}_{s,t}(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ on $[0, T]$ and*

$$\mathrm{Z}_{s,t} : \begin{cases} \mathbb{R}^3 \times \mathbb{R}^3 & \longrightarrow \mathbb{R}^3 \times \mathbb{R}^3 \\ (x, v) & \longmapsto \mathrm{Z}_{s,t}(x, v) := (\mathrm{X}(s; t, x, v), \mathrm{V}(s; t, x, v)) \end{cases}$$

is a diffeomorphism of $\mathbb{R}^3 \times \mathbb{R}^3$ whose inverse is given by $\mathrm{Z}_{s,t}^{-1} = \mathrm{Z}_{t,s}$ and whose Jacobian determinant is $e^{3(t-s)}$.

In what follows, for all $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$, we will sometimes use the notation

$$\mathrm{Z}_{s,t}(x, v) := (\mathrm{X}_{s,t}(x, v), \mathrm{V}_{s,t}(x, v)) := (\mathrm{X}(s; t, x, v), \mathrm{V}(s; t, x, v)). \tag{2.3.5}$$

2.3.2 A representation formula for the kinetic distribution

We keep the same assumptions and notations as in the previous subsection. We now define for any $t \geq 0$

$$\mathcal{O}^t := \bigcap_{\sigma \in [0,t]} \mathrm{Z}(\sigma; t, \Omega \times \mathbb{R}^3).$$

We easily see that

$$\mathcal{O}^t = \Big\{(x, v) \in \Omega \times \mathbb{R}^3 \mid \forall \sigma \in [0, t], \mathrm{X}(\sigma; t, x, v) \in \Omega\Big\}. \tag{2.3.6}$$

For $(x, v) \in \Omega \times \mathbb{R}^3$ and for any $t \geq 0$, we also define

$$\tau^+(t, x, v) := \sup\{s \geq t \mid \forall \sigma \in [t, s], \ \mathrm{X}(\sigma; t, x, v) \in \Omega\}.$$

One can observe that

$$\mathrm{Z}_{0,t}(\mathcal{O}^t) = \Big\{(x, v) \in \Omega \times \mathbb{R}^3 \mid \tau^+(0, x, v) > t\Big\}. \tag{2.3.7}$$

We now derive an important representation formula for the distribution function which solves the Vlasov equation in the weak sense.

Proposition 2.3.2. *Let f be the weak solution to the initial boundary value problem for the Vlasov equation, associated to the velocity field $u \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)) \cap \mathrm{L}^1(0,T;\mathrm{W}^{1,\infty}(\Omega))$ with initial condition f_0 and with absorption boundary condition. There holds*

$$f(t,x,v) = e^{3t}\mathbf{1}_{\mathcal{O}^t}(x,v)\, f_0(\mathrm{Z}_{0,t}(x,v)) \quad a.e. \tag{2.3.8}$$

This formula seems to be folklore but we have not been able to find a proof in the literature. For the sake of completeness, and because this will be useful in the later Sections 2.7-2.8, we give a complete proof of Proposition 2.3.2 in Section 2.B of the Appendix.

A first application of this formula is the proof of Lemma 2.2.1 that we had stated in Section 2.2.

Proof of Lemma 2.2.1. First suppose that u and f_0 are smooth enough. By Proposition 2.3.2, we have

$$f(t,x,v) = e^{3t}\mathbf{1}_{\mathcal{O}^t}(x,v)\, f_0(\mathrm{Z}_{0,t}(x,v)) \quad \text{a.e.}$$

therefore

$$\begin{aligned}\int_{\Omega\times\mathbb{R}^3} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v &= \int_{\Omega\times\mathbb{R}^3} e^{3t}\mathbf{1}_{\mathcal{O}^t}(x,v)\, f_0(\mathrm{Z}_{0,t}(x,v))\,\mathrm{d}x\,\mathrm{d}v \\ &= \int_{\Omega\times\mathbb{R}^3} \mathbf{1}_{\tau^+(0,x,v)>t}\, f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v,\end{aligned}$$

thanks to the change of variable $(x,v) \mapsto \mathrm{Z}_{t,0}(x,v)$ (see Proposition 2.3.1 and (2.3.7)). We thus deduce that

$$\int_{\Omega\times\mathbb{R}^3} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \leq \int_{\Omega\times\mathbb{R}^3} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

In the general case, we use a stability result coming from the DiPerna-Lions theory for transport equations (see Section 2.A of the Appendix): we consider a sequence of nonnegative distribution functions $(f_n)_n$ solutions to the Vlasov equation with the absorption boundary condition, associated to regularized velocity fields $(u_n)_n$ and regularized and truncated initial conditions $(f_{0,n})_n$, converging respectively to u and f_0. The associated characteristic curves Z^n for the Vlasov equation are classicaly defined and we have for all $n \in \mathbb{N}$

$$f_n(t,x,v) = e^{3t}\mathbf{1}_{\mathcal{O}^t_n}(x,v)\, f_{0,n}(\mathrm{Z}^n_{0,t}(x,v)),$$

where $\mathcal{O}^t_n$ is defined with respect to Z^n according to (2.3.6).

We also know that f is the strong limit in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^p(\Omega\times\mathbb{R}^3))$ for $1 \leq p < \infty$ and the weak-$\star$ limit in $\mathrm{L}^\infty([0,T]\times\Omega\times\mathbb{R}^3)$ of the sequence $(f_n)_n$. This is thus enough to pass to the limit in the inequality

$$\int_{\Omega\times\mathbb{R}^3} f_n(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \leq \int_{\Omega\times\mathbb{R}^3} f_{0,n}(x,v)\,\mathrm{d}x\,\mathrm{d}v,$$

which holds true thanks to the first part of the analysis. This concludes the proof because $f \in \mathscr{C}(\mathbb{R}^+;\mathrm{L}^1_{\mathrm{loc}}(\overline{\Omega}\times\mathbb{R}^3))$ (see 2.A in Appendix). $\square$

2.3.3 Change of variable in velocity and bounds on moments

In order to get global bounds on the moments ρ_f and j_f, we rely on a change of variable in velocity (inspired by [BD85]). As in [HKMM20, HK22], such a strategy can be used if the quantity $\|\nabla u\|_{\mathrm{L}^1(\mathbb{R}^+;\mathrm{L}^\infty(\Omega))}$ is small enough. We use the same notations as before.

Lemma 2.3.3. *Suppose* $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+, \mathrm{H}^1_0(\Omega)) \cap \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\Omega))$. *Fix* $\delta > 0$ *satisfying* $\delta e^\delta \leq 1/9$. *Then, for all times* $t \in \mathbb{R}^+$ *satisfying*

$$\int_0^t \|\nabla u(s)\|_{\mathrm{L}^\infty(\Omega)} \, \mathrm{d}s < \delta, \tag{2.3.9}$$

and for all $x \in \Omega$, *the map*

$$\Gamma_{t,x} : v \mapsto \mathrm{V}(0; t, x, v),$$

is a global $\mathscr{C}^1$*-diffeomorphism from* $\mathbb{R}^3$ *to itself satisfying furthermore*

$$\forall v \in \mathbb{R}^3, \quad |\det \mathrm{D}_v \Gamma_{t,x}(v)| \geq \frac{e^{3t}}{2}. \tag{2.3.10}$$

Proof. The proof is a straightforward adaptation of that of [HKMM20, Lemma 4.4]: here, the characteristic curves $\mathrm{Z} = (\mathrm{X}, \mathrm{V})$ are defined in $\mathbb{R}^3 \times \mathbb{R}^3$ but we can use the inequality

$$\int_0^t \|\nabla (Pu)(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3)} \, \mathrm{d}s \leq \int_0^t \|\nabla u(s)\|_{\mathrm{L}^\infty(\Omega)} \, \mathrm{d}s.$$

which follows from (2.3.4). The rest of the proof is then similar. □

As a consequence, we obtain the following proposition which is based on the representation formula for the distribution function.

Proposition 2.3.4. *Suppose* $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+, \mathrm{H}^1_0(\Omega)) \cap \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\Omega))$. *If the assumption (2.3.9) is satisfied at a time* $t \geq 0$, *then we have*

$$\|\rho_f(t)\|_{\mathrm{L}^\infty(\Omega)} \leq 2 I_q N_q(f_0),$$
$$\|j_f(t)\|_{\mathrm{L}^\infty(\Omega)} \leq 2 I_q e^{-t} \left(\int_0^t e^s \|u(s)\|_{\mathrm{L}^\infty(\Omega)} \, \mathrm{d}s + 1 \right) N_q(f_0),$$

where we recall that $N_q(f_0)$ *is defined is (2.1.10) and where*

$$I_q := \int_{\mathbb{R}^3} \frac{1 + |v|}{1 + |v|^q} \, \mathrm{d}v.$$

Proof. Note that because of the assumption (2.3.9) on u, we can define the characteristics curves for the Vlasov-Navier-Stokes system in a classical sense as in the previous subsection. We use the representation formula (2.3.8) to write that for all $x \in \Omega$

$$\rho_f(t, x) = e^{3t} \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t}(x, v) f_0(\mathrm{X}_{0,t}(x, v), \mathrm{V}_{0,t}(x, v)) \, \mathrm{d}v.$$

Now, using the change of variable $v \mapsto \Gamma_{t,x}(v) = \mathrm{V}(0; t, x, v)$, we get

$$\rho_f(t, x) = e^{3t} \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t}(x, \Gamma_{t,x}^{-1}(w)) f_0(\mathrm{X}_{0,t}(x, \Gamma_{t,x}^{-1}(w)), w) |\det \mathrm{D}_w (\Gamma_{t,x})^{-1}(w)| \, \mathrm{d}w,$$

and thanks to the bound (2.3.10), we obtain

$$\rho_f(t, x) \leq 2 \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t}(x, \Gamma_{t,x}^{-1}(w)) f_0(\mathrm{X}_{0,t}(x, \Gamma_{t,x}^{-1}(w)), w) \, \mathrm{d}w.$$

Now, thanks to the definition (2.3.6) of $\mathcal{O}^t$, $(x, \Gamma_{t,x}^{-1}(w)) \in \mathcal{O}^t$ if and only if

$$\forall \sigma \in [0, t], \mathrm{X}(\sigma; t, x, \Gamma_{t,x}^{-1}(w)) \in \Omega,$$

therefore

$$(1+|w|^q)\mathbf{1}_{\mathcal{O}^t}(x,\Gamma_{t,x}^{-1}(w))f_0(\mathrm{X}_{0,t}(x,\Gamma_{t,x}^{-1}(w)),w) \leq \sup_{(x,v)\in\Omega\times\mathbb{R}^3}(1+|v|^q)f_0(x,v) = N_q(f_0).$$

We thus deduce the desired inequality on ρ_f. We proceed in the same way for the bound on j_f, namely we start from the following representation formula

$$j_f(t,x) = e^{3t}\int_{\mathbb{R}^3}\Gamma_{t,x}^{-1}(w)\mathbf{1}_{\mathcal{O}^t}(x,\Gamma_{t,x}^{-1}(w))f_0(\mathrm{X}_{0,t}(x,\Gamma_{t,x}^{-1}(w)),w)|\det \mathrm{D}_w(\Gamma_{t,x})^{-1}(w)|\,\mathrm{d}w.$$

Using the formula

$$\mathrm{V}(0;t,x,v) = e^t v - \int_0^t e^\tau (Pu)(\tau,\mathrm{X}(\tau;t,x,v))\,\mathrm{d}\tau,$$

we infer that

$$w = e^t\Gamma_{t,x}^{-1}(w) - \int_0^t e^\tau (Pu)(\tau,\mathrm{X}(\tau;t,x,\Gamma_{t,x}^{-1}(w)))\,\mathrm{d}\tau,$$

which becomes

$$\begin{aligned}|\Gamma_{t,x}^{-1}(w)| &\leq e^{-t}\left(|w| + \int_0^t e^\tau \|(Pu)(\tau)\|_{\mathrm{L}^\infty(\mathbb{R}^3)}\,\mathrm{d}\tau\right)\\ &\leq e^{-t}(1+|w|)\left(1+\int_0^t e^\tau\|u(\tau)\|_{\mathrm{L}^\infty(\Omega)}\,\mathrm{d}\tau\right).\end{aligned}$$

Coming back to the representation formula for j_f, we can conclude exactly as in the previous case. □

In the following lemma, we study how the pointwise decay condition of Definition 2.1.6 can be propagated after times $t=0$.

Lemma 2.3.5. *Let $t_0>0$. If $N_q(f_0)<\infty$ and if $u\in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+;\mathrm{H}^1_0\cap\mathrm{L}^\infty(\Omega))$ then $N_q(f(t_0))<\infty$ with*

$$N_q(f(t_0)) \lesssim e^{3t_0}(1+\|u\|^q_{\mathrm{L}^1(0,t_0;\mathrm{L}^\infty(\Omega))})N_q(f_0).$$

Proof. In this proof, we apply the stability results coming from the DiPerna-Lions theory (see Appendix 2.A) since the vector field u has not enough regularity to define the characteristic curves in a classical sense. We omit the details and write the proof of the desired inequality with only sufficiently regular solutions and data (we refer to the proof of Lemma 2.2.1 where we have already explained this regularization argument).

We use the representation formula

$$f(t_0,x,v) = e^{3t_0}\mathbf{1}_{\mathcal{O}^{t_0}}(x,v)f_0(\mathrm{Z}_{0,t_0}(x,v)).$$

Furthermore, we write

$$\mathrm{V}(0;t_0,x,v) = e^{t_0}v - \int_0^{t_0} e^s(Pu)(s,\mathrm{X}(s;t_0,x,v))\,\mathrm{d}s,$$

to get

$$v = e^{-t_0}\mathrm{V}(0;t_0,x,v) + \int_0^{t_0} e^{s-t_0}(Pu)(s,\mathrm{X}(s;t_0,x,v))\,\mathrm{d}s,$$

and therefore obtain

$$|v| \leq |\mathrm{V}(0;t_0,x,v)| + \int_0^{t_0} \|u(s)\|_{\mathrm{L}^\infty(\Omega)}\, \mathrm{d}s.$$

Thus, we can estimate

$$(1 + |v|^q)f(t_0,x,v) \lesssim \Big(1 + |\mathrm{V}(0;t_0,x,v)|^q + \|u\|^q_{\mathrm{L}^1(0,t_0;\mathrm{L}^\infty(\Omega))}\Big) e^{3t_0} \mathbf{1}_{\mathcal{O}^{t_0}}(x,v) f_0(\mathrm{Z}(0;t_0,x,v)),$$

and the very definition of $N_q(f_0)$ leads to

$$(1+|v|^q)f(t_0,x,v) \lesssim e^{3t_0}(1+\|u\|^q_{\mathrm{L}^1(0,t_0;\mathrm{L}^\infty(\Omega))})N_q(f_0),$$

which concludes the proof.

□

By performing the same analysis as in the three previous results and by replacing the initial time $t=0$ by t_0 and f_0 by $f(t_0)$, we get the following statement.

Lemma 2.3.6. *Let $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+, \mathrm{H}^1_0(\Omega)) \cap \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\Omega))$. Let δ be fixed such that $\delta e^\delta \leq 1/9$. For all times $t \geq t_0 \geq 0$ such that*

$$\int_{t_0}^t \|\nabla u(s)\|_{\mathrm{L}^\infty(\Omega)}\, \mathrm{d}s < \delta, \tag{2.3.11}$$

we have

$$\|\rho_f(t)\|_{\mathrm{L}^\infty(\Omega)} \lesssim e^{3t_0} N_q(f_0)(1+\|u\|^q_{\mathrm{L}^1(0,t_0;\mathrm{L}^\infty(\Omega))}),$$
$$\|j_f(t)\|_{\mathrm{L}^\infty(\Omega)} \lesssim e^{3t_0-t}\left(\int_{t_0}^t e^s \|u(s)\|_{\mathrm{L}^\infty(\Omega)}\, \mathrm{d}s + 1\right) N_q(f_0)(1+\|u\|^q_{\mathrm{L}^1(0,t_0;\mathrm{L}^\infty(\Omega))}).$$

Remark 2.3.7. We will actually see later that for any weak solution (u,f) to the Vlasov-Navier-Stokes system, we have $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\Omega))$: this comes from the local estimates of Proposition 2.4.8 which are independent of this section.

2.4 Preparation for the bootstrap

This section aims at providing local in time controls on ρ_f and j_f by using the results of the previous section and at deriving a $\mathrm{L}^\infty\mathrm{H}^1 \cap \mathrm{L}^2\mathrm{H}^2$ estimate for the fluid velocity. These two types of estimates will be crucial in order to show Theorem 2.1.7.

We first introduce the following useful notations, which allows us to track the dependency on the initial data in the later estimates. This will be useful at the very end of the our proof of Theorem 2.1.7 because of the smallness condition (2.1.16).

Notation 2.4.1. *The notation $A \lesssim_0 B$ means*

$$A \leq \varphi\Big(1 + \|\nabla u_0\|_{\mathrm{L}^2(\Omega)} + N_q f_0 + \mathrm{E}(0)\Big) B, \tag{2.4.1}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is onto, continuous and nondecreasing, and $q > 3$ is the exponent given in the statement of Theorem 2.1.7.

From now on, we fix (u,f) a weak solution to the Vlasov-Navier-Stokes system in the sense of Definition 2.1.3 with admissible initial data (u_0,f_0) in the sense of Definition 2.1.1. Such a weak solution has for instance been constructed in [BGM17].

Notation 2.4.2. *We set*

$$F := j_f - \rho_f u,$$
$$S := F - (u \cdot \nabla)u.$$

First, we have the following estimate on the Brinkman force F.

Lemma 2.4.3. *For all $t \geq 0$, we have*

$$\int_0^t \|F(s)\|_{\mathrm{L}^2(\Omega)}^2 \,\mathrm{d}s \leq \mathrm{E}(0) \sup_{s\in[0,t]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\Omega)}.$$

Proof. Using Cauchy-Schwarz inequality, we get by dropping the time variable,

$$|F| = \left|\int_{\mathbb{R}^3} f(v-u)\,\mathrm{d}v\right| \leq \rho_f^{1/2} \left(\int_{\mathbb{R}^3} f|v-u|^2\,\mathrm{d}v\right)^{1/2},$$

therefore for almost every $s \in [0,t]$

$$\|F(s)\|_{\mathrm{L}^2(\Omega)}^2 \leq \|\rho_f(s)\|_{\mathrm{L}^\infty(\Omega)}\mathrm{D}(s) \leq \sup_{s\in[0,t]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\Omega)}\mathrm{D}(s),$$

where D is defined in (2.1.7). Then, we integrate the last inequality between 0 and t and conclude thanks to the energy inequality (2.1.9). □

2.4.1 Local estimates

We first recall standard interpolation estimates on the moments (see for instance [Ham98]) for which we refer to the notations of Definition 2.1.5.

Proposition 2.4.4. *Let $k > 0$ and let g be a nonnegative function in $\mathrm{L}^\infty(\mathbb{R}^+ \times \Omega \times \mathbb{R}^3)$. Then the following estimates hold for any $\ell \in [0,k]$ and a.e $(t,x) \in \mathbb{R}^+ \times \Omega$*

$$m_\ell g(t,x) \lesssim \left(\|g\|_{\mathrm{L}^\infty(\mathbb{R}^+\times\Omega\times\mathbb{R}^3)} + 1\right) m_k g(t,x)^{\frac{\ell+3}{k+3}},$$
$$\|m_\ell g(t)\|_{\mathrm{L}^{\frac{k+3}{\ell+3}}(\Omega)} \leq C_{k,\ell} \|g(t)\|_{\mathrm{L}^\infty(\Omega\times\mathbb{R}^3)}^{\frac{k-\ell}{k+3}} M_k g(t)^{\frac{\ell+3}{k+3}}.$$

Lemma 2.4.5. *Consider $\alpha \geq 2$ such that $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^{\alpha+3} \cap \mathrm{W}^{1,1}(\Omega))$ and $M_\alpha f_0 < \infty$. Then $M_\alpha f(t) < \infty$ for all $t > 0$ and we have*

$$M_\alpha f(t) \lesssim \left(M_\alpha f_0^{\frac{1}{\alpha+3}} + e^{\frac{3t}{\alpha+3}} \int_0^t \|u(s)\|_{\mathrm{L}^{\alpha+3}(\Omega)} \mathrm{d}s\right)^{\alpha+3}. \tag{2.4.2}$$

Proof. The proof basically follows the same arguments as in [HKMM20, Lemma 4.2]. As in the proof of Lemma 2.2.1, we rely on the stability results of DiPerna-Lions theory on $\Omega \times \mathbb{R}^3$ (see Appendix 2.A). We do not detail the argument and we write the proof as if u and f_0 were smooth, so that the characteristic curves for the Vlasov equation are classicaly defined. Furthermore, by Proposition 2.3.2, we have

$$f(t,x,v) = e^{3t}\mathbf{1}_{\mathcal{O}^t}(x,v)\, f_0(\mathrm{Z}_{0,t}(x,v)),$$

therefore

$$\begin{aligned} M_\alpha f(t) &= \int_{\Omega\times\mathbb{R}^3} |v|^\alpha e^{3t} \mathbf{1}_{\mathcal{O}^t}(x,v)\, f_0(\mathrm{Z}_{0,t}(x,v))\,\mathrm{d}x\,\mathrm{d}v \\ &= \int_{\Omega\times\mathbb{R}^3} |\mathrm{V}_{t,0}(x,v)|^\alpha \mathbf{1}_{\tau^+(0,x,v)>t}\, f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v, \end{aligned}$$

thanks to the change of variable $(x,v)\mapsto \mathrm{Z}_{t,0}(x,v)$ (see Proposition 2.3.1 and (2.3.7)).

Since $\alpha \geq 2$, we also have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}|\mathrm{V}_{s,0}(x,v))|^\alpha &= \alpha \frac{\mathrm{d}}{\mathrm{d}s}[\mathrm{V}_{s,0}(x,v))]\cdot \mathrm{V}_{s,0}(x,v)|\mathrm{V}(s;0,x,v)|^{\alpha-2} \\ &= \alpha\,[u(s,\mathrm{X}_{s,0}(x,v)) - \mathrm{V}_{s,0}(x,v)]\cdot \mathrm{V}_{s,0}(x,v)|\mathrm{V}_{s,0}(x,v)|^{\alpha-2}, \end{aligned}$$

from which we infer that

$$|\mathrm{V}_{t,0}(x,v))|^\alpha \leq |v|^\alpha + \alpha\int_0^t [u(s,\mathrm{X}_{s,0}(x,v))]\cdot \mathrm{V}_{s,0}(x,v)|\mathrm{V}_{s,0}(x,v)|^{\alpha-2}\,\mathrm{d}s.$$

By Fubini Theorem, we get

$$\begin{aligned} M_\alpha f(t) \leq &\int_{\Omega\times\mathbb{R}^3} |v|^\alpha \mathbf{1}_{\tau^+(0,x,v)>t}\, f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \\ &+ \alpha\int_0^t \int_{\Omega\times\mathbb{R}^3} |u(s,\mathrm{X}_{s,0}(x,v))||\mathrm{V}_{s,0}(x,v)|\mathrm{V}_{s,0}(x,v)|^{\alpha-2}|\mathbf{1}_{\tau^+(0,x,v)>t}\, f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v. \end{aligned}$$

Using the reverse change of variable in the last integral, we get

$$\begin{aligned} M_\alpha f(t) &\leq M_\alpha f_0 + \alpha \int_0^t\int_\Omega |u(s,x)| m_{\alpha-1} f(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ &\leq M_\alpha f_0 + \alpha\int_0^t \|u(s)\|_{\mathrm{L}^{\alpha+3}(\Omega)} \|m_{\alpha-1} f(s)\|_{\mathrm{L}^{\frac{\alpha+3}{\alpha+2}}(\Omega)}\,\mathrm{d}s, \end{aligned}$$

thanks to Hölder's inequality. The rest of the proof is then similar to that of [HKMM20, Lemma 4.2], by using the interpolation estimate on the moments (Proposition 2.4.4) with $(\ell,k)=(\alpha-1,\alpha)$, the rough control

$$\|f(t)\|_{\mathrm{L}^\infty(\Omega\times\mathbb{R}^3)} \leq e^{3t}\|f_0\|_{\mathrm{L}^\infty(\Omega\times\mathbb{R}^3)},$$

and a Grönwall's lemma in time. □

Lemma 2.4.6. *If $M_3 f_0 < +\infty$, we have $M_3 f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+)$. Moreover, there exists a continuous non-negative and nondecreasing function η such that*

$$\|\rho_f(t)\|_{\mathrm{L}^2(\Omega)} + \|j_f(t)\|_{\mathrm{L}^{3/2}(\Omega)} \lesssim_0 \eta(t).$$

Proof. Thanks to the estimate (2.4.2) with $\alpha = 3$, we obtain

$$M_3 f(t) \lesssim \left({M_3 f_0}^{1/6} + e^{t/2}\int_0^t \|u(s)\|_{\mathrm{L}^6(\Omega)}\mathrm{d}s\right)^6,$$

as in [HKMM20, Lemma 4.2]. Then, the rest of the proof is similar to that of [HKMM20, Lemma 4.3]: we use Sobolev embedding and the Poincaré inequality on $\mathrm{H}^1_0(\Omega)$ to write $\|u(s)\|_{\mathrm{L}^6(\Omega)} \lesssim \|\nabla u(s)\|_{\mathrm{L}^2(\Omega)}$ so that the Cauchy-Schwarz inequality yields

$$\int_0^t \|u(s)\|_{\mathrm{L}^6(\Omega)}\mathrm{d}s \lesssim t^{1/2}\left(\int_0^t \|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)}\mathrm{d}s\right)^{1/2} \leq t^{1/2}\mathrm{E}(0)^{1/2},$$

where we have used the energy inequality (2.1.9). □

In the following lemma, we slightly improve the rough bound on ρ_f and j_f which was given in Lemma 2.4.6.

Lemma 2.4.7. *There exists a continuous nondecreasing function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $s \geq 0$*

$$\|\rho_f(s)\|_{\mathrm{L}^3(\Omega)} + \|j_f(s)\|_{\mathrm{L}^{9/4}(\Omega)} \lesssim \eta(s). \tag{2.4.3}$$

Moreover,

$$j_f - \rho_f u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\Omega)).$$

Proof. In the proof we denote by η a generic positive nondecreasing continuous function which may vary from line to line.

Recall that in (2.1.15), we suppose $M_2 f_0 < \infty$ and $M_6 f_0 < \infty$ so that $M_3 f_0 \lesssim M_2 f_0 + M_6 f_0 < \infty$. We first write that the velocity field u is solution of the following Stokes system

$$\begin{cases} \partial_t u + Au = \mathbb{P}S, \\ \operatorname{div}_x u = 0, \\ u_{|\partial\Omega} = 0, \\ u(0) = u_0, \end{cases} \tag{2.4.4}$$

with

$$S := j_f - \rho_f u - (u \cdot \nabla)u,$$

where $\mathbb{P}$ stands for the Leray projection on vector fields and where A is the Stokes operator (see Section 2.E of the Appendix). To estimate this source term, we first infer from Lemma 2.4.6 that

$$\|\rho_f(t)\|_{\mathrm{L}^2(\Omega)} + \|j_f(t)\|_{\mathrm{L}^{3/2}(\Omega)} \lesssim_0 \eta(t).$$

The previous estimate, with the Hölder's inequality and the energy inequality (2.1.9), allows us to obtain the following inequalities for all $t \geq 0$

$$\int_0^t \|j_f(s)\|^2_{\mathrm{L}^{3/2}(\Omega)} \mathrm{d}s \lesssim_0 \eta(t),$$

$$\int_0^t \|\rho_f(s)u(s)\|^2_{\mathrm{L}^{3/2}(\Omega)} \mathrm{d}s \leq \int_0^t \|u(s)\|^2_{\mathrm{L}^6(\Omega)} \|\rho_f(s)\|^2_{\mathrm{L}^2(\Omega)} \mathrm{d}s \lesssim_0 \eta(t),$$

where we have again used the Sobolev inequality $\|u(s)\|_{\mathrm{L}^6(\Omega)} \lesssim \|\nabla u(s)\|_{\mathrm{L}^2(\Omega)}$. Thus, we get

$$j_f - \rho_f u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^{3/2}(\Omega)).$$

Then, since $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\Omega)) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}^1(\Omega))$ and again by Sobolev embedding, we get

$$(u \cdot \nabla)u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^1(\Omega)) \quad \text{and} \quad (u \cdot \nabla)u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^{3/2}(\Omega)),$$

so that $(u \cdot \nabla)u \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^q(\Omega))$ with

$$\frac{1}{p} = \frac{\theta + 1}{2}, \qquad \frac{1}{q} = \frac{3 - \theta}{3}, \qquad 0 \leq \theta \leq 1.$$

We choose $\theta = 2/3$ which yields $(u \cdot \nabla)u \in \mathrm{L}^{6/5}_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^{9/7}(\Omega))$. We end up with

$$\mathbb{P}S = \mathbb{P}[j_f - \rho_f u - (u \cdot \nabla)u] \in \mathrm{L}^{6/5}_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^{9/7}_{\mathrm{div}}(\Omega)),$$

by the continuity of the operator $\mathbb{P}$ from $\mathrm{L}^{9/7}(\Omega)$ to $\mathrm{L}^{9/7}_{\mathrm{div}}(\Omega)$. Thanks to this integrability of the source term and the assumption (2.1.15), we can use the maximal $\mathrm{L}^p - \mathrm{L}^q$ regularity result for the Stokes system (see Section 2.E of the Appendix) to deduce that

$$u \in \mathrm{L}^{6/5}_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{W}^{2,9/7}(\Omega)).$$

By the Sobolev embedding $\mathrm{W}^{2,9/7}(\Omega) \hookrightarrow \mathrm{L}^9(\Omega)$, we finally end up with

$$u \in \mathrm{L}^{6/5}_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^9(\Omega)).$$

In particular, we obtain $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^9(\Omega))$. With the estimate (2.4.2) of Lemma 2.4.5, we first have

$$M_6 f(t) \lesssim \left(M_6 {f_0}^{1/9} + e^{\frac{t}{3}} \int_0^t \|u(s)\|_{\mathrm{L}^9(\Omega)} \mathrm{d}s \right)^9 \lesssim \eta(t),$$

which can be used together with the interpolation estimate (2.4.4) for moments with $k = 6$ and $\ell \in \{0,1\}$ to get

$$\|\rho_f(t)\|_{\mathrm{L}^3(\Omega)} + \|j_f(t)\|_{\mathrm{L}^{9/4}(\Omega)} \lesssim \eta(t).$$

We can now combine the last estimate with Hölder's inequality and the energy inequality (2.1.9) to write for all $t \geq 0$

$$\int_0^t \|j_f(s)\|^2_{\mathrm{L}^2(\Omega)} \mathrm{d}s \lesssim \int_0^t \|j_f(s)\|^2_{\mathrm{L}^{9/4}(\Omega)} \mathrm{d}s \lesssim \eta(t),$$

and

$$\int_0^t \|\rho_f(s) u(s)\|^2_{\mathrm{L}^2(\Omega)} \mathrm{d}s \leq \int_0^t \|u(s)\|^2_{\mathrm{L}^6(\Omega)} \|\rho_f(s)\|^2_{\mathrm{L}^3(\Omega)} \mathrm{d}s$$
$$\lesssim \int_0^t \|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)} \|\rho_f(s)\|^2_{\mathrm{L}^3(\Omega)} \mathrm{d}s \lesssim \eta(t).$$

This concludes the proof of Lemma 2.4.7. □

The next proposition tells us that any weak solution u to the VNS system belongs to the space $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\Omega))$. This result has been already proved in [HKMM20, Proposition 5.1] in the case of the torus but the proof seems to be specific to the periodic setting.

On a bounded domain, this type of integrability of Leray solutions to the Navier-Stokes equations is actually a general property which can be deduced from [FMRT01, p104-106] combined with the end of the proof of [RRS16, Lemma 8.15]. At first sight, the result stated in these references holds under the general condition that the source term of the Navier-Stokes equations belongs to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\Omega))$. Even this is not (yet) the case for the Brinkman force $j_f - \rho_f u$, we will easily adapt the proof.

Proposition 2.4.8. *We have*

$$u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\Omega)), \tag{2.4.5}$$
$$\rho_f, j_f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\Omega)). \tag{2.4.6}$$

More precisely, there exists a continuous nondecreasing function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|u\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\Omega))} \lesssim \eta(t), \tag{2.4.7}$$
$$\|\rho_f\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty(\Omega))} + \|j_f\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty(\Omega))} \lesssim_0 \eta(t). \tag{2.4.8}$$

Proof. • Let $T > 0$ be fixed. We want to show that $u \in \mathrm{L}^1(0,T;\mathrm{L}^\infty(\Omega))$. As already said, some parts of the proof mimic those of [FMRT01, p104-106] but we will introduce some modifications.

First, thanks to Lemma 2.4.7, we know that the source term $j_f - \rho_f u$ of the Navier-Stokes equations satisfied by u belongs to $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$. So, we can apply the theory of epochs of regularity (see [FMRT01, II - Section 7]) for u on $[0,T]$. Note that this fact is valid because, by our assumption, u is a Leray solution to the Navier-Stokes equations which satisfies the strong energy inequality.

We get the existence of a subset $\sigma_T \subset [0,T]$ of full measure in $[0,T]$ with $\sigma_T = \bigcup_i]a_i, b_i[$ and where u is a strong solution on each $]a_i, b_i[$ (namely, $u \in \mathrm{L}^\infty_{\mathrm{loc}}(a_i, b_i; \mathrm{H}^1(\Omega)) \cap \mathrm{L}^2_{\mathrm{loc}}(a_i, b_i; \mathrm{H}^2(\Omega))$). On each interval $]a_i, b_i[$, we can take the $\mathrm{L}^2(\Omega)$ inner product (denoted by $\langle \cdot, \cdot \rangle$) of the Navier-Stokes equations with Au, where A stands for the Stokes operator on $\mathrm{L}^2(\Omega)$, in order to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2_{\mathrm{L}^2(\Omega)} + 2\|Au\|^2_{\mathrm{L}^2(\Omega)} + 2\langle \mathbb{P}(u\cdot\nabla)u, Au\rangle = 2\langle \mathbb{P}(j_f - \rho_f u), Au\rangle \quad \text{on} \quad]a_i, b_i[,$$

where we have used the fact that $\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2_{\mathrm{L}^2(\Omega)} = 2\langle \partial_t u, Au\rangle$ (see [RRS16, Lemma 6.7]). Here and in what follows, we omit the time variable for the sake of clarity. Then, we combine the Agmon inequality (see Proposition 2.D.2 in the Appendix), the Poincaré inequality and the elliptic estimate $\|u\|_{\mathrm{H}^2(\Omega)} \lesssim \|Au\|_{\mathrm{L}^2(\Omega)}$ to obtain

$$\begin{aligned}|\langle \mathbb{P}(u\cdot\nabla)u, Au\rangle| &\leq \|u\|_{\mathrm{L}^\infty(\Omega)}\|\nabla u\|_{\mathrm{L}^2(\Omega)}\|\mathbb{P}Au\|_{\mathrm{L}^2(\Omega)} \\ &\lesssim \|\nabla u\|^{3/2}_{\mathrm{L}^2(\Omega)}\|\mathbb{P}Au\|^{3/2}_{\mathrm{L}^2(\Omega)},\end{aligned}$$

using that the operator $\mathbb{P}$ is self-adjoint on $\mathrm{L}^2(\Omega)$. The Young inequality is now used twice to write

$$\begin{aligned}2|\langle \mathbb{P}(u\cdot\nabla)u, Au\rangle| &\leq C\|\nabla u\|^6_{\mathrm{L}^2(\Omega)} + \frac{3}{4}\|Au\|^2_{\mathrm{L}^2(\Omega)}, \\ 2|\langle \mathbb{P}(j_f - \rho_f u), Au\rangle| &\leq \frac{1}{2}\left(8C\|j_f - \rho_f u\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2}\|Au\|^2_{\mathrm{L}^2(\Omega)}\right),\end{aligned}$$

thanks to the continuity of the operator $\mathbb{P}$ on $\mathrm{L}^2(\Omega)$. Here, C is a positive constant independent of the time variable. All in all, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2_{\mathrm{L}^2(\Omega)} + \|Au\|^2_{\mathrm{L}^2(\Omega)} \lesssim \|j_f - \rho_f u\|^2_{\mathrm{L}^2(\Omega)} + \|\nabla u\|^6_{\mathrm{L}^2(\Omega)}, \tag{2.4.9}$$

on each interval $]a_i, b_i[$, where $\lesssim$ refers to a constant independent of the time variable and independent of i.

To deal with the Brinkman force $\|j_f - \rho_f u\|^2_{\mathrm{L}^2(\Omega)}$, we invoke the inequality (2.4.3) of Lemma 2.4.7 according to which there exists a continuous nondecreasing function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ (independent of T) such that for all $s \in [0,T]$

$$\|\rho_f(s)\|_{\mathrm{L}^3(\Omega)} + \|j_f(s)\|_{\mathrm{L}^{9/4}(\Omega)} \lesssim \eta(s). \tag{2.4.10}$$

For all $s \in [0,T]$, we can now estimate the L^2 norm of the Brinkman force $j_f(s) - \rho_f(s)u(s)$ in the following way,

$$\begin{aligned}\|j_f(s) - \rho_f(s)u(s)\|^2_{\mathrm{L}^2(\Omega)} &\lesssim \|j_f(s)\|^2_{\mathrm{L}^2(\Omega)} + \|\rho_f(s)u(s)\|^2_{\mathrm{L}^2(\Omega)} \\ &\lesssim \|j_f(s)\|^2_{\mathrm{L}^2(\Omega)} + \|\rho_f(s)\|^2_{\mathrm{L}^3(\Omega)}\|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)},\end{aligned}$$

where we have used the Hölder's inequality and the Sobolev embedding $\mathrm{H}^1_0(\Omega) \hookrightarrow \mathrm{L}^6(\Omega)$. So, thanks to (2.4.10), we obtain the following inequality for all $s \in [0,T]$

$$\|j_f(s) - \rho_f(s)u(s)\|^2_{\mathrm{L}^2(\Omega)} \lesssim \eta(s)^2 + \eta(s)^2\|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)} \leq C_T\left(1 + \|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)}\right).$$

Coming back to (2.4.9), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2_{\mathrm{L}^2(\Omega)} + \|Au\|^2_{\mathrm{L}^2(\Omega)} \leq C_T\left(1 + \|\nabla u\|^2_{\mathrm{L}^2(\Omega)} + \|\nabla u\|^6_{\mathrm{L}^2(\Omega)}\right),$$

on each $]a_i, b_i[$, where $C_T \geq 0$. We divide this inequality by $(1 + \|\nabla u\|^2_{\mathrm{L}^2(\Omega)})^2$ to find that

$$\frac{1}{(1 + \|\nabla u\|^2_{\mathrm{L}^2(\Omega)})^2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{(1 + \|\nabla u\|^2_{\mathrm{L}^2(\Omega)})^2}\|Au\|^2_{\mathrm{L}^2(\Omega)}$$
$$\leq C_T\left(\frac{1}{(1 + \|\nabla u\|^2_{\mathrm{L}^2(\Omega)})^2} + \frac{\|\nabla u\|^6_{\mathrm{L}^2(\Omega)}}{(1 + \|\nabla u\|^2_{\mathrm{L}^2(\Omega)})^2}\right),$$

which gives

$$\frac{1}{(1 + \|\nabla u\|^2_{\mathrm{L}^2(\Omega)})^2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{(1 + \|\nabla u\|^2_{\mathrm{L}^2(\Omega)})^2}\|Au\|^2_{\mathrm{L}^2(\Omega)}$$
$$\leq C_T\left(1 + \|\nabla u\|^2_{\mathrm{L}^2(\Omega)}\right), \quad (2.4.11)$$

on each $]a_i, b_i[$. We now integrate this inequality between a and b, with $a_i < a < b < b_i$, to get

$$\frac{1}{1 + \|\nabla u(a)\|^2_{\mathrm{L}^2(\Omega)}} - \frac{1}{1 + \|\nabla u(b)\|^2_{\mathrm{L}^2(\Omega)}} + \int_a^b \frac{\|Au(s)\|^2_{\mathrm{L}^2(\Omega)}}{(1 + \|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)})^2}\,\mathrm{d}s$$
$$\leq C_T\int_a^b\left(1 + \|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)}\right)\mathrm{d}s.$$

As $]a_i, b_i[$ is a maximal interval of strong regularity for u (if $b_i \neq T$), we know that $\|\nabla u(b)\|_{\mathrm{L}^2(\Omega)}$ goes to infinity when $b \to b_i$. Hence, if $b_i \neq T$, we have

$$\int_{a_i}^{b_i} \frac{\|Au(s)\|^2_{\mathrm{L}^2(\Omega)}}{(1 + \|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)})^2}\,\mathrm{d}s \leq C_T\int_{a_i}^{b_i}\left(1 + \|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)}\right)\mathrm{d}s.$$

We sum the inequality on all i (for which $b_i \neq T$) and use the fact that σ_T is of full measure in $[0,T]$ to obtain

$$\int_0^T \frac{\|Au(s)\|^2_{\mathrm{L}^2(\Omega)}}{(1 + \|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)})^2}\,\mathrm{d}s \leq C_T\int_0^T\left(1 + \|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)}\right)\mathrm{d}s + \frac{1}{1 + \|\nabla u(T)\|^2_{\mathrm{L}^2(\Omega)}}$$
$$\leq C_T(T + \mathrm{E}(0)) + 1.$$

We conclude the proof as in [FMRT01, p104-106] and [RRS16, Lemma 8.15] by writing

$$\int_0^T \|Au(s)\|^{2/3}_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s = \int_0^T \frac{\|Au(s)\|^{2/3}_{\mathrm{L}^2(\Omega)}}{(1 + \|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)})^{2/3}}(1 + \|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)})^{2/3}\,\mathrm{d}s$$
$$\leq \left(\int_0^T \frac{\|Au(s)\|^2_{\mathrm{L}^2(\Omega)}}{(1 + \|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)})^2}\,\mathrm{d}s\right)^{1/3}\left(\int_0^T (1 + \|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)})\,\mathrm{d}s\right)^{2/3}$$
$$\leq (C_T(T + \mathrm{E}(0)) + 1)^{1/3}\,(T + \mathrm{E}(0))^{2/3} < \infty,$$

and by using again the Agmon inequality (see Proposition 2.D.2 in the Appendix), the Poincaré inequality, the elliptic estimate $\|u\|_{\mathrm{H}^2(\Omega)} \lesssim \|Au\|_{\mathrm{L}^2(\Omega)}$ and the Young inequality, we finally deduce that

$$\begin{aligned}\int_0^T \|u(s)\|_{\mathrm{L}^\infty(\Omega)}\,\mathrm{d}s &\lesssim \int_0^T \|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s + \int_0^T \|Au(s)\|^{2/3}_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s \\ &\leq \mathrm{E}(0) + (C_T(T+\mathrm{E}(0))+1)^{1/3}\,(T+\mathrm{E}(0))^{2/3} < \infty.\end{aligned}$$

- To prove the last estimate (2.4.8), we use Lemma 2.3.5 to write that for all $s \in [0,t]$

$$\begin{aligned}\|\rho_f(s)\|_{\mathrm{L}^\infty(\Omega)} + \|j_f(s)\|_{\mathrm{L}^\infty(\Omega)} \lesssim N_q(f(s)) &\lesssim e^{3s}(1+\|u\|^q_{\mathrm{L}^1(0,s;\mathrm{L}^\infty(\Omega))})N_q(f_0) \\ &\leq e^{3t}(1+\|u\|^q_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\Omega))})N_q(f_0),\end{aligned}$$

so that the conclusion follows. □

2.4.2 Higher order energy estimates for the fluid velocity

We now state a smoothing property of the Vlasov-Navier-Stokes system for the fluid velocity u. We rely on the parabolic regularization for the Navier-Stokes equations: in short, there is a gain of regularity if the initial data and the source term, that is the Brinkman force $F := j_f - \rho_f u$, are small enough.

Proposition 2.4.9. *There exist universal constants $C_1, C_2 > 0$ such that the following holds. Assume that for some $T > 0$, one has*

$$\|\nabla u_0\|^2_{\mathrm{L}^2(\Omega)} + C_1 \int_0^T \|F(s)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s \leq \frac{1}{\sqrt{8C_1C_2}}. \tag{2.4.12}$$

Then one has for all $0 \leq t \leq T$

$$\|\nabla u(t)\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2}\int_0^t \|Au(s)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s \lesssim \|\nabla u_0\|^2_{\mathrm{L}^2(\Omega)} + \mathrm{E}(0) \sup_{s\in[0,t]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\Omega)}, \tag{2.4.13}$$

where $\lesssim$ only depends on C_1 and C_2, and where A stands for the Stokes operator on $\mathrm{L}^2(\Omega)$.

Proof. The estimate is a direct consequence of the parabolic regularization for the Navier-Stokes system with source $F = j_f - \rho_f u$, that we state in Theorem 2.F.1 in Section 2.F of the Appendix, together with the estimate on the Brinkman force of Lemma 2.4.3. □

Remark 2.4.10. Thanks to the smallness assumption (2.1.16), we can actually ensure that

$$\|\nabla u_0\|^2_{\mathrm{L}^2(\Omega)} \leq \frac{1}{2\sqrt{8C_1C_2}}. \tag{2.4.14}$$

By choosing an appropriate function φ in (2.1.16), we can also reduce $\|\nabla u_0\|^2_{\mathrm{L}^2(\Omega)}$ and $\mathrm{E}(0)$ in the sequel if necessary.

Remark 2.4.11. By Proposition 2.4.8, the r.h.s of (2.4.13) is finite. Using the elliptic estimate $\|u\|_{\mathrm{H}^2(\Omega)} \lesssim \|Au\|_{\mathrm{L}^2(\Omega)}$, we infer that

$$u \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega)) \cap \mathrm{L}^2(0,T;\mathrm{H}^2(\Omega)),$$

and in particular

$$\nabla u \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)).$$

Thus, we get the following estimate

$$\|u\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^6(\Omega))} \lesssim \|\nabla u\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^2(\Omega))} \lesssim \|\nabla u_0\|^2_{\mathrm{L}^2(\Omega)} + \mathrm{E}(0) \sup_{s\in[0,t]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\Omega)}. \tag{2.4.15}$$

In order to ensure that the smallness condition (2.4.12) is satisfied for all times, we now introduce the following terminology, which has been already used in [HKMM20] and [HK22] to take advantage of the parabolic regulazation for the fluid.

Definition 2.4.12 (Strong existence time)**.** *A real number $T \geq 0$ is a* strong existence *time whenever (2.4.12) holds.*

Lemma 2.4.13. *The smallness condition (2.1.16) of Theorem 2.1.7 ensures that $T = 1$ is a strong existence time in the sense of Definition 2.4.12.*

Proof. Recall the meaning of the notation $\lesssim_0$ from Notation 2.4.1. We use Lemma 2.4.3 and the local estimate (2.4.8) to write

$$\int_0^1 \|F(s)\|^2_{\mathrm{L}^2(\Omega)} \, \mathrm{d}s \leq \mathrm{E}(0) \sup_{s\in[0,1]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\Omega)} \lesssim_0 \mathrm{E}(0) \leq \frac{1}{2C_1\sqrt{8C_1C_2}},$$

where we have used the assumption (2.1.16). Combining this inequality with (2.4.14) leads to the result. □

2.5 Estimate for the second derivatives of the fluid velocity

In this section, we provide a crucial estimate on the second order derivatives of u: to do so, we look at the convection term $(u \cdot \nabla) u$ and at the Brinkman force $F = j_f - \rho_f u$ as source terms for the Navier-Stokes equations. We rely on some maximal $\mathrm{L}^p\mathrm{L}^q$ regularity for the Stokes equation on a bounded domain in order to get estimates involving moments of f.

We first introduce the following useful notations involving the moments of the kinetic distribution.

Definition 2.5.1. *We set for all $t \geq 1$*

$$\mathrm{M}_{\rho_f}(t) := \sup_{s\in[1,t]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\Omega)}, \quad \mathrm{M}_{j_f}(t) := \sup_{s\in[1,t]} \|j_f(s)\|_{\mathrm{L}^\infty(\Omega)},$$
$$\mathrm{M}_{\rho_f, j_f}(t) := \mathrm{M}_{\rho_f}(t) + \mathrm{M}_{j_f}(t).$$

Remark 2.5.2. Note that thanks to Proposition 2.4.8, we can control $\rho_f(s)$ and $j_f(s)$ on $[0,1]$ in $\mathrm{L}^\infty(\Omega)$. Therefore, if $t > 1$, we will make a constant use of

$$\sup_{[0,t]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\Omega)} + \sup_{[0,t]} \|j_f(s)\|_{\mathrm{L}^\infty(\Omega)} \lesssim_0 1 + \mathrm{M}_{\rho_f}(t).$$

Proposition 2.5.3. *Let $a, b, r \in]1, \infty[$ and $\lambda > 0$ fixed. For all $t \geq 1/2$ and all $0 < q \leq a, b$, we have*

$$\int_{1/2}^t e^{-\lambda s} \|\mathrm{D}^2 u(s)\|^q_{\mathrm{L}^r(\Omega)} \, \mathrm{d}s \lesssim_0 \Phi(\lambda) \left(1 + \|(u\cdot\nabla)u\|^q_{\mathrm{L}^a(0,t;\mathrm{L}^r(\Omega))} + \|F\|^q_{\mathrm{L}^b(0,t;\mathrm{L}^r(\Omega))}\right),$$

where $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is nonincreasing.

Proof. We introduce w_1 et w_2 which are the unique divergence-free solutions on $[0, +\infty[$ to the following Cauchy problems

$$\begin{cases} \partial_t w_1 + A w_1 = -\mathbb{P}(u\cdot\nabla)u, \\ w_1(0) = 0, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t w_2 + A w_2 = \mathbb{P}F, \\ w_2(0) = 0, \end{cases} \tag{2.5.1}$$

where A stands for the Stokes operator on $\mathrm{L}^2(\Omega)$ and where $\mathbb{P}$ is the Leray projection on vector fields.

Thanks to the maximal regularity for the Stokes system and the continuity of the operator $\mathbb{P}$ on $\mathrm{L}^r(\Omega)$ (see Section 2.E of the Appendix), we infer the following estimates for all $t \geq 0$

$$\left(\int_0^t \|\mathrm{D}^2 w_1(s)\|^a_{\mathrm{L}^r(\Omega)}\,\mathrm{d}s\right)^{1/a} \lesssim \left(\int_0^t \|(u\cdot\nabla)\,u\|^a_{\mathrm{L}^r(\Omega)}\,\mathrm{d}s\right)^{1/a}, \tag{2.5.2}$$

$$\left(\int_0^t \|\mathrm{D}^2 w_2(s)\|^b_{\mathrm{L}^r(\Omega)}\,\mathrm{d}s\right)^{1/b} \lesssim \left(\int_0^t \|F(s)\|^b_{\mathrm{L}^r(\Omega)}\,\mathrm{d}s\right)^{1/b}. \tag{2.5.3}$$

On the other hand, if we set $\widetilde{u} := u-(w_1+w_2)$, we have $\widetilde{u}(t) = e^{-tA}u_0 \in D(A^s)$ for all $s \geq 0$. Let $(a_k)_{k\geq 0}$ be an Hilbertian basis of $\mathrm{L}^2_{\mathrm{div}}(\Omega)$ made up of eigenfunctions of A with associated positive eigenvalues $(\lambda_k)_{k\geq 0}$ such that the sequence $(\lambda_k)_{k\geq 0}$ is nondecreasing and $\lambda_k \overset{k\to+\infty}{\longrightarrow} +\infty$. We can write in $\mathrm{L}^2(\Omega)$

$$u_0(x) = \sum_{k=0}^{\infty} c_k a_k, \quad c_k := \langle u_0, a_k\rangle_{\mathrm{L}^2(\Omega)},$$

and the continuity of e^{-tA} on $\mathrm{L}^2(\Omega)$ yields

$$\widetilde{u}(t,x) = \sum_{k=0}^{\infty} c_k e^{-\lambda_k t} a_k, \quad t \geq 0.$$

Then, we use the fact for all $t \geq 0$ and $\ell \geq 0$, we have

$$\|\widetilde{u}(t)\|_{\mathrm{H}^\ell(\Omega)} \lesssim \|A^{\ell/2}\widetilde{u}(t)\|_{\mathrm{L}^2(\Omega)},$$

(see e.g: [RRS16, Chapter 2 - p69]) so that we get for all $t \geq 0$ and $\ell \geq 1$

$$\|\widetilde{u}(t)\|^2_{\mathrm{H}^\ell(\Omega)} \lesssim \sum_{k=0}^{\infty} |c_k|^2 \lambda_k^\ell e^{-2\lambda_k t}.$$

Note that there exists a constant $C > 0$ independent of k and t such that

$$\lambda_k^\ell e^{-2\lambda_k t} \leq C e^{-\lambda_k t}, \quad t \geq 1/2, \quad \ell \geq 1.$$

Indeed, for all $t \geq 1/2$

$$\lambda_k^\ell e^{-\lambda_k t} \leq \lambda_k^\ell e^{-\lambda_k/2},$$

and the r.h.s of this inequality tends to 0 as $\to +\infty$ because $\lambda_k \overset{k\to+\infty}{\longrightarrow} +\infty$. Thus, for all $t \geq 1/2$ and $\ell \geq 1$, we have the following estimate

$$\|\widetilde{u}(t)\|^2_{\mathrm{H}^\ell(\Omega)} \lesssim \left(\sum_{k=0}^{\infty} |c_k|^2\right) e^{-\lambda_1 t} = \|u_0\|^2_{\mathrm{L}^2(\Omega)} e^{-\lambda_1 t},$$

where we have used the Plancherel-Parseval theorem. We therefore have, for all $t \geq 1/2$ and $\ell \geq 1$

$$\int_{1/2}^t \|\widetilde{u}(s)\|^q_{\mathrm{H}^\ell(\Omega)}\,\mathrm{d}s \lesssim \|u_0\|^q_{\mathrm{L}^2(\Omega)} \int_{1/2}^{+\infty} e^{-q\lambda_1 s/2}\,\mathrm{d}s \lesssim \|u_0\|^q_{\mathrm{L}^2(\Omega)}. \tag{2.5.4}$$

By using the estimate (2.5.4) with ℓ large enough, together with Sobolev embedding, we thus get

$$\left(\int_{1/2}^t \|\mathrm{D}^2\widetilde{u}(s)\|^q_{\mathrm{L}^r(\Omega)}\,\mathrm{d}s\right)^{1/q} \lesssim_0 1. \tag{2.5.5}$$

We then write $u = w_1 + w_2 + \widetilde{u}$ and it follows that

$$\|\mathrm{D}^2 u(s)\|^q_{\mathrm{L}^r(\Omega)} \lesssim \|\mathrm{D}^2 w_1(s)\|^q_{\mathrm{L}^r(\Omega)} + \|\mathrm{D}^2 w_2(s)\|^q_{\mathrm{L}^r(\Omega)} + \|\mathrm{D}^2 \widetilde{u}(s)\|^q_{\mathrm{L}^r(\Omega)}.$$

We have just dealt with the last term. For the other ones, and for $a \neq q, b \neq q$, we use Hölder's inequality, which is justified since $\frac{a}{q}, \frac{b}{q} \geq 1$, to write

$$\int_{1/2}^t e^{-\lambda s}\|\mathrm{D}^2 w_1(s)\|^q_{\mathrm{L}^r(\Omega)}\,\mathrm{d}s \leq \left(\int_0^t e^{-\lambda s \frac{a}{a-q}}\,\mathrm{d}s\right)^{1-q/a}\left(\int_0^t \|\mathrm{D}^2 w_1(s)\|^a_{\mathrm{L}^r(\Omega)}\,\mathrm{d}s\right)^{q/a}, \tag{2.5.6}$$

$$\int_{1/2}^t e^{-\lambda s}\|\mathrm{D}^2 w_2(s)\|^q_{\mathrm{L}^r(\Omega)}\,\mathrm{d}s \leq \left(\int_0^t e^{-\lambda s \frac{b}{b-q}}\,\mathrm{d}s\right)^{1-q/b}\left(\int_0^t \|\mathrm{D}^2 w_2(s)\|^b_{\mathrm{L}^r(\Omega)}\,\mathrm{d}s\right)^{q/b}. \tag{2.5.7}$$

The first integral in the r.h.s of (2.5.6) (resp of (2.5.7)) is equal to $\frac{a-q}{a}\frac{1}{\lambda}$ (resp equal to$\frac{b-q}{b}\frac{1}{\lambda}$) and these expression are indeed nonnegative and nonincreasing in λ. If $a = q$, we have to to replace the previous inequality by

$$\int_{1/2}^t e^{-\lambda s}\|\mathrm{D}^2 w_1(s)\|^q_{\mathrm{L}^r(\Omega)}\,\mathrm{d}s \leq e^{-\lambda/2}\left(\int_0^t \|\mathrm{D}^2 w_1(s)\|^a_{\mathrm{L}^r(\Omega)}\,\mathrm{d}s\right)^{q/a},$$

and in a similar way for the term with w_2 if $b = q$.

Combining the inequalities (2.5.2), (2.5.3) et (2.5.5), we end up with

$$\begin{aligned}
&\int_{1/2}^t e^{-\lambda s}\|\mathrm{D}^2 u(s)\|^q_{\mathrm{L}^r(\Omega)}\,\mathrm{d}s \\
&\qquad \lesssim (\lambda^{-1} + e^{-\lambda/2})\left(\|\mathrm{D}^2 w_1\|^q_{\mathrm{L}^a(0,t;\mathrm{L}^r(\Omega))} + \|\mathrm{D}^2 w_2\|^q_{\mathrm{L}^b(0,t;\mathrm{L}^r(\Omega))}\right) + \|\mathrm{D}^2\widetilde{u}\|^q_{\mathrm{L}^q(1/2,t;\mathrm{L}^r(\Omega))} \\
&\qquad\qquad \lesssim_0 \Phi(\lambda)\left(1 + \|(u\cdot\nabla)u\|^q_{\mathrm{L}^a(0,t;\mathrm{L}^r(\Omega))} + \|F\|^q_{\mathrm{L}^b(0,t;\mathrm{L}^r(\Omega))}\right),
\end{aligned}$$

for some nonincreasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$.

□

Remark 2.5.4. Note that we can also state a variant of the result of Proposition 2.5.3 under the form

$$\int_1^t e^{-\lambda s}\|\mathrm{D}^2 u(s)\|^q_{\mathrm{L}^r(\Omega)}\,\mathrm{d}s \lesssim_0 \Psi(\lambda)\left(1 + \|(u\cdot\nabla)u\|^q_{\mathrm{L}^a(1/2,t;\mathrm{L}^r(\Omega))} + \|F\|^q_{\mathrm{L}^b(1/2,t;\mathrm{L}^r(\Omega))}\right),$$

for all $t \geq 1$ and for some nonincreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$. To do so, we have to change the previous proof in the following way: we define w_1 and w_2 as solutions to the Cauchy problems (2.5.1) with $w_1(1/2) = w_2(1/2) = 0$ and we still set $\widetilde{u} := u - (w_1 + w_2)$. We thus have $\widetilde{u}(t) = e^{-tA}u(1/2)$ and we can write

$$\widetilde{u}(t) = \sum_{k=1}^{\infty} \gamma_k e^{-\lambda_k(t-1/2)} a_k, \quad \gamma_k := \langle u(1/2), a_k\rangle_{\mathrm{L}^2(\Omega)}.$$

The rest of a the proof is then similar.

In the two following lemmas, we give $\mathrm{L}^p\mathrm{L}^q$ estimates on the convection term and on the Brinkman force $F = j_f - \rho_f u$. These results are obtained in the very same way as in the case of the torus

[HKMM20, Lemmas 6.2 - 6.3] by using interpolation arguments and the parabolic regularization of (2.4.13) which is valid for strong times. Indeed, Remark 2.4.11 entails

$$\begin{aligned}\|u\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^6(\Omega))} &\lesssim \|\nabla u_0\|^2_{\mathrm{L}^2(\Omega)} + \mathrm{E}(0)\sup_{s\in[0,t]}\|\rho_f(s)\|_{\mathrm{L}^\infty(\Omega)} \\ &\lesssim (\|\nabla u_0\|^2_{\mathrm{L}^2(\Omega)} + \mathrm{E}(0))(1+\sup_{s\in[0,t]}\|\rho_f(s)\|_{\mathrm{L}^\infty(\Omega)}) \\ &\lesssim_0 1+\sup_{s\in[0,t]}\|\rho_f(s)\|_{\mathrm{L}^\infty(\Omega)}.\end{aligned}$$

This yields the following results.

Lemma 2.5.5. *There exist $a \in (2,4)$ and $r_a \in (2,3)$ such that for all strong existence times $t \geq 0$*

$$\|(u\cdot\nabla)u\|_{\mathrm{L}^a(0,t;\mathrm{L}^{r_a}(\Omega))} \lesssim_0 1 + \mathrm{M}_{\rho_f,j_f}(t).$$

Lemma 2.5.6. *For all $b > 4$, there exists $r_b > 4$ such that for all strong existence times $t \geq 0$*

$$\|F\|_{\mathrm{L}^b(0,t;\mathrm{L}^{r_b}(\Omega))} \lesssim_0 1 + \mathrm{M}_{\rho_f,j_f}(t)^{\frac{3}{2}-\frac{2}{b}}.$$

2.6 End of the proof of Theorem 2.1.7

We are now able to set up the bootstrap procedure we have mentioned in the end of the introduction. In order to get a control on ∇u, we interpolate the higher regularity estimates with the poinstwise $\mathrm{L}^2(\Omega)$ bound on u provided by the exponential decay of the total kinetic energy.

We first state the following result on ∇u, which is non-uniform in time for the moment.

Lemma 2.6.1. *For any strong existence time $t \geq 1$, one has*

$$\nabla u \in \mathrm{L}^1(1,t;\mathrm{L}^\infty(\Omega)).$$

Proof. Let $t \geq 1$ be a strong existence time. Since the trace of u is 0 on $\partial\Omega$, we use the Gagliardo-Nirenberg-Sobolev inequality (see Theorem 2.D.1 in Appendix) with $(j,m,q) = (1,2,2)$ to write

$$\|\nabla u(s)\|_{\mathrm{L}^p(\Omega)} \lesssim \|\mathrm{D}^2 u(s)\|^\alpha_{\mathrm{L}^r(\Omega)}\|u(s)\|^{1-\alpha}_{\mathrm{L}^2(\Omega)}, \qquad s \geq 1/2,$$

for all $p \in [1,\infty]$ and $r \in [1,\infty]$ satisfying the relation

$$\frac{1}{p} = \frac{1}{3} + \alpha\left(\frac{1}{r} - \frac{2}{3}\right) + \frac{1-\alpha}{2}, \tag{2.6.1}$$

and where $\alpha \in [1/2, 1[$. With the energy inequality (2.1.9) and a Hölder inequality in time, we get

$$\int_{1/2}^t \|\nabla u(s)\|^c_{\mathrm{L}^p(\Omega)}\,\mathrm{d}s \leq \mathrm{E}(0)^{c\frac{1-\alpha}{2}} c^{-1}(e^{ct} - e^{\frac{c}{2}}) \int_{1/2}^t e^{-cs}\|\mathrm{D}^2 u(s)\|^{c\alpha}_{\mathrm{L}^r(\Omega)}\,\mathrm{d}s,$$

which turns into

$$\begin{aligned}&\int_{1/2}^t \|\nabla u(s)\|^c_{\mathrm{L}^p(\Omega)}\,\mathrm{d}s \\ &\qquad\lesssim_0 \mathrm{E}(0)^{c\frac{1-\alpha}{2}} c^{-1}(e^{ct} - e^{\frac{c}{2}})\Phi(c)\left(1 + \|(u\cdot\nabla)u\|^{c\alpha}_{\mathrm{L}^a(0,t;\mathrm{L}^r(\Omega))} + \|F\|^{c\alpha}_{\mathrm{L}^b(0,t;\mathrm{L}^r(\Omega))}\right),\end{aligned}$$

thanks to Proposition 2.5.3, for all $c \in [1, +\infty[$ such that $c\alpha \leq a, b$ and exponents $1 < a, b < \infty$. Now using Lemma 2.5.5 and Lemma 2.5.6, we obtain a, b, r_a, r_b such that $b > 4 > a > 2$ and $r := \min(r_a, r_b) > 2$ and for which we can write

$$\int_{1/2}^{t} \|\nabla u(s)\|_{\mathrm{L}^p(\Omega)}^c \,\mathrm{d}s \lesssim_0 \gamma(t),$$
$$\gamma(t) := \mathrm{E}(0)^{c\frac{1-\alpha}{2}} c^{-1}(e^{ct} - e^{\frac{c}{2}})\Phi(c)\left[1 + (1 + \mathrm{M}_{\rho_f, j_f}(t))^{c\alpha} + \left(1 + \mathrm{M}_{\rho_f, j_f}(t)^{\frac{3}{2}-\frac{2}{b}}\right)^{c\alpha}\right], \tag{2.6.2}$$

provided that $\alpha \in [1/2, 1[$ and $p \in [1, \infty]$ satisfy $\alpha c \leq \min(a, b)$ and the relation (2.6.1).

However, reaching $p = \infty$ is not possible yet because the relation (2.6.1) would imply

$$\alpha = 5\left(7 - \frac{6}{r}\right)^{-1},$$

so that the condition $\alpha \in [1/2, 1[$ is actually equivalent to the condition $r > 3$. This last inequality is not *a priori* satisfied by our choice $r = \min(r_a, r_b)$. Nevertheless, we will replace r_a by a larger exponent. To do so, we first use (2.6.2) with $c = a < b$ and by carefully looking at the relation (2.6.1) when α is close enough to 1, we see there exists $\alpha \in [1/2, 1[$ related to $p > 6$ by (2.6.1) and such that

$$\|\nabla u\|_{\mathrm{L}^a(1/2,t;\mathrm{L}^p(\Omega))} \lesssim_0 \gamma(t).$$

Since $p > 6$, we now use Hölder's inequality to find $\tilde{r}_a := 6p/(p-6) > 3$ and write

$$\begin{aligned}\left(\int_{1/2}^{t} \|(u\cdot\nabla)\,u(s)\|_{\mathrm{L}^{\tilde{r}_a}(\Omega)}^a \,\mathrm{d}s\right)^{1/a} &\leq \|u\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\Omega))}\|\nabla u\|_{\mathrm{L}^a(1/2,t;\mathrm{L}^p(\Omega))} \\ &\leq \gamma(t)\|u\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^6(\Omega))} \\ &\leq \gamma(t)\left(\|\nabla u_0\|_{\mathrm{L}^2(\Omega)}^2 + \mathrm{E}(0)\eta(t)\right),\end{aligned}$$

thanks to the estimates (2.4.15) and (2.4.8).

It allows us to replace Lemma 2.5.5 by the previous inequality so that we can get $\tilde{r}_a > 3$ instead of r_a. Now, this yields $\tilde{r} := \min(r_b, \tilde{r}_a) > 3$ and by taking

$$\tilde{\alpha} = 5\left(7 - \frac{6}{\tilde{r}}\right)^{-1},$$

we have

$$\tilde{\alpha} \in [1/2, 1[, \quad 0 = \frac{1}{3} + \tilde{\alpha}\left(\frac{1}{\tilde{r}} - \frac{2}{3}\right) + \frac{1-\tilde{\alpha}}{2}.$$

Arguing as in the beginning of the proof, we now use Remark 2.5.4 with $\tilde{r} > 3$, $c = 1$ and $\tilde{\alpha}$ and we get

$$\int_{1}^{t} \|\nabla u(s)\|_{\mathrm{L}^\infty(\Omega)} \,\mathrm{d}s \lesssim_0 \tilde{\gamma}(t),$$

where

$$\tilde{\gamma}(t) := \mathrm{E}(0)^{\frac{1-\tilde{\alpha}}{2}}(e^t - e)\Phi(1)\left[1 + (1 + \mathrm{M}_{\rho_f, j_f}(t))^{\tilde{\alpha}} + \left(1 + \mathrm{M}_{\rho_f, j_f}(t)^{\frac{3}{2}-\frac{2}{b}}\right)^{\tilde{\alpha}}\right].$$

Because of (2.4.8), the quantity $\tilde{\gamma}(t)$ is finite so that this concludes the proof.

□

In order to set up a bootstrap argument, we naturally introduce the following quantity:

$$t^\star := \sup \left\{ \text{strong existence times } t \in \mathbb{R}^+ \text{ such that } \int_1^t \|\nabla u(s)\|_{\mathrm{L}^\infty(\Omega)} \, \mathrm{d}s < \delta \right\}, \tag{2.6.3}$$

where $\delta > 0$ is taken small enough in order to satisfy $\delta e^\delta \leq 1/9$. Our main goal is now to show that $t^\star = +\infty$.

Lemma 2.6.2. *We have $t^\star > 1$ and for any $t < t^\star$, the estimate $\mathrm{M}_{\rho_f, j_f}(t) \lesssim_0 1$ holds.*

Proof. By reducing $\mathrm{E}(0)$ in Lemma 2.4.3 and by the same proof as in Lemma 2.4.13 , we observe that $t = 1 + \varepsilon$ is still a strong existence time for ε small enough. Thus, thanks to Lemma 2.6.1, we can find ε small enough such that $1+\varepsilon$ is a strong existence time and such that $\|\nabla u\|_{\mathrm{L}^1(1,1+\varepsilon;\mathrm{L}^\infty(\Omega))} < \delta$. It therefore implies that $t^\star > 1$.

Now, we take $t \in [1, t^\star[$ and we write it as $t = t^\star - r$ with $r > 0$. By definition of $t^\star$, there exists a time $\tilde{t}$ such that $t = t^\star - r < \tilde{t} < t^\star$ and such that $\tilde{t}$ is a strong existence satisfying $\|\nabla u\|_{\mathrm{L}^1(1,\tilde{t};\mathrm{L}^\infty(\Omega))} < \delta$. Now, we can use Lemma 2.3.6 with $t_0 = 1$ and the estimate (2.4.6) for u to get

$$\|\rho_f(t)\|_{\mathrm{L}^\infty(\Omega)} \lesssim N_q(f_0)(1 + \|u\|^q_{\mathrm{L}^1(0,1;\mathrm{L}^\infty(\Omega))}) \lesssim_0 N_q(f_0)(1 + \eta(1)^q),$$

therefore $\mathrm{M}_{\rho_f}(t) \lesssim_0 1$, uniformly in t. Similarly, for j_f, we have

$$\begin{aligned}
\|j_f(t)\|_{\mathrm{L}^\infty(\Omega)} &\lesssim e^{-t} \left(\int_1^t e^s \|u(s)\|_{\mathrm{L}^\infty(\Omega)} \, \mathrm{d}s + 1 \right) N_q(f_0)(1 + \|u\|^q_{\mathrm{L}^1(0,1;\mathrm{L}^\infty(\Omega))}) \\
&\lesssim_0 e^{-t} \left(\int_1^t e^s \|u(s)\|_{\mathrm{L}^\infty(\Omega)} \, \mathrm{d}s + 1 \right).
\end{aligned}$$

The Sobolev embedding $\mathrm{H}^2(\Omega) \hookrightarrow \mathrm{L}^\infty(\Omega)$ and the elliptic estimate $\|u\|_{\mathrm{H}^2(\Omega)} \lesssim \|Au\|_{\mathrm{L}^2(\Omega)}$ ensure that for all $t \in [1, t^\star[$

$$e^{-t} \int_1^t e^s \|u(s)\|_{\mathrm{L}^\infty(\Omega)} \, \mathrm{d}s \lesssim e^{-t} \int_1^t e^s \|Au(s)\|_{\mathrm{L}^2(\Omega)} \, \mathrm{d}s.$$

Thanks to Cauchy-Schwarz inequality, we get for all $t \in [1, t^\star[$

$$\begin{aligned}
e^{-t} \int_1^t e^s \|u(s)\|_{\mathrm{L}^\infty(\Omega)} \, \mathrm{d}s &\leq e^{-t} \left(\int_1^t e^{2s} \, \mathrm{d}s \right)^{1/2} \left(\int_1^t \|Au(s)\|^2_{\mathrm{L}^2(\Omega)} \, \mathrm{d}s \right)^{1/2} \\
&\lesssim \left(\int_1^t \|Au(s)\|^2_{\mathrm{L}^2(\Omega)} \, \mathrm{d}s \right)^{1/2} \\
&\lesssim_0 \left(1 + \sup_{s \in [0,t]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\Omega)} \right)^{1/2} \\
&\lesssim_0 \left(1 + \mathrm{M}_{\rho_f}(t) \right)^{1/2},
\end{aligned}$$

where we have used the parabolic estimate (2.4.13) (which holds since $t < \tilde{t}$). This concludes the proof as we have already proved that $\mathrm{M}_{\rho_f}(t) \lesssim_0 1$. □

Remark 2.6.3. Note that the estimate we have just proved is uniform in time. Therefore, by considering for example a sequence $(t_n)_n$ of strong existence times with $\|\nabla u\|_{\mathrm{L}^1(1,t_n;\mathrm{L}^\infty(\Omega))} < \delta$ such that $t_n \to t^\star$ and $t_n \leq t^\star$, we get $\mathrm{M}_{\rho_f, j_f}(t^\star) \lesssim_0 1$.

Proposition 2.6.4. *If $t^\star < \infty$, there exists $\gamma > 0$ such that the following estimate holds*

$$\int_1^{t^\star} \|\nabla u(s)\|_{\mathrm{L}^\infty(\Omega)}\,\mathrm{d}s \lesssim_0 \mathrm{E}(0)^\gamma. \tag{2.6.4}$$

Proof. We modify the proof of Lemma 2.6.1 and take advantage of the exponential decay of the kinetic energy on $[1,t^\star]$ in order to get uniform in time estimates. We start again using the Gagliardo-Nirenberg-Sobolev inequality (see Theorem 2.D.1 in Appendix) with $(j,m,q)=(1,2,2)$ to write

$$\|\nabla u(s)\|_{\mathrm{L}^p(\Omega)} \lesssim \|\mathrm{D}^2 u(s)\|^\alpha_{\mathrm{L}^r(\Omega)} \|u(s)\|^{1-\alpha}_{\mathrm{L}^2(\Omega)}, \qquad s \geq 1/2,$$

for all $p \in [1,\infty]$ and $r \in [1,\infty]$ satisfying the relation (2.6.1) and where $\alpha \in [1/2, 1[$. Now, since $t^\star < \infty$, we can combine Proposition 2.2.4 on $[0,t^\star]$ and Lemma 2.6.2 to get a rate $\lambda^\star$ such that $\mathrm{E}(t) \leq e^{-\lambda^\star t}\mathrm{E}(0)$ on $[0,t^\star]$. By looking at the definition of the kinetic energy and by setting $\lambda := \lambda^\star(1-\alpha)/2$, we have

$$\|\nabla u(s)\|_{\mathrm{L}^p(\Omega)} \lesssim \mathrm{E}(0)^{\frac{1-\alpha}{2}} e^{-\lambda s} \|\mathrm{D}^2 u(s)\|^\alpha_{\mathrm{L}^r(\Omega)}, \qquad s \in [1/2, t^\star]. \tag{2.6.5}$$

This inequality is the key to get uniform in time estimates.

Moreover, by taking a sequence $(t_n)_n$ of strong existence times with $\|\nabla u\|_{\mathrm{L}^1(1,t_n;\mathrm{L}^\infty(\Omega))} < \delta$ such that $t_n \to t^\star$ and $t_n \leq t^\star$, we see that estimate (2.4.13) holds for $t^\star$ thanks to Lemma 2.6.2 so that the statements of Proposition 2.5.3 and Lemmas 2.5.5-2.5.6 still hold at time $t = t^\star$.

We now perfom the same arguments as in the proof of Lemma 2.6.1: namely, we replace the inequality (2.6.2) by the inequality (2.6.5). We eventually end up with a control on ∇u of the form (2.6.4) and this concludes the proof. □

In view of Lemma 2.6.2, it remains to show the following statement.

Proposition 2.6.5. *We have $t^\star = +\infty$.*

Proof. By contradiction, let assume that $t^\star < +\infty$. We will get a contradiction by proving (provided that $\mathrm{E}(0)$ is small enough) the existence of a strong existence time $t > t^\star$ such that the estimate $\|\nabla u\|_{\mathrm{L}^1(1,t;\mathrm{L}^\infty(\Omega))} < \delta$ holds.

- We first prove that there exists a strong existence time larger than $t^\star$. Recall that we work under the assumption $\|\nabla u_0\|^2_{\mathrm{L}^2(\Omega)} \leq (2\sqrt{8C_1C_2})^{-1}$ (see (2.4.14)). Recall also that the estimate (2.4.3) yields

$$\int_0^t \|F(s)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s \lesssim_0 \mathrm{E}(0)\left(1 + \mathrm{M}_{\rho_f, j_f}(t)\right),$$

for all $t \in [1,t^\star]$, so that Lemma 2.6.2, Remark 2.6.3 and the Notation (2.4.1) for the symbol $\lesssim_0$ provide the existence of a nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ which is onto and such that

$$\int_0^{t^\star} \|F(s)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s \leq \mathrm{E}(0)\varphi\Big(1 + \|\nabla u_0\|_{\mathrm{L}^2(\Omega)} + N_q f_0 + \mathrm{E}(0)\Big).$$

We thus infer that

$$\begin{aligned}
\|\nabla u_0\|^2_{\mathrm{L}^2(\Omega)} &+ C_1\int_0^{t^\star} \|F(s)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s \\
&\leq \|\nabla u_0\|^2_{\mathrm{L}^2(\Omega)} + C_1\mathrm{E}(0)\varphi\Big(1 + \|\nabla u_0\|_{\mathrm{L}^2(\Omega)} + N_q f_0 + \mathrm{E}(0)\Big) \\
&\leq \frac{1}{2\sqrt{8C_1C_2}} + C_1\mathrm{E}(0)\varphi\Big(1 + \|\nabla u_0\|_{\mathrm{L}^2(\Omega)} + N_q f_0 + \mathrm{E}(0)\Big).
\end{aligned}$$

Thanks to the smallness condition (2.1.16), we can choose $\mathrm{E}(0)$ and $\|\nabla u_0\|_{\mathrm{L}^2(\Omega)}$ small enough so that

$$\|\nabla u_0\|_{\mathrm{L}^2(\Omega)}^2 + C_1 \int_0^{t^\star} \|F(s)\|_{\mathrm{L}^2(\Omega)}^2 \,\mathrm{d}s < \frac{1}{\sqrt{8C_1C_2}}.$$

Using again the integrability (2.4.3) of $s \mapsto \|F(s)\|_{\mathrm{L}^2(\Omega)}^2$ on $[0,T]$ for all $T > 0$, we obtain by continuity a strong existence time (strictly) larger than $t^\star$.

- Now, we turn to the existence of a strong existence time larger than $t^\star$ which satisfies (2.6.3).

We use the uniform control of Proposition 2.6.4 to get the existence of a nondecreasing continuous and onto function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_1^{t^\star} \|\nabla u(s)\|_{\mathrm{L}^\infty(\Omega)} \leq \varphi\Big(1 + \|\nabla u_0\|_{\mathrm{L}^2(\Omega)} + N_q f_0 + \mathrm{E}(0)\Big)\mathrm{E}(0)^\gamma,$$

with $\gamma > 0$. A smallness condition such as (2.1.16) again ensures that

$$\int_1^{t^\star} \|\nabla u(s)\|_{\mathrm{L}^\infty(\Omega)} \leq \frac{\delta}{2}.$$

Thanks to Lemma 2.6.1, a continuity argument shows that there exists a strong existence time $t > t^\star$ such that

$$\int_1^{t} \|\nabla u(s)\|_{\mathrm{L}^\infty(\Omega)} < \delta.$$

This is a contradiction with the definition of $t^\star$. Therefore we must have $t^\star = +\infty$ and the proof of Theorem 2.1.7 is finally complete. □

2.7 Further description of the asymptotic local density

This section aims at providing a further description of the asymptotic behavior of f in the space variable, that is to say at proving Theorem 2.1.10. Indeed, we have obtained the existence of a spatial profile ρ^∞ in an abstract framework in Corollary 2.1.9. However, a careful study of the particle trajectory will bring more information about this asymptotic state. We will see that, in full generality, the description of this profile will depend on the whole evolution of the system.

We first refer to Section 2.3 where we have defined the characteristic curves $s \mapsto (\mathrm{X}(s), \mathrm{V}(s))$ in $\mathbb{R}^3 \times \mathbb{R}^3$ for the Vlasov equation (associated to the natural extension Pu for the fluid velocity u). We will make a constant use of the notations used in this section.

Suppose that $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}^1_0 \cap \mathrm{W}^{1,\infty}(\Omega))$. The characteristic curves for the Vlasov equation are given for all $t, s \geq 0$ by

$$\begin{cases} \mathrm{X}(s;t,x,v) = x + (1 - e^{-s+t})v + \displaystyle\int_t^s (1 - e^{\tau - s})(Pu)(\tau, \mathrm{X}(\tau;t,x,v))\,\mathrm{d}\tau, \\ \mathrm{V}(s;t,x,v) = e^{-s+t}v + \displaystyle\int_t^s e^{\tau - s}(Pu)(\tau, \mathrm{X}(\tau;t,x,v))\,\mathrm{d}\tau. \end{cases} \tag{2.7.1}$$

We will also use the notation $\mathrm{Z}_{s,t}$ defined in (2.3.5).

We then recall the following representation formula for a given weak solution (u, f) to the Vlasov-Navier-Stokes system (see Proposition 2.3.2). We have

$$f(t,x,v) = e^{3t}\mathbf{1}_{\mathcal{O}^t}(x,v)\, f_0(\mathrm{X}(0;t,x,v), \mathrm{V}(0;t,x,v)), \tag{2.7.2}$$

where $\mathcal{O}^t$ is defined in (2.3.6).

In order to describe the asymptotic profile ρ^∞ of Corollary 2.1.9, we take a test function $\psi \in \mathscr{C}_c^\infty(\Omega)$ and we look at the following quantity for $t \geq 0$

$$\int_\Omega \rho_f(t,x)\psi(x)\,\mathrm{d}x.$$

Thanks to the previous representation formula (2.7.2) for the Vlasov equation, we can write for all $t > 0$

$$\begin{aligned}\int_\Omega \rho_f(t,x)\psi(x)\,\mathrm{d}x &= \int_{\Omega\times\mathbb{R}^3} f(t,x,v)\psi(x)\,\mathrm{d}x\,\mathrm{d}v \\ &= \int_{\Omega\times\mathbb{R}^3} e^{3t}\mathbf{1}_{\mathcal{O}^t}(x,v)\, f_0(\mathrm{X}(0;t,x,v),\mathrm{V}(0;t,x,v))\psi(x)\,\mathrm{d}x\,\mathrm{d}v.\end{aligned}$$

We now use the natural change of variable $z = \mathrm{Z}_{0,t}(x,v)$, by remembering that

$$\mathrm{Z}_{0,t}(\mathcal{O}^t) = \Big\{(x,v)\in\Omega\times\mathbb{R}^3 \ \mid\ \tau^+(0,x,v) > t\Big\},$$

where the forward exit time $\tau^+(0,x,v)$ is defined by

$$\tau^+(0,x,v) = \sup\left\{s \geq 0 \ \mid \forall \sigma \in [0,s],\ \mathrm{X}(\sigma;0,x,v) \in \Omega\right\},$$

(see Section 2.3.2). We infer that for all $t > 0$

$$\int_\Omega \rho_f(t,x)\psi(x)\,\mathrm{d}x = \int_{\Omega\times\mathbb{R}^3} \mathbf{1}_{\tau^+(0,x,v)>t}\, f_0(x,v)\psi(\mathrm{X}_{t,0}(x,v))\,\mathrm{d}x\,\mathrm{d}v,$$

as in the proof of Proposition 2.3.2 in Section 2.B. We now remark that for any $t > 0$, $\mathbf{1}_{\tau^+(0,x,v)>t} = 0$ if $x \notin \Omega$. We thus get for all $t > 0$

$$\int_\Omega \rho_f(t,x)\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathbf{1}_{\tau^+(0,x,v)>t} f_0(x,v)\psi(\mathrm{X}_{t,0}(x,v))\,\mathrm{d}x\,\mathrm{d}v. \tag{2.7.3}$$

To go further, we need to understand the behavior of the family of curves $\mathrm{X}_{t,0}(x,v)$ when $t \to \infty$. This is the aim of the following Lemma, which is in the same spirit as [HKMM20, Lemma 8.1].

Lemma 2.7.1. *There exists $\delta > 0$ such that if $u \in \mathrm{L}^1(\mathbb{R}^+;\mathrm{H}^1_0 \cap \mathrm{W}^{1,\infty}(\Omega))$ with*

$$\int_0^\infty \|\nabla u(s)\|_{\mathrm{L}^\infty(\Omega)}\,\mathrm{d}s \leq \delta, \tag{2.7.4}$$

then the family of functions $(x,v)\mapsto \mathrm{X}_{t,0}(x,v)$ converges in $\mathscr{C}^1(\mathbb{R}^3\times\mathbb{R}^3)$, when $t\to+\infty$, towards $\mathrm{X}_\infty : \mathbb{R}^3\times\mathbb{R}^3 \mapsto \mathrm{X}_\infty(x,v)\in\mathbb{R}^3$ with

$$\mathrm{X}_\infty(x,v) = x + v + \int_0^\infty (Pu)(\tau,\mathrm{X}_{\tau,0}(x,v))\,\mathrm{d}\tau.$$

Proof. In order to show that the family of curves $(\mathrm{X}_{t,0}(x,v))_t$ has a limit in $\mathscr{C}^1(\mathbb{R}^3\times\mathbb{R}^3)$ when $t\to+\infty$, we will show that it satisfies a local Cauchy's criterion when $t\to+\infty$.

By coming back to the expression (2.7.1), we first write

$$\mathrm{X}_{t,0}(x,v) = x + v + \int_0^\infty \mathbf{1}_{\tau\leq t}(Pu)(\tau,\mathrm{X}_{\tau,0}(x,v))\,\mathrm{d}\tau + \varepsilon(t,x,v), \tag{2.7.5}$$

where

$$\varepsilon(t,x,v) := e^{-t}v - \int_0^\infty e^{\tau-t}\mathbf{1}_{\tau\le t}(Pu)(\tau,\mathrm{X}_{\tau,0}(x,v))\,\mathrm{d}\tau.$$

Henceforth, we will use the notation $\mathrm{X}_{\tau,0}(z) := \mathrm{X}_{\tau,0}(x,v)$, where $z=(x,v)\times\mathbb{R}^3\times\mathbb{R}^3$ as usual.

To show that a Cauchy's criterion is satisfied, we compute the difference between two expressions (2.7.5) at t_1 and t_2 with $0<t_1<t_2$ and we get

$$\begin{aligned}|\mathrm{X}_{t_2,0}(z)-\mathrm{X}_{t_1,0}(z)| &\le \int_{t_1}^{t_2}|(Pu)(\tau,\mathrm{X}_{\tau,0}(z))|\,\mathrm{d}\tau + |\varepsilon(t_2,x,v)-\varepsilon(t_1,x,v)|\\ &\le \int_{t_1}^{t_2}\|(Pu)(\tau)\|_{\mathrm{L}^\infty(\mathbb{R}^3)}\,\mathrm{d}\tau + |\varepsilon(t_2,x,v)-\varepsilon(t_1,x,v)|\\ &\le \int_{t_1}^{t_2}\|u(\tau)\|_{\mathrm{L}^\infty(\Omega)}\,\mathrm{d}\tau + |\varepsilon(t_2,x,v)-\varepsilon(t_1,x,v)|.\end{aligned}$$

We now fix a compact set $K\subset\mathbb{R}^3\times\mathbb{R}^3$. The previous computation thus yields

$$\begin{aligned}\sup_{z\in K}|\mathrm{X}_{t_2,0}(z)-\mathrm{X}_{t_1,0}(z)| &\le \int_{t_1}^{t_2}\|u(\tau)\|_{\mathrm{L}^\infty(\Omega)}\,\mathrm{d}\tau + \sup_{z\in K}|\varepsilon(t_2,z)-\varepsilon(t_1,z)|\\ &:= \mathrm{(I)}+\mathrm{(II)},\end{aligned}$$

and it remains to see how the terms (I) and (II) behave when $\min(t_1,t_2)\to+\infty$.

Thanks to Poincaré inequality on $\mathrm{H}^1_0(\Omega)\cap\mathrm{W}^{1,\infty}(\Omega)$ and to the assumption (2.7.4), we observe that $t\mapsto\|u(t)\|_{\mathrm{L}^\infty(\Omega)}$ is integrable on $\mathbb{R}^+$. We thus infer that the term (I) converges to 0 when $\min(t_1,t_2)\to+\infty$.

For the term (II), we remark that for all $t>0$

$$\sup_{z\in K}|\varepsilon(t,z)| \lesssim_K e^{-t} + \int_0^\infty e^{\tau-t}\mathbf{1}_{\tau\le t}\|(Pu)(\tau)\|_{\mathrm{L}^\infty(\mathbb{R}^3)}\,\mathrm{d}\tau.$$

Since $e^{\tau-t}\mathbf{1}_{\tau\le t}\|(Pu)(\tau)\|_{\mathrm{L}^\infty(\mathbb{R}^3)} \xrightarrow{t\to+\infty} 0$ for almost any $\tau\in\mathbb{R}^+$ and since it is controlled by the integrable function $\tau\mapsto\|u(\tau)\|_{\mathrm{L}^\infty(\Omega)}$ on $\mathbb{R}^+$, we can apply the dominated convergence theorem to show that the previous r.h.s converges to 0 when $t\to+\infty$. Thanks to the triangular inequality, the term (II) finally tends to 0 when $\min(t_1,t_2)\to+\infty$.

Therefore we have shown that

$$\sup_{z\in K}|\mathrm{X}_{t_2,0}(z)-\mathrm{X}_{t_1,0}(z)| \longrightarrow 0 \quad\text{when}\quad \min(t_1,t_2)\longrightarrow+\infty.$$

This yields the existence of a mapping $(s,z)\mapsto\mathrm{X}_\infty(x,v)\in\mathbb{R}^3$ which is the uniform limit in $\mathscr{C}(\mathbb{R}^3\times\mathbb{R}^3)$ of the maps $z\mapsto\mathrm{X}_{t,0}(z)$ quand $t\to+\infty$. Again thanks to the dominated convergence theorem, we can pass to the limit when $t\to+\infty$ in the expression (2.7.1) to get, as announced

$$\mathrm{X}_\infty(x,v) = x+v+\int_0^\infty (Pu)(\tau,\mathrm{X}_{\tau,0}(x,v))\,\mathrm{d}\tau.$$

For the moment, the limit $\mathrm{X}_\infty(x,v)$ is only continuous in its variables (x,v) as a uniform limit of such functions. A (local) Cauchy's criterion for the family of derivatives in space-velocity $z\mapsto\mathrm{D}_z\mathrm{X}_{t,0}(z)$ will be actually satisfied when $t\to+\infty$. Indeed, we differentiate the expression (2.7.1) to get

$$\mathrm{D}_z\mathrm{X}_{t,0}(z) = 2\mathrm{A} + \int_0^\infty \mathbf{1}_{\tau\le t}\nabla(Pu)(\tau,\mathrm{X}_{\tau,0}(z))\mathrm{D}_z\mathrm{X}_{\tau,0}(z)\,\mathrm{d}\tau + \widetilde{\varepsilon}(t,z),$$

where

$$\widetilde{\varepsilon}(t,z) := e^{-t}\mathrm{B} - \int_0^\infty e^{\tau - t}\mathbf{1}_{\tau\le t}\nabla(Pu)(\tau,\mathrm{X}_{\tau,0}(z))\mathrm{D}_z\mathrm{X}_{\tau,0}(z)\,\mathrm{d}\tau,$$

$$\mathrm{A} := \Big(\mathrm{I}_3 \mid \mathrm{I}_3\Big) \in \mathrm{M}_{3,6}(\mathbb{R}),$$

$$\mathrm{B} := \Big(0_{\mathrm{M}_{3,3}(\mathbb{R})} \mid \mathrm{I}_3\Big) \in \mathrm{M}_{3,6}(\mathbb{R}),$$

this identity being valid thanks to the integrability of $\tau \mapsto \|\nabla(Pu)(\tau)\|_{\mathrm{L}^\infty(\mathbb{R}^3)}$ on $\mathbb{R}^+$. As before, if $t_1 < t_2$, we have the following inequality

$$|\mathrm{D}_z\mathrm{X}_{t_2,0}(z) - \mathrm{D}_z\mathrm{X}_{t_1,0}(z)| \le \int_{t_1}^{t_2} |\nabla(Pu)(\tau,\mathrm{X}_{\tau,0}(z))\mathrm{D}_z\mathrm{X}_{\tau,0}(z)|\,\mathrm{d}\tau + |\widetilde{\varepsilon}(t_2,z) - \widetilde{\varepsilon}(t_1,z)|.$$

We thus need to derive some bounds on $\mathrm{D}_z\mathrm{X}_{\tau,0}(z)$. We have for all $0 \le s \le t$

$$\begin{aligned}|\mathrm{D}_x\mathrm{X}_{s,0}(z)| &\le 1 + \int_0^\infty \mathbf{1}_{\tau\le t}(1 - e^{\tau-t})|\nabla(Pu)(\tau,\mathrm{X}_{\tau,0}(z)||\mathrm{D}_x\mathrm{X}_{\tau,0}(z)|\,\mathrm{d}\tau \\ &\le 1 + \sup_{0\le\tau\le t}|\mathrm{D}_x\mathrm{X}_{\tau,0}(z)| \int_0^\infty \|\nabla(Pu)(\tau)\|_{\mathrm{L}^\infty(\mathbb{R}^3)}\,\mathrm{d}\tau \\ &\le 1 + \sup_{0\le\tau\le t}|\mathrm{D}_x\mathrm{X}_{\tau,0}(z)| \int_0^\infty \|\nabla u(\tau)\|_{\mathrm{L}^\infty(\Omega)}\,\mathrm{d}\tau.\end{aligned}$$

Thanks to the assumption (2.7.4) on ∇u, we obtain

$$\sup_{0\le\tau\le t}|\mathrm{D}_x\mathrm{X}_{\tau,0}(z)| \le \frac{1}{1-\delta}. \tag{2.7.6}$$

We observe that the very same procedure leads to

$$\sup_{0\le\tau\le t}|\mathrm{D}_v\mathrm{X}_{\tau,0}(z)| \le \frac{1}{1-\delta}. \tag{2.7.7}$$

Imposing $\delta \le 1/2$, the two previous r.h.s become smaller that 2 and we will use this uniform bound in what follows. Again, we fix a compact set $K \subset \mathbb{R}^3 \times \mathbb{R}^3$. We have

$$\begin{aligned}\sup_{z\in K}|\mathrm{D}_z\mathrm{X}_{t_2,0}(z) - \mathrm{D}_z\mathrm{X}_{t_1,0}(z)| &\le \int_{t_1}^{t_2} \|\nabla(Pu)(\tau)\|_{\mathrm{L}^\infty(\mathbb{R}^3)}\,\mathrm{d}\tau + \sup_{z\in K}|\widetilde{\varepsilon}(t_2,z) - \widetilde{\varepsilon}(t_1,z)| \\ &\le \int_{t_1}^{t_2} \|\nabla u(\tau)\|_{\mathrm{L}^\infty(\Omega)}\,\mathrm{d}\tau + \sup_{z\in K}|\widetilde{\varepsilon}(t_2,z) - \widetilde{\varepsilon}(t_1,z)| \\ &:= (\widetilde{\mathrm{I}}) + (\widetilde{\mathrm{II}}).\end{aligned}$$

As before, the first term $(\widetilde{\mathrm{I}})$ tends to 0 when $\min(t_1,t_2)$ goes to $+\infty$ because of the integrability assumption (2.7.4).

For the second term $(\widetilde{\mathrm{II}})$, we proceed as before and we write for all $t > 0$

$$\sup_{z\in K}|\widetilde{\varepsilon}(t,z)| \le e^{-t} + \int_0^\infty e^{\tau-t}\mathbf{1}_{\tau\le t}\|\nabla(Pu)(\tau)\|_{\mathrm{L}^\infty(\mathbb{R}^3)}\,\mathrm{d}\tau.$$

Again, the dominated convergence theorem shows that this expression tends to 0 when t tends to $+\infty$. and this implies that the term $(\widetilde{\mathrm{II}})$ converges to 0 when $\min(t_1,t_2)$ goes to 0.

All in all, we have shown that

$$\sup_{z\in K}|\mathrm{D}_z\mathrm{X}_{t_2,0}(z) - \mathrm{D}_z\mathrm{X}_{t_1,0}(z)| \longrightarrow 0 \quad \text{when} \quad \min(t_1,t_2) \longrightarrow +\infty.$$

Cauchy's criterion then applies for the derivatives and this concludes the proof. □

In what follows, we use the notation $X_{t,0,v}(x) := X(t;0,x,v)$ and $X_{\infty,v}(x) := X_\infty(x,v)$.

Lemma 2.7.2. *Under the assumption (2.7.4), for all $v \in \mathbb{R}^3$ and $t \in \mathbb{R}^+$, the maps $X_{t,0,v} : x \mapsto X_{t,0,v}(x)$ and the map $X_{\infty,v} : x \mapsto X_\infty(x,v)$ are $\mathscr{C}^1$-diffeomorphisms from $\mathbb{R}^3$ to $\mathbb{R}^3$.*

Proof. The proof follows the same steps as that of [HKMM20, Lemma 8.2]. Indeed, from the expression (2.7.1), we get

$$D_x X_{t,0,v}(x) - I_3 = \int_0^\infty \mathbf{1}_{\tau \le t}(1 - e^{\tau - t})\nabla(Pu)(\tau, X_{\tau,0}(x,v)) D_x X_{\tau,0}(x,v)\, d\tau,$$

and then, thanks to the bound (2.7.6), we can write

$$\|D_x X_{t,0,v} - I_3\|_\infty \le \frac{\delta}{1-\delta},$$

so that up to taking δ small enough in the assumption (2.7.4), we can obtain

$$\|D_x X_{t,0,v} - I_3\|_\infty \le \frac{1}{9}. \tag{2.7.8}$$

A variant of the global inversion theorem about perturbation of the identity mapping (see [HKMM20, Lemma 9.4]) leads to the conclusion. □

In order to pass to the limit in (2.7.3) when $t \to +\infty$, we need to determine the limit of $\mathbf{1}_{\tau^+(0,x,v)>t}$ when $t \to +\infty$.

Lemma 2.7.3. *For almost every $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$, we have*

$$\mathbf{1}_{\tau^+(0,x,v)>t} \underset{t\to+\infty}{\longrightarrow} \mathbf{1}_{\mathcal{O}_\infty},$$

where $\mathcal{O}_\infty := \left\{(x,v) \in \Omega \times \mathbb{R}^3 \mid \forall t \ge 0,\ X(t;0,x,v) \in \Omega\right\}$.

Proof. First, we note that for every $(x,v) \in \Omega \times \mathbb{R}^3$, we have $\tau^+(0,x,v) > 0$ while for every $(x,v) \in \Omega^c \times \mathbb{R}^3$, we have $\tau^+(0,x,v) = 0$ and $X(0;0,x,v) \in \Omega^c$. We then focus on the following alternative when $(x,v) \in \Omega \times \mathbb{R}^3$. If $\tau^+(0,x,v) = +\infty$, then $t \mapsto \mathbf{1}_{\tau^+(0,x,v)>t}$ is constant equal to 1 for all $t > 0$ therefore for all $\sigma \ge 0$, $X(\sigma;0,x,v) \in \Omega$. If $\tau^+(0,x,v) \ne +\infty$, then $\mathbf{1}_{\tau^+(0,x,v)>t} \underset{t\to+\infty}{\longrightarrow} 0$ and we have $X(\tau^+(0,x,v);0,x,v) \in \partial\Omega$ so that the conclusion holds. □

We can now conclude the proof of Theorem 2.1.10 by passing to the limit in the expression (2.7.3) when $t \to +\infty$. Indeed, thanks to Lemma 2.7.1 and 2.7.3 and since $f_0 \in L^1(\Omega \times \mathbb{R}^3)$, we can use the dominated convergence theorem when $t \to +\infty$ to get

$$\int_\Omega \rho_f(t,x)\psi(x)\, dx \underset{t\to+\infty}{\longrightarrow} \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\infty}(x,v) f_0(x,v)\psi(X_{\infty,v}(x))\, dx\, dv.$$

We then use the reverse change of variable $x = X_{\infty,v}^{-1}(y)$ for all velocity $v \in \mathbb{R}^3$ in the previous integral thanks to Lemma 2.7.2 in order to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3\times\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\infty}(x,v) f_0(x,v)\psi(X_\infty(x,v))\, dx\, dv \\
&= \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\infty}(X_{\infty,v}^{-1}(x),v) f_0(X_{\infty,v}^{-1}(x),v) |\det D_x X_{\infty,v}^{-1}(x)| \psi(x)\, dx\, dv \\
&= \int_\Omega \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\infty}(X_{\infty,v}^{-1}(x),v) f_0(X_{\infty,v}^{-1}(x),v) |\det D_x X_{\infty,v}^{-1}(x)| \psi(x)\, dx\, dv,
\end{aligned}$$

because ψ is compactly supported in Ω. Thanks to the convergence $\rho_f(t) \overset{t\to+\infty}{\longrightarrow} \rho^\infty$ in $\mathrm{H}^{-1}(\Omega)$ of Corollary 2.1.9, we thus get by uniqueness of the limit that for a.e $x \in \Omega$,

$$\rho^\infty(x) = \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\infty}(\mathrm{X}_{\infty,v}^{-1}(x), v) f_0(\mathrm{X}_{\infty,v}^{-1}(x), v) |\det \mathrm{D}_x \mathrm{X}_{\infty,v}^{-1}(x)| \, \mathrm{d}v.$$

Note that the meaning of the previous indicator function is the following. If $(x, v) \in \Omega \times \mathbb{R}^3$,

$$\begin{aligned}(\mathrm{X}_{\infty,v}^{-1}(x), v) \in \mathcal{O}_\infty &\Leftrightarrow \mathrm{X}_{\infty,v}^{-1}(x) \in \Omega, \quad \text{and} \quad \forall t \geq 0, \quad \mathrm{X}(t; 0, \mathrm{X}_{\infty,v}^{-1}(x), v) \in \Omega \\ &\Leftrightarrow \exists! y \in \Omega, \quad \mathrm{X}_{\infty,v}(y) = x, \quad \text{and} \quad \forall t \geq 0, \quad \mathrm{X}(t; 0, y, v) \in \Omega.\end{aligned}$$

This concludes the proof of Theorem 2.1.10.

2.8 Asymptotic profiles with a prescribed mass

As already explained in the introduction, the asymptotic profile ρ^∞ has a total mass which is not known *a priori*, because $t \mapsto \|\rho_f(t)\|_{\mathrm{L}^1(\Omega)}$ can be decreasing. We can actually find some class of initial data for which any prescribed mass (which is less than of equal to the initial mass 1) will be achieved, that is Proposition 2.1.12.

In order to do so, we first study two scenarios for the support of the initial density, namely we impose $\operatorname{supp} f_0 \subset \mathrm{K}_1 \times \mathrm{K}_2$ with the following alternative:

- $\mathrm{K}_1 \times \mathrm{K}_2$ is a compact of $\Omega \times \mathbb{R}^3$, with K_1 at strictly positive distance from $\partial\Omega$, say included in a small interior ball $\mathrm{B}_x(a, \varepsilon) \subset \Omega$, and with $\mathrm{K}_2 \subset \mathrm{B}_v(0, R)$ for a very small $R > 0$;

- $\mathrm{K}_1 \times \mathrm{K}_2$ is a compact of $\Omega \times \mathbb{R}^3$, with K_1 included in some ball very close to the boundary of Ω, and with $\mathrm{K}_2 \subset \mathrm{B}_v(0, R)$ for a very small $R > 0$ such that all the velocities $v \in \mathrm{K}_2$ are pointing outside Ω.

Thanks to the representation formula (2.3.8), we observe that for all $t \geq 0$, $f(t)$ has a support in space and velocity which is transported by the flow from the initial support of f_0. More precisely, we have for all $t \geq 0$

$$\operatorname{supp} f(t) \subset \Big(\mathrm{X}(t; 0, \operatorname{supp} f_0) \times \mathrm{V}(t; 0, \operatorname{supp} f_0)\Big) \cap \mathcal{O}^t \subset \mathbb{R}^3 \times \mathbb{R}^3.$$

Our main idea is that small initial velocities combined with a small spatial support localized inside the domain will lead to a limit profile which is compactly supported in Ω; while small initial velocities pointing in the outgoing direction with a spatial support close to the boundary will give a profile vanishing in Ω.

In this section, we shall use several times the DiPerna-Lions theory for transport equation in the same fashion as that of the proof of Lemma 2.3.5: this allows us to define the characteric curves in a classical sense. Note that this procedure is actually only required on the interval of time $[0, 1]$ because the smallness condition (2.1.16) will ensure that $u \in \mathrm{L}^1(1, +\infty; \mathrm{W}_0^{1,\infty}(\Omega))$ (see the proof of Propositions 2.8.1-2.8.5).

2.8.1 The case of initial data localized far from $\partial\Omega$

We first investigate the case in which the initial support in velocity is included in a small ball, leading to a situation where particles stay confined in the domain Ω, and far from the boundary.

Proposition 2.8.1. *Let (u_0, f_0) be an admissible initial condition in the sense of Definition 2.1.1 satisfying*

$$\mathrm{supp}\, f_0 \subset \mathrm{B}_x(a,\varepsilon) \times \mathrm{B}_v(0,R) \subset \Omega \times \mathbb{R}^3,$$

where $\varepsilon, R > 0$ satisfy the geometric condition

$$d(\overline{\mathrm{B}}_x(a,\varepsilon), \partial\Omega) > 0, \quad 2R < d(\overline{\mathrm{B}}_x(a,\varepsilon), \partial\Omega). \tag{2.8.1}$$

If the smallness condition (2.1.16) is satisfied, then, for any weak solution (u, f) to the Vlasov-Navier-Stokes system with initial data (u_0, f_0), there exists $d = d(\varepsilon, R, \Omega) > 0$ such that we have for all $t \geq 0$

$$\mathrm{supp}\, f(t) \subset \mathrm{B}_x(a,d) \times \mathbb{R}^3 \subsetneq \Omega \times \mathbb{R}^3.$$

Furthermore, there exists $\rho^\infty \in \mathrm{L}^\infty_c(\Omega)$ with $\int_\Omega \rho^\infty(x)\,\mathrm{d}x = 1$ and such that

$$\mathrm{W}_1\left(f(t), \rho^\infty \otimes \delta_{v=0}\right) \underset{t\to+\infty}{\longrightarrow} 0,$$

exponentially fast.

We state two lemmas that give estimates for the size of the image of the initial support $\mathrm{B}_x(a,\varepsilon) \times \mathrm{B}_v(0,R)$ under the flow $t \mapsto \mathrm{X}(t;0,x,v)$. This is achieved thanks to an appropriate control on the $\mathrm{L}^1\mathrm{L}^\infty$ norm of u.

Lemma 2.8.2. *Let $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}^1_0(\Omega)) \cap \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{W}^{1,\infty}(\Omega))$. For all $t \geq 0$, for all $x \in \Omega$ and for all $v \in \mathrm{B}(0,R)$, we have*

$$\begin{aligned}
|\mathrm{V}(t;0,x,v) - v| &\leq \|u\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\Omega))} + R,\\
|\mathrm{X}(t;0,x,v) - x| &\leq 2\|u\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\Omega))} + R.
\end{aligned}$$

Moreover, we have

$$\mathrm{V}(t;0,x,v) \in \mathrm{B}\left(0, \|u\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\Omega))} + 2R\right),$$

and if $x \in \mathrm{B}(a,\varepsilon)$, we have

$$\mathrm{X}(t;0,x,v) \in \mathrm{B}\left(a, 2\|u\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\Omega))} + R + \varepsilon\right).$$

Proof. Recall that we have the formula for $t \geq 0$

$$\begin{aligned}
\mathrm{V}(t;0,x,v) &= e^{-t}v + \int_0^t e^{\tau - t}(Pu)(\tau, \mathrm{X}(\tau;0,x,v))\,\mathrm{d}\tau,\\
\mathrm{X}(t;0,x,v) &= x + v - \mathrm{V}(t;0,x,v) + \int_0^t (Pu)(\tau, \mathrm{X}(\tau;0,x,v))\,\mathrm{d}\tau.
\end{aligned}$$

Therefore, for $t \geq 0$, the triangular inequality first leads to

$$\begin{aligned}
|\mathrm{V}(t;0,x,v) - v| &\leq |\mathrm{V}(t;0,x,v) - e^{-t}v| + |e^{-t}v - v|\\
&\leq \|Pu\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{R}^3))} + (1 - e^{-t})|v|\\
&\leq \|u\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\Omega))} + R,
\end{aligned}$$

if $v \in \mathrm{B}_v(0,R)$. Then, in a similar way we get

$$\begin{aligned}
|\mathrm{X}(t;0,x,v) - x| &\leq \|Pu\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{R}^3))} + |\mathrm{V}(t;0,x,v) - v|\\
&\leq \|Pu\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{R}^3))} + \|u\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\Omega))} + R\\
&\leq 2\|u\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\Omega))} + R.
\end{aligned}$$

Finally, the triangular inequality gives the last statements. □

Corollary 2.8.3. *Let $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}^1_0(\Omega)) \cap \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{W}^{1,\infty}(\Omega))$. Let $\varepsilon > 0$ and $a \in \Omega$ such that $\mathrm{B}_x(a,\varepsilon) \subset \Omega$ with $d(\overline{\mathrm{B}}_x(a,\varepsilon), \partial\Omega) > 0$ and $R > 0$ such that*

$$2R < d(\overline{\mathrm{B}}_x(a,\varepsilon), \partial\Omega). \tag{2.8.2}$$

Suppose that the velocity field u satisfies

$$\|u\|_{\mathrm{L}^1(\mathbb{R}^+;\mathrm{L}^\infty(\Omega))} \leq \delta, \tag{2.8.3}$$

where

$$\delta := \frac{1}{2}\left(\frac{d(\overline{\mathrm{B}}_x(a,\varepsilon), \partial\Omega)}{2} - R\right).$$

Then for all $T \geq 0$ and for all $(x,v) \in \mathrm{B}_x(a,\varepsilon) \times \mathrm{B}_v(0,R)$, we have

$$\mathrm{X}(T; 0, x, v) \in \Omega, \tag{2.8.4}$$

and more precisely

$$\mathrm{X}(T; 0, x, v) \in \mathrm{B}\Big(a, \varepsilon + R + 2\delta\Big) \subsetneq \Omega. \tag{2.8.5}$$

Proof. Let $T \geq 0$. We first apply Lemma 2.8.2 until time T to see that for all $\sigma \in [0,T]$

$$\mathrm{X}(\sigma; 0, x, v) \in \mathrm{B}\left(a, \varepsilon + R + 2\|u\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\Omega))}\right),$$

if $(x,v) \in \mathrm{B}_x(a,\varepsilon) \times \mathrm{B}_v(0,R)$.

Then, thanks to the global assumption (2.8.3), we get for all $\sigma \in [0,T]$ and for all $(x,v) \in \mathrm{B}_x(a,\varepsilon) \times \mathrm{B}_v(0,R)$

$$\mathrm{X}(\sigma; 0, x, v) \in \mathrm{B}(a, L),$$

where

$$L := \varepsilon + R + 2\delta.$$

Thus, the point is just to ensure that

$$\mathrm{B}(a, L) \subsetneq \Omega.$$

Since $d(a, \partial\Omega) = \varepsilon + d(\overline{\mathrm{B}}_x(a,\varepsilon), \partial\Omega)$, the inequality $L < d(a, \partial\Omega)$ will be satisfied if we prove that

$$R + 2\delta < d(\overline{\mathrm{B}}_x(a,\varepsilon), \partial\Omega).$$

By the definition of δ, we have indeed

$$R + 2\delta = \frac{d(\overline{\mathrm{B}}_x(a,\varepsilon), \partial\Omega)}{2} < d(\overline{\mathrm{B}}_x(a,\varepsilon), \partial\Omega),$$

and thus infer that

$$\mathrm{B}(a, L) \subsetneq \Omega, \quad \text{with} \quad d(\mathrm{B}(a, L), \partial\Omega) > 0,$$

leading to the conclusion (2.8.4). □

It means that, for a well chosen small support of f_0, the support of $f(t)$ stays inside $\Omega\times\mathbb{R}^3$ and far from the boundary as long as $\|u\|_{\mathrm{L}^1_T\mathrm{L}^\infty_x(\Omega)}$ is small enough for all positive times. This assumption will be actually essentially satisfied by using the previous results of the bootstrap argument of Section 2.6.

We eventually turn to the proof of Proposition 2.8.1.

Proof. • Recall that we work with an initial distribution f_0 such that $\operatorname{supp} f_0 \subset \mathrm{B}_x(a,\varepsilon)\times \mathrm{B}_v(0,R)\subset\Omega\times\mathbb{R}^3$ and under the assumption (2.8.1), namely

$$2R < d(\overline{\mathrm{B}}_x(a,\varepsilon),\partial\Omega),$$

which is exactly the previous condition (2.8.2).

First, we define $\delta=\delta(\varepsilon,R,\Omega)>0$ as in (2.8.3). We then use the fact that $u\in\mathrm{L}^1(0,1,\mathrm{L}^\infty(\Omega))$ (see Proposition 2.4.8) together with the dominated convergence theorem to obtain $\underline{t}\in]0,1[$ such that the condition

$$\|u\|_{\mathrm{L}^1(0,\underline{t};\mathrm{L}^\infty(\Omega))}<\frac{\delta}{2},$$

holds.

Furthermore, from the assumption (2.1.16) on the initial data, we can perform the same analysis as in the bootstrap argument of Section 2.6 to get that the quantity $\|\nabla u\|_{\mathrm{L}^1(\underline{t},+\infty;\mathrm{L}^\infty(\Omega))}$ is as small as we want if we reduce $\mathrm{E}(0)$ and $\|\nabla u_0\|_{\mathrm{L}^2(\Omega)}$. Thanks to the Poincaré inequality on $\mathrm{H}^1_0\cap\mathrm{W}^{1,\infty}(\Omega)$, we see that we are able to get the control

$$\|u\|_{\mathrm{L}^1(\underline{t},+\infty;\mathrm{L}^\infty(\Omega))}\leq C_\Omega\|\nabla u\|_{\mathrm{L}^1(\underline{t},+\infty;\mathrm{L}^\infty(\Omega))}<\frac{\delta}{2},$$

(this is possible because we work on an interval of time far from 0). Combining these two pieces, we obtain the fact that the global condition (2.8.3) is satisfied. Thanks to Corollary 2.8.3 together with the inclusion

$$\operatorname{supp} f(t)\subset \mathrm{X}(t;0,\operatorname{supp} f_0)\times\mathrm{V}(t;0,\operatorname{supp} f_0),$$

we get for all $t\in\mathbb{R}^+$

$$\operatorname{supp} f(t)\subsetneq\Omega\times\mathbb{R}^3.$$

In particular, with (2.8.5) of Corollary 2.8.3, we have for all $t\in\mathbb{R}^+$

$$\operatorname{supp} f(t)\subset\mathrm{B}_x\Big(a,\varepsilon+R+2\delta\Big)\times\mathbb{R}^3.$$

□

Remark 2.8.4. In such a situation where the particles never leave the domain, we observe that the trace of f vanishes for all times therefore there is conservation of the mass. Thus, it is now possible to prove a convergence of the type

$$\mathrm{W}_{1,\overline{\Omega}\times\mathbb{R}^3}\left(f(t),\rho^\infty\otimes\delta_{v=0}\right)\leq\mathrm{E}(0)^{1/2}C_\lambda\exp(-\lambda t),$$

in Corollary 2.1.9, where $\mathrm{W}_{1,\overline{\Omega}\times\mathbb{R}^3}$ stands for the 1-Wasserstein distance on $\overline{\Omega}\times\mathbb{R}^3$, and with

$$\mathrm{W}_{1,\overline{\Omega}}\left(\rho_f(t),\rho^\infty\right)\overset{+\infty}{\longrightarrow}0, \tag{2.8.6}$$

where $\mathrm{W}_{1,\overline{\Omega}}$ stands for the 1-Wasserstein distance on $\overline{\Omega}$, as it was already observed in the case of the torus (see [HKMM20]). Indeed, thanks to the vanishing trace of f, we can now take smooth test functions on $\overline{\Omega}$, whose Lipschitz constant is less than 1, in the proof of Proposition 2.2.5 because there are no boundary terms.

In particular, the previous convergence (2.8.6) and the fact that the support of each $\rho_f(t)$ is uniformly included in a ball imply that the limit ρ^∞ is also compactly supported in the same ball.

2.8.2 The case of initial data localised near $\partial\Omega$

Here, we deal with the situation where all particles escape from the domain Ω after a finite time so that the kinetic distribution vanishes uniformly after this time. This will be the case if starting in a thin neighborhood of the boundary $\partial\Omega$, with small initial velocities pointing towards outgoing direction.

First, we introduce several notations. For $r, R > 0$ such that $r < R$, we define the annulus

$$\mathscr{C}(0, r, R) := \{v \in \mathbb{R}^3 \mid r < |v| < R\}.$$

For $\gamma > 0$, let us also set

$$\mathcal{V}_\gamma(\partial\Omega) := \{x \in \Omega \mid d(x, \partial\Omega) < \gamma\}.$$

Since Ω is smooth, we know (see e.g. [BF12]) that for γ small enough (depending on Ω), there exists a projection map on the boundary $p : \mathcal{V}_\gamma(\partial\Omega) \to \partial\Omega$ such that $|x - p(x)| = d(x, \partial\Omega)$ for all $x \in \mathcal{V}_\gamma(\partial\Omega)$. The map p is Lipschitz continuous and satisfies the identity

$$\forall x \in \mathcal{V}_\gamma(\partial\Omega), \quad p(x) = x + \delta(x) n(p(x)),$$

where $n(y)$ refers to the outward unit normal at $y \in \partial\Omega$. Furthermore, since Ω is smooth, we can always find an open ball B_ℓ of radius $\ell > 0$ and a smooth map $\phi \in \mathscr{C}^1(\mathbb{R}^d)$ such that

$$y \in B_\ell \cap \overline{\Omega}^c \iff \phi(y) < 0,$$
$$y \in B_\ell \cap \partial\Omega \iff \phi(y) = 0.$$

For γ small enough, we can always suppose that

$$p : \mathcal{V}_\gamma(\partial\Omega) \cap B_{\ell/4} \longrightarrow \partial\Omega \cap B_{\ell/2}.$$

We also consider a fixed exterior tubular neighborhood of $\partial\Omega$ of width $\sigma > 0$.

We can now state the following proposition.

Proposition 2.8.5. *Let (u_0, f_0) be an admissible initial condition in the sense of Definition 2.1.1 and take any weak solution (u, f) to the Vlasov-Navier-Stokes system with initial data (u_0, f_0). There exist $\gamma > 0$, $R = R(\Omega) > 0$ and $r = r(\Omega) > 0$ small enough such that if*

$$\operatorname{supp} f_0 \subset \Big\{(x, v) \in \Big(\mathcal{V}_\gamma(\partial\Omega) \cap B_{\ell/4}\Big) \times \mathscr{C}(0, r, R) \mid v \cdot n(p(x)) > 0\Big\}, \tag{2.8.7}$$

then the following holds. If the smallness condition (2.1.16) is satisfied, then for any weak solution (u, f) to the Vlasov-Navier-Stokes system with initial data (u_0, f_0), we have

$$\forall t \geq \log(2), \quad f(t) = 0 \ a.e. \tag{2.8.8}$$

Remark 2.8.6. The time $t = \log(2)$ has been choosen to simplify the subsequent proof and must be understood as a finite time after which the kinetic distribution vanishes. Note also we can indeed make the contribution of f_0 to $\mathrm{E}(0)$ small

Proof of Proposition 2.8.5. First, we write the formula (2.7.1) under the form

$$\mathrm{X}(\log(2), 0, x, v) = x - p(x) + \int_0^{\log(2)} (1 - e^{\tau - \log(2)})(Pu)(\tau, \mathrm{X}(\tau, 0, x, v))\, \mathrm{d}s + p(x) + \frac{1}{2}v,$$

for all $x \in \mathcal{V}_\gamma(\partial\Omega) \times \mathbb{R}^3$. For $x \in \mathcal{V}_\gamma(\partial\Omega) \cap B_\ell$ and $|v| \leq R$ small enough, we have

$$\phi\left(p(x) + \frac{1}{2}v\right) = -\frac{1}{2}|\nabla\phi_k(p(x))| v \cdot n(p(x)) + o(|v|).$$

Since $\nabla\phi(p(x)) \neq 0$, taking $v \in \mathbb{R}^3$ such that $v \cdot n(p(x)) > 0$ yields $\phi\left(p(x) + \frac{1}{2}v\right) < 0$ for $|v| \leq R$ small enough (uniformly in x thanks to Taylor-Lagrange inequality), hence

$$p(x) + \frac{1}{2}v \notin \Omega.$$

Furthermore, by relying on the exterior tubular neighborhood of Ω of size σ, we can always find $r > 0$ and reduce R so that the previous (x, v) with $|v| > r$ satisfy

$$0 < \frac{\sigma}{k(r,R)} < d\left(p(x) + \frac{1}{2}v, \partial\Omega\right) < \sigma, \quad p(x) + \frac{1}{2}v \notin \Omega.$$

for some $k(r,R) > 1$. We now treat the remaining part of $\mathrm{X}(\log(2), 0, x, v)$ as follows. We have to make the quantity

$$x - p(x) + \int_0^{\log(2)} (1 - e^{\tau - \log(2)})(Pu)(\tau, \mathrm{X}(\tau, 0, x, v))$$

small enough, in order to make it as a small perturbation of $p(x) + \frac{1}{2}v$, entailing $\mathrm{X}(\log(2), 0, x, v) \notin \overline{\Omega}$. But this remainder is bounded by

$$\gamma + \int_0^{+\infty} \|u(\tau)\|_{\mathrm{L}^\infty(\Omega)} \,\mathrm{d}\tau. \tag{2.8.9}$$

since $x \in \mathcal{V}_\gamma(\partial\Omega)$. We use the fact that $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+, \mathrm{L}^\infty(\Omega))$ (see Proposition 2.4.8) together with the dominated convergence theorem to obtain a time $\bar{t} > 0$ such that $\|u\|_{\mathrm{L}^1(0,\bar{t};\mathrm{L}^\infty(\Omega))} < \sigma/8$. Then, if we reduce $\mathrm{E}(0)$ and $\|\nabla u_0\|_{\mathrm{L}^2(\Omega)}$, we can perform the same analysis as in the bootstrap argument of Section 2.6 to get that the quantity $\|\nabla u\|_{\mathrm{L}^1(\underline{t},+\infty;\mathrm{L}^\infty(\Omega))}$ is as small as we want. Combining this argument with the Poincaré inequality on $\mathrm{H}^1_0 \cap \mathrm{W}^{1,\infty}(\Omega)$, we get

$$\|u\|_{\mathrm{L}^1(\underline{t},+\infty;\mathrm{L}^\infty(\Omega))} \leq C_\Omega \|\nabla u\|_{\mathrm{L}^1(\underline{t},+\infty;\mathrm{L}^\infty(\Omega))} < \frac{\sigma}{8}.$$

We thus have $\|u\|_{\mathrm{L}^1(0,+\infty;\mathrm{L}^\infty(\Omega))} \leq \sigma/4$. Choosing $\gamma < \sigma/4$, we can now make the quantity (2.8.9) less than $\sigma/2$, and we end up with

$$\forall (x,v) \in \operatorname{supp} f_0, \quad \mathrm{X}(\log(2), 0, x, v) \notin \overline{\Omega}. \tag{2.8.10}$$

Let us now conclude the proof. Thanks to the representation formula for the distribution function and by using the same change of variable as in the proof of Proposition 2.3.2, we have for any $t \geq \log(2)$

$$\begin{aligned}
\int_{\Omega \times \mathbb{R}^3} f(t,x,v) \,\mathrm{d}x\mathrm{d}v &= \int_{\mathcal{O}^t} e^{3t} f_0(\mathrm{Z}_{0,t}(x,v)) \,\mathrm{d}x \,\mathrm{d}v \\
&= \int_{\Omega \times \mathbb{R}^3} \mathbf{1}_{\tau^+(0,z) > t} \, f_0(z) \,\mathrm{d}z \\
&\leq \int_{\operatorname{supp} f_0} \mathbf{1}_{\tau^+(0,z) > \log(2)} \, f_0(z) \,\mathrm{d}z,
\end{aligned}$$

because $f_0 \geq 0$ *a.e.* Since $f \geq 0$ *a.e*, it is enough to show that

$$\forall z \in \operatorname{supp} f_0, \quad \tau^+(0,z) \leq \log(2),$$

for the integral $\int_{\Omega\times\mathbb{R}^3} f(t,x,v)\,\mathrm{d}x\mathrm{d}v$ to vanish for all $t \geq \log(2)$. By definition of τ^+ and the continuity in time of the trajectory, the previous condition is indeed satisfied because of (2.8.10). It therefore implies that $f \equiv 0$ almost everywhere after time $\log(2)$, which is what we wanted to prove.

□

Remark 2.8.7. In such a scenario, we have $\rho^\infty = 0$ in the statement of Corollary 2.1.9.

2.8.3 Proof of Proposition 2.1.12

The aim of this subsection is to provide a proof of Proposition 2.1.12. We want to show that, for a fixed $\alpha \in [0,1]$, there exists an initial data (u_0, f_0) which gives rise to weak solutions whose kinetic part concentrates in velocity, with a spatial asymptotic profile of total mass α.

To do so, in view of the two previous sections, we consider $a \in \Omega$, $\varepsilon, R_{int} > 0$ and $\gamma, R_{ext}, r_{ext} > 0$ small enough given by Proposition 2.8.5 such that $\mathrm{B}_x(a,\varepsilon) \cap \mathcal{V}_\gamma(\partial\Omega) = \emptyset$ and an initial admissible data (u_0, f_0) in the sense of (2.1.15) such that

$$\operatorname{supp} f_0 \subset \mathrm{B}_x(a,\varepsilon) \times \mathrm{B}_v(0,R_{int}) \sqcup \Big\{(x,v) \in \Big(\mathcal{V}_\gamma(\partial\Omega) \cap B_{\ell/4}\Big) \times \mathscr{C}(0, r_{ext}, R_{ext}) \mid v \cdot n(p(x)) > 0\Big\},$$

$$\int_{\mathrm{B}_x(a,\varepsilon)\times\mathrm{B}_v(0,R_{int})} f_0\,\mathrm{d}x\,\mathrm{d}v = \alpha, \quad \int_{\operatorname{supp} f_0 \setminus \mathrm{B}_x(a,\varepsilon)\times\mathrm{B}_v(0,R_{int})} f_0\,\mathrm{d}x\,\mathrm{d}v = 1-\alpha,$$

$$\mathrm{B}_x(a,\varepsilon) \subset \Omega, \quad d(\overline{\mathrm{B}}_x(a,\varepsilon), \partial\Omega) > 0, \quad 2R_{int} < d(\overline{\mathrm{B}}_x(a,\varepsilon), \partial\Omega).$$

Furthermore, we consider that the initial kinetic energy $\mathrm{E}(0)$ and $\|\nabla u_0\|_{\mathrm{L}^2(\Omega))}$ are small enough in the sense of (2.1.16), which is always possible for the kinetic part f_0 up to reducing R_{int} and R_{ext} if necessary.

We then take (u, f) a weak solution to the Vlasov-Navier-Stokes system starting at (u_0, f_0). We write

$$f_0 = f_0 \mathbf{1}_{|v|<R_{int}} + f_0 \mathbf{1}_{|v|\geq R_{ext}} := f_0^{int} + f_0^{ext},$$

and we denote by f^{int} (resp. f^{ext}) the renormalized solution to the Vlasov equation associated to the field u and starting at f_0^{int} (resp. starting at f_0^{ext}). Thanks to the well-posedness and the linearity of the Vlasov equation (for a fixed field u), we get

$$f = f^{int} + f^{ext}.$$

From Propositions 2.8.1 and 2.8.5, we get

$$\exists c > 0, \quad \forall t \geq 0, \quad d(\operatorname{supp} f^{int}(t), \partial\Omega) \geq c > 0,$$
$$\forall t \geq \log(2), \quad f^{ext}(t) = 0.$$

Furthermore, since $f(t) = f^{int}(t)$ after time $t \geq \log(2)$, we can apply Remark 2.8.4 so that there exists $\rho^\infty \in \mathrm{L}^\infty(\Omega)$ compactly supported in Ω such that

$$\mathrm{W}_{1,\overline{\Omega}}(\rho_f(t), \rho^\infty) \underset{t\to+\infty}{\longrightarrow} 0.$$

In particular, this weak convergence yields

$$\int_\Omega \rho_f(t)\,\mathrm{d}x \underset{t\to+\infty}{\longrightarrow} \int_\Omega \rho^\infty\,\mathrm{d}x.$$

But since we have conservation of the mass for the interior part f^{int}, we get for all $t \geq \log(2)$, omitting the variables,

$$\int_{\Omega} \rho_f(t)\,\mathrm{d}x = \int_{\Omega\times\mathbb{R}^3} f(t)\,\mathrm{d}x\,\mathrm{d}v = \int_{\Omega\times\mathbb{R}^3} f^{int}(t)\,\mathrm{d}x\,\mathrm{d}v = \int_{\Omega\times\mathbb{R}^3} f_0^{int}\,\mathrm{d}x\,\mathrm{d}v = \alpha,$$

from which we infer that the asymptotic profile ρ^∞ satisfies

$$\int_{\Omega} \rho^\infty\,\mathrm{d}x = \alpha.$$

The proof of Proposition 2.1.12 is therefore complete.

Appendix

2.A Boundary value problem in $\Omega \times \mathbb{R}^3$ for the kinetic equation

Theorem 2.A.1. *Take $f_0 \in \mathrm{L}^1 \cap \mathrm{L}^\infty(\Omega\times\mathbb{R}^3)$ and a vector field $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{W}^{1,1}(\Omega))$. Consider the following kinetic boundary value problem on $\Omega\times\mathbb{R}^3$.*

$$\begin{aligned} \partial_t f + v\cdot\nabla_x f + \mathrm{div}_v((u-v)f) = 0,\\ f_{|t=0} = f_0,\\ f = 0, \text{ on } \Sigma^-. \end{aligned}$$

Then we have, for all fixed $T>0$

- *Well-posedness: There exists a unique $f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^1\cap\mathrm{L}^\infty(\Omega\times\mathbb{R}^3))$ which is a weak solution of the previous Cauchy problem. Furthermore,*

$$f \in \mathscr{C}(\mathbb{R}^+;\mathrm{L}^p_{\mathrm{loc}}(\overline{\Omega}\times\mathbb{R}^3)),$$

for all $p\in[1,\infty)$ and the function f has a trace on $\partial\Omega\times\mathbb{R}^3$ defined in the following sense: there exists a unique element $\gamma f \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T]\times\partial\Omega\times\mathbb{R}^3)$ such that for any test function $\psi\in\mathscr{C}^\infty([0,T]\times\overline{\Omega}\times\mathbb{R}^3)$ with compact support in velocity, and for all $0\leq t_1\leq t_2\leq T$

$$\begin{aligned} &\int_{t_1}^{t_2}\int_{\Omega\times\mathbb{R}^3} f(t,x,v)\,[\partial_t\psi + v\cdot\nabla_x\psi + (u-v)\cdot\nabla_v\psi]\,(t,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\ &\quad= \int_{\Omega\times\mathbb{R}^3} f(t_2,x,v)\psi(t_2,x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_{\Omega\times\mathbb{R}^3} f(t_1,x,v)\psi(t_1,x,v)\,\mathrm{d}v\,\mathrm{d}x\\ &\qquad + \int_{t_1}^{t_2}\int_{\partial\Omega\times\mathbb{R}^3} [(\gamma f)\psi(t,x,v)]\,v\cdot n(x)\,\mathrm{d}v\,\mathrm{d}\sigma(x)\,\mathrm{d}t. \end{aligned}$$

- *Stability: If*

$$u_n \to u \quad in \quad \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^1(\Omega)) \quad and \quad f_{0,n}\to f_0 \quad in \quad \mathrm{L}^1_{\mathrm{loc}}(\Omega\times\mathbb{R}^3),$$

the corresponding sequence of solutions $(f_n)_n$ satisfies for all $p<\infty$,

$$f_n \longrightarrow f \quad in\ \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^p(\Omega\times\mathbb{R}^3)).$$

Such a result can be found in [BGM17, Theorem 3.2 - Proposition 3.2] for the well-posedness and renormalization properties and in [BF12, Theorem VI.1.9] for the stability property.

2.B Proof of Proposition 2.3.2

This Section aims at giving a proof for the representation formula (2.3.8), which holds for the weak solution to the Vlasov equation. We use the notations and definitions of Section 2.3.

For $(x,v) \in \Omega \times \mathbb{R}^3$ and for any $t \geq 0$, we first define

$$\tau^+(t,x,v) := \sup\{s \geq t \ \mid \forall \sigma \in [t,s],\ \mathrm{X}(\sigma;t,x,v) \in \Omega\}.$$

If $t \geq 0$ is fixed, we recall that (see (2.3.6))

$$\mathscr{O}^t := \Big\{(x,v) \in \Omega \times \mathbb{R}^3 \ \mid \forall \sigma \in [0,t],\ \mathrm{X}(\sigma;t,x,v) \in \Omega\Big\}.$$

If $t \geq 0$ is fixed, we observe that

$$\mathrm{Z}_{0,t}(\mathscr{O}^t) = \Big\{(x,v) \in \Omega \times \mathbb{R}^3 \ \mid \ \tau^+(0,x,v) > t\Big\} = \bigcap_{\sigma \in [0,t]} \mathrm{Z}_{0,\sigma}(\Omega \times \mathbb{R}^3).$$

By continuity, we have $\mathrm{X}_{\tau^+(0,z),0}(z) \in \partial\Omega$ if $z \in \Omega \times \mathbb{R}^3$ and $\tau^+(0,z) < +\infty$. More precisely, we have the following result.

Lemma 2.B.1. *For $z = (x,v) \in \Omega \times \mathbb{R}^3$, if $\tau^+(0,z) < +\infty$ then we have*

$$\mathrm{Z}_{\tau^+(0,z),0}(z) = (\mathrm{X}_{\tau^+(0,z),0}(z), \mathrm{V}_{\tau^+(0,z),0}(z)) \in \Sigma^+ \cup \Sigma^0. \tag{2.B.1}$$

Proof. Let us suppose by contradiction that $\mathrm{V}_{\tau^+(0,z),0}(z) \cdot n(\mathrm{X}_{\tau^+(0,z),0}(z)) < 0$. Note that since $z \in \Omega \times \mathbb{R}^3$, we have $\tau^+(0,z) > 0$. Since Ω is smooth, there exists $r > 0$ and $\Psi \in \mathscr{C}^1(\mathrm{B}(\mathrm{X}_{\tau^+(0,z),0}(z), r))$ such that for all $y \in \mathbb{R}^3$,

$$y \in \mathrm{B}\left(\mathrm{X}_{\tau^+(0,z),0}(z), r\right) \cap \Omega \Leftrightarrow \Psi(y) > 0,$$
$$y \in \mathrm{B}\left(\mathrm{X}_{\tau^+(0,z),0}(z), r\right) \cap \partial\Omega \Leftrightarrow \Psi(y) = 0.$$

For all $\tau > 0$ close to $\tau^+ := \tau^+(0,z)$, we have

$$\begin{aligned}\Psi\left(\mathrm{X}_{\tau,0}(z)\right) &= \Psi\left(\mathrm{X}_{\tau^+,0}(z)\right) + (\tau - \tau^+)\dot{\mathrm{X}}_{\tau^+,0}(z) \cdot \nabla\Psi\left(\mathrm{X}_{\tau^+,0}(z)\right) + o(\tau - \tau^+) \\ &= (\tau^+ - \tau)\mathrm{V}_{\tau^+,0}(z) \cdot n\left(\mathrm{X}_{\tau^+,0}(z)\right)\left|\nabla\Psi\left(\mathrm{X}_{\tau^+,0}(z)\right)\right| + o(\tau - \tau^+).\end{aligned}$$

Thus, for $\tau > 0$ close to τ^+ with $\tau < \tau^+$, we get $\Psi\left(\mathrm{X}_{\tau,0}(z)\right) < 0$. This means there exists $\varepsilon > 0$ such that for all $\tau \in (\tau^+(0,z) - \varepsilon, \tau^+(0,z)) \subset \mathbb{R}^+$, $\mathrm{X}_{\tau,0}(z) \notin \overline{\Omega}$, which is in contradiction with the definition of $\tau^+(0,z)$. □

We now turn to the proof of Proposition 2.3.2. Since the Vlasov equation has a unique solution when u is fixed (see Appendix 2.A), we have to check that

$$\begin{aligned}\mathbb{R}^+ \times \Omega \times \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ (t, \quad x, \quad v) &\longmapsto e^{3t}\mathbf{1}_{\mathscr{O}^t}(x,v) f_0(\mathrm{Z}_{0,t}(x,v)),\end{aligned}$$

is a weak solution to this equation associated to the velocity field u and starting at f_0. Let us fix $T > 0$. We thus take $\varphi \in \mathscr{C}_c^\infty([0,T] \times \overline{\Omega} \times \mathbb{R}^3)$ such that $\varphi(T) = 0$ and vanishing on $[0,T] \times (\Sigma^+ \cup \Sigma_0)$ and we want to show that

$$\int_0^T \int_{\Omega \times \mathbb{R}^3} e^{3s}\mathbf{1}_{\mathscr{O}^s}(x,v) f_0(\mathrm{Z}_{0,s}(x,v))\Big[\partial_t\varphi + v \cdot \nabla_x\varphi + (u - v) \cdot \nabla_v\varphi\Big](s,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s$$
$$= -\int_{\Omega \times \mathbb{R}^3} f_0(x,v)\varphi(0,x,v)\,\mathrm{d}v\,\mathrm{d}x.$$

First, we use the fact that

$$\mathrm{Z}_{0,s}(\mathcal{O}^s) = \Big\{(x,v) \in \Omega \times \mathbb{R}^3 \ \mid \ \tau^+(0,x,v) > s\Big\},$$

to write

$$\begin{aligned}
&\int_0^T \int_{\Omega\times\mathbb{R}^3} e^{3s}\mathbf{1}_{\mathcal{O}^s}(x,v) f_0(\mathrm{Z}_{0,s}(x,v))\Big[\partial_t\varphi + v\cdot\nabla_x\varphi + (u-v)\cdot\nabla_v\varphi\Big](s,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\
&\quad = \int_0^T \int_{\mathcal{O}^s} e^{3s} f_0(\mathrm{Z}_{0,s}(x,v))\Big[\partial_t\varphi + v\cdot\nabla_x\varphi + (u-v)\cdot\nabla_v\varphi\Big](s,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\
&\quad = \int_0^T \int_{\Omega\times\mathbb{R}^3} \mathbf{1}_{\tau^+(0,z)>s} e^{3s} f_0(z)\Big[\partial_t\varphi + v\cdot\nabla_x\varphi + (u-v)\cdot\nabla_v\varphi\Big](s,\mathrm{Z}_{s,0}(z))\mathrm{J}_s(z)\,\mathrm{d}z\,\mathrm{d}s,
\end{aligned}$$

where we have used the change of variable $z = \mathrm{Z}_{0,s}(x,v)$ and where $\mathrm{J}_s(z)$ stands for the Jacobian of the map $z \mapsto \mathrm{Z}_{s,0}(z)$, whose value is $\mathrm{J}_s(z) = e^{-3s}$ for all $s \geq 0$. Furthermore, by using

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big[\varphi(s,\mathrm{Z}_{s,0}(z))\Big] = \Big[\partial_t\varphi + v\cdot\nabla_x\varphi + (u-v)\cdot\nabla_v\varphi\Big](s,\mathrm{Z}_{s,0}(z)),$$

we get

$$\begin{aligned}
&\int_0^T \int_{\Omega\times\mathbb{R}^3} \mathbf{1}_{\tau^+(0,z)>s} e^{3s} f_0(z)\Big[\partial_t\varphi + v\cdot\nabla_x\varphi + (u-v)\cdot\nabla_v\varphi\Big](s,\mathrm{Z}_{s,0}(z))\mathrm{J}_s(z)\,\mathrm{d}z\,\mathrm{d}s \\
&\quad = \int_0^T \int_{\Omega\times\mathbb{R}^3} \mathbf{1}_{\tau^+(0,z)>s} f_0(z)\frac{\mathrm{d}}{\mathrm{d}s}\Big[\varphi(s,\mathrm{Z}_{s,0}(z))\Big]\,\mathrm{d}z\,\mathrm{d}s \\
&\quad = \int_0^T \int_{\Omega\times\mathbb{R}^3} \mathbf{1}_{\tau^+(0,z)>s} \frac{\mathrm{d}}{\mathrm{d}s}\Big[f_0(z)\varphi(s,\mathrm{Z}_{s,0}(z))\Big]\,\mathrm{d}z\,\mathrm{d}s \\
&\quad = \int_{\Omega\times\mathbb{R}^3} \left\{\int_0^{\min(T,\tau^+(0,z))} \frac{\mathrm{d}}{\mathrm{d}s}\Big[f_0(z)\varphi(s,\mathrm{Z}_{s,0}(z))\Big]\,\mathrm{d}s\right\}\,\mathrm{d}z,
\end{aligned}$$

where we have used Fubini's Theorem. We then write

$$\int_{\Omega\times\mathbb{R}^3} \left\{\int_0^{\min(T,\tau^+(0,z))} \frac{\mathrm{d}}{\mathrm{d}s}\Big[f_0(z)\varphi(s,\mathrm{Z}_{s,0}(z))\Big]\,\mathrm{d}s\right\}\,\mathrm{d}z = (\mathrm{I}) + (\mathrm{II}),$$

where

$$\begin{aligned}
(\mathrm{I}) &:= \int_{\Omega\times\mathbb{R}^3} \mathbf{1}_{\tau^+(0,z)>T}\left\{\int_0^{\min(t,\tau^+(0,z))} \frac{\mathrm{d}}{\mathrm{d}s}\Big[f_0(z)\varphi(s,\mathrm{Z}_{s,0}(z))\Big]\,\mathrm{d}s\right\}\,\mathrm{d}z, \\
(\mathrm{II}) &:= \int_{\Omega\times\mathbb{R}^3} \mathbf{1}_{\tau^+(0,z)\leq T}\left\{\int_0^{\min(t,\tau^+(0,z))} \frac{\mathrm{d}}{\mathrm{d}s}\Big[f_0(z)\varphi(s,\mathrm{Z}_{s,0}(z))\Big]\,\mathrm{d}s\right\}\,\mathrm{d}z.
\end{aligned}$$

First, we have

$$(\mathrm{I}) = \int_{\Omega\times\mathbb{R}^3} \mathbf{1}_{\tau^+(0,z)>T} f_0(z)\varphi(T,\mathrm{Z}_{T,0}(z))\,\mathrm{d}z - \int_{\Omega\times\mathbb{R}^3} \mathbf{1}_{\tau^+(0,z)>T} f_0(z)\varphi(0,z)\,\mathrm{d}z.$$

Since $\varphi(T) = 0$, the first integral vanishes and we obtain

$$(\mathrm{I}) = -\int_{\Omega\times\mathbb{R}^3} \mathbf{1}_{\tau^+(0,z)>T} f_0(z)\varphi(0,z)\,\mathrm{d}z.$$

For the second term, we have

$$(\mathrm{II}) = \int_{\Omega\times\mathbb{R}^3} \mathbf{1}_{\tau^+(0,z)\leq T} f_0(z)\varphi(\tau^+(0,z), \mathrm{Z}_{\tau^+(0,z),0}(z))\,\mathrm{d}z - \int_{\Omega\times\mathbb{R}^3} \mathbf{1}_{\tau^+(0,z)\leq T} f_0(z)\varphi(0,z)\,\mathrm{d}z.$$

Thanks to (2.B.1) and to the fact that φ vanishes on $\Sigma^+\cup\Sigma^0$, we see that the first integral is actually 0.

Eventually, gathering all the previous pieces, we end up with

$$(\mathrm{I}) + (\mathrm{II}) = -\int_{\Omega\times\mathbb{R}^3} f_0(x,v)\varphi(0,x,v)\,\mathrm{d}v\,\mathrm{d}x.$$

The proof of Proposition (2.3.2) is finally complete.

2.C The Wasserstein distance

In this section, X stands for a separable and complete subset of $\mathbb{R}^d$ or $\mathbb{R}^d\times\mathbb{R}^d$ (in the previous sections, we used $X=\overline{\Omega}$ or $X=\overline{\Omega}\times\mathbb{R}^d$ where Ω is an open subset of $\mathbb{R}^d$).

We recall the definition of the 1-Wasserstein distance on X and the useful and classical Monge-Kantorovich formula (see [Dud18, Section 11.8] for instance).

Definition 2.C.1. *For all $m>0$, we define $\mathcal{M}_{1,m}(X)$ the set of positive measures μ on X such that*

$$\int_X |x|\,\mathrm{d}\mu(x) < \infty, \qquad \mu(X) = m.$$

Definition 2.C.2. *For all $m>0$, if μ et ν are two measures belonging to $\mathcal{M}_{1,m}(X)$, we define the Wasserstein distance $\mathrm{W}_1(\mu,\nu)$ as the quantity*

$$\mathrm{W}_1(\mu,\nu) := \inf_{\gamma\in\Pi(\mu,\nu)} \int_{X\times X} |x-x'|\,\mathrm{d}\gamma(x,x'),$$

where $\Pi(\mu,\nu)$ stands for the set of positive measures on $X\times X$ whose first marginal is μ and second marginal ν.

Proposition 2.C.3. *Fix $m>0$. Given $(\mu_n)_n\in\mathcal{M}_{1,m}(X)^{\mathbb{N}}$ and $\mu\in\mathcal{M}_{1,m}(X)$, the two following statements are equivalent*

(i) For all $f\in\mathscr{C}_b(X)$,

$$\int_X (f(z)+|z|)\,\mathrm{d}\mu_n(z) \overset{n\to+\infty}{\longrightarrow} \int_X (f(z)+|z|)\,\mathrm{d}\mu(z).$$

(ii) $(\mathrm{W}_1(\mu_n,\mu))_n \overset{n\to+\infty}{\longrightarrow} 0$.

Theorem 2.C.4 (Duality formula of Monge-Kantorovich)**.** *If μ et ν are two measures belonging to $\mathcal{M}_{1,m}(X)$, we have the following formula*

$$\mathrm{W}_1(\mu,\nu) = \sup\left\{\left|\int_X \psi(x)\mathrm{d}\mu(x) - \int_X \psi(x)\mathrm{d}\nu(x)\right| \;\middle|\; \psi\in\mathrm{Lip}(X),\quad \|\nabla\psi\|_\infty\leq 1\right\}.$$

2.D Gagliardo-Nirenberg-Sobolev inequality and Agmon inequality on a bounded domain of $\mathbb{R}^3$

Theorem 2.D.1 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Let Ω be a smooth bounded domain of $\mathbb{R}^d$. Let $1 \leq p, q, r \leq \infty$ and $m \in \mathbb{N}$. Suppose $j \in \mathbb{N}$ and $\alpha \in [0, 1]$ satisfy the relations*

$$\frac{1}{p} = \frac{j}{d} + \left(\frac{1}{r} - \frac{m}{d}\right)\alpha + \frac{1-\alpha}{q},$$
$$\frac{j}{m} \leq \alpha \leq 1,$$

with the exception $\alpha < 1$ if $m - j - d/r \in \mathbb{N}$.

Then for all $g \in \mathrm{L}^q(\Omega)$, if $\mathrm{D}^m g \in \mathrm{L}^r(\Omega)$, we have $\mathrm{D}^j g \in \mathrm{L}^p(\Omega)$ with the estimate

$$\|\mathrm{D}^j g\|_{\mathrm{L}^p(\Omega)} \lesssim \|\mathrm{D}^m g\|^{\alpha}_{\mathrm{L}^r(\Omega)} \|g\|^{1-\alpha}_{\mathrm{L}^q(\Omega)} + \|g\|_{\mathrm{L}^s(\Omega)},$$

where $1 \leq s \leq \max\{q, r\}$ and where $\lesssim$ only depends on Ω.

Moreover, if g has a vanishing trace at $\partial\Omega$, we can drop the last term $\|g\|_{\mathrm{L}^s(\Omega)}$ in the r.h.s of the inequality.

This result can be found in [CM12, Thm 1.5.2].

Proposition 2.D.2 (Agmon inequality)**.** *Let Ω be a smooth bounded domain of $\mathbb{R}^3$. For all $u \in \mathrm{H}^2(\Omega)$, we have the inequality*

$$\|u\|_{\mathrm{L}^\infty(\Omega)} \lesssim \|u\|^{1/2}_{\mathrm{H}^1(\Omega)} \|u\|^{1/2}_{\mathrm{H}^2(\Omega)},$$

where $\lesssim$ only depends on Ω.

We refer to [CF88, Lemma 4.10] for a proof.

2.E Maximal $\mathrm{L}^p\mathrm{L}^q$ regularity for the Stokes system on a bounded domain

Let Ω be a smooth bounded domain of $\mathbb{R}^d$ and $1 < q < \infty$. Each vector field $u \in \mathrm{L}^q(\Omega)$ is uniquely decomposed as

$$u = \widetilde{u} + \nabla p,$$
$$\widetilde{u} \in \mathrm{L}^q_{\mathrm{div}}(\Omega),\ p \in \mathrm{L}^q(\Omega),\ \nabla p \in \mathrm{L}^q(\Omega),$$

where $\mathrm{L}^q_{\mathrm{div}}(\Omega)$ stands for the closure in $\mathrm{L}^q(\Omega)$ of $\mathscr{D}_{\mathrm{div}}(\Omega)$. In this so called Helmoltz decomposition, we recall that the projection $\mathbb{P}_q : u \mapsto \widetilde{u}$ is continuous from $\mathrm{L}^q(\Omega)$ to $\mathrm{L}^q_{\mathrm{div}}(\Omega)$.

For $1 < q < \infty$, we consider the following Stokes operator:

$$A_q := -\mathbb{P}_q \Delta u, \quad D(A_q) := \mathrm{L}^q_{\mathrm{div}}(\Omega) \cap \mathrm{W}^{1,q}_0(\Omega) \cap \mathrm{W}^{2,q}(\Omega).$$

We also set

$$\mathrm{D}^{1-\frac{1}{s},s}_q(\Omega) := \left(D(A_q), \mathrm{L}^q_{\mathrm{div}}(\Omega)\right)_{1/s,s},$$

where $(\ ,\ )_{1/s,s}$ refers to the real interpolation space of exponents $(1/s, s)$.

Theorem 2.E.1. *Consider $0 < T \leq \infty$ and $1 < q, s < \infty$. Then, for every $u_0 \in D_q^{1-\frac{1}{s},s}(\Omega)$ which is divergence free and $f \in \mathrm{L}^s(0,T;\mathrm{L}^q_{\mathrm{div}}(\Omega))$, there exists a unique solution u of the Stokes system*

$$\begin{aligned} \partial_t u + A_q u &= f, \\ u(0,x) &= u_0(x), \end{aligned}$$

satisfying

$$\begin{aligned} &u \in \mathrm{L}^s(0,T';D(A_q)) \text{ for all finite } T' \leq T, \\ &\partial_t u \in \mathrm{L}^s(0,T;\mathrm{L}^q(\Omega)), \end{aligned}$$

and

$$\|\partial_t u\|_{\mathrm{L}^s(0,T;\mathrm{L}^q(\Omega))} + \|\mathrm{D}^2 u\|_{\mathrm{L}^s(0,T;\mathrm{L}^q(\Omega))} \leq C\left(\|u_0\|_{D_q^{1-\frac{1}{s},s}(\Omega)} + \|f\|_{\mathrm{L}^s(0,T;\mathrm{L}^q(\Omega))}\right),$$

where $C = C(q,s,\Omega)$. Furthermore, if $u_0 \in \mathrm{H}^1_{\mathrm{div}}(\Omega)$ and if $s \in (1,2)$, the statement holds and we can replace $\|u_0\|_{D_q^{1-1/s,s}(\Omega)}$ by $\|u_0\|_{\mathrm{H}^1_0(\Omega)}$ in the right hand side of the previous inequality.

A proof of this result and further references on the theory can be found in [GS91]. The last statement about the fact that $\mathrm{H}^1_{\mathrm{div}}(\Omega) = D(A^{\frac{1}{2}}) \hookrightarrow D_q^{1-1/s,s}(\Omega)$ for $s \in (1,2)$ comes from [GS91, Remark 2.5].

2.F Parabolic regularization for the Navier-Stokes system with a source term on a bounded domain

Theorem 2.F.1. *Let Ω be a regular bounded domain of $\mathbb{R}^3$. There exist two universal constants $C_1, C_2 > 0$ such that the following holds. Consider $u_0 \in \mathrm{H}^1_{\mathrm{div}}(\Omega)$ and $F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\Omega))$ and $T > 0$ such that*

$$\|\nabla u_0\|^2_{\mathrm{L}^2(\Omega)} + C_1 \int_0^T \|F(s)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s \leq \frac{1}{\sqrt{8C_1C_2}}. \tag{2.F.1}$$

Then, there exists on $[0,T]$ a unique Leray solution to the Navier-Stokes equations with initial data u_0 and source F. This solution u belongs to $\mathrm{L}^\infty(0,T;\mathrm{H}^1_{\mathrm{div}}(\Omega)) \cap \mathrm{L}^2(0,T;\mathrm{H}^2(\Omega))$ and satisfies for a.e. $0 \leq t \leq T$

$$\|\nabla u(t)\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2}\int_0^t \|A_2 u(s)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s \leq \|\nabla u_0\|^2_{\mathrm{L}^2(\Omega)} + C_1 \int_0^t \|F(s)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s, \tag{2.F.2}$$

where A_2 stand for the Stokes operator on $\mathrm{L}^2(\Omega)$.

Proof. First, if such a Leray solution to the Navier-Stokes equations exists, we have directly $u \in \mathrm{L}^4(0,T;\mathrm{H}^1_{\mathrm{div}}(\Omega))$, which corresponds to a classical tridimensional case of weak-strong uniqueness (see for instance [CDGG06, Theorem 3.3]). Hence, it remains to prove that such a solution does exist on $[0,T]$. We proceed in the following way.

We rely on an approximation procedure by regularising the data F and u_0 and by considering a standard Galerkine approximation $(u_N)_N$ of the corresponding Navier-Stokes system. The classical idea is to obtain the desired parabolic estimations on the sequence $(u_N)_N$ on $[0,T]$. Combining these new estimates with the classical energy estimates for the Leray solution of the Navier-Stokes system on $[0,T]$, we can use a compactness argument to produce a solution with the $\mathrm{L}^\infty(0,T;\mathrm{H}^1_{\mathrm{div}}(\Omega)) \cap \mathrm{L}^2(0,T;\mathrm{H}^2(\Omega))$ regularity and which satisfies the estimate (2.F.2) on $[0,T]$.

Thus, in the following, it is sufficient to work with a smooth solution u and with smooth data. We first apply the Leray projection $\mathbb{P}$ to the Navier-Stokes equations and multiply this by Au to obtain, as in the proof of Proposition 2.4.8

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2_{\mathrm{L}^2(\Omega)} + \|Au\|^2_{\mathrm{L}^2(\Omega)} \leq C_1\|F\|^2_{\mathrm{L}^2(\Omega)} + C_1\|\nabla u\|^6_{\mathrm{L}^2(\Omega)}, \tag{2.F.3}$$

where $C_1 > 0$. Moreover, by Poincaré inequality, there exists an another universal constant $C_2 > 0$ such that $\|\nabla u\|^2_{\mathrm{L}^2(\Omega)} \leq C_2\|Au\|^2_{\mathrm{L}^2(\Omega)}$. So we rewrite the inequality (2.F.3) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2}\|Au\|^2_{\mathrm{L}^2(\Omega)} \leq C_1\|F\|^2_{\mathrm{L}^2(\Omega)} + C_1\|\nabla u\|^6_{\mathrm{L}^2(\Omega)} - \frac{1}{2C_2}\|\nabla u\|^2_{\mathrm{L}^2(\Omega)},$$

on $[0,T]$. Now, if we set

$$x(t) := \|\nabla u(t)\|^2_{\mathrm{L}^2(\Omega)}, \qquad y(t) := \|Au(t)\|^2_{\mathrm{L}^2(\Omega)}, \qquad z(t) := \|F(t)\|^2_{\mathrm{L}^2(\Omega)}, \tag{2.F.4}$$

and if we integrate the previous inequality between 0 and t, we infer that for almost every $0 \leq t \leq T$,

$$x(t) + \frac{1}{2}\int_0^t y(s)\,\mathrm{d}s \leq x(0) + \int_0^t C_1 z(s)\,\mathrm{d}s + \int_0^t C_1 x(s)\left(x(s)^2 - \frac{1}{2C_1C_2}\right)\mathrm{d}s. \tag{2.F.5}$$

To obtain (2.F.2), it is sufficient to prove that $x(t) \leq 1/\sqrt{2C_1C_2}$ for $t \in [0,T)$. To do so, we remark that the assumption (2.F.1) precisely corresponds to

$$x(0) + \int_0^t C_1 z(s)\,\mathrm{d}s \leq \frac{1}{2\sqrt{2C_1C_2}},$$

and in particular, $x(0) \leq 1/2\sqrt{2C_1C_2}$. So, by continuity of $t \mapsto x(t)$, there exists a maximal time $T_1 \in]0,T]$ such that $x(t) \leq 1/\sqrt{2C_1C_2}$ on $[0,T_1[$. If $T = T_1$, there is nothing to do. So, we argue by contradiction by assuming that $T_1 < T$. We thus have for all $0 \leq s \leq T_1$

$$x(s)^2 - \frac{1}{2C_1C_2} \leq 0,$$

and the inequality (2.F.5) with $t = T_1$ turns into

$$x(T_1) + \frac{1}{2}\int_0^{T_1} y(s)\,\mathrm{d}s \leq x(0) + \int_0^{T_1} C_1 z(s)\,\mathrm{d}s, \leq x(0) + \int_0^{T} C_1 z(s)\,\mathrm{d}s,$$

which implies in particular

$$x(T_1) \leq \frac{1}{2\sqrt{2C_1C_2}} < \frac{1}{\sqrt{2C_1C_2}}.$$

Again by continuity of $t \mapsto x(t)$, there exists $\varepsilon > 0$ such that for all $t \in [T_1, T_1+\varepsilon]$, $x(t) \leq 1/2\sqrt{C_1C_2}$, which is a contradiction with the definition of T_1. It thus implies that for all $t \in [0,T)$, $x(t) \leq 1/\sqrt{2C_1C_2}$ and we finally end up with the following inequality on $[0,T]$

$$x(t) + \frac{1}{2}\int_0^t y(s)\,\mathrm{d}s \leq x(0) + \int_0^t C_1 z(s)\,\mathrm{d}s \leq \frac{1}{2\sqrt{2C_1C_2}}.$$

By a view of the definition (2.F.4), this finally proves the inequality (2.F.2). □

Chapter 3

Long-time behaviour for the Vlasov-Navier-Stokes system with gravity on a half-space

Based on the article [Ert21], prepublished and submitted in a journal.

3.1 Introduction

Fluid-kinetic systems aim at modelling the collective motion of a dispersed phase of small particles immersed within a fluid. In such systems, also called spray models, the dispersed phase is described at a mesoscopic level by a distribution function solving a kinetic equation while the evolution of macroscopic quantities for the fluid is governed by fluid mechanics equations.

Among the wide family of fluid-kinetic systems (see the pioneering works [Wil85, O'R81]), one can consider the so-called *thin spray* models where the particles volume fraction is small compared to that of the surrounding fluid. In this context, an interesting prototype is the incompressible Vlasov-Navier-Stokes system, coupling in a nonlinear way the fluid and kinetic equations through a drag term. The later one depends on the fluid unknowns and on the density function and allows for an exchange of momentum between the fluid and the particles. Beyond its mathematical interest, this system also appears in the study of the transport and deposition of a therapeutic aerosol in the airflows contained in the human upper airways (see [BGLM15]).

In this chapter, we are interested in the following Vlasov-Navier-Stokes system set in $\mathbb{R}^3_+ \times \mathbb{R}^3$:

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v[f(u-v) + fG] = 0, \qquad (t,x,v) \in \mathbb{R}^*_+ \times \mathbb{R}^3_+ \times \mathbb{R}^3, \tag{3.1.1}$$

$$\partial_t u + (u \cdot \nabla) u - \Delta u + \nabla p = \int_{\mathbb{R}^3} f(t,x,v)(v - u(t,x))\,\mathrm{d}v, \qquad (t,x) \in \mathbb{R}^*_+ \times \mathbb{R}^3_+, \tag{3.1.2}$$

$$\mathrm{div}\, u = 0, \qquad (t,x) \in \mathbb{R}^*_+ \times \mathbb{R}^3_+. \tag{3.1.3}$$

Here $\mathbb{R}^3_+ := \mathbb{R}^2 \times (0, +\infty)$ is the tridimensional half-space while $G := (0,0,-g) \in \mathbb{R}^3$ is a given vector, with $g > 0$ the constant gravitational acceleration. In these equations, $u = u(t,x) \in \mathbb{R}^3$ and $p = p(t,x) \in \mathbb{R}$ stand for the velocity field and pressure of the fluid, while $f = f(t,x,v) \in \mathbb{R}^+$ is the distribution function of the particles in the phase space $\mathbb{R}^3_+ \times \mathbb{R}^3$. Here, the particles undergo the friction force produced by the surrounding fluid, as well as the effect of gravity. Thus, using the Stokes law, the resultant force exerted on the particles is the sum of the drag and weight/buoyancy, that is

$$u(t,x) - v + G,$$

and the Vlasov equation (3.1.1) is thus coupled to the Navier-Stokes equations (3.1.2)-(3.1.3). Note that we have implicitely considered spherical particles of same radius and that the mass density of the particles is greater than that of the fluid (because of the positive coefficient before the vector G - see e.g. [CG06]).

A coupling term is also added in the Navier-Stokes equations (3.1.2)-(3.1.3), where a forcing term appears in the r.h.s and stems from the retroaction of the particles on the fluid. This source term is usually called the *Brinkman force* and can be rewritten as

$$\int_{\mathbb{R}^3} f(v-u)\,\mathrm{d}v = j_f - \rho_f u, \tag{3.1.4}$$

where

$$\rho_f(t,x) := \int_{\mathbb{R}^3} f(t,x,v)\,\mathrm{d}v,$$
$$j_f(t,x) := \int_{\mathbb{R}^3} v f(t,x,v)\,\mathrm{d}v.$$

Note that in the previous Navier-Stokes equations (3.1.2)-(3.1.3), the density and viscosity of the fluid are assumed to be constant and both chosen equal to 1, while the external gravity force

$G = -\nabla\Phi$, with $\Phi = gz$, has been absorbed in the pressure term. In short, the equations (3.1.1)-(3.1.2)-(3.1.3) account for the description of a cloud of fine particles sedimenting in an ambient incompressible viscous fluid.

The main goal of this chapter is to study the asymptotics in large time of small-data solutions to the Vlasov-Navier-Stokes system, relying on some specific boundary conditions that we shall detail below. The analysis of this system has been explored in different directions over the past two decades.

The Cauchy problem. Concerning the existence theory of global weak solutions to the system (3.1.1)-(3.1.2)-(3.1.3) (without the gravity force), different settings and boundary conditions for the distribution function have been adressed, depending on the spatial domain: a fixed bounded domain with specular reflexion in [ABdMB97], the flat torus $\mathbb{T}^3$ in [BDGM09], time-dependent domain with absorption boundary condition in [BGM17, BMM20], or a 2D rectangle with partly absorbing boundary condition in [GHKM18]. Local strong solutions can also be considered as in [CK15] (for inhomogeneous fluid equations), as well as blow-up in finite time of classic solutions in [Cho17].

Note that the additional term involving the gravity has only been taken into account for the Vlasov-Stokes system on bounded domain with specular reflexion in [Ham98], or for the Vlasov equation coupled to the stationary Stokes system with a regular and compactly supported initial distribution function on $\mathbb{R}^3$ in [Hö18].

The 2D case. In two dimensions, more results are available for this fluid-kinetic system, essentially because of the study of the Navier-Stokes system which is more favorable in this context: for instance, uniqueness of 2D-global weak solutions is proven for the whole space or the torus case in [HKMMM20]. The controllabillity of the system is also explored in dimension 2 in [Moy16].

Link to other models. In the spirit of Hilbert's 6th problem of axiomatization of physics, fluid-kinetic models can be linked to other systems of ODEs and PDEs. Deriving rigorously the Vlasov-Navier-Stokes system from "first laws" appears as an important issue, but remains essentially an outstanding open problem for the whole system. We refer to Section 1.3.3 on the Introduction for more details.

Through *hydrodynamic limits* of the Vlasov-Navier-Stokes system, one can also seek to derive some systems involving only averaged quantities: more precisely, high friction regimes of the system have been shown to lead to Navier-Stokes type systems. These asymptotic regimes have been first considered in [GJV04a, GJV04b] for the Vlasov-Fokker-Planck-Navier-Stokes equations, where the effect of Brownian motion is added in the equation of the distribution function.

Without diffusion in velocity in the Vlasov equation, one of these limits has been handled in [Hö18] for the Vlasov-(steady)Stokes system with gravity in the whole space. Very recently, the question raised by these different regimes has been adressed in [HKMar] for the full system (3.1.1)-(3.1.2)-(3.1.3) on the torus (without the gravity force).

Large time behavior. The *large time dynamics* of the Vlasov-Navier-Stokes system, which is the main issue of this chapter, has very recently received a particular attention. This natural question is studied for the first time by Jabin in [Jab00b] for a reduced kinetic model. In the absence of dissipative mechanism in the Vlasov equation (like a Fokker-Planck operator allowing to consider smooth equilibria, see [GHMZ10]), the sole effect of the drag force in the system should lead to nontrivial equilibria which are singular. More precisely, one expects a monokinetic behavior for the distribution function of the particles (that is to say, a convergence towards a Dirac mass in velocity). A conditional result accounting for this phenomenon in the Vlasov-Navier-Stokes system has been provided by Choi and Kwon in [CK15]. In short, it requires a global bound in time which is not *a priori* satisfied by global weak solutions to the system. More recently, this extra assumption has been removed for initial data which are in some sense close to equilibrium: the first complete result stems from the article [HKMM20] of Han-Kwan, Moussa and Moyano where the authors work in a periodic setting and in a framework *à la* Fujita-Kato. In the same spirit, such a monokinetic behavior of weak solutions has been obtained for small data in the whole space case by Han-Kwan in [HK22] and then extended to the case of bounded domains with absorption boundary conditions in [EHKM21] (see the Chapter 2). In short, these results are all based on a remarkable energy-dissipation inequality satisfied by weak solutions to the system.

Unlike this series of works, the existence and stability of regular equilibria has been obtained by Glass, Han-Kwan and Moussa in [GHKM18] for a particular 2D bounded domain with partly absorbing and injection boundary conditions. Finally, let us emphasize the fact that the high friction limit tackled by Han-Kwan and Michel in [HKMar] is closely linked to the monokinetic behavior we have mentioned just before, and in particular to the techniques used in [HKMM20, HK22].

Main contribution of this chapter. In the continuation of these previous works, the main goal of this chapter is the study of the large time dynamics of global weak solutions to the system (3.1.1)-(3.1.2)-(3.1.3). Its originality lies in dealing with a fluid-kinetic system on an unbounded domain with boundary, where one considers absorption boundary conditions for the distribution function, together with an additional gravity force term.

Loosely speaking, the presence of a gravity force may ruin the decay of the energy of the system. At first sight, this prevents the use of exactly the same techniques as in [HKMM20, HK22, EHKM21]. However, it is actually possible to take advantage of the absorption at the boundary to analyse the large time behavior of global weak solutions starting close to equilibrium.

Before going further, we give several definitions and set notations about the system that we will consider in this chapter. In what follows, we will sometimes refer to (3.1.1)-(3.1.2)-(3.1.3) as the *VNS system*. Along this chapter, we will make a constant use of the notation

$$\mathbb{R}^3_+ := \mathbb{R}^2 \times (0, +\infty).$$

First, the VNS system is supplemented with the following initial conditions for u and f:

$$u_{|t=0} = u_0 \text{ in } \mathbb{R}^3_+, \tag{3.1.5}$$

$$f_{|t=0} = f_0 \text{ in } \mathbb{R}^3_+ \times \mathbb{R}^3. \tag{3.1.6}$$

We prescribe the following Dirichlet boundary conditions for the fluid:

$$u(t, \cdot) = 0, \text{ on } \partial\mathbb{R}^3_+ = \mathbb{R}^2 \times \{0\}. \tag{3.1.7}$$

We also need to introduce the following outgoing/incoming phase-space boundaries:

$$\Sigma^\pm := \Big\{(x, v) \in \partial\mathbb{R}^3_+ \times \mathbb{R}^3 \mid \pm v \cdot n(x) > 0\Big\}, \tag{3.1.8}$$

$$\Sigma_0 := \Big\{(x, v) \in \partial\mathbb{R}^3_+ \times \mathbb{R}^3 \mid v \cdot n(x) = 0\Big\}, \tag{3.1.9}$$

$$\Sigma := \Sigma^+ \sqcup \Sigma^- \sqcup \Sigma_0 = \partial\mathbb{R}^3_+ \times \mathbb{R}^3, \tag{3.1.10}$$

where $n(x)$ stands for the normal vector to the boundary $\partial\mathbb{R}^3_+$ at point x. We observe that

$$\Sigma^\pm = \Big\{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x_3 = 0, \ \pm v_3 < 0\Big\},$$

$$\Sigma_0 = \Big\{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x_3 = 0, \ v_3 = 0\Big\}.$$

Then, we prescribe the following absorption boundary conditions for the distribution function:

$$f(t,\cdot,\cdot) = 0, \text{ on } \Sigma^-, \tag{3.1.11}$$

meaning that particles reaching transversally the physical boundary $\{x_3 = 0\}$ are absorbed.

Several functionals play an important role in the study of the VNS system. We introduce the following ones.

Definition 3.1.1. *1. The* ***kinetic energy*** *of the Vlasov-Navier-Stokes system is defined for all $t \geq 0$ as:*

$$\mathrm{E}(t) := \frac{1}{2}\|u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \frac{1}{2}\int_{\mathbb{R}^3_+\times\mathbb{R}^3} f(t,x,v)|v|^2 \,\mathrm{d}x\,\mathrm{d}v. \tag{3.1.12}$$

2. The ***dissipation*** *of the Vlasov-Navier-Stokes system (without gravity) is defined for all $t \geq 0$ as:*

$$\mathrm{D}(t) := \int_{\mathbb{R}^3_+\times\mathbb{R}^3} f(t,x,v)|u(t,x)-v|^2 \,\mathrm{d}x\,\mathrm{d}v + \|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}. \tag{3.1.13}$$

3. The ***dissipation with gravity*** *of the Vlasov-Navier-Stokes system is defined for all $t \geq 0$ as:*

$$\mathrm{D}_G(t) := \int_{\mathbb{R}^3_+\times\mathbb{R}^3} f(t,x,v)|u(t,x)-v|^2 \,\mathrm{d}x\,\mathrm{d}v + \|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} - \int_{\mathbb{R}^3_+} G\cdot j_f(t,x)\,\mathrm{d}x. \tag{3.1.14}$$

At a formal level, the VNS system enjoys an energy-dissipation structure involving the previous functionals. More precisely, smooth solutions to the system satisfy the following *a priori* estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) + \mathrm{D}_G(t) \leq 0.$$

Thus, there is a variation of energy coming from the dissipation inside of the fluid and because of the friction between the particles and the fluid, but the gravity force induces an additional term leading to a potential non-decay of the kinetic energy E: indeed, we expect the term $G\cdot j_f > 0$ to be positive at least after some time because the particles should ultimately fall in the same direction as the gravity.

We denote by $\mathscr{D}_{\mathrm{div}}(\mathbb{R}^3_+)$ the set of smooth $\mathbb{R}^3$ valued divergence free vector-fields having compact support in $\mathbb{R}^3_+$. The closures of $\mathscr{D}_{\mathrm{div}}(\mathbb{R}^3_+)$ in $\mathrm{L}^2(\mathbb{R}^3_+)$ and in $\mathrm{H}^1(\mathbb{R}^3_+)$ are respectively denoted by $\mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^3_+)$ and by $\mathrm{H}^1_{0,\mathrm{div}}(\mathbb{R}^3_+)$. We write $\mathrm{H}^{-1}_{\mathrm{div}}(\mathbb{R}^3_+)$ for the dual of the later.

We now define the class of admissible initial data for the VNS system.

Definition 3.1.2 (Initial condition)**.** *We shall say that a couple (u_0, f_0) is an admissible initial condition if:*

$$u_0 \in \mathrm{L}^2(\mathbb{R}^3_+), \quad \mathrm{div}_x\, u_0 = 0, \tag{3.1.15}$$

$$f_0 \in \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^3_+\times\mathbb{R}^3), \tag{3.1.16}$$

$$f_0 \geq 0, \quad \int_{\mathbb{R}^3_+\times\mathbb{R}^3} f_0 \,\mathrm{d}x\,\mathrm{d}v = 1, \tag{3.1.17}$$

$$(x,v) \mapsto f_0(x,v)|v|^2 \in \mathrm{L}^1(\mathbb{R}^3_+\times\mathbb{R}^3). \tag{3.1.18}$$

We then introduce some notations about the moments of any phase-space distribution function.

Definition 3.1.3. *For any $\alpha \geq 0$ and any measurable function $g : \mathbb{R}^+ \times \mathbb{R}^3_+ \times \mathbb{R}^3 \to \mathbb{R}^+$, we set*

$$m_\alpha g(t,x) := \int_{\mathbb{R}^3} |v|^\alpha g(t,x,v)\,\mathrm{d}v,$$
$$M_\alpha g(t) := \int_{\mathbb{R}^3_+\times\mathbb{R}^3} |v|^\alpha g(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{R}^3_+} m_\alpha g(t,x)\,\mathrm{d}x.$$

In our approach, we shall rely on some decay assumptions satisfied by the initial distribution function f_0. We thus introduce the following quantities.

Definition 3.1.4. *For any $q > 0$, $m > 0$ and $r \geq 1$, we set*

$$N_q(f_0) := \|(1+|v|^q)f_0\|_{\mathrm{L}^\infty(\mathbb{R}^3_+\times\mathbb{R}^3)}, \tag{3.1.19}$$

$$K_{q,r}(f_0) := \Big\|(1+|v|^q)\|f_0(\cdot,v)\|_{\mathrm{L}^r_x(\mathbb{R}^3_+)}\Big\|_{\mathrm{L}^\infty_v(\mathbb{R}^3)}, \tag{3.1.20}$$

$$H_{q,m}(f_0) := \Big\|(1+|v|^q)\|(1+x_3^m)f_0(\cdot,v)\|_{\mathrm{L}^\infty_x(\mathbb{R}^3_+)}\Big\|_{\mathrm{L}^1_v(\mathbb{R}^3)}, \tag{3.1.21}$$

$$F_{q,m,r}(f_0) := \Big\|(1+|v|^q)\|(1+x_3^m)f_0(\cdot,v)\|_{\mathrm{L}^r_x(\mathbb{R}^3_+)}\Big\|_{\mathrm{L}^1_v(\mathbb{R}^3)}. \tag{3.1.22}$$

We will consider weak solutions to the Vlasov equation with gravity force (3.1.1), with the boundary condition (3.1.11) and the previous initial conditions, which are defined as follows.

Definition 3.1.5 (Weak solutions to the Vlasov equation)**.** *Let $\mathrm{U} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^3_+)$ and an initial distribution f_0 satisfying (3.1.16)-(3.1.17)-(3.1.18). We say that a nonnegative function $f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^3_+ \times \mathbb{R}^3))$ is a weak solution to the Vlasov equation (3.1.1) with force field U, with boundary condition (3.1.11) and with initial condition f_0 if, for all $\Psi \in \mathscr{D}([0,T] \times \overline{\mathbb{R}^3_+} \times \mathbb{R}^3)$ with $\Psi(T,\cdot) = 0$ and vanishing on $\mathbb{R}^+ \times (\Sigma^+ \cup \Sigma_0)$, one has*

$$\int_0^T \int_{\mathbb{R}^3_+\times\mathbb{R}^3} f(t,x,v)\left[\partial_t\Psi + v\cdot\nabla_x\Psi + (\mathrm{U}-v)\cdot\nabla_v\Psi\right](t,x,v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t$$
$$= -\int_{\mathbb{R}^3_+\times\mathbb{R}^3} f_0(x,v)\Psi(0,x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

Weak solutions to the Vlasov equations enter in the framework of the DiPerna-Lions theory for transport equations (in the phase space $\mathbb{R}^3_+ \times \mathbb{R}^3$). We refer to Section 3.A in the Appendix for more details, where we recall in particular the classic stability property of renormalized solutions which we will constantly use throughout this chapter.

We will also consider weak solutions for the full Vlasov-Navier-Stokes with the boundary conditions (3.1.7)-(3.1.11) and the initial conditions described in (3.1.2), in the following sense.

Definition 3.1.6 (Weak solutions with strong energy inequality for the VNS system)**.** *Consider an admissible initial condition (u_0, f_0) in the sense of Definition 3.1.2. A global weak solution to the Vlasov-Navier-Stokes system with boundary condition (3.1.7)-(3.1.11) and with initial condition (u_0, f_0) is a pair (u, f) such that:*

$$u \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^3_+)) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}^1_{0,\mathrm{div}}(\mathbb{R}^3_+)), \tag{3.1.23}$$

$$\mathrm{div}_x\, u = 0, \tag{3.1.24}$$

$$f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^3_+ \times \mathbb{R}^3)), \tag{3.1.25}$$

$$j_f - \rho_f u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}^{-1}_{\mathrm{div}}(\mathbb{R}^3_+)), \tag{3.1.26}$$

and such that the following holds. The distribution function f is a weak solution to the Vlasov equation with force field $G+u$ with initial condition f_0 in the sense of Definition 3.1.5 and the velocity field u is a Leray solution to the Navier-Stokes equations with initial condition u_0, that is for all $\Phi \in \mathscr{C}^1([0,T];\mathscr{D}_{\mathrm{div}}(\mathbb{R}^3_+))$ such that $\Phi(T,\cdot)=0$, we have

$$\int_0^T \int_{\mathbb{R}^3_+} [u \cdot \partial_t \Phi + (u \otimes u) : \nabla_x \Phi - \nabla_x u : \nabla_x \Phi](t,x)\, \mathrm{d}x\, \mathrm{d}t \\ = -\int_{\mathbb{R}^3_+} u_0(x)\cdot \Phi(0,x)\, \mathrm{d}x - \int_0^T \int_{\mathbb{R}^3_+ \times \mathbb{R}^3} f(t,x,v)\,[v-u(t,x)]\cdot \Phi(t,x)\, \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}t, \quad (3.1.27)$$

and the strong energy inequality holds for the Navier-Stokes equations: for any $t \geq 0$ and almost every $0 \leq s \leq t$ (including $s=0$)

$$\|u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + 2\int_s^t \|\nabla u(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \mathrm{d}\tau \\ \leq \|u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + 2\int_s^t \int_{\mathbb{R}^3_+} (j_f(\tau,x) - \rho_f u(\tau,x))\cdot u(\tau,x)\, \mathrm{d}x\, \mathrm{d}\tau. \quad (3.1.28)$$

Furthermore, the following energy estimate holds for the Vlasov-Navier-Stokes system: for any $t \geq 0$ and almost every $0 \leq s \leq t$ (including $s=0$)

$$\mathrm{E}(t) + \int_s^t \mathrm{D}(\tau)\mathrm{d}\tau \leq \mathrm{E}(s) + \int_s^t \int_{\mathbb{R}^3_+ \times \mathbb{R}^3} G \cdot v f(\tau,x,v)\, \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}\tau, \quad (3.1.29)$$

where the energy E and dissipation D have been defined in (3.1.13).

Note that the last integral in the r.h.s of the inequality (3.1.28) actually makes sense because of Sobolev embedding and the fact that $j_f - \rho_f u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{6/5}(\mathbb{R}^3_+))$ (see Section 3.B in the Appendix).

Such global weak solutions can be obtained through an approximation procedure which seems to be classic by now (see e.g. [BGM17, GHKM18, BMM20]). Since the half-space/gravity framework has not been explicitly treated in the former literature, we provide some rather sketchy elements of proof about the Cauchy problem in Section 3.B, with a particular insight on the obtention of the strong energy inequalities (3.1.28) and (3.1.29).

In Section 3.2, we present the main result obtained in this chapter. As will be explained later on, the approach that we will use to prove this result shares some similar features with the ones introduced in [HKMM20, HK22, EHKM21, GHKM18]. We will detail the strategy which has been set up in these works and in comparison, we will describe the method we need in our case.

3.2 Main results

First, let us provide an informal statement of the main theorem of this chapter. Under some smallness assumption on the initial data (u_0, f_0), we shall prove that the fluid velocity u and the local density of particles ρ_f decay to 0 in large time, in the following sense: any global weak solution (u,f) to the VNS system with small data satisfies for all $t>0$ and $r \in [1,+\infty]$

$$\|u(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \frac{1}{(1+t)^{3/4}}, \quad \|\rho_f(t)\|_{\mathrm{L}^r(\mathbb{R}^3_+)} \lesssim \frac{1}{(1+t)^k},$$

for some $k > 0$, where $\lesssim$ depends on the initial data. For instance, the exponent k will be related to the decay of f_0 in space-velocity.

Before stating our results, we define some quantities based on the regularity and decay of the initial data (u_0, f_0). The notations we use here are introduced in Definitions 3.1.3–3.1.4 and in Section 3.D of the Appendix. If $p, s > 0$ are given, we set

$$\begin{aligned}
\mathcal{E}_{p,s} &:= \mathrm{W}_0^{1,\frac{9}{7}}(\mathbb{R}^3_+) \cap \mathrm{D}_3^{\frac{1}{2},2}(\mathbb{R}^3_+) \cap \mathrm{D}_3^{1-\frac{1}{s},s}(\mathbb{R}^3_+) \cap \mathrm{D}_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+), \\
\mathscr{E}(0) &:= \|u_0\|_{\mathrm{H}^1 \cap \mathrm{D}_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+)} + \mathrm{E}(0) + \|u_0\|_{\mathrm{L}^1(\mathbb{R}^3_+)} + M_6 f_0,
\end{aligned}$$

and for $q, m > 0$

$$\mathscr{N}_{q,m}(f_0) := N_q(f_0) + H_{q,m}(f_0) + \max_{r \in \{1,3\}} \{K_{q,r}(f_0) + F_{q,m,r}(f_0)\}.$$

The main result of this chapter reads as follows.

Theorem 3.2.1. *Let $s \in (2,3)$, $q \geq 7$ and $m \geq 4$. There exist $p \in \left(3, \frac{3(2+s)}{4}\right)$ and $\mathrm{C} > 0$ such that the following holds. Let (u_0, f_0) be an admissible initial condition in the sense of Definition 3.1.2 satisfying*

$$\begin{aligned}
&u_0 \in \mathrm{H}^1_{0,\mathrm{div}}(\mathbb{R}^3_+) \cap \mathrm{L}^1(\mathbb{R}^3_+) \cap \mathcal{E}_{p,s}, \\
&M_6 f_0 + \mathscr{N}_{q,m}(f_0) < \infty.
\end{aligned} \tag{3.2.1}$$

There exists a constant $C_0 = C_0(\mathscr{E}(0), \mathscr{N}_{q,m}(f_0)) > 0$ such that if

$$C_0 \times \left(\|u_0\|_{\mathrm{H}^1 \cap \mathrm{L}^1(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,m}(f_0)\right) < \mathrm{C}, \tag{3.2.2}$$

then the following holds: there exists a constant $\Lambda_0 = \Lambda_0\left(\|u_0\|_{\mathrm{H}^1 \cap \mathrm{L}^1(\mathbb{R}^3_+)}, N_q(f_0), H_{0,m}(f_0)\right) > 0$ such that any global Leray solution (u, f) to the Vlasov-Navier-Stokes system with initial data (u_0, f_0) (in the sense of Definition 3.1.6) satisfies for all $t > 0$, $\overline{q} \geq q$ and $k \in [0, \overline{q} - 3)$

$$\|u(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \frac{\Lambda_0}{(1+t)^{3/4}}, \tag{3.2.3}$$

$$\|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \leq C_{k,\overline{q},g} \frac{N_{\overline{q}}(f_0) + H_{0,k}(f_0)}{(1+t)^k}, \tag{3.2.4}$$

for some constant $C_{k,\overline{q},g} > 0$.

In particular, this shows that the more the initial data f_0 decays in the phase space, the more the local density ρ_f enjoys some decay in time.

We can also prove the following result for the decay of $\rho_f(t)$ in $\mathrm{L}^r(\mathbb{R}^3_+)$.

Proposition 3.2.2. *Consider the same assumptions (3.2.1) and (3.2.2) of Theorem 3.2.1, with the same set of exponents (s, p, q, m). Let $r \in [1, +\infty)$. Then for all $t > 0$, $\overline{q} \geq q$ and $k \in [0, \overline{q} - 3)$, we have*

$$\|\rho_f(t)\|_{\mathrm{L}^r(\mathbb{R}^3_+)} \leq C_{k,\overline{q},r,g} \frac{K_{\overline{q},r}(f_0) + F_{0,k,r}(f_0)}{(1+t)^k}, \tag{3.2.5}$$

for some constant $C_{k,q,r,g} > 0$.

Our result also holds for any moment $m_\ell f$ of f (see Definition 3.1.3).

Proposition 3.2.3. *Consider the same assumptions* (3.2.1) *and* (3.2.2) *of Theorem 3.2.1, with the same set of exponents* (s, p, q, m). *Let* $r \in [1, +\infty)$ *and* $\ell > 0$. *Then the following holds:*

- *for all* $t > 0$, $\overline{q} \geq q$ *and* $k \in [0, \overline{q} - 3)$, *we have*

$$\|m_\ell f(t)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \leq C_{k,\overline{q},g,\ell} \frac{N_{\overline{q}}(f_0) + H_{\ell,k}(f_0)}{(1+t)^k},$$

for some constant $C_{k,\overline{q},g,\ell} > 0$;

- *for all* $t > 0$, $\overline{q} \geq q$ *and* $k \in [0, \overline{q} - \ell - 3)$ *such that* $\ell + 3 < \overline{q}$, *we have*

$$\|m_\ell f(t)\|_{\mathrm{L}^r(\mathbb{R}^3_+)} \leq C_{k,\overline{q},r,g,\ell} \frac{K_{\overline{q},r}(f_0) + F_{\ell,k,r}(f_0)}{(1+t)^k},$$

for some constant $C_{k,\overline{q},r,g,\ell} > 0$.

The two previous propositions will be direct consequences of our proof of Theorem 3.2.1.

Remark 3.2.4. As a byproduct of our analysis, we will prove along the way that if f_0 is compactly supported in velocity and in the third direction in space, that is if

$$\operatorname{supp} f_0 \subset (\mathbb{R}^2 \times (0, L)) \times \mathrm{B}(0, R),$$

for some finite $L, R, > 0$, then there exists a finite time $T = T(L, R, g)$ such that $f(t) \equiv 0$ for every $t > T$.

Remark 3.2.5. In view of the results of [HKMM20, HK22], one could hope for a monokinetic behavior of the distribution function in large time: in other words, $f(t)$ should concentrate in velocity to a Dirac mass supported at G. However, the estimate (3.2.5) shows in particular that $\|f(t)\|_{\mathrm{L}^1(\mathbb{R}^3_+ \times \mathbb{R}^3)} \underset{t \to +\infty}{\longrightarrow} 0$ so that the previous singular behavior does not occur. This is due to the absorption of the particles at the boundary, which is combined to the presence of the gravity force.

Let us explain the main strategy that has been already devised and used to study the large time behavior of the VNS system in [HKMM20, HK22, EHKM21] (which are gravity-less cases). Roughly speaking, under some smallness assumption on the initial data that we will detail below, [HKMM20, HK22, EHKM21] have proven that the fluid velocity u tends to a constant when $t \to +\infty$, while the distribution function converges towards a Dirac mass in velocity. The later weak convergence is in particular measured thanks to the 1-Wasserstein distance on the phase space. All of these works heavily rely on the decay of a well-chosen energy functional which essentially controls the convergence of u and f. The choice of such a functional may depend on the domain. As explained before, two principal spatial frameworks have been explored.

• *The case of bounded domains*: as already mentioned in the introduction, Han-Kwan, Moussa and Moyano have tackled the large time behavior of global weak solutions to the system set on the torus $\mathbb{T}^3$ in [HKMM20], while the case of bounded domains with absorption boundary condition has been studied in [EHKM21] (see the Chapter 2). The main strategy is the following. In the torus, the quantity which plays a crucial role in the large time dynamics is the so-called *modulated energy* introduced by Choi and Kwon in [CK15]. It is defined as

$$\mathcal{E}(t) := \frac{1}{2} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f(t,x,v) |v - \langle j_f(t) \rangle|^2 \, \mathrm{d}x \, \mathrm{d}v + \frac{1}{2} \int_{\mathbb{T}^3} |u(t,x) - \langle u(t) \rangle|^2 \, \mathrm{d}x + \frac{1}{4} |\langle j_f(t) \rangle - \langle u(t) \rangle|^2,$$

where $\langle \cdot \rangle$ stands for the spatial average on $\mathbb{T}^3$. On a smooth bounded domain $\Omega \subset \mathbb{R}^3$, the key functional is the kinetic energy E_Ω itself, defined as in (3.1.12) where $\mathbb{R}^3_+$ is replaced by Ω. In both cases, the decay of the energies $\mathcal{E}$ and E_Ω is based on the following formal energy-dissipation identities

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + \mathrm{D}_{\mathbb{T}^3}(t) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}_\Omega(t) + \mathrm{D}_\Omega(t) = 0, \tag{3.2.6}$$

where $\mathrm{D}_{\mathbb{T}^3}$ or D_Ω are defined as in (3.1.13) where $\mathbb{R}^3_+$ is replaced by $\mathbb{T}^3$ or Ω. Under the assumption that the global bound $\rho_f \in \mathrm{L}^\infty(\mathbb{R}^+; \mathrm{L}^\infty(Q))$ holds, where $Q = \mathbb{T}^3$ or $Q = \Omega$, one can show that an exponential decay of the energy is satisfied, namely

$$\forall t \geq 0, \quad \mathcal{E}(t) \leq C_\lambda e^{-\lambda t}\mathcal{E}(0), \quad \mathrm{E}_\Omega(t) \leq C'_\lambda e^{-\lambda t}\mathrm{E}_\Omega(0), \tag{3.2.7}$$

for some constants $\lambda, C_\lambda, C'_\lambda > 0$. This mainly comes from Poincaré(-Wirtinger) inequality. Then, a straightening change of variable in velocity shows that a sufficient condition for obtaining the previous bound on ρ_f is an estimate on the Lipschitz seminorm of u, that is

$$\int_0^\infty \|\nabla u(s)\|_{\mathrm{L}^\infty(Q)}\,\mathrm{d}s \ll 1.$$

A bootstrap procedure has to be performed in order to ensure such a global control: the main idea is to interpolate the pointwise conditional decay (3.2.7) with higher order parabolic regularity estimates for the fluid velocity. In short, the previous approach requires a smallness assumption on the initial data of the type

$$\mathcal{E}(0) + \|u_0\|_{\mathrm{H}^1(Q)} \ll 1, \quad \text{or} \quad \mathrm{E}_\Omega(0) + \|u_0\|_{\mathrm{H}^1(Q)} \ll 1,$$

and one can even replace the previous norm for u_0 by $\|u_0\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)}$ in the torus case. Furthermore, one can describe the structure of the final spatial density, which depends on the whole evolution of the system.

• *The case of the whole space*: in this case studied in [HK22], Han-Kwan has shown that the crucial functional to consider is the kinetic energy $\mathrm{E}_{\mathbb{R}^3}$ itself, defined as in (3.1.12) where $\mathbb{R}^3_+$ is replaced by $\mathbb{R}^3$. This energy satisfies the same formal energy identity (3.2.6) as above. A bound of the type $\rho_f \in \mathrm{L}^\infty(\mathbb{R}^+; \mathrm{L}^\infty(\mathbb{R}^3))$ now provides a decay of the form

$$\forall t \geq 0, \quad \mathrm{E}_{\mathbb{R}^3}(t) \leq \frac{\varphi_\alpha(\mathrm{E}_{\mathbb{R}^3}(0) + \|u_0\|_{\mathrm{L}^1(\mathbb{R}^3)})}{(1+t)^\alpha}, \quad \text{for all} \quad \alpha \in]0, 3/2[,$$

for some function φ_α. Here, the polynomial decay is the best one can hope for because of the absence of a Poincaré inequality (with respect to the Lebesgue measure) on domains unbounded in any direction. In short, it corresponds to the decay of the solutions to the heat equation on the whole space, with almost the same rate. As before, the second step of the analysis is a bootstrap analysis which aim is to obtain the control on ρ_f, relying on the same sufficient control of $\|\nabla u\|_{\mathrm{L}^1(\mathbb{R}^+;\mathrm{L}^\infty(\mathbb{R}^3))}$. Since the energy decay is only polynomial and since the Brinkman force is not decaying, the use of dissipation functionals of higher order is needed[1]. It essentially leads to the study of weighted in time estimates for the second order derivatives (in space) of the fluid velocity. Here, this procedure is applied for small initial data, in the sense that

$$\mathrm{E}_{\mathbb{R}^3}(0) + \|u_0\|_{\mathrm{H}^1(\mathbb{R}^3)} + \|f_0\|_{\mathrm{L}^1_v(\mathbb{R}^3;\mathrm{L}^\infty_x(\mathbb{R}^3))} \ll 1.$$

[1]This family of identities has also found a powerful application in the study of the hydrodynamic limits of the VNS system in the torus performed by Han-Kwan and Michel in [HKMar].

Let us emphasize the fact that the case of a bounded domain with specific boundary conditions has been considered by Glass, Han-Kwan and Moussa in [GHKM18]. More precisely, the VNS system is set on a bidimensional rectangle $\Omega := (-L, L) \times (-1, 1)$: the fluid velocity satisfies a Dirichlet boundary condition corresponding to a Poiseuille flow while the distribution function obeys to mixed absorption/injection boundary conditions. Compared to the previous state of the art, this particular framework leads to a somewhat other type of asymptotic behavior. Indeed, relying on a key geometric control condition (the so-called *exit geometric condition*), it has been shown that one can get the existence and asymptotic stability of non-trivial smooth equilibria for the system.

Main strategy. Let us explain the main approach used in this chapter. As mentioned before, the study of the large time dynamics of the system (3.1.1)-(3.1.2)-(3.1.3) is in the same spirit as that of the previous works [HKMM20, HK22, EHKM21]. However, a main obstacle comes from the presence of the additional gravity force term in the Vlasov equation (3.1.1), which creates an extra term in the r.h.s of the energy inequality (3.1.29). This breaks one of the main structural tools of the analysis of the gravity-less case because it rules out the decay of the total kinetic energy E. We thus need to base our study upon an additional mechanism which is at stake in the system. What comes into play here is the absorption of the particles at the boundary, coming from the boundary condition (3.1.11) for the distribution function f. To understand the crucial role of this phenomenon, we will use the Lagrangian structure of the Vlasov equation. We shall define the characteristic curves (X, V) for the Vlasov equation as the solutions of the following differential system:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{X}(s;t,x,v) = \mathrm{V}(s;t,x,v), \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}(s;t,x,v) = u(s,\mathrm{X}(s;t,x,v)) + G - \mathrm{V}(s;t,x,v), \end{cases}$$

with $(\mathrm{X}(t;t,x,v), \mathrm{V}(t;t,x,v)) = (x, v)$ and where u has been extended by 0 outside the half-space. Introducing

$$\mathcal{O}^t = \left\{(x,v) \in \mathbb{R}^3_+ \times \mathbb{R}^3 \mid \forall \sigma \in [0,t], \quad \mathrm{X}(\sigma;t,x,v) \in \mathbb{R}^3_+\right\},$$

the method of characteristics shall provide the following representation formula:

$$f(t,x,v) = e^{3t}\mathbf{1}_{\mathcal{O}^t}(x,v)\, f_0(\mathrm{X}(0;t,x,v), \mathrm{V}(0;t,x,v)). \tag{3.2.8}$$

In view of this expression, a certain decay in time of the moments of f should be satisfied along the evolution of the system, provided that f_0 enjoys some decay in the phase space.

In order to take advantage of the absorption, we shall rely on an *exit geometric condition*, reminiscent of the work of Glass, Han-Kwan and Moussa in [GHKM18]. In short, we ask that all the characteristic curves starting from a compact set leave the half-space before a fixed time. Such type of condition is in the spirit of the celebrated work [BLR92] of Bardos, Lebeau and Rauch on the wave-equation, where they introduced the so-called *geometric control condition*.

Here, the main idea to propagate this condition will be to compare the coupled Vlasov equation to the Vlasov equation without fluid velocity and only governed by the gravity. The characteristic curves $(\mathrm{X}^g, \mathrm{V}^g)$ for this simplified equation are defined as the solutions of

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{X}^g(s;t,x,v) = \mathrm{V}^g(s;t,x,v), \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}^g(s;t,x,v) = G - \mathrm{V}^g(s;t,x,v), \end{cases}$$

with $(\mathrm{X}(t;t,x,v), \mathrm{V}(t;t,x,v)) = (x, v)$. We will then show that the exit geometric condition holds for all times for the VNS system, by using the particular geometry of the domain and the simple form of $(\mathrm{X}^g, \mathrm{V}^g)$. This will essentially require a control of the form

$$\int_0^\infty \|u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\, \mathrm{d}s \ll 1. \tag{3.2.9}$$

By combining the previous absorption phenomenon with a decay of the initial distribution function f_0 itself, we shall be able to obtain decay in time estimates of the moments of f. The argument will be based on the representation formula (3.2.8) and on a change of variable in velocity, namely $v \mapsto \mathrm{V}(0; t, x, v)$ (which was already used in [HKMM20, HK22, EHKM21]). This procedure will be allowed if we can ensure a control of the Lipschitz seminorm of the fluid velocity, that is

$$\int_0^\infty \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\, \mathrm{d}s \ll 1. \tag{3.2.10}$$

Thus, it turns out that the presence of the gravity force is eventually favorable for our purpose and leads to the decay estimates of the moments stated in Theorem 3.2.1 and Propositions 3.2.2 and 3.2.3.

Thanks to the specific form of the Brinkman force $j_f - \rho_f u$ in the Navier-Stokes equations, we shall obtain pointwise in time estimates in various norms for this source term. Hence, as a solution of the Navier-Stokes equations with a sufficiently decaying forcing term, the fluid velocity u will enjoy a polynomial convergence towards 0. This is essentially the result bearing on u in Theorem 3.2.1.

Therefore, our main guiding line will be the obtention of decay in times estimates for the moments of f. We will base our proof on a bootstrap argument, mainly directly taken from [HK22].

Outline of the chapter. According to the previous strategy, let us describe how this chapter is organised.

- In view of the arguments above, it makes sense to first consider the Navier-Stokes system having a source term (i.e. the Brinkman force (3.1.4)) with polynomial decay in time. It turns out that this assumption falls within the scope of the work of Wiegner in [Wie87] for the Navier-Stokes equations with a decaying source term on $\mathbb{R}^3$. *Modulo* an adaptation to the half-space case, this entails a polynomial decay of the L^2 norm of the fluid velocity. This conditional Theorem 3.3.1 is contained in **Section 3.3**.

- Before going further, we shall need to state some preliminaries gathered in **Section 3.4**. They are necessary for a local in time analysis as well as for the subsequent bootstrap argument. We obtain rough bounds on the moments of f, ensuring short time controls. We also derive some H^1 estimates for the fluid velocity and define the notion of *strong existence time* for the Navier-Stokes system.

- As explained before, the absorption of the particles at the boundary will be the key effect leading to global decay in time for the moments of f. In **Section 3.5**, we introduce the aforementioned crucial *exit geometric condition* coming from [GHKM18] and analyse its effect on the system. In particular, this enables us to track which proportion of the support of the initial distribution has disappeared from the system at any given time.

- **Section 3.6** is devoted to the bootstrap argument, which aims to achieve the global controls (3.2.9) and (3.2.10). Along the bootstrap, the previous absorption phenomenon is shown to lead to the desired decay estimates of the moments. We first show that the Brinkman force satisfies a suitable pointwise decay. Polynomial weighted in time estimates for the fluid velocity are then obtained. This will allow to close the bootstrap argument thanks to an interpolation procedure.

In the rest of the chapter, we will use the standard notation $A \lesssim B$ for $A \leq cB$ for some $c > 0$ which is independent of A, B and that may change from line to line.

3.3 Conditional large time behavior of the fluid velocity

The main goal of this short section is to show some conditional results about the polynomial decay of the L^2 norm of any Leray solution u to the Navier-Stokes equation with a source term $F = F(t, x)$, that is

$$\begin{cases} \partial_t u + (u \cdot \nabla_x) u - \Delta_x u + \nabla_x p = F, \\ \operatorname{div}_x u = 0, \\ u_{|x_3=0} = 0, \\ u_{|t=0} = u_0. \end{cases} \tag{3.3.1}$$

We shall require that u satisfies the strong energy inequality, that is for any $t \geq 0$ and almost every $0 \leq s \leq t$ (including $s = 0$), we have

$$\|u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + 2 \int_s^t \|\nabla u(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \mathrm{d}\tau \leq \|u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + 2 \int_s^t \int_{\mathbb{R}^3_+} F(\tau, x) \cdot u(\tau, x) \, \mathrm{d}x \, \mathrm{d}\tau. \tag{3.3.2}$$

The decay of such a solution will hold if F satisfies a conditional pointwise decay in $\mathrm{L}^2(\mathbb{R}^3_+)$ and is somewhat imposed by the decay of the Stokes semigroup on $\mathbb{R}^3_+$. The main result reads as follows.

Theorem 3.3.1. *Let $u_0 \in \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^3_+) \cap \mathrm{L}^1(\mathbb{R}^3_+)$ and $F \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{R}^3_+))$. Let u be a global Leray solution to the Navier-Stokes system (3.3.1) with strong energy inequality (3.3.2), associated to the initial data u_0 and the source term F. Let $T > 0$ and assume that*

$$\forall t \in [0, T], \quad \|F(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \frac{\mathrm{C}}{(1+t)^{7/4}}, \tag{3.3.3}$$

for some constant $\mathrm{C} > 0$ independent of T. Then there exists a continuous nonnegative function Ψ cancelling at 0 and independent of T such that

$$\forall t \in [0, T], \quad \|u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \frac{\Psi\left(\|u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|u_0\|^2_{\mathrm{L}^1(\mathbb{R}^3_+)} + \mathrm{C}\right)}{(1+t)^{3/2}}. \tag{3.3.4}$$

Here, we will follow Wiegner [Wie87] and Borchers and Miyakawa [BM88], relying on the Fourier splitting method of Schonbeck [Sch86]. More precisely, assuming a decay of the type (3.3.3) for the source term F means that the desired result for the large time behavior of the fluid velocity enters in the framework of [Wie87]. Since we work in an unbounded domain with boundaries, we shall adapt this method written in the whole space case, thanks to a spectral decomposition of the Stokes operator and the use of its fractional powers, as in [BM88]. This will entail a polynomial decay of the fluid velocity similar to the one without a source term in the equations, and whose rate is roughly speaking the same as that of the unsteady Stokes equations.

Coming back to the Vlasov-Navier-Stokes system, we shall consider the Brinkman force $j_f - \rho_f u$ as a fixed source term in the Navier-Stokes equations. This means that we shall use Theorem 3.3.1 with

$$F = j_f - \rho_f u.$$

Note that compared to [HK22] where the conditional decay of the energy E was related to the whole VNS system, our result concerns the decay of u as a solution of the Navier-Stokes equations only and is independent of the coupling with the Vlasov equation: we only use the strong energy inequality (3.1.28) for the Navier-Stokes system with a given source term (even if it may depend on u and f). Therefore, the assumption (3.3.3) on the decay of this source term makes the situation simpler. We will be able to prove that this strong decay does occur thanks to the absorption phenomenon along a bootstrap procedure in Section 3.6. Here, we do not require a bound of the type $\rho_f \in \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3_+))$ as in [HK22] and the rate of convergence is slightly better.

The combination of [Wie87] and [BM88] for the proof of Theorem 3.3.1 may appear as a classic result for the Navier-Stokes system: for the reader's convenience, we only write the proof in Section 3.E of the Appendix.

In view of the conditional Theorem 3.3.1, obtaining decay in time for the moments of f will be the main goal of the rest of this chapter.

3.4 Preliminaries for the bootstrap procedure

Thanks to a bootstrap argument, we will prove that there exists $\mathrm{M} > 0$ such that

$$\forall t \in \mathbb{R}^+, \quad \|j_f(t) - \rho_f u(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \frac{\mathrm{M}}{(1+t)^{7/4}}.$$

Then, by the conditional Theorem 3.3.1, this will imply the first statement of Theorem 3.2.1. Along the way, we shall rely on the following bounds and estimates for the local density ρ_f:

$$\|\rho_f\|_{\mathrm{L}^\infty(\mathbb{R}^+;\mathrm{L}^\infty(\mathbb{R}^3_+))} \lesssim 1,$$
$$\forall t \in \mathbb{R}^+, \forall r \geq 1, \quad \|\rho_f(t)\|_{\mathrm{L}^r(\mathbb{R}^3_+)} \lesssim \frac{1}{(1+t)^k}, \quad \text{for some } k > 0,$$

which will essentially lead to the second part of Theorem 3.2.1.

In this section, we collect several useful information in order to be able to set up a boostrap procedure in Section 3.6. We start by recalling some basic facts about the Lagrangian structure of the Vlasov equation (3.1.1). A careful analysis of the characteristic curves will indeed be required to deal with the absorption at the boundary. It also enables us to consider a straightening change of variable in velocity. Then, we derive local in time estimates for the moments ρ_f and j_f, as well as for the fluid velocity in $\mathrm{L}^1\mathrm{L}^\infty$. This will offer short time controls on these quantities. Thanks to a smoothing property of the Navier-Stokes system, we finally obtain $\mathrm{L}^2\mathrm{H}^2 \cap \mathrm{L}^\infty\mathrm{H}^1$ estimates for the fluid velocity. This requires the introduction of the so-called *strong existence times* and eventually entails a local in time $\mathrm{L}^1\mathrm{W}^{1,\infty}$ regularity for the fluid velocity.

3.4.1 Characteristic curves for the Vlasov equation: representation formula and change of variable

Given a time-dependent vector field u on $\mathbb{R}^+ \times \mathbb{R}^3_+$, a time $t \in \mathbb{R}^+$ and a point $(x,v) \in \mathbb{R}^3_+ \times \mathbb{R}^3$, we define the characteristic curves $s \in \mathbb{R}^+ \mapsto (\mathrm{X}(s;t,x,v), \mathrm{V}(s;t,x,v)) \in \mathbb{R}^3 \times \mathbb{R}^3$ for the Vlasov equation (associated to u) as the solution of the following system of ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{X}(s;t,x,v) = \mathrm{V}(s;t,x,v), \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}(s;t,x,v) = (Pu)(s,\mathrm{X}(s;t,x,v)) + G - \mathrm{V}(s;t,x,v), \\ \quad \mathrm{X}(t;t,x,v) = x, \\ \quad \mathrm{V}(t;t,x,v) = v. \end{cases} \tag{3.4.1}$$

Here, P is the linear extension operator continuous from $\mathrm{L}^\infty(\mathbb{R}^3_+)$ to $\mathrm{L}^\infty(\mathbb{R}^3)$ and from $\mathrm{W}^{1,\infty}_0(\mathbb{R}^3_+)$ to $\mathrm{W}^{1,\infty}(\mathbb{R}^3)$ defined by

$$\forall x \in \mathbb{R}^d, \quad (Pw)(x) := \begin{cases} w(x) & \text{if } x \in \mathbb{R}^3_+, \\ 0 & \text{if } x \in \mathbb{R}^3 \setminus \mathbb{R}^3_+, \end{cases} \tag{3.4.2}$$

and which satisfies

$$\forall w \in \mathrm{L}^\infty(\mathbb{R}^3_+), \ \|Pw\|_{\mathrm{L}^\infty(\mathbb{R}^3)} = \|w\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}, \tag{3.4.3}$$

$$\forall w \in \mathrm{W}^{1,\infty}_0(\mathbb{R}^3_+), \quad \|\nabla(Pw)\|_{\mathrm{L}^\infty(\mathbb{R}^3)} \leq \|\nabla w\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}. \tag{3.4.4}$$

Also, we will use the convention

$$(Pu)(t,\cdot) = P(u(t,\cdot)),$$

as well as the notation

$$\mathrm{X}^s_t(x,v) := \mathrm{X}(s;t,x,v), \quad \mathrm{V}^s_t(x,v) := \mathrm{V}(s;t,x,v).$$

Let $T > 0$ be fixed and suppose

$$u \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\mathbb{R}^3_+)) \cap \mathrm{L}^1(0,T;\mathrm{W}^{1,\infty}(\mathbb{R}^3_+)).$$

We can apply the Cauchy-Lipschitz theorem to show the following proposition.

Proposition 3.4.1. *Given $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ and a time $t \in [0,T]$, the system (3.4.1) admits a unique solution $s \mapsto \mathrm{Z}^s_t(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ on $[0,T]$ and*

$$\mathrm{Z}^s_t : \begin{cases} \mathbb{R}^3 \times \mathbb{R}^3 & \longrightarrow \mathbb{R}^3 \times \mathbb{R}^3 \\ (x,v) & \longmapsto \mathrm{Z}^s_t(x,v) := (\mathrm{X}(s;t,x,v), \mathrm{V}(s;t,x,v)) \end{cases}$$

is a (bi-Lipschitz) diffeomorphism of $\mathbb{R}^3 \times \mathbb{R}^3$ whose inverse is given by $(\mathrm{Z}^s_t)^{-1} = \mathrm{Z}^t_s$ and whose Jacobian determinant is $e^{3(t-s)}$.

In this context, the characteristic curves for the Vlasov equation are classically defined (at least) until time T and are given for all $s,t \in [0,T]$ by

$$\begin{cases} \mathrm{X}(s;t,x,v) = x + (1-e^{-s+t})v + (s-t+e^{-s+t}-1)G + \displaystyle\int_t^s (1-e^{\tau-s})(Pu)(\tau,\mathrm{X}(\tau;t,x,v))\,\mathrm{d}\tau, \\ \mathrm{V}(s;t,x,v) = e^{-s+t}v + (1-e^{-s+t})G + \displaystyle\int_t^s e^{\tau-s}(Pu)(\tau,\mathrm{X}(\tau;t,x,v))\,\mathrm{d}\tau. \end{cases} \tag{3.4.5}$$

Starting from a point $(x,v) \in \mathbb{R}^3_+ \times \mathbb{R}^3$ at time t, the curve $\mathrm{X}(s;t,x,v)$ remains during a certain interval of time in the half-space $\mathbb{R}^3_+$. This naturally leads to the following definitions, already considered in Chapter 2

Definition 3.4.2. *For $(x,v) \in \mathbb{R}^3_+ \times \mathbb{R}^3$ and for any $t \geq 0$, we set*

$$\tau^+(t,x,v) := \sup\left\{ s \geq t \ \mid \forall \sigma \in [t,s], \ \mathrm{X}(\sigma;t,x,v) \in \mathbb{R}^3_+ \right\}. \tag{3.4.6}$$

We also define

$$\mathcal{O}^t = \left\{ (x,v) \in \mathbb{R}^3_+ \times \mathbb{R}^3 \mid \forall \sigma \in [0,t], \quad \mathrm{X}(\sigma;t,x,v) \in \mathbb{R}^3_+ \right\}. \tag{3.4.7}$$

We state two basic results whose proof can be found in the Appendix of Chapter 2. The second one is a representation formula for the weak solution to the Vlasov equation (3.1.1), where one has to take into account the absorption boundary condition (3.1.11) satisfied by the distribution function.

Lemma 3.4.3. *For $z = (x, v) \in \mathbb{R}^3_+ \times \mathbb{R}^3$, if $\tau^+(0, z) < +\infty$ then we have*

$$\mathrm{Z}_0^{\tau^+(0,z)}(z) = \left(\mathrm{X}_0^{\tau^+(0,z)}(z), \mathrm{V}_0^{\tau^+(0,z)}(z)\right) \in \Sigma^+ \cup \Sigma^0, \tag{3.4.8}$$

where Σ^+ and Σ^0 are defined in (3.1.8). Furthermore, we have for all $t \geq 0$

$$\mathrm{Z}_t^0(\mathcal{O}^t) = \left\{(x, v) \in \mathbb{R}^3_+ \times \mathbb{R}^3 \mid \tau^+(0, x, v) > t\right\}.$$

Proposition 3.4.4. *Let f be the weak solution to the Vlasov equation*

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v\,(f(u - v) + fG) = 0, \\ f_{|t=0} = f_0, \\ f = 0 \text{ on } \Sigma^-, \end{cases}$$

associated to a velocity field $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}^1_0(\mathbb{R}^3_+)) \cap \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{W}^{1,\infty}(\mathbb{R}^3_+))$ with initial condition f_0 and with absorption boundary condition. There holds

$$f(t, x, v) = e^{3t}\mathbf{1}_{\mathcal{O}^t}(x, v)\, f_0(\mathrm{Z}_t^0(x, v)) \quad a.e. \tag{3.4.9}$$

Recall that we aim at obtaining a sufficient decay in time of the Brinkman force $j_f - \rho_f u$ and more generally of the moments of f. The representation formula (3.4.9) will be our main starting point: elaborating on the same strategy as that of [HKMM20, HK22, EHKM21, HKMar] (see also the Chapter 2), we shall rely on a straightening change of variable in velocity (and then in space) in this formula. Nevertheless, such a procedure requires a smallness assumption on the quantity $\|\nabla u\|_{\mathrm{L}^1(\mathbb{R}^+;\mathrm{L}^\infty(\mathbb{R}^3_+))}$ and obtaining this control will be at the core of Section 3.6. We emphasize the fact that we also need the help of the absorption at the boundary in order to recover the desired decay in time of the moments.

In view of the following formulas

$$\mathrm{V}(0; t, x, v) = e^t v + (1 - e^t)G - \int_0^t e^\tau (Pu)(\tau, \mathrm{X}(\tau; t, x, v))\,\mathrm{d}\tau,$$

$$\mathrm{D}_v\mathrm{V}(0; t, x, v) - e^t\mathrm{Id} = -\int_0^t e^\tau \nabla(Pu)(\tau, \mathrm{X}(\tau; t, x, v))\mathrm{D}_v\mathrm{X}(\tau; t, x, v))\,\mathrm{d}\tau,$$

and following closely the arguments of [HKMM20], we infer several statements which read as follows.

Lemma 3.4.5. *Suppose $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+, \mathrm{H}^1_0(\mathbb{R}^3_+)) \cap \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\mathbb{R}^3_+))$. Fix $\delta > 0$ satisfying $\delta e^\delta < 1/9$. Then, for all times $t \in \mathbb{R}^+$ satisfiying*

$$\int_0^t \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s < \delta, \tag{3.4.10}$$

and for all $x \in \mathbb{R}^3_+$, the map

$$\Gamma_{t,x} : v \mapsto \mathrm{V}(0; t, x, v),$$

is a global $\mathscr{C}^1$-diffeomorphism from $\mathbb{R}^3$ to itself satisfying furthermore

$$\forall v \in \mathbb{R}^3, \quad |\det \mathrm{D}_v\Gamma_{t,x}(v)| \geq \frac{e^{3t}}{2}. \tag{3.4.11}$$

Thanks to

$$|\Gamma_{t,x}^{-1}(w)| \leq e^{-t}\left[|w| + (e^t - 1)|G| + \int_0^t e^{\tau}\|Pu(\tau)\|_{\mathrm{L}^\infty(\mathbb{R}^3)}\,\mathrm{d}\tau\right], \tag{3.4.12}$$

$$|v| \leq |\mathrm{V}(0;t,x,v)| + (1-e^{-t})|G| + \int_0^t \|Pu(\tau)\|_{\mathrm{L}^\infty(\mathbb{R}^3)}\,\mathrm{d}\tau, \tag{3.4.13}$$

we also have the following result.

Lemma 3.4.6. *Let $t_0 > 0$ and $q > 0$. If $N_q(f_0) < \infty$ and if $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{H}^1_0 \cap \mathrm{L}^\infty(\mathbb{R}^3_+))$ then $N_q(f(t_0)) < \infty$ with*

$$N_q(f(t_0)) \lesssim e^{3t_0}(1 + |G|^q + \|u\|^q_{\mathrm{L}^1(0,t_0;\mathrm{L}^\infty(\mathbb{R}^3_+))})N_q(f_0).$$

As we shall see later, we will also perform a change of variable in space after the previous change of variable in velocity.

Lemma 3.4.7. *Consider the same assumptions as in Lemma 3.4.5. For any $t \in \mathbb{R}^+$ satisfying*

$$\int_0^t \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s \leq \delta,$$

and for any $v \in \mathbb{R}^3$, the map

$$\Lambda_{t,v} : x \longrightarrow \mathrm{X}^0_t(x, \Gamma^{-1}_{t,x}(v)) \tag{3.4.14}$$

is a global $\mathscr{C}^1$-diffeomorphism from $\mathbb{R}^3$ to itself satisfying

$$\forall x \in \mathbb{R}^3, \quad |\det \mathrm{D}_x\Lambda_{t,v}(x)| \geq \frac{1}{2}. \tag{3.4.15}$$

Proof. From

$$\Gamma^{-1}_{t,x}(w) = e^{-t}w + (1-e^t)G + \int_0^t e^{\tau - t}(Pu)(\tau, \mathrm{X}^\tau_t(x, \Gamma^{-1}_{t,x}(w))\,\mathrm{d}\tau,$$

we infer that

$$\Lambda_{t,w}(x) = x + (e^{-t}-1)w + (1 - t + e^{-t})G + \int_0^t (e^{\tau - t} - 1)(Pu)(\tau, \mathrm{X}^\tau_t(x, \Gamma^{-1}_{t,x}(w)))\,\mathrm{d}\tau.$$

We then refer to [HKMar, Lemma 3.26], the proof of which can be exactly adapted to the case of the characteristic curves with an additional gravity term. □

3.4.2 Local in time estimates

In this subsection, we derive some local in time estimates for the moments and velocity field. Recalling the quantities of Definition 3.1.4, we introduce the following useful notations, which allows us to track down the dependency on the initial data in the later estimates. In view of the smallness condition (3.2.2), this will enable us to set up a bootstrap strategy in the proof of Theorem 3.2.1.

Notation 3.4.8. *The notation $m \lesssim_0 M$ means that there exist some exponent $p > 3$ and a continuous increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$m \leq \varphi\Big(1 + \mathscr{E}(0) + \mathscr{N}_{q,m}(f_0)\Big)M, \tag{3.4.16}$$

where (q,m) *are the exponents introduced in Theorem 3.2.1 and where*

$$\mathscr{E}(0) = \|u_0\|_{\mathrm{H}^1\cap \mathrm{D}_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+)} + \mathrm{E}(0) + \|u_0\|_{\mathrm{L}^1(\mathbb{R}^3_+)} + M_6 f_0,$$
$$\mathscr{N}_{q,m}(f_0) = N_q(f_0) + H_{q,m}(f_0) + \max_{r\in\{1,3\}} \{K_{q,r}(f_0) + F_{q,m,r}(f_0)\}.$$

Here, $\mathrm{D}_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+)$ *refers to the space defined in (3.D.1), while the semi-norms involved in* $\mathscr{N}_{q,m}(f_0)$ *have been set in Definition 3.1.4. Note that in the end of the bootstrap argument, we shall be able to consider only the largest exponents* q *and* m *which are involved in the estimates.*

Until the end of this Section 3.4, we consider a fixed global weak solution (u,f) **to the Vlasov-Navier-Stokes system, in the sense of Definition 3.1.6 and associated to an admissible initial data** (u_0,f_0) **satisfying (3.2.1).**

We first state the following lemma, entailing some rough bounds on the kinetic distribution. Note that the first one is direct consequence of the absorption boundary condition (3.1.11).

Lemma 3.4.9. *For all* $t\geq 0$*, we have*

$$\int_{\mathbb{R}^3_+\times\mathbb{R}^3} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \leq \int_{\mathbb{R}^3_+\times\mathbb{R}^3} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v, \tag{3.4.17}$$

$$\|f(t)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+\times\mathbb{R}^3)} \leq e^{3t}\|f_0\|_{\mathrm{L}^\infty(\mathbb{R}^3_+\times\mathbb{R}^3)}. \tag{3.4.18}$$

Proof. We rely on the strong stability results from DiPerna-Lions theory about transport equations on $\mathbb{R}^3_+\times\mathbb{R}^3$ (see 3.A in the Appendix). In short, it allows to prove the desired estimate for a sequence of distributions (f_n) associated to a sequence of regularized initial data $(f_{0,n})_n$ and an approximating sequence of fluid velocities $(u_n)_n$. In particular, the associated characteristic curves (3.4.1) are defined in a classic way. The strong stability property of renormalized solutions to the Vlasov equation is then used to recover the estimate for the original solution (u,f). We do not detail the argument and we write the proof as if u and f_0 were smooth. By Proposition 3.4.4, we have

$$f(t,x,v) = e^{3t}\mathbf{1}_{\mathcal{O}^t}(x,v)\,f_0(\mathrm{Z}^0_t(x,v)),$$

therefore

$$\begin{aligned}\int_{\mathbb{R}^3_+\times\mathbb{R}^3} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v &= \int_{\mathbb{R}^3_+\times\mathbb{R}^3} e^{3t}\mathbf{1}_{\mathcal{O}^t}(x,v)\,f_0(\mathrm{Z}^0_t(x,v))\,\mathrm{d}x\,\mathrm{d}v \\ &= \int_{\mathbb{R}^3_+\times\mathbb{R}^3} \mathbf{1}_{\tau^+(0,x,v)>t}\,f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v,\end{aligned}$$

thanks to the change of variable $(x,v)\mapsto \mathrm{Z}^t_0(x,v)$ (see Proposition 3.4.1 and Lemma 3.4.3). The inequality (3.4.17) follows, as well as (3.4.18). □

An application of Hölder's inequality implies the following result.

Lemma 3.4.10. *Let* $p>1$*. For all* $t\geq 0$*, we have*

$$\|(j_f-\rho_f u)(t)\|_{\mathrm{L}^p(\mathbb{R}^3_+)} \leq \|\rho_f(t)\|^{\frac{p-1}{p}}_{\mathrm{L}^\infty(\mathbb{R}^3_+))}\left(\int_{\mathbb{R}^3_+\times\mathbb{R}^3} f(t,x,v)|v-u(t,x)|^p\,\mathrm{d}x\,\mathrm{d}v\right)^{1/p}. \tag{3.4.19}$$

Along the way, we will also need the simple following sublinear Grönwall's lemma.

Lemma 3.4.11. *Let $y \in \mathscr{C}(\mathbb{R}^+;\mathbb{R}^+)$ and $\mathrm{h} \in \mathscr{C}(\mathbb{R}^+;\mathbb{R}^+)$ such that for all $t \geq 0$*

$$y(t) \leq y_0 + \int_0^t \mathrm{h}(s) y(s)^\beta \,\mathrm{d}s,$$

where $\beta \in (0,1)$ and $y_0 \in \mathbb{R}^+$. Then for all $t \geq 0$

$$y(t) \leq \left(y_0^{1-\beta} + (1-\beta) \int_0^t \mathrm{h}(s)\,\mathrm{d}s \right)^{\frac{1}{1-\beta}}.$$

We now state several rough bounds on the moments of the distribution function f.

Lemma 3.4.12. *For all $t \geq 0$, we have*

$$\|j_f\|_{\mathrm{L}^1(0,t;\mathrm{L}^1(\mathbb{R}^3_+))} \lesssim \mathrm{E}(0)^{\frac{1}{2}} t + t^2,$$

In particular, the map $t \mapsto \|j_f\|_{\mathrm{L}^1(0,t;\mathrm{L}^1(\mathbb{R}^3_+))}$ belongs to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+)$.

Proof. We use the energy inequality (3.1.29) and the Cauchy-Schwarz inequality to write that for all $t \geq 0$

$$\mathrm{E}(t) \leq \mathrm{E}(0) + g \int_0^t \int_{\mathbb{R}^3_+ \times \mathbb{R}^3} |v| f(s,x,v) \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}s \leq \mathrm{E}(0) + g \int_0^t \mathrm{E}(s)^{\frac{1}{2}} \,\mathrm{d}s,$$

thanks to (3.4.17) and the normalization of f_0 given by (3.1.17). The sub-linear version of Grönwall's Lemma 3.4.11 with $\beta = 1/2$ implies

$$\mathrm{E}(t) \leq \left(\mathrm{E}(0)^{\frac{1}{2}} + \frac{g}{2} t \right)^2.$$

The Cauchy-Schwarz inequality again reads

$$\int_0^t \int_{\mathbb{R}^3_+ \times \mathbb{R}^3} |v| f(s,x,v) \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}s \leq \int_0^t \mathrm{E}(s)^{\frac{1}{2}} \,\mathrm{d}s \leq t\mathrm{E}(0)^{\frac{1}{2}} + g\frac{t^2}{4},$$

and this concludes the proof. □

Lemma 3.4.13. *If $F := j_f - \rho_f u$ then for all $t \geq 0$, we have*

$$\int_0^t \|F(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s \lesssim \sup_{s \in [0,t]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \left[\mathrm{E}(0) + \mathrm{E}(0)^{\frac{1}{2}} t + t^2 \right].$$

Proof. Using Lemma 3.4.10 with $p = 2$, we obtain for all $s \in [0,t]$

$$\|F(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \|\rho_f(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \mathrm{D}(s),$$

where the dissipation D has been defined in (3.1.13). Then, we integrate the last inequality between 0 and t and use the energy inequality (3.1.29) to obtain

$$\int_0^t \|F(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s \leq \sup_{s \in [0,t]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \left[\mathrm{E}(0) + g \int_0^t \int_{\mathbb{R}^3_+} |j_f| \,\mathrm{d}\tau \,\mathrm{d}x \right],$$

which concludes the proof, thanks to Lemma 3.4.12. □

We recall standard interpolation estimates on the moments of any kinetic distribution, where we use the notations introduced in Definition 3.1.3.

Proposition 3.4.14. *Let* h *be a nonnegative function in* $\mathrm{L}^\infty(\mathbb{R}^+ \times \mathbb{R}^3_+ \times \mathbb{R}^3)$*. Then we have for all* $t \geq 0$

$$\forall\, 0 \leq b \leq c, \quad \forall \ell \in [b,c], \quad M_\ell \mathrm{h}(t) \leq M_b \mathrm{h}(t)^{\frac{c-\ell}{c-b}} M_c \mathrm{h}(t)^{\frac{\ell-b}{c-b}},$$

$$\forall k > 0, \quad \forall \ell \in [0,k], \quad \|m_\ell \mathrm{h}(t)\|_{\mathrm{L}^{\frac{k+3}{\ell+3}}(\mathbb{R}^3_+)} \leq C_{k,\ell} \|\mathrm{h}(t)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+\times\mathbb{R}^3)}^{\frac{k-\ell}{k+3}} M_k \mathrm{h}(t)^{\frac{\ell+3}{k+3}},$$

for some universal constant $C_{k,\ell} > 0$.

We now provide a pointwise estimate for the moments of f, solution to the Vlasov equation.

Lemma 3.4.15. *Suppose that* $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^{a+3} \cap \mathrm{W}^{1,1}(\mathbb{R}^3_+))$ *and* $M_a f_0 < \infty$ *for all* $a \in [2,\alpha]$*, for some* $\alpha \geq 2$*. Then for all* $a \in [2,\alpha]$ *and for all* $t > 0$*,* $M_a f(t) < \infty$ *and* $M_a f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+)$*. Furthermore, if* $T > 0$ *then for all* $t \in [0,T]$

$$M_\alpha f(t) \leq \left(\left[M_\alpha f_0 + \alpha g \int_0^T M_{\alpha-1} f(s)\,\mathrm{d}s \right]^{\frac{1}{\alpha+3}} + e^{\frac{3t}{\alpha+3}} \|f_0\|_{\mathrm{L}^\infty(\mathbb{R}^3_+\times\mathbb{R}^3)}^{\frac{1}{\alpha+3}} \int_0^t \|u(s)\|_{\mathrm{L}^{\alpha+3}(\mathbb{R}^3_+)} \mathrm{d}s \right)^{\alpha+3}. \tag{3.4.20}$$

Proof. As in the proof of the bounds (3.4.17)–(3.4.18), we rely on the strong stability results from DiPerna-Lions theory about transport equations on $\mathbb{R}^3_+ \times \mathbb{R}^3$. We do not detail the argument and we write the proof as if u and f_0 were smooth. By Proposition 3.4.4, we have

$$f(t,x,v) = e^{3t} \mathbf{1}_{\mathcal{O}^t}(x,v)\, f_0(\mathrm{Z}^0_t(x,v)),$$

so that

$$M_\alpha f(t) = \int_{\mathbb{R}^3_+\times\mathbb{R}^3} |v|^\alpha e^{3t} \mathbf{1}_{\mathcal{O}^t}(x,v)\, f_0(\mathrm{Z}^0_t(x,v))\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^3_+\times\mathbb{R}^3} |\mathrm{V}^t_0(x,v)|^\alpha \mathbf{1}_{\tau^+(0,x,v)>t}\, f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v,$$

thanks to the change of variable $(x,v) \mapsto \mathrm{Z}^t_0(x,v)$ (see Proposition 3.4.1 and Lemma 3.4.3).

In view of the first estimate of Proposition 3.4.14, it is sufficient to prove the formula (3.4.20) for $\alpha \geq 2$ being an integer. We then argue by induction on α. For $\alpha = 2$, we observe

$$\frac{\mathrm{d}}{\mathrm{d}s} |\mathrm{V}^s_0(x,v))|^2 = 2 \frac{\mathrm{d}}{\mathrm{d}s} [\mathrm{V}^s_0(x,v))] \cdot \mathrm{V}^s_0(x,v) = 2\,[u(s, \mathrm{X}^s_0(x,v)) - \mathrm{V}^s_0(x,v) + G] \cdot \mathrm{V}^s_0(x,v),$$

from which we infer that

$$|\mathrm{V}^t_0(x,v))|^2 \leq |v|^2 + 2 \int_0^t [u(s, \mathrm{X}^s_0(x,v)) + G] \cdot \mathrm{V}^s_0(x,v)\,\mathrm{d}s.$$

By Fubini Theorem, we obtain

$$M_2 f(t) \leq \int_{\mathbb{R}^3_+\times\mathbb{R}^3} |v|^2 \mathbf{1}_{\tau^+(0,x,v)>t}\, f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v + 2g \int_0^t \int_{\mathbb{R}^3_+\times\mathbb{R}^3} |\mathrm{V}^s_0(x,v)| \mathbf{1}_{\tau^+(0,x,v)>t}\, f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v$$
$$+ 2 \int_0^t \int_{\mathbb{R}^3_+\times\mathbb{R}^3} |u(s, \mathrm{X}^s_0(x,v))| |\mathrm{V}^s_0(x,v)| \mathbf{1}_{\tau^+(0,x,v)>t}\, f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

Using the reverse change of variable $\mathrm{Z}_0^t(x,v) \mapsto (x,v)$ in the two last integrals, we get

$$\begin{aligned} M_2 f(t) &\leq M_2 f_0 + 2g \int_0^t M_1 f(s)\, \mathrm{d}s + 2 \int_0^t \int_{\mathbb{R}^3_+} |u(s,x)| m_1 f(s,x)\, \mathrm{d}x\, \mathrm{d}s \\ &\leq M_2 f_0 + 2g \int_0^t M_1 f(s)\, \mathrm{d}s + 2 \int_0^t \|u(s)\|_{\mathrm{L}^5(\mathbb{R}^3_+)} \|m_1 f(s)\|_{\mathrm{L}^{\frac{5}{4}}(\mathbb{R}^3_+)}\, \mathrm{d}s, \end{aligned}$$

thanks to Hölder's inequality. Furthermore, by Proposition 3.4.14 with $(\ell, k) = (1,2)$ and the rough control provided by 3.4.18, we get

$$\|m_1 f(s)\|_{\mathrm{L}^{\frac{5}{4}}(\mathbb{R}^3_+)} \lesssim \|f(s)\|^{\frac{1}{5}}_{\mathrm{L}^\infty(\mathbb{R}^3_+ \times \mathbb{R}^3)} M_2 f(s)^{\frac{4}{5}} \leq \|f_0\|^{\frac{1}{5}}_{\mathrm{L}^\infty(\mathbb{R}^3_+ \times \mathbb{R}^3)} e^{\frac{3s}{5}} M_2 f(s)^{\frac{4}{5}},$$

where $\lesssim$ is independent of s. We thus infer that for all $t \in [0,T]$

$$M_2 f(t) \leq M_2 f_0 + 2g \int_0^T M_1 f(s)\, \mathrm{d}s + 2 \|f_0\|^{\frac{1}{5}}_{\mathrm{L}^\infty(\mathbb{R}^3_+ \times \mathbb{R}^3)} \int_0^t e^{\frac{3s}{5}} \|u(s)\|_{\mathrm{L}^5(\mathbb{R}^3_+)} M_2 f(s)^{\frac{4}{5}}\, \mathrm{d}s.$$

Using the Grönwall's lemma stated in Lemma 3.4.11 with $\beta = 4/5$ entails

$$M_2 f(t) \lesssim \left(\left[M_2 f_0 + 2g \int_0^T M_1 f(s)\, \mathrm{d}s \right]^{\frac{1}{5}} + \frac{2}{5} e^{\frac{3t}{5}} \|f_0\|^{\frac{1}{5}}_{\mathrm{L}^\infty(\mathbb{R}^3_+ \times \mathbb{R}^3)} \int_0^t \|u(s)\|_{\mathrm{L}^5(\mathbb{R}^3_+)} \mathrm{d}s \right)^5.$$

This yields the result for $\alpha = 2$ (indeed, note that the previous r.h.s is finite because of Lemma 3.4.12). Now, if $M_{\alpha-1} f(t) < \infty$ and $M_{\alpha-1} f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+)$, we perform the same analysis as before. Since $\alpha \geq 2$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} |\mathrm{V}_0^s(x,v))|^\alpha &= \alpha \frac{\mathrm{d}}{\mathrm{d}s} [\mathrm{V}_0^s(x,v))] \cdot \mathrm{V}_0^s(x,v) \times |\mathrm{V}(s;0,x,v)|^{\alpha-2} \\ &= \alpha \left[u(s, \mathrm{X}_0^s(x,v)) - \mathrm{V}_0^s(x,v) + G \right] \cdot \mathrm{V}_0^s(x,v) |\mathrm{V}_0^s(x,v)|^{\alpha-2}, \end{aligned}$$

which entails

$$|\mathrm{V}_0^t(x,v))|^\alpha \leq |v|^\alpha + \alpha \int_0^t \left[u(s, \mathrm{X}_0^s(x,v)) + G \right] \cdot \mathrm{V}_0^s(x,v) |\mathrm{V}_0^s(x,v)|^{\alpha-2}\, \mathrm{d}s.$$

As before, we obtain

$$M_\alpha f(t) \leq M_\alpha f_0 + \alpha g \int_0^t M_{\alpha-1} f(s)\, \mathrm{d}s + \alpha \int_0^t \|u(s)\|_{\mathrm{L}^{\alpha+3}(\mathbb{R}^3_+)} \|m_{\alpha-1} f(s)\|_{\mathrm{L}^{\frac{\alpha+3}{\alpha+2}}(\mathbb{R}^3_+)}\, \mathrm{d}s.$$

Using Proposition 3.4.14 with $(\ell, k) = (\alpha - 1, \alpha)$ and the inequality 3.4.18, we get

$$\|m_{\alpha-1} f(s)\|_{\mathrm{L}^{\frac{\alpha+3}{\alpha+2}}(\mathbb{R}^3_+)} \lesssim \|f_0\|^{\frac{1}{\alpha+3}}_{\mathrm{L}^\infty} e^{\frac{3s}{\alpha+3}} M_\alpha f(s)^{\frac{\alpha+2}{\alpha+3}}.$$

This yields for all $t \in [0,T]$

$$M_\alpha f(t) \leq M_\alpha f_0 + \alpha g \int_0^T M_{\alpha-1} f(s)\, \mathrm{d}s + \alpha \|f_0\|^{\frac{1}{\alpha+3}}_{\mathrm{L}^\infty(\mathbb{R}^3_+ \times \mathbb{R}^3)} \int_0^t e^{\frac{3s}{\alpha+3}} \|u(s)\|_{\mathrm{L}^{\alpha+3}(\mathbb{R}^3_+)} M_\alpha f(s)^{\frac{\alpha+2}{\alpha+3}}\, \mathrm{d}s.$$

Thanks to Lemma 3.4.11 with $\beta = \frac{\alpha+2}{\alpha+3}$, we obtain the conclusion. □

Lemma 3.4.16. *We have $M_3 f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+)$. Moreover, for all finite $T>0$, there exists a continuous nonnegative and nondecreasing function $\varphi_{\mathrm{E}(0),M_3 f_0,T,\|f_0\|_{\mathrm{L}^\infty_{x,v}}}$ (increasing in all its parameters) such that for all $t\in[0,T]$*

$$\|\rho_f(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|j_f(t)\|_{\mathrm{L}^{3/2}(\mathbb{R}^3_+)} \leq \varphi_{\mathrm{E}(0),M_3 f_0,T,\|f_0\|_{\mathrm{L}^\infty_{x,v}}}(t).$$

Proof. Since $M_2 f_0 < \infty$ and $M_6 f_0 < \infty$ by (3.2.1), we have $M_3 f_0 \lesssim M_2 f_0 + M_6 f_0 < \infty$. Furthermore, u is a Leray solution so that by the Sobolev embedding, we can apply Lemma 3.4.15 with $\alpha = 3$ and we deduce that $M_3 f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+)$. Furthermore, the estimate (3.4.20) yields for all $t\in[0,T]$

$$M_3 f(t) \lesssim \left(\left[M_3 f_0 + 3g\int_0^T M_2 f(s)\,\mathrm{d}s\right]^{\frac{1}{6}} + \frac{e^{\frac{t}{2}}}{2}\|f_0\|^{\frac{1}{6}}_{\mathrm{L}^\infty(\mathbb{R}^3_+\times\mathbb{R}^3)}\int_0^t \|u(s)\|_{\mathrm{L}^6(\mathbb{R}^3_+)}\mathrm{d}s\right)^6.$$

The Sobolev embedding on $\mathrm{H}^1_0(\mathbb{R}^3_+)$ and the Cauchy-Schwarz inequality lead to

$$\int_0^t \|u(s)\|_{\mathrm{L}^6(\mathbb{R}^3_+)}\mathrm{d}s \lesssim t^{1/2}\left(\int_0^t \|\nabla u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\mathrm{d}s\right)^{1/2} \lesssim T^{1/2}\left[\mathrm{E}(0) + \mathrm{E}(0)^{\frac{1}{2}}T + T^2\right]^{1/2},$$

where we have used the energy inequality (3.1.29) and Lemma 3.4.12. For the same reasons, we also have

$$\int_0^T M_2 f(s)\,\mathrm{d}s \lesssim \mathrm{E}(0) + \mathrm{E}(0)^{\frac{1}{2}}T + T^2.$$

This implies that there exists a continuous nondecreasing function $\varphi_{\mathrm{E}(0),M_3 f_0,T,\|f_0\|_{\mathrm{L}^\infty_{x,v}}}$ (nonnegative and increasing in all its parameters) such that for all $t\in[0,T]$, we have

$$M_3 f(t) \leq \varphi_{\mathrm{E}(0),M_3 f_0,T,\|f_0\|_{\mathrm{L}^\infty_{x,v}}}(t). \tag{3.4.21}$$

Using Proposition 3.4.14 on interpolation of moments of the distribution function f with $k=3$ and $\ell\in\{0,1\}$, together with 3.4.18, we get

$$\|\rho_f(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \|f(t)\|^{1/2}_{\mathrm{L}^\infty(\mathbb{R}^3_+)} M_3 f(t)^{1/2} \leq \|f_0\|^{1/2}_{\mathrm{L}^\infty(\mathbb{R}^3_+)} e^{3t/2} M_3 f(t)^{1/2},$$
$$\|j_f(t)\|_{\mathrm{L}^{3/2}(\mathbb{R}^3_+)} \lesssim \|f(t)\|^{1/3}_{\mathrm{L}^\infty(\mathbb{R}^3_+)} M_3 f(t)^{2/3} \leq \|f_0\|^{1/3}_{\mathrm{L}^\infty(\mathbb{R}^3_+)} e^{t} M_3 f(t)^{2/3},$$

which yields the conclusion, thanks to the bound (3.4.21). □

We now prove that the source term in the Navier-Stokes equations, namely the Brinkman force $j_f - \rho_f u$, belongs to $\mathrm{L}^2\mathrm{L}^2$ (locally in time). The strategy of proof is very similar to that of Lemma 2.4.7 in Chapter 2 (with a minor adaptation to the half-space case) and details are thus omitted. Let us only emphasize that the result mainly follows from the combination of Lemma 3.4.12, Lemma 3.4.16, Lemma 3.4.14 and the maximal regularity property for the Stokes system (see Section 3.D in the Appendix) which can be applied in that case because $u_0 \in \mathrm{W}^{1,9/7}_0(\mathbb{R}^3_+)$ and $M_6 f_0 < \infty$ (see the assumption (3.2.1)).

Lemma 3.4.17. *We have*

$$j_f - \rho_f u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\mathbb{R}^3_+)).$$

We are then in position to state the following local in time integrability results of the Leray solution (u, f).

Proposition 3.4.18. *We have*

$$u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\mathbb{R}^3_+)), \tag{3.4.22}$$

and if $N_d(f_0) < \infty$ for some $d > 4$, then

$$\rho_f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\mathbb{R}^3_+)),$$
$$j_f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\mathbb{R}^3_+)).$$

More precisely, there exists a continuous nondecreasing function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|u\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{R}^3_+))} \lesssim \eta(t), \tag{3.4.23}$$

$$\|\rho_f\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty(\mathbb{R}^3_+))} + \|j_f\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty(\mathbb{R}^3_+))} \lesssim N_d(f_0)\eta(t). \tag{3.4.24}$$

Proof. Let $T > 0$. The proof of the fact that $u \in \mathrm{L}^1(0, T; \mathrm{L}^\infty(\mathbb{R}^3_+))$ is mostly directly taken from the arguments used in Proposition 2.4.8 of Chapter 2 and mainly relies on the theory of epochs of regularity for the Leray solutions to the Navier-Stokes equations. Owing to [Hey80, Theorem 8] and [Hey88, Remark 4] (which are valid since the strong energy inequality (3.1.28) is satisfied by the weak solutions that we consider), we know there exists a subset $\sigma_T \subset [0, T]$ of full measure in $[0, T]$ with $\sigma_T = \bigsqcup_i]a_i, b_i[$ (the union being countable) and for which $u \in \mathrm{L}^\infty_{\mathrm{loc}}(a_i, b_i; \mathrm{H}^1(\mathbb{R}^3_+)) \cap \mathrm{L}^2_{\mathrm{loc}}(a_i, b_i; \mathrm{H}^2(\mathbb{R}^3_+))$ and $\|\nabla u(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \underset{t \to b_i^-}{\longrightarrow} +\infty$ for all i. Furthermore, the function $t \mapsto \|\nabla u(t)\|^2$ is absolutely continuous on each interval $]a_i, b_i[$ (see e.g. [RRS16]).

Now, we can take the $\mathrm{L}^2(\mathbb{R}^3_+)$ inner product of (3.1.2) with Au on each interval $]a_i, b_i[$, where $A = -\mathbb{P}\delta_x u$ stands for the Stokes operator on $\mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^3_+)$ and $\mathbb{P}$ is the Leray projection on divergence-free vector field (see in the Appendix 3.D). We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + 2\|Au\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + 2\langle \mathbb{P}(u \cdot \nabla)u, Au\rangle = 2\langle \mathbb{P}(j_f - \rho_f u), Au\rangle \quad \text{on} \quad]a_i, b_i[,$$

where we have dropped the time variable. In order to estimate the term $\langle \mathbb{P}(u \cdot \nabla)u, Au\rangle$, we use the Gagliardo-Nirenberg-Sobolev inequality for the function ∇u with the exponents $(p, j, r, m, \alpha) = (3, 0, 2, 1, 1/2)$, which reads as

$$\|\nabla u\|_{\mathrm{L}^3(\mathbb{R}^3_+)} \lesssim \|\mathrm{D}^2 u\|^{1/2}_{\mathrm{L}^2(\mathbb{R}^3_+)} \|\nabla u\|^{1/2}_{\mathrm{L}^2(\mathbb{R}^3_+)},$$

and we combine this inequality with the Hölder's inequality to write

$$\begin{aligned} |\langle \mathbb{P}(u \cdot \nabla)u, Au\rangle| &\leq \|u\|_{\mathrm{L}^6(\mathbb{R}^3_+)} \|\nabla u\|_{\mathrm{L}^3(\mathbb{R}^3_+)} \|Au\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \\ &\lesssim \|\nabla u\|^{3/2}_{\mathrm{L}^2(\mathbb{R}^3_+)} \|Au\|^{3/2}_{\mathrm{L}^2(\mathbb{R}^3_+)}. \end{aligned}$$

Note that we have used [Gal11, Theorem IV.3.2] on each (a_i, b_i). Thanks to the Young inequality, we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|Au\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \|j_f - \rho_f u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|\nabla u\|^6_{\mathrm{L}^2(\mathbb{R}^3_+)}, \tag{3.4.25}$$

on each interval $]a_i, b_i[$, where $\lesssim$ is independent of the time variable and independent of i. Dividing this inequality by $(1 + \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)})^2$, we get

$$\frac{1}{(1 + \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)})^2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \frac{1}{(1 + \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)})^2} \|Au\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \|j_f - \rho_f u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}, \quad (3.4.26)$$

on each interval $]a_i, b_i[$. Integrating and summing over the previous epoch of regularities, an using in particular the fact that $\|\nabla u(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \underset{t \to b_i^-}{\longrightarrow} +\infty$ for all i, we can perform the same exact computations as in the proof of Proposition 2.4.8 in Chapter 2 and end up with

$$\int_0^T \frac{\|Au(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}}{(1 + \|\nabla u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)})^2} \,\mathrm{d}s \leq \int_0^T \left(\|j_f(s) - \rho_f u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|\nabla u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \right) \mathrm{d}s + 1,$$

from which we infer that

$$\int_0^T \|Au(s)\|^{2/3}_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s \leq \left(\int_0^T \frac{\|Au(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}}{(1 + \|\nabla u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)})^2} \,\mathrm{d}s \right)^{1/3} \left(\int_0^T (1 + \|\nabla u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}) \,\mathrm{d}s \right)^{2/3} < \infty,$$

due to Proposition 3.4.17 and to the fact that u is a Leray solution to the Navier-Stokes equations. Using the Gagliardo-Nirenberg-Sobolev inequality with exponents $(p, j, r, m, \alpha) = (\infty, 0, 2, 2, 1/2)$ and Sobolev embedding, we deduce that

$$\int_0^T \|u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \,\mathrm{d}s \lesssim \int_0^T \|\nabla u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s + \int_0^T \|Au(s)\|^{2/3}_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s < \infty,$$

from the same reasons as before, therefore this proves (3.4.22). The last estimate (3.4.24) is eventually obtained by observing that for $d > 4$

$$\begin{aligned} \|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} + \|j_f(t)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} &\lesssim N_d(f(t)) \\ &\lesssim e^{3t}(1 + |G|^q + \|u\|^q_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{R}^3_+))}) N_d(f_0) \\ &\lesssim N_q(f_0)\eta(t), \end{aligned}$$

thanks to Lemma 3.4.6 and (3.4.22). □

3.4.3 Strong existence times and higher order energy estimates

Along the bootstrap procedure, we shall need H^1 energy estimates for the fluid velocity u, which is *a priori* only a Leray solution to the Navier-Stokes equations. In order to consider higher regularity for this solution, we rely on a parabolic smoothing property of the (Vlasov-)Navier-Stokes system. We will be able to propagate this regularity if the contribution of the source term, that is the Brinkman force $F := j_f - \rho_f u$, and the initial data, are small enough.

Proposition 3.4.19. *There exists a universal constant $\mathrm{C}_\star$ such that the following holds. Assume that for some $T > 0$, one has*

$$\|u_0\|^2_{\mathrm{H}^1(\mathbb{R}^3_+)} + \int_0^T \|F(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s + \int_0^T \|F(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s < \mathrm{C}_\star. \quad (3.4.27)$$

Then one has

$$u \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(\mathbb{R}^3_+)) \cap \mathrm{L}^2(0,T;\mathrm{H}^2(\mathbb{R}^3_+)),$$

and for all $t \in [0,T]$

$$\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \frac{1}{2}\int_0^t \|\mathrm{D}^2 u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s \lesssim \|\nabla u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \int_0^t \|F(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s, \tag{3.4.28}$$

where $\lesssim$ *only depends on* $\mathrm{C}_\star$.

Proof. The estimate is a direct consequence of the parabolic regularization for the Navier-Stokes system with source $F = j_f - \rho_f u$, that we state in Theorem 3.F.1 in Section 3.F of the Appendix. □

Remark 3.4.20. By Lemma 3.4.13, the r.h.s of (3.4.28) is finite. In particular, if the condition (3.4.27) holds for some T, then for all $t \in [0,T]$

$$\|u\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^6(\mathbb{R}^3_+))} \lesssim \|\nabla u\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^2(\mathbb{R}^3_+))} \lesssim \|\nabla u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \sup_{s\in[0,t]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \left[\mathrm{E}(0) + \mathrm{E}(0)^{\frac{1}{2}} t + t^2\right].$$

In order to ensure that the smallness condition (3.4.27) is satisfied for all times, we now introduce the following terminology, which has been already used in [HKMM20, HK22] to take advantage of the parabolic regularization for the fluid velocity.

Definition 3.4.21 (Strong existence time)**.** *A real number* $T \geq 0$ *is a* strong existence *time whenever (3.4.27) holds.*

In the remaining part of this section, we state a local in time $\mathrm{L}^1\mathrm{W}^{1,\infty}$ regularity result for the fluid velocity. Note that for the moment, we are only interested in obtaining non-uniform in time estimates. Of course, quantitative and uniform in time estimates based on the polynomial decay of the kinetic energy will require an additional analysis.

Corollary 3.4.22. *For any finite strong existence time* $t > 0$ *and for any* $p \in [1,6]$*, we have*

$$j_f - \rho_f u \in \mathrm{L}^p(0,t;\mathrm{L}^p(\mathbb{R}^3_+)).$$

Proof. Thanks to Lemma 3.4.16 and (3.4.24), we have

$$j_f \in \mathrm{L}^\infty(0,t;\mathrm{L}^{3/2}(\mathbb{R}^3_+)) \cap \mathrm{L}^\infty(0,t;\mathrm{L}^\infty(\mathbb{R}^3_+)) \hookrightarrow \mathrm{L}^r(0,t;\mathrm{L}^r(\mathbb{R}^3_+)),$$

for any $r \in [3/2,+\infty]$ by interpolation. For the same reasons, we also have

$$\rho_f \in \mathrm{L}^\infty(0,t;\mathrm{L}^2(\mathbb{R}^3_+)) \cap \mathrm{L}^\infty(0,t;\mathrm{L}^\infty(\mathbb{R}^3_+)) \hookrightarrow \mathrm{L}^r(0,t;\mathrm{L}^r(\mathbb{R}^3_+)),$$

for any $r \in [2,+\infty]$. In addition, by (3.6.16), we have $u \in \mathrm{L}^\infty(0,t;\mathrm{L}^q(\mathbb{R}^3_+))$ for any $q \in [2,6]$, because t is a strong existence time. So, the Hölder's inequality yields

$$\rho_f u \in \mathrm{L}^p(0,t;\mathrm{L}^p(\mathbb{R}^3_+)), \quad \frac{1}{p} = \frac{1}{r} + \frac{1}{q}, \quad r \in [2,+\infty], \quad q \in [2,6].$$

This leads to the condition $p \in [1,6]$ and then to the conclusion. □

The next result is similar to that of [HKMar, Lemma 3.28]. We detail the proof for the sake of completeness, highlighting the role of Assumption (3.2.1).

Proposition 3.4.23. *Consider the exponent s given in Assumption (3.2.1). There exists $p \in \left(3, \frac{3(2+s)}{4}\right)$ such that for any finite strong existence time $t > 0$, we have*

$$(u \cdot \nabla)u \in \mathrm{L}^p(0,t;\mathrm{L}^p(\mathbb{R}^3_+)). \tag{3.4.29}$$

Proof. For any $(a, b, r_1, r_2) \in (1, +\infty)^4$, we can use interpolation inequalities to write

$$\|(u\cdot\nabla)u\|_{\mathrm{L}^a(0,t;\mathrm{L}^b(\mathbb{R}^3_+))} \leq \|u\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^6(\mathbb{R}^3_+))}\|\nabla u\|^{1-\frac{r_1}{a}}_{\mathrm{L}^\infty(0,t;\mathrm{L}^2(\mathbb{R}^3_+))}\|\nabla u\|^{\frac{r_1}{a}}_{\mathrm{L}^{r_1}(0,t;\mathrm{L}^{r_2}(\mathbb{R}^3_+))}, \tag{3.4.30}$$

provided that $r_1 \leq a$, $2 \leq b \leq r_2$ and

$$\frac{1}{b} = \frac{2}{3} + \frac{r_1}{a}\left(\frac{1}{r_2} - \frac{1}{2}\right). \tag{3.4.31}$$

Taking $(a, b, r_1, r_2) = (2, 3, 2, 6)$ in (3.4.30), the Sobolev embedding and Proposition 3.4.19 imply

$$(u\cdot\nabla)u \in \mathrm{L}^2(0,t;\mathrm{L}^3(\mathbb{R}^3_+)).$$

Owing to the maximal regularity for the Stokes system (see Section 3.D in the Appendix) and to Corollary 3.4.22 which gives $j_f - \rho_f u \in \mathrm{L}^2(0,t;\mathrm{L}^3(\mathbb{R}^3_+))$, as well as on the assumption (3.2.1), we obtain

$$u \in \mathrm{L}^2(0,t;\mathrm{W}^{2,3}(\mathbb{R}^3_+)).$$

So, by the Sobolev embedding and since $\nabla u \in \mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}^3_+))$, we infer that for all $r \in [2, +\infty)$

$$\nabla u \in \mathrm{L}^2(0,t;\mathrm{L}^r(\mathbb{R}^3_+)).$$

Coming back to the inequality (3.4.30) with $(b, r_1) = (3, 2)$, $r_2 \geq 2$ (which means $a = \widetilde{a} = 3(r_2 - 2)r_2^{-1} \in [2, 3)$), we now get

$$(u\cdot\nabla)u \in \mathrm{L}^{\widetilde{a}}(0,t;\mathrm{L}^3(\mathbb{R}^3_+)).$$

By Corollary 3.4.22, we also have $j_f - \rho_f u \in \mathrm{L}^{\widetilde{a}}(0,t;\mathrm{L}^3(\mathbb{R}^3_+))$, therefore an other application of the maximal regularity for the Stokes system implies that for all $\widetilde{a} \in [2, 3)$

$$u \in \mathrm{L}^{\widetilde{a}}(0,t;\mathrm{W}^{2,3}(\mathbb{R}^3_+)),$$

if $u_0 \in \mathrm{D}_3^{1-\frac{1}{\widetilde{a}},\widetilde{a}}(\mathbb{R}^3_+)$. So, under this assumption, the Sobolev embedding implies that for all $\widetilde{a} \in [2, 3)$ and for all $r \in [2, +\infty]$, we have

$$\nabla u \in \mathrm{L}^{\widetilde{a}}(0,t;\mathrm{L}^r(\mathbb{R}^3_+)).$$

This allows to apply the estimate (3.4.30) with $a = b = p > 3$ and $\widetilde{a} = r_1 \in [2, 3)$ (and also $p \leq r_2$) so that

$$(u\cdot\nabla)u \in \mathrm{L}^p(0,t;\mathrm{L}^p(\mathbb{R}^3_+)).$$

The relation (3.4.31) reads as

$$r_2 = \frac{6r_1}{3r_1 + 6 - 4p}, \quad p \leq r_2,$$

therefore this turns into

$$p < \frac{3(r_1 + 2)}{4}.$$

Since $r_1 = 2$ leads to the limiting case $p < 3$, we see that taking $p > 3$ close enough to 3 and $\widetilde{a} = r_1 \in (2, 3)$ ensures the previous condition. Thus, the assumption (3.2.1) is compatible with the previous analysis and this concludes the proof. ∎

We finally deduce the following non-uniform in time result.

Corollary 3.4.24. *For any finite strong existence time $t > 0$, we have*

$$\nabla u \in \mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{R}^3_+)).$$

Proof. Let $t > 0$ be a finite strong existence time. We consider the exponent $p > 3$ given Proposition 3.4.29, and which also appears in the Assumption (3.2.1). We invoke the Gagliardo-Nirenberg-Sobolev inequality (see Theorem 3.C.1 in the Appendix) which yields

$$\|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \lesssim \|\mathrm{D}^2 u(s)\|^{\beta_p}_{\mathrm{L}^p(\mathbb{R}^3_+)} \|u(s)\|^{1-\beta_p}_{\mathrm{L}^2(\mathbb{R}^3_+)}, \quad s \in [0,t],$$

where $\beta_p := \dfrac{5p}{7p-6}$. Combining the energy inequality (3.1.29) and Lemma 3.4.12, we get

$$\begin{aligned}
\int_0^t \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s &\lesssim \int_0^t \|\mathrm{D}^2 u(s)\|^{\beta_p}_{\mathrm{L}^p(\mathbb{R}^3_+)} \|u(s)\|^{1-\beta_p}_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s \\
&\lesssim \|u\|^{1-\beta_p}_{\mathrm{L}^\infty(0,t;\mathrm{L}^2(\mathbb{R}^3_+))} \int_0^t \|\mathrm{D}^2 u(s)\|^{\beta_p}_{\mathrm{L}^p(\mathbb{R}^3_+)}\,\mathrm{d}s \\
&\lesssim \left(\mathrm{E}(0) + \mathrm{E}(0)^{\frac{1}{2}} t + t^2\right)^{\frac{(1-\beta_p)}{2}} \int_0^t \|\mathrm{D}^2 u(s)\|^{\beta_p}_{\mathrm{L}^p(\mathbb{R}^3_+)}\,\mathrm{d}s \\
&\lesssim \left(\mathrm{E}(0) + \mathrm{E}(0)^{\frac{1}{2}} t + t^2\right)^{\frac{(1-\beta_p)}{2}} t^{1-\frac{p}{\beta_p}} \|\mathrm{D}^2 u\|^{\beta_p}_{\mathrm{L}^p(0,t;\mathrm{L}^p(\mathbb{R}^3_+))},
\end{aligned}$$

where we have used the Hölder's inequality in the last line. Furthermore, thanks to the maximal $\mathrm{L}^p\mathrm{L}^p$ regularity for the Stokes system, we have

$$\|\mathrm{D}^2 u\|_{\mathrm{L}^p(0,t;\mathrm{L}^p(\mathbb{R}^3_+))} \lesssim \|u_0\|_{D_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+)} + \|j_f - \rho_f u\|_{\mathrm{L}^p(0,t;\mathrm{L}^p(\mathbb{R}^3_+))} + \|(u\cdot\nabla)u\|_{\mathrm{L}^p(0,t;\mathrm{L}^p(\mathbb{R}^3_+))} < \infty,$$

thanks to the assumption (3.2.1), Corollary 3.4.22 and Proposition 3.4.29. This allows to conclude the proof. □

Remark 3.4.25. By Corollary 3.6.4 and Corollary 3.4.24, we get $u \in \mathrm{L}^1(0,t;\mathrm{W}^{1,\infty}_0(\mathbb{R}^3_+))$ when $t > 0$ is a finite strong existence time, so that the characteristic curves for the Vlasov equation are classically defined on $(0,t)$ and the representation formula (3.4.9) can be applied.

3.5 Exit geometric condition and absorption

The main goal of this section is to describe precisely the effect of the absorption boundary condition (3.1.11) satisfied by the distribution function f solution to the Vlasov equation (3.1.1). In short, we will study the time of absorption when one starts from a compact support for the initial distribution function f_0. The simple geometry of the flat boundary will allow us to base our study upon the characterics curves for the Vlasov equation.

As explained in the introduction, we rely on different ideas mainly taken from the work of Glass, Han-Kwan and Moussa in [GHKM18] (but which lead to different types of results). In some sense, the adaptation to the half-space case is less tedious because we only deal with a flat boundary. Here, the set $\mathrm{B}_v(R)$ refers to $\mathrm{B}(0,R) \subset \mathbb{R}^3$ with $R \in [0,+\infty]$.

We first introduce the so-called *exit geometric condition.*

Definition 3.5.1. *Let $L, R > 0$. We say that a vector field $\mathrm{U} : \mathbb{R}^+ \times \mathbb{R}^3_+ \to \mathbb{R}^3$ satisfies the exit geometric condition (EGC) in time T with respect to $(\mathbb{R}^2 \times (0, L)) \times \mathrm{B}_v(R)$ if*

$$\sup_{(x,v)\in (\mathbb{R}^2\times(0,L))\times\mathrm{B}_v(R)} \tau^+_{\mathrm{U}}(0, x, v) < T, \tag{3.5.1}$$

where τ^+_{U} refers to Definition (3.4.2) for the characteristic curves $(\mathrm{X}_{\mathrm{U}}, \mathrm{V}_{\mathrm{U}})$ of the Vlasov equation associated to a velocity field U in (3.4.1), that is the solution to

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{X}_{\mathrm{U}}(s; t, x, v) = \mathrm{V}_{\mathrm{U}}(s; t, x, v), \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}_{\mathrm{U}}(s; t, x, v) = P\mathrm{U}(s, \mathrm{X}_{\mathrm{U}}(s; t, x, v)) - \mathrm{V}_{\mathrm{U}}(s; t, x, v) + G, \\ \quad \mathrm{X}_{\mathrm{U}}(t; t, x, v) = x, \\ \quad \mathrm{V}_{\mathrm{U}}(t; t, x, v) = v. \end{cases} \tag{3.5.2}$$

A direct consequence of an EGC satisfied by a velocity field is the following.

Proposition 3.5.2. *Suppose that a velocity field $\mathrm{U} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{W}^{1,\infty}_0(\mathbb{R}^3_+))$ satisfies an EGC in time $T > 0$ with respect to $(\mathbb{R}^2 \times (0, L)) \times \mathrm{B}_v(R)$ for some fixed $L, R > 0$. Then, if f is the solution to the Vlasov equation*

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v\,(f(\mathrm{U} - v) + fG) = 0, \\ \qquad\qquad f_{|t=0} = f_0, \\ \qquad\qquad f = 0, \ \text{ on } \Sigma^-, \end{cases}$$

with initial data f_0, we have for almost every $(x, v) \in \mathbb{R}^3_+ \times \mathbb{R}^3$ and any $t > T$

$$\begin{aligned} f(t, x, v) &= e^{3t}\mathbf{1}_{\mathcal{O}^t_{\mathrm{U}}}(x, v)\mathbf{1}_{|\mathrm{V}^0_t(x,v)|>R}\, f_0(\mathrm{X}^0_t(x, v), \mathrm{V}^0_t(x, v)) \\ &\quad + e^{3t}\mathbf{1}_{\mathcal{O}^t_{\mathrm{U}}}(x, v)\mathbf{1}_{|\mathrm{V}^0_t(x,v)|\leq R}\,\mathbf{1}_{\mathrm{X}^0_t(x,v)_3>L}\, f_0(\mathrm{X}^0_t(x, v), \mathrm{V}^0_t(x, v)), \end{aligned} \tag{3.5.3}$$

where $(\mathrm{X}, \mathrm{V}) = (\mathrm{X}_{\mathrm{U}}, \mathrm{V}_{\mathrm{U}})$ and

$$\mathcal{O}^t_{\mathrm{U}} = \left\{(x, v) \in \mathbb{R}^3_+ \times \mathbb{R}^3 \mid \forall\sigma \in [0, t], \quad \mathrm{X}_{\mathrm{U}}(\sigma; t, x, v) \in \mathbb{R}^3_+\right\},$$

Proof. We drop the dependency un U. Let $t > T$. Observe that the regularity of u allows us to manipulate the characteristic curves for the Vlasov equation in a classic sense on $[0, t]$. From the representation formula (3.4.9), we have

$$f(t, x, v) = e^{3t}\mathbf{1}_{\mathcal{O}^t}(x, v)\,\mathbf{1}_{|\mathrm{V}^0_t(x,v)|\leq R}\,\mathbf{1}_{\mathrm{X}^0_t(x,v)_3\leq L}\, f_0(\mathrm{X}^0_t(x, v), \mathrm{V}^0_t(x, v)) + \widetilde{f}(t, x, v),$$

where $\widetilde{f}(t, x, v)$ denotes the expression in the r.h.s of (3.5.3). We thus have to prove that the first term of the previous equality vanishes. By Proposition 3.4.1, we use the change of variable $(x, v) \mapsto (\mathrm{X}(0; t, x, v), \mathrm{V}(0; t, x, v))$ together with the fact that

$$(\mathrm{X}, \mathrm{V})(0; t, \mathcal{O}^t) = \left\{(x, v) \in \mathbb{R}^3_+ \times \mathbb{R}^3 \mid \tau^+(0, x, v) > t\right\}.$$

and get

$$\begin{aligned} &\int_{\mathbb{R}^3_+\times\mathbb{R}^3} e^{3t}\mathbf{1}_{\mathcal{O}^t}(x, v)\,\mathbf{1}_{|\mathrm{V}^0_t(x,v)|\leq R}\,\mathbf{1}_{\mathrm{X}^0_t(x,v)_3\leq L}\, f_0(\mathrm{X}^0_t(x, v), \mathrm{V}^0_t(x, v))\,\mathrm{d}x\,\mathrm{d}v \\ &= \int_{\mathbb{R}^3_+\times\mathbb{R}^3} \mathbf{1}_{\tau^+(0,x,v)>t}\,\mathbf{1}_{|v|\leq R}\,\mathbf{1}_{x_3\leq L}\, f_0(x, v)\,\mathrm{d}x\,\mathrm{d}v \\ &= \int_{x_3\leq L,\ |v|\leq R} \mathbf{1}_{\tau^+(0,x,v)>t}\, f_0(x, v)\,\mathrm{d}x\,\mathrm{d}v. \end{aligned}$$

By definition of the EGC in time $T > 0$ with respect to $(\mathbb{R}^2 \times (0, L)) \times \mathrm{B}_v(R)$, we have $\tau^+(0, x, v) < T$ for all $(x, v) \in (\mathbb{R}^2 \times (0, L)) \times \mathrm{B}_v(R)$ therefore the last integral is actually zero because $t > T$. Since it is true for all $t > T$ and since f_0 is nonnegative, this concludes the proof. □

The main idea that we follow now is to compare the Vlasov equation with velocity field u (solution to the Navier-Stokes equations) to the "free" Vlasov equation without coupling. We thus consider the following characteristic curves $(\mathrm{X}^g, \mathrm{V}^g)$ for the Vlasov equation associated with the vector field $(x, v) \mapsto (v, G - v)$, namely

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{X}^g(s; t, x, v) = \mathrm{V}^g(s; t, x, v), \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}^g(s; t, x, v) = G - \mathrm{V}^g(s; t, x, v), \\ \quad \mathrm{X}^g(t; t, x, v) = x, \\ \quad \mathrm{V}^g(t; t, x, v) = v. \end{cases} \tag{3.5.4}$$

These are the equations satisfied by the characteristic curves associated to a trivial velocity field (i.e $\mathrm{U} = 0$) in the Vlasov equation and where the particles only undergo the effect of the gravity force G, without being coupled to a surrounding fluid. We have the formulas

$$\begin{cases} \mathrm{X}^g(t; 0, x, v) = x + (1 - e^{-t})v + (t + e^{-t} - 1)G, \\ \mathrm{V}^g(t; 0, x, v) = e^{-t}v + (1 - e^{-t})G, \end{cases} \tag{3.5.5}$$

and in particular, because $G = (0, 0, -g)$, we have

$$\begin{cases} \mathrm{X}^g(t; 0, x, v)_3 = x_3 + (1 - e^{-t})v_3 - (t + e^{-t} - 1)g, \\ \mathrm{V}^g(t; 0, x, v)_3 = -g + e^{-t}(v_3 + g). \end{cases}$$

Remark 3.5.3. In view of the property (3.4.8), we observe that for the characteristic curves $\mathrm{Z}^g := (\mathrm{X}^g, \mathrm{V}^g)$, we have

$$\Big\{(x, v) \in (\mathbb{R}^2 \times (0, L)) \times \mathbb{R}^3 \mid \tau_g^+(0, x, v) = \infty \text{ or } \mathrm{Z}^g(\tau^+(0, x, v); 0, x, v) \in \Sigma^0\Big\} = \emptyset,$$

where τ_g^+ refers to the forward exit time associated to the curves $(\mathrm{X}^g, \mathrm{V}^g)$. This means that the curve X^g leaves the domain with a transversal exit. In addition, if $(x, v) \in (\mathbb{R}^2 \times (0, L)) \times \mathbb{R}^3$ is fixed and if $\mathrm{V}^g(t; 0, x, v)_3 < 0$ then $\mathrm{V}^g(s; 0, x, v)_3 < 0$ for all $s \geq t \geq 0$.

As we shall see in the end of the current section, quantitative information about the EGC are easily available for the trivial velocity field. The following stability result, which is directly inspired by [GHKM18], will thus enable us to show that any velocity field solution to the Navier-Stokes equations satisfies the EGC in some finite time, provided that its $\mathrm{L}^1\mathrm{L}^\infty$ norm is small enough.

Lemma 3.5.4. *Let $\alpha > 0$. There exists a constant $\kappa_\alpha > 0$ such that the following holds. Suppose that the trivial vector field (related to $(\mathrm{X}^g, \mathrm{V}^g)$) satisfies the EGC with respect to $(\mathbb{R}^2 \times (0, L)) \times \mathrm{B}_v(0, R)$ in time $T > 0$, where $L, R > 0$ are given. Then, any vector field $\mathrm{U} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{W}^{1,\infty}_0(\mathbb{R}^3_+))$ such that*

$$\int_0^{T+\alpha} \|\mathrm{U}(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s \leq \kappa_\alpha, \tag{3.5.6}$$

satisfies the EGC in time $T + \alpha$ with respect to $(\mathbb{R}^2 \times (0, L)) \times \mathrm{B}_v(0, R)$.

Proof. For all $(x, v) \in (\mathbb{R}^2 \times (0, L)) \times \mathrm{B}_v(0, R)$ and $(s, t) \in \mathbb{R}^+ \times \mathbb{R}^+$, we consider the characteristic curves $(\mathrm{X}_{\mathrm{U}}(s, t, x, v), \mathrm{V}_{\mathrm{U}}(s, t, x, v))$ (resp. $(\mathrm{X}^g(s, t, x, v), \mathrm{V}^g(s, t, x, v))$) associated to U (resp. to the trivial velocity field). We first set

$$(\mathrm{Y}, \mathrm{W}) := (\mathrm{X}_{\mathrm{U}} - \mathrm{X}^g, \mathrm{V}_{\mathrm{U}} - \mathrm{V}^g),$$

which satisfy the following equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{Y}(s; 0, x, v) = \mathrm{W}(s; 0, x, v), \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{W}(s; 0, x, v) = (P\,\mathrm{U})(s, \mathrm{X}_{\mathrm{U}}(s; 0, x, v)) - \mathrm{W}(s; 0, x, v), \\ \quad \mathrm{Y}(0; 0, x, v) = 0, \\ \quad \mathrm{W}(0; 0, x, v) = 0. \end{cases}$$

We now fix $(x, v) \in (\mathbb{R}^2 \times (0, L)) \times \mathrm{B}_v(0, R)$. We observe that we have for all $t \in \mathbb{R}^+$

$$\mathrm{Y}(t; 0, x, v) = \int_0^t (1 - e^{\tau - t})(P\mathrm{U})(\tau, \mathrm{X}_{\mathrm{U}}(\tau, 0, x, v)\,\mathrm{d}\tau,$$

so that for all $t \in [0, T + \alpha]$

$$|\mathrm{Y}(t; 0, x, v)| \leq \int_0^t (1 - e^{\tau - t})\|(P\mathrm{U})(\tau)\|_{\mathrm{L}^\infty(\mathbb{R}^3)}\,\mathrm{d}\tau \leq \int_0^{T+\alpha} \|\mathrm{U}(\tau)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}\tau, \tag{3.5.7}$$

because of the property (3.4.3).

Furthermore, thanks to the EGC satisfied by the trivial velocity field in time T and Remark 3.5.3, we have $\mathrm{X}^g(T; 0, x, v)_3 < 0$ and $\mathrm{V}^g(T; 0, x, v)_3 < 0$. Hence,

$$\begin{aligned} \mathrm{X}^g(T + \alpha; 0, x, v)_3 &= \mathrm{X}^g(T + \alpha; T, \mathrm{X}^g(T; 0, x, v)_3, \mathrm{V}^g(T; 0, x, v)_3)_3 \\ &= \mathrm{X}^g(T; 0, x, v)_3 + (1 - e^{-\alpha})(\mathrm{V}^g(T; 0, x, v)_3 + g) - \alpha g \\ &< (1 - \alpha - e^{-\alpha})g < 0, \end{aligned} \tag{3.5.8}$$

because $\alpha > 0$. If we set $\eta_\alpha := (e^{-\alpha} + \alpha - 1)g > 0$ and $\kappa_\alpha := \eta_\alpha/2$, we see that if U satisfies the condition

$$\int_0^{T+\alpha} \|\mathrm{U}(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s \leq \kappa_\alpha,$$

then the estimate (3.5.7) turns into

$$\sup_{t \in [0, T+\alpha]} |\mathrm{Y}(t; 0, x, v)| \leq \frac{\eta_\alpha}{2}. \tag{3.5.9}$$

In view of (3.5.8) and Remark 3.5.3, we get the existence of $t_0 = t_0(x, v) \in (0, T + \alpha]$ such that $\mathrm{X}^g(t_0; 0, x, v) \notin \{h \in \mathbb{R}^3 \mid h_3 > -\eta_\alpha\}$. From (3.5.9), we deduce that $\mathrm{X}_{\mathrm{U}}(t_0; 0, x, v) \notin \mathbb{R}^3_+$ and therefore $\tau^+_{\mathrm{U}}(0, x, v) < T + \alpha$. As it is true for any $(x, v) \in (\mathbb{R}^2 \times (0, L)) \times \mathrm{B}_v(R)$, this means by definition that the EGC is satisfied for U in time $T + \alpha$ with respect to $(\mathbb{R}^2 \times (0, L)) \times \mathrm{B}_v(0, R)$. □

Then, in view of the simple and explicit form of the characteristic curves $(\mathrm{X}^g, \mathrm{V}^g)$ for the Vlasov equation associated to a trivial velocity field, we can easily obtain precise information on the EGC satisfied for this velocity field. Indeed, for all $(x, v) \in (\mathbb{R}^2 \times (0, L)) \times \mathrm{B}_v(0, R)$ and for all $t \geq 0$, we have

$$\begin{aligned} \mathrm{X}^g(t; 0, x, v)_3 &= x_3 + (1 - e^{-t})(v_3 + g) - tg \\ &< L + |v| + g - tg \\ &< L + R + g - tg. \end{aligned}$$

It naturally leads to the following definition and properties about the EGC for the trivial vector field and the characteristic curves $(\mathrm{X}^g, \mathrm{V}^g)$.

Definition 3.5.5. *We set*

$$t_0(L, R) := \frac{L + R + g}{g}, \tag{3.5.10}$$

$$t_0 := t_0(1, 1). \tag{3.5.11}$$

Lemma 3.5.6. *If L, R are given, the trivial vector field $\mathrm{U} \equiv 0$ (associated to $(\mathrm{X}^g, \mathrm{V}^g)$) satisfies the EGC in time $t_0(L, R)$ with respect to $(\mathbb{R}^2 \times (0, L)) \times \mathrm{B}_v(R)$.*

Proof. Assume that $\mathrm{X}^g(t, 0, x, v)_3 > 0$ for all $t \in [0, t_0(L, R)]$ and $(x, v) \in (\mathbb{R}^2 \times (0, L)) \times \mathrm{B}_v(0, R)$. Then

$$\begin{aligned} 0 &< x_3 + (1 - e^{-t})(v_3 + g) - tg \\ &< L + R + g - tg \\ &= g(t_0(L, R) - t), \end{aligned}$$

so that we get a contradiction by taking $t = t_0(L, R)$. □

Remark 3.5.7. We note that $t_0 = t_0(1, 1)$ does not depend on the initial data. This time only has to be seen as a reference time after which we will use the absorption phenomenon. The subsequent analysis could have been performed by replacing $t_0(1, 1)$ by $t_0(\kappa, \kappa)$ for any $\kappa > 0$.

The following result is, in some sense, of reverse nature: given a time $t > t_0$, we describe which proportion of the initial velocities will lead to absorption before time t. More precisely, we state the following lemma.

Lemma 3.5.8. *There exist some continuous increasing functions $\mathrm{L}_g : [t_0, +\infty) \to \mathbb{R}^+$ and $\mathrm{R}_g : [t_0, +\infty) \to \mathbb{R}^+$ such that for all $t > t_0$ the trivial velocity vector field (associated to $(\mathrm{X}^g, \mathrm{V}^g)$) satisfies the EGC in time t with respect to $(\mathbb{R}^2 \times (0, 1 + \mathrm{L}_g(t))) \times \mathrm{B}_v(0, 1 + \mathrm{R}_g(t))$. Furthermore, there exist $C_g, \underline{C}_g, \overline{C}_g > 0$ such that for all $s > t_0$*

$$\frac{1}{1 + \mathrm{L}_g(s)} \leq \frac{C_g}{1 + s},$$

$$\frac{\underline{C}_g}{1 + s} \leq \frac{1}{1 + \mathrm{R}_g(s)} \leq \frac{\overline{C}_g}{1 + s}.$$

Proof. Since the characteristic curve X^g is a vertical line, a necessary and sufficient condition ensuring an EGC in time t with respect to $(\mathbb{R}^2 \times (0, 1 + \mathrm{L}_g(t))) \times \mathrm{B}_v(0, 1 + \mathrm{R}_g(t))$ (for some positive functions L_g and R_g to be determined) is that for all $(x, v) \in (\mathbb{R}^2 \times (0, 1 + \mathrm{L}_g(t))) \times \mathrm{B}_v(0, 1 + \mathrm{R}_g(t))$, the following inequality holds:

$$\mathrm{X}^g(t; 0, x, v)_3 = x_3 + (1 - e^{-t})(v_3 + g) - tg < 0. \tag{3.5.12}$$

Indeed, one can easily show that the real function $s \mapsto \mathrm{X}^g(s;0,x,v)_3$ becomes strictly decreasing after its first cancellation. We then set

$$\mathrm{L}_g(s) := \frac{1}{2}(sg - (1-e^{-s})g) - 1, \quad \mathrm{R}_g(s) := \frac{1}{2}\left(\frac{sg}{1-e^{-s}} - g\right) - 1.$$

Now, if $0 < x_3 < 1 + \mathrm{L}_g(t)$ and $|v| < 1 + \mathrm{R}_g(t)$, we observe that

$$x_3 + (1-e^{-t})v_3 \leq x_3 + (1-e^{-t})|v| < tg - (1-e^{-t})g,$$

therefore we have $\mathrm{X}^g(t;0,x,v)_3 < 0$. It remains to show that L_g and R_g are positive on $[t_0, +\infty[$. As they are nondecreasing functions, we only have to prove that $\mathrm{L}_g(t_0) > 0$ and $\mathrm{R}_g(t_0) > 0$: recalling the explicit Definition 3.5.5 of the time $t_0 = 1 + 2g^{-1}$, we have

$$\mathrm{L}_g(t_0) = \frac{1}{2}\left((1+\frac{2}{g})g - (1-e^{-(1+\frac{2}{g})})g\right) - 1 = \frac{1}{2}e^{-(1+\frac{2}{g})}g > 0,$$
$$\mathrm{R}_g(t_0) = \frac{1}{2}\left(\frac{(1+\frac{2}{g})g}{1-e^{-(1+\frac{2}{g})}} - g\right) - 1 = \frac{2}{1-e^{-(1+\frac{2}{g})}} - 1 + g\left(\frac{1}{1-e^{-(1+\frac{2}{g})}} - \frac{1}{2}\right) > 1 + \frac{g}{2} > 0,$$

which is the desired claim. Concerning the last part, a direct computation shows that for all $s \geq t_0$, we have

$$\frac{1+s}{1+\mathrm{L}_g(s)} \leq \frac{1+t_0}{1+\mathrm{L}_g(t_0)}, \quad \frac{1+s}{1+\mathrm{R}_g(s)} \leq \frac{1+t_0}{1+\mathrm{R}_g(t_0)},$$

and that $s \mapsto \dfrac{1+\mathrm{R}_g(s)}{1+s}$ is bounded from above on $[t_0, +\infty)$. The proof is then complete. □

3.6 The bootstrap argument

In this section, we provide a proof of Theorem 3.2.1, relying on the absorption effect highlighted in Section 3.5. Our strategy is based on a bootstrap argument reminiscent of the ideas of [HKMM20, HK22]. Roughly speaking, we will prove that one can propagate the controls (3.4.10) and (3.5.6) for the velocity field u.

3.6.1 Initialization of the bootstrap procedure

In order to set up a boostrap procedure, we introduce the following quantities.

Definition 3.6.1. *We set $T_0 := t_0 + 1$ where t_0 is given in Definition 3.5.5.*

Definition 3.6.2. *We consider $\delta_0 > 0$ which satisfies $\delta_0 e^{\delta_0} < 1/9$ and $\delta_0 < \kappa_{1/2}$, where $\kappa_{1/2}$ is given in Lemma 3.5.4.*

Let (u, f) be a global weak solution to the Vlasov-Navier-Stokes system in the sense of Definition 3.1.6 and associated to an admissible initial data (u_0, f_0) satisfying (3.2.1). We start with the following lemma.

Lemma 3.6.3. *We have*

$$\int_0^{T_0} \|j_f(s) - \rho_f u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \, \mathrm{d}s \lesssim N_q(f_0)\eta(T_0)\left[\mathrm{E}(0) + T_0^2\right], \tag{3.6.1}$$

for some nondecreasing positive continuous function η and the exponent q appearing in ssumption (3.2.1). Furthermore, under the smallness assumption (3.2.2), the time T_0 is a strong existence time, in the sense of Definition 3.4.21.

Proof. We first set $F := j_f - \rho_f u$. From Lemma 3.4.13 and (3.4.24) in Proposition 3.4.18, we infer that

$$\int_0^{T_0} \|F(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s \leq \sup_{s\in[0,T_0]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \left[\mathrm{E}(0) + \mathrm{E}(0)T_0 + T_0^2\right]$$
$$\lesssim N_q(f_0)\eta(T_0)\left[\mathrm{E}(0) + T_0^2\right].$$

Thanks to the Cauchy-Schwarz inequality, we also have

$$\int_0^{T_0} \|F(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s \leq T_0^{1/2}\left(\int_0^{T_0} \|F(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s\right)^{1/2},$$

therefore the inequality (3.6.1) now entails

$$\|u_0\|^2_{\mathrm{H}^1(\mathbb{R}^3_+)} + \int_0^{T_0} \left[\|F(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|F(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\right]\,\mathrm{d}s$$
$$\lesssim \|u_0\|^2_{\mathrm{H}^1(\mathbb{R}^3_+)} + N_q(f_0)\eta(T_0)\left[\mathrm{E}(0) + \mathrm{E}(0)T_0 + T_0^2\right]$$
$$+ N_q(f_0)^{1/2}T_0^{1/2}\eta(T_0)^{1/2}\left[\mathrm{E}(0) + T_0^2\right]^{1/2}.$$

If $\mathrm{C}_\star$ refers to the universal constant given in Proposition 3.4.19, we can thus use the smallness assumption (3.2.2) to ensure that

$$\|u_0\|^2_{\mathrm{H}^1(\mathbb{R}^3_+)} + \int_0^{T_0} \left[\|F(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|F(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\right]\,\mathrm{d}s < \mathrm{C}_\star,$$

which means that T_0 is a strong existence time. □

Corollary 3.6.4. *Under the smallness assumption* (3.2.2)*, we have*

$$\int_0^{T_0} \|u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s < \frac{\delta_0}{2}. \tag{3.6.2}$$

Proof. By Lemma 3.6.3, we know that T_0 is a strong existence time therefore the parabolic regularization of the Navier-Stokes equations stated in Proposition 3.4.19 holds for u on $[0, T_0]$. Namely, we get

$$u \in \mathrm{L}^\infty(0, T_0; \mathrm{H}^1_{\mathrm{div}}(\mathbb{R}^3_+)) \cap \mathrm{L}^2(0, T_0; \mathrm{H}^2(\mathbb{R}^3_+)),$$

and there exists $\widetilde{C} > 0$ such that for all $t \in [0, T_0]$

$$\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \int_0^t \|\mathrm{D}^2 u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s \leq \widetilde{C}\left(\|u_0\|^2_{\mathrm{H}^1(\mathbb{R}^3_+)} + \|j_f - \rho_f u\|^2_{\mathrm{L}^2(0,T_0;\mathrm{L}^2(\mathbb{R}^3_+))}\right). \tag{3.6.3}$$

We then use the Gagliardo-Nirenberg-Sobolev inequality (see Theorem 3.C.1 in the Appendix) with $(p, r, q, j, m) = (\infty, 2, 6, 0, 2)$ so that there exists a universal constant $C_0 > 0$ such that

$$\|u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \leq C_0 \|\mathrm{D}^2(s)\|^{1/2}_{\mathrm{L}^2(\mathbb{R}^3_+)} \|u(s)\|^{1/2}_{\mathrm{L}^6(\mathbb{R}^3_+)}.$$

By Sobolev's embedding, we infer from (3.6.3) that

$$\int_0^{T_0} \|u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s \leq C_0 \int_0^{T_0} \|\mathrm{D}^2 u(s)\|^{1/2}_{\mathrm{L}^2(\mathbb{R}^3_+)} \|\nabla u(s)\|^{1/2}_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s$$
$$\leq C_0 T_0^{3/4} \|\nabla u\|^{1/2}_{\mathrm{L}^\infty(0,T_0;\mathrm{L}^2(\mathbb{R}^3_+))} \left(\int_0^{T_0} \|\mathrm{D}^2 u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s\right)^{1/4}$$
$$\leq C_0 T_0^{3/4} \widetilde{C}^{1/16} \left(\|u_0\|^2_{\mathrm{H}^1(\mathbb{R}^3_+)} + \|j_f - \rho_f u\|^2_{\mathrm{L}^2(0,T_0;\mathrm{L}^2(\mathbb{R}^3_+))}\right)^{1/16}$$
$$\lesssim C_0 T_0^{3/4} \widetilde{C}^{1/16} \left(\|u_0\|^2_{\mathrm{H}^1(\mathbb{R}^3_+)} + N_q(f_0)\eta(T_0)\left[\mathrm{E}(0) + \mathrm{E}(0)T_0 + T_0^2\right]\right)^{1/16},$$

thanks to (3.6.1). We can now proceed exactly in the same way as in the proof of Lemma 3.6.3: since T_0 is fixed, we can use the smallness condition (3.2.2) to ensure that

$$\int_0^{T_0} \|u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\, \mathrm{d}s < \frac{\delta_0}{2}.$$

This concludes the proof of the corollary. □

In order to set up the bootstrap argument, we introduce

$$t^\star = \sup\left\{\text{strong existence times } t \geq T_0 \;\middle|\; \int_{T_0}^{t} \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\, \mathrm{d}s < \delta_0,\ \int_{T_0}^{t} \|u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\, \mathrm{d}s < \frac{\delta_0}{2}\right\}. \tag{3.6.4}$$

Our main goal is now to show that $t^\star = +\infty$.

Lemma 3.6.5. *We have $t^\star > T_0$.*

Proof. In view of Lemma 3.6.3, we see that using one more time the smallness assumption (3.2.2), the time $t = T_0 + \varepsilon_0$ is a strong existence time, if $\varepsilon_0 > 0$ is fixed. Now, we observe that a continuity argument, Proposition 3.4.24 and (3.4.22) ensure that

$$\int_{T_0}^{T_0+\varepsilon_0} \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\, \mathrm{d}s < \delta_0, \quad \int_{T_0}^{T_0+\varepsilon_0} \|u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\, \mathrm{d}s < \frac{\delta_0}{2},$$

if ε_0 is small enough. By definition of $t^\star$, this concludes the proof. □

We then argue by contradiction and shall assume from now on that $t^\star < \infty$.

3.6.2 Absorption and decay in time of the moments

We eventually explain how one can take advantage of the absorption effect at the boundary. Recall first the concept of EGC introduced in Definition 3.5.1. In view of the bounds satisfied before $t^\star$, the vector field u will satisfy an EGC at time $t < t^\star$ with respect to some compact depending on t. This will imply a (polynomial) decay in time of the moments of f and then of the Brinkman force $j_f - \rho_f u$.

Recalling the notations introduced in Definition 3.1.3 and Definition 3.1.4, we state the following fundamental lemma which highlights the precise link between absorption and decay.

Lemma 3.6.6. *Let $t \in (T_0, t^\star)$ and $r \in [1, \infty)$. Suppose that $k_1, k_2, \ell \in \mathbb{R}^+$ and $q > 3$ satisfy*

$$q > k_1 + \ell + 3,$$

Assume that the velocity field u satisfies an EGC in time t with respect to $(\mathbb{R}^2 \times (0, 1 + \mathrm{L}(t))) \times \mathrm{B}_v(0, 1 + \mathrm{R}(t))$ for some nondecreasing functions $\mathrm{L} : \mathbb{R}^+ \to \mathbb{R}^+$ and $\mathrm{R} : \mathbb{R}^+ \to \mathbb{R}^+$. Then we have for almost every $x \in \mathbb{R}^3_+$

$$m_\ell f(t,x) \lesssim \frac{N_q(f_0)}{(1+\mathrm{R}(t))^{k_1}} + \frac{H_{\ell,k_2}(f_0)}{(1+\mathrm{L}(t))^{k_2}}, \tag{3.6.5}$$

$$\|m_\ell f(t)\|_{\mathrm{L}^r(\mathbb{R}^3_+)} \lesssim \frac{K_{q,r}(f_0)}{(1+\mathrm{R}(t))^{k_1}} + \frac{F_{\ell,k_2,r}(f_0)}{(1+\mathrm{L}(t))^{k_2}}, \tag{3.6.6}$$

where $\lesssim$ only depends on $k_1, k_2, q, \ell, g, \delta_0, r$.

Proof. The whole proof is based on the fact that

$$f(t,x,v) = f^{\natural}(t,x,v) + f^{\flat}(t,x,v), \tag{3.6.7}$$

where

$$f^{\natural}(t,x,v) := e^{3t}\mathbf{1}_{\mathcal{O}^t}(x,v)\,\mathbf{1}_{|\mathrm{V}_t^0(x,v)|>1+\mathrm{R}(t)}\,f_0(\mathrm{X}_t^0(x,v),\mathrm{V}_t^0(x,v)), \tag{3.6.8}$$

$$f^{\flat}(t,x,v) := e^{3t}\mathbf{1}_{\mathcal{O}^t}(x,v)\,\mathbf{1}_{|\mathrm{V}_t^0(x,v)|\leq 1+\mathrm{R}(t)}\,\mathbf{1}_{\mathrm{X}_t^0(x,v)_3>1+\mathrm{L}(t)}\,f_0(\mathrm{X}_t^0(x,v),\mathrm{V}_t^0(x,v)). \tag{3.6.9}$$

Indeed, because of the EGC satisfied by u in time t with respect to $(\mathbb{R}^2\times(0,1+\mathrm{L}(t)))\times\mathrm{B}_v(0,1+\mathrm{R}(t))$, we can apply Proposition 3.5.2 so that (3.6.7) holds. The decay of the moments of $f^{\natural}$ and $f^{\flat}$ is then studied separetely.

• For the term $f^{\natural}$ defined in (3.6.8), we use the admissible change of variable $v\mapsto \mathrm{V}_t^0(x,v) = \Gamma_{t,x}(v)$ given by Lemma 3.4.5, whose Jacobian inverse is bounded by $2e^{-3t}$. We get

$$\begin{aligned}
m_\ell f^{\natural}(t,x) &= \int_{\mathbb{R}^3} |v|^\ell f^{\natural}(t,x,v)\,\mathrm{d}v\\
&= \int_{\mathbb{R}^3} |v|^\ell e^{3t}\mathbf{1}_{\mathcal{O}^t}(x,v)\,\mathbf{1}_{|\mathrm{V}_t^0(x,v)|>1+\mathrm{R}(t)}\,f_0(\mathrm{X}_t^0(x,v),\mathrm{V}_t^0(x,v))\,\mathrm{d}v\\
&= \int_{\mathbb{R}^3} |\Gamma_{t,x}^{-1}(w)|^\ell e^{3t}\mathbf{1}_{\mathcal{O}^t}(x,\Gamma_{t,x}^{-1}(w))\mathbf{1}_{|w|>1+\mathrm{R}(t)}\,f_0(\Lambda_{t,w}(x),w)|\det \mathrm{D}_w\Gamma_{t,x}^{-1}(w)|\,\mathrm{d}w\\
&\lesssim \frac{1}{(1+\mathrm{R}(t))^{k_1}}\int_{\mathbb{R}^3} |w|^{k_1}|\Gamma_{t,x}^{-1}(w)|^\ell \mathbf{1}_{\mathcal{O}^t}(x,\Gamma_{t,x}^{-1}(w))\,f_0(\Lambda_{t,w}(x),w)\,\mathrm{d}w,
\end{aligned}$$

where we have used the notations of Lemma 3.4.7, namely $\Lambda_{t,w}(x) = \mathrm{X}_t^0(x,\Gamma_{t,x}^{-1}(w))$. Thanks to the inequality (3.4.12), we also have

$$\begin{aligned}
|\Gamma_{t,x}^{-1}(w)| &\leq |w| + (1-e^{-t})g + \int_0^t e^{\tau-t}\|Pu(\tau)\|_{\mathrm{L}^\infty(\mathbb{R}^3)}\,\mathrm{d}\tau,\\
&\leq |w| + g + \delta_0,
\end{aligned}$$

and because of the Definition (3.4.7) of $\mathcal{O}^t$, we see that if $(x,\Gamma_{t,x}^{-1}(w))\in\mathcal{O}^t$ then $\Lambda_{t,w}(x)\in\mathbb{R}^3_+$. Therefore we obtain

$$\begin{aligned}
m_\ell f^{\natural}(t,x) &\lesssim \frac{1}{(1+\mathrm{R}(t))^{k_1}}\int_{\mathbb{R}^3}\Big[|w|^{k_1+\ell}+|w|^{k_1}\Big]\,\mathbf{1}_{\mathcal{O}^t}(x,\Gamma_{t,x}^{-1}(w))\,f_0(\Lambda_{t,w}(x),w)\,\mathrm{d}w\\
&= \frac{1}{(1+\mathrm{R}(t))^{k_1}}\int_{\mathbb{R}^3}\frac{|w|^{k_1+\ell}+|w|^{k_1}}{1+|w|^q}\mathbf{1}_{\mathcal{O}^t}(x,\Gamma_{t,x}^{-1}(w))(1+|w|^q)\,f_0(\Lambda_{t,w}(x),w)\,\mathrm{d}w\\
&\leq \frac{N_q(f_0)}{(1+\mathrm{R}(t))^{k_1}}\int_{\mathbb{R}^3}\frac{|w|^{k_1+\ell}+|w|^{k_1}}{1+|w|^q}\,\mathrm{d}w\\
&\lesssim \frac{N_q(f_0)}{(1+\mathrm{R}(t))^{k_1}},
\end{aligned}$$

for $q>k_1+\ell+3$. This entails the contribution from $f^{\natural}$ in the inequality (3.6.5). Concerning its contribution to the inequality (3.6.6), we combine the previous change of variable in velocity with the Minkowski's integral inequality (see [HLP88, Theorem 202]) and the bound (3.4.12) to write with the notations of Lemma 3.4.7

$$\begin{aligned}\|m_\ell f^\natural(t)\|_{\mathrm{L}^r(\mathbb{R}^3_+)} &= \left[\int_{\mathbb{R}^3_+}\left(\int_{\mathbb{R}^3}|v|^\ell f^\natural(t,x,v)\,\mathrm{d}v\right)^r \mathrm{d}x\right]^{1/r}\\
&\lesssim \frac{1}{(1+\mathrm{R}(t))^{k_1}}\left[\int_{\mathbb{R}^3_+}\left(\int_{\mathbb{R}^3}|w|^{k_1}|\Gamma^{-1}_{t,x}(w)|^\ell \mathbf{1}_{\mathcal{O}^t}(x,\Gamma^{-1}_{t,x}(w))\,f_0(\Lambda_{t,w}(x),w)\,\mathrm{d}w\right)^r \mathrm{d}x\right]^{1/r}\\
&\leq \frac{1}{(1+\mathrm{R}(t))^{k_1}}\int_{\mathbb{R}^3}|w|^{k_1}\left(\int_{\mathbb{R}^3_+}|\Gamma^{-1}_{t,x}(w))|^{r\ell}\mathbf{1}_{\mathcal{O}^t}(x,\Gamma^{-1}_{t,x}(w))\,f_0(\Lambda_{t,w}(x),w)^r\,\mathrm{d}x\right)^{1/r}\mathrm{d}w\\
&\leq \frac{1}{(1+\mathrm{R}(t))^{k_1}}\int_{\mathbb{R}^3}\left[|w|^{k_1+\ell}+|w|^{k_1}\right]\left(\int_{\mathbb{R}^3_+}\mathbf{1}_{\mathcal{O}^t}(x,\Gamma^{-1}_{t,x}(w))\,f_0(\Lambda_{t,w}(x),w)^r\,\mathrm{d}x\right)^{1/r}\mathrm{d}w.\end{aligned}$$

Performing the admissible change of variable $x \mapsto \Lambda_{t,w}(x)$ in the interior integral, with a Jacobian inverse bounded by 2 (see Lemma 3.4.7), we get

$$\begin{aligned}\|m_\ell f^\natural(t)\|_{\mathrm{L}^r(\mathbb{R}^3_+)} &\lesssim \frac{1}{(1+\mathrm{R}(t))^{k_1}}\int_{\mathbb{R}^3}\left[|w|^{k_1+\ell}+|w|^{k_1}\right]\\
&\qquad\left(\int_{\mathbb{R}^3}\mathbf{1}_{\Lambda^{-1}_{t,w}(x)\in\mathbb{R}^3_+}\mathbf{1}_{\mathcal{O}^t}\left(\Lambda^{-1}_{t,w}(x),\Gamma^{-1}_{t,\Lambda^{-1}_{t,w}(x)}(w)\right)\,f_0(x,w)^r\,\mathrm{d}x\right)^{1/r}\mathrm{d}w\\
&\lesssim \frac{1}{(1+\mathrm{R}(t))^{k_1}}\int_{\mathbb{R}^3}\frac{|w|^{k_1+\ell}+|w|^{k_1}}{1+|w|^q}(1+|w|^q)\|f_0(\cdot,w)\|_{\mathrm{L}^r(\mathbb{R}^3_+)}\,\mathrm{d}w\\
&\leq \frac{K_{q,r}(f_0)}{(1+\mathrm{R}(t))^{k_1}}\int_{\mathbb{R}^3}\frac{|w|^{k_1+\ell}+|w|^{k_1}}{1+|w|^q}\,\mathrm{d}w\\
&\lesssim \frac{K_{q,r}(f_0)}{(1+\mathrm{R}(t))^{k_1}},\end{aligned}$$

for $q > k_1+\ell+3$.

• For the term $f^\flat$ defined in (3.6.9), we also perform the change of variable $v \mapsto \mathrm{V}^0_t(x,v) = \Gamma_{t,x}(v)$ which entails

$$\begin{aligned}m_\ell f^\flat(t,x) &= \int_{\mathbb{R}^3}|v|^\ell f^\flat(t,x,v)\,\mathrm{d}v\\
&= \int_{\mathbb{R}^3}|v|^\ell e^{3t}\mathbf{1}_{\mathcal{O}^t}(x,v)\,\mathbf{1}_{|\mathrm{V}^0_t(x,v)|\leq 1+\mathrm{R}(t)}\,\mathbf{1}_{\mathrm{X}^0_t(x,v)_3>1+\mathrm{L}(t)}\,f_0(\mathrm{X}^0_t(x,v),\mathrm{V}^0_t(x,v))\,\mathrm{d}v\\
&= \int_{\mathbb{R}^3}|\Gamma^{-1}_{t,x}(w)|^\ell e^{3t}\mathbf{1}_{\mathcal{O}^t}(x,\Gamma^{-1}_{t,x}(w))\,\mathbf{1}_{|w|\leq 1+\mathrm{R}(t)}\,\mathbf{1}_{\Lambda_{t,w}(x)_3>1+\mathrm{L}(t)}\\
&\qquad\qquad f_0(\Lambda_{t,w}(x),w)|\det \mathrm{D}_w\Gamma^{-1}_{t,x}(w)|\,\mathrm{d}w\\
&\lesssim \frac{1}{(1+\mathrm{L}(t))^{k_2}}\int_{\mathbb{R}^3}(|w|^\ell+1)\mathbf{1}_{|w|\leq 1+\mathrm{R}(t)}\,\mathbf{1}_{\mathcal{O}^t}(x,\Gamma^{-1}_{t,x}(w))|\Lambda_{t,w}(x)_3|^{k_2}\,f_0(\Lambda_{t,w}(x),w)\,\mathrm{d}w,\end{aligned}$$

so that

$$m_\ell f^\flat(t,x) \lesssim \frac{H_{\ell,k_2}(f_0)}{(1+\mathrm{L}(t))^{k_2}}.$$

Here, we have again used the fact that if $(x,\Gamma^{-1}_{t,x}(w)) \in \mathcal{O}^t$ then $\Lambda_{t,w}(x) \in \mathbb{R}^3_+$. This provides the contribution of $f^\flat$ to the estimate (3.6.5). To get its contribution to the inequality (3.6.6), we perform exactly the same computations as for $f^\natural$ to write

$$\begin{aligned}
\|m_\ell f^\flat(t)\|_{\mathrm{L}^r(\mathbb{R}^3_+)} &= \left[\int_{\mathbb{R}^3_+}\left(\int_{\mathbb{R}^3}|v|^\ell f^\flat(t,x,v)\,\mathrm{d}v\right)^r\mathrm{d}x\right]^{1/r} \\
&\lesssim \frac{1}{(1+\mathrm{L}(t))^{k_2}}\int_{|w|\leq 1+\mathrm{R}(t)}\left[|w|^\ell+1\right] \\
&\qquad \left(\int_{\mathbb{R}^3}\mathbf{1}_{\Lambda^{-1}_{t,w}(x)\in\mathbb{R}^3_+}\mathbf{1}_{\mathcal{O}^t}\left(\Lambda^{-1}_{t,w}(x),\Gamma^{-1}_{t,\Lambda^{-1}_{t,w}(x)}(w)\right)|x_3|^{k_2 r}f_0(x,w)^r\,\mathrm{d}x\right)^{1/r}\mathrm{d}w \\
&\lesssim \frac{1}{(1+\mathrm{L}(t))^{k_2}}\int_{|w|\leq 1+\mathrm{R}(t)}\left[|w|^\ell+1\right]\|x_3^{k_2}f_0(\cdot,w)\|_{\mathrm{L}^r(\mathbb{R}^3_+)}\,\mathrm{d}w,
\end{aligned}$$

so that

$$\|m_\ell f^\flat(t)\|_{\mathrm{L}^r(\mathbb{R}^3_+)} \lesssim \frac{F_{\ell,k_2,r}(f_0)}{(1+\mathrm{L}(t))^{k_2}}.$$

Gathering all the pieces together, we have proven inequalities (3.6.5) and (3.6.6). This achieves the proof. □

Of course, the previous lemma is only available if we could ensure that a certain EGC is satisfied by the velocity field u at time t. Thanks to Lemma 3.5.8 stated in Section 3.5, we are able to obtain such a condition.

Proposition 3.6.7. *For any $t\in(T_0,t^\star)$, the velocity field u satisfies an EGC in time t with respect to $(\mathbb{R}^2\times(0,1+\mathrm{L}(t))\times\mathrm{B}_v(0,1+\mathrm{R}(t))$ for some nondecreasing continuous functions $\mathrm{L}:[T_0,+\infty)\to\mathbb{R}^+$ and $\mathrm{R}:[T_0,+\infty)\to\mathbb{R}^+$ satisfying for all $s>T_0$*

$$\frac{1}{1+\mathrm{L}(s)}\lesssim\frac{1}{1+s},\quad \frac{1}{1+\mathrm{R}(s)}\lesssim\frac{1}{1+s},$$

where $\lesssim$ only depends on T_0 and g.

In particular, if $r\in[1,\infty)$, $k_1,k_2,\ell\in\mathbb{R}^+$ and $q>3$ satisfy

$$q>k_1+\ell+3,$$

then for almost every $x\in\mathbb{R}^3_+$

$$m_\ell f(t,x)\lesssim\frac{N_q(f_0)}{(1+t)^{k_1}}+\frac{H_{\ell,k_2}(f_0)}{(1+t)^{k_2}}, \tag{3.6.10}$$

$$\|m_\ell f(t)\|_{\mathrm{L}^r(\mathbb{R}^3_+)}\lesssim\frac{K_{q,r}(f_0)}{(1+t)^{k_1}}+\frac{F_{\ell,k_2,r}(f_0)}{(1+t)^{k_2}}, \tag{3.6.11}$$

where $\lesssim$ only depends on $k_1,k_2,q,\ell,g,\delta_0$ and T_0.

Proof. Thanks to Lemma 3.5.8, we know that if $t-1/2>t_0$, the trivial velocity field satisfies an EGC in time $t-1/2$ with respect to $(\mathbb{R}^2\times(0,1+\widetilde{\mathrm{L}}(t-1/2))\times\mathrm{B}_v(0,1+\widetilde{\mathrm{R}}(t-1/2))$ for some continuous nondecreasing functions $\widetilde{\mathrm{L}}:[t_0,+\infty)\to\mathbb{R}^+$ and $\widetilde{\mathrm{R}}:[t_0,+\infty)\to\mathbb{R}^+$ such that for any $s>t_0$

$$\frac{1}{1+\widetilde{\mathrm{L}}(s)}\lesssim\frac{1}{1+s},\quad \frac{1}{1+\widetilde{\mathrm{R}}(s)}\lesssim\frac{1}{1+s}, \tag{3.6.12}$$

where $\lesssim$ depends on t_0 and g. Furthermore, by definition of δ_0, we have

$$\int_0^t \|u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s < \delta_0 < \kappa_{1/2},$$

so that, owing to Lemma 3.5.4, we get the fact that the velocity field u satisfies an EGC in time t with respect to $(\mathbb{R}^2 \times (0, 1+\widetilde{\mathrm{L}}(t-1/2)) \times \mathrm{B}_v(0, 1+\widetilde{\mathrm{R}}(t-1/2))$. We have $t > T_0 = t_0+1 > t_0+1/2$ so that if we set $\mathrm{L}(s) := \widetilde{\mathrm{L}}(s-1/2)$ and $\mathrm{R}(s) := \widetilde{\mathrm{R}}(s-1/2)$, we observe that the velocity field u satisfies an EGC in time t with respect to $(\mathbb{R}^2 \times (0, 1+\mathrm{L}(t)) \times \mathrm{B}_v(0, 1+\mathrm{R}(t))$. Furthermore, by (3.6.12), for all $s > t_0+1/2$

$$\frac{1}{1+\mathrm{L}(s)} \lesssim \frac{1}{1+s}, \quad \frac{1}{1+\mathrm{R}(s)} \lesssim \frac{1}{1+s}.$$

The last part of the statement readily comes from the previous estimates and the inequalities (3.6.5)-(3.6.6) in Lemma 3.6.6. $\square$

We now fix $T \in (T_0, t^\star)$. Since the decay provided by the absorption can only be considered after time T_0, we will split the study of the decay estimates between $[0, T_0]$ and $[T_0, T]$. In particular, the local in time estimates of Section 3.4 will help us to treat the part on $[0, T_0]$.

In view of the assumptions (3.2.1) and (3.2.2) on the initial data, we will not be very precise about the exact range of exponents used when the quantities $N_q(f_0), H_{q,m}(f_0), K_{q,r}(f_0), F_{q,m,r}(f_0)$ of Definition 3.1.4 may appear. Note that they are nondecreasing when the parameters q and m increase. In the last part of the current section, we will consider the maximum of the finite family of indices we have used previously. Moreover, these indices may sometimes depend on the exponent γ that we use to quantify the decay in time but this one will be somehow fixed in the last step of the bootstrap. Thus, all the following results must be understood as if we take the involved exponents large enough. Let us also recall the symbol $\lesssim_0$ introduced in Notation 3.4.8.

Corollary 3.6.8. *There exists $\varepsilon > 0$ small enough such that the following holds. Let $p \in (3, 3+\varepsilon)$. There exist nondecreasing positive functions φ_2 and φ_p vanishing at 0 such that for any $k > 0$ satisfying $q > k+3$, we have*

$$\forall t \in [0,T], \quad \|j_f(t) - \rho_f u(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim_0 \frac{\varphi_2(N_q(f_0) + H_{0,k}(f_0))}{(1+t)^{\frac{k}{2}}}, \tag{3.6.13}$$

and if $\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^6(\mathbb{R}^3_+))} \lesssim_0 1$, we have

$$\forall t \in [0,T], \quad \|j_f(t) - \rho_f u(t)\|_{\mathrm{L}^p(\mathbb{R}^3_+)} \lesssim_0 \frac{\varphi_p(N_q(f_0) + H_{0,k}(f_0))}{(1+t)^{k\frac{p-1}{p}}}. \tag{3.6.14}$$

Proof. We separate the study between $[0, T_0]$ and $[T_0, T]$.

- If $t \in [T_0, T]$, we use Lemma 3.4.10 and get for all $r \geq 2$

$$\begin{aligned}\|(j_f - \rho_f u)(t)\|_{\mathrm{L}^r(\mathbb{R}^3_+)} &\leq \|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+))}^{\frac{r-1}{r}} \left(\int_{\mathbb{R}^3_+ \times \mathbb{R}^3} f(t,x,v)|v - u(t,x)|^r \,\mathrm{d}x\,\mathrm{d}v \right)^{1/r} \\ &\lesssim \|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+))}^{\frac{r-1}{r}} \left(\int_{\mathbb{R}^3_+ \times \mathbb{R}^3} f(t,x,v)|v|^r \,\mathrm{d}x\,\mathrm{d}v + \int_{\mathbb{R}^3_+} \rho_f(t,x)|u(t,x)|^r \,\mathrm{d}x \right)^{1/r}.\end{aligned}$$

We first focus on the estimate (3.6.13). We have

$$
\begin{aligned}
&\|(j_f-\rho_f u)(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\\
&\lesssim \|m_0 f(t)\|^{\frac12}_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\left(\|m_2 f(t)\|_{\mathrm{L}^1(\mathbb{R}^3_+)}+\|m_0 f(t)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\mathrm{E}(t)\right)^{1/2}\\
&\lesssim \|m_0 f(t)\|^{\frac12}_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\Bigg(\|m_2 f(t)\|_{\mathrm{L}^1(\mathbb{R}^3_+)}\\
&\qquad\qquad + \|m_0 f(t)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\left[\mathrm{E}(0)+\mathrm{E}(0)^{1/2}T_0+T_0^2+\int_{T_0}^T \|j_f(\tau)\|_{\mathrm{L}^1(\mathbb{R}^3_+)}\,\mathrm{d}\tau\right]\Bigg)^{1/2},
\end{aligned}
$$

where we have used the energy inequality (3.1.29) and Lemma 3.4.12. The estimates (3.6.10) and (3.6.11) of Proposition 3.6.7 then imply

$$
\begin{aligned}
\|(j_f-\rho_f u)(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq &\left(\frac{N_q(f_0)}{(1+t)^k}+\frac{H_{0,k}(f_0)}{(1+t)^k}\right)^{1/2}\\
&\times\Bigg(\frac{K_{q_1,1}(f_0)}{(1+t)^{k_1}}+\frac{F_{2,k_1,1}(f_0)}{(1+t)^{k_1}}+\left(\frac{N_{q_2}(f_0)}{(1+t)^{k_2}}+\frac{H_{0,k_2}(f_0)}{(1+t)^{k_2}}\right)\\
&\quad\times\left[\mathrm{E}(0)+\mathrm{E}(0)^{1/2}T_0+T_0^2+\int_{T_0}^T\left(\frac{K_{q_3,1}(f_0)}{(1+\tau)^{k_3}}+\frac{F_{1,k_3,1}(f_0)}{(1+\tau)^{k_3}}\right)\mathrm{d}\tau\right]\Bigg)^{1/2},
\end{aligned}
$$

provided that $q>k+3$, $q_1>k_1+5$, $q_2>k_2+3$ and $q_3>k_3+4$. Thus, we obtain for all $t\in[T_0,T]$

$$
\|(j_f-\rho_f u)(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim_0 \frac{N_q(f_0)^{1/2}}{(1+t)^{k/2}}+\frac{H_{0,k}(f_0)^{1/2}}{(1+t)^{k/2}}.
$$

if we can choose $k_1,k_2>0$ and $k_3>1$, which is possible in view of the assumptions (3.2.1) of Theorem 3.2.1.

We now treat the estimate (3.6.14), assuming $\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^6(\mathbb{R}^3_+))}\lesssim_0 1$. By Hölder's inequality, we now have for $p\in(3,3+\varepsilon)$

$$
\begin{aligned}
&\|(j_f-\rho_f u)(t)\|_{\mathrm{L}^p(\mathbb{R}^3_+)}\\
&\lesssim \|\rho_f(t)\|^{\frac{p-1}{p}}_{\mathrm{L}^\infty(\mathbb{R}^3_+))}\left(\int_{\mathbb{R}^3_+\times\mathbb{R}^3} f(t,x,v)|v|^p\,\mathrm{d}x\,\mathrm{d}v+\int_{\mathbb{R}^3_+}\rho_f(t,x)|u(t,x)|^p\,\mathrm{d}x\,\mathrm{d}v\right)^{1/p}\\
&\leq \|m_0 f(t)\|^{\frac{p-1}{p}}_{\mathrm{L}^\infty(\mathbb{R}^3_+))}\left(\|m_p f(t)\|_{\mathrm{L}^1(\mathbb{R}^3_+)}+\|u\|^p_{\mathrm{L}^\infty(0,T;\mathrm{L}^6(\mathbb{R}^3_+))}\|m_0 f(t)\|_{\mathrm{L}^{r_\varepsilon(p)}(\mathbb{R}^3_+)}\right)^{1/p},
\end{aligned}
$$

where $2<r_\varepsilon(p)<6/(3-\varepsilon)$. Thanks to $\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^6(\mathbb{R}^3_+))}\lesssim_0 1$, the estimates (3.6.10)-(3.6.11) of Proposition 3.6.7 and (3.2.1), we obtain in the same way as before that

$$
\|(j_f-\rho_f u)(t)\|_{\mathrm{L}^p(\mathbb{R}^3_+)} \lesssim_0 \frac{N_q(f_0)^{\frac{p-1}{p}}}{(1+t)^{k\frac{p-1}{p}}}+\frac{H_{0,k}(f_0)^{\frac{p-1}{p}}}{(1+t)^{k\frac{p-1}{p}}},
$$

for $q>k+3$ and $t\in[T_0,T]$. Note that the the treament of the term $\|m_p f(t)\|_{\mathrm{L}^1(\mathbb{R}^3_+)}$ via (3.6.11) requires $q\geq 7$, which is allowed by (3.2.1).

• If $t\in[0,T_0]$, and for the proof of (3.6.13), we use the triangular inequality and the Hölder's inequality to write

$$
\begin{aligned}
\|j_f(t)-\rho_f u(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} &\leq \|j_f(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}+\|\rho_f u(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\\
&\leq \|j_f(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}+\|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\|u(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}.
\end{aligned}
$$

Hence, by interpolation, Lemma 3.4.12 and Lemma 3.4.16, we get

$$\begin{aligned}\|j_f(t)-\rho_f u(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} &\leq \|j_f(t)\|^{3/4}_{\mathrm{L}^{3/2}(\mathbb{R}^3_+)}\|j_f(t)\|^{1/4}_{\mathrm{L}^\infty(\mathbb{R}^3_+)} + \|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}[\mathrm{E}(0)+\mathrm{E}(0)^{1/2}T_0+T_0^2]\\ &\leq \phi(T_0)\|j_f\|^{1/4}_{\mathrm{L}^\infty(0,T_0;\mathrm{L}^\infty(\mathbb{R}^3_+))} + \|\rho_f\|_{\mathrm{L}^\infty(0,T_0;\mathrm{L}^\infty(\mathbb{R}^3_+))}[\mathrm{E}(0)+\mathrm{E}(0)^{1/2}T_0+T_0^2].\end{aligned}$$

The local estimates (3.4.24) then provides

$$\begin{aligned}\|j_f(t)-\rho_f u(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} &\lesssim \phi(T_0)(N_q(f_0)\eta(T_0))^{1/4} + (N_q(f_0)\eta(T_0))[\mathrm{E}(0)+\mathrm{E}(0)^{1/2}T_0+T_0^2]\\ &\lesssim_0 N_q(f_0)^{1/4}+N_q(f_0)\\ &\lesssim_0 \frac{\varphi(N_q(f_0))}{(1+T_0)^{k/2}},\end{aligned}$$

for some nonincreasing function φ vanishing at 0, provided that $N_q(f_0)$ is taken small enough by the smallness assumption (3.2.2). Concerning the estimate (3.6.14) under the assumption $\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^6(\mathbb{R}^3_+))} \lesssim_0 1$, we proceed as before by writing

$$\begin{aligned}\|j_f(t)-\rho_f u(t)\|_{\mathrm{L}^p(\mathbb{R}^3_+)} &\leq \|j_f(t)\|_{\mathrm{L}^p(\mathbb{R}^3_+)} + \|\rho_f u(t)\|_{\mathrm{L}^p(\mathbb{R}^3_+)}\\ &\leq \|j_f(t)\|_{\mathrm{L}^p(\mathbb{R}^3_+)} + \|\rho_f(t)\|_{\mathrm{L}^{\widetilde{r_\varepsilon}}(\mathbb{R}^3_+)}\|u(t)\|_{\mathrm{L}^6(\mathbb{R}^3_+)},\end{aligned}$$

where $6<\widetilde{r_\varepsilon}<\dfrac{18+6\varepsilon}{3-\varepsilon}$. By interpolation, this turns into

$$\begin{aligned}\|(j_f-\rho_f u)\|^p_{\mathrm{L}^p(0,T_0;\mathrm{L}^p(\mathbb{R}^3_+))} &\leq \|j_f\|^{3/2}_{\mathrm{L}^\infty(0,T_0;\mathrm{L}^{3/2}(\mathbb{R}^3_+))}\|j_f\|^{p-3/2}_{\mathrm{L}^\infty(0,T_0;\mathrm{L}^\infty(\mathbb{R}^3_+))}\\ &\quad+\|\rho_f\|^{p/\widetilde{r_\varepsilon}}_{\mathrm{L}^\infty(0,T_0;\mathrm{L}^1(\mathbb{R}^3_+))}\|\rho_f\|^{p-p/\widetilde{r_\varepsilon}}_{\mathrm{L}^\infty(0,T_0;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|u\|^p_{\mathrm{L}^\infty(0,T_0;\mathrm{L}^6(\mathbb{R}^3_+))}\\ &\lesssim_0 N_q(f_0)^{p-3/2}+N_q(f_0)^{p-p/\widetilde{r_\varepsilon}},\end{aligned}$$

thanks to the Lemma 3.4.16, Lemma 3.4.17, the bound (3.4.24) and $\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^6(\mathbb{R}^3_+))} \lesssim_0 1$. By taking $N_q(f_0)$ small enough, we get for all $t\in[0,T_0]$

$$\|j_f(t)-\rho_f u(t)\|_{\mathrm{L}^p(\mathbb{R}^3_+)} \lesssim_0 \frac{\varphi(N_q(f_0))}{(1+T_0)^{k\frac{p-1}{p}}}.$$

Combining the estimates on $[0,T_0]$ and on $[T_0,T]$, we finally obtain the result. □

Corollary 3.6.9. *For all $0\leq t\leq T$ and for any $k>1$ such that $q>k+3$, we have*

$$\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}+\frac{1}{2}\int_0^t \|\mathrm{D}^2u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s \lesssim_0 \|u_0\|^2_{\mathrm{H}^1(\mathbb{R}^3_+)}+\varphi_2(N_q(f_0)+H_{0,k}(f_0)). \tag{3.6.15}$$

Furthermore, the following estimate holds for all $0\leq t\leq T$

$$\|u\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^6(\mathbb{R}^3_+))} \lesssim \|\nabla u\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^2(\mathbb{R}^3_+))} \lesssim_0 \|u_0\|^2_{\mathrm{H}^1(\mathbb{R}^3_+)}+\varphi_2(N_q(f_0)+H_{0,k}(f_0)), \tag{3.6.16}$$

and in particular, the estimate (3.6.14) holds true.

Proof. As we have $T\in(T_0,t^\star)$ and in view of (3.6.1), we only treat the case where $t\in[T_0,T]$. Since the time t is a strong existence time (see Definition 3.4.21), Proposition 3.4.19 entails the following estimate

$$\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}+\frac{1}{2}\int_0^t \|\mathrm{D}^2u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \|\nabla u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}+\|j_f-\rho_f u\|^2_{\mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}^3_+))}.$$

Applying Corollary 3.6.8 with $k > 1$ leads to

$$\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \frac{1}{2}\int_0^t \|\mathrm{D}^2 u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim_0 \|\nabla u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \varphi_2(N_q(f_0) + \mathrm{H}_{0,k}(f_0)),$$

which is the first part of the result. The other statement is then a consequence of the Sobolev embedding. □

3.6.3 Estimates with a polynomial weight in time

We recall that we have fixed $T \in (T_0, t^\star)$. As in [HK22], the guiding line in order to obtain polynomial weighted estimates for the fluid velocity u is to derive a Stokes system satisfied by a weighted version of u, in such a way that the maximal regularity result of Section 3.D can be applied. Note that since we do not have enough information about the regularity of $u(T_0)$ with respect to the interpolation spaces defined in (3.D.1), we rely on this result on the whole interval $[0, T]$.

In what follows, the function Ψ is a generic continuous positive and increasing function that may change from one line to another. Let us also recall the symbol $\lesssim_0$ introduced in Notation 3.4.8.

We first state the following $\mathrm{L}^2\mathrm{H}^2$ regularity result which is reminiscent of [HK22].

Lemma 3.6.10. *For all* $\gamma \in (0, 3/4)$, *we have*

$$\|(1+t)^\gamma \mathrm{D}^2 u\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} \lesssim_0 \Psi\left(\|u_0\|_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,k}(f_0)\right) + \|u_0\|_{\mathrm{H}^1(\mathbb{R}^3_+)}. \tag{3.6.17}$$

Proof. Let $\gamma \in (0, 3/4)$ and set $U := (1+t)^\gamma u$. We observe that U satisfies the following Stokes system on $[0, T]$

$$\begin{cases} \partial_t U + A_2 U = S(u, f), \\ U_{|x_3=0} = 0, \\ U_{|t=0} = u_0, \end{cases} \tag{3.6.18}$$

where A_2 stands for the Stokes operator on $\mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^3_+)$ and where

$$S(u, f) := (1+t)^\gamma \mathbb{P}(j_f - \rho_f u) - (1+t)^\gamma \mathbb{P}\,(u\cdot\nabla)\,u + \gamma(1+t)^{\gamma-1} u. \tag{3.6.19}$$

According to the maximal regularity result in $\mathrm{L}^2\mathrm{L}^2$ for the Stokes system (see Section 3.D in the Appendix), we get

$$\|\partial_t U\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} + \|\mathrm{D}^2\, U\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} \lesssim \|S(u, f)\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} + \|u_0\|_{\mathrm{H}^1(\mathbb{R}^3_+)},$$

where $\lesssim$ is independent of T, u and f.

We now estimate each term in the modified source term $S(u, f)$. Owing to Corollary 3.6.8 with $k = 7/2$ and Theorem 3.3.1, we have

$$(1+t)^{\gamma-1}\|u(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \frac{\Psi\left(\|u_0\|_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,k}(f_0)\right)}{(1+t)^{3/4-\gamma+1}}, \tag{3.6.20}$$

which is bounded independently of T in $\mathrm{L}^2(0, T)$ because $\gamma \in (0, 3/4)$. For the term coming from the Brinkman force $j_f - \rho_f u$, we use the continuity of the Leray projection on $\mathrm{L}^2(\mathbb{R}^3_+)$ and the estimate (3.6.13) of Corollary 3.6.8 which entail

$$\begin{aligned} \|(1+t)^\gamma \mathbb{P}(j_f - \rho_f u)\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} &\lesssim_0 \int_0^T (1+s)^{2\gamma}\|j_f(s) - \rho_f u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s \\ &\lesssim_0 \varphi_2(N_q(f_0) + H_{0,k}(f_0)) \int_0^T \frac{1}{(1+s)^{k-2\gamma}}\,\mathrm{d}s \\ &\lesssim_0 \Psi\left(\|u_0\|_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,k}(f_0)\right), \end{aligned} \tag{3.6.21}$$

for $q > k+3 > 4+3/2$. For the convection term $(u\cdot\nabla)u$, we proceed as in the proof of Theorem 3.F.1 in the Appendix (more precisely, see (3.F.5)-(3.F.6)) to obtain, with the Young inequality, that

$$\|(u\cdot\nabla)u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq C\eta^{-4}\|u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\|\nabla u(t)\|^4_{\mathrm{L}^2(\mathbb{R}^3_+)}\|\nabla u(t)\|^4_{\mathrm{L}^2(\mathbb{R}^3_+)} + \frac{3\eta^{4/3}}{4}\|\mathrm{D}^2 u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)},$$

where $\eta > 0$ can be taken as small as we want. We then use the Gagliardo-Nirenberg-Sobolev inequality to write

$$\|\nabla u(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \|u(t)\|^{1/2}_{\mathrm{L}^2(\mathbb{R}^3_+)}\|\mathrm{D}^2 u(t)\|^{1/2}_{\mathrm{L}^2(\mathbb{R}^3_+)},$$

therefore this yields

$$\begin{aligned}\|(1+t)^\gamma\mathbb{P}(u\cdot\nabla)u\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} \\ \leq \int_0^T (1+t)^{2\gamma}\left[C\eta^{-4}\|u(t)\|^4_{\mathrm{L}^2(\mathbb{R}^3_+)}\|\nabla u(t)\|^4_{\mathrm{L}^2(\mathbb{R}^3_+)} + \frac{3\eta^{4/3}}{4}\right]\|\mathrm{D}^2 u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}t.\end{aligned}$$

Furthermore, owing to Theorem 3.3.1 and the estimate (3.6.16), we can write

$$\begin{aligned}\|u(t)\|^4_{\mathrm{L}^2(\mathbb{R}^3_+)}\|\nabla u(t)\|^4_{\mathrm{L}^2(\mathbb{R}^3_+)} &\leq \Psi\left(\|u_0\|_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,k}(f_0)\right)^4\|\nabla u\|^4_{\mathrm{L}^\infty(0,t;\mathrm{L}^2(\mathbb{R}^3_+))} \\ &\lesssim_0 \Psi\left(\|u_0\|_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,k}(f_0)\right)^4.\end{aligned}$$

Using the smallness assumption (3.2.2), we can first choose η and then $\|u_0\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,k}(f_0)$ small enough so that

$$\|(1+t)^\gamma\mathbb{P}(u\cdot\nabla)u\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} \leq \frac{1}{2}\|(1+t)^\gamma\mathrm{D}^2u\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))}. \tag{3.6.22}$$

We eventually combine (3.6.20), (3.6.21) together with (3.6.22) so that we get

$$\|(1+t)^\gamma\mathrm{D}^2u\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} \lesssim_0 \Psi\left(\|u_0\|_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,k}(f_0)\right) + \|u_0\|_{\mathrm{H}^1(\mathbb{R}^3_+)},$$

which concludes the proof. □

By interpolation, the previous weighted L^2 maximal parabolic estimates allow us to obtain improved L^p estimates of the weighted source term $S(u,f)$ defined in (3.6.19), for $p > 3$ (close enough to 3). We refer to [HK22] where the proof of the two following results can be found in the whole space case and apply *mutatis mutandis* to the half-space case thanks to Theorem 3.3.1, Corollary 3.6.8 and the estimate (3.6.17).

Corollary 3.6.11. *Let $p > 3$. For all $\gamma \in (0, \frac{17}{8} - \frac{7}{4p})$, we have*

$$\|(1+t)^{\gamma-1}u\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \lesssim_0 \Psi\left(\|u_0\|_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,k}(f_0)\right) + \|u_0\|_{\mathrm{H}^1(\mathbb{R}^3_+)}.$$

Corollary 3.6.12. *There exists $\varepsilon > 0$ such that for all $p \in (3, 3+\varepsilon)$, the following holds. For all $\gamma > 0$, we have*

$$\begin{aligned}\|(1+t)^\gamma(u\cdot\nabla)u\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \\ \lesssim_0 \Psi\left(\|u_0\|_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,k}(f_0)\right)\|(1+t)^\gamma\mathrm{D}^2u\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))}.\end{aligned}$$

We are now in position to provide some $\mathrm{L}^p\mathrm{L}^p$ estimates for the weighted source term

$$S(u,f) = (1+t)^{\gamma}\mathbb{P}(j_f - \rho_f u) - (1+t)^{\gamma}\mathbb{P}\,(u\cdot\nabla)\,u + \gamma(1+t)^{\gamma-1}u,$$

with $p>3$ (close enough to 3) and $\gamma>0$ large enough, thanks to the next lemma.

Lemma 3.6.13. *There exists $\varepsilon>0$ such that for all $p\in(3,3+\varepsilon)$, the following holds. For all $\gamma\in(0,\frac{17}{8}-\frac{7}{4p})$, we have*

$$\|S(u,f)\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \lesssim_0 \left[\Psi\left(\|u_0\|_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,k}(f_0)\right) + \|u_0\|_{\mathrm{H}^1(\mathbb{R}^3_+)}\right] \times\left[1+\|(1+t)^{\gamma}\mathrm{D}^2u\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))}\right].$$

Proof. We first use Corollary 3.6.11 and Corollary 3.6.12 to estimate the contribution of $(1+t)^{\gamma}\mathbb{P}\,(u\cdot\nabla)\,u - \gamma(1+t)^{\gamma-1}u$, that is

$$\begin{aligned}\|(1+t)^{\gamma}\mathbb{P}\,(u\cdot\nabla)\,u + \gamma(1+t)^{\gamma-1}u\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))}&\\ \lesssim_0 \left[\Psi\left(\|u_0\|_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,k}(f_0)\right) + \|u_0\|_{\mathrm{H}^1(\mathbb{R}^3_+)}\right]&\\ \times\left[1+\|(1+t)^{\gamma}\mathrm{D}^2u\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))}\right],&\end{aligned}$$

for $p\in(3,3+\varepsilon)$, where we have also used the continuity of the Leray projection of $\mathrm{L}^p(\mathbb{R}^3_+)$.

In order to treat the term coming from the Brinkman force, we use the estimate (3.6.14) of Corollary 3.6.8 (which is valid in view of (3.6.16)). We get

$$\begin{aligned}\|(1+t)^{\gamma}\mathbb{P}(j_f-\rho_f u)\|^p_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} &\lesssim_0 \int_0^T (1+s)^{p\gamma}\|j_f(s)-\rho_f u(s)\|^p_{\mathrm{L}^p(\mathbb{R}^3_+)}\,\mathrm{d}s\\ &\lesssim_0 \varphi_p(N_q(f_0)+H_{0,k}(f_0))\int_0^T \frac{1}{(1+s)^{k(p-1)-p\gamma}}\,\mathrm{d}s\\ &\lesssim_0 \Psi\left(\|u_0\|_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,k}(f_0)\right),\end{aligned}\tag{3.6.23}$$

for k large enough with respect to $\gamma\in(0,\frac{17}{8}-\frac{7}{4p})$. More precisely, we have to choose k such that

$$q > k+3 > \frac{1+p\gamma}{p-1}+3.$$

Since $\frac{1+p\gamma}{p-1} < \frac{17}{16}p - \frac{3}{8} < \frac{17}{16}\frac{3(2+s)}{4} - \frac{3}{8}$ with $s\in(2,3)$ (see (3.2.1)), this procedure is allowed by the assumption (3.2.1) of Theorem 3.2.1 on the exponents q and m. Gathering all the pieces together, we end up with

$$\|S(u,f)\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \lesssim_0 \left[\Psi\left(\|u_0\|_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,k}(f_0)\right) + \|u_0\|_{\mathrm{H}^1(\mathbb{R}^3_+)}\right] \times\left[1+\|(1+t)^{\gamma}\mathrm{D}^2u\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))}\right],$$

and this concludes the proof. □

Corollary 3.6.14. *There exists $\varepsilon>0$ such that for all $p\in(3,3+\varepsilon)$, the following holds. For all $\gamma\in(0,\frac{17}{8}-\frac{7}{4p})$, we have*

$$\|(1+t)^{\gamma}\mathrm{D}^2u\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \lesssim_0 1.$$

Proof. We apply the maximal $\mathrm{L}^p\mathrm{L}^p$ regularity for the Stokes system (see Section 3.D in the Appendix) satisfied by $\mathrm{U} := (1+t)^\gamma u$:

$$\begin{cases} \partial_t U + A_p U = S(u,f), \\ U_{|x_3=0} = 0, \\ U_{|t=0} = u_0, \end{cases} \tag{3.6.24}$$

where A_p stands for the Stokes operator in $\mathrm{L}^p_{\mathrm{div}}(\mathbb{R}^3_+)$. This entails

$$\|(1+t)^\gamma \mathrm{D}^2 u\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \lesssim \|u_0\|_{\mathrm{D}_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+)} + \|S(u,f)\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))},$$

where $\lesssim$ involves a universal constant and where $\mathrm{D}_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+)$ has been defined in (3.D.1). Recall the meaning of the notation $\lesssim_0$ from Definition 3.4.8. Using Lemma 3.6.13, we then have

$$\begin{aligned} \|(1+t)^\gamma \mathrm{D}^2 u\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \leq \|u_0\|_{\mathrm{D}_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+)} & \\ + \psi\Big(1+\mathscr{E}(0)+\mathscr{N}_{q,m}(f_0)\Big) & \\ \times \Big[\Psi\Big(\|u_0\|_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,k}(f_0)\Big) + \|u_0\|_{\mathrm{H}^1(\mathbb{R}^3_+)}\Big] & \\ \times \Big[1 + \|(1+t)^\gamma \mathrm{D}^2 u\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))}\Big]. & \end{aligned}$$

Using the smallness assumption (3.2.2), we can ensure that

$$\psi\Big(1+\mathscr{E}(0)+\mathscr{N}_{q,m}(f_0)\Big) \times \Big[\Psi\Big(\|u_0\|_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,k}(f_0)\Big) + \|u_0\|_{\mathrm{H}^1(\mathbb{R}^3_+)}\Big] < \frac{1}{2},$$

and this allows to absorb the term $\|(1+t)^\gamma \mathrm{D}^2 u\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))}$ in the l.h.s. This concludes the proof. □

3.6.4 End of the proof of Theorem 3.2.1

We are now able to close the bootstrap argument. Recall that $T \in (T_0, t^\star)$, where $t^\star$ is given in Definition 3.6.4.

Corollary 3.6.15. *The following inequalities hold*

$$\int_{T_0}^T \|u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s \lesssim_0 \Psi\Big(\|u_0\|_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,k}(f_0)\Big),$$

$$\int_{T_0}^T \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s \lesssim_0 \Psi\Big(\|u_0\|_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,k}(f_0)\Big).$$

Proof. We use the Gagliardo-Nirenberg-Sobolev inequality twice (see Section 3.C.1 in the Appendix) to write that for $p > 3$ (to be determined later)

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \lesssim \|\mathrm{D}^2 u\|^{\alpha_p}_{\mathrm{L}^p(\mathbb{R}^3_+)} \|u\|^{1-\alpha_p}_{\mathrm{L}^2(\mathbb{R}^3_+)}, \quad \alpha_p := \frac{3p}{7p-6},$$

$$\|\nabla u\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \lesssim \|\mathrm{D}^2 u\|^{\beta_p}_{\mathrm{L}^p(\mathbb{R}^3_+)} \|u\|^{1-\beta_p}_{\mathrm{L}^2(\mathbb{R}^3_+)}, \quad \beta_p := \frac{5p}{7p-6},$$

where $\lesssim$ hides a universal constant. Thanks to the Hölder's inequality in time, we then get for all $\gamma > 0$

$$\begin{aligned}\int_{T_0}^{T} \|u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s &\lesssim \int_0^T \|\mathrm{D}^2 u(s)\|_{\mathrm{L}^p(\mathbb{R}^3_+)}^{\alpha_p}\|u(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}^{1-\alpha_p}\,\mathrm{d}s \\ &\leq \|(1+t)^\gamma \mathrm{D}^2 u\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))}^{\alpha_p}\left(\int_0^T (1+t)^{-\gamma\frac{p\alpha_p}{p-\alpha_p}}\|u(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}^{(1-\alpha_p)\frac{p}{p-\alpha_p}}\,\mathrm{d}s\right)^{\frac{p-\alpha_p}{p}},\end{aligned} \tag{3.6.25}$$

as well as

$$\int_{T_0}^{T} \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s \lesssim \|(1+t)^\gamma \mathrm{D}^2 u\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))}^{\beta_p}\left(\int_0^T (1+t)^{-\gamma\frac{p\beta_p}{p-\beta_p}}\|u(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}^{(1-\beta_p)\frac{p}{p-\beta_p}}\,\mathrm{d}s\right)^{\frac{p-\beta_p}{p}}. \tag{3.6.26}$$

We also note that we have

$$\frac{p-\alpha_p}{p\alpha_p} = \frac{7}{3} - \frac{3}{p}, \quad \frac{p-\beta_p}{p\beta_p} = \frac{7}{5} - \frac{11}{5p}.$$

In view of Corollary 3.6.14, we thus choose $p > 3$ close to 3 (remembering the assumption (3.2.1)) and take γ close to $\frac{17}{8} - \frac{7}{4p}$ (which is strictly greater than $\frac{7}{3} - \frac{3}{p}$ and $\frac{7}{5} - \frac{11}{p}$ for p close to 3) so that

$$\gamma\frac{p\alpha_p}{p-\alpha_p} > 1, \quad \gamma\frac{p\beta_p}{p-\beta_p} > 1.$$

Observe that we can first choose such p and γ and then perform the whole analysis of the two previous subsections (using a finite number of quantities defined in Definition 3.1.4, as well as a finite number of times the smallness assumption (3.2.2), whether the estimates depend on γ or not). We then take the largest exponent involved in these quantities. We can now come back to (3.6.25) and (3.6.26), and use the uniform (in time) inequality $\|u(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \Psi\left(\|u_0\|_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + N_q(f_0) + H_{0,m}(f_0)\right)$ on $[0,T]$ with the previous exponents (ensuring uniform in time bound), therefore we get the desired results thanks to Corollary 3.6.14. □

Recall that we have assumed that $t^\star < \infty$.

End of the proof of Theorem 3.2.1. As before, we use the notation $F := j_f - \rho_f u$. Applying Corollary 3.6.8 with $k > 1$, we get

$$\int_0^{t^\star} \|F(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}^2\,\mathrm{d}s \lesssim_0 \varphi_2(N_q(f_0) + H_{0,k}(f_0)),$$

while combining the Cauchy-Schwarz inequality with Corollary 3.6.8 with $k > 3$ (since $q \geq 7$) yields

$$\begin{aligned}\int_0^{t^\star} \|F(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s &\leq \left(\int_0^{t^\star} \frac{\mathrm{d}s}{(1+s)^2}\right)^{1/2}\left(\int_0^{t^\star} (1+s)^2\|F(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}^2\,\mathrm{d}s\right)^{1/2} \\ &\lesssim_0 \varphi_2(N_q(f_0) + H_{0,k}(f_0)).\end{aligned}$$

By the smallness assumption (3.2.2), we can thus ensure that

$$\|u_0\|_{\mathrm{H}^1(\mathbb{R}^3_+)}^2 + \int_0^{t^\star} \left[\|F(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}^2 + \|F(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\right]\mathrm{d}s < \frac{\mathrm{C}_\star}{2}, \tag{3.6.27}$$

where $\mathrm{C}_\star$ refers to the universal constant from Proposition 3.4.19. Since $F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\mathbb{R}^3_+)) \hookrightarrow \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\mathbb{R}^3_+))$ by Proposition 3.4.17, a continuity argument shows that there exists a strong existence time strictly larger than $t^\star$. Furthermore, by an appropriate choice of the data in (3.2.2), we can use Corollary 3.6.15 to ensure

$$\int_{T_0}^{t^\star} \|u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s < \frac{\delta}{4}, \quad \int_{T_0}^{t^\star} \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s < \frac{\delta}{2}.$$

According to Proposition 3.4.18 and Proposition 3.4.24, this means that there exists a strong existence time $t > t^\star$ such that

$$\int_{T_0}^{t} \|u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s < \frac{\delta}{2}, \quad \int_{T_0}^{t} \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s < \delta.$$

This is a contradiction with the very definition of $t^\star$ therefore we necessarily have $t^\star = +\infty$. The proof of Theorem 3.2.1 is now complete. □

Appendix

3.A DiPerna-Lions theory in $\mathbb{R}^3_+ \times \mathbb{R}^3$

Theorem 3.A.1. *Let $\chi \in \mathscr{C}^\infty(\mathbb{R})$ such that $|\chi(z)| \leq |z|$ and $\chi' \in \mathrm{L}^\infty(\mathbb{R})$. Take $f_0 \in \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^3_+ \times \mathbb{R}^3)$, $G \in \mathbb{R}^3$ and a vector field $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{W}^{1,1}(\mathbb{R}^3_+))$. Consider the following boundary value problem on $\mathbb{R}^3_+ \times \mathbb{R}^3$:*

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v(f\chi(u-v) + fG) = 0, \\ f_{|t=0} = f_0, \\ f = 0, \ \text{on } \Sigma^-, \end{cases} \tag{3.A.1}$$

where $\chi(Z) = (\chi(Z_1), \chi(Z_2), \chi(Z_3))$ for all $Z \in \mathbb{R}^3$. Then we have, for all fixed $T > 0$

- *Well-posedness: There exists a unique $f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^3_+ \times \mathbb{R}^3))$ which is a weak solution of the Cauchy problem (3.A.1). Furthermore,*

$$f \in \mathscr{C}(\mathbb{R}^+;\mathrm{L}^p_{\mathrm{loc}}(\overline{\mathbb{R}^3_+} \times \mathbb{R}^3)),$$

for all $p \in [1,\infty)$ and the function f has a trace on $\partial\mathbb{R}^3_+ \times \mathbb{R}^3$ defined in the following sense: there exists a unique element $\gamma f \in \mathrm{L}^\infty([0,T] \times \partial\mathbb{R}^3_+ \times \mathbb{R}^3)$ such that for any test function $\psi \in \mathscr{C}^\infty([0,T] \times \overline{\mathbb{R}^3_+} \times \mathbb{R}^3)$ with compact support in space and velocity, and for all $0 \leq t_1 \leq t_2 \leq T$

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathbb{R}^3_+ \times \mathbb{R}^3} & f(t,x,v)\left[\partial_t\psi + v\cdot\nabla_x\psi + (\chi(u(t,x)-v)+G)\cdot\nabla_v\psi\right](t,x,v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\ &= \int_{\mathbb{R}^3_+ \times \mathbb{R}^3} f(t_2,x,v)\psi(t_2,x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^3_+ \times \mathbb{R}^3} f(t_1,x,v)\psi(t_1,x,v)\,\mathrm{d}x\,\mathrm{d}v \\ &\quad + \int_{t_1}^{t_2}\int_{\partial\mathbb{R}^3_+ \times \mathbb{R}^3} \left[(\gamma f)\psi(t,x,v)\right] v\cdot n(x)\,\mathrm{d}\sigma(x)\,\mathrm{d}v\,\mathrm{d}t.\end{aligned}$$

- *Stability: If*

$$u_n \longrightarrow u \quad \text{in} \quad \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^1(\mathbb{R}^3_+)) \quad \text{and} \quad f_{0,n} \longrightarrow f_0 \quad \text{in} \quad \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^3_+ \times \mathbb{R}^3),$$

the corresponding sequence of solutions (f_n) satisfies for all $p \in [1,\infty)$,

$$f_n \longrightarrow f \quad \text{in } \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^p(\mathbb{R}^3_+ \times \mathbb{R}^3)).$$

Such a result can be found in [BGM17, Theorem 3.2 - Proposition 3.2] for the well-posedness and renormalization properties and in [BF12, Theorem VI.1.9] for the stability property.

3.B The Cauchy problem for the Vlasov-Navier-Stokes system in the half-space

Since the existence of global weak solutions to the Vlasov-Navier-Stokes system on a half-space with a gravity force has not been explicitly written in the literature, we provide some rather short elements of proof. We especially focus on the difficulties which may appear when obtaining the strong energy inequalities satisfied by these solutions (namely, the inequalities (3.1.28) and (3.1.29)).

Theorem 3.B.1 (Existence of weak solutions)**.** *Consider (u_0, f_0) a pair of admissible initial conditions in the sense of Definition 3.1.2. Then there exists a global weak solution (u, f) in the sense of Definition 3.1.6 to the system (3.1.1)-(3.1.3) with boundary conditions (3.1.7)-(3.1.11) and with initial data (u_0, f_0).*

The proof of this result follows a now classic method for those non linear coupled problems: first, we introduce an approximated problem of the whole system. For each fixed $n \in \mathbb{N}^*$, an appropriate fixed point procedure gives the existence of (f_n, u_n) which are solutions to the following regularized system on each interval $[0, T]$

$$\partial_t f_n + v \cdot \nabla_x f_n + \mathrm{div}_v(f_n \chi_n(u_n - v) + f_n G) = 0, \tag{3.B.1}$$

$$\partial_t u_n + (\mathrm{J}_n u_n \cdot \nabla) u_n - \Delta u_n + \nabla p_n = \int_{\mathbb{R}^3} f_n \chi_n(v - u_n)\, \mathrm{d}v, \tag{3.B.2}$$

$$\mathrm{div}\, u_n = 0, \tag{3.B.3}$$

$$(f_n, u_n)_{|t=0} = (f_0^n, \mathrm{J}_n u_0), \tag{3.B.4}$$

where $T > 0$ is arbitrary and where we consider the following operator, regularizations and data, and their respective convergences when n goes to infinity:

- $\chi_n \underset{n\to+\infty}{\longrightarrow} \mathrm{Id}_{\mathbb{R}}$, where for all $n \in \mathbb{N}\backslash\{0\}$, $\chi_n \in \mathscr{D}(\mathbb{R})$ is an odd, increasing and bounded function satisfying $|\chi(z)| \leq |z|$, $\|\chi_n'\|_{\mathrm{L}^\infty(\mathbb{R})} \leq 1$ and $z\chi_n(z) \geq 0$. Above, χ_n is applied componentwise.
- $f_0^n \underset{n\to+\infty}{\longrightarrow} f_0$ strongly in $\mathrm{L}^p(\mathbb{R}^3_+ \times \mathbb{R}^3)$ for all $1 \leq p < \infty$ and weakly-$*$ in $\mathrm{L}^\infty(\mathbb{R}^3_+ \times \mathbb{R}^3)$, and where $f_0^n = \eta^n f_0$ with $(\eta^n)_n$ a family of positive functions compactly supported in velocity such that $0 \leq \eta^n \leq 1$ and increasing to 1.
- for all $n \in \mathbb{N} \setminus \{0\}$, the operator J_n refers to the Yosida approximation of the identity defined by

  $$\mathrm{J}_n := (\mathrm{I} + n^{-1} A_2)^{-1},$$

  where A_2 is the Stokes operator on $\mathrm{L}^2(\mathbb{R}^3_+)$ (see Section 3.D in the Appendix). The operator J_n acts on $\mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^3_+)$ and satisfies: for all $v \in \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^3_+)$, $\mathrm{J}_n u \in D(A_2)$, $\|\mathrm{J}_n v\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \|v\|_{\mathrm{L}^2(\mathbb{R}^3_+)}$ and $\mathrm{J}_n v \underset{n\to+\infty}{\longrightarrow} v$ in $\mathrm{L}^2(\mathbb{R}^3_+)$.

In the previous construction, u_n is a strong solution to the Navier-Stokes equations while f_n is a weak solution to the Vlasov equation which is compactly supported in velocity. Note that the presence of the gravity force is harmless in this procedure.

Then, we pass to the limit in the previous approximated problem by compactness arguments, in order to recover solution to the whole system of equations. To do so, we shall obtain energy estimates which are independent of n: all in all, it can be proven that $(u_n)_n$ is weakly compact in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\mathbb{R}^3_+))\cap\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{H}^1(\mathbb{R}^3_+))$ while (f_n) is weakly-$*$ compact in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^\infty(\mathbb{R}^3_+\times\mathbb{R}^3))$. Furthermore, a standard application of the Aubin-Lions lemma ensures that $(u_n)_n$ is strongly compact in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3_+))$ and this also implies that, up to an additional extraction, $(u_n)_n$ converges almost everywhere. Note that the treament of the gravity term G requires an application of Grönwall's lemma for the quantity $M_2 f_n$, which turns out to be bounded itself in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+)$.

Theses convergences are enough to pass to the limit in the weak formulation of the Vlasov equation and of the Navier-Stokes equations because each term converges in the sense of distributions (see e.g. [BMM20] for a proof, in a more involved context). An additional diagonal extraction along increasing intervals of the type $[0,T_j]$ with $T_j\to+\infty$ allows to obtain global solutions.

We then explain how one can recover the strong energy inequality (3.1.28) for the Navier-Stokes equations, as well as the strong energy inequality (3.1.29) for the Vlasov-Navier-Stokes system: since the case $s=0$ is classic (at least for the Navier-Stokes equations with a given source term, see e.g [BF12]), we restrict ourselves to the case $s>0$ and we pay attention to the particular treatment of the coupling terms.

We first write the strong energy inequality (3.1.28) for the Navier-Stokes equations. It seems that the construction of solutions satisfying this strong energy inequality (and not only the same inequality with $s=0$ and $t\geq 0$) has long remained an open problem for general unbounded domains, due to the interaction of the pressure with the distant boundaries [Hey90] (see however a general modern treatment in [FKS05]). Nevertheless, the strategy applied in [MS88] for exterior domains actually extends for the half-space case. We thus refer to [MS88] where the proof of (3.1.28) for the Navier-Stokes equations with a forcing term is written down explicitly: namely, it consists in obtaining a localised energy inequality between almost any times $s<t$, by multiplying the regularized version of the Navier-Stokes equations by $2u_n\varphi_k$ (where $\varphi_k:=\varphi(\cdot/k)$ with $\varphi\in\mathscr{D}(\mathbb{R}^3)$ satisfying $0\leq\varphi\leq 1$, $\varphi(x)=1$ for $|x|\leq 1$ and $\varphi(x)=0$ for $|x|\geq 2$), integrating on $(s,t)\times\mathbb{R}^3_+$, performing integration by parts, and then understanding the behavior of the remaining terms, especially for the pressure ones. All in all, this reads as

$$\int_{\mathbb{R}^3_+}|u_n(t)|^2\phi_k\,\mathrm{d}x+2\int_s^t\int_{\mathbb{R}^3_+}|\nabla u_n|^2\phi_k\,\mathrm{d}x\,\mathrm{d}\tau \\ \leq \mathrm{R}^{k,n}_{s,t}+\int_{\mathbb{R}^3_+}|u_n(s)|^2\phi_k\,\mathrm{d}x+2\int_s^t\int_{\mathbb{R}^3_+}\varphi_k u_n\cdot S_{\chi_n}(u_n,f_n)\,\mathrm{d}x\,\mathrm{d}\tau, \tag{3.B.5}$$

with

$$S_{\chi_n}(u_n,f_n):=\int_{\mathbb{R}^3}f_n\chi_n(v-u_n)\,\mathrm{d}v,$$

and where $\mathrm{R}^{k,n}_{s,t}$ is a remainder term such that $\lim\limits_{k\to+\infty}\limsup\limits_{n\to+\infty}\mathrm{R}^{k,n}_{s,t}=0$ (see [MS88] for a proof).

Thus, the main difficulty essentially lies in the treatment of the last term in the r.h.s of the inequality (3.B.5). First, since $(u_n)_n$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\mathbb{R}^3_+))\cap\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^6(\mathbb{R}^3_+))$, an interpolation argument shows that $(u_n)_n$ is bounded in $\mathrm{L}^r_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^s(\mathbb{R}^3_+))$ if

$$\frac{2}{r}+\frac{3}{s}=\frac{3}{2},\quad 2\leq s\leq 6,\quad 2\leq r\leq\infty.$$

For every n, we have in particular $u_n\in\mathrm{L}^{20/9}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^5(\mathbb{R}^3_+))\hookrightarrow\mathrm{L}^{19/9}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^5(\mathbb{R}^3_+))$. Furthermore, there exists $r_1>19/9$ and $s_1>5$ such that $(u_n)_n$ is bounded in $\mathrm{L}^{r_1}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{s_1}(\mathbb{R}^3_+))$. Since (u_n)

converges almost everywhere towards u, a corollary of Vitali convergence theorem shows that $u_n \underset{n\to+\infty}{\longrightarrow} u$ in $\mathrm{L}^{19/9}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^5_{\mathrm{loc}}(\mathbb{R}^3_+))$.

We also know that $(M_2 f_n)_n$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+)$ so that Proposition 3.4.14 and the maximum principle (3.4.18) ensure that $(j_{f_n})_n$ in bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{5/4}(\mathbb{R}^3_+))$ while $(\rho_{f_n})_n$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{5/3}(\mathbb{R}^3_+))\cap\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^1(\mathbb{R}^3_+))$, and thus bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{p_\theta}(\mathbb{R}^3_+))$ where $p_\theta = \frac{5}{3+2\theta}$ with $\theta\in[0,1]$. As there exist $s_2<6$ close enough to 6 and $\theta\in[0,1]$ such that $p_\theta^{-1}+s_2^{-1}=4/5$ and $(u_n)_n$ is bounded in $\mathrm{L}^{r_2}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{s_2}(\mathbb{R}^3_+))$ for some $r_2>2>19/10=(19/9)'$, Hölder's inequality shows that $(\rho_{f_n}u_n)_n$ is bounded in $\mathrm{L}^{19/10}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{5/4}(\mathbb{R}^3_+))$, as well as $(j_{f_n})_n$. By the properties of χ_n, we observe that for all n

$$|S_{\chi_n}(u_n,f_n)|\leq j_{f_n}+|u_n|\rho_{f_n},$$

therefore the previous bound entails that $(S_{\chi_n}(u_n,f_n))_n$ converges weakly in $\mathrm{L}^{19/10}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{5/4}(\mathbb{R}^3_+))$ to some limit S, up to extraction. Since $(S_{\chi_n}(u_n,f_n))_n$ also converges to $j_f-\rho_f u$ in the sense of distributions, we can identify $S=j_f-\rho_f u$.

All in all, combining the strong convergence of $(u_n)_n$ and the weak convergence of $(S_{\chi_n}(u_n,f_n))_n$ we have just obtained, we get for all fixed k

$$\int_s^t\int_{\mathbb{R}^3_+}\varphi_k u_n\cdot S_{\chi_n}(u_n,f_n)\,\mathrm{d}x\,\mathrm{d}\tau \underset{n\to+\infty}{\longrightarrow}\int_s^t\int_{\mathbb{R}^3_+}\varphi_k u\cdot(j_f-\rho_f u)\,\mathrm{d}x\,\mathrm{d}\tau,$$

and by the dominated convergence theorem, we can then pass to the limit when $k\to+\infty$ in the inequality (3.B.5).

In this procedure, one has to first pick one $s>0$ such that $u_n(s)\underset{n\to+\infty}{\longrightarrow} u(s)$ in $\mathrm{L}^2(K)$ for any compact $K\subset\mathbb{R}^3$ and the inequality (3.1.28) eventually holds for almost any $s>0$ (including $s=0$) and almost any $t\geq s$. Standard lower-semicontinuity arguments allow one to extend this inequality to any $t\geq s$.

Finally, we come back to the proof of the energy inequality (3.1.29) for the Vlasov-Navier-Stokes system: in view of (3.1.28), we only need to prove that for all $0\leq s\leq t$

$$M_2f(t)+2\int_s^t M_2f(\tau)\,\mathrm{d}\tau\leq M_2f(s)+2\int_s^t\int_{\mathbb{R}^3_+\times\mathbb{R}^3}f(\tau,x,v)v\cdot[u(t,x)+G]\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau. \tag{3.B.6}$$

We proceed as follows: mimicking the beginning of the proof of Lemma 3.4.15 for $\alpha=2$ (which does not use the energy inequality (3.1.29) but only the regularity of u), we obtain the fact that $(t,x,v)\mapsto|v|^2 f\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^1(\mathbb{R}^3_+\times\mathbb{R}^3))$. We now consider $\psi_n(v):=\theta_n(|v|^2)$ where $\theta_n(z):=n\theta(z/n)$ is defined thanks to a positive function $\theta\in\mathscr{D}(\mathbb{R}^+)$ equal to the identity function on $[0,1]$, less than this function on $(1,+\infty)$ and with a derivative uniformly bounded by 1. After a suitable localization in space that we do not detail here, the function ψ_n is an admissible test function in the weak formulation of the Vlasov equation with a trace (see Section 3.A) and this implies

$$\begin{aligned}\int_s^t\int_{\mathbb{R}^3_+\times\mathbb{R}^3}&f(\tau,x,v)2\theta'\left(\frac{|v|^2}{n}\right)v\cdot[(u(t,x)-v+G)]\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&=\int_{\mathbb{R}^3_+\times\mathbb{R}^3}f(t,x,v)\theta_n(|v|^2)\,\mathrm{d}v\,\mathrm{d}x-\int_{\mathbb{R}^3_+\times\mathbb{R}^3}f(s,x,v)\theta_n(|v|^2)\,\mathrm{d}x\,\mathrm{d}v\\
&\qquad+\int_s^t\int_{\Sigma^+}\Big[(\gamma f)\theta_n(|v|^2)\Big]\,v\cdot n(x)\,\mathrm{d}\sigma(x)\,\mathrm{d}v\,\mathrm{d}\tau\\
&\geq\int_{\mathbb{R}^3_+\times\mathbb{R}^3}f(t,x,v)\theta_n(|v|^2)\,\mathrm{d}v\,\mathrm{d}x-\int_{\mathbb{R}^3_+\times\mathbb{R}^3}f(s,x,v)\theta_n(|v|^2)\,\mathrm{d}x\,\mathrm{d}v,\end{aligned}$$

because of the boundary condition (3.1.11). Since $\theta_n(|v|^2) \underset{n\to+\infty}{\longrightarrow} |v|^2$ with $\theta_n(|v|^2) \leq |v|^2$, and $\theta'(|v|^2/n) \underset{n\to+\infty}{\longrightarrow} 1$ with $\theta'(|v|^2/n) \leq 1$, we can use the dominated convergence theorem to pass to the limit when $n \to +\infty$ in the previous inequality: indeed, $M_2 f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+)$ while $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^5(\mathbb{R}^3_+))$ and $m_1 f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^{5/4}(\mathbb{R}^3_+))$ by interpolation. Adding (3.1.28) to (3.B.6), we eventually get the energy inequality (3.1.29).

3.C Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}^3_+$

Theorem 3.C.1. *Let $1 \leq p, q, r \leq \infty$ and $m \in \mathbb{N}$. Suppose $j \in \mathbb{N}$ and $\alpha \in [0,1]$ satisfy the relations*

$$\frac{1}{p} = \frac{j}{3} + \left(\frac{1}{r} - \frac{m}{3}\right)\alpha + \frac{1-\alpha}{q},$$
$$\frac{j}{m} \leq \alpha \leq 1,$$

with the exception $\alpha < 1$ if $m - j - d/r \in \mathbb{N}$.

Then for all $g \in \mathrm{L}^q(\mathbb{R}^3_+)$, if $\mathrm{D}^m g \in \mathrm{L}^r(\mathbb{R}^3_+)$, we have $\mathrm{D}^j g \in \mathrm{L}^p(\mathbb{R}^3_+)$ with the estimate

$$\|\mathrm{D}^j g\|_{\mathrm{L}^p(\mathbb{R}^3_+)} \lesssim \|\mathrm{D}^m g\|^{\alpha}_{\mathrm{L}^r(\mathbb{R}^3_+)} \|g\|^{1-\alpha}_{\mathrm{L}^q(\mathbb{R}^3_+)},$$

where $\lesssim$ refers to a constant only depending on the dimension d.

This result can be found in [CM12, Thm 1.5.2], at least for the case of the whole space $\mathbb{R}^3$. In the case of the half-space $\mathbb{R}^3_+$, we can rely on the existence of an extension operator mapping functions defined on $\mathbb{R}^3_+$ to functions defined on $\mathbb{R}^3$ and which is continuous for the topology of Sobolev spaces. More precisely, for all $m \in \mathbb{N} \setminus \{0\}$ and $\ell \in [1, \infty)$, there exists an operator $E : \mathrm{W}^{m,\ell}(\mathbb{R}^3_+) \longrightarrow \mathrm{W}^{m,\ell}(\mathbb{R}^3)$ (which is independent of ℓ) such that for all $v \in \mathrm{W}^{m,\ell}(\mathbb{R}^3_+)$, we have

- $(Ev)_{|\mathbb{R}^3_+} = v$,
- for all $i \in \{0, \cdots, m\}$, $\|\mathrm{D}^i(Ev)\|_{\mathrm{L}^\ell(\mathbb{R}^3)} \lesssim \|\mathrm{D}^i v\|_{\mathrm{L}^\ell(\mathbb{R}^3_+)}$.

This result can be found in (the proofs of) [DDE12, Section 2.3.3] and can be combined with the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}^3$ to deduce Theorem 3.C.1.

3.D Maximal $\mathrm{L}^p\mathrm{L}^q$ regularity for the Stokes system on $\mathbb{R}^3_+$

Let $1 < q < \infty$ be fixed. Given a vector field $u \in \mathrm{L}^q(\mathbb{R}^3_+)$, this can be uniquely decomposed as

$$u = \widetilde{u} + \nabla p,$$
$$\widetilde{u} \in \mathrm{L}^q_{\mathrm{div}}(\mathbb{R}^3_+),\ p \in \mathrm{L}^q(\mathbb{R}^3_+),\ \nabla p \in \mathrm{L}^q(\mathbb{R}^3_+),$$

where $\mathrm{L}^q_{\mathrm{div}}(\mathbb{R}^3_+)$ stands for the closure in $\mathrm{L}^q(\mathbb{R}^3_+)$ of $\mathscr{D}_{\mathrm{div}}(\mathbb{R}^3_+)$. We recall that the projection $\mathbb{P}_q : u \mapsto \widetilde{u}$ is continuous from $\mathrm{L}^q(\mathbb{R}^3_+)$ to $\mathrm{L}^q_{\mathrm{div}}(\mathbb{R}^3_+)$.

For $1 < q < \infty$, we consider the following Stokes operator

$$A_q := -\mathbb{P}_q \Delta, \quad D(A_q) := \mathrm{L}^q_{\mathrm{div}}(\mathbb{R}^3_+) \cap \mathrm{W}^{1,q}_0(\mathbb{R}^3_+) \cap \mathrm{W}^{2,q}(\mathbb{R}^3_+).$$

We also set

$$\mathrm{D}_q^{1-\frac{1}{s},s}(\mathbb{R}^3_+) := \Big(D(A_q), \mathrm{L}^q_{\mathrm{div}}(\mathbb{R}^3_+)\Big)_{1/s,s}, \tag{3.D.1}$$

where $(\ ,\ )_{1/s,s}$ refers to the real interpolation space of exponents $(1/s, s)$. In the case of the Stokes operator A_q, which generates an analytic semigroup e^{-tA_q}, the quantity

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^3_+)} + \left(\int_0^\infty \|A_q e^{-tA_q} u\|^s_{\mathrm{L}^q(\mathbb{R}^3_+)} \,\mathrm{d}t\right)^{1/s} \tag{3.D.2}$$

defines an equivalent norm on $\mathrm{D}_q^{1-\frac{1}{s},s}(\mathbb{R}^3_+)$ (see [Lun18, Chapter 5]).

The main result is the following and can be found with further references in [GS91].

Theorem 3.D.1. *Consider $0 < T \leq \infty$ and $1 < q, s < \infty$. Then, for every $u_0 \in \mathrm{D}_q^{1-\frac{1}{s},s}(\mathbb{R}^3_+)$ which is divergence free and $f \in \mathrm{L}^s(0,T;\mathrm{L}^q_{\mathrm{div}}(\mathbb{R}^3_+))$, there exists a unique solution u of the Stokes system*

$$\begin{cases} \partial_t u + A_q u = f, \\ u_{|x_3=0} = 0, \\ u(0,x) = u_0(x), \end{cases}$$

satisfying

$$u \in \mathrm{L}^s(0,T';D(A_q)) \text{ for all finite } T' \leq T,$$

and

$$\|\partial_t u\|_{\mathrm{L}^s(0,T;\mathrm{L}^q(\mathbb{R}^3_+))} + \|\mathrm{D}^2 u\|_{\mathrm{L}^s(0,T;\mathrm{L}^q(\mathbb{R}^3_+))} \leq C\left(\|u_0\|_{\mathrm{D}_q^{1-\frac{1}{s},s}(\mathbb{R}^3_+)} + \|f\|_{\mathrm{L}^s(0,T;\mathrm{L}^q(\mathbb{R}^3_+))}\right),$$

where $C = C(q,s) > 0$.

Furthermore, if $u_0 \in \mathrm{W}^{1,q}_0(\mathbb{R}^3_+) \cap \mathrm{L}^q_{\mathrm{div}}(\mathbb{R}^3_+)$ and if $s \in (1,2]$, the statement holds and we can replace $\|u_0\|_{\mathrm{D}_q^{1-1/s,s}(\mathbb{R}^3_+)}$ by $\|u_0\|_{\mathrm{W}^{1,q}_0(\mathbb{R}^3_+)}$ in the right hand side of the previous inequality.

For the sake of completeness, we bring some precisions concerning the last statement of the theorem (even if a related fact can be found in [GS91, Remark 2.5]). Suppose that $u \in D(A_q^{\frac{1}{2}}) \cap \mathrm{L}^q_{\mathrm{div}}(\mathbb{R}^3_+)$. If $s \in (1,2)$, we write $\frac{1}{2} = 1 - \frac{1}{s} + \varepsilon$ with $\varepsilon = \frac{2-s}{2s} > 0$ (so that $1 - s\varepsilon < 1$). Since $A_q = A$ generates an analytic semigroup, we have by standard functional calculus manipulations (see e.g. [Haa06, Chapter 3])

$$\begin{aligned} \int_0^\infty \|Ae^{-tA}u\|^s_{\mathrm{L}^q(\mathbb{R}^3_+)} \,\mathrm{d}t &= \int_0^1 \|Ae^{-tA}u\|^s_{\mathrm{L}^q(\mathbb{R}^3_+)} \,\mathrm{d}t + \int_1^\infty \|Ae^{-tA}u\|^s_{\mathrm{L}^q(\mathbb{R}^3_+)} \,\mathrm{d}t \\ &= \int_0^1 \|A^{\frac{1}{s}-\varepsilon} e^{-tA} A^{\frac{1}{2}} u\|^s_{\mathrm{L}^q(\mathbb{R}^3_+)} \,\mathrm{d}t + \int_1^\infty \|Ae^{-tA}u\|^s_{\mathrm{L}^q(\mathbb{R}^3_+)} \,\mathrm{d}t \\ &\lesssim \|A^{\frac{1}{2}}u\|^s_{\mathrm{L}^q(\mathbb{R}^3_+)} \int_0^1 \frac{1}{t^{1-s\varepsilon}} \,\mathrm{d}t + \|u\|^s_{\mathrm{L}^q(\mathbb{R}^3_+)} \int_1^\infty \frac{1}{t^s} \,\mathrm{d}t < \infty, \end{aligned}$$

where we have used the fact that the function $x \mapsto x^{\frac{1}{s}-\varepsilon} e^{-tx}$ and $x \mapsto xe^{-tx}$ are respectively bounded on $\mathbb{R}^+$ by (a constant times) $t^{\varepsilon - \frac{1}{s}}$ and t^{-1}.

In the case where $s = 2$, we proceed in a similar way and write

$$\begin{aligned} \int_0^\infty \|Ae^{-tA}u\|^2_{\mathrm{L}^q(\mathbb{R}^3_+)} \,\mathrm{d}t &\leq \int_0^1 \|A^{\frac{1}{2}} e^{-tA} A^{\frac{1}{2}} u\|^s_{\mathrm{L}^q(\mathbb{R}^3_+)} \,\mathrm{d}t + \int_1^\infty \|Ae^{-tA}u\|^2_{\mathrm{L}^q(\mathbb{R}^3_+)} \,\mathrm{d}t \\ &\lesssim \|A^{\frac{1}{2}}u\|^2_{\mathrm{L}^q(\mathbb{R}^3_+)} \int_0^1 \frac{e^{-\frac{1}{t}}}{t} \,\mathrm{d}t + \|u\|^2_{\mathrm{L}^q(\mathbb{R}^3_+)} \int_1^\infty \frac{1}{t^2} \,\mathrm{d}t < \infty, \end{aligned}$$

where we have used the fact that the function $x \mapsto \sqrt{x} e^{-tx}$ is bounded by $t^{-1/2} e^{-\frac{1}{2t}}$.

Thus, the conclusion follows because $D(A_q^{\frac{1}{2}}) = \mathrm{W}^{1,q}_0(\mathbb{R}^3_+) \cap \mathrm{L}^q_{\mathrm{div}}(\mathbb{R}^3_+)$ (see [BM88]).

3.E Conditional decay of the energy: proof of Theorem 3.3.1

Let u be a Leray solution (with strong energy inequality (3.3.2)) to the Navier-Stokes system (3.3.1) with source term F and initial data u_0. We provide some elements of proof concerning the conditional decay of the L^2 norm of u on $[0,T]$, provided that the forcing term F satisfies

$$\forall t \in [0,T], \quad \|F(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \frac{\mathrm{C}}{(1+t)^{7/4}}, \tag{3.E.1}$$

for some constant C. As explained in Section 3.3, this enters within the scope of the situation treated in [Wie87], with the adaptation to the half-space case coming from [BM88].

We first recall the following $\mathrm{L}^r\mathrm{L}^q$ estimates for the semigroup generated by the Stokes operator A_q on $\mathrm{L}^q_{\mathrm{div}}(\mathbb{R}^3_+)$ (see Section 3.D in the Appendix). A proof can be found in [BM88].

Lemma 3.E.1. *Let $a \in \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^3_+) \cap \mathrm{L}^r(\mathbb{R}^3_+)$ for some $r \in [1,\infty]$. Then for all $t \geq 0$*

$$\|e^{-tA_q}a\|_{\mathrm{L}^q(\mathbb{R}^3_+)} \leq \frac{C}{t^{\frac{3}{2}(\frac{1}{r}-\frac{1}{q})}}\|a\|_{\mathrm{L}^r(\mathbb{R}^3_+)}, \tag{3.E.2}$$

where C in independent of a and t, provided either $1 < r \leq q < \infty$, or $1 \leq r < q \leq \infty$.

Proof of Theorem 3.3.1. Consider a given time-dependent smooth positive cut-off function $g : \mathbb{R}^+ \to (0,+\infty)$. Since the Stokes operator $A = A_2$ on $\mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^3_+)$ is a self-adjoint and positive operator (see Section 3.D in the Appendix), we can use a spectral decomposition of this operator thanks to a resolution of the identity $(E_\lambda)_{\lambda\geq 0}$ (see [Soh12, Section 3.2]), namely

$$A = \int_0^{+\infty} \lambda \, \mathrm{d}E_\lambda.$$

By standard functional calculus rules and the Plancherel-Parseval theorem, we can write for all $\tau \in [0,T]$

$$\begin{aligned}\int_{\mathbb{R}^3_+} |\nabla u(\tau,x)|^2 \,\mathrm{d}x = \|A^{1/2}u(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} &= \int_0^{+\infty} \lambda \, \|\mathrm{d}E_\lambda u(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \\ &\geq \int_{\sqrt{\lambda}>g(\tau)} \lambda \, \|\mathrm{d}E_\lambda u(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \\ &\geq g^2(\tau)\|u(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} - g^2(\tau)\int_0^{g^2(\tau)} \|\mathrm{d}E_\lambda u(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}.\end{aligned}$$

Then, we can combine the strong energy inequality (3.1.28) for the Navier-Stokes equations with the Cauchy-Schwarz inequality to get the following key inequality: for all $t \in [0,T]$ and almost every $s \in [0,t]$ (including $s = 0$)

$$\begin{aligned}\|u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \int_s^t g^2(\tau)\|u(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}\tau \leq \|u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \int_s^t g^{-2}(\tau)\|F(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}\tau \\ + 2\int_s^t g^2(\tau)\int_0^{g^2(\tau)} \|\mathrm{d}E_\lambda u(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}\tau.\end{aligned} \tag{3.E.3}$$

Before going further, we introduce the notation $E_{\leq g^2(\tau)} := \mathbf{1}_{[0,g^2(t)]}(A)$ (in the sense of the bounded functional calculus). In order to estimate the last term in the r.h.s of (3.E.3), we fix $\tau \in [s,t]$ and follow [BM90]: we choose $\psi \in \mathscr{D}_{\mathrm{div}}(\mathbb{R}^3_+)$ and then take $\varphi(\sigma) := e^{-(\tau-\sigma)A}E_{\leq g^2(\tau)}\psi$ as a test function in the weak formulation (3.1.27) of the Navier-Stokes equations between time $\underline{\tau}$ and τ, for some $\underline{\tau} \leq \tau$ (see e.g. [Mas84] for a discussion about this available procedure). This yields

$$\int_{\underline{\tau}}^{\tau} \Big\langle (u\cdot\nabla)u(\sigma), \varphi(\sigma)\Big\rangle \,\mathrm{d}\sigma = \int_{\underline{\tau}}^{\tau} \Big\langle F(\sigma), \varphi(\sigma)\Big\rangle \,\mathrm{d}\sigma - \Big\langle u(\tau), E_{\leq g^2(\tau)}\psi\Big\rangle + \Big\langle u(\underline{\tau}), e^{-(\tau-\underline{\tau})A} E_{\leq g^2(\tau)}\psi\Big\rangle. \tag{3.E.4}$$

We now take $\underline{\tau} = 0$ and by observing that $(u\cdot\nabla)u = \operatorname{div}(u\otimes u)$, we have

$$\int_0^{\tau} \Big\langle (u\cdot\nabla)u(\sigma), e^{-(\tau-\sigma)A} E_{\leq g^2(\tau)}\psi\Big\rangle \,\mathrm{d}\sigma = -\int_0^{\tau} \Big\langle u(\sigma), u(\sigma)\cdot\nabla E_{\leq g^2(\tau)} e^{-(\tau-\sigma)A}\psi\Big\rangle \,\mathrm{d}\sigma,$$

therefore (3.E.4) yields

$$\begin{aligned}\Big|\Big\langle u(\tau), E_{\leq g^2(\tau)}\psi\Big\rangle\Big| &\leq \Big|\Big\langle u_0, e^{-\tau A} E_{\leq g^2(\tau)}\psi\Big\rangle\Big| + \int_0^{\tau} \Big|\Big\langle F(\sigma), E_{\leq g^2(\tau)} e^{-(\tau-\sigma)A}\psi\Big\rangle\Big| \,\mathrm{d}\sigma \\ &\quad + \int_0^{\tau} \Big|\Big\langle u(\sigma), u(\sigma)\cdot\nabla E_{\leq g^2(\tau)} e^{-(\tau-\sigma)A}\psi\Big\rangle\Big| \,\mathrm{d}\sigma \\ &:= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}).\end{aligned}$$

For (I) and (II), we use Lemma 3.E.1 and get

$$\begin{aligned}(\mathrm{I}) &\lesssim \|\mathbf{1}_{[0,g^2(\tau)]}(A) e^{-\tau A} u_0\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \|\psi\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \|e^{-\tau A} u_0\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \|\psi\|_{\mathrm{L}^2(\mathbb{R}^3_+)}, \\ (\mathrm{II}) &\leq \int_0^{\tau} \|F(\sigma)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \|E_{\leq g^2(\tau)} e^{-(\tau-\sigma)A}\psi\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}\sigma \lesssim \|\psi\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \int_0^{\tau} \|F(\sigma)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}\sigma,\end{aligned}$$

where $\lesssim$ refers to a universal constant. For (III), we rely on the following lemma, which can be found in the proof of [BM88, Lemma 4.3].

Lemma 3.E.2. *For any $w \in \mathrm{H}^1_{\mathrm{div}}(\mathbb{R}^3_+)$, $z \in \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^3_+)$ and $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$, we have for all $\tau \geq 0$*

$$\Big\langle w, w\cdot\nabla E_{\leq\gamma(\tau)} z\Big\rangle \leq C\gamma^{\frac{5}{4}}(\tau) \|w\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \|z\|_{\mathrm{L}^2(\mathbb{R}^3_+)},$$

where $\langle\cdot,\cdot\rangle$ stands for the inner product on $\mathrm{L}^2(\mathbb{R}^3_+)$ and where C is a universal constant.

We apply this Lemma with $w = u(\sigma)$, $z = e^{-(\tau-\sigma)A}\psi$ and $\gamma = g^2$ to obtain

$$(\mathrm{III}) \lesssim \|\psi\|_{\mathrm{L}^2(\mathbb{R}^3_+)} g^{5/2}(\tau) \int_0^t \|u(\sigma)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}\sigma.$$

Since all the previous estimates are valid for any $\psi \in \mathscr{D}_{\mathrm{div}}(\mathbb{R}^3_+)$, we infer that

$$\begin{aligned}&\int_0^{g^2(\tau)} \|\mathrm{d}E_\lambda u(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \\ &\leq C\|e^{-\tau A}u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + C\left(\int_0^{\tau} \|F(r)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}r\right)^2 + Cg(\tau)^5 \left(\int_0^{\tau} \|u(r)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}r\right)^2,\end{aligned}$$

where C stands for a universal constant. In view of (3.E.3), we end up with the following differential inequality: for almost all $s \geq 0$ and for all $t \geq s$

$$\begin{aligned}\|u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \int_s^t g^2(\tau)\|u(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}\tau &\leq \|u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + 2\int_s^t Cg(\tau)^7 \left(\int_0^{\tau} \|u(r)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}r\right)^2 \mathrm{d}\tau \\ &\quad + 2\int_s^t Cg^2(\tau)\Bigg[\|e^{-\tau A}u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + g^{-4}(\tau)\|F(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \\ &\qquad + \left(\int_0^{\tau} \|F(r)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}r\right)^2\Bigg] \mathrm{d}\tau.\end{aligned}$$

We then rely on the following Grönwall-like inequality (see [BM90]).

Lemma 3.E.3. *Let $t > 0$. Let $y \in \mathrm{L}^\infty(0,t)$ and $\beta \in \mathrm{L}^1(0,t)$ be nonnegative functions. Suppose that the following differential inequality holds: for almost all $0 \le s < t$*

$$y(t) + \int_s^t \mathrm{g}(\tau)y(\tau)\,\mathrm{d}\tau \le y(s) + \int_s^t \beta(\tau)\,\mathrm{d}\tau,$$

where $\tau \mapsto \mathrm{g}(\tau)$ is a smooth positive function on $[0,t]$. Then we have for almost all $0 \le s < t$

$$\exp\left(\int_0^t \mathrm{g}(\tau)\,\mathrm{d}\tau\right) y(t) \le \exp\left(\int_0^s \mathrm{g}(\tau)\,\mathrm{d}\tau\right) y(s) + \int_s^t \exp\left(\int_0^\tau \mathrm{g}(r)\,\mathrm{d}r\right)\beta(\tau)\,\mathrm{d}\tau.$$

Applying this lemma, we finally obtain

$$\begin{aligned}
&\|u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \exp\left(\int_0^t g^2(\tau)\,\mathrm{d}\tau\right) \\
&\le \|u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + C\int_0^t g^2(\tau)\left[\|e^{-\tau A}u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + g^{-4}(\tau)\|F(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\right]\exp\left(\int_0^\tau g^2(r)\,\mathrm{d}r\right)\mathrm{d}\tau \\
&\quad + C\int_0^t g^2(\tau)\left(\int_0^\tau \|F(\sigma)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}\sigma\right)^2 \exp\left(\int_0^\tau g^2(r)\,\mathrm{d}r\right)\mathrm{d}\tau \\
&\quad + C\int_0^t g^7(\tau)\left(\int_0^\tau \|u(\sigma)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}\sigma\right)^2 \exp\left(\int_0^\tau g^2(r)\,\mathrm{d}r\right)\mathrm{d}\tau.
\end{aligned} \tag{3.E.5}$$

We now take

$$g^2(t) := \frac{\alpha}{1+t}, \qquad \exp\left(\int_0^t g^2(\tau)\,\mathrm{d}\tau\right) = (1+t)^\alpha,$$

where $\alpha > 0$ and this means that for all $t \in [0,T]$

$$\begin{aligned}
\|u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}(1+t)^\alpha \lesssim \|u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} &+ \int_0^t (1+\tau)^{\alpha-1}\left[\|e^{-\tau A}u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + (1+\tau)^2\|F(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\right]\mathrm{d}\tau \\
&+ \int_0^t (1+\tau)^{\alpha-1}\left(\int_0^\tau \|F(\sigma)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}\sigma\right)^2\mathrm{d}\tau \\
&+ \int_0^t (1+\tau)^{\alpha-7/2}\left(\int_0^\tau \|u(\sigma)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}\sigma\right)^2\mathrm{d}\tau,
\end{aligned} \tag{3.E.6}$$

where $\lesssim$ is independent of t. Using Lemma 3.E.1, we get for all $\tau > 0$

$$\|e^{-\tau A}u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \le \frac{C}{(1+\tau)^{3/2}}\left(\|u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|u_0\|^2_{\mathrm{L}^1(\mathbb{R}^3_+)}\right),$$

where C is independent of τ and u. Observe now that, according to the assumption (3.E.1), the contribution of the source term $(1+\tau)^2\|F(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \left(\int_0^\tau \|F(r)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}r\right)^2$ is as $(1+\tau)^{-3/2}$ so that for all $t > 0$

$$\begin{aligned}
\|u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \int_0^t (1+\tau)^{\alpha-1}&\left[\|e^{-\tau A}u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + g^{-4}(\tau)\|F(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \left(\int_0^\tau \|F(r)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}r\right)^2\right]\mathrm{d}\tau \\
&\lesssim \|u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \int_0^t \frac{1}{(1+\tau)^{1-\alpha+3/2}}\,\mathrm{d}\tau, \\
&\lesssim \|u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + (1+t)^{\alpha-3/2},
\end{aligned} \tag{3.E.7}$$

provided that $\alpha \neq 3/2$, and where $\lesssim$ depends on $\|u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|u_0\|^2_{\mathrm{L}^1(\mathbb{R}^3_+)} + \mathrm{C}$, for the constant C appearing in (3.E.1).

Let us assume that on $[0,T]$, we have

$$\|u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \frac{1}{(1+t)^\beta}, \tag{3.E.8}$$

for some $\beta > 0$, different from 1. From this estimate, we deduce the following inequality

$$\int_0^t (1+\tau)^{\alpha-7/2} \left(\int_0^\tau \|u(r)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}r \right)^2 \mathrm{d}\tau \lesssim \begin{cases} 1 & \text{if } \beta < 1, \ \alpha - 2\beta - 1/2 < 0, \\ (1+t)^{\alpha-2\beta-1/2} & \text{if } \beta < 1, \ \alpha - 2\beta - 1/2 \geq 0, \\ 1 & \text{if } \beta > 1. \end{cases} \tag{3.E.9}$$

Now, we start with $\beta = 0$: the a priori estimate (3.E.8) indeed holds with this choice of exponent because the energy inequality (3.1.28) can be rewritten, together with (3.E.1), as

$$\begin{aligned} \|u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + 2\int_0^t \|\nabla u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \mathrm{d}s &\lesssim \|u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \int_0^t \|F(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s \\ &\lesssim \|u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \int_0^t (1+s)^{-7/4} \,\mathrm{d}s \\ &\lesssim \|u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + 1. \end{aligned}$$

We then take $\alpha = 1$ and use (3.E.6), (3.E.7) and (3.E.9) to obtain

$$(1+t)\|u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim 1 + (1+t)^{-1/2} + (1+t)^{1/2},$$

therefore

$$\|u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim (1+t)^{-1/2}.$$

This means that the a priori estimate (3.E.8) now holds with $\beta = 1/2$. We then take $\alpha = 2$ and combining again (3.E.6), (3.E.7) and (3.E.9) yields

$$\|u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim (1+t)^{-2} + (1+t)^{-3/2} \lesssim (1+t)^{-3/2}.$$

Note that in view of (3.E.7) and (3.E.9), we cannot improve the exponent in the previous estimate. The proof of Theorem 3.3.1 is therefore complete. □

3.F Parabolic regularization for the Navier-Stokes system on $\mathbb{R}^3_+$

Theorem 3.F.1. *For all $T > 0$, there exists a universal constant $\mathrm{C}_\star > 0$ such that the following holds. Consider $u_0 \in \mathrm{H}^1_{\mathrm{div}}(\mathbb{R}^3_+)$ and $F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{R}^3_+))$ and $T > 0$ such that*

$$\|u_0\|^2_{\mathrm{H}^1(\mathbb{R}^3_+)} + \int_0^T \left[\|F(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|F(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \right] \mathrm{d}s \leq \mathrm{C}_\star. \tag{3.F.1}$$

Then, there exists on $[0,T]$ a unique Leray solution to the Navier-Stokes system

$$\begin{cases} \partial_t u + (u \cdot \nabla_x) u - \Delta_x u + \nabla_x p = F, \\ \mathrm{div}_x\, u = 0, \\ u_{|x_3=0} = 0, \\ u_{|t=0} = u_0, \end{cases}$$

with initial data u_0 *and source* F*. This solution* u *belongs to* $\mathrm{L}^\infty(0,T;\mathrm{H}^1_{\mathrm{div}}(\mathbb{R}^3_+))\cap\mathrm{L}^2(0,T;\mathrm{H}^2(\mathbb{R}^3_+))$ *and satisfies for almost every* $t\in[0,T]$

$$\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}+\frac{1}{2}\int_0^t\|\mathrm{D}^2u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s\leq\widetilde{\mathrm{C}}\left(\|u_0\|^2_{\mathrm{H}^1(\mathbb{R}^3_+)}+\|F\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))}\right),\tag{3.F.2}$$

for some universal constant $\widetilde{\mathrm{C}}>0$.

Proof. We argue in two different steps. First, if such a Leray solution to the Navier-Stokes equations exists, it satisfies in particular $u\in\mathrm{L}^4(0,T;\mathrm{H}^1_{\mathrm{div}}(\mathbb{R}^3_+))$, which is a well-known case of weak-strong uniqueness for this system (see e.g. [CDGG06, Theorem 3.3], which holds for general domains). Thus, this solution will be equal to any Leray solution of the Navier-Stokes system with source F and initial value u_0.

In a second step, we have to prove that such a solution indeed exists on $[0,T]$: to do so, we rely on a standard approximation procedure by regularizing the data F and u_0 and by smoothing the convection term in the momentum equation (see e.g. [MS88] where the Yosida operator is used, as in Section 3.B). We essentially obtain a sequence $(u_N)_N$ of solutions to regularized Navier-Stokes systems. This sequence will classically be bounded in $\mathrm{L}^\infty\mathrm{L}^2\cap\mathrm{L}^2\mathrm{H}^1$ and will satisfy additional parabolic estimates coming from (3.F.1). A compactness argument allows to get the existence of a Leray solution in $\mathrm{L}^\infty(0,T;\mathrm{H}^1_{\mathrm{div}}(\mathbb{R}^3_+))\cap\mathrm{L}^2(0,T;\mathrm{H}^2(\mathbb{R}^3_+))$ and which satisfies the estimate (3.F.2) on $[0,T]$.

Hence in the following, we deal with a smooth solution u with smooth data u_0 and we are looking for parabolic estimates for the velocity field u. The following strategy is based on an idea which is well-known in the context of the inhomogeneous incompressible Navier-Stokes equations (see [LS78, DM19]): we first apply the Leray projection $\mathbb{P}$ to the Navier-Stokes equations and then multiply by $\partial_t u$ to obtain, after integration by parts

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}+\|\partial_t u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}=-\langle(u\cdot\nabla)u,\partial_t u\rangle_{\mathrm{L}^2(\mathbb{R}^3_+)}+\langle F,\partial_t u\rangle_{\mathrm{L}^2(\mathbb{R}^3_+)},$$

where we have dropped the time variable. We then use the Young inequality twice in order to absorb $\|\partial_t u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}$ in the l.h.s, that is on $[0,T]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}+\|\partial_t u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\leq 2\int_{\mathbb{R}^3_+}|(u\cdot\nabla)u|^2\,\mathrm{d}x+2\int_{\mathbb{R}^3_+}|F|^2\,\mathrm{d}x.\tag{3.F.3}$$

In order to estimate D^2u, we rewrite the Navier-Stokes system satisfied by (u,p) as the following stationary Stokes system in the half-space $\mathbb{R}^3_+$ with source term $F-(u\cdot\nabla)u-\partial_t u$

$$\begin{cases}-\Delta u+\nabla p=F-(u\cdot\nabla)u-\partial_t u,\\ \operatorname{div}u=0,\\ u_{|x_3=0}=0.\end{cases}$$

According to [Gal11, Theorem IV.3.2], there exists a universal constant $C>0$ such that on $[0,T]$

$$\|\mathrm{D}^2u\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\leq C\|F-(u\cdot\nabla)u-\partial_t u\|_{\mathrm{L}^2(\mathbb{R}^3_+)},$$

and thus, denoting by $C>0$ another universal constant, we have on $[0,T]$

$$\|\mathrm{D}^2u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\leq C\left(\|F\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}+\|(u\cdot\nabla)u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}+\frac{1}{2}\|\partial_t u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\right).$$

Coming back to (3.F.3), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \frac{1}{2}\|\partial_t u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \frac{1}{C}\|\mathrm{D}^2 u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq 3\int_{\mathbb{R}^3_+} |(u\cdot\nabla)u|^2\,\mathrm{d}x + 3\int_{\mathbb{R}^3_+} |F|^2\,\mathrm{d}x. \tag{3.F.4}$$

We now provide a bound for the convective term on [0,T] by using consecutively the Cauchy-Schwarz inequality, interpolation inequality for Lebesgue spaces and the Gagliardo-Nirenberg-Sobolev inequality (see Theorem 3.C.1) to write

$$\begin{aligned}\int_{\mathbb{R}^3_+} |(u\cdot\nabla)u|^2\,\mathrm{d}x &\leq \|u\|^2_{\mathrm{L}^4(\mathbb{R}^3_+)}\|\nabla u\|^2_{\mathrm{L}^4(\mathbb{R}^3_+)} \\ &\leq \|u\|^{1/2}_{\mathrm{L}^2(\mathbb{R}^3_+)}\|u\|^{3/2}_{\mathrm{L}^6(\mathbb{R}^3_+)}\|\nabla u\|^{1/2}_{\mathrm{L}^2(\mathbb{R}^3_+)}\|\nabla u\|^{3/2}_{\mathrm{L}^6(\mathbb{R}^3_+)} \\ &\lesssim \|u\|^{1/2}_{\mathrm{L}^2(\mathbb{R}^3_+)}\|\nabla u\|^{3/2}_{\mathrm{L}^2(\mathbb{R}^3_+)}\|\nabla u\|^{1/2}_{\mathrm{L}^2(\mathbb{R}^3_+)}\|\mathrm{D}^2 u\|^{3/2}_{\mathrm{L}^2(\mathbb{R}^3_+)}.\end{aligned} \tag{3.F.5}$$

Thanks to Young inequality, we deduce the existence of a universal constant $\overline{C} > 0$ such that

$$3\|(u\cdot\nabla)u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \overline{C}\|u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\|\nabla u\|^8_{\mathrm{L}^2(\mathbb{R}^3_+)} + \frac{1}{2C}\|\mathrm{D}^2 u\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}. \tag{3.F.6}$$

We integrate (3.F.4) between 0 and $t \in [0,T]$ and get

$$\begin{aligned}&\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \frac{1}{2C}\int_0^t \|\mathrm{D}^2 u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s \\ &\qquad\leq \|\nabla u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + 3\int_0^t \|F(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s + \overline{C}\int_0^t \|u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\|\nabla u(s)\|^8_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s.\end{aligned}$$

If we set

$$x(t) := \|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \frac{1}{2C}\int_0^t \|\mathrm{D}^2 u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s, \tag{3.F.7}$$

$$h(t) := \|\nabla u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + 3\int_0^t \|F(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s, \tag{3.F.8}$$

$$g(s) := \overline{C}\|u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\|\nabla u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}, \tag{3.F.9}$$

then the previous inequality can be written as

$$x(t) \leq h(t) + \int_0^t g(s)x(s)^3\,\mathrm{d}s,$$

for all $t \in [0,T]$. We are now in position to apply Bihari's lemma, that we recall now (see [Dan85] for a proof).

Lemma 3.F.2. *Let x, g and h three positive continuous functions on $\mathbb{R}^+$, such that h is increasing . Let w a continuous submultiplicative and nondecreasing function on $\mathbb{R}^+$ such that $w(u) > 0$ for all $u > 0$. Suppose that for all $t \geq 0$*

$$x(t) \leq h(t) + \int_0^t g(s)w(x(s))\,\mathrm{d}s. \tag{3.F.10}$$

If the function

$$W(u) := \int_1^u \frac{\mathrm{d}s}{w(s)}, \quad u \in (0,a) \quad \textit{for some } a > 0,$$

is a bijection on $(0, a)$*, then the following inequality*

$$x(t) \leq h(t) W^{-1}\left(\int_0^t g(s) \frac{w(h(s)}{h(s))} \,\mathrm{d}s\right) \tag{3.F.11}$$

holds for all $t \geq 0$ *such that* $\int_0^t g(s) \frac{w(h(s))}{h(s)} \,\mathrm{d}s \in (0, a)$*.*

We use this lemma with the function $w(s) = s^3$ associated to

$$W(u) = \int_1^u \frac{\mathrm{d}s}{s^3} = \frac{1}{2}\left(1 - \frac{1}{u^2}\right), \quad u > 0,$$

and whose inverse, defined on $(0, 1/2)$, is $W^{-1}(v) = (1 - 2v)^{-1/2}$. The inequality (3.F.11) reads on $[0, T]$ as

$$x(t) \leq h(t) \left(1 - 2\int_0^t g(s) h^2(s) \mathrm{d}s\right)^{-1/2}, \tag{3.F.12}$$

provided that for all $t \in [0, T]$, we have $1 - 2\int_0^t g(s) h^2(s) \mathrm{d}s > 0$. Thus, this is enough to ensure for instance that

$$1 - 2\int_0^T g(s) h^2(s) \mathrm{d}s > 1/2, \tag{3.F.13}$$

is satisfied in order to get $x(t) \leq h(t)$ for all $t \in [0, T]$. We observe that

$$\int_0^T g(s) h^2(s) \mathrm{d}s \leq \left[\|\nabla u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + 3\int_0^T \|F(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s\right]^2 \int_0^T g(s) \mathrm{d}s,$$

and

$$\int_0^T g(s) \mathrm{d}s \leq \overline{C} \|u\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} \int_0^T \|\nabla u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \mathrm{d}s.$$

Furthermore, the following energy inequality for the Navier-Stokes system with source F and initial data u_0 is satisfied by u

$$\|u\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} + 2\int_0^T \|\nabla u(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \mathrm{d}s \leq \widetilde{C} \left[\|u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \left(\int_0^T \|F(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s\right)^2\right],$$

where $\widetilde{C} > 0$ is a universal constant. The condition (3.F.13) can thus be satisfied provided that the quantity

$$\left[\|\nabla u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \int_0^T \|F(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s\right]^2 \left[\|u_0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \left(\int_0^T \|F(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s\right)^2\right]^2$$

is small enough. We observe that an assumption of the type of (3.F.1) can indeed provide such a smallness. Now, in view of the definitions (3.F.7)-(3.F.8), the inequality $x(t) \leq h(t)$ for all $t \in [0, T]$ brings to the conclusion (3.F.2) and this completes the proof. □

Chapter 4

Global hydrodynamic limit towards a Boussinesq-Navier-Stokes system

Based on the article [Ert22], prepublished and submitted in a journal.

4.1 Introduction

In this work, we consider the following Vlasov-Navier-Stokes system set in the half-space:

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v[f(u-v) - fe_3] = 0, \qquad t>0, \quad (x,v) \in \mathbb{R}^3_+ \times \mathbb{R}^3, \tag{4.1.1}$$

$$\partial_t u + (u \cdot \nabla_x)u - \Delta_x u + \nabla_x p = \int_{\mathbb{R}^3} f(v-u)\,\mathrm{d}v, \qquad t>0, \quad x \in \mathbb{R}^3_+, \tag{4.1.2}$$

$$\mathrm{div}_x\, u = 0, \qquad t>0, \quad x \in \mathbb{R}^3_+, \tag{4.1.3}$$

with

$$\mathbb{R}^3_+ := \mathbb{R}^2 \times (0,+\infty), \quad e_3 := (0,0,1).$$

This system of equations accounts for the evolution of a cloud of droplets within an ambient viscous fluid. It belongs to the wide family of fluid-kinetic models (see e.g. [O'R81, Wil85, Des10]). More precisely, the particles are described thanks to a distribution function $f(t,\cdot,\cdot) \in \mathbb{R}^+$ on the phase space $\mathbb{R}^3_+ \times \mathbb{R}^3$ while the fluid is described by its velocity $u(t,\cdot) \in \mathbb{R}^3$ and pressure $p(t,\cdot) \in \mathbb{R}$. The surrounding fluid is assumed to be incompressible, homogeneous and viscous, and the monodispersed phase of particles is sufficiently dilute that collisions can be neglected. Here, the third term in the Vlasov equation (4.1.1) asserts for the the acceleration undergone by the particles and which comes from the action of the fluid (drag term $u-v$) and the gravity (external gravity force $-e_3$). The particles also act on the fluid by retroaction, inducing a source term

$$F(t,x) = \int_{\mathbb{R}^3} f(t,x,v)(v-u(t,x))\,\mathrm{d}v$$

in the Navier-Stokes equations (4.1.2). This term is referred to as the *Brinkman force*. Note also that the gravity force in the Navier-Stokes equations (4.1.2) is absorbed in the pressure term. The physical constants are all normalized in (4.1.1)–(4.1.2)–(4.1.3).

The system is endowed with the initial conditions

$$u(0,x) = u_0(x), \quad f(0,x,v) = f_0(x,v), \quad (x,v) \in \mathbb{R}^3_+ \times \mathbb{R}^3.$$

The boundary conditions for the Vlasov-Navier-Stokes system read as follows: we prescribe the homogeneous Dirichlet boundary conditions for the fluid

$$u(t,\cdot) = 0, \text{ on } \partial\mathbb{R}^3_+ = \mathbb{R}^2 \times \{0\}. \tag{4.1.4}$$

We also introduce the following outgoing/incoming phase-space boundaries:

$$\Sigma^{\pm} := \left\{(x,v) \in \partial\mathbb{R}^3_+ \times \mathbb{R}^3 \mid \pm v \cdot n(x) > 0\right\},$$
$$\Sigma_0 := \left\{(x,v) \in \partial\mathbb{R}^3_+ \times \mathbb{R}^3 \mid v \cdot n(x) = 0\right\},$$
$$\Sigma := \Sigma^+ \sqcup \Sigma^- \sqcup \Sigma_0 = \partial\mathbb{R}^3_+ \times \mathbb{R}^3,$$

where $n(x)$ is the outer-pointing normal vector to the boundary $\partial\mathbb{R}^3_+$ at point x. We prescribe the absorption boundary conditions for the distribution function:

$$f(t,\cdot,\cdot) = 0, \text{ on } \Sigma^-. \tag{4.1.5}$$

In this chapter, we are interested in a particular hydrodynamic limit of the system (4.1.1)–(4.1.2)–(4.1.3). Roughly speaking, it corresponds to a high-field regime where the particle volume fraction is small compared to the one of the fluid, and where the Stokes number is small. Physically, this means that the particles tend to follow the ambient fluid and that the inertial effects are not very important. After a nondimensionalization based on physical quantities appearing in the system that we shall detail in the subsequent Section 4.1.1, it consists[1] in

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \dfrac{1}{\varepsilon}\mathrm{div}_v[f(u - v - e_3)] = 0, \\ \partial_t u + (u \cdot \nabla_x)u - \Delta_x u + \nabla_x p = \displaystyle\int_{\mathbb{R}^3} f(v-u)\,\mathrm{d}v, \\ \mathrm{div}_x\, u = 0, \end{cases} \tag{4.1.6}$$

for some parameter $0 < \varepsilon \ll 1$. Our main goal is to justify an approximation of the Vlasov-Navier-Stokes system under this regime. This should lead, in some sense to be made precise later, to an hydrodynamic system of the form

$$\begin{cases} \partial_t \rho + \mathrm{div}_x[\rho(u - e_3)] = 0, \\ \partial_t u + (u \cdot \nabla_x)u - \Delta_x u + \nabla_x p = -\rho e_3, \\ \mathrm{div}_x\, u = 0, \end{cases} \tag{4.1.7}$$

which is a Boussinesq-Navier-Stokes type system without diffusivity. The question of this rigorous passage to the limit has been raised as an open problem by Han-Kwan and Michel in [HKMar]. In the current chapter, we establish the global derivation of (4.1.7) from (4.1.6). A rough version of our main result if the following.

Theorem 4.1.1. *Assume that $(u^0_\varepsilon, f^0_\varepsilon)$ are smooth and small enough initial data, uniformly in ε. If f^0_ε is decaying fast enough with respect to x and v then any solution $(u_\varepsilon, f_\varepsilon)$ to the system (4.1.6) with initial data $(u^0_\varepsilon, f^0_\varepsilon)$ satisfies for any $T > 0$*

$$u_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} u \ \text{ in } \mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^3_+)) \quad \text{and} \quad \int_{\mathbb{R}^3} f_\varepsilon\,\mathrm{d}v \underset{\varepsilon \to 0}{\longrightarrow} \rho \ \text{ in } \mathrm{L}^\infty(0,T;\mathrm{H}^{-1}(\mathbb{R}^3_+)),$$

where (u, ρ) is a strong solution to (4.1.7).

We refer to Subsection 4.1.4 for a more precise version of the statements (see Theorems 4.1.10–4.1.11). As we shall explain later on, this result is *not* a consequence of the hydrodynamic limits of the Vlasov-Navier-Stokes system studied by Han-Kwan and Michel in [HKMar]. Their analysis (in the gravity-less case, i.e. without the term $-e_3$ in (4.1.1)) only allows for a local in time derivation of (4.1.7) starting from (4.1.6). Roughly speaking, this comes from a continuous injection of energy in the system coming from the additional gravity term. Our main contribution is thus to justify the previous limit for *arbitrarily large* times. Let us point out that this result will be obtained thanks to the the combined mechanism of gravity *and* absorption boundary condition on the half-space.

4.1.1 Scaling procedure

Let us write the Vlasov-Navier-Stokes system in a dimensionless form. First, we come back to the equations without normalization of the physical quantities, which read

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v[f\Gamma(t,x,v)] = 0, \\ \rho_F(\partial_t u + (u \cdot \nabla_x)u) - \mu\Delta_x u + \nabla_x p = \mathfrak{F}(t,x), \\ \mathrm{div}_x\, u = 0. \end{aligned}$$

[1] An alternative nondimensionalization can also be found in the PhD thesis [Höf20] of Richard Höfer.

where we assume that the fluid is of dynamic viscosity $\mu > 0$ and mass density $\rho_F > 0$. Furthermore, we consider monodispersed spherical particles of radius $a > 0$ and of mass density $\rho_P > 0$. We suppose that the acceleration $\Gamma(t, x, v)$ in the Vlasov equation satisfies

$$\frac{4}{3}\pi a^3 \rho_P \Gamma(t, x, v) = \mathrm{F}_d(t, x, v) + \mathrm{F}_G,$$

where F_d is the drag force stemming from the action from the fluid on the particles and F_G is the resultant of weight/buoyancy. The particles act on the fluid by retroaction so that the source term (Brinkman force) $\mathfrak{F}(t, x)$ in the Navier-Stokes equations can be expressed as

$$\mathfrak{F}(t, x) = -\int_{\mathbb{R}^3} \mathrm{F}_d(t, x, v)\,\mathrm{d}v.$$

Let us precise the structure of the terms F_d and F_G. If $G := -ge_3$ stands for the gravity field (with $g > 0$ and $e_3 := (0, 0, 1)$) then, according to Stokes law, we have

$$\begin{aligned} \mathrm{F}_d(t, x, v) &:= 6\pi\mu a(u(t, x) - v), \\ \mathrm{F}_G &:= \frac{4}{3}\pi a^3(\rho_P - \rho_F)G, \end{aligned}$$

so that

$$\Gamma(t, x, v) = \frac{9}{2}\frac{\mu}{\rho_P a^2}(u(t, x) - v) + \left(1 - \frac{\rho_F}{\rho_P}\right) G,$$

while

$$\mathfrak{F}(t, x) = -\int_{\mathbb{R}^3} \mathrm{F}_d(t, x, v)\,\mathrm{d}v = \int_{\mathbb{R}^3} 6\pi\mu a f(t, x, v)(v - u(t, x))\,\mathrm{d}v.$$

Making all the physical parameters appear in the equations (4.1.1)–(4.1.2)–(4.1.3), and thanks to the previous expressions of the forces, the system can be rewritten as

$$\begin{aligned} \rho_F(\partial_t u + (u \cdot \nabla_x)u) - \mu\Delta u + \nabla p &= \int_{\mathbb{R}^3} 6\pi\mu a f(t, x, v)(v - u(t, x))\,\mathrm{d}v, \\ \operatorname{div} u &= 0, \\ \partial_t f + v \cdot \nabla_x f + \left(1 - \frac{\rho_F}{\rho_P}\right) G \cdot \nabla_v f + \operatorname{div}_v \left[\frac{9}{2}\frac{\mu}{\rho_P a^2}(u - v) f\right] &= 0. \end{aligned}$$

We define the *Stokes relaxation time* as

$$\tau_S := \frac{2a^2\rho_P}{9\mu},$$

so that

$$\int_{\mathbb{R}^3} 6\pi\mu a f(t, x, v)(v - u(t, x))\,\mathrm{d}v = \frac{\rho_P}{\tau_S}\int_{\mathbb{R}^3} \frac{4}{3}\pi a^3 f(t, x, v)(v - u(t, x))\,\mathrm{d}v.$$

Then, we introduce L and T the typical length and time of observation. We also introduce the typical thermal velocity of the particles $\sqrt{\theta}$ defined by

$$\sqrt{\theta} := \left(\frac{kT}{M}\right)^{1/2}, \quad M = \frac{4}{3}\pi a^3 \rho_P,$$

(this expression does not matter very much in the subsequent analysis), which can be compared to

$$U := \frac{L}{T}.$$

Then, we perform a nondimensionalization procedure:

$$t = T\bar{t}, \quad x = L\bar{x}, \quad v = \sqrt{\theta}\bar{v}, \tag{4.1.8}$$

$$u(T\bar{t}, L\bar{x}) = U\bar{u}(\bar{t},\bar{x}), \quad p(T\bar{t}, L\bar{x}) = U^2\bar{p}(\bar{t},\bar{x}), \tag{4.1.9}$$

$$\bar{f}(\bar{t},\bar{x},\bar{v}) = \frac{L^3}{N_L}\sqrt{\theta}^3 f(T\bar{t}, L\bar{x}, \sqrt{\theta}\bar{v}), \tag{4.1.10}$$

where N_L is the typical number of particles in a cube of size L.

Let us explain the equality (4.1.10): we want $\bar{f}$ to be dimensionless while $f(t,x,v)\mathrm{d}x\mathrm{d}v$ is a number of particles. Thus, the equality

$$f(t,x,v)\mathrm{d}x\mathrm{d}v = \frac{N_L}{L^3\sqrt{\theta}^3}\bar{f}(\bar{t},\bar{x},\bar{v})L^3\sqrt{\theta}^3\mathrm{d}\bar{x}\mathrm{d}\bar{v} = N_L\bar{f}(\bar{t},\bar{x},\bar{v})\mathrm{d}\bar{x}\mathrm{d}\bar{v}$$

makes sense.

Let us continue with (4.1.8)–(4.1.9)–(4.1.10). We obtain

$$\begin{cases} \dfrac{1}{T}\partial_{\bar{t}}\bar{f} + \dfrac{\sqrt{\theta}}{L}\bar{v}\cdot\nabla_{\bar{x}}\bar{f} + \dfrac{1}{\sqrt{\theta}}\left(1-\dfrac{\rho_F}{\rho_P}\right)G\cdot\nabla_{\bar{v}}\bar{f} + \dfrac{1}{\sqrt{\theta}}\mathrm{div}_{\bar{v}}\left[\dfrac{9}{2}\dfrac{\mu}{\rho_P a^2}\left(U\bar{u}-\sqrt{\theta}\bar{v}\right)\bar{f}\right] = 0, \\ \rho_F\left(\dfrac{L}{T^2}\partial_{\bar{t}}\bar{u} + \dfrac{L}{T^2}(\bar{u}\cdot\nabla_{\bar{x}})\bar{u}\right) - \dfrac{\mu}{LT}\Delta_{\bar{x}}\bar{u} + \dfrac{L}{T^2}\nabla_{\bar{x}}\bar{p} \\ \qquad = \dfrac{4}{3}\pi a^3\dfrac{\rho_P}{\tau_S}\displaystyle\int_{\mathbb{R}^3}\dfrac{N_L}{L^3\sqrt{\theta}^3}\bar{f}(\bar{t},\bar{x},\bar{v})\left(\sqrt{\theta}\bar{v}-U\bar{u}(\bar{t},\bar{x})\right)\sqrt{\theta}^3\,\mathrm{d}\bar{v}, \\ \mathrm{div}_{\bar{x}}\,\bar{u} = 0. \end{cases}$$

We have used the fact that $\mathrm{d}v = \sqrt{\theta}^3\mathrm{d}\bar{v}$. We simplify the previous equalities and we use the expression of the Stokes relaxation time to end up with

$$\begin{cases} \partial_{\bar{t}}\bar{f} + T\dfrac{\sqrt{\theta}}{L}\bar{v}\cdot\nabla_{\bar{x}}\bar{f} + T\dfrac{1}{\sqrt{\theta}}\left(1-\dfrac{\rho_F}{\rho_P}\right)G\cdot\nabla_{\bar{v}}\bar{f} + \dfrac{T}{\tau_S}\mathrm{div}_{\bar{v}}\left[\left(\dfrac{U}{\sqrt{\theta}}\bar{u}-\bar{v}\right)\bar{f}\right] = 0, \\ \partial_{\bar{t}}\bar{u} + (\bar{u}\cdot\nabla_{\bar{x}})\bar{u} - \dfrac{2}{9}\dfrac{a^2}{L^2}\dfrac{T}{\tau_S}\dfrac{\rho_P}{\rho_F}\Delta_{\bar{x}}\bar{u} + \dfrac{1}{\rho_F}\nabla_{\bar{x}}\bar{p} = \dfrac{4}{3}\pi a^3\dfrac{N_L}{L^3}\dfrac{\rho_P}{\rho_F}\dfrac{T}{\tau_S}\displaystyle\int_{\mathbb{R}^3}\bar{f}(\bar{t},\bar{x},\bar{v})\left(\dfrac{\sqrt{\theta}}{U}\bar{v}-\bar{u}(\bar{t},\bar{x})\right)\mathrm{d}\bar{v}, \\ \mathrm{div}_{\bar{x}}\,\bar{u} = 0. \end{cases}$$

Let us define the following quantities:

- $\phi := N_L\dfrac{4}{3}\pi\dfrac{a^3}{L^3}$ which is the particle volume fraction;
- $\mathbf{A} = \dfrac{\sqrt{\theta}}{U}$;
- $\mathbf{B}_1 = \dfrac{T}{\tau_S}$, which is the inverse of the *Stokes number* St defined as

$$\mathrm{St} := \frac{2a^2\rho_P U}{9\mu L};$$

- $\mathbf{B}_2 = \dfrac{V_S}{\sqrt{\theta}} \dfrac{T}{\tau_S} = \dfrac{V_S}{\sqrt{\theta}} \mathbf{B}_1$ where V_S is the *Stokes settling velocity* defined as

$$V_S := \tau_S \left(1 - \frac{\rho_F}{\rho_P}\right) g.$$

  We have $V_S \geq 0$ if and only if $\rho_P > \rho_F$. Note that $G/g = -e_3$.

- $\mathbf{C} = \phi \mathbf{B}_1 \dfrac{\rho_P}{\rho_F}$;

- $\mathbf{F} = \dfrac{2}{9} \dfrac{a^2}{L^2} \mathbf{B}_1 \dfrac{\rho_P}{\rho_F}$.

Dropping the $\overline{\cdot}$, the Vlasov-Navier-Stokes system can now be rewritten under a dimensionless form which is

$$\begin{cases} \partial_t f + \mathbf{A} v \cdot \nabla_x f - \mathbf{B}_2 e_3 \cdot \nabla_v f + \mathbf{B}_1 \mathrm{div}_v \left[f \left(\frac{1}{\mathbf{A}} u - v \right) \right] = 0, \\ \partial_t u + (u \cdot \nabla_x) u - \mathbf{F} \Delta_x u + \nabla_x p = \mathbf{C} \displaystyle\int_{\mathbb{R}^3} f(t,x,v)(\mathbf{A} v - u(t,x)) \, \mathrm{d}v, \\ \mathrm{div}_x \, u = 0. \end{cases}$$

In view of the previous expressions, we only have to impose $\phi, \mathbf{A}, \mathbf{B}_1, \mathbf{B}_2$ and $\mathbf{F}$. We choose the following regime:

$$\phi = \varepsilon, \quad \mathbf{A} = 1, \quad \mathbf{B}_1 = \frac{1}{\varepsilon}, \quad \mathbf{B}_2 = \frac{1}{\varepsilon}, \quad \mathbf{F} = 1,$$

which leads to

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \frac{1}{\varepsilon} \mathrm{div}_v [f(u(t,x) - v) - f e_3] = 0, \\ \partial_t u + (u \cdot \nabla_x) u - \Delta_x u + \nabla_x p = \displaystyle\int_{\mathbb{R}^3} f(t,x,v)(v - u(t,x)) \, \mathrm{d}v, \\ \mathrm{div}_x \, u = 0, \end{cases}$$

as announced in the previous subsection. It corresponds to the situation where

- the particle volume fraction ϕ is small (meaning that $a \ll L$ or N_L is large), which falls within the scope of the so-called *thin spray* model (see e.g. [Des10]). Note that the assumption $a \ll L$ seems to be physically relevant;

- $\tau_S \ll T$, meaning that the Stokes relaxion time is small compared to the observation time, i.e. the Stokes number St is small. Physically, this means that the particles tend to follow the ambient fluid and that the inertial effects are not very important. Let us note that the regime where $\mathrm{St} \to 0$ is also investigated in [Hö18] in the case of the Vlasov-Stokes equations, but with a different choice of timescale leading to a somewhat different expression of St;

- $a \ll L$, which also means that the condition $\mathbf{F} = 1$ does not imply that the mass density of the particles is small compared to that of the fluid (namely, we only have $\rho_P > \rho_F$).

4.1.2 Formal limit

Let us formally derive the limit macroscopic system (4.1.7). For any $\varepsilon > 0$, we consider a solution $(u_\varepsilon, f_\varepsilon)$ to the system

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \dfrac{1}{\varepsilon}\mathrm{div}_v\left[f_\varepsilon(u_\varepsilon - v - e_3)\right] = 0, & t>0, \quad (x,v) \in \mathbb{R}^3_+ \times \mathbb{R}^3, \\ \partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla_x) u_\varepsilon - \Delta_x u_\varepsilon + \nabla_x p_\varepsilon = j_\varepsilon - \rho_\varepsilon u_\varepsilon, & t>0, \quad x \in \mathbb{R}^3_+, \\ \mathrm{div}_x\, u_\varepsilon = 0, & t>0, \quad x \in \mathbb{R}^3_+, \end{cases} \tag{VNS$_\varepsilon$}$$

where

$$\rho_\varepsilon := \int_{\mathbb{R}^3} f_\varepsilon(t,x,v)\,\mathrm{d}v, \quad j_\varepsilon := \int_{\mathbb{R}^3} v f_\varepsilon(t,x,v)\,\mathrm{d}v.$$

This system is supplemented with initial conditions and with boundary conditions similar to (4.1.4)–(4.1.5).

Assume that

$$u_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} u, \quad \rho_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \rho.$$

Integrating the Vlasov equation on $\mathbb{R}^3$ in velocity yields the conservation of mass while multiplying the Vlasov equation by v and then integrating on $\mathbb{R}^3$ in velocity yields the conservation of momentum for the particles: this reads

$$\begin{cases} \partial_t \rho_\varepsilon + \mathrm{div}_x\, j_\varepsilon = 0, \\ \partial_t j_\varepsilon + \mathrm{div}_x \left(\displaystyle\int_{\mathbb{R}^3} v \otimes v f_\varepsilon \,\mathrm{d}v \right) = \dfrac{1}{\varepsilon}\left(\rho_\varepsilon(u_\varepsilon - e_3) - j_\varepsilon\right). \end{cases}$$

Assuming the following convergence

$$j_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} j,$$

we deduce that we must have

$$\rho_\varepsilon(u_\varepsilon - e_3) - j_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} 0.$$

We thus formally get

$$j = \rho(u - e_3),$$

and then

$$\partial_t \rho + \mathrm{div}_x[\rho(u - e_3)] = 0,$$

as well as the source term $-\rho e_3$ in the Navier-Stokes equations. Note that we have dealt with the convergence of products in a formal way. As we will see in more detail later on, the rigorous convergence of $j_\varepsilon - \rho_\varepsilon u_\varepsilon + \rho_\varepsilon e_3$ will be a crucial issue of the analysis.

Concerning the boundary conditions satisfied by (ρ, u), recall that u_ε satisfies a Dirichlet boundary condition (4.1.4) and that f_ε satisfies an absorption boundary condition (4.1.5). We thus hope for

$$u(t,\cdot)_{|x_3=0} = 0,$$

at the limit $\varepsilon \to 0$, without any boundary condition for the density ρ. Indeed, the transport equation satisfied by $\rho(t,x)$ in (4.1.11) requires a condition on the subset of $\{x_3 = 0\}$ which is

$$\Gamma^-(t) := \left\{ x \in \mathbb{R}^2 \times \{0\} \mid [u(t,x) - e_3] \cdot n(x) < 0 \right\}.$$

But this set is empty since $u(t,\cdot)_{|\mathbb{R}^2\times\{0\}} = 0$ and $(-e_3)\cdot n(x) = (-e_3)\cdot(-e_3) = 1$.

All in all, the formal limit of (VNS$_\varepsilon$) as $\varepsilon \to 0$ is the following Boussinesq-Navier-Stokes type system set on $\mathbb{R}^3_+$:

$$\begin{cases} \partial_t \rho + \mathrm{div}_x[\rho(u-e_3)] = 0, & t>0, \quad x\in\mathbb{R}^3_+, \\ \partial_t u + (u\cdot\nabla_x)u - \Delta_x u + \nabla_x p = -\rho e_3, & t>0, \quad x\in\mathbb{R}^3_+, \\ \mathrm{div}_x\, u = 0, & t>0, \quad x\in\mathbb{R}^3_+, \end{cases} \tag{4.1.11}$$

with the boundary condition

$$u(t,\cdot) = 0, \text{ on } \mathbb{R}^2\times\{0\}.$$

At the limit, the system thus consists in a transport of the local density of particles by the flow of the fluid and the gravity, while the action of the particles appears as a forcing term in the Navier-Stokes equations, in the direction of the gravity field. In short, the velocities of the particles align on the sum of the fluid velocity and gravity field.

Let us comment on the system (4.1.11) we have just obtained. It formally ressembles to the classical Boussinesq-Navier-Stokes system (without diffusivity), which appears in the literature with a vector field u in the transport equation on ρ, and not $u-e_3$. However, since it essentially shares the same features, we shall refer to (4.1.11) as a Boussinesq-Navier-Stokes type system.

The Boussinesq-Navier-Stokes system is a standard geophysical fluid dynamics model (see [Sal98, Val17, Maj03]). From the analysis point of view, and because its 2D version retains several key features of 3D incompressible models (see e.g. [MB02]), it has recently received significant attention. The *existence theory* (in the less-diffusivity case) has been for instance developed in [HL05, Cha06, AH07, HK07, DP08, HR10], while *stability* of hydrodynamic equilibria is studied in [DWZZ18, TWZZ20, MSHZ20, DS21]. We also refer to the so-called *temperature patch (or front) problem* adressed in [DZ17, GGJ20, CMX21]. Note that there is no diffusivity in the transport equation on the density, which makes the mathematical analysis much more challenging than the thermal diffusion case.

Note that when the Navier-Stokes equations are replaced by the (steady) Stokes equations in (4.1.11) (i.e. neglecting the self-advection term), we obtain

$$\begin{cases} \partial_t \rho + \mathrm{div}_x[\rho(u-e_3)] = 0, & t>0, \quad x\in\mathbb{R}^3_+, \\ -\Delta_x u + \nabla_x p = -\rho e_3, & t>0, \quad x\in\mathbb{R}^3_+, \\ \mathrm{div}_x\, u = 0, & t>0, \quad x\in\mathbb{R}^3_+, \end{cases} \tag{4.1.12}$$

which is classicaly referred to as the Transport-Stokes system and which appears as an interesting model of sedimentation. On the whole space, this system has been obtained by Höfer in [Hö18] from the Vlasov-Stokes system, and by considering the same scaling as ours for the hydrodynamical limit.

If starting at the microscopic level (with a N–solid particle system coupled with a fluid equation), one can seek to recover the related mesoscopic and macroscopic models. Up to our knowledge, the best results only deal with the direct passage to the macroscopic system (4.1.12), by working in a dilute regime where the inertia of the particles is somehow neglected (see the work of Mecherbet [Mec19], Höfer [Höf18] but also the recent papers of Höfer and Schubert [HS21, HS22] and the related mean-field techniques). For further results on the Transport-Stokes system (4.1.12), we refer to [Mec20, Leb22, MS88, GI22, GGBS22, Cob23, DGL23].

The derivation of the Vlasov-Navier-Stokes system (4.1.1)–(4.1.2)–(4.1.3) from microscopic laws is however an outstanding open problem and results in that direction are still fragmentory. We refer to Section 1.3.3 fn the Introduction for more details.

4.1.3 Definitions and notations

Until the end of this work, we shall refer to the system (VNS$_\varepsilon$) as the VNS system. Recall that for all $\varepsilon > 0$, we have set

$$\rho_\varepsilon(t,x) := \int_{\mathbb{R}^3} f_\varepsilon(t,x,v)\,\mathrm{d}v, \quad j_\varepsilon(t,x) := \int_{\mathbb{R}^3} v f_\varepsilon(t,x,v)\,\mathrm{d}v, \quad t \geq 0, \quad x \in \mathbb{R}^3_+.$$

We denote by $\mathscr{D}_{\mathrm{div}}(\mathbb{R}^3_+)$ the set of smooth $\mathbb{R}^3$ valued divergence free vector-fields having compact support in $\mathbb{R}^3_+$. For all $q \in (1,+\infty)$, the closures of $\mathscr{D}_{\mathrm{div}}(\mathbb{R}^3_+)$ in $\mathrm{L}^q(\mathbb{R}^3_+)$ and in $\mathrm{H}^1(\mathbb{R}^3_+)$ are respectively denoted by $\mathrm{L}^q_{\mathrm{div}}(\mathbb{R}^3_+)$ and by $\mathrm{H}^1_{0,\mathrm{div}}(\mathbb{R}^3_+)$. We write $\mathrm{H}^{-1}_{\mathrm{div}}(\mathbb{R}^3_+)$ for the dual of the latter.

If $q \in (1,+\infty)$ is given, any vector field $u \in \mathrm{L}^q(\mathbb{R}^3_+)$ can be uniquely decomposed as

$$\begin{aligned} &u = \widetilde{u} + \nabla p, \\ &\widetilde{u} \in \mathrm{L}^q_{\mathrm{div}}(\mathbb{R}^3_+),\ p \in \mathrm{L}^q(\mathbb{R}^3_+),\ \nabla p \in \mathrm{L}^q(\mathbb{R}^3_+), \end{aligned}$$

We recall that the Leray projection $\mathbb{P}_q : u \mapsto \widetilde{u}$ is continuous from $\mathrm{L}^q(\mathbb{R}^3_+)$ to $\mathrm{L}^q_{\mathrm{div}}(\mathbb{R}^3_+)$.

Considering the following Stokes operator

$$A_q := -\mathbb{P}_q\Delta, \quad D(A_q) := \mathrm{L}^q_{\mathrm{div}}(\mathbb{R}^3_+) \cap \mathrm{W}^{1,q}_0(\mathbb{R}^3_+) \cap \mathrm{W}^{2,q}(\mathbb{R}^3_+),$$

we also set for $s \in (1,+\infty)$

$$\mathrm{D}_q^{1-\frac{1}{s},s}(\mathbb{R}^3_+) := \Big(D(A_q), \mathrm{L}^q_{\mathrm{div}}(\mathbb{R}^3_+)\Big)_{1/s,s}, \tag{4.1.13}$$

where $(\ ,\ )_{1/s,s}$ refers to the real interpolation space of exponents $(1/s,s)$. In the case of the Stokes operator A_q, which generates an analytic semigroup e^{-tA_q}, the quantity

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^3_+)} + \left(\int_0^\infty \|A_q e^{-tA_q} u\|^s_{\mathrm{L}^q(\mathbb{R}^3_+)}\,\mathrm{d}t\right)^{1/s}$$

defines an equivalent norm on $\mathrm{D}_q^{1-\frac{1}{s},s}(\mathbb{R}^3_+)$ (see [Lun18, Chapter 5]).

We will rely on the following maximal regularity result for the Stokes operator (see [GS91] and the Section 3.D of the Appendix of Chapter 3).

Theorem 4.1.2. *Consider $0 < T \leq \infty$ and $1 < q, s < \infty$. Then, for every $u_0 \in \mathrm{D}_q^{1-\frac{1}{s},s}(\mathbb{R}^3_+)$ which is divergence free and $f \in \mathrm{L}^s(0,T;\mathrm{L}^q_{\mathrm{div}}(\mathbb{R}^3_+))$, the unique solution u of the Stokes system*

$$\begin{cases} \partial_t u + A_q u = f, \\ \quad u_{|x_3=0} = 0, \\ \quad u(0,x) = u_0(x), \end{cases}$$

satisfying, for some $C = C(q,s) > 0$,

$$\|\partial_t u\|_{\mathrm{L}^s(0,T;\mathrm{L}^q(\mathbb{R}^3_+))} + \|\mathrm{D}^2_x u\|_{\mathrm{L}^s(0,T;\mathrm{L}^q(\mathbb{R}^3_+))} \leq C\left(\|u_0\|_{\mathrm{D}_q^{1-\frac{1}{s},s}(\mathbb{R}^3_+)} + \|f\|_{\mathrm{L}^s(0,T;\mathrm{L}^q(\mathbb{R}^3_+))}\right),$$

Next, we define several functionals which are crucial in the analysis of the VNS system.

Definition 4.1.3. *The* ***kinetic energy*** *of the VNS system is defined by*

$$\mathrm{E}_\varepsilon(t) := \frac{1}{2}\|u_\varepsilon(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \frac{\varepsilon}{2}\int_{\mathbb{R}^3_+\times\mathbb{R}^3} |v|^2 f_\varepsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}v. \tag{4.1.14}$$

The ***potential energy*** *of the VNS system is defined by*

$$\mathrm{E}^{\mathrm{p}}_\varepsilon(t) := \int_{\mathbb{R}^3_+\times\mathbb{R}^3} x_3 f_\varepsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^3_+} x_3\rho_\varepsilon(t,x)\,\mathrm{d}x. \tag{4.1.15}$$

We finally define the ***total energy*** *of the VNS system as*

$$\mathcal{E}_\varepsilon(t) := \mathrm{E}_\varepsilon(t) + \mathrm{E}^{\mathrm{p}}_\varepsilon(t), \tag{4.1.16}$$

and the ***dissipation*** *of the VNS system as*

$$\mathrm{D}_\varepsilon(t) := \|\nabla_x u_\varepsilon(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \int_{\mathbb{R}^3_+\times\mathbb{R}^3} |v - u_\varepsilon(t,x)|^2\, f_\varepsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}v. \tag{4.1.17}$$

Formally, we can multiply the Navier-Stokes equations in (VNS_ε) by u_ε and then integrate on $\mathbb{R}^3_+$ with suitable integrations by parts (using the divergence free condition). We can also multiply the Vlasov equation in (VNS_ε) by $|v|^2/2$ and by x_3 and then integrate on $\mathbb{R}^3_+\times\mathbb{R}^3$ with suitable integrations by parts (and using the absorption boundary condition (4.1.5)). All in all, we formally obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}_\varepsilon(t) + \mathrm{D}_\varepsilon(t) \leq -\int_{\mathbb{R}^3_+\times\mathbb{R}^3} v\cdot e_3 f_\varepsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}v, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}^{\mathrm{p}}_\varepsilon(t) \leq \int_{\mathbb{R}^3_+\times\mathbb{R}^3} v\cdot e_3 f_\varepsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

We now define the class of admissible initial data for the VNS system.

Definition 4.1.4 (Initial condition)**.** *Let $\varepsilon > 0$. We shall say that a couple $(u^0_\varepsilon, f^0_\varepsilon)$ is an admissible initial condition if*

$$u^0_\varepsilon \in \mathrm{L}^2(\mathbb{R}^3_+), \quad \mathrm{div}_x\, u^0_\varepsilon = 0, \tag{4.1.18}$$

$$f^0_\varepsilon \in \mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R}^3_+\times\mathbb{R}^3), \tag{4.1.19}$$

$$f^0_\varepsilon \geq 0, \quad \int_{\mathbb{R}^3_+\times\mathbb{R}^3} f^0_\varepsilon(x,v)\,\mathrm{d}x\,\mathrm{d}v = 1, \tag{4.1.20}$$

$$(x,v)\mapsto |v|^2 f^0_\varepsilon(x,v) \in \mathrm{L}^1(\mathbb{R}^3_+\times\mathbb{R}^3), \tag{4.1.21}$$

$$(x,v)\mapsto x_3 f^0_\varepsilon(x,v) \in \mathrm{L}^1(\mathbb{R}^3_+\times\mathbb{R}^3). \tag{4.1.22}$$

In the rest of this chapter, we will consider global weak solutions to the VNS system which satisfy an energy-dissipation inequality and which are defined in the following sense[2].

Definition 4.1.5 (Weak solutions)**.** *Let $\varepsilon > 0$. Given an admissible initial condition $(u^0_\varepsilon, f^0_\varepsilon)$ in the sense of Definition 4.1.4. we say that a pair $(u_\varepsilon, f_\varepsilon)$ is a global weak solution to the Vlasov-Navier-Stokes system with boundary conditions (4.1.4)-(4.1.5) and with initial condition $(u^0_\varepsilon, f^0_\varepsilon)$ if*

$$u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^3_+)) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{H}^1_{0,\mathrm{div}}(\mathbb{R}^3_+)),$$

$$f_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R}^3_+\times\mathbb{R}^3)),$$

$$j_\varepsilon - \rho_\varepsilon u_\varepsilon \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{H}^{-1}_{\mathrm{div}}(\mathbb{R}^3_+)),$$

and if

[2] We refer to the Section 3.B in the Appendix of Chapter 3 for more details about the construction of such global weak solutions (for any $\varepsilon > 0$ fixed). The introduction of this reference also provides further information on the Cauchy problem for the VNS system.

- u_ε *is a Leray solution to the Navier-Stokes equations satisfying for any* $t \geq 0$ *and almost every* $0 \leq s \leq t$ *(including* $s = 0$*)*

$$\|u_\varepsilon(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + 2\int_s^t \|\nabla u_\varepsilon(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\mathrm{d}\tau \\ \leq \|u_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + 2\int_s^t \int_{\mathbb{R}^3_+} (j_\varepsilon(\tau,x) - \rho_\varepsilon u_\varepsilon(\tau,x)) \cdot u_\varepsilon(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau.$$

- f_ε *is a renormalized nonnegative solution to the Vlasov equation with absorption boundary condition.*

Furthermore, for any $t \geq 0$ *and almost every* $0 \leq s \leq t$ *(including* $s = 0$*), the following inequality holds for all* $\varepsilon > 0$*:*

$$\mathrm{E}_\varepsilon(t) + \int_s^t \mathrm{D}_\varepsilon(\tau)\,\mathrm{d}\tau \leq \mathrm{E}_\varepsilon(s) - \int_s^t \int_{\mathbb{R}^3_+\times\mathbb{R}^3} v\cdot e_3 f_\varepsilon(\tau,x,v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau. \tag{4.1.23}$$

The notion of renormalized solutions (in the sense of DiPerna and Lions for transport equations [DL89c]) for the Vlasov equation allows to consider the trace of f_ε at the boundary of the half-space (see [Mis00b]). This also provides some strong stability properties of such solutions (see Section 3.A in the Appendix of Chapter 3 for further detail)

As we shall prove later on, the energy inequality (4.1.23) will be improved in Subsection 4.3.1, by adding the contribution of the potential energy $\mathrm{E}^{\mathrm{p}}_\varepsilon$. The new energy-dissipation inequality shall read

$$\mathcal{E}_\varepsilon(t) + \int_s^t \mathrm{D}_\varepsilon(\tau)\,\mathrm{d}\tau \leq \mathcal{E}_\varepsilon(s).$$

We refer to Lemma 4.3.1 and to (4.3.1) for more details about the obtention of this structural inequality.

Notation 4.1.6. *In the whole chapter, the notation* $A \lesssim B$ *will always denote the fact that there exists a universal constant* $M > 0$ *independent of all the parameters such that*

$$A \leq MB.$$

4.1.4 Assumptions and main results

Let $(u^0_\varepsilon)_{\varepsilon>0}$ and $(f^0_\varepsilon)_{\varepsilon>0}$ be a family of admissible initial data in the sense of Definition 4.1.4. We introduce the following set of assumptions.

Assumption 4.1.7 (Regularity and decay assumption)**.** *We assume that:*

- *for any* $\varepsilon > 0$*, we have*

$$u^0_\varepsilon \in \mathrm{H}^1_0(\mathbb{R}^3_+) \cap \mathrm{L}^1(\mathbb{R}^3_+); \tag{\textbf{A1-a}}$$

- *there exist* $p_0 > 3$*,* $s \in (2,3)$ *and* $p \in (3, p_0)$ *such that*

$$\forall \varepsilon > 0, \qquad u^0_\varepsilon \in \mathrm{D}_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+) \cap \mathrm{D}_3^{\frac{1}{2},2}(\mathbb{R}^3_+) \cap \mathrm{D}_3^{1-\frac{1}{s},s}(\mathbb{R}^3_+), \tag{\textbf{A1-b}}$$

where we refer to (4.1.13) *for the definition of the previous spaces;*

- *for any* $\varepsilon > 0$*, we have*

$$|v|^6 f_\varepsilon^0 \in \mathrm{L}^1(\mathbb{R}^3_+ \times \mathbb{R}^3); \tag{A1-c}$$

- *there exists* $q > 3$ *such that for any* $\varepsilon > 0$

$$(1 + x_3^q)(1 + |v|^q) f_\varepsilon^0 \in \mathrm{L}^1(\mathbb{R}^3; \mathrm{L}^\infty \cap \mathrm{L}^1(\mathbb{R}^3_+)). \tag{A1-d}$$

Assumption 4.1.8 (Uniform boundedness assumption)**.** *We assume that there exists* $M > 1$ *such that:*

- *for any* $\varepsilon > 0$*, we have*

$$\begin{aligned} \|u_\varepsilon^0\|_{\mathrm{L}^1 \cap \mathrm{H}^1 \cap \mathrm{D}_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+)} &\leq M, \\ \|(1 + x_3^q)(1 + |v|^q)\, f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3; \mathrm{L}^\infty \cap \mathrm{L}^1(\mathbb{R}^3_+))} &\leq M; \end{aligned} \tag{A2-a}$$

where the exponents (p, q) *refer to the ones introduced in Assumption 4.1.7.*

- *the total energy satisfies*

$$\forall \varepsilon > 0, \qquad \mathcal{E}_\varepsilon(0) < M. \tag{A2-b}$$

Assumption 4.1.9 (Smallness assumption)**.** *We assume that:*

- *if* $\mathrm{C}_\star > 0$ *is the universal constant given by Proposition 4.3.7, then*

$$\forall \varepsilon > 0, \qquad \|u_\varepsilon^0\|_{\mathrm{H}^1(\mathbb{R}^3_+)} < \left(\frac{\mathrm{C}_\star}{2}\right)^{1/2}; \tag{A3-a}$$

- *there exist* $\eta > 0$ *and* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$*, we have*

$$\mathcal{E}_\varepsilon(0) + \|(1 + |v|) f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3; \mathrm{L}^2(\mathbb{R}^3_+))} + \|(|v|^{1+\iota} + x_3^{1+\iota}) f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3; \mathrm{L}^2(\mathbb{R}^3_+))} < \eta, \tag{A3-b}$$

for some $\iota > 0$.

Of course, one can make the previous assumptions with only $\varepsilon > 0$ small enough. Furthermore, the choice of the exponent q can be made more explicit and actually depends on the exponent p. For the sake of readibility, we do note give a precise value and we refer to the proofs where the assumptions will be used. Note that we do not assume that $\varepsilon \mapsto \mathcal{E}_\varepsilon(0)$ tends to 0 when $\varepsilon \to 0$.

The main results of our work read as follows. Consider $(u_\varepsilon, f_\varepsilon)$ a global weak solution to the Vlasov-Navier-Stokes system associated to an admissible initial data $(u_\varepsilon^0, f_\varepsilon^0)$ with $\varepsilon > 0$.

Theorem 4.1.10. *Under Assumptions* ***4.1.7–4.1.8–4.1.9****, and assuming that*

$$u_\varepsilon^0 \underset{\varepsilon \to 0}{\rightharpoonup} u^0 \text{ in } w\text{-}\mathrm{L}^2(\mathbb{R}^3_+) \qquad \text{and} \qquad \rho_\varepsilon^0 \underset{\varepsilon \to 0}{\rightharpoonup} \rho^0 \text{ in } w^*\text{-}\mathrm{L}^\infty(\mathbb{R}^3_+),$$

where

$$u^0 \in \mathrm{L}^2(\mathbb{R}^3_+), \quad \mathrm{div}_x\, u^0 = 0, \quad \rho^0 \in \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^3_+), \quad \rho^0 \geq 0,$$

then, up to a subsequence, $(u_\varepsilon)_{\varepsilon>0}$ *converges to* u *in* $\mathrm{L}^2(0,T;\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3_+))$, $(\rho_\varepsilon)_{\varepsilon>0}$ *converges weakly-*$*$ *to* ρ *in* $\mathrm{L}^\infty((0,T)\times\mathbb{R}^3_+)$ *for any* $T>0$*, where* (ρ,u) *is a solution of*

$$\begin{cases} \partial_t\rho + \operatorname{div}_x[\rho(u-e_3)] = 0, \\ \partial_t u + (u\cdot\nabla_x)u - \Delta_x u + \nabla_x p = -\rho e_3, \\ \operatorname{div}_x u = 0, \\ \rho|_{t=0} = \rho^0, \quad u|_{t=0} = u^0, \\ u(t,\cdot)_{|x_3=0} = 0, \end{cases} \tag{4.1.24}$$

with

$$u \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{H}^1_{\mathrm{div}}(\mathbb{R}^3_+)) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{H}^2(\mathbb{R}^3_+)) \cap \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{W}^{1,\infty}(\mathbb{R}^3_+)), \tag{4.1.25}$$

$$\rho \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^\infty(\mathbb{R}^3_+\times\mathbb{R}^3)), \quad \rho \geq 0. \tag{4.1.26}$$

Theorem 4.1.11. *Let* (u,ρ) *be any global solution to the system* (4.1.24) *with the regularity* (4.1.25)*–*(4.1.26)*. Under Assumptions* ***4.1.7****–****4.1.8****–****4.1.9****, there exist* $\varepsilon_0>0$*,* $\omega>0$ *and* $C_M>0$ *such that if* $T>0$*, then for all* $t\in[0,T]$ *and* $\varepsilon\in(0,\varepsilon_0)$

$$\begin{aligned} &\|u_\varepsilon(t)-u(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|\rho_\varepsilon(t)-\rho(t)\|_{\mathrm{H}^{-1}(\mathbb{R}^3_+)} \\ &\qquad\qquad \lesssim e^{C_M(1+T)}\left(\|u^0_\varepsilon-u^0\|_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|\rho^0_\varepsilon-\rho^0\|_{\mathrm{H}^{-1}(\mathbb{R}^3_+)} + \varepsilon^{\frac{1}{2}}(1+T)^{\frac{7}{10}}M^\omega\right). \end{aligned}$$

Remark 4.1.12. Let us clarify the two previous results with respect to the limit system (4.1.24).

1. Theorem 4.1.10 implies a theorem of existence of a global strong solution for the Boussinesq-Navier-Stokes system (4.1.24). Of course, the smallness assumption of the initial data $(u^0_\varepsilon)_\varepsilon$ contained in Assumption **4.1.9** (which is uniform in ε) is implicitely transferred to the initial data u^0 that we choose, thanks to a weak-compactness argument (up to an additional subsequence).

2. Theorem 4.1.11 implies a theorem of uniqueness of global strong solutions for the Boussinesq-Navier-Stokes system (4.1.24), stated in a very weak form. More precisely, this theorem proves that there is a unique solution to the system for a class of initial conditions (u^0,ρ^0) which can be approached in $\mathrm{L}^2(\mathbb{R}^3_+)\times\mathrm{H}^{-1}(\mathbb{R}^3_+)$ by a sequence $(u^0_\varepsilon,\rho^0_\varepsilon)$ satisfying Assumptions **4.1.7**–**4.1.8**–**4.1.9**. This is somehow related to a smallness assumption of the initial data (u^0,ρ^0) (recall that we work in dimension 3).

   Of course, this result is far from being optimal: we refer to [DP08] for further results, stated and proved in the case of $\mathbb{R}^3$.

Remark 4.1.13. Let us also point out that our analysis somehow requires solutions to the Navier Stokes equations with *high-regularity*. Our proof eventually being based on a weak-compactness argument, such control for the VNS system (uniform in ε) shall be transferred to the solution of the limit system. It explains the regularity obtained in (4.1.25).

It seems to be an open and natural problem to obtain the same kind of results by relaxing the regularity and smallness assumptions 4.1.7 and 4.1.9 (i.e. only considering weak solution to (VNS_ε) in the sense of Definition 4.1.5). Note indeed that the limit system (4.1.11) admits global weak Leray solutions [DP08].

Let us finally highlight the **main original difficulties** that arise in the justification of the hydrodynamic limit of (VNS_ε) towards (4.1.11).

- First, the proof of Han-Kwan and Michel in [HKMar] (for the gravity-less case) cannot directly lead to a global in time result in our context. Indeed, their analysis is crucially based on the *exponential decay* in time of the total (modulated) kinetic energy of the system set on $\mathbb{T}^3$ (i.e. a twisted version of E_ε). This somehow virtually provides any integrability in time, allowing for global results at some point. However, the decay of the energy on an unbounded domain is not exponential but only polynomial in time, as shown in [HK22]. Furthermore, this decay is actually *not guaranteed* when a constant gravity force is added, because an additional contribution to the energy has to be considered (see (4.1.23) and Remark 4.3.4). Roughly speaking, the effect of gravity somehow destroys the remarkable energy decay displayed by the Vlasov-Navier-Stokes system without gravity.
- On the whole space, it is unclear if the global solutions to the Boussinesq-Navier-Stokes system enjoy some decay in time estimates: more precisely, global existence results for that system such as [DP08, HR10] only provide at most exponential upper bounds for the fluid velocity. More recent results [BS12, BM17, KW20] seem to indicate that obtaining decay in time in the whole space case is actually not possible and that a growth phenomenon can occur.
- Contrary to the the Vlasov-Stokes case with gravity in [Hö18], the presence of the nonlinearity in the 3d Navier-Stokes equations is of course a new difficulty. As a matter of fact, the strategy used in [Hö18] is strongly based on the linearity of the Stokes equations, as well as on some explicit representation formula for the solution. In addition, the nonlinearity formally forbids any direct long-time results requiring additional regularity, because of the coupling between f and u. Note that we also do not require any regularity assumption on the kinetic part, which only belongs to some Lebesgue spaces.

The previous observations somehow justify the choice of the half-space setting in this article, combined with appropriate boundary conditions (see (4.1.4)–(4.1.5)). Let us mention that on the half-space, the energy decay of the limiting system can be expected (see e.g [HS14]).

The outcome is also the fact that a delicate bootstrap argument ensuring uniformity in ε is required. This should prevent some possible growth in time of the solutions. We refer to Subsection 4.1.6 for more details about the strategy of proof and the main mechanism allowing to overcome the previous difficulties.

In particular, we provide a particular spatial framework where we can positively answer the question of the global hydrodynamic limit raised in [HKMar] in the presence of gravity. Note that treating the case of the whole space still seems to be an open problem.

4.1.5 Broad panorama on hydrodynamic limits for fluid-kinetic systems

In this subsection, we draw a review of results concerning the derivation of hydrodynamical systems from fluid-kinetic models. Let us emphasize the fact that this contains different variants of the Vlasov-Navier-Stokes system, with different scalings in term of ε.

- Some high friction regime (similar to the one considered in this chapter) has been studied in the seminal work of Jabin in [Jab00a]: a Vlasov equation without coupling is considered and the fluid velocity is recovered in terms of a convolution of a moment of f_ε with a smooth kernel (in order to mimick Stokes flow). It highlights the monokinetic behavior $f_\varepsilon(t) \overset{\varepsilon\to 0}{\to} \rho(t)\otimes\delta_{v=U(t)}$ which is at stake in the system. The case of a given fluid velocity field with extra-integrability condition is considered in [Jab02]. Goudon and Poupaud [GP04] then treated the case of a very thin spray for the particles, in which the fluid velocity is given as a fixed random field. In [Gou01], the case of two different regimes is handled by Goudon for a coupled Burgers-Vlasov system: the asymptotic result heavily relies on the one-dimensional setting (see also [CWY20] for a 1D compressible model).

- In the same time, fluid kinetic couplings where a *Fokker-Planck term* smoothes out the kinetic equation (namely, a term $\Delta_v f$ is added on the left-hand side of the Vlasov equation) have been extensively studied: the two main pioneering results have been obtained by Goudon, Jabin and Vasseur in [GJV04a, GJV04b] in a domain without boundary (and without gravity). In short, their proof is based on the obtention of global entropy bounds for the system. In this context, the distribution function f_ε tends to converge towards a (local) *Maxwellian* which parameters are solutions to the limit equations. This kind of relaxation is highly linked to the smoothing in velocity in the kinetic equation and does not appear if the Fokker-Planck operator is absent. The scaling considered in [GJV04a] (called the *light particles* regime by the authors) leads to an advection-diffusion system referred to as the Smoluchowski-Navier-Stokes system, while the scaling considered in [GJV04b] (called the *fine particles* regime by the authors) leads to an inhomogeneous incompressible Navier-Stokes system. The second one requires the use of *relative entropy* methods. These results have also been extended by Mellet and Vasseur in [MV11] for a compressible Navier-Stokes system and by Su and Yao in [SY20] for a non-homogeneous system.

  We also refer to the formal analysis performed by Carrillo and Goudon in [CG06] and by Carrillo, Goudon and Laffitte in [CGL08] for the same system (but with the Euler equations instead of Navier-Stokes ones) where general external potentials and boundary conditions are discussed.

- In the case where there is no extra-dissipation in velocity in the kinetic equation (and with the dimension different from 1), few results were known until now. A fine particles regime was derived in [BDM14] for a two phase Vlasov-Navier-Stokes system, but still at a formal level. In the direction of the Vlasov-Navier-Stokes system with gravity, the first result has been obtained, to the best of our knowledge, by Höfer in [Hö18]: he considered the so-called *inertialess limit* for the Vlasov equation coupled with the steady Stokes system in $\mathbb{R}^3$ and for compactly supported and regular initial distribution functions. His proof relies on a trajectorial analysis and leads to the derivation of the Transport-Stokes system (4.1.12). As mentioned previously, we adopt the same scaling in the current chapter.

- Very recently, Han-Kwan and Michel have proposed a framework in [HKMar] to rigorously handle the high-friction limit of the full Vlasov-Navier-Stokes system in a tridimensional periodic setting (without gravity). They consider three differents regimes for the system, leading to hydrodynamical systems in the limit: following their denomination, *light particles* and *light and fast particles* regimes lead to Transport-Navier-Stokes system, while the *fine particles* regime leads to an inhomogeneous Navier-Stokes system. These asymptotic models mainly come from a convergence of the distribution function f_ε towards a Dirac mass in velocity centered at the fluid velocity limit when $\varepsilon \to 0$.

  As we shall explain later on, their techniques are related to the recent progress concerning the large time behavior of the Vlasov-Navier-Stokes system on the tridimensional torus, performed by Han-Kwan, Moussa and Moyano in [HKMM20]. As mentioned in [HKMar], the proof seems to be suitable for a local in time hydrodynamical limit in the gravity case (at least on the torus), that is a local in time derivation of the Boussinesq-Navier-Stokes system (4.1.11). The interesting question of the global in time derivation of (4.1.11) (for unbounded domains) thus appears as a natural extension of [HKMar], which left this problem as open. As explained above, serious difficulties due the gravity effect arise when looking for a global in time derivation.

- Let us finally mention a specific case adressed by Moussa and Sueur in [MS13], for a two-dimensional coupling between Vlasov and Euler equations, with gyroscopic effects: in the massless limit for the particles, one recovers the incompressible Euler system, through techniques close to the ones deviced by Brenier in [Bre00] for the study of the so-called *gyrokinetic limit* of the Vlasov-Poisson system.

Let us again emphasize the fact that the hydrodynamical limits for the Vlasov-Navier-Stokes system (without smoothing in the kinetic equation) is closely linked to the monokinetic behavior of the distribution function f_ε when $\varepsilon \to 0$, that is a convergence towards a Dirac distribution in velocity. It strongly differs from the Vlasov-Fokker-Planck case where there is a formal convergence towards a local Maxwellian.

Fluid-kinetic systems, such as the Vlasov-Navier-Stokes system, are not the only ones where Dirac masses can appear. Indeed, there are some other Vlasov-type equations displaying singular asymptotic regimes with respect to a small parameter.

- A famous example is the quasineutral limit of the Vlasov-Poisson system, corresponding to a regime where the ratio of the Debye length over the typical observation scale is small. In the first part of the important work [Bre00], Brenier shows that the solutions of the system have a monokinetic behavior with a velocity solution to the incompressible Euler equations, provided that the initial distribution also converges towards a Dirac mass in velocity. This link between the Vlasov-Poisson system and equations from fluid dynamics (see e.g. [Bre89]) is still a very active field of research. We refer to [Mas01, HK11, HKI17b] for further results in that direction and to [HK17] for related works.

- Another interesting example of such monokinetic behavior is given by the kinetic Cucker-Smale equation describing flocking dynamics without Brownian noise. It has been shown in [FK19] (see also [CC20]) that the solutions to this system converge to a monokinetic distribution with associated density and velocity satisfying a pressureless Euler system with nonlocal flocking dissipation.

- We also refer to some recent works about the spatially-extended FitzHugh-Nagumo system, which is a mean-field kinetic model describing a neural network as the number of neurons goes to infinity. In [Cre20, CFF19], the authors consider a regime of strong local interaction between neurons, which asymptotically leads to a somewhat monokinetic distribution in some of the variables. At the limit, one obtains a macroscopic model which is a reaction-diffusion system (see also [BF23]). This hydrodynamic limit shares some similarities with the one described for the kinetic Cucker-Smale model.

4.1.6 General strategy of proof

We now describe the guiding lines for the proof of Theorems 4.1.10–4.1.11. For the sake of conciseness, the ideas we will present are partly formal and we refer to the related sections of this chapter for rigorous details.

Owing to the analysis of [HKMar], one shall expect that the following convergences should hold, at least in a weak sense:

$$u_\varepsilon(t) \underset{\varepsilon\to 0}{\longrightarrow} u(t), \quad f_\varepsilon(t) \underset{\varepsilon\to 0}{\longrightarrow} \rho(t) \otimes \delta_{v=u(t)-e_3},$$

where (ρ, u) is a solution to the Boussinesq-Navier-Stokes system (4.1.11). Such convergences will make the formal analysis of Subsection 4.1.2 rigorous and will lead to the result of Theorem 4.1.10. Again, the kinetic equation handled here is not of Fokker-Planck type so that the framework we consider here is different from the one studied in [GJV04a, GJV04b].

In view of the expected previous singular limit, we are thus looking for uniform bounds in ε for ρ_ε and u_ε. From the Vlasov-Navier-Stokes system, we observe that $(\rho_\varepsilon, u_\varepsilon)$ satisfies the following system

$$\begin{cases} \partial_t \rho_\varepsilon + \mathrm{div}_x\, j_\varepsilon = 0, \\ \partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla_x) u_\varepsilon - \Delta_x u_\varepsilon + \nabla_x p_\varepsilon = j_\varepsilon - \rho_\varepsilon u_\varepsilon, \\ \mathrm{div}_x\, u_\varepsilon = 0, \end{cases} \tag{4.1.27}$$

therefore a weak compactness argument shall enable us to pass to the limit when $\varepsilon \to 0$ and to recover (4.1.11).

Classical energy estimates for the Navier-Stokes equations should provide uniform bounds for u_ε, at least in the energy space where the global Leray solutions belong. However, uniform bounds on ρ_ε are not directly given by (4.1.27) and this constitutes one of the main obstacles of the analysis. In addition, one has to look at the convergence of j_ε when $\varepsilon \to 0$, which is expected to be towards $\rho(u-e_3)$. These convergences will allow us to pass to the limit in the source term (i.e. the Brinkman force in the Navier-Stokes equations) and in the the first equation of (4.1.27).

Hence, inspired by the strategy performed by Han-Kwan and Michel in [HKMar], we aim at obtaining for any $T > 0$

- some uniform bounds on the local density ρ_ε in $\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3_+))$. This issue was already at stake in [HKMM20, HK22, EHKM21] for the study of the large time behavior of the system in different spatial contexts - see also the Chapters 2-3.
- some results of convergence of $F_\varepsilon := j_\varepsilon - \rho_\varepsilon u_\varepsilon$ in $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))$ when $\varepsilon \to 0$.

4.1.6.1 A Lagrangian framework

To do so, we rely on the *Lagrangian structure* of the Vlasov equation satisfied by f_ε, introducing the characteristic curves $s \mapsto (\mathrm{X}_\varepsilon(s;t,x,v), \mathrm{V}_\varepsilon(s;t,x,v))$ starting at (x,v) at time t and associated to this equation, namely

$$\begin{cases} \dot{\mathrm{X}}_\varepsilon(s;t,x,v) = \mathrm{V}_\varepsilon(s;t,x,v), \\ \dot{\mathrm{V}}_\varepsilon(s;t,x,v) = \dfrac{1}{\varepsilon}\Big(u_\varepsilon(s, \mathrm{X}_\varepsilon(s;t,x,v)) - e_3 - \mathrm{V}_\varepsilon(s;t,x,v)\Big), \\ \mathrm{X}_\varepsilon(t;t,x,v) = x, \\ \mathrm{V}_\varepsilon(t;t,x,v) = v, \end{cases} \tag{4.1.28}$$

where $u_\varepsilon(s,\cdot)$ has been by extended 0 outside the half-space $\mathbb{R}^3_+$. From the method of characteristics for the Vlasov equation, we can infer a key representation formula for the solution f_ε. In view of the absorption boundary condition (4.1.5) satisfied by $f_\varepsilon(t)$, this function will vanish on the points (x,v) of the phase space such that the trajectory $\sigma \mapsto \mathrm{X}_\varepsilon(\sigma;t,x,v)$ has left the half-space on $[0,t]$.

Thanks to the formulas

$$\rho_\varepsilon(t,x) = \int_{\mathbb{R}^3} f_\varepsilon(t,x,v)\,\mathrm{d}v, \quad F_\varepsilon(t,x) = \int_{\mathbb{R}^3} f_\varepsilon(t,x,v)\,(v - u_\varepsilon(t,x))\,\mathrm{d}v,$$

the representation of f_ε in terms of the characteristic curves will be the starting point towards the desired bound on ρ_ε and the convergence of F_ε when $\varepsilon \to 0$.

We will first perform a change of variable in velocity of the form

$$v \mapsto \mathrm{V}_\varepsilon(0;t,x,v)$$

in the previous integrals. This procedure will be allowed and will yield a control on ρ_ε, provided that

$$\int_0^T \|\nabla_x u_\varepsilon(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s \ll 1. \tag{4.1.29}$$

Note that this idea, reminiscent of the work of Bardos and Degond in [BD07], has recently provided many results on the large time behavior of the VNS system [HKMM20, HK22, EHKM21, Ert21] as well as on its high friction limits [Hö18, HKMar]. Note that obtaining a control like (4.1.29) indicates that some decay of u_ε seems to be required.

A more careful study of the Brinkman force F_ε (still based on the trajectories and the previous change of variable, following [HKMar]) will show that an additional control similar to (4.1.29) actually ensures the convergence of $F_\varepsilon + \rho_\varepsilon e_3$ towards 0 when $\varepsilon \to 0$. This sufficient control will be satisfied if one obtains *some decay in time of* u_ε.

4.1.6.2 Towards decay in time estimates

As the energy inequality (4.1.23) includes the presence of a gravity term, this only paves the way for a short time result with respect to (4.1.29). This observation has already been made by Han-Kwan and Michel in [HKMar] where the question of the global in time derivation of (4.1.11) was left as an open problem (on the whole space).

In the spirit of Chapter 3, we first look for a *conditional decay in time* result for u_ε, which is a somewhat general property of the Navier-Stokes system and which basically requires some decay of the Brinkman force $F_\varepsilon = j_\varepsilon - \rho_\varepsilon u_\varepsilon$. Following Wiegner [Wie87] and Borchers and Miyakawa [BM88], one can prove that the polynomial decay

$$\|u_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \frac{1}{(1+s)^\alpha} \tag{4.1.30}$$

holds for any $\alpha \in [0, 3/4]$, on a time interval where the source term $F_\varepsilon = j_\varepsilon - \rho_\varepsilon u_\varepsilon$ in the Navier-Stokes equations satisfies the following pointwise decay estimates:

$$\|F_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \frac{1}{(1+s)^{\frac{7}{4}}}. \tag{4.1.31}$$

Ensuring (4.1.31) appears as one the main goals of the analysis[3].

To obtain (4.1.29) with $T = +\infty$, we will rely on a *bootstrap argument*. The main idea is to interpolate higher order energy estimates for u_ε with pointwise estimates on its L^2_x norm. Such higher order estimates involving $\mathrm{D}^2_x u_\varepsilon$ are handled thanks to the maximal regularity theory for the unsteady Stokes system. Because of the slow polynomial decay (4.1.30) of the L^2_x norm of u_ε, obtaining integrability results for large time is not directly possible. To overcome this issue, we will look for polynomial weighted in time estimates, following the techniques of [HK22] and of Chapter 3.

Roughly speaking, this whole procedure mainly amounts in controlling the Brinkman force $F_\varepsilon = j_\varepsilon - \rho_\varepsilon u_\varepsilon$ with some uniform in ε bounds and with some decay in time estimates (entailing in particular (4.1.31)).

[3] We also refer to Remark 4.3.4 for a discussion about the possible use of the improved energy-dissipation inequality (4.3.1) including the potential energy.

4.1.6.3 The use of the absorption at the boundary

To go further in the analysis of the Brinkman force, we rely on the so-called *exit geometric condition*, defined for the Vlasov equation. This crucial notion stems from the work of Glass, Han-Kwan and Moussa in [GHKM18]. It has been used in Chapter 3 for the study of the large time dynamics of the VNS system on the half-space with gravity, following ideas similar to the one we will enforce here. Let us emphasize the very strong influence of the *geometric control condition* introduced by Bardos, Lebeau and Rauch in their celebrated work [BLR92] on the wave equation: the introduction of the exit geometric condition in our context is really reminiscent of this controllability result.

Roughly speaking, this condition referred to as EGC asks for the particles trajectory starting from a compact set to leave the half-space before a fixed time. More precisely, this compact refers to a product of $[0, L]$ in the third spatial direction in $\mathbb{R}^3_+$ by a ball $\overline{\mathrm{B}}(0, R)$ in velocity. This condition is of course related to the vector field u_ε defining the solutions to the system (4.1.28). The main consequence of an exit geometric condition satisfied by u_ε, in a time T and with respect to $(\mathbb{R}^2 \times [0, L]) \times \overline{\mathrm{B}}(0, R)$, is that for all $t > T$, we have

$$\mathbf{1}_{\mathrm{X}_\varepsilon(0;t,x,v)_3 \leq L}\, \mathbf{1}_{|\mathrm{V}_\varepsilon(0;t,x,v)| \leq R}\, f_\varepsilon(t, x, v) = 0.$$

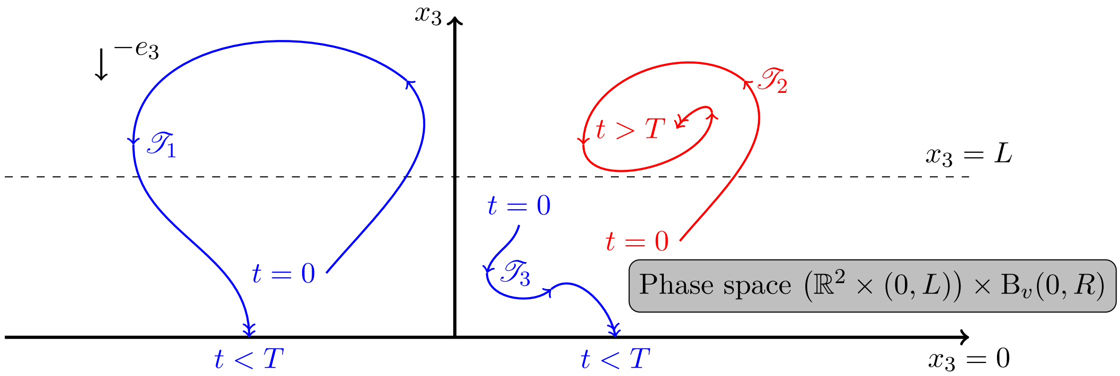

Figure 4.1.1: EGC not satisfied in time T w.r.t $(\mathbb{R}^2 \times (0, L)) \times \mathrm{B}_v(0, R)$ (traj. $\mathscr{T}_2$ not absorbed before $t = T$)

This comes from the absorption condition (4.1.5) satisfied by f_ε at the boundary of the half-space. In short, this cancellation of the distribution function paves the way for the obtention of decay estimates.

Of course, a major issue is to ensure such a condition and to propagate it throughout the evolution of the system. One can find on Figure 4.1.1 an example of a situation where the exit geometric condition in some time T is not satisfied. To ensure it, we will rely on a comparison principle for the system (4.1.28), by looking at the free evolution of the particles still undergoing the gravity effect but without the influence of the fluid (i.e. with $u_\varepsilon \equiv 0$). The analysis of this modified system of ODEs can be adressed explicitly, the trajectories being more or less straight lines in the physical space. This method, in the spirit of the one devised in Chapter 3, shows that an exit geometric condition is available at any time to (4.1.28), provided that

$$\int_0^{+\infty} \|u_\varepsilon(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\, \mathrm{d}s \ll 1. \tag{4.1.32}$$

This constitutes the key condition to use the geometry of the problem through absorption at the boundary. Let us point out that the presence of the (linear) gravity term in the VNS is *required* to perform such a strategy.

Building on the previous ideas, one can hope for some decay of the Brinkman force F_ε. Our method will follow the one introduced in [HKMar] and is based on a careful splitting and desingularization (in ε) of F_ε[4]. Combined with the exit geometric condition, obtaining some decay in time estimates should be possible if the initial distribution function f_ε^0 itself enjoys some decay in the phase space. This mainly explains the required mixed-moment type assumption **A2-a**. We refer to Section 4.4 for more details about this procedure.

All in all, the previous analysis mainly shows that it comes down to obtain

$$\int_0^{+\infty} \|u_\varepsilon(s)\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^3_+)} \, \mathrm{d}s \ll 1, \tag{4.1.33}$$

at least for ε small enough. We shall follow a bootstrap strategy to ensure such a control. We will rely on the smallness condition (**A3-b**) to initialize the procedure, as well as to propagate all the estimates until any time.

4.1.7 Outline of the chapter

The chapter is organized as follows.

- In **Section 4.2**, we derive some useful information on the system through its Lagrangian structure. We define the key concept of exit geometric condition and provide some of its useful properties. We also show how the control (4.1.33) actually ensures both the change of variable in velocity and the propagation of the exit geometric condition. Furthermore, we provide a first splitting of the Brinkman force leading to its conditional convergence when ε tends to 0. This requires a first careful desingularization of this term with respect to $\varepsilon \to 0$.

- In **Section 4.3**, we collect several preliminar regularity results and estimates on global weak solutions to the VNS system, as well as a conditional theorem of convergence. First, we derive an improvement of the energy-dissipation inequality satisfied by weak solutions, by considering the total energy (4.1.16) of the system. Next, we state the aforementioned conditional polynomial decay of the fluid kinetic energy, which is valid provided that the source term itself enjoys some pointwise decay. This enables us to determine sufficient conditions leading to a proof of Theorem 4.1.10 and which highlight the need of decay in time estimates (see Proposition 4.3.5). We also define the notion of *strong existence time* for the Vlasov-Navier-Stokes system, which allows to consider higher order energy estimates for the fluid velocity, provided that the initial data and the source term in the Navier-Stokes equations are small enough. The previous discussion on our strategy then leads to a bootstrap procedure which consists in obtaining the control (4.1.33) for ε small enough.

- **Section 4.4** provides a family of fine estimates on the Brinkman force, based on the same decomposition as in Section 4.2. We aim at obtaining a pointwise decay in time of the L^2_x norm of this force (see (4.1.31)), as well as some weighted in time bounds in $\mathrm{L}^p_t\mathrm{L}^p_x$. These polynomial in time estimates are built by propagating the exit geometric condition. Uniform bounds (independent of time and of ε) are obtained thanks to Assumption **4.1.8** and by relying on the conditional polynomial decay of u_ε.

[4]While studying the large time behavior of the VNS system with gravity in $\mathbb{R}^3_+$ in Chapter 3 (corresponding to $\varepsilon = 1$), it is possible to deal independently with each moment $\int_{\mathbb{R}^3} |v|^k f_\varepsilon(t,x,v)\,\mathrm{d}v$. In particular, it turns out to be sufficient to treat the moments of order 0 and 1 to directly get some control on the Brinkman force $j_f - \rho_f u$, because there is no singularity in the expression of the reverse velocity curve.

– The bootstrap argument takes place in **Section 4.5**. A careful interpolation procedure combined with weigthed in time estimates allows to prove that (4.1.33) holds for ε small enough. Passing to the limit in the VNS system is then a direct consequence of Proposition 4.3.5 and of all the previous uniform estimates on u_ε and ρ_ε which will be valid on $\mathbb{R}^+$. This will entail Theorem 4.1.10 and the Boussinesq-Navier-Stokes system will thus be recovered at any time. The quantitave rates of (strong) convergence announced in Theorem 4.1.11 are then obtained by performing direct energy estimates.

Remark 4.1.14. In the rest of the chapter, we shall state many properties which hold true only for ε small enough: we will refer to some range of $\varepsilon \in (0, \varepsilon_0)$ where $\varepsilon_0 > 0$ may change from one statement to another and can be reduced if necessary. Furthermore, we will always refer to $M > 1$ as a constant involved in estimates which are uniform with respect to the parameter ε and bearing on the initial data (see Assumption **4.1.8**).

4.2 Particle trajectories

4.2.1 Lagrangian structure for the Vlasov equation

We first define the characteristic curves associated to Vlasov equation, which is a transport equation in the phase-space $\mathbb{R}^3_+ \times \mathbb{R}^3$. This provides a useful representation formula for the distribution function f_ε which takes into account the absorption boundary condition (4.1.5).

Let $u_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^3_+ \to \mathbb{R}^3$ be a given time-dependent vector field, with $\varepsilon > 0$ fixed. For $t \geq 0$ and $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$, we consider the solution $s \mapsto (\mathrm{X}_\varepsilon(s; t, x, v), \mathrm{V}_\varepsilon(s; t, x, v))$ to the following system of ordinary differential equations:

$$\begin{cases} \dot{\mathrm{X}}_\varepsilon(s; t, x, v) = \mathrm{V}_\varepsilon(s; t, x, v), \\ \dot{\mathrm{V}}_\varepsilon((s; t, x, v) = \dfrac{1}{\varepsilon}\Big((Pu_\varepsilon)(s, \mathrm{X}_\varepsilon(s; t, x, v)) - e_3 - \mathrm{V}_\varepsilon(s; t, x, v)\Big), \\ \mathrm{X}_\varepsilon(t; t, x, v) = x, \\ \mathrm{V}_\varepsilon(t; t, x, v) = v, \end{cases} \tag{4.2.1}$$

where the dot means derivative along the first variable. Here, the operator P refers to the extension operator by 0 outside the half-space, such that

$$P : \mathrm{W}^{1,\infty}_0(\mathbb{R}^3_+) \longrightarrow \mathrm{W}^{1,\infty}(\mathbb{R}^3)$$

is bounded. In what follows, we shall use the harmless notation

$$\forall t \geq 0, \quad (Pu_\varepsilon)(t, \cdot) = P(u_\varepsilon(t, \cdot)),$$

because the operator P does not act on the time variable. When there will be no ambiguity, the curves $(\mathrm{X}_\varepsilon, \mathrm{V}_\varepsilon)$ will always refer to solutions to (4.2.1), that is, associated to a velocity field u_ε solution to the Navier-Stokes equations in the VNS system.

Proposition 4.2.1. *Let $\varepsilon > 0$. Suppose that $u_\varepsilon \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{W}^{1,\infty}(\mathbb{R}^3_+))$. Then for any $s, t \geq 0$, the mapping*

$$(x, v) \mapsto (\mathrm{X}_\varepsilon(s; t, x, v), \mathrm{V}_\varepsilon(s; t, x, v))$$

is a diffeomorphism of $\mathbb{R}^6$, whose Jacobian value is $e^{\frac{3(t-s)}{\varepsilon}}$ and whose inverse is

$$(x, v) \mapsto (\mathrm{X}_\varepsilon(t; s, x, v), \mathrm{V}_\varepsilon(t; s, x, v)).$$

A solution to the previous system satisfies for all $s, t \geq 0$ and $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$

$$\begin{cases} \mathrm{X}_\varepsilon(s;t,x,v) = x + \varepsilon(1 - e^{\frac{t-s}{\varepsilon}})v - \left(s - t + \varepsilon e^{\frac{t-s}{\varepsilon}} - \varepsilon\right)e_3 + \displaystyle\int_t^s (1 - e^{\frac{\tau-s}{\varepsilon}})(Pu_\varepsilon)(\tau, \mathrm{X}_\varepsilon(\tau;t,x,v))\,\mathrm{d}\tau, \\ \mathrm{V}_\varepsilon(s;t,x,v) = e^{\frac{t-s}{\varepsilon}}v - (1 - e^{\frac{t-s}{\varepsilon}})e_3 + \dfrac{1}{\varepsilon}\displaystyle\int_t^s e^{\frac{\tau-s}{\varepsilon}}(Pu_\varepsilon)(\tau, \mathrm{X}_\varepsilon(\tau;t,x,v))\,\mathrm{d}\tau. \end{cases} \tag{4.2.2}$$

In order to take into account the absorption boundary condition (4.1.5) that must be satisfied by the distribution function f_ε, we introduce the following backward and forward exit times.

Definition 4.2.2. *For $(x, v) \in \mathbb{R}^3_+ \times \mathbb{R}^3$ and for any $t \geq 0$, we set*

$$\tau^+_\varepsilon(t, x, v) := \sup\left\{s \geq t \mid \forall \sigma \in [t, s],\ \mathrm{X}_\varepsilon(\sigma;t,x,v) \in \mathbb{R}^3_+\right\}. \tag{4.2.3}$$

We also define

$$\mathcal{O}^t_\varepsilon := \left\{(x, v) \in \mathbb{R}^3_+ \times \mathbb{R}^3 \mid \forall \sigma \in [0, t],\ \mathrm{X}_\varepsilon(\sigma;t,x,v) \in \mathbb{R}^3_+\right\}.$$

We can now state the following proposition giving a crucial representation formula for the distribution function f_ε (see for instance the Appendix of Chapter 2).

Proposition 4.2.3. *Let $\varepsilon > 0$. If $(u_\varepsilon, f_\varepsilon)$ is a weak solution to the Vlasov-Navier-Stokes system in the sense of Definition 4.1.5 with $u_\varepsilon \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{W}^{1,\infty}(\mathbb{R}^3_+))$ then*

$$f_\varepsilon(t, x, v) = e^{\frac{3t}{\varepsilon}}\,\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, v)\,f^0_\varepsilon(\mathrm{X}_\varepsilon(0;t,x,v), \mathrm{V}_\varepsilon(0;t,x,v)), \tag{4.2.4}$$

where f^0_ε refers to the admissible initial condition for the Vlasov equation in the sense of Definition 4.1.4.

For later purposes, note that we have the following identity

$$(\mathrm{X}_\varepsilon, \mathrm{V}_\varepsilon)(0;t,\mathcal{O}^t_\varepsilon) = \left\{(x, v) \in \mathbb{R}^3_+ \times \mathbb{R}^3 \mid \tau^+_\varepsilon(0, x, v) > t\right\},$$

therefore for any integrable function ψ on $\mathbb{R}^3_+ \times \mathbb{R}^3$, we can write

$$\begin{aligned} \int_{\mathbb{R}^3_+ \times \mathbb{R}^3} \psi(x, v) f_\varepsilon(t, x, v)\,\mathrm{d}x\,\mathrm{d}v &= \int_{\mathcal{O}^t_\varepsilon} \psi(x, v) e^{\frac{3t}{\varepsilon}} f^0_\varepsilon(\mathrm{X}_\varepsilon(0;t,x,v), \mathrm{V}_\varepsilon(0;t,x,v))\,\mathrm{d}x\,\mathrm{d}v \\ &= \int_{\mathbb{R}^3_+ \times \mathbb{R}^3} \psi(\mathrm{X}_\varepsilon(t;0,x,v), \mathrm{V}_\varepsilon(t;0,x,v))\,\mathbf{1}_{\tau^+_\varepsilon(0,x,v) > t}\,f^0_\varepsilon(x, v)\,\mathrm{d}x\,\mathrm{d}v. \end{aligned}$$

4.2.2 Changes of variable in velocity and space

Next, we derive some sufficient conditions on the velocity field u_ε so that a change of variable with respect to the characteristic curve in velocity holds true. As explained in the introduction, this straightening change of variable has been intensively used for the study of the large time behavior of the system (see [HKMM20, HK22, EHKM21, Ert21]), as well as for its hydrodynamic limits in [Hö18, HKMar]. In short, this change of variable is valid until time T if a control of the type $\|\nabla_x u_\varepsilon\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{R}^3_+))} \ll 1$ can be ensured. Thanks to the previous representation formula, this will mainly entail a bound on the local density ρ_ε of the type

$$\|\rho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3_+))} \lesssim \|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}.$$

As we shall see later on, a change of variable in space will also be performed after the change of variable in velocity. The same control on $\nabla_x u_\varepsilon$ actually also makes such a change of variable admissible.

The first change of variable is the straightening change of variable in velocity.

Lemma 4.2.4. *Let $\varepsilon > 0$. Assume that there exists some time $t > 0$ such that*

$$\int_0^t \|\nabla_x u_\varepsilon(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s < \delta, \tag{4.2.5}$$

where $\delta > 0$ satisfies $\delta e^\delta < \frac{1}{9}$. Then for any $x \in \mathbb{R}^3$, the mapping

$$\Gamma_\varepsilon^{t,x} : v \mapsto \mathrm{V}_\varepsilon(0; t, x, v)$$

is a $\mathscr{C}^1$-diffeomorphism from $\mathbb{R}^3$ to itself with the bound

$$\forall v \in \mathbb{R}^3, \quad \det\left(\mathrm{D}_v\, \Gamma_\varepsilon^{t,x}(v)\right) \gtrsim e^{\frac{3t}{\varepsilon}},$$

where $\gtrsim$ refers to a universal constant.

Proof. We refer to Lemma 3.4.5 in Chapter 3, having in mind

$$\mathrm{V}_\varepsilon(0; t, x, v) = e^{\frac{t}{\varepsilon}} v - (1 - e^{\frac{t}{\varepsilon}}) e_3 - \frac{1}{\varepsilon}\int_0^t e^{\frac{\tau}{\varepsilon}} (P u_\varepsilon)(\tau, \mathrm{X}_\varepsilon(\tau; t, x, v))\,\mathrm{d}\tau,$$

which is directly inferred from (4.2.2). □

We can now present the first main consequence of the change of variable in velocity, namely a uniform bound for ρ_ε in $\mathrm{L}^\infty_t \mathrm{L}^\infty_x$.

Corollary 4.2.5. *Let $\varepsilon > 0$. Assume that there exists some time $t > 0$ such that the condition (4.2.5) holds. Then we have*

$$\|\rho_\varepsilon\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty(\mathbb{R}^3_+))} \lesssim \|f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}.$$

Proof. We basically follow [HK22, Lemma 3.2]: we use the representation formula (4.2.4) to write

$$\begin{aligned}
\rho_\varepsilon(t,x) &= \int_{\mathbb{R}^3} f_\varepsilon(t,x,v)\,\mathrm{d}v \\
&= \int_{\mathbb{R}^3} e^{\frac{3t}{\varepsilon}}\, \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,v)\, f_\varepsilon^0(\mathrm{X}_\varepsilon(0;t,x,v), \mathrm{V}_\varepsilon(0;t,x,v))\,\mathrm{d}v \\
&= \int_{\mathbb{R}^3} e^{\frac{3t}{\varepsilon}}\, \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x, [\Gamma_\varepsilon^{t,x}]^{-1}(w))\, f_\varepsilon^0(\mathrm{X}_\varepsilon(0;t,x,[\Gamma_\varepsilon^{t,x}]^{-1}(w)), w) |\det \mathrm{D}_w\, [\Gamma_\varepsilon^{t,x}]^{-1}(w)|\,\mathrm{d}w \\
&\lesssim \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x, [\Gamma_\varepsilon^{t,x}]^{-1}(w))\, f_\varepsilon^0(\mathrm{X}_\varepsilon(0;t,x,[\Gamma_\varepsilon^{t,x}]^{-1}(w)), w)\,\mathrm{d}w,
\end{aligned}$$

thanks to the change of variable $v \mapsto \Gamma_\varepsilon^{t,x}(v)$ and the bound of Lemma 4.2.4. Since

$$(x, [\Gamma_\varepsilon^{t,x}]^{-1}(w)) \in \mathcal{O}_\varepsilon^t \Longrightarrow \mathrm{X}_\varepsilon(0;t,x,[\Gamma_\varepsilon^{t,x}]^{-1}(w)) \in \mathbb{R}^3_+,$$

we obtain the bound

$$\rho_\varepsilon(t,x) \lesssim \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x, [\Gamma_\varepsilon^{t,x}]^{-1}(w)) \|\, f_\varepsilon^0(\cdot, w)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}w \leq \|f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))},$$

which is the claimed result. □

Remark 4.2.6. From the expression (4.2.2), we can also deduce the following important formula

$$[\Gamma_\varepsilon^{t,x}]^{-1}(w) = e^{-\frac{t}{\varepsilon}} w - (1 - e^{-\frac{t}{\varepsilon}}) e_3 + \frac{1}{\varepsilon}\int_0^t e^{\frac{\tau - t}{\varepsilon}} (P u_\varepsilon)(\tau, \mathrm{X}_\varepsilon(\tau; t, x, [\Gamma_\varepsilon^{t,x}]^{-1}(w)))\,\mathrm{d}\tau.$$

Note that the same strategy as in Corollary 4.2.5 cannot be applied for higher-order moments in velocity because this will make some $[\Gamma_\varepsilon^{t,x}]^{-1}(w)$ appear, which diverges with ε. In Subsection 4.2.3 and Section 4.4, we will perform a desingularization procedure, based on the fine structure of the Brinkman force, to get rid of this problem.

Notation 4.2.7. *For $\varepsilon > 0$, $s, t \geq 0$ and $(x, w) \in \mathbb{R}^6$, we set*

$$\widetilde{\mathrm{X}}_\varepsilon^{s;t}(x, w) := \mathrm{X}_\varepsilon(s; t, x, [\Gamma_\varepsilon^{t,x}]^{-1}(w)).$$

Whene it is necessary, we whall also use the notation $\widetilde{\mathrm{X}}_\varepsilon^{s;t,w}(x) := \widetilde{\mathrm{X}}_\varepsilon^{s;t}(x, w)$.

We finally state a lemma about a change of variable in space along the trajectories. It will be applied after the previous change of variable in velocity and will be useful when considering L^p norm in space of the Brinkman force. We refer to Lemma 3.4.7 in Chapter 3.

Lemma 4.2.8. *Let $\varepsilon > 0$. Assume that there exists some time $t > 0$ such that the condition (4.2.5) holds. Then for any $0 \leq s \leq t$ and any $w \in \mathbb{R}^3$, the mapping*

$$x \mapsto \widetilde{\mathrm{X}}_\varepsilon^{s;t,w}(x)$$

is a $\mathscr{C}^1$-diffeomorphism from $\mathbb{R}^3$ to itself and its Jacobian determinant satisfies the following bound from below

$$\forall x \in \mathbb{R}^3, \quad \det\left(\mathrm{D}_x \widetilde{\mathrm{X}}_\varepsilon^{s;t,w}(x)\right) \gtrsim 1, \tag{4.2.6}$$

where $\gtrsim$ refers to a universal constant.

4.2.3 Towards the convergence of the Brinkman force when $\varepsilon \to 0$

Recall the notation

$$F_\varepsilon := j_\varepsilon - \rho_\varepsilon u_\varepsilon = \int_{\mathbb{R}^3} f_\varepsilon (v - u_\varepsilon) \,\mathrm{d}v$$

for the Brinkman force. The main goal of this subsection is to identify sufficient conditions leading to the convergence of the Brinkman force when $\varepsilon \to 0$. More precisely, we aim at proving a convergence of $F_\varepsilon + \rho_\varepsilon e_3$ towards 0 in $\mathrm{L}^2_t \mathrm{L}^2_x$ when $\varepsilon \to 0$, based on a careful decomposition of this expression.

To do so, we will rely on the tools of Subsections 4.2.1 and 4.2.2 which are based on particles trajectories. As announced in Subsection 4.2.2, we shall make an intensive use of the following change of variable in velocity:

$$v \mapsto \Gamma_\varepsilon^{t,x}(v) = \mathrm{V}_\varepsilon(0; t, x, v),$$

which should be combined to the representation formula (4.2.4).

In the rest of this chapter, we will make a constant use of the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}^3_+$, that we recall now (see the Section 2.D of the Appendix in Chapter 3).

Theorem 4.2.9. *Let $(p, q, r) \in [1, +\infty]^3$ and $m \in \mathbb{N}$. Suppose $j \in \mathbb{N}$ and $\alpha \in [0, 1]$ satisfy the relations*

$$\frac{1}{p} = \frac{j}{3} + \left(\frac{1}{r} - \frac{m}{3}\right)\alpha + \frac{1 - \alpha}{q},$$
$$\frac{j}{m} \leq \alpha \leq 1,$$

with the exception $\alpha < 1$ if $m - j - d/r \in \mathbb{N}$. There exists $C > 0$ such that for all $g \in \mathrm{L}^q(\mathbb{R}^3_+)$ with $\mathrm{D}^m g \in \mathrm{L}^r(\mathbb{R}^3_+)$, we have

$$\|\mathrm{D}^j g\|_{\mathrm{L}^p(\mathbb{R}^3_+)} \leq C \|\mathrm{D}^m g\|^{\alpha}_{\mathrm{L}^r(\mathbb{R}^3_+)} \|g\|^{1-\alpha}_{\mathrm{L}^q(\mathbb{R}^3_+)}.$$

The very first step of the strategy is to write

$$\begin{aligned}
&F_\varepsilon(t,x)\\
&= e^{\frac{3t}{\varepsilon}} \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,v) f^0_\varepsilon(\mathrm{X}_\varepsilon(0;s,x,v), \mathrm{V}_\varepsilon(0;s,x,v))\,(v-u_\varepsilon(t,x))\,\mathrm{d}v\\
&= e^{\frac{3t}{\varepsilon}} \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w)) f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)\left([\Gamma^{t,x}_\varepsilon]^{-1}(w)-u_\varepsilon(t,x)\right)|\det \mathrm{D}_w[\Gamma^{t,x}_\varepsilon]^{-1}(w)|\,\mathrm{d}w.
\end{aligned} \tag{4.2.7}$$

As highlighted before, the crucial quantity

$$[\Gamma^{t,x}_\varepsilon]^{-1}(w) = e^{-\frac{t}{\varepsilon}}w - (1-e^{-\frac{t}{\varepsilon}})e_3 + \frac{1}{\varepsilon}\int_0^t e^{\frac{\tau-t}{\varepsilon}}(Pu_\varepsilon)(\tau,\mathrm{X}_\varepsilon(\tau;t,x,[\Gamma^{t,x}_\varepsilon]^{-1}(w)))\,\mathrm{d}\tau$$

appearing above is singular with respect to the convergence $\varepsilon \to 0$.

Desingularization procedure with respect to ε.

In order to get rid of the factor ε^{-1}, we shall perform an integration by parts in time thanks to the exponential factor (coming from friction in the system) appearing in the last integral. This key idea comes from the strategy devised by Han-Kwan and Michel in [HKMar], already inspired by the work of Han-Kwan in [HK22]. At least formally, we have for all $(t,x,w) \in \mathbb{R}^+ \times \mathbb{R}^3_+ \times \mathbb{R}^3$

$$\begin{aligned}
[\Gamma^{t,x}_\varepsilon]^{-1}(w) &= e^{-\frac{t}{\varepsilon}}w - (1-e^{-\frac{t}{\varepsilon}})e_3 + \frac{1}{\varepsilon}\int_0^t e^{\frac{\tau-t}{\varepsilon}}(Pu_\varepsilon)(\tau,\widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w))\,\mathrm{d}\tau\\
&= e^{-\frac{t}{\varepsilon}}w - (1-e^{-\frac{t}{\varepsilon}})e_3 + (Pu_\varepsilon)(t,\widetilde{\mathrm{X}}^{t;t}_\varepsilon(x,w)) - e^{\frac{t}{\varepsilon}}(Pu_\varepsilon)(0,\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w))\\
&\quad - \int_0^t e^{\frac{\tau-t}{\varepsilon}}\partial_\tau[Pu_\varepsilon](\tau,\widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w))\,\mathrm{d}\tau\\
&\quad - \int_0^t e^{\frac{\tau-t}{\varepsilon}}\left(\mathrm{V}_\varepsilon(\tau;t,x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))\cdot\nabla_x\right)[Pu_\varepsilon](\tau,\widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w))\,\mathrm{d}\tau,
\end{aligned}$$

so that

$$\begin{aligned}
[\Gamma^{t,x}_\varepsilon]^{-1}(w) &= e^{-\frac{t}{\varepsilon}}(w+e_3-(Pu_\varepsilon)(0,\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w))) + u_\varepsilon(t,x) - e_3\\
&\quad - \int_0^t e^{\frac{\tau-t}{\varepsilon}}P[\partial_\tau u_\varepsilon](\tau,\widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w))\,\mathrm{d}\tau\\
&\quad - \int_0^t e^{\frac{\tau-t}{\varepsilon}}\left(\mathrm{V}_\varepsilon(\tau;t,x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))\cdot\nabla_x\right)[Pu_\varepsilon](\tau,\widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w))\,\mathrm{d}\tau.
\end{aligned} \tag{4.2.8}$$

As we shall see, the analysis of the last term will require the following formula, coming from the expression of $[\Gamma^{t,x}_\varepsilon]^{-1}(w)$ and $\mathrm{V}_\varepsilon(s;t,x,v)$ from Subsections 4.2.1–4.2.2:

$$\begin{aligned}
\mathrm{V}_\varepsilon(s;t,x,[\Gamma^{t,x}_\varepsilon]^{-1}(w)) &= e^{\frac{t-s}{\varepsilon}}[\Gamma^{t,x}_\varepsilon]^{-1}(w) - (1-e^{\frac{t-s}{\varepsilon}})e_3 + \frac{1}{\varepsilon}\int_t^s e^{\frac{\tau-s}{\varepsilon}}(Pu_\varepsilon)(\tau,\widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w))\,\mathrm{d}\tau\\
&= e^{\frac{-s}{\varepsilon}}(w+e_3) - e_3 + \frac{1}{\varepsilon}\int_0^s e^{\frac{\tau-s}{\varepsilon}}(Pu_\varepsilon)(\tau,\widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w))\,\mathrm{d}\tau.
\end{aligned} \tag{4.2.9}$$

Remark 4.2.10 (Justification of the integration by parts)**.** The previous computations are still partly formal if we do not assume some additional regularity of u_ε. The intervals of time with which we will systematically work in the following will in fact justify this procedure. We refer to Remark 4.3.13 for the introduction of the related strong existence times.

The identity (4.2.8) on $[\Gamma^{t,x}_\varepsilon]^{-1}(w)$ combined with (4.2.7) then provides the following lemma, which is the starting point for our analysis.

Lemma 4.2.11. *For any $\varepsilon > 0$, if u_ε is smooth and if the change of variable in velocity is admissible until time T, then for all $(t,x) \in (0,T) \times \mathbb{R}^3_+$*

$$|F_\varepsilon(t,x) + \rho_\varepsilon e_3| \lesssim G^0_\varepsilon(t,x) + G^1_\varepsilon(t,x) + G^2_\varepsilon(t,x),$$

where

$$G^0_\varepsilon(t,x) := e^{-\frac{t}{\varepsilon}} \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w)) f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w), w) \Big| w + e_3 - (Pu_\varepsilon)(0, \widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w)) \Big| \,\mathrm{d}w,$$

$$G^1_\varepsilon(t,x) := \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w)) \int_0^t e^{\frac{\tau - t}{\varepsilon}} f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w), w) \Big| \partial_\tau [Pu_\varepsilon](\tau, \widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w)) \Big| \,\mathrm{d}\tau \,\mathrm{d}w,$$

$$G^2_\varepsilon(t,x) := \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w)) \int_0^t e^{\frac{\tau - t}{\varepsilon}} f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w), w)$$
$$\times \Big| \Big(\mathrm{V}_\varepsilon(\tau; t, x, [\Gamma^{t,x}_\varepsilon]^{-1}(w)) \cdot \nabla_x \Big) [Pu_\varepsilon](\tau, \widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w)) \Big| \,\mathrm{d}\tau \,\mathrm{d}w.$$

We will estimate the three previous terms separately.

Lemma 4.2.12. *Let $T > 0$ such that the condition (4.2.5) holds at time T. Assume that for all $\varepsilon > 0$, we have*

$$\|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3; \mathrm{L}^\infty(\mathbb{R}^3_+))} + \|(1+|v|^2) f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3_+ \times \mathbb{R}^3)} \le M,$$
$$\|u^0_\varepsilon\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \le M,$$

for some $M > 1$ independent of ε. There exists $\mu_2 > 0$ such that for all $\varepsilon > 0$

$$\|G^0_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)} \lesssim \varepsilon^{\frac{1}{2}} M^{\mu_2}.$$

Proof. Thanks to Hölder's inequality, we have

$$\int_0^T \int_{\mathbb{R}^3_+} |G^0_\varepsilon(t,x)|^2 \,\mathrm{d}t \,\mathrm{d}x \le \int_0^T e^{-\frac{2t}{\varepsilon}} \|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3; \mathrm{L}^\infty(\mathbb{R}^3_+))} \int_{\mathbb{R}^3_+ \times \mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w)) f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w), w)$$
$$\times \Big| w + e_3 - (Pu_\varepsilon)(0, \widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w)) \Big|^2 \,\mathrm{d}x \,\mathrm{d}w \,\mathrm{d}t.$$

In the last inequality, we have used the following fact: for all $y \in \mathbb{R}^3$ and $w \in \mathbb{R}^3$ we have

$$(y, [\Gamma^{t,y}_\varepsilon]^{-1}(w)) \in \mathcal{O}^t_\varepsilon \Longrightarrow \forall \sigma \in [0,t],\ \mathrm{X}_\varepsilon(\sigma; t, y, [\Gamma^{t,y}_\varepsilon]^{-1}(w)) \in \mathbb{R}^3_+. \tag{4.2.10}$$

Then, isolating the integral in space, we perform the change of variable in space $x \mapsto \widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w)$ which is valid thanks to Lemma 4.2.8, and we get for all $t \in [0,T]$ and $w \in \mathbb{R}^3$

$$\int_{\mathbb{R}^3_+} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w)) f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w), w) \Big| w + e_3 - (Pu_\varepsilon)(0, \widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w)) \Big|^2 \,\mathrm{d}x$$
$$= \int_{\mathbb{R}^3} \mathbf{1}_{\mathbb{R}^3_+}([\widetilde{\mathrm{X}}^{0;t,w}_\varepsilon]^{-1}(x)) \mathbf{1}_{\mathcal{O}^t_\varepsilon}([\widetilde{\mathrm{X}}^{0;t,w}_\varepsilon]^{-1}(x), [\Gamma^{t,[\widetilde{\mathrm{X}}^{0;t,w}_\varepsilon]^{-1}(x)}_\varepsilon]^{-1}(w))$$
$$\times f^0_\varepsilon(x,w) \Big| w + e_3 - (Pu_\varepsilon)(0,x) \Big|^2 |\mathrm{D}_x [\widetilde{\mathrm{X}}^{0;t,w}_\varepsilon]^{-1}(x)| \,\mathrm{d}x,$$

therefore, thanks to (4.2.6), we get

$$\int_0^T \int_{\mathbb{R}^3_+} |G^0_\varepsilon(t,x)|^2 \,\mathrm{d}t \,\mathrm{d}x \lesssim \int_0^T e^{-\frac{2t}{\varepsilon}} \|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3; \mathrm{L}^\infty(\mathbb{R}^3_+))}$$
$$\times \Big[\|(1+|v|^2) f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3_+ \times \mathbb{R}^3)} + \|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3; \mathrm{L}^\infty(\mathbb{R}^3_+))} \|u^0_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \Big] \,\mathrm{d}t$$
$$\le \varepsilon M^2 + \varepsilon M^4,$$

and this concludes the proof since $M > 1$. □

Lemma 4.2.13. *Let $T > 0$ such that the condition (4.2.5) holds at time T. Assume that for all $\varepsilon > 0$*

$$\|f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \leq M,$$

for some $M > 1$ independent of ε. For all $\varepsilon > 0$, we have

$$\|G_\varepsilon^1\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)} \lesssim \varepsilon M \|\partial_t u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)}.$$

Proof. By the Hölder's inequality in space and time, we get

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3_+} |G_\varepsilon^1(t,x)|^2 \,\mathrm{d}t\,\mathrm{d}x \\
&\leq \varepsilon \|f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \int_0^T \int_{\mathbb{R}^3} \int_0^t e^{\frac{\tau-t}{\varepsilon}} \\
&\qquad \times \left(\int_{\mathbb{R}^3_+} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x, [\Gamma_\varepsilon^{t,x}]^{-1}(w)) f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w) \left|\partial_\tau [P u_\varepsilon](\tau, \widetilde{\mathrm{X}}_\varepsilon^{\tau;t}(x,w))\right|^2 \mathrm{d}x \right) \mathrm{d}\tau\,\mathrm{d}w\,\mathrm{d}t.
\end{aligned}$$

thanks to Fubini theorem and the same procedure as in the proof of Lemma 4.2.12. For the integral in space, we use Lemma 4.2.8 and perform the change of variable $x' = \widetilde{\mathrm{X}}_\varepsilon^{\tau;t}(x,w)$ by first observing that (4.2.10) entails

$$([\widetilde{\mathrm{X}}_\varepsilon^{\tau;t,w}]^{-1}(x), [\Gamma_\varepsilon^{t,x}]^{-1}(w)) \in \mathcal{O}_\varepsilon^t \Longrightarrow \widetilde{\mathrm{X}}_\varepsilon^{0;t,w}([\widetilde{\mathrm{X}}_\varepsilon^{\tau;t,w}]^{-1}(x)) \in \mathbb{R}^3_+.$$

We get for all $w \in \mathbb{R}^3$

$$\begin{aligned}
&\int_{\mathbb{R}^3_+} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x, [\Gamma_\varepsilon^{t,x}]^{-1}(w)) f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w) \left|\partial_\tau [P u_\varepsilon](\tau, \widetilde{\mathrm{X}}_\varepsilon^{\tau;t}(x,w))\right|^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \mathbf{1}_{[\widetilde{\mathrm{X}}_\varepsilon^{\tau;t,w}]^{-1}(x)\in\mathbb{R}^3_+} \mathbf{1}_{\mathcal{O}_\varepsilon^t}([\widetilde{\mathrm{X}}_\varepsilon^{\tau;t,w}]^{-1}(x), [\Gamma_\varepsilon^{t,x}]^{-1}(w)) \\
&\qquad\qquad \times f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t,w}([\widetilde{\mathrm{X}}_\varepsilon^{\tau;t,w}]^{-1}(x)),w) \left|\partial_\tau [P u_\varepsilon](\tau,x)\right|^2 |\det(\mathrm{D}_x[\Xi_\varepsilon^{\tau;t,w}]^{-1}(x))| \,\mathrm{d}x \\
&\lesssim \|f_\varepsilon^0(\cdot,w)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \|\partial_\tau [P u_\varepsilon](\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3)}.
\end{aligned}$$

All in all, this yields

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3_+} |G_\varepsilon^1(t,x)|^2 \,\mathrm{d}t\,\mathrm{d}x \\
&\lesssim \varepsilon \|f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \int_0^T \int_{\mathbb{R}^3} \int_0^t e^{\frac{\tau-t}{\varepsilon}} \|f_\varepsilon^0(\cdot,w)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \|\partial_\tau [P u_\varepsilon](\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3)} \,\mathrm{d}\tau\,\mathrm{d}w\,\mathrm{d}t \\
&\leq \varepsilon^2 \|f_\varepsilon^0\|^2_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \|\partial_t u_\varepsilon\|^2_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)},
\end{aligned}$$

which concludes the proof. □

Lemma 4.2.14. *Let $T > 0$ such that the condition (4.2.5) holds at time T. Assume that for all $\varepsilon > 0$*

$$\|(1+|v|^2) f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \leq M,$$

for some $M > 1$ independent of ε. Then for all $\varepsilon > 0$

$$\begin{aligned}
\|G_\varepsilon^2\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+} &\lesssim \varepsilon^{\frac{3}{4}} M \|u_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} \|\mathrm{D}_x^2 u_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)} T^{\frac{1}{2}} \\
&\quad + \varepsilon M \|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} \|\nabla_x u_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{L}^\infty(\mathbb{R}^3_+))}.
\end{aligned}$$

Proof. In view of the formula (4.2.9), we have

$$
\begin{aligned}
|G_\varepsilon^2(t,x)| \leq & \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w)) \int_0^t e^{\frac{-t}{\varepsilon}} f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w)(1+|w|)\Big|\nabla_x[Pu_\varepsilon](s,\widetilde{\mathrm{X}}_\varepsilon^{s;t}(x,w))\Big| \,\mathrm{d}s\,\mathrm{d}w \\
& + \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w)) \int_0^t e^{\frac{s-t}{\varepsilon}} f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w)\Big|\nabla_x[Pu_\varepsilon](s,\widetilde{\mathrm{X}}_\varepsilon^{s;t}(x,w))\Big| \,\mathrm{d}s\,\mathrm{d}w \\
& + \frac{1}{\varepsilon}\int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w)) \int_0^t \int_0^s e^{\frac{s-t}{\varepsilon}} e^{\frac{\tau-s}{\varepsilon}} f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w) \\
& \qquad\qquad \times \Big|(Pu_\varepsilon)(\tau,\widetilde{\mathrm{X}}_\varepsilon^{\tau;t}(x,w))\Big| \Big|\nabla_x[Pu_\varepsilon](s,\widetilde{\mathrm{X}}_\varepsilon^{s;t}(x,w))\Big| \,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}w \\
& = \mathrm{I}(t,x)+\mathrm{II}(t,x)+\mathrm{III}(t,x).
\end{aligned}
$$

We now estimate each term separately. By Fubini theorem and Hölder inequality (in velocity), we have

$$
\begin{aligned}
|\mathrm{I}(t,x)|^2 \lesssim\, & e^{\frac{-2t}{\varepsilon}} \Bigg[\int_0^t \left(\int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))(1+|w|^2) f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w)\,\mathrm{d}w \right)^{\frac12} \\
& \qquad \times \left(\int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w)) f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w)\Big|\nabla_x[Pu_\varepsilon](s,\widetilde{\mathrm{X}}_\varepsilon^{s;t}(x,w))\Big|^2 \mathrm{d}w \right)^{\frac12} \mathrm{d}s \Bigg]^2 \\
\leq\, & \|(1+|v|^2) f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} e^{\frac{-2t}{\varepsilon}} \Bigg[\int_0^t \varepsilon e^{\frac{s}{\varepsilon}}(1-e^{-\frac{t}{\varepsilon}}) \\
& \times \left(\int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w)) f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w)\Big|\nabla_x[Pu_\varepsilon](s,\widetilde{\mathrm{X}}_\varepsilon^{s;t}(x,w))\Big|^2 \mathrm{d}w \right)^{\frac12} \frac{e^{-\frac{s}{\varepsilon}}\,\mathrm{d}s}{\varepsilon(1-e^{-\frac{t}{\varepsilon}})} \Bigg]^2 \\
\leq\, & \varepsilon \|(1+|v|^2) f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \\
& \times \int_0^t e^{\frac{2(s-t)}{\varepsilon}} \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w)) f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w)\Big|\nabla_x[Pu_\varepsilon](s,\widetilde{\mathrm{X}}_\varepsilon^{s;t}(x,w))\Big|^2 \mathrm{d}w\,\mathrm{d}s.
\end{aligned}
$$

Here, we have used Jensen inequality for the probability space

$$
\left((0,t),\ \frac{e^{-\frac{s}{\varepsilon}}\,\mathrm{d}s}{\varepsilon(1-e^{-\frac{t}{\varepsilon}})} \right).
$$

Then, we perform the change of variable in space $x' = \widetilde{\mathrm{X}}_\varepsilon^{s;t}(x,w)$ allowed by Lemma 4.2.8 together with Fubini Theorem to get

$$
\begin{aligned}
\int_{\mathbb{R}^3_+} |\mathrm{I}(t,x)|^2\,\mathrm{d}x \lesssim\, & \varepsilon \|(1+|v|^2) f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \\
& \times \int_0^t e^{\frac{2(s-t)}{\varepsilon}} \Bigg(\int_{\mathbb{R}^3_+\times\mathbb{R}^3} \mathbf{1}_{[\widetilde{\mathrm{X}}_\varepsilon^{\tau;t,w}]^{-1}(x)\in\mathbb{R}^3_+} \mathbf{1}_{\mathcal{O}_\varepsilon^t}([\widetilde{\mathrm{X}}_\varepsilon^{\tau;t,w}]^{-1}(x),[\Gamma_\varepsilon^{t,x}]^{-1}(w)) \\
& \qquad \times f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t,w}([\widetilde{\mathrm{X}}_\varepsilon^{\tau;t,w}]^{-1}(x)),w)\Big|\nabla_x[Pu_\varepsilon](s,x)\Big|^2 \mathrm{d}x\,\mathrm{d}w \Bigg) \mathrm{d}s \\
\leq\, & \varepsilon \|(1+|v|^2) f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \|f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \\
& \times \int_0^t e^{\frac{2(s-t)}{\varepsilon}} \|\nabla_x[Pu_\varepsilon](s)\|^2_{\mathrm{L}^2(\mathbb{R}^3)}\,\mathrm{d}s.
\end{aligned}
$$

Note that since $u_\varepsilon(s) \in \mathrm{H}^1_0(\mathbb{R}^3_+)$, we have $\nabla_x[Pu_\varepsilon](s) = \mathbf{1}_{\mathbb{R}^3_+}\nabla_x u_\varepsilon(s)$. Thanks to the Gagliardo-Nirenberg-Sobolev inequality (see Theorem 4.2.9) and the energy inequality (4.3.1), we have

$$\|\nabla_x u_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \|\mathrm{D}^2_x u_\varepsilon(s)\|^{\frac{1}{2}}_{\mathrm{L}^2(\mathbb{R}^3_+)}\|u_\varepsilon(s)\|^{\frac{1}{2}}_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \|u_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^3_+))}\|\mathrm{D}^2_x u_\varepsilon(s)\|^{\frac{1}{2}}_{\mathrm{L}^2(\mathbb{R}^3_+)},$$

from which we infer

$$\int_{\mathbb{R}^3_+}|\mathrm{I}(t,x)|^2\,\mathrm{d}x \lesssim \varepsilon\|(1+|v|^2)f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}$$
$$\times\|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^3_+))}\int_0^t e^{\frac{2(s-t)}{\varepsilon}}\|\mathrm{D}^2_x u_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s.$$

Since

$$\int_0^t e^{\frac{2(s-t)}{\varepsilon}}\|\mathrm{D}^2_x u_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s \leq \|\mathrm{D}^2_x u_\varepsilon\|_{\mathrm{L}^2((0,t)\times\mathbb{R}^3_+)}\left(\int_0^t e^{\frac{4(s-t)}{\varepsilon}}\,\mathrm{d}s\right)^{\frac{1}{2}},$$

we obtain

$$\|\mathrm{I}(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \varepsilon^{\frac{3}{4}}\|(1+|v|^2)f^0_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|f^0_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|u_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^3_+))}\|\mathrm{D}^2_x u_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^2((0,t)\times\mathbb{R}^3_+)},$$

and an integration in time yields

$$\|\mathrm{I}\|_{\mathrm{L}^2(0;T,\mathrm{L}^2(\mathbb{R}^3_+)}$$
$$\lesssim \varepsilon^{\frac{3}{4}}\|(1+|v|^2)f^0_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|f^0_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|u_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^3_+))}\|\mathrm{D}^2_x u_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)}T^{\frac{1}{2}}.$$

Let us turn to the control of $\mathrm{II}(t)$. The very same procedure as for $\mathrm{I}(t)$ leads to

$$|\mathrm{II}(t,x)|^2$$
$$\leq \|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}$$
$$\times\left[\int_0^t e^{\frac{2(s-t)}{\varepsilon}}\left(\int_{\mathbb{R}^3}\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)\left|\nabla_x[Pu_\varepsilon](s,\widetilde{\mathrm{X}}^{s;t}_\varepsilon(x,w))\right|^2\mathrm{d}w\right)^{\frac{1}{2}}\mathrm{d}s\right]^2$$
$$= \|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\left[\int_0^t\frac{\epsilon}{2}(1-e^{-\frac{2t}{\varepsilon}})\right.$$
$$\left.\left(\int_{\mathbb{R}^3}\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)\left|\nabla_x[Pu_\varepsilon](s,\widetilde{\mathrm{X}}^{s;t}_\varepsilon(x,w))\right|^2\mathrm{d}w\right)^{\frac{1}{2}}\frac{e^{\frac{2(s-t)}{\varepsilon}}\,\mathrm{d}s}{\frac{\epsilon}{2}(1-e^{-\frac{2t}{\varepsilon}})}\right]^2$$
$$\lesssim \varepsilon\|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}$$
$$\times\int_0^t e^{\frac{2(s-t)}{\varepsilon}}\left(\int_{\mathbb{R}^3}\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)\left|\nabla_x[Pu_\varepsilon](s,\widetilde{\mathrm{X}}^{s,t}_\varepsilon(x,w))\right|^2\mathrm{d}w\right)\mathrm{d}s.$$

The same change of variable in space as in the case of $\mathrm{I}(t)$ and the same computations give us

$$\int_{\mathbb{R}^3_+}|\mathrm{II}(t,x)|^2\,\mathrm{d}x \lesssim \varepsilon\|f^0_\varepsilon\|^2_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^3_+))}\|\mathrm{D}^2_x u_\varepsilon\|_{\mathrm{L}^2((0,t)\times\mathbb{R}^3_+)}\left(\int_0^t e^{\frac{4(s-t)}{\varepsilon}}\,\mathrm{d}s\right)^{\frac{1}{2}},$$

and then

$$\|\mathrm{II}(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \varepsilon^{\frac{3}{4}}\|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|u_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^3_+))}\|\mathrm{D}^2_x u_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^2((0,t)\times\mathbb{R}^3_+)}.$$

We thus obtain

$$\|\mathrm{II}\|_{\mathrm{L}^2(0;T,\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \varepsilon^{\frac{3}{4}} \|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \|u_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} \|\mathrm{D}^2_x u_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)} T^{\frac{1}{2}}.$$

For the last term $\mathrm{III}(t)$, we start writing

$$\begin{aligned}
|\mathrm{III}(t,x)|^2 &= \Bigg| \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w)) f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w) \int_0^t e^{\frac{s-t}{\varepsilon}} \Big|\nabla_x [P u_\varepsilon](s, \widetilde{\mathrm{X}}^{s;t}_\varepsilon(x,w))\Big| \\
&\qquad \times \left(\int_0^s e^{\frac{\tau-s}{\varepsilon}} \Big| (P u_\varepsilon)(\tau, \widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w)) \Big| \,\mathrm{d}\tau \right) \mathrm{d}s \,\mathrm{d}w \Bigg|^2 \\
&\le \Bigg| \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w)) f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w) \int_0^t e^{\frac{s-t}{\varepsilon}} \Big|\nabla_x [P u_\varepsilon](s, \widetilde{\mathrm{X}}^{s;t}_\varepsilon(x,w))\Big| \\
&\qquad \times \left(\int_0^s e^{\frac{\tau-s}{\varepsilon}} \,\mathrm{d}\tau \right)^{\frac{1}{2}} \left(\int_0^s e^{\frac{\tau-s}{\varepsilon}} \Big| (P u_\varepsilon)(\tau, \widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w)) \Big|^2 \,\mathrm{d}\tau \right)^{\frac{1}{2}} \mathrm{d}s \,\mathrm{d}w \Bigg|^2
\end{aligned}$$

hence we deduce

$$\begin{aligned}
&|\mathrm{III}(t,x)|^2 \\
&\lesssim \varepsilon^{-1} \Bigg[\int_{\mathbb{R}^3} \int_0^t \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w)) f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w) e^{\frac{s-t}{\varepsilon}} \Big|\nabla_x [P u_\varepsilon](s, \widetilde{\mathrm{X}}^{s;t}_\varepsilon(x,w))\Big| \\
&\qquad \times \left(\int_0^s e^{\frac{\tau-s}{\varepsilon}} \Big| (P u_\varepsilon)(\tau, \widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w)) \Big|^2 \,\mathrm{d}\tau \right)^{\frac{1}{2}} \mathrm{d}s \,\mathrm{d}w \Bigg]^2 \\
&\le \varepsilon^{-1} \left(\int_0^t \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w)) f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w) e^{\frac{s-t}{\varepsilon}} \Big|\nabla_x [P u_\varepsilon](s, \widetilde{\mathrm{X}}^{s;t}_\varepsilon(x,w))\Big|^2 \mathrm{d}s \,\mathrm{d}w \right) \\
&\times \left(\int_0^t \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w)) f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w) e^{\frac{s-t}{\varepsilon}} \left(\int_0^s e^{\frac{\tau-s}{\varepsilon}} \Big| (P u_\varepsilon)(\tau, \widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w)) \Big|^2 \,\mathrm{d}\tau \right) \mathrm{d}s \,\mathrm{d}w \right),
\end{aligned}$$

where we have used Hölder's inequality in velocity and time in the last inequality. By (4.2.10), we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3_+} |\mathrm{III}(t,x)|^2 \,\mathrm{d}x \,\mathrm{d}t \\
&\lesssim \varepsilon^{-1} \|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \left[\int_0^T \left(\int_0^t e^{\frac{s-t}{\varepsilon}} \|\nabla_x u_\varepsilon(s)\|^2_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \,\mathrm{d}s \right) \mathrm{d}t \right] \\
&\quad \times \sup_{t\in(0,T)} \Bigg\{ \int_0^t e^{\frac{s-t}{\varepsilon}} \int_0^s e^{\frac{\tau-s}{\varepsilon}} \\
&\qquad \times \int_{\mathbb{R}^3_+\times\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w)) f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w) \Big| (P u_\varepsilon)(\tau, \widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w)) \Big|^2 \,\mathrm{d}x \,\mathrm{d}w \,\mathrm{d}\tau \,\mathrm{d}s \Bigg\}.
\end{aligned}$$

For the term between braces, we perform the change of variable $x' = \widetilde{\mathrm{X}}_\varepsilon^{\tau;t}(x,w)$ and we get

$$\int_0^t e^{\frac{s-t}{\varepsilon}} \int_0^s e^{\frac{\tau-s}{\varepsilon}} \int_{\mathbb{R}^3_+\times\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w)) f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w) \left|(Pu_\varepsilon)(\tau,\widetilde{\mathrm{X}}_\varepsilon^{\tau;t}(x,w))\right|^2 \,\mathrm{d}x\,\mathrm{d}w\,\mathrm{d}\tau\,\mathrm{d}s$$
$$\lesssim \|f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \int_0^t e^{\frac{s-t}{\varepsilon}} \int_0^s e^{\frac{\tau-s}{\varepsilon}} \|u_\varepsilon(\tau)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}\tau\,\mathrm{d}s$$
$$\lesssim \varepsilon^2 \||v|^k f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \|u_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^3_+))}.$$

For the term in brackets, we write

$$\int_0^T \left(\int_0^t e^{\frac{s-t}{\varepsilon}} \|\nabla_x u_\varepsilon(s)\|^2_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s\right)\mathrm{d}t \le \varepsilon \int_0^T \|\nabla_x u_\varepsilon(s)\|^2_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s.$$

This yields

$$\int_0^T \int_{\mathbb{R}^3_+} |\mathrm{III}(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t \lesssim \varepsilon^2 \|f_\varepsilon^0\|^2_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \|u_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} \int_0^T \|\nabla_x u_\varepsilon(s)\|^2_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s.$$

We obtain the result by gathering all the terms together. □

In view of the previous uniform bounds, we get the following result which eventually quantifies the convergence for $j_\varepsilon - \rho_\varepsilon u_\varepsilon$ when $\varepsilon \to 0$.

Corollary 4.2.15. *Let $((u_\varepsilon, f_\varepsilon))_{\varepsilon>0}$ be a family of global weak solutions to the VNS system which are smooth. Let $T>0$ such that the condition (4.2.5) holds at time T. Assume that for all $\varepsilon>0$, we have*

$$\|u_\varepsilon^0\|_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|(1+|v|^2) f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty\cap\mathrm{L}^1(\mathbb{R}^3_+))} \le M,$$

for some $M>1$ independent of ε. Then there exists $\mu_2>0$ such that for all $\varepsilon>0$, we have

$$\begin{aligned}\|j_\varepsilon - \rho_\varepsilon u_\varepsilon + \rho_\varepsilon e_3\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)} \lesssim\ & \varepsilon^{\frac12} M^{\mu_2} + \varepsilon M \|\partial_t u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)} \\ & + \varepsilon^{\frac34} M \|u_\varepsilon\|^{\frac12}_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} \|\mathrm{D}_x^2 u_\varepsilon\|^{\frac12}_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)} T^{\frac12} \\ & + \varepsilon M \|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} \|\nabla_x u_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{L}^\infty(\mathbb{R}^3_+))}.\end{aligned}$$

4.2.4 Exit geometric condition and absorption on the half-space

We eventually define and study the *exit geometric condition*, ensuring that particles starting in a given area of the phase-space leave the half-space before a prescribed time. This will be the key tool to analyse the absorption effect at the boundary, of crucial importance in Section 4.4. This notion is reminiscent of an important idea used in [GHKM18] and has also been revisited in Chapter 3 (from [Ert21]) for the study of the large time behavior. In short, it requires a control of the type $\|u_\varepsilon\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{R}^3_+))} \ll 1$ in order to hold true and is truly based on the presence of the gravity term in the kinetic equation.

Definition 4.2.16. *Let $\varepsilon>0$ and $L,R>0$. We say that a vector field $\mathrm{U}\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{W}^{1,\infty}_0(\mathbb{R}^3_+))$ satisfies the exit geometric condition (EGC) in time $T\ge0$ with respect to $(\mathbb{R}^2\times(0,L))\times\mathrm{B}(0,R)$ if*

$$\sup_{\substack{x\in\mathbb{R}^2\times(0,L)\\ v\in\mathrm{B}(0,R)}} \tau^+_{\mathrm{U},\varepsilon}(0,x,v) < T, \tag{4.2.11}$$

where $\tau^+_{\mathrm{U},\varepsilon}$ refers to Definition (4.2.2) for the characteristic curves $(\mathrm{X}_{\mathrm{U},\varepsilon},\mathrm{V}_{\mathrm{U},\varepsilon})$ of the Vlasov equation associated to a velocity field U in (4.2.1).

In what follows, we shall say that U satisfies $\mathrm{EGC}^{L,R}_\varepsilon(T)$.

As a consequence of the representation formula (4.2.4), we obtain the following proposition which highlights the effect of an EGC on the solution to the Vlasov equation. We refer to Proposition 3.5.2 in Chapter 3 for a proof.

Proposition 4.2.17. *Suppose that a velocity field* $\mathrm{U} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{W}^{1,\infty}_0(\mathbb{R}^3_+))$ *satisfies* $\mathrm{EGC}^{L,R}_\varepsilon(T)$ *for some fixed* $L, R > 0$. *Then, if* f_ε *is the solution to the Vlasov equation associated to* U, *we have for almost every* $(x,v) \in \mathbb{R}^3_+ \times \mathbb{R}^3$ *and any* $t > T$

$$\begin{aligned} f_\varepsilon(t,x,v) \leq e^{\frac{3t}{\varepsilon}}\, \mathbf{1}_{\mathcal{O}^t_{\varepsilon,\mathrm{U}}}(x,v)\, \mathbf{1}_{|\mathrm{V}^{0;t}_{\varepsilon,\mathrm{U}}(x,v)|>R}\, f^0_\varepsilon(\mathrm{X}^{0;t}_{\varepsilon,\mathrm{U}}(x,v), \mathrm{V}^{0;t}_{\varepsilon,\mathrm{U}}(x,v)) \\ + e^{\frac{3t}{\varepsilon}}\, \mathbf{1}_{\mathcal{O}^t_{\varepsilon,\mathrm{U}}}(x,v)\, \mathbf{1}_{\mathrm{X}^{0;t}_{\varepsilon,\mathrm{U}}(x,v)_3>L}\, f^0_\varepsilon(\mathrm{X}^{0;t}_{\varepsilon,\mathrm{U}}(x,v), \mathrm{V}^{0;t}_{\varepsilon,\mathrm{U}}(x,v)), \end{aligned} \tag{4.2.12}$$

where

$$\mathcal{O}^t_{\varepsilon,\mathrm{U}} := \left\{(x,v) \in \mathbb{R}^3_+ \times \mathbb{R}^3 \mid \forall \sigma \in [0,t],\ \mathrm{X}_\varepsilon(\sigma;t,x,v) \in \mathbb{R}^3_+\right\}.$$

The main task is now to find a sufficient condition which can ensure that a vector field satisfies an EGC. We rely on a stability principle, comparing the whole system of curves (4.2.1) for the Vlasov equation with velocity field u_ε (solution to the Navier-Stokes equations) to the same version without the fluid velocity. We thus consider the following characteristic curves $(\mathrm{X}^g_\varepsilon, \mathrm{V}^g_\varepsilon)$ for the Vlasov equation associated with the vector field $(x,v) \mapsto (v, -e_3 - v)$:

$$\begin{cases} \dot{\mathrm{X}}^g_\varepsilon(s;t,x,v) = \mathrm{V}^g_\varepsilon(s;t,x,v), & \dot{\mathrm{V}}^g_\varepsilon(s;t,x,v) = \dfrac{1}{\varepsilon}\left(-e_3 - \mathrm{V}^g_\varepsilon(s;t,x,v)\right), \\ \mathrm{X}^g_\varepsilon(t;t,x,v) = x, & \mathrm{V}^g_\varepsilon(t;t,x,v) = v. \end{cases} \tag{4.2.13}$$

This corresponds to the free evolution of the particles, without coupling with the fluid phase and with the sole presence of the gravity field. In view of the simpler form of that system, we hope for precise information on the absorption at the boundary for (4.2.1). Indeed, we have

$$\begin{cases} \mathrm{X}^g_\varepsilon(t;s,x,v) = x + \varepsilon(1 - e^{\frac{s-t}{\varepsilon}})(v + e_3) - (t-s)e_3, \\ \mathrm{V}^g_\varepsilon(t;s,x,v) = e^{\frac{s-t}{\varepsilon}}(v + e_3) - e_3, \end{cases} \tag{4.2.14}$$

so that

$$\begin{cases} \mathrm{X}^g_\varepsilon(t;s,x,v)_3 = x_3 + \varepsilon(1 - e^{\frac{s-t}{\varepsilon}})(v_3 + 1) - (t-s), \\ \mathrm{V}^g_\varepsilon(t;s,x,v)_3 = e^{\frac{s-t}{\varepsilon}}(v_3 + 1) - 1. \end{cases}$$

In particular, we will precisely quantify how one can ensure an EGC for (4.2.14). Coming back to the full system will be possible thanks to the following stability result, mainly inspired from [GHKM18] and from Chapter 3. It will allow us to transfer to (4.2.1) any EGC satisfied in finite time, provided that the $\mathrm{L}^1_t\mathrm{L}^\infty_x$ norm of the vector field U defining the curves is small enough.

Lemma 4.2.18. *Let* $\alpha > 0$. *There exists a constant* $\kappa_\alpha > 0$ *such that the following holds for all* $\varepsilon \in (0,1)$. *Suppose that the trivial vector field (related to* $(\mathrm{X}^g_\varepsilon, \mathrm{V}^g_\varepsilon)$*) satisfies* $\mathrm{EGC}^{L,R}_\varepsilon(T)$, *where* $L, R > 0$ *are given. Then, any vector field* $\mathrm{U} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{W}^{1,\infty}_0(\mathbb{R}^3_+))$ *such that*

$$\int_0^{T+\alpha} \|\mathrm{U}(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\, \mathrm{d}s \leq \kappa_\alpha, \tag{4.2.15}$$

satisfies $\mathrm{EGC}^{L,R}_\varepsilon(T+\alpha)$.

Proof. The proof of Lemma 3.5.4 from Chapter 3 actually applies *mutatis mutandis* when one assumes $\varepsilon \in (0,1)$. □

Before going further, we also observe that for all $(x,v) \in \left(\mathbb{R}^2 \times (0,L)\right) \times \mathrm{B}(0,R)$ and for all $t \geq 0$, we have

$$\begin{aligned}
\mathrm{X}^g_\varepsilon(t;0,x,v)_3 &= x_3 + \varepsilon(1-e^{-\frac{t}{\varepsilon}})(v_3+1) - t \\
&\leq L + \varepsilon(1-e^{-\frac{t}{\varepsilon}})(1+R) - t \\
&< L + \varepsilon(1+R) - t.
\end{aligned}$$

We thus infer the following lemma.

Lemma 4.2.19. *Let $\varepsilon > 0$. If $L, R > 0$ are given, the trivial vector field $\mathrm{U} \equiv 0$ (associated to $(\mathrm{X}^g_\varepsilon, \mathrm{V}^g_\varepsilon)$) satisfies $\mathrm{EGC}^{L,R}_\varepsilon(t^g_\varepsilon(L,R))$ where*

$$t^g_\varepsilon(L,R) := L + \varepsilon(1+R). \tag{4.2.16}$$

Definition 4.2.20. *For $\varepsilon > 0$ and $L, R > 0$, we set for $s > 0$*

$$\ell^L_\varepsilon(s) := \frac{1}{2}(s - \varepsilon(1-e^{-\frac{s}{\varepsilon}})) - L, \quad r^R_\varepsilon(s) := \frac{1}{2}\left(\frac{s}{\varepsilon(1-e^{-\frac{s}{\varepsilon}})} - 1\right) - R.$$

We observe that for any $\varepsilon > 0$

$$\ell^L_\varepsilon(t^g_\varepsilon(L,R)) > -L, \quad r^R_\varepsilon(t^g_\varepsilon(L,R)) > -R.$$

Furthermore, the functions ℓ^L_ε and r^R_ε are increasing on $\mathbb{R}^+$ and diverge towards $+\infty$ when $t \to +\infty$.

Useful information about the EGC for the free system (4.2.13) of curves $(\mathrm{X}^g_\varepsilon, \mathrm{V}^g_\varepsilon)$ are then gathered in the following lemma.

Lemma 4.2.21. *Let $\varepsilon \in (0,1)$. Let $L, R > 0$ such that*

$$t^g_\varepsilon(L,R) < t_0$$

for some $t_0 > 0$ independent of ε. Then for all $t \geq t_0$ the trivial vector field $\mathrm{U} \equiv 0$ (associated to $(\mathrm{X}^g_\varepsilon, \mathrm{V}^g_\varepsilon)$) satisfies $\mathrm{EGC}^{L+\ell^L_\varepsilon(t), R+r^R_\varepsilon(t)}_\varepsilon(t)$. Furthermore, there exists $C = C(t_0) > 0$ independent of ε such that

$$\forall s \geq t_0, \quad \frac{1}{L+\ell^L_\varepsilon(s)} \leq \frac{C}{1+s}, \quad \frac{1}{R+r^R_\varepsilon(s)} \leq \frac{C}{1+s}.$$

Proof. In view of the previous remark, we have

$$\forall t \geq t_0, \quad \ell^L_\varepsilon(t_0) > -L, \quad r^R_\varepsilon(t_0) > -R.$$

Now, let $t \geq t_0$. We observe that if $x_3 \in (0, L+\ell^L_\varepsilon(t))$ and $|v| \in [0, R+r^R_\varepsilon(t))$ then

$$\mathrm{X}^g_\varepsilon(t;0,x,v)_3 = x_3 + \varepsilon(1-e^{-\frac{s}{\varepsilon}})(v_3+1) - t < 0.$$

This implies that the trivial vector field $\mathrm{U} \equiv 0$ satisfies $\mathrm{EGC}^{L+\ell^L_\varepsilon(t), R+r^R_\varepsilon(t)}_\varepsilon(t)$. Indeed, the function $s \mapsto \mathrm{X}^g(s;0,x,v)_3$ is strictly decreasing after the first time it vanishes. Finally, tedious but basic computations show that the functions

$$s \mapsto \frac{1+s}{L+\ell^L_\varepsilon(s)}, \quad \text{and} \quad s \mapsto \frac{1+s}{R+r^R_\varepsilon(s)}$$

are positive and nonincreasing on $[t_0, +\infty)$ therefore we have

$$\forall s \geq t_0, \quad \frac{1+s}{L+\ell_\varepsilon^L(s)} \leq \frac{1+t_0}{L+\ell_\varepsilon^L(t_0)}, \quad \frac{1+s}{L+r_\varepsilon^R(s)} \leq \frac{1+t_0}{L+r_\varepsilon^R(t_0)}.$$

A Taylor expansion at $\varepsilon \to 0$ then shows that the two previous r.h.s are continuous and uniformly bounded by some constant independent of $\varepsilon \in (0,1)$: there exists $C(t_0) > 0$, independent of $\varepsilon > 0$ such that

$$\forall s \geq t_0, \quad \frac{1+s}{L+\ell_\varepsilon^L(s)} \leq C(t_0), \quad \frac{1+s}{L+r_\varepsilon^R(s)} \leq C(t_0).$$

The proof is then complete. □

Remark 4.2.22. We also have the following link between two EGC related to different parameters ε: if $t \geq t_0 > t^g_{\varepsilon_0}(L,R)$ then Lemma 4.2.21 ensures that $\mathrm{U} = 0$ satisfies $\mathrm{EGC}_{\varepsilon_0}^{L+\ell_{\varepsilon_0}^L(t), R+r_{\varepsilon_0}^R(t)}(t)$ and in addition, $\mathrm{U} = 0$ satisfies $\mathrm{EGC}_{\varepsilon}^{L+\ell_{\varepsilon_0}^L(t), R+r_{\varepsilon_0}^R(t)}(t)$ for any

$$\varepsilon \in \left(0, \frac{\varepsilon_0}{2\varepsilon_0+1}\right).$$

Indeed, if ε is given in the previous interval we know from Lemma 4.2.19 that $\mathrm{U} = 0$ satisfies

$$\mathrm{EGC}_{\varepsilon}^{L+\ell_{\varepsilon_0}^L(t), R+r_{\varepsilon_0}^R(t)}\Big(t^g_\varepsilon(L+\ell_{\varepsilon_0}^L(t), R+r_{\varepsilon_0}^R(t))\Big).$$

It thus remains to prove that

$$t^g_\varepsilon(L+\ell_{\varepsilon_0}^L(t), R+r_{\varepsilon_0}^R(t)) < t,$$

that is

$$\frac{L+\ell_{\varepsilon_0}^L(t)}{t} + \varepsilon\frac{1+R+r_{\varepsilon_0}^R(t)}{t} < 1.$$

We observe that $s \mapsto \dfrac{L+\ell_{\varepsilon_0}^L(s)}{s}$ is increasing on $\mathbb{R}^+$ and tends to $\dfrac{1}{2}$ as $t \to +\infty$, while $s \mapsto \dfrac{1+R+r_{\varepsilon_0}^R(s)}{s}$ is increasing on $\mathbb{R}^+$ and tends to $\dfrac{1}{2\varepsilon_0}$ as $t \to +\infty$, therefore

$$\frac{L+\ell_{\varepsilon_0}^L(t)}{t} + \varepsilon\frac{1+R+r_{\varepsilon_0}^R(t)}{t} < \frac{1}{2} + \varepsilon\frac{2\varepsilon_0+1}{2\varepsilon_0} < 1,$$

by our choice of ε_0.

4.3 Preliminary results on the solutions to the Vlasov-Navier-Stokes system

In this section, we mainly exhibit sufficient conditions ensuring the convergence of $(u_\varepsilon, \rho_\varepsilon)$ when $\varepsilon \to 0$, as well as some several non-uniform (in ε and T) estimates for the VNS system paving the way for a local in time analysis.

- In Subsection 4.3.1, we derive an improvement of the energy-dissipation inequality (4.1.23) by considering the contribution of the potential energy. We also state a conditional result about the polynomial decay of the fluid kinetic energy, whenever the Brinkman force enjoys some pointwise decay in L^2_x. We finally show how one can obtain the conclusion of Theorem 4.1.10, assuming some non-trivial controls on u_ε.

- Subsection 4.3.2 then introduces the definition of the so-called *strong existence times* which are useful to propagate extra regularity on the weak solutions to the Navier-Stokes equations. Such intervals of strong existence times make some additional integrability results on the system available.
- In Subsection 4.3.3, we mainly introduce the bootstrap strategy which will be at the heart of the proof of Theorems 4.1.10–4.1.11. To do so, we consider the greatest time until which the controls (4.2.5) and (4.2.15) on u_ε hold true.

4.3.1 Decay of the energy functionals and conditional results

First recall the definition (4.1.15) of the potential energy $\mathrm{E}^{\mathrm{p}}_\varepsilon$. We will use this functional to balance the last term coming from the gravity field in (4.1.23), thanks to the following lemma.

Lemma 4.3.1. *Let $\left((u_\varepsilon, f_\varepsilon)\right)_{\varepsilon>0}$ be a family of global weak solutions to the Vlasov-Navier-Stokes system. For any $t \geq 0$ and almost every $0 \leq s \leq t$ (including $s = 0$), we have for all $\varepsilon > 0$*

$$\mathrm{E}^{\mathrm{p}}_\varepsilon(t) \leq \mathrm{E}^{\mathrm{p}}_\varepsilon(s) + \int_s^t \int_{\mathbb{R}^3_+ \times \mathbb{R}^3} v_3 f_\varepsilon(\tau, x, v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau.$$

Proof. We rely on the strong stability results from DiPerna-Lions theory about transport equations, f_ε being a renormalized solution to the Vlasov equation (see e.g. [Mis00b]). We thus write the proof as if u_ε and f^0_ε were smooth and compactly supported. In particular, the characteristics curves are classicaly defined. Thanks to the representation formula given in Lemma 4.2.4, we have

$$f_\varepsilon(t,x,v) = e^{\frac{3t}{\varepsilon}} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,v) f^0_\varepsilon(\mathrm{X}_\varepsilon(0;t,x,v), \mathrm{V}_\varepsilon(0;t,x,v)).$$

This yields

$$\begin{aligned}\mathrm{E}^{\mathrm{p}}_\varepsilon(t) &= e^{\frac{3t}{\varepsilon}} \int_{\mathcal{O}^t_\varepsilon} x_3 f^0_\varepsilon(\mathrm{X}_\varepsilon(0;t,x,v), \mathrm{V}_\varepsilon(0;t,x,v))\,\mathrm{d}x\,\mathrm{d}v \\ &= \int_{\mathbb{R}^3_+ \times \mathbb{R}^3} \mathrm{X}_\varepsilon(t;0,x,v)_3\, \mathbf{1}_{\tau^+(0,x,v)>t}\, f^0_\varepsilon(x,v)\,\mathrm{d}x\,\mathrm{d}v,\end{aligned}$$

by the change of variables $(x,v) \mapsto (\mathrm{X}_\varepsilon(t;0,x,v), \mathrm{V}_\varepsilon(t;0,x,v))$ (see Proposition 4.2.1). In view of

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{X}_\varepsilon(\tau;0,x,v)_3 = \mathrm{V}_\varepsilon(\tau;0,x,v)_3,$$

we know that

$$\mathrm{X}_\varepsilon(t;0,x,v)_3 = \mathrm{X}_\varepsilon(s;0,x,v)_3 + \int_s^t \mathrm{V}_\varepsilon(\tau,0,x,v)_3\,\mathrm{d}\tau,$$

therefore for all $s < t$, we have by Fubini theorem

$$\begin{aligned}\mathrm{E}^{\mathrm{p}}_\varepsilon(t) &\leq \int_{\mathbb{R}^3_+ \times \mathbb{R}^3} \mathrm{X}_\varepsilon(s;0,x,v)_3 \mathbf{1}_{\tau^+(0,x,v)>t} f^0_\varepsilon(x,v)\,\mathrm{d}x\,\mathrm{d}v \\ &\qquad + \int_s^t \int_{\mathbb{R}^3_+ \times \mathbb{R}^3} \mathrm{V}_\varepsilon(\tau;0,x,v)_3\, \mathbf{1}_{\tau^+(0,x,v)>t}\, f^0_\varepsilon(x,v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau \\ &\leq \int_{\mathbb{R}^3_+ \times \mathbb{R}^3} \mathrm{X}_\varepsilon(s;0,x,v)_3\, \mathbf{1}_{\tau^+(0,x,v)>s}\, f^0_\varepsilon(x,v)\,\mathrm{d}x\,\mathrm{d}v \\ &\qquad + \int_s^t \int_{\mathbb{R}^3_+ \times \mathbb{R}^3} \mathrm{V}_\varepsilon(\tau;0,x,v)_3\, \mathbf{1}_{\tau^+(0,x,v)>\tau}\, f^0_\varepsilon(x,v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau.\end{aligned}$$

Performing the reverse changes of variable in the two last integrals, we eventually obtain the result. □

Combining Lemma 4.3.1 with the energy-dissipation inequality (4.1.23) satisfied by any weak solution to the system (in the sense of Definition 4.1.5), we obtain the following result.

Proposition 4.3.2. *Let $((u_\varepsilon, f_\varepsilon))_{\varepsilon>0}$ be a family of global weak solutions to the Vlasov-Navier-Stokes system. For any $t \geq 0$ and almost every $0 \leq s \leq t$ (including $s = 0$), we have for all $\varepsilon > 0$*

$$\mathcal{E}_\varepsilon(t) + \int_s^t \mathrm{D}_\varepsilon(\tau)\,\mathrm{d}\tau \leq \mathcal{E}_\varepsilon(s). \tag{4.3.1}$$

Following the seminal result of Wiegner [Wie87] and Borchers and Miyakawa [BM88], we now derive the following conditional result concerning the large time behavior of the fluid kinetic energy. Note that this result only deals with the Navier-Stokes part of the system, treating the Brinkman force as a fixed source term. We refer to Theorem 3.3.1 of Chapter 3.

Theorem 4.3.3. *Let $((u_\varepsilon, f_\varepsilon))_{\varepsilon>0}$ be a family of global weak solutions to the Vlasov-Navier-Stokes system. Let $T > 0$ and assume that there exists $\varepsilon_0 > 0$ such that*

$$\forall \varepsilon \in (0, \varepsilon_0), \quad \forall s \in [0, T], \quad \|j_\varepsilon(s) - \rho_\varepsilon u_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \frac{K}{(1+s)^{7/4}}, \tag{4.3.2}$$

for some some $K > 0$ independent of T and ε. Then there exists a nonnegative nondecreasing continuous function Ψ cancelling at 0 and independent of T and ε such that

$$\forall \varepsilon \in (0, \varepsilon_0), \quad \forall s \in [0, T], \quad \|u_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \frac{\Psi\left(\|u^0_\varepsilon\|^2_{\mathrm{L}^1 \cap \mathrm{L}^2(\mathbb{R}^3_+)} + K\right)}{(1+t)^{\frac{3}{2}}}. \tag{4.3.3}$$

Remark 4.3.4. In view of the improved energy-dissipation inequality (4.3.1), it may be reasonable to obtain a conditional (polynomial) decay result on the total energy $\mathcal{E}_\varepsilon$, which takes into account the whole coupling between the Vlasov equation and the Navier-Stokes equations. In the gravity-less case and in the whole space, this idea has been empowered by Han-Kwan in [HK22] under the condition of a (uniform) bound on ρ_ε in $\mathrm{L}^\infty_t \mathrm{L}^\infty_x$. This strategy relies on a fine algebraic structure of the whole system. However, in the gravity case on the half-space, an adaptation of this result would require an additional assumption of the potential energy which reads

$$\forall s \in [0, T], \quad \mathrm{E}^{\mathrm{p}}_\varepsilon(s) \lesssim \frac{1}{(1+s)^{3/2}}.$$

Indeed, it seems difficult to control the dissipation of the system from below by a part of the potential energy therefore one must assume *a priori* that this energy has some decay.

That is why we shall rather use the conditional result stated in Theorem 4.3.3, which only deals with the decay of the kinetic energy of the fluid part. As we shall see later on, this will be enough for our purpose.

We now state a conditional proposition which emphasizes some sufficient conditions leading to the proof of Theorem 4.1.10. We mainly combine a classical weak-compactness type argument with the conditional convergence of the Brinkman force provided by Corollary 4.2.15.

Proposition 4.3.5. *Let $((u_\varepsilon, f_\varepsilon))_{\varepsilon>0}$ be a family of global weak solutions to the Vlasov-Navier-Stokes system which are smooth and such that $u_\varepsilon \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{W}^{1,\infty}_0(\mathbb{R}^3_+))$. Let $T > 0$. Assume that* *for all $\varepsilon > 0$, we have*

$$\|\nabla_x u_\varepsilon\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{R}^3_+))} < \delta, \tag{C1}$$

$$\|\partial_t u_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} \leq M, \tag{C2}$$

$$\mathcal{E}_\varepsilon(0) + \|(1+|v|^2) f^0_\varepsilon\|_{\mathrm{L}^\infty\cap\mathrm{L}^1(\mathbb{R}^3_+\times\mathbb{R}^3)} \leq M, \tag{C3}$$

$$\|\nabla_x u_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{L}^\infty(\mathbb{R}^3_+))} \leq C_T, \tag{C4}$$

where $M > 1$ is independent of ε and T, where $C_T > 0$ is independent of ε and where $0 < \delta e^\delta < 1/9$. Then the convergence results stated in Theorem 4.1.10 hold true on $[0, T]$.

Proof. As the sequence $(u_\varepsilon)_\varepsilon$ is bounded in $\mathrm{L}^2(0,T;\mathrm{H}^1_0(\mathbb{R}^3_+))$ (thanks to the energy inequality (4.3.1) and (**C3**)) there exists $u \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\mathbb{R}^3_+))$ such that, up to a subsequence that we shall not denote here, we have

$$u_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} u, \quad \text{in} \quad w\text{-}\mathrm{L}^2(0,T;\mathrm{H}^1_0(\mathbb{R}^3_+)).$$

On the other hand, the sequence $(\rho_\varepsilon)_\varepsilon$ is bounded in $\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3_+))$ thanks to Corollary 4.2.5 and (**C1**)–(**C3**). Therefore there exists $\rho \in \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3_+))$ such that, up to a subsequence, we have

$$\rho_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \rho, \quad \text{in} \quad w^*\text{-}\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3_+)).$$

Now, let K be a compact subset of $\mathbb{R}^3_+$. By the Aubin-Lions lemma (which holds because $(\partial_t u_\varepsilon)_\varepsilon$ is bounded in $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))$ thanks to (**C2**)), we deduce that, up to another extraction, $(u_\varepsilon)_\varepsilon$ converges strongly to u in $\mathrm{L}^2((0,T)\times K)$. In particular, we have the convergence of the product

$$\rho_\varepsilon u_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \rho u, \quad \text{in} \quad w\text{-}\mathrm{L}^2((0,T)\times K).$$

In a second part, we use the conservation of mass for the particles which yields

$$\partial_t \rho_\varepsilon + \mathrm{div}_x\, j_\varepsilon = 0, \quad \text{in} \quad \mathscr{D}'([0,T)\times\mathbb{R}^3_+). \tag{4.3.4}$$

Using Corollary 4.2.15, we observe that the Brinkman force $(F_\varepsilon)_\varepsilon = (j_\varepsilon - \rho_\varepsilon u_\varepsilon)_\varepsilon$ converges to $-\rho e_3$ in $\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)$ when $\varepsilon \to 0$ (because of the energy inequality (4.3.1) and (**C1**)–(**C2**)–(**C3**)–(**C4**)) and thus in $\mathrm{L}^2((0,T)\times K)$. This implies that

$$j_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \rho(u - e_3), \quad \text{in} \quad w\text{-}\mathrm{L}^2((0,T)\times K).$$

This last convergence and the previous convergence of $(\rho_\varepsilon)_\varepsilon$ allow one to pass to the limit in the equation (4.3.4): we get

$$\partial_t \rho + \mathrm{div}_x\, [\rho(u - e_3)] = 0, \quad \text{in} \quad \mathscr{D}'((0,T)\times\mathbb{R}^3_+).$$

Finally, the aforementioned strong convergence of $(u_\varepsilon)_\varepsilon$ towards u is enough to classically pass to the limit in the divergence-free condition, in the l.h.s of the Navier-Stokes equations and in the source term $j_\varepsilon - \rho_\varepsilon u_\varepsilon$ thanks to the previous convergence of the Brinkman force. □

Remark 4.3.6. The previous result provides a guideline for the proof of Theorem 4.1.10 in large time.

In view of the Assumption **4.1.8** we shall make on the initial data, it goes without saying that the conditional hypothesis (**C1**) and (**C4**) of Proposition 4.3.5 are the most difficult to obtain and constitute the main issue of the analysis.

Thanks to an interpolation argument of the type

$$\|\nabla_x u_\varepsilon(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \lesssim \|\mathrm{D}^2_x u_\varepsilon(s)\|^{\beta_p}_{\mathrm{L}^p(\mathbb{R}^3_+)} \|u_\varepsilon(s)\|^{1-\beta_p}_{\mathrm{L}^2(\mathbb{R}^3_+)}, \quad p > 3, \quad \beta_p \in (0,1),$$

we will essentially prove that a decay of u_ε under the form

$$\forall s \in [0,T], \quad \|u_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \frac{1}{(1+t)^{\frac{3}{2}}}$$

shall imply (**C1**) and (**C4**) on $(0,T)$. In view of Theorem 4.3.3, this will be ensured provided that the Brinkman force satisfies a decay like

$$\forall s \in [0,T], \quad \|j_\varepsilon(s) - \rho_\varepsilon u_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \frac{K}{(1+s)^{7/4}}, \tag{C5}$$

where $K > 0$ is independent of T and ε.

Because of the slow decay in time of u_ε, we will actually need a refined argument requiring, through maximal regularity estimates (see Theorem 4.1.2), a polynomial decay in time of the Brinkman force of the form

$$\|(1+t)^\gamma (j_\varepsilon - \rho_\varepsilon u_\varepsilon)\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \lesssim K, \quad p > 3. \tag{C6}$$

In short, we have

$$(\mathbf{C5}) \text{ and } (\mathbf{C6}) \text{ on } (0,T) \Longrightarrow (\mathbf{C1}) \text{ and } (\mathbf{C4}) \text{ on } (0,T).$$

We refer to the beginning of Subsection 4.5.2 for more details about the previous implication.

According to the conditional Proposition 4.3.5, ensuring (**C5**)–(**C6**) is the key of our analysis and will lead to a proof of Theorem 4.1.10. The main mechanism leading to (**C5**)–(**C6**) will be the combination of the absorption boundary condition and the gravity effect.

4.3.2 Local estimates and strong existence times

We now introduce the notion of interval of strong existence for the Navier-Stokes equations. It will enable us to consider higher regularity estimates for the system, which will be crucial in the final bootstrap strategy. It is based on a parabolic smoothing effect for the equations and roughly states the instantaneous gain of two derivatives (in space) for a solution to the Navier-Stokes equations. This requires that the forcing term (i.e. the Brinkman force $j_\varepsilon - \rho_\varepsilon u_\varepsilon$) and the initial data enjoy some smallness properties. We refer to (a small variant of) the result stated in the Section 3.F in the Appendix of Chapter 3 for a proof.

Recall the notation

$$F_\varepsilon = j_\varepsilon - \rho_\varepsilon u_\varepsilon$$

for the Brinkman force, $(u_\varepsilon, f_\varepsilon)$ being any global weak solution to the VNS system.

Proposition 4.3.7. *There exists a universal constant* $\mathrm{C}_\star$ *such that the following holds. Consider a family of global weak solutions* $\left((u_\varepsilon, f_\varepsilon)\right)_{\varepsilon>0}$ *to the Vlasov-Navier-Stokes system. Assume that for some* $T > 0$*, one has*

$$\|u^0_\varepsilon\|^2_{\mathrm{H}^1(\mathbb{R}^3_+)} + \int_0^T \|F_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s + \int_0^T \|F_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s < \mathrm{C}_\star. \tag{4.3.5}$$

Then one has

$$\begin{aligned} u_\varepsilon &\in \mathrm{L}^\infty(0,T;\mathrm{H}^1(\mathbb{R}^3_+)) \cap \mathrm{L}^2(0,T;\mathrm{H}^2(\mathbb{R}^3_+)), \\ \partial_t u_\varepsilon &\in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+)), \end{aligned}$$

and for all $t \in [0,T]$

$$\|\nabla_x u_\varepsilon(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \int_0^t \|\mathrm{D}^2_x u_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s + \int_0^t \|\partial_t u_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s \\ \lesssim \|\nabla_x u^0_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \int_0^t \|F_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s, \quad (4.3.6)$$

where $\lesssim$ *only depends on* $\mathrm{C}_\star$.

Recall that we work under Assumption **A3-a** on the initial data ensuring that

$$\forall \varepsilon > 0, \qquad \|u^0_\varepsilon\|^2_{\mathrm{H}^1(\mathbb{R}^3_+)} < \frac{\mathrm{C}_\star}{2}.$$

In order to prove that the smallness condition (4.3.5) is satisfied for all times, we now introduce the notion of *strong existence times* for the Vlasov-Navier-Stokes system.

Definition 4.3.8 (Strong existence time). *Let* $\varepsilon > 0$. *A real number* $T \geq 0$ *is a* strong existence time *(for a global weak solution* $(u_\varepsilon, f_\varepsilon)$*) whenever the inequality*

$$\int_0^T \|F_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s + \int_0^T \|F_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s < \frac{\mathrm{C}_\star}{2},$$

holds.

Definition 4.3.9. *For all* $\varepsilon > 0$ *and* $T > 0$, *we set*

$$\Upsilon^0_\varepsilon(T) := \|u^\varepsilon_0\|^2_{\mathrm{H}^1(\mathbb{R}^3_+)} + \int_0^T \|F_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s + \int_0^T \|F_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s.$$

Remark 4.3.10. If T is a strong existence time in the sense of Definition 4.3.8, then

$$\forall \varepsilon > 0,\ \forall t \in [0,T], \quad \Upsilon^0_\varepsilon(t) < \mathrm{C}_\star,$$

which is therefore a uniform bound in ε and t. Thus, in what follows, we shall use the harmless notation Υ^0_ε without mentioning the time t.

Remark 4.3.11. Note that the parabolic smoothing used in [HKMar] for the torus case is rather based on the standard Fujita-Kato type smallness assumption for the Navier-Stokes system, namely requiring that the initial data u^0_ε is small in $\dot{\mathrm{H}}^{\frac{1}{2}}(\mathbb{T}^3)$. It should be possible to relax the H^1 assumption of (4.3.5) but, for the sake of simplicity, we have preferred avoiding such technical details.

A straightforward reformulation of Proposition 4.3.7 combined with Sobolev embedding leads to the following result.

Corollary 4.3.12. *For any finite strong existence times* $T > 0$ *of a global weak solution* $(u_\varepsilon, f_\varepsilon)$ *, we have*

$$\|\partial_t u_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} + \|\mathrm{D}^2_x u_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} \lesssim 2\mathrm{C}_\star,$$

$$\|u_\varepsilon\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^6(\mathbb{R}^3_+))} \lesssim \|\nabla_x u_\varepsilon\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^2(\mathbb{R}^3_+))} \lesssim \mathrm{C}_\star.$$

Remark 4.3.13. Let us explain how one can now make the computation of Subsection 4.2.3 rigorous, justifying in particular the integration by parts in time at the heart of the desingularization in ε of the Brinkman force (see (4.2.8)). The main point is that we shall perform this procedure on intervals of time which are strong existence times so that $\partial_t u_\varepsilon \in \mathrm{L}^2_T\mathrm{L}^2_x$ on theses intervals, in view of Proposition 4.3.7. The exponential factor in the integral involved in $\Gamma_\varepsilon^{t,x}$ being as smooth as we want, the computation is allowed thanks to well known properties of Sobolev functions in time with value in Banach spaces.

Dealing with strong existence times also provides some useful integrability estimates on the solutions to the Vlasov-Navier-Stokes system. We shall use the following ones.

Corollary 4.3.14. *Let T be a finite strong existence time of a global weak solution $(u_\varepsilon, f_\varepsilon)$. Then*

- *for any $p \in [1,6]$, we have*

$$j_\varepsilon - \rho_\varepsilon u_\varepsilon \in \mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+));$$

- *there exists $\varsigma > 0$ and $\mu > 0$ such that for all $p \in (3, 3+\varsigma)$, we have for all $t \in (0,T)$*

$$\|(u_\varepsilon \cdot \nabla_x)u_\varepsilon(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim (\Upsilon_\varepsilon^0)^{\frac{1}{2}}\mathcal{E}_\varepsilon(0)^{\frac{1}{4}}\|\mathrm{D}^2_x u_\varepsilon(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)},$$

$$\|(u_\varepsilon \cdot \nabla_x)u_\varepsilon(t)\|_{\mathrm{L}^p(\mathbb{R}^3_+)} \lesssim (\Upsilon_\varepsilon^0)^{\varsigma_p}\mathcal{E}_\varepsilon(0)^{\mu}\|\mathrm{D}^2_x u_\varepsilon(t)\|_{\mathrm{L}^p(\mathbb{R}^3_+)},$$

  for some $\varsigma_p > 0$;

- *for the exponent p given in Assumption **4.1.7**, we have*

$$(u_\varepsilon \cdot \nabla_x)u_\varepsilon \in \mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+)), \tag{4.3.7}$$

$$\partial_t u_\varepsilon,\, \mathrm{D}^2_x u_\varepsilon \in \mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+)); \tag{4.3.8}$$

- *we have*

$$\nabla_x u_\varepsilon \in \mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{R}^3_+)). \tag{4.3.9}$$

Proof. We refer to [HK22] and Chapter 3. Note that we use Assumption **A1-b** to ensure such integrability results. □

4.3.3 Bootstrap procedure

Before setting up a bootstrap procedure, we aim at obtaining further local in time integrability results. We mainly refer to Section 3.F of Chapter 3. Indeed, the proofs performed in Chapter 3 are the same, with a fixed parameter ε appearing at some points. All the local in time estimates actually blow up with $\varepsilon \to 0$ but this is harmless since we shall not use quantitative estimates for the moment, arguing only with integrability properties.

Proposition 4.3.15. *Let $\varepsilon > 0$. Suppose that $|v|^6 f^0_\varepsilon \in \mathrm{L}^1(\mathbb{R}^3_+ \times \mathbb{R}^3) < \infty$. For any global weak solution $(u_\varepsilon, f_\varepsilon)$, we have*

$$F_\varepsilon \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\mathbb{R}^3_+)), \tag{4.3.10}$$

$$u_\varepsilon \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^\infty(\mathbb{R}^3_+)), \tag{4.3.11}$$

$$\rho_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^\infty(\mathbb{R}^3_+)). \tag{4.3.12}$$

Remark 4.3.16. In particular, under Assumption **4.1.7**, for all $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that

$$\left(1+\sqrt{T_\varepsilon}\right)\int_0^{T_\varepsilon} \|F_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s < \frac{\mathrm{C}_\star}{2},$$

and this also means that for all $\varepsilon > 0$, there exists a positive strong existence time.

Until the end of this work, we consider a fixed family $((u_\varepsilon, f_\varepsilon))_{\varepsilon>0}$ of global weak solutions to the Vlasov-Navier-Stokes system, in the sense of Definition 4.1.5, associated to an admissible initial data $(u^0_\varepsilon, f^0_\varepsilon)$ and satisfying Assumptions 4.1.7–4.1.8–4.1.9.

In view of the conditional Proposition 4.3.5 and Remark 4.3.6, we will follow a strategy based on a bootstrap argument and which requires the following definition.

Definition 4.3.17. *Let $\alpha \in (0,1)$ be fixed. For any $\varepsilon > 0$, we set*

$$t^\star_\varepsilon := \sup\left\{\textit{strong existence times } t>0 \textit{ such that } \int_0^t \|u_\varepsilon(s)\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^3_+)}\,\mathrm{d}s < \delta^\star\right\}, \tag{4.3.13}$$

where $\delta^\star := \min(\kappa_\alpha, \delta)$ is defined in the following way: κ_α refers to the constant of Lemma 4.2.18 and δ is choosen such that $\delta e^\delta < \frac{1}{9}$ (see in particular Lemma 4.2.4).

Lemma 4.3.18. *For all $\varepsilon > 0$, we have $t^\star_\varepsilon > 0$.*

Proof. According to Remark 4.3.16, there exists a strong existence time $T_\varepsilon > 0$. By (4.3.9) of Corollary 4.3.14, we know that $\nabla_x u_\varepsilon \in \mathrm{L}^1(0,T_\varepsilon;\mathrm{L}^\infty(\mathbb{R}^3_+))$ while $u_\varepsilon \in \mathrm{L}^1(0,T_\varepsilon;\mathrm{L}^\infty(\mathbb{R}^3_+))$ by (4.3.11), therefore a continuity in time argument shows there exists $\underline{T_\varepsilon} \in (0,T_\varepsilon)$ such that

$$\int_0^{\underline{T_\varepsilon}} \|u_\varepsilon(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s < \frac{\delta^\star}{2}, \quad \int_0^{\underline{T_\varepsilon}} \|\nabla_x u_\varepsilon(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s < \frac{\delta^\star}{2}.$$

Since $\underline{T_\varepsilon}$ is still a strong existence time, this concludes the proof by definition of $t^\star_\varepsilon$. □

Our main goal is now to show that $t^\star_\varepsilon = +\infty$, at least for any ε small enough. This will require a finite number of use of Assumptions **4.1.8**–**4.1.9** bearing on the initial data (see in the following Sections 4.4–4.5) and we will be able to consider global weak solutions arising from such initial data.

4.4 Estimates and decay of the Brinkman force

The purpose of this section is twofold: having in mind the strategy described in Subsection 4.1.6 and in the end of Subsection 4.3.1 (see in particular Remark 4.3.6), we want to provide

- pointwise decay in time estimates for F_ε in L^2_x, thanks to the absorption effect at the boundary. Since we do not yet have access to the conditional decay in time of u_ε in L^2_x provided by Theorem 4.3.3, we shall rely on the energy inequality (4.3.1). This first step is performed in Subsection 4.4.1. Note that we shall start by this very first procedure in order to ensure the polynomial decay of the kinetic energy of the fluid afterwards.

- decay in time estimates for F_ε in $\mathrm{L}^p_t\mathrm{L}^p_x$, thanks to the absorption effect at the boundary and the polynomial decay in time of u_ε in L^2_x provided by Subsection 4.4.1 and Theorem 4.3.3. These estimates are derived in Subsection 4.4.2.

The main starting point in order to establish such estimates is the use of the Lagrangian framework of Section 4.2. We shall refine the computations of Subsection 4.2.3 for the Brinkman force. Note that the statements of that subsection only provided bounds ensuring the convergence of $F_\varepsilon + \rho_\varepsilon e_3$ when $\varepsilon \to 0$.

As in Subsection 4.2.3, we start writing

$$F_\varepsilon(t,x) = e^{\frac{3t}{\varepsilon}} \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w)) f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w) \left([\Gamma^{t,x}_\varepsilon]^{-1}(w) - u_\varepsilon(t,x)\right) |\det \mathrm{D}_w [\Gamma^{t,x}_\varepsilon]^{-1}(w)| \, \mathrm{d}w,$$

and perform a splitting of the integral thanks to the identity (4.2.8) on $[\Gamma^{t,x}_\varepsilon]^{-1}$. In view of Definition 4.3.17 and Lemma 4.2.4, this procedure will be valid for times $t < t^\star_\varepsilon$.

To go further and obtain some decay estimates from the previous expression of F_ε, we shall rely on the absorption condition at the boundary which is encoded in the indicator $\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w))$. Our strategy is crucially based upon the exit geometric condition of Subsection 4.2.4. It will provide some quantitative decay in time estimates for F_ε, thanks to the decay in the phase space of f^0_ε itself.

Use of the absorption.
In view of Lemma 4.2.21, assume that there exists $T_0 > 0$ such that for ε small enough, we have $T_0 < t^\star_\varepsilon$ and such that for all $t \in (T_0, t^\star_\varepsilon)$

$$u_\varepsilon \ \text{ satisfies } \ \mathrm{EGC}^{1+\mathrm{L}(t),1+\mathrm{R}(t)}_\varepsilon(t), \tag{4.4.1}$$

for some continuous and positive functions L and R satisfying

$$\forall t \in (T_0, t^\star_\varepsilon), \quad \frac{1}{1+\mathrm{L}(t)} \lesssim \frac{1}{1+t}, \quad \frac{1}{1+\mathrm{R}(t)} \lesssim \frac{1}{1+t}. \tag{4.4.2}$$

Then, according to Proposition 4.2.17, we can write

$$f_\varepsilon(t,x,v) \leq f^\natural_\varepsilon(t,x,v) + f^\flat_\varepsilon(t,x,v),$$

where

$$f^\natural_\varepsilon(t,x,v) := e^{\frac{3t}{\varepsilon}} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,v) \, \mathbf{1}_{|\mathrm{V}_\varepsilon(0;t,x,v)|>1+\mathrm{R}(t)} \, f^0_\varepsilon(\mathrm{X}_\varepsilon(0;t,x,v), \mathrm{V}_\varepsilon(0;t,x,v)),$$
$$f^\flat_\varepsilon(t,x,v) := e^{\frac{3t}{\varepsilon}} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,v) \, \mathbf{1}_{\mathrm{X}_\varepsilon(0;t,x,v)_3>1+\mathrm{L}(t)} \, f^0_\varepsilon(\mathrm{X}_\varepsilon(0;t,x,v), \mathrm{V}_\varepsilon(0;t,x,v)).$$

From the previous splitting, we can infer

$$\begin{aligned} |F_\varepsilon(t,x)| &\leq \int_{\mathbb{R}^3} f^\natural_\varepsilon(t,x,v)|v-u_\varepsilon(t,x)| \, \mathrm{d}v + \int_{\mathbb{R}^3} f^\flat_\varepsilon(t,x,v)|v-u_\varepsilon(t,x)| \, \mathrm{d}v \\ &= \int_{\mathbb{R}^3} e^{\frac{3t}{\varepsilon}} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,v) \, \mathbf{1}_{|\mathrm{V}_\varepsilon(0;t,x,v)|>1+\mathrm{R}(t)} \, f^0_\varepsilon(\mathrm{X}_\varepsilon(0;t,x,v), \mathrm{V}_\varepsilon(0;t,x,v))|v-u_\varepsilon(t,x)| \, \mathrm{d}v \\ &\quad + \int_{\mathbb{R}^3} e^{\frac{3t}{\varepsilon}} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,v) \, \mathbf{1}_{\mathrm{X}_\varepsilon(0;t,x,v)_3>1+\mathrm{L}(t)} \, f^0_\varepsilon(\mathrm{X}_\varepsilon(0;t,x,v), \mathrm{V}_\varepsilon(0;t,x,v))|v-u_\varepsilon(t,x)| \, \mathrm{d}v. \end{aligned}$$

Arguing exactly as in Subsection 4.2.3, we have the following splitting lemma for the Brinkman force F_ε, which in the same spirit as that of Lemma 4.2.11..

Lemma 4.4.1. *Assume that* (4.4.1)–(4.4.2) *hold with respect to a time* T_0. *For any* $\varepsilon > 0$, *if* $T \in (T_0, t^\star_\varepsilon)$ *is a strong existence time, then for all* $(t,x) \in (T_0,T) \times \mathbb{R}^3_+$

$$|F_\varepsilon(t,x)| \lesssim \sum_{i=0}^{2} F^{\natural,i}_\varepsilon(t,x) + \sum_{i=0}^{2} F^{\flat,i}_\varepsilon(t,x),$$

where

$$
\begin{aligned}
&F_\varepsilon^{\natural,0}(t,x) \\
&:= e^{-\frac{t}{\varepsilon}} \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))\, \mathbf{1}_{|w|>1+\mathrm{R}(t)}\, f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w) \Big| w+e_3-(Pu_\varepsilon)(0,\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w)) \Big| \, \mathrm{d}w \\
&\quad + \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))\, \mathbf{1}_{|w|>1+\mathrm{R}(t)}\, f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w)\, \mathrm{d}w, \\
&F_\varepsilon^{\natural,1}(t,x) \\
&:= \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))\, \mathbf{1}_{|w|>1+\mathrm{R}(t)} \int_0^t e^{\frac{\tau-t}{\varepsilon}} f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w) \Big| \partial_\tau [Pu_\varepsilon](\tau,\widetilde{\mathrm{X}}_\varepsilon^{\tau;t}(x,w)) \Big| \, \mathrm{d}\tau\, \mathrm{d}w, \\
&F_\varepsilon^{\natural,2}(t,x) := \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))\, \mathbf{1}_{|w|>1+\mathrm{R}(t)} \int_0^t e^{\frac{\tau-t}{\varepsilon}} f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w) \\
&\qquad\qquad \times \Big| \Big(\mathrm{V}_\varepsilon(\tau;t,x,[\Gamma_\varepsilon^{t,x}]^{-1}(w)) \cdot \nabla_x \Big) [Pu_\varepsilon](\tau,\widetilde{\mathrm{X}}_\varepsilon^{\tau;t}(x,w)) \Big| \, \mathrm{d}\tau\, \mathrm{d}w,
\end{aligned}
$$

and

$$
\begin{aligned}
&F_\varepsilon^{\flat,0}(t,x) \\
&:= e^{-\frac{t}{\varepsilon}} \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))\, \mathbf{1}_{\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w)_3>1+\mathrm{L}(t)}\, f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w) \Big| w+e_3-(Pu_\varepsilon)(0,\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w)) \Big| \, \mathrm{d}w \\
&\quad + \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))\, \mathbf{1}_{\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w)_3>1+\mathrm{L}(t)}\, f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w)\, \mathrm{d}w, \\
&F_\varepsilon^{\flat,1}(t,x) \\
&:= \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))\, \mathbf{1}_{\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w)_3>1+\mathrm{L}(t)} \int_0^t e^{\frac{\tau-t}{\varepsilon}} f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w) \Big| \partial_\tau [Pu_\varepsilon](\tau,\widetilde{\mathrm{X}}_\varepsilon^{\tau;t}(x,w)) \Big| \, \mathrm{d}\tau\, \mathrm{d}w, \\
&F_\varepsilon^{\flat,2}(t,x) := \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))\, \mathbf{1}_{\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w)_3>1+\mathrm{L}(t)} \int_0^t e^{\frac{\tau-t}{\varepsilon}} f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w) \\
&\qquad\qquad \times \Big| \Big(\mathrm{V}_\varepsilon(\tau;t,x,[\Gamma_\varepsilon^{t,x}]^{-1}(w)) \cdot \nabla_x \Big) [Pu_\varepsilon](\tau,\widetilde{\mathrm{X}}_\varepsilon^{\tau;t}(x,w)) \Big| \, \mathrm{d}\tau\, \mathrm{d}w.
\end{aligned}
$$

We shall estimate the contribution of all these terms, by performing a change of variable in space based on Lemma 4.2.8.

Choice of the initial time.
In this entire section, we consider a time $0 \leq T_0 < t_\varepsilon^\star$ (for ε small enough) such that all the subsequent estimates are performed on $(T_0, t_\varepsilon^\star)$. There are mainly two cases:

- if $T_0 = 0$, we do not use the absorption effect and do not rely on the exit geometric condition (namely by dropping the indicators involving $\mathrm{L}(t)$ and $\mathrm{R}(t)$ from the previous formulas).
- if $0 < T_0 < t_\varepsilon^\star$ is such that the general conditions of absorption (4.4.1) and (4.4.2) are satisfied, we shall refer to T_0 as *a starting time of absorption* (with respect to the functions L and R). Of course, this is the interesting case in which we hope for some decay estimates to hold (in large time). Note that if we restrict ourselves to uniform in time bounds (without any decay), this procedure comes back to the case $T_0 = 0$.

Later on, we shall specify a time after which the absorption effect can be handled (see Definition 4.5.5). In what follows, we will mainly state all the results on $(T_0, t_\varepsilon^\star)$ where T_0 is a starting time of absorption. When it is useful for our purpose, we shall mention the result in the case $T_0 = 0$.

4.4.1 Pointwise in time estimates of the Brinkman force in L^2_x

In the current subsection, we are not yet allowed to use the conditional decay of the kinetic energy of the fluid stated in Theorem 4.3.3. Nevertheless, we will obtain pointwise decay estimates for the Brinkman force, the main tool being the energy inequality (4.3.1).

Lemma 4.4.2. *For all $t \in (T_0, t_\varepsilon^\star)$ and any $k \geq 0$, we have*

$$\begin{aligned}
&\|F_\varepsilon^{\natural,0}(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \\
&\lesssim \frac{e^{\frac{-t}{\varepsilon}}}{(1+t)^k} \||v|^k f_\varepsilon^0\|^{\frac{1}{2}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \Big[\|(1+|v|^{k+2}) f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3_+\times\mathbb{R}^3)} + \||v|^k f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \|u_\varepsilon^0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\Big]^{\frac{1}{2}} \\
&\quad + \frac{1}{(1+t)^k} \||v|^k f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))}
\end{aligned}$$

and

$$\begin{aligned}
&\|F_\varepsilon^{\flat,0}(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \\
&\lesssim \frac{e^{\frac{-t}{\varepsilon}}}{(1+t)^k} \|x_3^k f_\varepsilon^0\|^{\frac{1}{2}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \Big[\|(1+|v|^2) x_3^k f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3_+\times\mathbb{R}^3)} + \|x_3^k f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \|u_\varepsilon^0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\Big]^{\frac{1}{2}} \\
&\quad + \frac{1}{(1+t)^k} \|x_3^k f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))}.
\end{aligned}$$

Proof. We focus on the proof of the first estimate, the proof of the second one being similar. We have for all $t \in (T_0, t_\varepsilon^\star)$

$$\begin{aligned}
\|F_\varepsilon^{\natural,0}(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \frac{e^{-\frac{t}{\varepsilon}}}{(1+R(t))^k} \Bigg[\int_{\mathbb{R}^3_+} \Bigg| \int_{\mathbb{R}^3} &\mathbf{1}_{\mathcal{O}_\varepsilon^t}(x, [\Gamma_\varepsilon^{t,x}]^{-1}(w)) |w|^k f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w), w) \\
&\times \Big| w + e_3 - (Pu_\varepsilon)(0, \widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w))\Big| \, \mathrm{d}w \Bigg|^2 \mathrm{d}x\Bigg]^{\frac{1}{2}} \\
+ \frac{1}{(1+R(t))^k} \Bigg[\int_{\mathbb{R}^3_+} \Bigg(\int_{\mathbb{R}^3} &\mathbf{1}_{\mathcal{O}_\varepsilon^t}(x, [\Gamma_\varepsilon^{t,x}]^{-1}(w)) |w|^k f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w), w) \, \mathrm{d}w\Bigg)^2 \mathrm{d}x\Bigg]^{\frac{1}{2}}.
\end{aligned}$$

For the first term, we apply Hölder's inequality in velocity and write as in Lemma 4.2.12

$$\begin{aligned}
&\int_{\mathbb{R}^3_+} \left(\int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x, [\Gamma_\varepsilon^{t,x}]^{-1}(w)) |w|^k f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w), w) \Big| w + e_3 - (Pu_\varepsilon)(0, \widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w))\Big| \, \mathrm{d}w\right)^2 \mathrm{d}x \\
&\leq \||v|^k f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \\
&\qquad \times \int_{\mathbb{R}^3_+\times\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x, [\Gamma_\varepsilon^{t,x}]^{-1}(w)) |w|^k f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w), w) \Big| w + e_3 - (Pu_\varepsilon)(0, \widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w))\Big|^2 \mathrm{d}w \, \mathrm{d}x.
\end{aligned}$$

We then perform the change of variable in space $x \mapsto \widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w)$ (see Lemma 4.2.8) and get for all $w \in \mathbb{R}^3$

$$\begin{aligned}
&\int_{\mathbb{R}^3_+} \left(\int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x, [\Gamma_\varepsilon^{t,x}]^{-1}(w)) |w|^k f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w), w) \Big| w + e_3 - (Pu_\varepsilon)(0, \widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w))\Big| \, \mathrm{d}w\right)^2 \mathrm{d}x \\
&\qquad \leq \||v|^k f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \Big[\|(1+|v|^{k+2}) f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3_+\times\mathbb{R}^3)} + \||v|^k f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \|u_\varepsilon^0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\Big].
\end{aligned}$$

as in Lemma 4.2.12. We have thus obtained the claimed estimate coming from the first term.

For the second term, we apply the generalized Minkowski inequality (see e.g. [HLP88]) and get

$$\left[\int_{\mathbb{R}^3_+}\left(\int_{\mathbb{R}^3}\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))|w|^k f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)\,\mathrm{d}w\right)^2\mathrm{d}x\right]^{\frac{1}{2}}$$
$$\leq \int_{\mathbb{R}^3}|w|^k\left(\int_{\mathbb{R}^3_+}\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)^2\,\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}w \lesssim \int_{\mathbb{R}^3}|w|^k\|f^0_\varepsilon(\cdot,w)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}w,$$

where we have performed the same procedure as above thanks to the change of variable in space $x\mapsto\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w)$.

Adding the two previous contributions concludes the proof of the lemma, thanks to (4.4.2). □

Remark 4.4.3. There is a variant of the previous proof concerning the treatment of the first term and which leads to a slightly different conclusion. We first write

$$\int_{\mathbb{R}^3_+}\left(\int_{\mathbb{R}^3}\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))|w|^k f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)\left|w+e_3-(Pu_\varepsilon)(0,\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w))\right|\mathrm{d}w\right)^2\mathrm{d}x$$
$$\lesssim \int_{\mathbb{R}^3_+}\left(\int_{\mathbb{R}^3}\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))|w|^k f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)|w+e_3|\,\mathrm{d}w\right)^2\mathrm{d}x$$
$$+\int_{\mathbb{R}^3_+}\left(\int_{\mathbb{R}^3}\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))|w|^k f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)\left|(Pu_\varepsilon)(0,\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w))\right|\mathrm{d}w\right)^2\mathrm{d}x.$$

For the first of these two terms, we apply the generalized Minkowski inequality and obtain, using again the change of variable in space $x\mapsto\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w)$

$$\left(\int_{\mathbb{R}^3_+}\left(\int_{\mathbb{R}^3}\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))|w|^k f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)|w+e_3|\,\mathrm{d}w\right)^2\mathrm{d}x\right)^{\frac{1}{2}}$$
$$\leq\int_{\mathbb{R}^3}(1+|w|)|w|^k\left(\int_{\mathbb{R}^3_+}\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)^2\,\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}w$$
$$\lesssim\int_{\mathbb{R}^3}(1+|w|)|w|^k\|f^0_\varepsilon(\cdot,w)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}w.$$

For the second term, we proceed as in the original proof. We end up with the following conclusion:

$$\|F^{\natural,0}_\varepsilon(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\lesssim\frac{e^{\frac{-t}{\varepsilon}}}{(1+t)^k}\Big[\|(1+|v|^{k+1})f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))}+\||v|^kf^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|u^0_\varepsilon\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\Big]$$
$$+\frac{1}{(1+t)^k}\||v|^kf^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))},$$

and, in a similar way

$$\|F^{\flat,0}_\varepsilon(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\lesssim\frac{e^{\frac{-t}{\varepsilon}}}{(1+t)^k}\Big[\|(1+|v|)\,x_3^k\,f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))}+\|x_3^kf^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|u^0_\varepsilon\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\Big]$$
$$+\frac{1}{(1+t)^k}\|x_3^k\,f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))}.$$

Lemma 4.4.4. *For all $t \in (T_0, t_\varepsilon^\star)$ and any $k > 0$, we have*

$$\|F_\varepsilon^{\natural,1}(t)\|_{L^2(\mathbb{R}^3_+)} \lesssim \frac{\varepsilon^{\frac{1}{2}}}{(1+t)^k}\||v|^k f_\varepsilon^0\|_{L^1(\mathbb{R}^3;L^\infty(\mathbb{R}^3_+))}\|\partial_\tau u_\varepsilon\|_{L^2(0,t;L^2(\mathbb{R}^3_+))},$$
$$\|F_\varepsilon^{\flat,1}(t)\|_{L^2(\mathbb{R}^3_+)} \lesssim \frac{\varepsilon^{\frac{1}{2}}}{(1+t)^k}\|x_3^k f_\varepsilon^0\|_{L^1(\mathbb{R}^3;L^\infty(\mathbb{R}^3_+))}\|\partial_\tau u_\varepsilon\|_{L^2(0,t;L^2(\mathbb{R}^3_+))}.$$

Proof. We focus on the treatment of the first estimate, the second one being similar. We first use the Hölder's inequality in velocity and time in Lemma 4.2.13 and get

$$\begin{aligned}
&\int_{\mathbb{R}^3_+} |F_\varepsilon^{\natural,1}(t,x)|^2 \,\mathrm{d}x \\
&\leq \frac{1}{(1+\mathrm{R}(t))^{2k}}\int_{\mathbb{R}^3_+}\left(\int_{\mathbb{R}^3}\int_0^t e^{\frac{\tau-t}{\varepsilon}}\mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))|w|^k f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w)\,\mathrm{d}\tau\,\mathrm{d}w\right) \\
&\quad\times\left(\int_{\mathbb{R}^3}\int_0^t e^{\frac{\tau-t}{\varepsilon}}\mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))|w|^k f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w)\left|\partial_\tau[Pu_\varepsilon](\tau,\widetilde{\mathrm{X}}_\varepsilon^{\tau;t}(x,w))\right|^2\mathrm{d}\tau\,\mathrm{d}w\right)\mathrm{d}x \\
&\leq \frac{\varepsilon\||v|^k f_\varepsilon^0\|_{L^1(\mathbb{R}^3;L^\infty(\mathbb{R}^3_+))}}{(1+t)^{2k}} \\
&\quad\times\int_{\mathbb{R}^3}|w|^k\int_0^t e^{\frac{\tau-t}{\varepsilon}}\left(\int_{\mathbb{R}^3_+}\mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w)\right. \\
&\qquad\qquad\qquad\left.\times\left|\partial_\tau[Pu_\varepsilon](\tau,\widetilde{\mathrm{X}}_\varepsilon^{\tau;t}(x,w))\right|^2\mathrm{d}x\right)\mathrm{d}\tau\,\mathrm{d}w,
\end{aligned}$$

thanks to Fubini theorem and the same procedure as in the proof of Lemma 4.4.2. By the change of variable $x' = \widetilde{\mathrm{X}}_\varepsilon^{\tau;t}(x,w)$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3_+}|F_\varepsilon^{\natural,1}(t,x)|^2\,\mathrm{d}x \\
&\lesssim \frac{\varepsilon\||w|^k f_\varepsilon^0\|_{L^1(\mathbb{R}^3;L^\infty(\mathbb{R}^3_+))}}{(1+\mathrm{R}(t))^{2k}}\int_{\mathbb{R}^3}\int_0^t e^{\frac{\tau-t}{\varepsilon}}|w|^k\|f_\varepsilon^0(\cdot,w)\|_{L^\infty(\mathbb{R}^3_+)}\|\partial_\tau[Pu_\varepsilon](\tau)\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau\,\mathrm{d}w \\
&\leq \frac{\varepsilon\||w|^k f_\varepsilon^0\|^2_{L^1(\mathbb{R}^3;L^\infty(\mathbb{R}^3_+))}}{(1+\mathrm{R}(t))^{2k}}\|P[\partial_t u_\varepsilon]\|^2_{L^2((0,t)\times\mathbb{R}^3)},
\end{aligned}$$

which leads to the conclusion thanks to the definition of the extension operator P. □

Lemma 4.4.5. *For all $t \in (T_0, t_\varepsilon^\star)$ and any $k_1, k_2 \geq 0$, we have*

$$\begin{aligned}
&\|F_\varepsilon^{\natural,2}(t)\|_{L^2(\mathbb{R}^3_+)} \\
&\lesssim \frac{1}{(1+t)^{k_1}}\Big[\varepsilon^{\frac{3}{4}}\|(1+|v|^2)|v|^{k_1}f_\varepsilon^0\|^{\frac{1}{2}}_{L^1(\mathbb{R}^3;L^\infty(\mathbb{R}^3_+))}\||v|^{k_1}f_\varepsilon^0\|^{\frac{1}{2}}_{L^1(\mathbb{R}^3;L^\infty(\mathbb{R}^3_+))}\mathcal{E}_\varepsilon(0)^{\frac{1}{4}}\|\mathrm{D}_x^2 u_\varepsilon\|^{\frac{1}{2}}_{L^2((0,t)\times\mathbb{R}^3_+)} \\
&\qquad\qquad + \varepsilon^{\frac{3}{4}}\||v|^{k_1}f_\varepsilon^0\|_{L^1(\mathbb{R}^3;L^\infty(\mathbb{R}^3_+))}\mathcal{E}_\varepsilon(0)^{\frac{1}{4}}\|\mathrm{D}_x^2 u_\varepsilon\|^{\frac{1}{2}}_{L^2((0,t)\times\mathbb{R}^3_+)}\Big] \\
&\quad + \frac{1}{(1+t)^{k_2-\frac{1}{4}}}\||v|^{k_2}f_\varepsilon^0\|_{L^1(\mathbb{R}^3;L^\infty(\mathbb{R}^3_+))}\mathcal{E}_\varepsilon(0)^{\frac{1}{2}}\|u_\varepsilon\|^{\frac{1}{2}}_{L^\infty(0,t;L^6(\mathbb{R}^3_+))}\|\mathrm{D}_x^2 u_\varepsilon\|^{\frac{1}{2}}_{L^2(0,t;L^2(\mathbb{R}^3_+))},
\end{aligned}$$

and

$$\begin{aligned}
&\|F_\varepsilon^{\flat,2}(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\\
&\lesssim \frac{1}{(1+t)^{k_1}}\Big[\varepsilon^{\frac34}\|(1+|v|^2)x_3^{k_1}f_\varepsilon^0\|^{\frac12}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|x_3^{k_1}f_\varepsilon^0\|^{\frac12}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\mathcal{E}_\varepsilon(0)^{\frac14}\|\mathrm{D}_x^2u_\varepsilon\|^{\frac12}_{\mathrm{L}^2((0,t)\times\mathbb{R}^3_+)}\\
&\qquad\qquad+\varepsilon^{\frac34}\|x_3^{k_1}f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\mathcal{E}_\varepsilon(0)^{\frac14}\|\mathrm{D}_x^2u_\varepsilon\|^{\frac12}_{\mathrm{L}^2((0,t)\times\mathbb{R}^3_+)}\Big]\\
&\quad+\frac{1}{(1+t)^{k_2-\frac14}}\|x_3^{k_2}f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\mathcal{E}_\varepsilon(0)^{\frac12}\|u_\varepsilon\|^{\frac12}_{\mathrm{L}^\infty(0,t;\mathrm{L}^6(\mathbb{R}^3_+))}\|\mathrm{D}_x^2u_\varepsilon\|^{\frac12}_{\mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}^3_+))}.
\end{aligned}$$

Proof. We focus on the first estimate as in the previous proofs. Thanks to the indicator in velocity in the definition of $F_\varepsilon^{\natural,2}$, we can write

$$\|F_\varepsilon^{\natural,2}(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\lesssim\frac{1}{(1+t)^k}\Big[\|\mathrm{I}^\natural(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}+\|\mathrm{II}^\natural(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}+\|\mathrm{III}^\natural(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\Big],$$

where

$$\begin{aligned}
\mathrm{I}^\natural(t,x):=\int_{\mathbb{R}^3}\mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))|w|^k\int_0^t e^{\frac{-t}{\varepsilon}}f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w)(1+|w|)&\\
\times\left|\nabla_x[Pu_\varepsilon](s,\widetilde{\mathrm{X}}_\varepsilon^{s;t}(x,w))\right|\mathrm{d}s\,\mathrm{d}w&,\\
\mathrm{II}^\natural(t,x):=\int_{\mathbb{R}^3}\mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))|w|^k\int_0^t e^{\frac{s-t}{\varepsilon}}f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w)\left|\nabla_x[Pu_\varepsilon](s,\widetilde{\mathrm{X}}_\varepsilon^{s;t}(x,w))\right|\mathrm{d}s\,\mathrm{d}w&,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{III}^\natural(t,x):=\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))|w|^k\int_0^t\int_0^s e^{\frac{s-t}{\varepsilon}}e^{\frac{\tau-s}{\varepsilon}}f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w)&\\
\times\left|(Pu_\varepsilon)(\tau,\widetilde{\mathrm{X}}_\varepsilon^{\tau;t}(x,w))\right|\left|\nabla_x[Pu_\varepsilon](s,\widetilde{\mathrm{X}}_\varepsilon^{s;t}(x,w))\right|\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}w&.
\end{aligned}$$

Here, we have used the formula (4.2.9). The two first terms $\mathrm{I}^\natural(t)$ and $\mathrm{II}^\natural(t)$ are handled in the same way in the proof of Lemma 4.2.14, taking into account the additional polynomial in velocity of degree k. By the energy inequality (4.3.1), we obtain

$$\begin{aligned}
\|\mathrm{I}^\natural(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}&\lesssim\varepsilon^{\frac34}\|(1+|v|^2)|v|^kf_\varepsilon^0\|^{\frac12}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\||v|^kf_\varepsilon^0\|^{\frac12}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\mathcal{E}_\varepsilon(0)^{\frac14}\|\mathrm{D}_x^2u_\varepsilon\|^{\frac12}_{\mathrm{L}^2((0,t)\times\mathbb{R}^3_+)},\\
\|\mathrm{II}^\natural(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}&\lesssim\varepsilon^{\frac34}\||v|^kf_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\mathcal{E}_\varepsilon(0)^{\frac14}\|\mathrm{D}_x^2u_\varepsilon\|^{\frac12}_{\mathrm{L}^2((0,t)\times\mathbb{R}^3_+)}.
\end{aligned}$$

For the last term $\mathrm{III}^\natural(t)$, we start writing

$$\begin{aligned}
&|\mathrm{III}^\natural(t,x)|^2\\
&\leq\varepsilon^{-1}\left(\int_0^t\int_{\mathbb{R}^3}\mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))|w|^kf_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w)e^{\frac{s-t}{\varepsilon}}\left|\nabla_x[Pu_\varepsilon](s,\widetilde{\mathrm{X}}_\varepsilon^{s;t}(x,w))\right|^2\mathrm{d}s\,\mathrm{d}w\right)\\
&\left(\int_0^t\int_{\mathbb{R}^3}\mathbf{1}_{\mathcal{O}_\varepsilon^t}(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))|w|^kf_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w),w)e^{\frac{s-t}{\varepsilon}}\left(\int_0^s e^{\frac{\tau-s}{\varepsilon}}\left|(Pu_\varepsilon)(\tau,\widetilde{\mathrm{X}}_\varepsilon^{\tau;t}(x,w))\right|^2\mathrm{d}\tau\right)\mathrm{d}s\,\mathrm{d}w\right),
\end{aligned}$$

where we have used the same manipulations than in Lemma 4.2.14. Since

$$(x,[\Gamma_\varepsilon^{t,x}]^{-1}(w))\in\mathcal{O}_\varepsilon^t\Longrightarrow\forall s\in[0,t],\ \widetilde{\mathrm{X}}_\varepsilon(s;t,x,w)\in\mathbb{R}^3_+,$$

we have

$$\begin{aligned}
&\int_{\mathbb{R}^3_+} |\mathrm{III}^\natural(t,x)|^2 \,\mathrm{d}x \\
&\lesssim \varepsilon^{-1} \| |v|^k f^0_\varepsilon \|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \left(\int_0^t e^{\frac{s-t}{\varepsilon}} \left(\int_0^s e^{\frac{\tau-s}{\varepsilon}} \|u_\varepsilon(\tau)\|^2_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \,\mathrm{d}\tau \right) \mathrm{d}s \right) \\
&\times \left(\int_0^t \int_{\mathbb{R}^3_+\times\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w)) |w|^k f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w), w) e^{\frac{s-t}{\varepsilon}} \left| \nabla_x [P u_\varepsilon](s, \widetilde{\mathrm{X}}^{s;t}_\varepsilon(x,w)) \right|^2 \mathrm{d}x \,\mathrm{d}w \,\mathrm{d}s \right).
\end{aligned}$$

As above, we then use the change of variable in space $x' = \widetilde{\mathrm{X}}^{s;t}_\varepsilon(x,w)$ given by Lemma 4.2.8 and we obtain

$$\begin{aligned}
\int_{\mathbb{R}^3_+} |\mathrm{III}^\natural(t,x)|^2 \,\mathrm{d}x \lesssim \varepsilon^{-1} \| |v|^k f^0_\varepsilon \|^2_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} &\left(\int_0^t e^{\frac{s-t}{\varepsilon}} \left(\int_0^s e^{\frac{\tau-s}{\varepsilon}} \|u_\varepsilon(\tau)\|^2_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \,\mathrm{d}\tau \right) \mathrm{d}s \right) \\
&\times \left(\int_0^t e^{\frac{s-t}{\varepsilon}} \|\nabla_x u_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s \right).
\end{aligned}$$

Using

$$\|u_\varepsilon(\tau)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \lesssim \|\mathrm{D}^2_x u_\varepsilon(\tau)\|^{\frac12}_{\mathrm{L}^2(\mathbb{R}^3_+)} \|u_\varepsilon(\tau)\|^{\frac12}_{\mathrm{L}^6(\mathbb{R}^3_+)},$$

we end up with

$$\begin{aligned}
&\int_{\mathbb{R}^3_+} |\mathrm{III}^\natural(t,x)|^2 \,\mathrm{d}x \\
&\lesssim \varepsilon^{-1} \| |v|^k f^0_\varepsilon \|^2_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \left(\int_0^t e^{\frac{s-t}{\varepsilon}} \left(\int_0^s \|\mathrm{D}^2_x u_\varepsilon(\tau)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \|u_\varepsilon(\tau)\|_{\mathrm{L}^6(\mathbb{R}^3_+)} \,\mathrm{d}\tau \right) \mathrm{d}s \right) \\
&\quad \times \left(\int_0^t e^{\frac{s-t}{\varepsilon}} \|\nabla_x u_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s \right) \\
&\lesssim \varepsilon^{-1} \| |v|^k f^0_\varepsilon \|^2_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \left(\int_0^t e^{\frac{s-t}{\varepsilon}} \|u_\varepsilon\|_{\mathrm{L}^\infty(0,s;\mathrm{L}^6(\mathbb{R}^3_+))} \sqrt{s} \|\mathrm{D}^2_x u_\varepsilon\|_{\mathrm{L}^2(0,s;\mathrm{L}^2(\mathbb{R}^3_+))} \,\mathrm{d}s \right) \\
&\quad \times \left(\int_0^t \mathrm{D}_\varepsilon(s) \,\mathrm{d}s \right) \\
&\lesssim \varepsilon^{-1} \| |v|^k f^0_\varepsilon \|^2_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \mathcal{E}_\varepsilon(0) \|u_\varepsilon\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^6(\mathbb{R}^3_+))} \|\mathrm{D}^2_x u_\varepsilon\|_{\mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}^3_+))} \left(\int_0^t e^{\frac{s-t}{\varepsilon}} \sqrt{s} \,\mathrm{d}s \right) \\
&\lesssim \| |v|^k f^0_\varepsilon \|^2_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \mathcal{E}_\varepsilon(0) \|u_\varepsilon\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^6(\mathbb{R}^3_+))} \|\mathrm{D}^2_x u_\varepsilon\|_{\mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}^3_+))} \sqrt{t},
\end{aligned}$$

where we have used the energy inequality (4.3.1) (recall the Definition (4.1.17) of the dissipation). This entails

$$\|\mathrm{III}^\natural(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \| |v|^k f^0_\varepsilon \|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \mathcal{E}_\varepsilon(0)^{\frac12} \|u_\varepsilon\|^{\frac12}_{\mathrm{L}^\infty(0,t;\mathrm{L}^6(\mathbb{R}^3_+))} \|\mathrm{D}^2_x u_\varepsilon\|^{\frac12}_{\mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}^3_+))} (1+t)^{\frac14}.$$

We obtain the final result by combining all the terms together. □

Corollary 4.4.6. *For $\varepsilon \in (0,1)$ and under Assumption 4.1.8, we have for any strong existence time $t \in (T_0, t^\star_\varepsilon)$*

$$\|F_\varepsilon(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \frac{M^{\varpi_2}}{(1+t)^{\frac74}},$$

for some universal constant $\varpi_2 > 0$, which is independent of ε and T_0.

Proof. In what follows, the exponents k, k_1 and k_2 are generic and will be choosen in the end of the proof. In view of Lemma 4.4.2 and 4.4.4, Assumption **4.1.8** entails

$$\sum_{i=0}^{1}\|F_\varepsilon^{\natural,i}(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} + \sum_{i=0}^{1}\|F_\varepsilon^{\flat,i}(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \frac{M^{\nu_2}}{(1+t)^k}\left(1+\|\partial_t u_\varepsilon\|_{\mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}^3_+))}\right)$$
$$\lesssim \frac{M^{\nu_2}}{(1+t)^k},$$

for some $\nu_2 > 0$, thanks to Corollary 4.3.12. Furthermore, using Lemma 4.4.5 and Assumption **4.1.8**, there exists $\omega_2 > 0$ such that

$$\|F_\varepsilon^{\natural,2}(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|F_\varepsilon^{\flat,2}(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \frac{M^{\omega_2}}{(1+t)^{k_1}}\left(1+\|\mathrm{D}_x^2 u_\varepsilon\|^{\frac12}_{\mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}^3_+))}\right)$$
$$+\frac{M^{\omega_2}}{(1+t)^{k_2-\frac14}}\|u_\varepsilon\|^{\frac12}_{\mathrm{L}^\infty(0,t;\mathrm{L}^6(\mathbb{R}^3_+))}\|\mathrm{D}_x^2 u_\varepsilon\|^{\frac12}_{\mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}^3_+))}$$
$$\lesssim \frac{M^{\omega_2}}{(1+t)^{k_1}} + \frac{M^{\omega_2}}{(1+t)^{k_2-\frac14}},$$

by Corollary 4.3.12. Choosing $k = k_1 = 7/4$ and $k_2 = 2$ eventually yields the result, by taking the corresponding constant $M > 1$ in Assumption **4.1.8**. □

Remark 4.4.7. The estimates provided by Lemmas 4.4.2–4.4.4–4.4.5 with $k = k_1 = k_2 = 0$ can be extended to the interval $(0, t_\varepsilon^\star)$ (considering $T_0 = 0$), because neither an exit geometric condition nor absorption are required. In view of Remark 4.4.3, we can write for all $t \in (0, t_\varepsilon^\star)$

$$\|F_\varepsilon(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \|(1+|v|)f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))} + \|f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|u_\varepsilon^0\|_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))}$$
$$+\varepsilon^{\frac12}\|f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|\partial_\tau u_\varepsilon\|_{\mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}^3_+))}$$
$$+\Big[\varepsilon^{\frac34}\|(1+|v|^2)f_\varepsilon^0\|^{\frac12}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|f_\varepsilon^0\|^{\frac12}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\mathcal{E}_\varepsilon(0)^{\frac14}\|\mathrm{D}_x^2 u_\varepsilon\|^{\frac12}_{\mathrm{L}^2((0,t)\times\mathbb{R}^3_+)}$$
$$+\varepsilon^{\frac34}\|f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\mathcal{E}_\varepsilon(0)^{\frac14}\|\mathrm{D}_x^2 u_\varepsilon\|^{\frac12}_{\mathrm{L}^2((0,t)\times\mathbb{R}^3_+)}\Big]$$
$$+(1+t)^{\frac14}\|f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\mathcal{E}_\varepsilon(0)^{\frac12}\|u_\varepsilon\|^{\frac12}_{\mathrm{L}^\infty(0,t;\mathrm{L}^6(\mathbb{R}^3_+))}\|\mathrm{D}_x^2 u_\varepsilon\|^{\frac12}_{\mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}^3_+))}.$$

4.4.2 Polynomial decay estimates of the Brinkman force in $\mathrm{L}^p_t\mathrm{L}^p_x$

We aim at deriving uniform in time estimates in $\mathrm{L}^p_t\mathrm{L}^p_x$ for some weighted in time version of the Brinkman force (namely, of the form $(1+t)^k F_\varepsilon$). Thanks to Corollary 4.4.6, we are now in position to use the polynomial decay in time of u_ε provided by Theorem 4.3.3. We state a series of lemmas based on the decomposition of Lemma 4.4.1.

The strategy of proofs is mainly inspired by the one performed in Subsection 4.4.1, with an additional integration in time here. For the sake of readibility, we state the result and the proof of Lemmas 4.4.8–4.4.9–4.4.10 is postponed to Appendix 4.A.

Lemma 4.4.8. *Let $T \in (T_0, t_\varepsilon^\star)$ and $r \in [2,\infty)$. For any $k, k_1, k_2 \geq 0$ satisfying $k_1 > k$ and $k_2 > k + \frac1r$, we have*

$$\|(1+t)^k F_\varepsilon^{\natural,0}\|_{\mathrm{L}^r((T_0,T)\times\mathbb{R}^3_+)} \lesssim \varepsilon^{\frac1r}\||v|^{k_1}f_\varepsilon^0\|^{\frac{r-1}{r}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\Big[\|(1+|v|^{k_1+r})f_\varepsilon^0\|^{\frac1r}_{\mathrm{L}^1(\mathbb{R}^3_+\times\mathbb{R}^3)}$$
$$+\||v|^{k_1}f_\varepsilon^0\|^{\frac1r}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|u_\varepsilon^0\|_{\mathrm{L}^r(\mathbb{R}^3_+)}\Big]$$
$$+\||v|^{k_2}\|f_\varepsilon^0\|^{\frac1r}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^r(\mathbb{R}^3_+))},$$

and

$$\|(1+t)^k F_\varepsilon^{\flat,0}\|_{\mathrm{L}^r((T_0,T)\times\mathbb{R}^3_+)} \lesssim \varepsilon^{\frac{1}{r}} \|x_3^{k_1} f_\varepsilon^0\|^{\frac{r-1}{r}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \Big[\|(1+|v|^r) x_3^{k_1} f_\varepsilon^0\|^{\frac{1}{r}}_{\mathrm{L}^1(\mathbb{R}^3_+\times\mathbb{R}^3)} + \|x_3^{k_1} f_\varepsilon^0\|^{\frac{1}{r}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \|u_\varepsilon^0\|_{\mathrm{L}^r(\mathbb{R}^3_+)}\Big] + \|x_3^{k_2} f_\varepsilon^0\|^{\frac{1}{r}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^r(\mathbb{R}^3_+))}.$$

Furthermore, if $k = k_1 = k_2 = 0$*, we can replace* T_0 *by* 0 *in the previous result.*

Lemma 4.4.9. *Let* $T \in (T_0, t_\varepsilon^\star)$ *and* $r \in [2,\infty)$*. For any* $k \geq 0$*, we have*

$$\|(1+t)^k F_\varepsilon^{\natural,1}\|_{\mathrm{L}^r(T_0,T;\mathrm{L}^r(\mathbb{R}^3_+))} \lesssim \varepsilon \||v|^k f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \|\partial_t u_\varepsilon\|_{\mathrm{L}^r(0,T;\mathrm{L}^r(\mathbb{R}^3_+))},$$

$$\|(1+t)^k F_\varepsilon^{\flat,1}\|_{\mathrm{L}^r(T_0,T;\mathrm{L}^r(\mathbb{R}^3_+))} \lesssim \varepsilon \|x_3^k f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \|\partial_t u_\varepsilon\|_{\mathrm{L}^r(0,T;\mathrm{L}^r(\mathbb{R}^3_+))}.$$

Furthermore, if $k = 0$*, we can replace* T_0 *by* 0 *in the previous result.*

Recall that the estimate obtained in Corollary 4.4.6 holds true on $(T_0, t_\varepsilon^\star)$. Until the end of the section, we will refer to a generic nonnegative continuous and nondecreasing function Ψ, which is independent of ε (and related to Theorem 4.3.3).

Lemma 4.4.10. *Let* $T \in (T_0, t_\varepsilon^\star)$ *and* $r \in (3,+\infty)$*. If we assume that*

$$\forall s \in [0,T], \quad \|F_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \frac{1}{(1+s)^{7/4}}, \tag{4.4.3}$$

then for any $k \geq 0$*,*

$$\begin{aligned}\|(1+t)^k F_\varepsilon^{\natural,2}\|_{\mathrm{L}^r((T_0,T)\times\mathbb{R}^3_+)} &\lesssim \varepsilon \|(1+|v|^{\frac{r}{r-1}})|v|^k f_\varepsilon^0\|^{\frac{r-1}{(r(1-\alpha_r)}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \||v|^k f_\varepsilon^0\|^{\frac{1}{r(1-\alpha_r)}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \\ &\quad\times \Psi\left(\|u_\varepsilon^0\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac{1}{2}} \\ &\quad+ \varepsilon \||v|^k f_\varepsilon^0\|^{\frac{1}{1-\alpha_r}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \Psi\left(\|u_\varepsilon^0\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac{1}{2}} \\ &\quad+ \varepsilon (\Upsilon_\varepsilon^0)^{\frac{1}{1-\beta_r}} \||v|^k f_\varepsilon^0\|^{\frac{1}{1-\beta_r}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \Psi\left(\|u_\varepsilon^0\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac{1}{2}} \\ &\quad+ \varepsilon \|\mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^r((0,T)\times\mathbb{R}^3_+)}.\end{aligned}$$

for some $\alpha_r, \beta_r \in (0,1)$*. Furthermore, if* $k = 0$*, we can replace* T_0 *by* 0 *in the previous result.*

Dealing with the term $F_\varepsilon^{\flat,2}$ can be achieved in the same way. For the sake of conciseness, we do not detail the proof and directly state the result.

Lemma 4.4.11. *Let* $T \in (T_0, t_\varepsilon^\star)$ *and* $r \in (3,+\infty)$*. If we assume that*

$$\forall s \in [0,T], \quad \|F_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \frac{1}{(1+s)^{7/4}},$$

then for any $k \geq 0$,

$$\begin{aligned}\|(1+t)^k F_\varepsilon^{\flat,2}\|_{\mathrm{L}^r((T_0,T)\times\mathbb{R}^3_+)} \lesssim \varepsilon \|(1+|v|^{\frac{r}{r-1}}) x_3^k f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}^{\frac{r-1}{(r(1-\alpha_r)}} \|x_3^k f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}^{\frac{1}{r(1-\alpha_r)}} \\ \times \Psi\left(\|u_\varepsilon^0\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac12} \\ + \varepsilon \|x_3^k f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}^{\frac{1}{1-\alpha_r}} \Psi\left(\|u_\varepsilon^0\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac12} \\ + \varepsilon (\Upsilon_\varepsilon^0)^{\frac{1}{1-\beta_r}} \|x_3^k f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}^{\frac{1}{1-\beta_r}} \Psi\left(\|u_\varepsilon^0\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac12} \\ + \varepsilon \|\mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^r((0,T)\times\mathbb{R}^3_+)}.\end{aligned}$$

for some $\alpha_r, \beta_r \in (0,1)$. Furthermore, if $k=0$, we can replace T_0 by 0 in the previous result.

We now conclude this section by collecting all the previous estimates. Note that we obviously have

$$\begin{aligned}\|\partial_t u_\varepsilon\|_{\mathrm{L}^r(0,T;\mathrm{L}^r(\mathbb{R}^3_+))} + \|\mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^r((0,T)\times\mathbb{R}^3_+)} \\ \leq \|(1+t)^k \partial_t u_\varepsilon\|_{\mathrm{L}^r(0,T;\mathrm{L}^r(\mathbb{R}^3_+))} + \|(1+t)^k \mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^r((0,T)\times\mathbb{R}^3_+)}.\end{aligned}$$

By writing

$$\|(1+t)^k F_\varepsilon\|_{\mathrm{L}^r((T_0,T)\times\mathbb{R}^3_+)} \leq \sum_{i=0}^{2} \|(1+t)^k F_\varepsilon^{\natural,i}\|_{\mathrm{L}^r((T_0,T)\times\mathbb{R}^3_+)} + \sum_{i=0}^{2} \|(1+t)^k F_\varepsilon^{\flat,i}\|_{\mathrm{L}^r((T_0,T)\times\mathbb{R}^3_+)},$$

we can infer the following result.

Corollary 4.4.12. *Let $T \in (T_0, t_\varepsilon^\star)$ and $r \in (3,+\infty)$. Let $k \geq 0$ be fixed. There exists $\ell_{k,r} = \ell > 0$ such that if we assume*

$$\|(1+|v|^\ell)(1+x_3^\ell) f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty\cap\mathrm{L}^1(\mathbb{R}^3_+))} \leq M,$$

$$\|u_\varepsilon^0\|_{\mathrm{L}^r(\mathbb{R}^3_+)} \leq M,$$

for some $M > 1$, then the following holds. Under the assumption

$$\forall s \in [0,T], \quad \|F_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \frac{1}{(1+s)^{7/4}}, \tag{4.4.4}$$

we have

$$\begin{aligned}\|(1+t)^k F_\varepsilon\|_{\mathrm{L}^r((T_0,T)\times\mathbb{R}^3_+)} \lesssim \varepsilon^{\frac{1}{r}} M^{\omega_r} + M + \varepsilon\Big[1 + (\Upsilon_\varepsilon^0)^{\mu_r}\Big] M^{\omega_r} \Psi\left(\|u_\varepsilon^0\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac12} \\ + \varepsilon M \|(1+t)^k \partial_t u_\varepsilon\|_{\mathrm{L}^r((0,T)\times\mathbb{R}^3_+)} \\ + \varepsilon \|(1+t)^k \mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^r((0,T)\times\mathbb{R}^3_+)},\end{aligned}$$

for some $\omega_r, \mu_r > 0$. Furthermore, if $k=0$, we can replace T_0 by 0 in the previous result.

As a consequence, we are now in position to obtain higher order integrability results for u_ε, which are uniform in ε. Indeed, by the maximal $\mathrm{L}^p_t\mathrm{L}^p_x$ parabolic regularity theory for the Stokes system with $p \in (1,\infty)$ (see Theorem 4.1.2), we know that

$$\begin{aligned}\|\partial_t u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \|\mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \\ \lesssim \|F_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \|(u_\varepsilon\cdot\nabla_x)u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \|u_\varepsilon^0\|_{\mathrm{D}_p^{1-\frac1p,p}(\mathbb{R}^3_+))}.\end{aligned}$$

Hence, assuming that the pointwise decay of the Brinkman force (4.4.4) is satisfied on $(0,T)$ with $T \in (0, t_\varepsilon^\star)$, we can infer from Corollary 4.4.12 with $k = 0$ and Corollary 4.3.14 that for p close enough to 3, we have

$$\begin{aligned}\|\partial_t u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \|\mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \lesssim\ & \varepsilon^{\frac{1}{p}} M^{\omega_p} + M \\ & + \varepsilon\Big[1 + (\Upsilon_\varepsilon^0)^{\mu_p}\Big] M^{\omega_p} \Psi\left(\|u_\varepsilon^0\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac{1}{2}} \\ & + \varepsilon M \|\partial_t u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{R}^3_+)} \\ & + \varepsilon \|\mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{R}^3_+)} \\ & + (\Upsilon_\varepsilon^0)^{\widetilde{\mu}_p} \mathcal{E}_\varepsilon(0)^{\mu_p} \|\mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{R}^3_+)} \\ & + \|u_\varepsilon^0\|_{\mathrm{D}_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+))}.\end{aligned}$$

Taking ε and $\mathcal{E}_\varepsilon(0)$ small enough according to Assumption **A3-b** and the uniform bound from Assumption **4.1.8**, we can infer the following result.

Corollary 4.4.13. *There exists $\varepsilon_0 > 0$ and $p_0 > 3$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $p \in (3, p_0)$, the following holds. Let $T \in (0, t_\varepsilon^\star)$ and assume that*

$$\forall s \in [0,T], \quad \|F_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \frac{1}{(1+s)^{7/4}}. \tag{4.4.5}$$

Then

$$\|\partial_t u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \|\mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \lesssim M^{\omega_p},$$

for some $\omega_p > 0$.

Remark 4.4.14. Under the same assumptions than Corollary 4.4.13, the very same kind of manipulations lead to the following local and weighted control in time. There exists $\varepsilon_0 > 0$ and $p_0 > 3$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $p \in (3, p_0)$, we have for all $T \in (0, t_\varepsilon^\star)$

$$\begin{aligned}&\|(1+t)^k F_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{R}^3_+)} \\ &\qquad \lesssim (1+T)^k \Big(\varepsilon^{\frac{1}{p}} M^{\omega_p} + \varepsilon M^{\omega_p} + M + \varepsilon\Big[1 + (\Upsilon_\varepsilon^0)^{\mu_p}\Big] M^{\omega_p} \Psi\left(\|u_\varepsilon^0\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac{1}{2}}\Big).\end{aligned}$$

4.5 Bootstrap and convergence towards the Boussinesq-Navier-Stokes system

The main goal of this section is to complete the bootstrap argument which is needed in our study of the convergence to the Boussinesq-Navier-Stokes sytem. To do so, we will use the precise decay estimates obtained in Section 4.4, proving that such controls can be propagated until any time along the evolution. Recall that this mainly amounts to prove that for ε small enough, we have $t_\varepsilon^\star = +\infty$. Bearing on the uniform estimates on u_ε and ρ_ε derived from this strategy, we will be able to prove Theorems 4.1.10–4.1.11.

4.5.1 Initial horizon for the bootstrap procedure

We first set up the bootstrap procedure, as explained in Subsection 4.1.6. As the polynomial estimates of Subsection 4.4.2 are all based on the absorption effect at the boundary (through the propagation of the exit geometric condition), one must ensure that we are indeed allowed to use such an effect (see the discussion made just before Subsection 4.4.1).

We rely on the family of estimates of the previous section which hold without assuming any absorption at the boundary. The smallness condition contained in Assumption **4.1.9** allows to prove that, for ε small enough, the time $t^\star_\varepsilon$ is bounded from below by some uniform time after which one can use the exit geometric condition. We first consider the following definition.

Definition 4.5.1. *We set*

$$T_{\mathrm{abs}} := t^g_{\frac{1}{2}}(1,1),$$

where $t^g_{\frac{1}{2}}$ *is defined in* (4.2.16) *in Subsection 4.2.4.*

Roughly speaking, the following lemma is the counterpart of Corollary 4.4.6.

Lemma 4.5.2. *There exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$*, we have*

$$\forall t \in (0, \min(T_{\mathrm{abs}} + 10\alpha, t^\star_\varepsilon)), \quad \|F_\varepsilon(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \frac{M^{\varpi_2}}{(1 + T_{\mathrm{abs}} + 10\alpha)^{\frac{7}{4}}}.$$

where $\varpi_2 > 0$ *is the universal constant involved in Corollary 4.4.6 and where* α *has been fixed in Definition* (4.3.17)*.*

Proof. According to Remark 4.4.7, there exists $\omega > 0$ such that for all $t \in (0, \min(T_{\mathrm{abs}} + 10\alpha, t^\star_\varepsilon))$

$$\begin{aligned}
\|F_\varepsilon(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim\ & \|(1+|v|)f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))} + \|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\mathcal{E}_\varepsilon(0)^{\frac{1}{2}} + \|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))} \\
& + \varepsilon^{\frac{1}{2}}\|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|\partial_\tau u_\varepsilon\|_{\mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}^3_+))} \\
& + \Big[\varepsilon^{\frac{3}{4}}\|(1+|v|^2)f^0_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|f^0_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\mathcal{E}_\varepsilon(0)^{\frac{1}{4}}\|\mathrm{D}^2_x u_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^2((0,t)\times\mathbb{R}^3_+)} \\
& \qquad + \varepsilon^{\frac{3}{4}}\|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\mathcal{E}_\varepsilon(0)^{\frac{1}{4}}\|\mathrm{D}^2_x u_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^2((0,t)\times\mathbb{R}^3_+)}\Big] \\
& + (1+t)^{\frac{1}{4}}\|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\mathcal{E}_\varepsilon(0)^{\frac{1}{2}}\|u_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^\infty(0,t;\mathrm{L}^6(\mathbb{R}^3_+))}\|\mathrm{D}^2_x u_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}^3_+))}.
\end{aligned}$$

Since $t < t^\star_\varepsilon$, we can use the unifom bound (4.3.6) given by Proposition 4.3.7 and by Assumption **4.1.8** to get

$$\begin{aligned}
&\|F_\varepsilon(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \\
&\lesssim \|(1+|v|)f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))} + M\mathcal{E}_\varepsilon(0)^{\frac{1}{2}} + (\varepsilon^{\frac{3}{4}} + \varepsilon^{\frac{1}{2}})M^\omega + (1+t)^{\frac{1}{4}}\mathcal{E}_\varepsilon(0)^{\frac{1}{2}}M^\omega \\
&\lesssim \|(1+|v|)f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))} + M\mathcal{E}_\varepsilon(0)^{\frac{1}{2}} + (\varepsilon^{\frac{3}{4}} + \varepsilon^{\frac{1}{2}})M^\omega + (1+T_{\mathrm{abs}}+10\alpha)^{\frac{1}{4}}\mathcal{E}_\varepsilon(0)^{\frac{1}{2}}M^\omega.
\end{aligned}$$

Since $T_{\mathrm{abs}} + 10\alpha$ is fixed and does not depend on ε, owing to Assumption **A3-b**, we can first reduce $\varepsilon > 0$ and then $\|(1+|v|)f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))}$ and $\mathcal{E}_\varepsilon(0)^{\frac{1}{2}}$ so that

$$\|(1+|v|)f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))} + M\mathcal{E}_\varepsilon(0)^{\frac{1}{2}} + (\varepsilon^{\frac{3}{4}} + \varepsilon^{\frac{1}{2}})M^\omega + (1+T_{\mathrm{abs}}+10\alpha)^{\frac{1}{4}}\mathcal{E}_\varepsilon(0)^{\frac{1}{2}}M^\omega < \frac{M^{\varpi_2}}{(1+T_{\mathrm{abs}}+10\alpha)^{\frac{7}{4}}}.$$

This concludes the proof of the lemma. ☐

Thanks to the pointwise decay of the Brinkman force provided by Lemma 4.5.2, we are allowed to use the conditional decay of the kinetic energy stated in Theorem 4.3.3, but only on the interval $(0, \min(T_{\mathrm{abs}}+10\alpha, t_\varepsilon^\star))$ for the moment. In addition, this means that this can be used on this interval in the estimates of Subsection 4.4.2, with the exponent $k = 0$.

Lemma 4.5.3. *There exists ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$, and $T \in (0, \min(T_{\mathrm{abs}} + 10\alpha, t_\varepsilon^\star))$, we have*

$$\|F_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} \lesssim \varepsilon^{\frac{1}{2}} M^{\mu_2} + \varepsilon M \|\partial_t u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)} + \varepsilon \|\mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)} + \varepsilon \|\mathrm{D}_x^2 u_\varepsilon\|^{2\beta_p}_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} (1+T)^{\zeta_p} + \|f_\varepsilon^0\|^{\frac{1}{2}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))},$$

for any $p > 3$ and some $\beta_p \in (0,1)$ and $\zeta_p > 0$.

Proof. Note that we do not use the absorption at the boundary in the proof. We use the splitting of Lemma 4.2.11 by writing

$$F_\varepsilon = F_\varepsilon^0 + F_\varepsilon^1 + F_\varepsilon^2,$$

and we observe that the estimates given by Lemma 4.4.8 and Lemma 4.4.9 for $k = 0$ are unchanged and applied for $r = 2$, that is

$$\|F_\varepsilon^0\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)} \lesssim \varepsilon^{\frac{1}{2}} M^{\mu_2} + \|f_\varepsilon^0\|^{\frac{1}{2}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))},$$
$$\|F_\varepsilon^1\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)} \lesssim \varepsilon M \|\partial_t u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)}.$$

Compared to the proof of Corollary 4.4.10 for $p > 3$, we are not allowed to use the Gagliardo-Nirenberg-Sobolev inequality to treat $\|\nabla_x u_\varepsilon\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}$ involved in F_ε^2 in the same way. We still have

$$\|F_\varepsilon^2\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)} \lesssim \varepsilon M^{\widetilde{\mu}_2} + \varepsilon \|\mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)} + \|\mathrm{III}\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)},$$

where

$$\mathrm{III}(t,x) = \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x, [\Gamma_\varepsilon^{t,x}]^{-1}(w)) \int_0^t \int_0^s e^{\frac{s-t}{\varepsilon}} e^{\frac{\tau-s}{\varepsilon}} f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w), w) \times \left|(Pu_\varepsilon)(\tau, \widetilde{\mathrm{X}}_\varepsilon^{\tau;t}(x,w))\right| \left|\nabla_x [Pu_\varepsilon](s, \widetilde{\mathrm{X}}_\varepsilon^{s;t}(x,w))\right| \mathrm{d}\tau \, \mathrm{d}s \, \mathrm{d}w.$$

Similar computations as those of the proof of Corollary 4.4.10 give

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3_+} |\mathrm{III}(t,x)|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\lesssim \varepsilon^{-1} \|f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \left[\int_0^T \left(\int_0^t e^{\frac{s-t}{\varepsilon}} \|\nabla_x u_\varepsilon(s)\|^2_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \, \mathrm{d}s \right) \mathrm{d}t \right] \\
&\quad \times \sup_{t \in (0,T)} \left\{ \int_0^t e^{\frac{s-t}{\varepsilon}} \int_0^s e^{\frac{\tau-s}{\varepsilon}} \right. \\
&\qquad \left. \times \int_{\mathbb{R}^3_+ \times \mathbb{R}^3} \mathbf{1}_{\mathcal{O}_\varepsilon^t}(x, [\Gamma_\varepsilon^{t,x}]^{-1}(w)) f_\varepsilon^0(\widetilde{\mathrm{X}}_\varepsilon^{0;t}(x,w), w) \left|(Pu_\varepsilon)(\tau, \widetilde{\mathrm{X}}_\varepsilon^{\tau;t}(x,w))\right|^2 \mathrm{d}x \, \mathrm{d}w \, \mathrm{d}\tau \, \mathrm{d}s \right\}.
\end{aligned}$$

Again, for the term between braces we have

$$\begin{aligned}
&\int_0^t e^{\frac{s-t}{\varepsilon}} \int_0^s e^{\frac{\tau-s}{\varepsilon}} \int_{\mathbb{R}^3_+\times\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w)) f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w) \left|(Pu_\varepsilon)(\tau,\widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w))\right|^2 \mathrm{d}x\,\mathrm{d}w\,\mathrm{d}\tau\,\mathrm{d}s \\
&\lesssim \varepsilon^2 \|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}(\Upsilon^0_\varepsilon)^2 \\
&\lesssim \varepsilon^2 M.
\end{aligned}$$

The term between brackets is actually the only one requiring a slightly different treatment: using the Gagliardo-Nirenberg-Sobolev inequality (see Theorem 4.2.9), we can write that for $p>3$

$$\|\nabla_x u_\varepsilon(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \lesssim \|\mathrm{D}^2_x u_\varepsilon(s)\|^{\beta_p}_{\mathrm{L}^p(\mathbb{R}^3_+)} \|u_\varepsilon(s)\|^{1-\beta_p}_{\mathrm{L}^2(\mathbb{R}^3_+)}, \quad \beta_p = \frac{5p}{10p-6},$$

therefore, applying Hölder's inequality as well as the combination of Lemma 4.5.2 and Theorem 4.3.3, we get

$$\begin{aligned}
\int_0^T \left(\int_0^t e^{\frac{s-t}{\varepsilon}} \|\nabla_x u_\varepsilon(s)\|^2_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s\right)\mathrm{d}t &\lesssim \int_0^T \left(\int_0^t e^{\frac{s-t}{\varepsilon}} \|\mathrm{D}^2_x u_\varepsilon(s)\|^{2\beta_p}_{\mathrm{L}^p(\mathbb{R}^3_+)} \|u_\varepsilon(s)\|^{2(1-\beta_p)}_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s\right)\mathrm{d}t \\
&\lesssim \int_0^T \left(\int_0^t e^{\frac{s-t}{\varepsilon}} \|\mathrm{D}^2_x u_\varepsilon(s)\|^{2\beta_p}_{\mathrm{L}^p(\mathbb{R}^3_+)} \frac{1}{(1+s)^{2(1-\beta_p)\frac{3}{4}}}\,\mathrm{d}s\right)\mathrm{d}t \\
&= \int_0^T \left(\int_s^T e^{\frac{s-t}{\varepsilon}} \|\mathrm{D}^2_x u_\varepsilon(s)\|^{2\beta_p}_{\mathrm{L}^p(\mathbb{R}^3_+)} \frac{1}{(1+s)^{2(1-\beta_p)\frac{3}{4}}}\,\mathrm{d}t\right)\mathrm{d}s \\
&\leq \varepsilon \|\mathrm{D}^2_x u_\varepsilon\|^{2\beta_p}_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \left(\int_0^T \frac{1}{(1+s)^{\frac{p(1-\beta_p)}{p-2\beta_p}\frac{6}{4}}}\right)^{1-\frac{2\beta_p}{p}}.
\end{aligned}$$

Note that if $p \geq 3$ then

$$\frac{p}{2\beta_p} = \frac{7p-6}{10} \geq 1, \quad \frac{p(1-\beta_p)}{p-2\beta_p}\frac{6}{4} = \frac{2p^2-6p}{7p^2-16p}\frac{6}{4} \in \left(0,\frac{1}{2}\right).$$

We do not get a uniform bound in time for the last integral: we can just write

$$\int_0^T \left(\int_0^t e^{\frac{s-t}{\varepsilon}} \|\nabla_x u_\varepsilon(s)\|^2_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s\right)\mathrm{d}t \lesssim \varepsilon \|\mathrm{D}^2_x u_\varepsilon\|^{2\beta_p}_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))}(1+T)^{\zeta_p},$$

for some $\zeta_p \in (0,1)$. At the end of the day, we obtain the conclusion by gathering all the previous estimates together. □

Recall that $\alpha \in (0,1)$ has been be fixed once and for all before Definition 4.3.17. In the following result, we prove that the time $t^\star_\varepsilon$ is bounded from below by some time independent of ε, after which all the estimates based on the absorption effect will be available.

Corollary 4.5.4. *Under Assumption 4.1.9, there exists $\varepsilon_0 = \varepsilon_0(\alpha) > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$, we have*

$$T_{\mathrm{abs}} + 10\alpha < t^\star_\varepsilon.$$

Furthermore, we have

$$\int_0^{T_{\mathrm{abs}}+10\alpha} \|F_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s + \int_0^{T_{\mathrm{abs}}+10\alpha} \|F_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s < \frac{\mathrm{C}_\star}{4}, \tag{4.5.1}$$

$$\int_0^{T_{\mathrm{abs}}+10\alpha} \|u_\varepsilon(s)\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^3_+)}\,\mathrm{d}s < \frac{\delta^\star}{4}, \tag{4.5.2}$$

$$\forall t \in (0, T_{\mathrm{abs}}+10\alpha), \quad \|F_\varepsilon(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \frac{M^{\varpi_2}}{(1+T_{\mathrm{abs}}+10\alpha)^{\frac{7}{4}}}, \tag{4.5.3}$$

where $\varpi_2 > 0$ is the universal constant given in Corollary 4.4.6, where $\mathrm{C}_\star$ is the universal constant of Proposition 4.3.7 and where $\delta^\star$ has been introduced in Definition (4.3.17) of $t^\star_\varepsilon$.

In particular, $T_{\mathrm{abs}} + 10\alpha$ is a strong existence time.

Proof. By combining Lemma 4.5.3 and Corollary 4.3.12, we know that for $\varepsilon > 0$ small enough, we have for all $T \in (0, \min(T_{\mathrm{abs}} + 10\alpha, t^\star_\varepsilon))$

$$\begin{aligned}
\|F_\varepsilon\|_{\mathrm{L}^1\cap\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} &\lesssim (1+\sqrt{T})\|F_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} \\
&\lesssim (1+\sqrt{T})\Big(\varepsilon^{\frac{1}{2}}M^{\mu_2} + \varepsilon M\|\partial_t u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)} + \varepsilon\|\mathrm{D}^2_x u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)} \\
&\qquad + \varepsilon\|\mathrm{D}^2_x u_\varepsilon\|^{2\beta_p}_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))}(1+T)^{\zeta_p} + \|f^0_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))}\Big),
\end{aligned}$$

where $p \to 3^+$. We then use the uniform bounds of Corollary 4.3.12 and of Corollary 4.4.13. Note that this last corollary actually requires the decay of the Brinkman force on $[0,T]$, provided by Lemma 4.5.2. This entails, according to Assumption **4.1.8** and for ε small enough

$$\|F_\varepsilon\|_{\mathrm{L}^1\cap\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} \lesssim (1+\sqrt{T})\Big(\varepsilon^{\frac{1}{2}}M^{\widetilde{\mu}_2} + \varepsilon M^{\widetilde{\mu}_p}(1+T)^{\zeta_p} + \|f^0_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))}\Big). \tag{4.5.4}$$

Furthermore, owing to the Gagliardo-Nirenberg-Sobolev inequality (see Theorem 4.2.9), we can write for all $T \in (0, \min(T_{\mathrm{abs}} + 10\alpha, t^\star_\varepsilon))$

$$\begin{aligned}
\|u_\varepsilon\|_{\mathrm{L}^1(0,T;\mathrm{W}^\infty(\mathbb{R}^3_+))} &\lesssim \int_0^T \|\mathrm{D}^2_x u_\varepsilon(s)\|^{\alpha_p}_{\mathrm{L}^p(\mathbb{R}^3_+)}\|u_\varepsilon(s)\|^{1-\alpha_p}_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s + \int_0^T \|\mathrm{D}^2_x u_\varepsilon(s)\|^{\beta_p}_{\mathrm{L}^p(\mathbb{R}^3_+)}\|u_\varepsilon(s)\|^{1-\beta_p}_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s \\
&\lesssim T^{1-\alpha_p}\mathcal{E}_\varepsilon(0)^{1-\alpha_p}\|\mathrm{D}^2_x u_\varepsilon\|^{p\alpha_p}_{\mathrm{L}^p((0,T)\times\mathbb{R}^3_+)} + T^{1-\beta_p}\mathcal{E}_\varepsilon(0)^{1-\beta_p}\|\mathrm{D}^2_x u_\varepsilon\|^{p\beta_p}_{\mathrm{L}^p((0,T)\times\mathbb{R}^3_+)} \\
&\lesssim T^{1-\alpha_p}\mathcal{E}_\varepsilon(0)^{1-\alpha_p}M^{\omega_p} + T^{1-\beta_p}\mathcal{E}_\varepsilon(0)^{1-\beta_p}M^{\tilde{\omega}_p},
\end{aligned} \tag{4.5.5}$$

where $p \to 3^+$ and for some $(\alpha_p, \beta_p) \in (0,1)^2$, thanks to Corollary 4.4.13 and Assumption **4.1.8**.

We then proceed as follows. First, invoking Assumption **A3-b**, there exists $\varepsilon_0 > 0$ such that if we choose ε, $\mathcal{E}_\varepsilon(0)$ and $\|f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))}$ small enough, then for all $\varepsilon \in (0, \varepsilon_0)$

$$\begin{aligned}
M^{\omega_p}\mathcal{E}_\varepsilon(0)^{1-\alpha_p} + M^{\tilde{\omega}_p}\mathcal{E}_\varepsilon(0)^{1-\beta_p} &< \frac{\delta^\star}{8\max\{(T_{\mathrm{abs}}+10\alpha)^{1-\alpha_p}, (T_{\mathrm{abs}}+10\alpha)^{1-\beta_p}\}}, \\
\varepsilon^{\frac{1}{2}}M^{\widetilde{\mu}_2} + \|f^0_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))} &< \frac{1}{2\,(1+\sqrt{T_{\mathrm{abs}}+10\alpha})}\left(\frac{\mathrm{C}_\star}{4}\right)^{1/2}, \\
\varepsilon M^{\widetilde{\mu}_p} &< \frac{1}{2(1+T_{\mathrm{abs}}+10\alpha)^{\zeta_p}\,(1+\sqrt{T_{\mathrm{abs}}+10\alpha})}\left(\frac{\mathrm{C}_\star}{4}\right)^{1/2}.
\end{aligned}$$

Suppose now that there exists $\varepsilon \in (0, \varepsilon_0)$ such that $t_\varepsilon^\star \leq T_{\text{abs}} + 10\alpha$ (in particular, $t_\varepsilon^\star$ is finite and is a strong existence time). In view of Assumption **A3-a**, the two previous inequalities combined with the continuity of $s \mapsto \|F_\varepsilon\|_{\mathrm{L}^2(0,s;\mathrm{L}^2(\mathbb{R}^3_+))}$ and $s \mapsto \|u_\varepsilon\|_{\mathrm{L}^1(0,s;\mathrm{W}^\infty(\mathbb{R}^3_+))}$ at $t_\varepsilon^\star$ (according to the integrability properties (4.3.10), (4.3.11) and (4.3.9)) lead to a contradiction with the definition of $t_\varepsilon^\star$.

This means that for all $\varepsilon \in (0, \varepsilon_0)$, we have $t_\varepsilon^\star > T_{\text{abs}} + 10\alpha$ and this bound from below is independent of ε and can also be expressed as

$$\min(T_{\text{abs}} + 10\alpha, t_\varepsilon^\star) = T_{\text{abs}} + 10\alpha,$$

for all $\varepsilon \in (0, \varepsilon_0)$. The previous choice also yields (4.5.1) and (4.5.2).

The pointwise local estimate (4.5.3) for the Brinkman force is now a direct consequence of Lemma 4.5.2. This eventually concludes the proof. □

Starting the absorption.

Recall again that $\alpha \in (0, 1)$ has been fixed just before Definition 4.3.17 and is independent of ε. We then consider the associated $\varepsilon_0 = \varepsilon_0(\alpha)$ given by Corollary 4.5.4. We know that

$$\forall \varepsilon \in (0, \varepsilon_0), \quad T_{\text{abs}} + 10\alpha < t_\varepsilon^\star.$$

For any $\varepsilon \in (0, \varepsilon_0)$ and $t \in (T_{\text{abs}} + 10\alpha, t_\varepsilon^\star)$, we observe that

$$t^g_{\frac{1}{2}}(1, 1) = T_{\text{abs}} < T_{\text{abs}} + \alpha < T_{\text{abs}} + 9\alpha < t - \alpha.$$

According to Lemma 4.2.21, the trivial velocity field satisfies $\mathrm{EGC}_{1/2}^{1+\ell^1_{1/2}(t-\alpha), 1+r^1_{1/2}(t-\alpha)}(t-\alpha)$. Thus, by Remark 4.2.22, we obtain the fact that $\mathrm{EGC}_\varepsilon^{1+\ell^1_{1/2}(t-\alpha), 1+r^1_{1/2}(t-\alpha)}(t-\alpha)$ is satisfied by the trivial velocity field, provided that $\varepsilon < \frac{1}{4}$. Thus, if $\varepsilon \in (0, \min(\frac{1}{4}, \varepsilon_0))$ and $t \in (T_{\text{abs}} + 10\alpha, t_\varepsilon^\star)$ then we know that

$$\int_0^t \|u_\varepsilon(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \, \mathrm{d}s < \delta^\star,$$

therefore Lemma 4.2.18 entails that the vector field u_ε satisfies $\mathrm{EGC}_\varepsilon^{1+\ell^1_{1/2}(t-\alpha), 1+r^1_{1/2}(t-\alpha)}(t)$. In addition, again according to Lemma 4.2.21, we know that for all $t > T_{\text{abs}} + 10\alpha$

$$\frac{1}{1 + \ell^1_{1/2}(t-\alpha)} \lesssim \frac{1}{1 + t - \alpha} \lesssim \frac{1}{1+t}, \quad \frac{1}{1 + r^1_{1/2}(t-\alpha)} \lesssim \frac{1}{1 + t - \alpha} \lesssim \frac{1}{1+t},$$

where $\lesssim$ refers to a constant independent of ε and t.

Definition 4.5.5. *We set*

$$T_0 := T_{\text{abs}} + 10\alpha. \tag{4.5.6}$$

We can sum up our results as follows. We have $T_0 < t_\varepsilon^\star$ for ε small enough and the following lemma holds.

Lemma 4.5.6. *For all $\varepsilon \in (0, \min(\frac{1}{4}, \varepsilon_0))$ and for all $t \in (T_0, t_\varepsilon^\star)$, the vector field u_ε satisfies $\mathrm{EGC}_\varepsilon^{1+\ell(t), 1+r(t)}(t)$ for some continuous and positive functions ℓ and r independent of ε and satisfying*

$$\forall t \in (T_0, t_\varepsilon^\star), \quad \frac{1}{1 + \ell(t)} \leq \frac{C}{1+t}, \quad \frac{1}{1 + r(t)} \leq \frac{C}{1+t},$$

for some constant $C > 0$ independent of ε and t. Furthermore, according to (4.5.3) and Corollary 4.4.6, we have for all $T \in (T_0, t_\varepsilon^\star)$

$$\forall t \in [0, T], \quad \|F_\varepsilon(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \frac{M^{\varpi_2}}{(1+t)^{\frac{7}{4}}}.$$

Roughly speaking, all the estimates of Subsection 4.4.2 which involve the absorption effect (namely, with $k > 0$) are now admissible.

4.5.2 Weighted in time estimates

To obtain the fact that $t_\varepsilon^\star = +\infty$, our final technical argument is based on an interpolation procedure of the form

$$\|\nabla_x u_\varepsilon(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \lesssim \|\mathrm{D}_x^2 u_\varepsilon(s)\|_{\mathrm{L}^p(\mathbb{R}^3_+)}^{\beta_p} \|u_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}^{1-\beta_p}, \tag{4.5.7}$$

for $p > 3$ and $\beta_p \in (0, 1)$. According to the exponent involved in the polynomial decay of u_ε provided by Theorem 4.3.3, we can't directly recover integrability result for large times. We are thus looking for some polynomial weighted in time versions of (4.5.7). While dealing with the higher order terms by maximal parabolic estimates, we ultimately rely on the polynomial decay estimates of the Brinkman force.

Let us explain how one can obtain polynomial in time decay estimates for derivatives of u_ε. As explained in Subsection 4.1.6, our approach is based on the maximal parabolic regularity theory. Setting

$$\mathrm{U}_\varepsilon(t, x) := (1 + t)^\gamma u_\varepsilon(t, x), \quad \gamma \geq 0,$$

we observe that the function U_ε satisfies the following Stokes system on $(0, +\infty) \times \mathbb{R}^3_+$:

$$\begin{cases} \partial_t \mathrm{U}_\varepsilon - \Delta_x \mathrm{U}_\varepsilon + \nabla_x p_\varepsilon = S(u_\varepsilon, f_\varepsilon), \\ \qquad\qquad \mathrm{div}_x\, \mathrm{U}_\varepsilon = 0, \\ \qquad\qquad \mathrm{U}_{\varepsilon|x_3=0} = 0, \\ \qquad\qquad \mathrm{U}_\varepsilon(0, \cdot) = u_\varepsilon^0, \end{cases}$$

where

$$S(u_\varepsilon, f_\varepsilon) := (1 + t)^\gamma (j_\varepsilon - \rho_\varepsilon u_\varepsilon) - (1 + t)^\gamma (u_\varepsilon \cdot \nabla_x) u_\varepsilon + \gamma (1 + t)^{\gamma - 1} u_\varepsilon.$$

Applying the Leray projection $\mathbb{P}$, the previous system also reads as

$$\begin{cases} \partial_t \mathrm{U}_\varepsilon + A_p \mathrm{U}_\varepsilon = \mathbb{P} S(u_\varepsilon, f_\varepsilon), \\ \qquad \mathrm{U}_{\varepsilon|x_3=0} = 0, \\ \qquad \mathrm{U}_\varepsilon(0, \cdot) = u_\varepsilon^0, \end{cases}$$

where A_p refers to the Stokes operator on $\mathrm{L}^p(\mathbb{R}^3_+)$. The maximal $\mathrm{L}^p_t \mathrm{L}^p_x$ parabolic regularity theory for the Stokes system with $p \in (1, \infty)$ (see Theorem 4.1.2) leads to the following fact: for all $T > 0$, we have

$$\|\partial_t \mathrm{U}_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \|\mathrm{D}_x^2 \mathrm{U}_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \lesssim \|\mathbb{P} S(u_\varepsilon, f_\varepsilon)\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \|u_\varepsilon^0\|_{\mathrm{D}_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+)},$$

where $\lesssim$ is independent of T and ε. We thus obtain for all $T > 0$

$$\begin{aligned} &\|(1 + t)^\gamma \partial_t u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \|(1 + t)^\gamma \mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \\ &\qquad \lesssim \|\mathbb{P} S(u_\varepsilon, f_\varepsilon)\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \gamma \|(1 + t)^{\gamma - 1} u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \|u_\varepsilon^0\|_{\mathrm{D}_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+)}, \end{aligned}$$

and then, by continuity of the Leray projection on $\mathrm{L}^p(\mathbb{R}^3_+)$

$$\begin{aligned}&\|(1+t)^\gamma \partial_t u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \|(1+t)^\gamma \mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \\ &\lesssim \|(1+t)^\gamma F_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \|(1+t)^\gamma (u_\varepsilon \cdot \nabla_x) u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \gamma \|(1+t)^{\gamma-1} u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \\ &\quad + \|u_\varepsilon^0\|_{\mathrm{D}_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+)}. \end{aligned} \tag{4.5.8}$$

The guiding line is therefore to get some estimates on the three first terms of the r.h.s in the previous inequality.

We fix $T \in (T_0, t_\varepsilon^\star)$. Thanks to Lemma 4.5.6, we are allowed to use the decay estimate (4.3.3) of Theorem 4.3.3 on the interval $[0,T]$. This estimate explicitly involves some quantity $\Psi\left(\|u_\varepsilon^0\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M^{\varpi_2}\right)$, where Ψ is a nonnegative and nondecreasing continuous function. In view of Assumption **4.1.8**, we shall get rid of this dependency on the initial data in the estimates.

We first state two results dealing with two terms of the estimate (4.5.8). The proof of [HK22, Lemma 5.12 and Lemma 5.13] apply *mutatis mutandis* to the half-space case and we refer to this chapter for more details.

Corollary 4.5.7. *Let $p > 3$. There exists $\sigma > 0$ such that for all $\gamma \in (0, \frac{17}{8} - \frac{7}{4p})$ and $\varepsilon > 0$, we have*

$$\|(1+t)^{\gamma-1} u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \lesssim M^\sigma.$$

Corollary 4.5.8. *There exist $\varsigma > 0$ and $\mu > 0$ such that for all $p \in (3, 3+\varsigma)$ and $\varepsilon > 0$, we have*

$$\|(1+t)^\gamma (u_\varepsilon \cdot \nabla_x) u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \lesssim \mathcal{E}_\varepsilon(0)^\mu \|(1+t)^\gamma \mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))}.$$

In order to deal with the term involving the Brinkman force in the estimate (4.5.8), we shall rely on Corollary 4.4.12. We are thus in position to state the following result.

Lemma 4.5.9. *There exists $\varsigma > 0$ and $\varepsilon_0 > 0$ such that for all $p \in (3, 3+\varsigma)$ and $\varepsilon \in (0, \varepsilon_0)$, the following holds. For all $\gamma \in (0, \frac{17}{8} - \frac{7}{4p})$, we have*

$$\|(1+t)^\gamma \partial_t u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \|(1+t)^\gamma \mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \lesssim M^{\widetilde{\omega}_p},$$

for some $\widetilde{\omega}_p > 0$.

Proof. According to the weighted maximal parabolic estimate (4.5.8), we have

$$\begin{aligned}&\|(1+t)^\gamma \partial_t u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \|(1+t)^\gamma \mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \\ &\lesssim \|(1+t)^\gamma F_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \|(1+t)^\gamma (u_\varepsilon \cdot \nabla_x) u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \gamma \|(1+t)^{\gamma-1} u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \\ &\quad + \|u_\varepsilon^0\|_{\mathrm{D}_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+)}. \end{aligned}$$

For the first term in the r.h.s, we can invoke Corollary 4.4.12 and Remark 4.4.14 with the corresponding exponent p, splitting the quantity $\|(1+t)^\gamma F_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))}$ in two parts between the intervals $(0, T_0)$

and (T_0, T). The second and third term of the previous r.h.s are handled thanks to Corollary 4.5.7 and Corollary 4.5.8. We get

$$\begin{aligned}
&\|(1+t)^{\gamma}\partial_t u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \|(1+t)^{\gamma}\mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \\
&\qquad\qquad \lesssim \varepsilon^{\frac{1}{p}} M^{\omega_p} + M + \varepsilon\Big[1 + (\Upsilon_\varepsilon^0)^{\mu_p}\Big] M^{\omega_p}\Psi\left(\|u_\varepsilon^0\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac{1}{2}} \\
&\qquad\qquad\quad + \varepsilon M\|(1+t)^k\partial_t u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \varepsilon\|(1+t)^k \mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{R}^3_+)} \\
&\qquad\qquad\quad + \|(1+t)^{\gamma} F_\varepsilon\|_{\mathrm{L}^p(0,T_0;\mathrm{L}^p(\mathbb{R}^3_+))} \\
&\qquad\qquad\quad + \mathcal{E}_\varepsilon(0)^{\mu}\|(1+t)^{\gamma}\mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \gamma M^{\sigma} \\
&\qquad\qquad\quad + \|u_\varepsilon^0\|_{\mathrm{D}_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+)}.
\end{aligned}$$

Taking for example $\varepsilon \in (0, \frac{1}{2})$ such that

$$\varepsilon M < \frac{1}{2}, \quad \mathcal{E}_\varepsilon(0)^{\mu} < \frac{1}{4},$$

thanks to Assumption **A3-b**, the previous estimate yields a bound of the type

$$\begin{aligned}
&\|(1+t)^{\gamma}\partial_t u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \|(1+t)^{\gamma}\mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \\
&\lesssim M^{\omega_p} + M + \Big[1 + (\Upsilon_\varepsilon^0)^{\mu_p}\Big] M^{\omega_p}\Psi\left(\|u_\varepsilon^0\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac{1}{2}} + \gamma M^{\sigma} + \|u_\varepsilon^0\|_{\mathrm{D}_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+)} \\
&\quad + \|(1+t)^{\gamma} F_\varepsilon\|_{\mathrm{L}^p(0,T_0;\mathrm{L}^p(\mathbb{R}^3_+))} \\
&\lesssim M^{\omega_p} + M + \Big[1 + (\Upsilon_\varepsilon^0)^{\mu_p}\Big] M^{\omega_p}\Psi\left(\|u_\varepsilon^0\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac{1}{2}} + \gamma M^{\sigma} + \|u_\varepsilon^0\|_{\mathrm{D}_p^{1-\frac{1}{p},p}(\mathbb{R}^3_+)} \\
&\quad + (1+T_0)^{\gamma}\left(\varepsilon^{\frac{1}{p}} M^{\omega_r} + \varepsilon M^{\omega_p} + M + \varepsilon\Big[1 + (\Upsilon_\varepsilon^0)^{\mu_r}\Big] M^{\omega_r}\Psi\left(\|u_\varepsilon^0\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac{1}{2}}\right),
\end{aligned}$$

where we have used Remark 4.4.14. Again by Assumption **4.1.8**, this reads

$$\|(1+t)^{\gamma}\partial_t u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} + \|(1+t)^{\gamma}\mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \lesssim M^{\widetilde{\omega}_p},$$

where $\widetilde{\omega}_p > 0$. This yields the claimed estimate. □

4.5.3 Conclusion of the bootstrap argument

Recall the definition (4.3.17) of the time $t_\varepsilon^\star$. We eventually reach the following result.

Proposition 4.5.10. *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and for all $T \in (T_{\mathrm{abs}} + 10\alpha, t_\varepsilon^\star)$*

$$\int_{T_{\mathrm{abs}}+10\alpha}^{T} \|u_\varepsilon(s)\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^3_+)}\,\mathrm{d}s \lesssim \mathcal{E}_\varepsilon(0)^{\theta}.$$

for some $\theta > 0$, and where $\lesssim$ only depends on the constant M.

Proof. By the Gagliardo-Nirenberg-Sobolev inequality (see Theorem 4.2.9), we have for all $s \in (T_{\mathrm{abs}} + 10\alpha, t_\varepsilon^\star)$

$$\|u_\varepsilon(s)\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^3_+)} \lesssim \|\mathrm{D}_x^2 u_\varepsilon(s)\|^{\alpha_p}_{\mathrm{L}^p(\mathbb{R}^3_+)}\|u_\varepsilon(s)\|^{1-\alpha_p}_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|\mathrm{D}_x^2 u_\varepsilon(s)\|^{\beta_p}_{\mathrm{L}^p(\mathbb{R}^3_+)}\|u_\varepsilon(s)\|^{1-\beta_p}_{\mathrm{L}^2(\mathbb{R}^3_+)},$$

where

$$p > 3, \quad \alpha_p = \frac{3p}{7p-6}, \quad \beta_p = \frac{5p}{7p-6}.$$

Hence, for all $\gamma > 0$, we have by Hölder's inequality in time

$$\begin{aligned}
&\int_{T_{\mathrm{abs}}+10\alpha}^{T} \|u_\varepsilon(s)\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^3_+)} \,\mathrm{d}s \\
&\lesssim \int_{T_{\mathrm{abs}}+10\alpha}^{T} \|\mathrm{D}^2_x u_\varepsilon(s)\|^{\alpha_p}_{\mathrm{L}^p(\mathbb{R}^3_+)} \|u_\varepsilon(s)\|^{1-\alpha_p}_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s + \int_{T_{\mathrm{abs}}+10\alpha}^{T} \|\mathrm{D}^2_x u_\varepsilon(s)\|^{\beta_p}_{\mathrm{L}^p(\mathbb{R}^3_+)} \|u_\varepsilon(s)\|^{1-\beta_p}_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s \\
&\leq \int_0^T \left[\|(1+s)^\gamma \mathrm{D}^2_x u_\varepsilon(s)\|_{\mathrm{L}^p(\mathbb{R}^3_+)}\right]^{\alpha_p} (1+s)^{-\gamma\alpha_p} \|u_\varepsilon(s)\|^{1-\alpha_p}_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s \\
&\quad + \int_0^T \left[\|(1+s)^\gamma \mathrm{D}^2_x u_\varepsilon(s)\|_{\mathrm{L}^p(\mathbb{R}^3_+)}\right]^{\beta_p} (1+s)^{-\gamma\beta_p} \|u_\varepsilon(s)\|^{1-\beta_p}_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s \\
&\leq \|(1+t)^\gamma \mathrm{D}^2_x u_\varepsilon\|^{\alpha_p}_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \left(\int_0^T (1+s)^{-\gamma\frac{p\alpha_p}{p-\alpha_p}} \mathcal{E}_\varepsilon(s)^{\frac{1-\alpha_p}{2}\frac{p}{p-\alpha_p}} \,\mathrm{d}s\right)^{\frac{p-\alpha_p}{p}} \\
&\quad + \|(1+t)^\gamma \mathrm{D}^2_x u_\varepsilon\|^{\beta_p}_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \left(\int_0^T (1+s)^{-\gamma\frac{p\beta_p}{p-\beta_p}} \mathcal{E}_\varepsilon(s)^{\frac{1-\beta_p}{2}\frac{p}{p-\beta_p}} \,\mathrm{d}s\right)^{\frac{p-\beta_p}{p}},
\end{aligned}$$

therefore, by the energy-dissipation inequality (4.3.1)

$$\begin{aligned}
\int_{T_{\mathrm{abs}}+10\alpha}^{T} \|u_\varepsilon(s)\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^3_+)} \,\mathrm{d}s &\lesssim \mathcal{E}_\varepsilon(0)^{\frac{1-\alpha_p}{2}} \|(1+t)^\gamma \mathrm{D}^2_x u_\varepsilon\|^{\alpha_p}_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \left(\int_0^T (1+s)^{-\gamma\frac{p\alpha_p}{p-\alpha_p}} \,\mathrm{d}s\right)^{\frac{p-\alpha_p}{p}} \\
&\quad + \mathcal{E}_\varepsilon(0)^{\frac{1-\beta_p}{2}} \|(1+t)^\gamma \mathrm{D}^2_x u_\varepsilon\|^{\beta_p}_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \left(\int_0^T (1+s)^{-\gamma\frac{p\beta_p}{p-\beta_p}} \,\mathrm{d}s\right)^{\frac{p-\beta_p}{p}}.
\end{aligned}$$

Setting $\gamma_p := \frac{17}{8} - \frac{7}{4p}$, we observe that for $p > 3$, we have

$$\min\left(\frac{p\beta_p}{p-\beta_p}, \frac{p\alpha_p}{p-\alpha_p}\right) > \frac{1}{\gamma_p},$$

therefore we can pick some $\gamma \in (0, \gamma_p)$ (which depends on p) such that

$$\gamma\frac{p\beta_p}{p-\beta_p} > 1, \quad \gamma\frac{p\alpha_p}{p-\alpha_p} > 1.$$

The two previous integrals in time are thus bounded uniformly in T. Owing to the uniform bound of Lemma 4.5.9, we get

$$\int_{T_{\mathrm{abs}}+10\alpha}^{T} \|u_\varepsilon(s)\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^3_+)} \,\mathrm{d}s \lesssim \mathcal{E}_\varepsilon(0)^{\frac{1-\alpha_p}{2}} + \mathcal{E}_\varepsilon(0)^{\frac{1-\beta_p}{2}},$$

where $\lesssim$ only depends on the constant M. Taking the maximum of the two last quantities yields the result. □

Proposition 4.5.11. *There exists $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ then $t^\star_\varepsilon = +\infty$.*

Proof. According to the estimate (4.5.2) from Corollary 4.5.4 and to Proposition 4.5.10, we have for all $T \in (T_{\mathrm{abs}} + 10\alpha, t_\varepsilon^\star)$

$$\int_0^T \|u_\varepsilon(s)\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^3_+)} \, \mathrm{d}s < \frac{\delta^\star}{4} + \mathrm{A}\mathcal{E}_\varepsilon(0)^\theta,$$

for some $\theta > 0$ and $\mathrm{A} > 0$, provided that $\varepsilon > 0$ is small enough. Thus, according to the smallness condition of Assumption **A3-b**, we can ensure that for all $T \in (T_{\mathrm{abs}} + 10\alpha, t_\varepsilon^\star)$

$$\int_0^T \|u_\varepsilon(s)\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^3_+)} \, \mathrm{d}s < \frac{\delta^\star}{2}. \tag{4.5.9}$$

On the other hand, according to the estimate (4.5.1) from Corollary 4.5.4, we know that

$$\int_0^{T_{\mathrm{abs}}+10\alpha} \|F_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \, \mathrm{d}s + \int_0^{T_{\mathrm{abs}}+10\alpha} \|F_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \, \mathrm{d}s < \frac{\mathrm{C}_\star}{4}.$$

Using the inequality

$$\|F_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \|\rho_\varepsilon(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \mathrm{D}_\varepsilon(s),$$

and the energy dissipation inequality (4.3.1), we also have for all $T \in (T_{\mathrm{abs}} + 10\alpha, t_\varepsilon^\star)$

$$\int_{T_{\mathrm{abs}}+10\alpha}^T \|F_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \, \mathrm{d}s \leq \|\rho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3_+))} \mathcal{E}_\varepsilon(0) \leq \|f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \mathcal{E}_\varepsilon(0) \leq M\mathcal{E}_\varepsilon(0),$$

thanks to Corollary 4.2.5. Using Assumption **A3-b**, one can ensure

$$\int_{T_{\mathrm{abs}}+10\alpha}^T \|F_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \, \mathrm{d}s < \frac{\mathrm{C}_\star}{16}.$$

By the Cauchy-Schwarz inequality, we also have

$$\int_{T_{\mathrm{abs}}+10\alpha}^T \|F_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \, \mathrm{d}s \lesssim \left(\int_{T_{\mathrm{abs}}+10\alpha}^T (1+t)^\gamma \|F_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \, \mathrm{d}s \right)^{\frac{1}{2}},$$

for any $\gamma > 1$ (take for instance $\gamma = 1^+$ to be optimal). We then invoke the pointwise estimates obtained in Subsection 4.4.1: via Lemmas 4.4.2–4.4.4–4.4.5 and the uniform bound of Assumption **4.1.8**, we have for all $t \in (T_{\mathrm{abs}} + 10\alpha, T)$

$$\begin{aligned} \|F_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} &\lesssim \frac{e^{\frac{-2t}{\varepsilon}}}{(1+t)^{k_1}} M^{\iota_2} + \frac{1}{(1+t)^{2k_2}} \left[\||v|^{2k_2} f_\varepsilon^0\|^2_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))} + \|x_3^{2k_2} f_\varepsilon^0\|^2_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))} \right] \\ &\quad + \frac{\varepsilon}{(1+t)^{k_3}} M^{\iota_2} + \frac{\varepsilon^{\frac{3}{2}}}{(1+t)^{k_4}} M^{\iota_2} + \frac{\mathcal{E}_\varepsilon(0)}{(1+t)^{k_5 - \frac{1}{2}}} M^{\iota_2}, \end{aligned}$$

for some $\iota_2 > 0$ and where the exponents k_i $(i = 1, \cdots, 5)$ are choosen as follows, according to Assumption **4.1.8**: we take $k_1 = \gamma$, $2k_2 > 1 + \gamma$, $k_3 > 1 + \gamma$, $k_4 > 1 + \gamma$ and $k_5 > \frac{3}{2} + \gamma$. We end up with

$$\int_{T_{\mathrm{abs}}+10\alpha}^T \|F_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \, \mathrm{d}s \lesssim \varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{3}{4}} + \mathcal{E}_\varepsilon(0)^{\frac{1}{2}} + \||v|^{2k_2} f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))} + \|x_3^{2k_2} f_\varepsilon^0\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^2(\mathbb{R}^3_+))}.$$

According to Assumption **A3-b**, we can ensure that the last quantity is choosen so that

$$\int_{T_{\mathrm{abs}}+10\alpha}^{T} \|F_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} < \frac{\mathrm{C}_\star}{16}.$$

Gathering the previous estimates together, we get for all $T \in (0, t_\varepsilon^\star)$

$$\|u_\varepsilon^0\|^2_{\mathrm{H}^1(\mathbb{R}^3_+)} + \int_0^T \|F_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s + \int_0^T \|F_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s < \frac{7\mathrm{C}_\star}{8}, \tag{4.5.10}$$

thanks to Assumption **A3-a**. The estimates (4.5.9) and (4.5.10) therefore hold for $\varepsilon > 0$ small enough, say $\varepsilon \in (0, \varepsilon^\star)$.

Now assume that there exists $\varepsilon \in (0, \varepsilon^\star)$ such that $t_\varepsilon^\star < \infty$. Invoking the continuity of $s \mapsto \|F_\varepsilon\|_{\mathrm{L}^2\cap\mathrm{L}^1(0,s;\mathrm{L}^2(\mathbb{R}^3_+))}$ (given by (4.3.10)), one observes that the estimate (4.5.10) entails there exists a strong existence time strictly greater than $t_\varepsilon^\star$. An additional continuity argument (owing to the integrability results (4.3.11) and (4.3.9)) combined with the estimate (4.5.9) shows that there exists a strong existence time $T^\varepsilon > t_\varepsilon^\star$ which satisfies

$$\int_0^{T^\varepsilon} \|u_\varepsilon(s)\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^3_+)}\,\mathrm{d}s < \frac{3\delta^\star}{4}.$$

This is a contradiction with the definition of $t_\varepsilon^\star$ and this finally achieves the proof of the proposition. □

We now turn to the proof of Theorem 4.1.10, where we obtain the weak convergence of $(\rho_\varepsilon, u_\varepsilon)$ towards a solution (ρ, u) of the Boussinesq-Navier-Stokes system (4.1.11), assuming the convergence of the initial condition $(\rho_\varepsilon^0, u_\varepsilon^0)$.

As explained in the introduction, our proof is based on all the uniform (in ε) estimates that we have obtained as a byproduct of the previous boostrap strategy and which now hold on any interval of time since $t_\varepsilon^\star = +\infty$. More precisely, we work in the framework underlined by the conditional Proposition 4.3.5.

Proof of Theorem 4.1.10 . Let $T > 0$ be any fixed time. Let us show that the assumptions of Proposition 4.3.5 are satisfied for ε small enough.

- (**C1**) is satisfied thanks to Proposition 4.5.11 and the very definition of $t_\varepsilon^\star$.
- (**C2**) is satisfied in view of Corollary 4.3.12 and because any time is a strong existence time.
- (**C3**) is satisfied in view of Assumption **4.1.8** on the initial data.
- (**C4**) can be obtained by using the fact that

$$\forall s \geq 0, \quad \|u_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \frac{\Psi\left(\|u_\varepsilon^0\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)}{(1+s)^{\frac{3}{4}}}.$$

  Indeed, performing the same computations as in the end of Lemma 4.5.3, one can use the previous decay in time to prove that

$$\int_0^T \|\nabla_x u_\varepsilon(s)\|^2_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s \lesssim \|\mathrm{D}_x^2 u_\varepsilon\|^{2\beta_p}_{\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{R}^3_+))}(1+T)^{\zeta_p},$$

  for any $p > 3$ and some $(\beta_p, \zeta_p) \in (0,1) \times (0,+\infty)$. Owing to Lemma 4.5.9 for $p \to 3^+$ then ensures (**C4**).

Applying the conditional Proposition 4.3.5 on the interval $[0, T]$ yields the claimed convergence when $\varepsilon \to 0$, up to an extraction. A final diagonal extraction along increasing interval of times shows that we can find a common extraction which is valid for all times.

According to the estimate (4.5.8) of Proposition 4.3.7 which holds for all times, and to the very definition of $t_\varepsilon^\star = +\infty$, another weak compactness argument shows that the accumulation point u we have obtained before satisfies

$$u \in \mathrm{L}^\infty(\mathbb{R}^+; \mathrm{H}^1(\mathbb{R}^3_+)) \cap \mathrm{L}^2(\mathbb{R}^+; \mathrm{H}^2(\mathbb{R}^3_+)) \cap \mathrm{L}^1(\mathbb{R}^+; \mathrm{W}^{1,\infty}(\mathbb{R}^3_+)).$$

This eventually concludes the proof. □

Remark 4.5.12. Following precisely the exponent involved in the proof of 4.5.3 and which depend on $p \to 3^+$ (especially for the treatment of the term III), one can prove that there exists $\mu_2 > 0$ such that for all $\varepsilon > 0$ small enough, we have

$$\|j_\varepsilon - \rho_\varepsilon u_\varepsilon + \rho_\varepsilon e_3\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3_+)} \lesssim M^{\mu_2}\left(\varepsilon^{\frac{1}{2}} + \varepsilon + \varepsilon(1+T)^{\frac{1}{5}}\right),$$

provided that Assumptions **4.1.7–4.1.8–4.1.9** hold.

4.5.4 Rates of strong convergence

We finally deal with the proof of Theorem 4.1.11: namely, we are looking for some rate of convergence in order to quantify the result of Theorem 4.1.10, assuming strong convergence of the initial data. To do so, we use a proof based on energy estimates.

Proof of Theorem 4.1.11. Let (ρ, u) be any global strong solution of the Boussinesq-Navier-Stokes system (4.1.11) (namely, $(\rho, u) \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\mathbb{R}^3_+ \times \mathbb{R}^3)) \times \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}^1_{\mathrm{div}}(\mathbb{R}^3_+)) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}^2(\mathbb{R}^3_+)) \cap \mathrm{L}^1(\mathbb{R}^+; \mathrm{W}^{1,\infty}(\mathbb{R}^3_+))$): note that such a solution indeed exists thanks to the previous subsection. We set for any $\varepsilon > 0$

$$w_\varepsilon := u_\varepsilon - u, \quad \theta_\varepsilon := \rho_\varepsilon - \rho.$$

Let $T > 0$. We observe that $(w_\varepsilon, \theta_\varepsilon)$ satisfies (at least in the sense of distributions)

$$\begin{cases} \partial_t \theta_\varepsilon + \mathrm{div}_x\left[\theta_\varepsilon(u_\varepsilon - e_3)\right] = -\mathrm{div}_x\left[F_\varepsilon + \rho_\varepsilon e_3 + \rho(u_\varepsilon - u)\right], \\ \theta_{\varepsilon|t=0} = \rho_\varepsilon^0 - \rho^0, \end{cases} \tag{4.5.11}$$

and

$$\begin{cases} \partial_t w_\varepsilon + (u\cdot\nabla_x) w_\varepsilon - \Delta_x w_\varepsilon + \nabla_x(p_\varepsilon - p) = F_\varepsilon + \rho_\varepsilon e_3 - \theta_\varepsilon e_3 - (w_\varepsilon \cdot \nabla_x) u_\varepsilon, \\ \mathrm{div}_x\, w_\varepsilon = 0, \\ w_{\varepsilon|t=0} = u_\varepsilon^0 - u^0. \end{cases} \tag{4.5.12}$$

Since $u(t)$ is divergence free and all the solutions are strong for all times, a classical energy estimate in L^2_x for the system (4.5.12) leads to

$$\begin{aligned} \frac{1}{2}\|w_\varepsilon(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \int_0^t \|\nabla_x w_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s &\lesssim \frac{1}{2}\|w_\varepsilon^0\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} \\ &\quad + \int_0^t \|w_\varepsilon(s)\|^{\frac{1}{2}}_{\mathrm{L}^2(\mathbb{R}^3_+)} \|\nabla_x w_\varepsilon(s)\|^{\frac{3}{2}}_{\mathrm{L}^2(\mathbb{R}^3_+)} \|\nabla_x u_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s \\ &\quad + \int_0^t \int_{\mathbb{R}^3_+} \left|\left[F_\varepsilon + \rho_\varepsilon e_3 + \theta_\varepsilon e_3\right](s,x)\right| |w_\varepsilon(s,x)|\,\mathrm{d}x\,\mathrm{d}s, \end{aligned}$$

for any $t \in [0, T]$. Using Proposition 4.3.7 and Young inequality, we can write

$$\int_0^t \|w_\varepsilon(s)\|_{L^2(\mathbb{R}^3_+)}^{\frac{1}{2}} \|\nabla_x w_\varepsilon(t)\|_{L^2(\mathbb{R}^3_+)}^{\frac{3}{2}} \|\nabla_x u_\varepsilon(s)\|_{L^2(\mathbb{R}^3_+)} \,\mathrm{d}s \lesssim \frac{c^{-4}}{4}\int_0^t \|w_\varepsilon(s)\|_{L^2(\mathbb{R}^3_+)}^2 \,\mathrm{d}s + \frac{3c^{\frac{4}{3}}}{4}\int_0^t \|\nabla_x w_\varepsilon(s)\|_{L^2(\mathbb{R}^3_+)}^2 \,\mathrm{d}s,$$

for any $c > 0$. Choosing $c > 0$ small enough, we can absorb the last term and get for all $t \in [0, T]$

$$\begin{aligned}
&\frac{1}{2}\|w_\varepsilon(t)\|_{L^2(\mathbb{R}^3_+)}^2 + \frac{1}{2}\int_0^t \|\nabla_x w_\varepsilon(s)\|_{L^2(\mathbb{R}^3_+)}^2 \,\mathrm{d}s \\
&\lesssim \|w_\varepsilon^0\|_{L^2(\mathbb{R}^3_+)}^2 + \int_0^t \|w_\varepsilon(s)\|_{L^2(\mathbb{R}^3_+)}^2 \,\mathrm{d}s + \int_0^t\int_{\mathbb{R}^3_+} |[F_\varepsilon + \rho_\varepsilon e_3 - \theta_\varepsilon e_3](s,x)|\,|w_\varepsilon(s,x)| \,\mathrm{d}x\,\mathrm{d}s \\
&\leq \|w_\varepsilon^0\|_{L^2(\mathbb{R}^3_+)}^2 + \int_0^t \|w_\varepsilon(s)\|_{L^2(\mathbb{R}^3_+)}^2 \,\mathrm{d}s \\
&\quad + \int_0^t \|F_\varepsilon(s) + \rho_\varepsilon(s) e_3\|_{L^2(\mathbb{R}^3_+)} \|w_\varepsilon(s)\|_{L^2(\mathbb{R}^3_+)} \,\mathrm{d}s + \int_0^t \|\theta_\varepsilon(s)\|_{H^{-1}(\mathbb{R}^3_+)} \|w_\varepsilon(s)\|_{H^1_0(\mathbb{R}^3_+)} \,\mathrm{d}s \\
&\leq \|w_\varepsilon^0\|_{L^2(\mathbb{R}^3_+)}^2 + \frac{3}{2}\int_0^t \|w_\varepsilon(s)\|_{L^2(\mathbb{R}^3_+)}^2 \,\mathrm{d}s + \int_0^t \|F_\varepsilon(s) + \rho_\varepsilon(s) e_3\|_{L^2(\mathbb{R}^3_+)}^2 \,\mathrm{d}s \\
&\qquad + \frac{c^{-1}}{2}\int_0^t \|\theta_\varepsilon(s)\|_{H^{-1}(\mathbb{R}^3_+)}^2 \,\mathrm{d}s + \frac{c}{2}\int_0^t \|w_\varepsilon(s)\|_{H^1_0(\mathbb{R}^3_+)}^2 \,\mathrm{d}s,
\end{aligned}$$

for any $c > 0$, as above. Therefore, choosing again c small enough, we can absorb the gradient part of $\|w_\varepsilon(s)\|_{H^1_0(\mathbb{R}^3_+)}^2$ in the l.h.s, and this yields for all $t \in [0, T]$

$$\begin{aligned}
&\|w_\varepsilon(t)\|_{L^2(\mathbb{R}^3_+)}^2 \\
&\lesssim \|w_\varepsilon^0\|_{L^2(\mathbb{R}^3_+)}^2 + \|F_\varepsilon + \rho_\varepsilon e_3\|_{L^2(0,T;L^2(\mathbb{R}^3_+))}^2 + \int_0^t \|\theta_\varepsilon(s)\|_{H^{-1}(\mathbb{R}^3_+)}^2 \,\mathrm{d}s + \int_0^t \|w_\varepsilon(s)\|_{L^2(\mathbb{R}^3_+)}^2 \,\mathrm{d}s.
\end{aligned} \tag{4.5.13}$$

Hence, we need to derive an estimate on θ_ε in $L^2_t H^{-1}_x$. This is the purpose of the following lemma.

Lemma 4.5.13. *For all $s \in [0, T]$, we have*

$$\|\theta_\varepsilon(s)\|_{H^{-1}(\mathbb{R}^3_+)}^2 \lesssim \|\theta_\varepsilon^0\|_{H^{-1}(\mathbb{R}^3_+)}^2 + T\|F_\varepsilon + \rho_\varepsilon e_3\|_{L^2(0,T;L^2(\mathbb{R}^3_+))}^2 + T\int_0^s \|w_\varepsilon(\tau)\|_{L^2(\mathbb{R}^3_+)}^2 \,\mathrm{d}\tau. \tag{4.5.14}$$

Proof. Let us define the characteristic curves associated to the continuity equation (4.5.11): for $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^3$, we consider $s \mapsto \mathfrak{X}_\varepsilon(s; t, x)$ the solution to

$$\begin{cases} \dot{\mathfrak{X}}_\varepsilon(s; t, x) = (Pu_\varepsilon)(s, \mathfrak{X}_\varepsilon(s; t, x)) - e_3, \\ \mathfrak{X}_\varepsilon(t; t, x) = x, \end{cases} \tag{4.5.15}$$

where the dot means derivative along the first variable. Note that $[(Pu_\varepsilon)(s, x) - e_3] \cdot n(x) = 1 > 0$ for all $x \in \{x_3 = 0\}$ therefore

$$y \in \mathbb{R}^3_+ \implies \forall\, 0 \leq s \leq t, \quad \mathfrak{X}(s; t, y) \in \mathbb{R}^3_+,$$
$$y \in \{x_3 = 0\} \implies \forall\, t > 0, \quad \mathfrak{X}(t; 0, y) \notin \mathbb{R}^3_+.$$

In addition, since $\mathrm{div}_x[(Pu_\varepsilon)(s) - e_3] = 0$, we have $\det \mathrm{D}_x\mathfrak{X}(s; t, x) = 1$ and

$$\|\mathrm{D}_x\mathfrak{X}(s; t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq e^{\|\nabla_x (Pu_\varepsilon - e_3)\|_{L^1(0,T;L^\infty(\mathbb{R}^3))}} \lesssim 1, \tag{4.5.16}$$

since $t_\varepsilon^\star = +\infty$, at least for ε small enough. Our proof is thus based on the following identity: for any test function $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^3_+)$ and any $t \geq 0$, we have

$$\int_{\mathbb{R}^3_+} \theta_\varepsilon(t,x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^3_+} \theta_\varepsilon^0(\mathfrak{X}_\varepsilon(0;t,x))\varphi(x)\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^3_+} H_\varepsilon(s,x)\cdot\nabla_x[\varphi(\mathfrak{X}_\varepsilon(t;s,x))]\,\mathrm{d}x\,\mathrm{d}s, \tag{4.5.17}$$

where

$$H_\varepsilon := F_\varepsilon + \rho_\varepsilon e_3 + \rho(u_\varepsilon - u).$$

Let us fix $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^3_+)$ such that $\|\varphi\|_{\mathrm{H}^1(\mathbb{R}^3_+)} \leq 1$. We shall estimate the two terms in the r.h.s of (4.5.17).

- For the first term, we use the natural change of variable $x \mapsto \mathfrak{X}_\varepsilon(0;t,x)$ and get

$$\int_{\mathbb{R}^3_+} \theta_\varepsilon^0(\mathfrak{X}_\varepsilon(0;t,x))\varphi(x)\,\mathrm{d}x = \int_{\mathfrak{X}_\varepsilon(0;t,\mathbb{R}^3_+)} \theta_\varepsilon^0(y)\varphi(\mathfrak{X}_\varepsilon(t;0,y))\,\mathrm{d}y \leq \int_{\mathbb{R}^3_+} \theta_\varepsilon^0(y)\varphi(\mathfrak{X}_\varepsilon(t;0,y))\,\mathrm{d}y.$$

By the previous remark, we observe that $y \mapsto \varphi(\mathfrak{X}_\varepsilon(t;0,y))$ vanishes on $\{x_3 = 0\}$. It also belongs to $\mathrm{H}^1(\mathbb{R}^3_+)$. Indeed, since φ is compactly supported in $\mathbb{R}^3_+$, we have

$$\int_{\mathbb{R}^3_+} |\varphi(\mathfrak{X}_\varepsilon(t;0,y))|^2\,\mathrm{d}y = \int_{\mathfrak{X}_\varepsilon(t;0,\mathbb{R}^3_+)} |\varphi(x)|^2\,\mathrm{d}x = \int_{\mathfrak{X}_\varepsilon(t;0,\mathbb{R}^3_+)\cap\mathbb{R}^3_+} |\varphi(x)|^2\,\mathrm{d}x \leq \|\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)}.$$

Furthermore, by (4.5.16), we have

$$\begin{aligned}
\|\nabla_x[\varphi(\mathfrak{X}_\varepsilon(t;0))]\|_{\mathrm{L}^2(\mathbb{R}^3_+)} &= \|\mathrm{D}_x\mathfrak{X}_\varepsilon(t;0)\nabla_x\varphi(\mathfrak{X}_\varepsilon(t;0))\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \\
&\leq \|\mathrm{D}_x\mathfrak{X}_\varepsilon(t;0)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\|\nabla_x\varphi(\mathfrak{X}_\varepsilon(t;0))\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \\
&\lesssim \|\nabla_x\varphi\|_{\mathrm{L}^2(\mathbb{R}^3_+)},
\end{aligned}$$

as in the previous computation. This yields

$$\int_{\mathbb{R}^3_+} \theta_\varepsilon^0(\mathfrak{X}_\varepsilon(0;t,x))\varphi(x)\,\mathrm{d}x \leq \|\theta_\varepsilon^0\|_{\mathrm{H}^{-1}(\mathbb{R}^3_+)}.$$

- For the second term, we write

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^3_+} H_\varepsilon(s,x)\cdot\nabla_x[\varphi(\mathfrak{X}_\varepsilon(t;s,x))]\,\mathrm{d}x\,\mathrm{d}s &\leq \int_0^t \|H_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\|\nabla_x[\varphi(\mathfrak{X}_\varepsilon(t;s))]\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s \\
&\lesssim \|\nabla_x\varphi\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \int_0^t \|H_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s,
\end{aligned}$$

as above. By Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3_+} H_\varepsilon(s,x)\cdot\nabla_x[\varphi(\mathfrak{X}_\varepsilon(t;s,x))]\,\mathrm{d}x\,\mathrm{d}s \\
&\leq \sqrt{T}\|F_\varepsilon + \rho_\varepsilon e_3\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} + \sqrt{T}\|\rho(u_\varepsilon - u)\|_{\mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}^3_+))} \\
&\leq \sqrt{T}\|F_\varepsilon + \rho_\varepsilon e_3\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} + \sqrt{T}\|w_\varepsilon\|_{\mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}^3_+))},
\end{aligned}$$

since

$$\|\rho\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3_+))} \leq \liminf_{\varepsilon\to 0} \|\rho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3_+))} \lesssim 1,$$

by lower semicontinuity of the weak* convergence and the uniform bound given in Corollary (4.2.5) which holds for all times.

We obtain the conclusion of the lemma thanks to the identity (4.5.17). □

We can now conclude the proof of Theorem 4.1.11. We add the estimate (4.5.13) to the estimate (4.5.14) and obtain for all $t \in [0, T]$

$$\|w_\varepsilon(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|\theta_\varepsilon(t)\|^2_{\mathrm{H}^{-1}(\mathbb{R}^3_+)} \lesssim \|w^0_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|\theta^0_\varepsilon\|^2_{\mathrm{H}^{-1}(\mathbb{R}^3_+)} + (1+T)\|F_\varepsilon + \rho_\varepsilon e_3\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))} + (1+T)\int_0^t \left(\|w_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|\theta_\varepsilon(s)\|^2_{\mathrm{H}^{-1}(\mathbb{R}^3_+)}\right) \mathrm{d}s.$$

Grönwall's lemma gives for all $t \in [0, T]$

$$\|w_\varepsilon(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|\theta_\varepsilon(t)\|^2_{\mathrm{H}^{-1}(\mathbb{R}^3_+)} \lesssim e^{C_M(1+T)} \left(\|w^0_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{R}^3_+)} + \|\theta^0_\varepsilon\|^2_{\mathrm{H}^{-1}(\mathbb{R}^3_+)} + (1+T)\|F_\varepsilon + \rho_\varepsilon e_3\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^3_+))}\right).$$

Invoking Remark 4.5.12, we end up with the result claimed in Theorem 4.1.11. □

Appendix

4.A Proof of Lemmas 4.4.8–4.4.9–4.4.10

This appendix is devoted to the proof of Lemmas 4.4.8–4.4.9–4.4.10, which provide decay estimates of the Brinkman force in $\mathrm{L}^p_t\mathrm{L}^p_x$. Recall that we work under the polynomial decay in time of u_ε given by Theorem 4.3.3.

Proof of Lemma 4.4.8. We only focus on the first estimate of Lemma 4.4.8. By the triangular inequality, we have for all $t \in (T_0, T)$

$$\|F^{\natural,0}_\varepsilon(t)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \leq \frac{e^{-\frac{t}{\varepsilon}}}{(1+R(t))^k}\left[\int_{\mathbb{R}^3_+}\left|\int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w))|w|^{k_1} f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w) \times \left|w + e_3 - (Pu_\varepsilon)(0,\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w))\right| \mathrm{d}w\right|^r \mathrm{d}x\right]^{\frac{1}{r}}$$
$$+ \frac{1}{(1+R(t))^{k_2}}\left[\int_{\mathbb{R}^3_+}\left(\int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w))|w|^{k_2} f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)\,\mathrm{d}w\right)^r \mathrm{d}x\right]^{\frac{1}{r}}.$$

We apply Hölder inequality in velocity for the first term, as in the proof of Lemma 4.4.2, and we obtain

$$\int_{\mathbb{R}^3_+}\left(\int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w))|w|^{k_1} f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)\left|w + e_3 - (Pu_\varepsilon)(0,\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w))\right| \mathrm{d}w\right)^r \mathrm{d}x$$
$$\leq \int_{\mathbb{R}^3_+}\left(\int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w))|w|^{k_1} f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)\,\mathrm{d}w\right)^{r-1} \times \left(\int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}^t_\varepsilon}(x, [\Gamma^{t,x}_\varepsilon]^{-1}(w))|w|^{k_1} f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)\left|w + e_3 - (Pu_\varepsilon)(0,\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w))\right|^r \mathrm{d}w\right) \mathrm{d}x$$
$$\leq \||v|^{k_1} f^0_\varepsilon\|^{r-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\left[\|(1+|v|^{k_1+r})f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3_+\times\mathbb{R}^3)} + \||v|^{k_1} f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|u^0_\varepsilon\|^r_{\mathrm{L}^r(\mathbb{R}^3_+)}\right],$$

where we have followed the rest of the proof of Lemma 4.4.2. For the second term, we apply the generalized Minkowski inequality (see e.g. [HLP88]) and get

$$\left[\int_{\mathbb{R}^3_+}\left(\int_{\mathbb{R}^3}\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))|w|^{k_2}f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)\,\mathrm{d}w\right)^r\mathrm{d}x\right]^{\frac{1}{r}}$$
$$\leq\int_{\mathbb{R}^3}|w|^{k_2}\left(\int_{\mathbb{R}^3_+}\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)^r\,\mathrm{d}x\right)^{\frac{1}{r}}\mathrm{d}w\lesssim\int_{\mathbb{R}^3}|w|^{k_2}\|f^0_\varepsilon(\cdot,w)\|_{\mathrm{L}^r(\mathbb{R}^3_+)}\,\mathrm{d}w.$$

All in all, we obtain

$$\begin{aligned}\int_{T_0}^T\|(1+t)^kF^{\natural,0}_\varepsilon(t)\|^r_{\mathrm{L}^r(\mathbb{R}^3_+)}\,\mathrm{d}t&\lesssim\int_0^T\frac{e^{-r\frac{t}{\varepsilon}}}{(1+t)^{r(k_1-k)}}\||v|^{k_1}f^0_\varepsilon\|^{r-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\Big[\|(1+|v|^{k_1+r})f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3_+\times\mathbb{R}^3)}\\&\qquad+\||v|^{k_1}f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|u^0_\varepsilon\|^r_{\mathrm{L}^p(\mathbb{R}^3_+)}\Big]\,\mathrm{d}t\\&\quad+\int_0^T\frac{1}{(1+t)^{r(k_2-k)}}\||v|^{k_2}f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^r(\mathbb{R}^3_+))}\,\mathrm{d}t\\&\lesssim\varepsilon\||v|^{k_1}f^0_\varepsilon\|^{r-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\Big[\|(1+|v|^{k_1+r})f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3_+\times\mathbb{R}^3)}\\&\qquad+\||v|^{k_1}f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|u^0_\varepsilon\|^r_{\mathrm{L}^p(\mathbb{R}^3_+)}\Big]\\&\quad+\||v|^{k_2}f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^r(\mathbb{R}^3_+))},\end{aligned}$$

where the last inequality $\lesssim$ is independent of T, thanks to the choice of k_1 and k_2. This concludes the proof of the lemma. □

Proof of Lemma 4.4.9. As in the proof of Lemma 4.4.8, we only write the proof for the first estimate. Following the proof of Lemma 4.4.4, we get for any $t\in(T_0,T)$

$$\int_{\mathbb{R}^3_+}|F^{\natural,1}_\varepsilon(t,x)|^r\,\mathrm{d}x\lesssim\frac{\varepsilon^{r-1}\||w|^kf^0_\varepsilon\|^r_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}}{(1+t)^{kr}}\int_0^te^{\frac{\tau-t}{\varepsilon}}\|\partial_\tau u_\varepsilon(\tau)\|^r_{\mathrm{L}^r(\mathbb{R}^3)}\,\mathrm{d}\tau.$$

We then use Fubini theorem to get

$$\begin{aligned}\|(1+t)^kF^{\natural,1}_\varepsilon(t)\|^r_{\mathrm{L}^r(T_0,T;\mathrm{L}^r(\mathbb{R}^3_+))}&\lesssim\varepsilon^{r-1}\||v|^kf^0_\varepsilon\|^r_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\int_{T_0}^T\int_0^te^{\frac{\tau-t}{\varepsilon}}\|\partial_\tau u_\varepsilon(\tau)\|^r_{\mathrm{L}^r(\mathbb{R}^3)}\,\mathrm{d}\tau\,\mathrm{d}t\\&\lesssim\varepsilon^{r-1}\||v|^kf^0_\varepsilon\|^r_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\int_0^T\|\partial_\tau u_\varepsilon(\tau)\|^r_{\mathrm{L}^r(\mathbb{R}^3)}\left(\int_\tau^Te^{\frac{\tau-t}{\varepsilon}}\,\mathrm{d}t\right)\mathrm{d}\tau\\&\lesssim\varepsilon^r\||v|^kf^0_\varepsilon\|^r_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\|\partial_tu_\varepsilon\|^r_{\mathrm{L}^r(0,T;\mathrm{L}^r(\mathbb{R}^3_+))},\end{aligned}$$

and this concludes the proof. □

Proof of Lemma 4.4.10. First, we derive estimates which hold without using (4.4.3). We have for all $t\in(T_0,T)$

$$\|(1+t)^kF^{\natural,2}_\varepsilon(t)\|_{\mathrm{L}^r(\mathbb{R}^3_+)}\lesssim\|\mathrm{I}^\natural(t)\|_{\mathrm{L}^r(\mathbb{R}^3_+)}+\|\mathrm{II}^\natural(t)\|_{\mathrm{L}^r(\mathbb{R}^3_+)}+\|\mathrm{III}^\natural(t)\|_{\mathrm{L}^r(\mathbb{R}^3_+)},$$

where we refer to the proof of Lemma 4.4.5 for the definition of the previous terms. We have

$$\begin{aligned}|\mathrm{I}^\natural(t,x)|^r&\lesssim\varepsilon^{r-1}\|(1+|v|^{\frac{r}{r-1}})|v|^kf^0_\varepsilon\|^{r-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\\&\quad\times\int_0^te^{\frac{r(s-t)}{\varepsilon}}\int_{\mathbb{R}^3}\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))|w|^kf^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)\Big|\nabla_x[Pu_\varepsilon](s,\widetilde{\mathrm{X}}^{s;t}_\varepsilon(x,w))\Big|^r\,\mathrm{d}w\,\mathrm{d}s\end{aligned}$$

thanks to Jensen inequality, so that

$$\int_{T_0}^{T}\int_{\mathbb{R}^3_+}|\mathrm{I}^\natural(t,x)|^r\,\mathrm{d}x\,\mathrm{d}t \lesssim \varepsilon^{r-1}\|(1+|v|^{\frac{r}{r-1}})|v|^k f^0_\varepsilon\|^{r-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\||v|^k f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \times\int_{T_0}^{T}\int_0^t e^{\frac{r(s-t)}{\varepsilon}}\|\nabla_x[Pu_\varepsilon](s)\|^r_{\mathrm{L}^r(\mathbb{R}^3)}\,\mathrm{d}s\,\mathrm{d}t$$
$$\lesssim \varepsilon^{r}\|(1+|v|^{\frac{r}{r-1}})|v|^k f^0_\varepsilon\|^{r-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\||v|^k f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \times\int_0^T\|\nabla_x u_\varepsilon(s)\|^r_{\mathrm{L}^r(\mathbb{R}^3_+)}\,\mathrm{d}s.$$

For $\mathrm{II}^\natural$, we obtain in the same fashion

$$|\mathrm{II}^\natural(t,x)|^r$$
$$\leq \||v|^k f^0_\varepsilon\|^{r-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}$$
$$\times\left[\int_0^t e^{\frac{r(s-t)}{\varepsilon}}\left(\int_{\mathbb{R}^3}\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))|w|^k f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)\left|\nabla_x[Pu_\varepsilon](s,\widetilde{\mathrm{X}}^{s;t}_\varepsilon(x,w))\right|^r\mathrm{d}w\right)^{\frac{1}{r}}\mathrm{d}s\right]^r$$
$$\lesssim \varepsilon^{r-1}\|f^0_\varepsilon\|^{r-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}$$
$$\times\int_0^t e^{\frac{r(s-t)}{\varepsilon}}\left(\int_{\mathbb{R}^3}\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))|w|^k f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)\left|\nabla_x[Pu_\varepsilon](s,\widetilde{\mathrm{X}}^{s;t}_\varepsilon(x,w))\right|^r\mathrm{d}w\right)\mathrm{d}s,$$

therefore

$$\int_{T_0}^{T}\int_{\mathbb{R}^3_+}|\mathrm{II}^\natural(t,x)|^r\,\mathrm{d}x\,\mathrm{d}t \lesssim \varepsilon^{r-1}\|f^0_\varepsilon\|^{r-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\int_{T_0}^{T}\int_0^t e^{\frac{r(s-t)}{\varepsilon}}\|\nabla_x[Pu_\varepsilon](s)\|^r_{\mathrm{L}^r(\mathbb{R}^3)}\,\mathrm{d}s\,\mathrm{d}t$$
$$\lesssim \varepsilon^{r}\|f^0_\varepsilon\|^{r-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\int_0^T\|\nabla_x u_\varepsilon(s)\|^r_{\mathrm{L}^r(\mathbb{R}^3_+)}\,\mathrm{d}s.$$

For the third term $\mathrm{III}^\natural$, we have as in the proof of Lemma 4.4.5

$$\int_{T_0}^{T}\int_{\mathbb{R}^3_+}|\mathrm{III}^\natural(t,x)|^r\,\mathrm{d}x\,\mathrm{d}t$$
$$\lesssim \varepsilon^{-1}\||v|^k f^0_\varepsilon\|^{r-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\left[\int_0^T\left(\int_0^t e^{\frac{r(s-t)}{2(r-1)\varepsilon}}\|\nabla_x u_\varepsilon(s)\|^{\frac{r}{r-1}}_{\mathrm{L}^\infty(\mathbb{R}^3_+)}\,\mathrm{d}s\right)^{r-1}\mathrm{d}t\right]$$
$$\times\sup_{t\in(0,T)}\left\{\int_0^t e^{\frac{r(s-t)}{2\varepsilon}}\int_0^s e^{\frac{r(\tau-s)}{2\varepsilon}}\right.$$
$$\left.\times\int_{\mathbb{R}^3_+\times\mathbb{R}^3}\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))|w|^k f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)\left|(Pu_\varepsilon)(\tau,\widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w))\right|^r\mathrm{d}x\,\mathrm{d}w\,\mathrm{d}\tau\,\mathrm{d}s\right\}.$$

For the term between braces, we perform the change of variable $x'=\widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w)$ and we get

$$\int_0^t e^{\frac{r(s-t)}{2\varepsilon}}\int_0^s e^{\frac{r(\tau-s)}{2\varepsilon}}\int_{\mathbb{R}^3_+\times\mathbb{R}^3}\mathbf{1}_{\mathcal{O}^t_\varepsilon}(x,[\Gamma^{t,x}_\varepsilon]^{-1}(w))|w|^k f^0_\varepsilon(\widetilde{\mathrm{X}}^{0;t}_\varepsilon(x,w),w)\left|(Pu_\varepsilon)(\tau,\widetilde{\mathrm{X}}^{\tau;t}_\varepsilon(x,w))\right|^r\mathrm{d}x\,\mathrm{d}w\,\mathrm{d}\tau\,\mathrm{d}s$$
$$\lesssim \||w|^k f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}\int_0^t e^{\frac{r(s-t)}{2\varepsilon}}\int_0^s e^{\frac{r(\tau-s)}{2\varepsilon}}\|u_\varepsilon(\tau)\|^r_{\mathrm{L}^p(\mathbb{R}^3_+)}\,\mathrm{d}\tau\,\mathrm{d}s$$
$$\lesssim \varepsilon^2\||v|^k f^0_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))}(\Upsilon^0_\varepsilon)^r,$$

where we have used the fact that T is a strong existence time so that $\|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^p(\mathbb{R}^3_+))} \lesssim \Upsilon^0_\varepsilon$ for all $p \in [2,6]$ (see Corollary (4.3.14)). For the term in brackets, we write

$$\begin{aligned}
&\int_0^T \left(\int_0^t e^{\frac{r(s-t)}{2(r-1)\varepsilon}} \|\nabla_x u_\varepsilon(s)\|^{\frac{r}{r-1}}_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \,\mathrm{d}s \right)^{r-1} \mathrm{d}t \\
&= \int_0^T \left(\int_0^t \frac{2(r-1)\varepsilon}{r} (1 - e^{\frac{-rt}{2(r-1)\varepsilon}}) \|\nabla_x u_\varepsilon(s)\|^{\frac{r}{r-1}}_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \frac{e^{\frac{r(s-t)}{2(r-1)\varepsilon}} \,\mathrm{d}s}{\frac{2(r-1)\varepsilon}{r}(1 - e^{\frac{-rt}{2(r-1)\varepsilon}})} \right)^{r-1} \mathrm{d}t \\
&\lesssim \varepsilon^{r-2} \int_0^T \int_0^t \|\nabla_x u_\varepsilon(s)\|^r_{\mathrm{L}^\infty(\mathbb{R}^3_+)} e^{\frac{r(s-t)}{2\varepsilon}} \,\mathrm{d}s \,\mathrm{d}t = \varepsilon^{r-2} \int_0^T \|\nabla_x u_\varepsilon(s)\|^r_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \left(\int_s^T e^{\frac{r(s-t)}{2\varepsilon}} \,\mathrm{d}t \right) \mathrm{d}s,
\end{aligned}$$

where we have used the Jensen inequality in the probability space

$$\left((0,t), \ \frac{e^{\frac{r(s-t)}{2(r-1)\varepsilon}} \,\mathrm{d}s}{\frac{2(r-1)\varepsilon}{r}(1 - e^{\frac{-rt}{2(r-1)\varepsilon}})} \right).$$

This yields

$$\int_{T_0}^T \int_{\mathbb{R}^3_+} |\mathrm{III}^\natural(t,x)|^r \,\mathrm{d}x \,\mathrm{d}t \lesssim \varepsilon^r \| |v|^k f^0_\varepsilon \|^r_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} (\Upsilon^0_\varepsilon)^r \int_0^T \|\nabla_x u_\varepsilon(s)\|^r_{\mathrm{L}^\infty(\mathbb{R}^3_+)} \,\mathrm{d}s.$$

The proof then goes by taking advantage of the hypothesis (4.4.3). We shall make a constant use of the Gagliardo-Nirenberg-Sobolev inequality (see Theorem 4.2.9 in the Appendix)

$$\begin{aligned}
\|\nabla_x u_\varepsilon(s)\|_{\mathrm{L}^r(\mathbb{R}^3_+)} &\lesssim \|\mathrm{D}^2_x u_\varepsilon(s)\|^{\alpha_r}_{\mathrm{L}^r(\mathbb{R}^3_+)} \|u_\varepsilon(s)\|^{1-\alpha_r}_{\mathrm{L}^2(\mathbb{R}^3_+)}, \quad r \geq 2, \quad \alpha_r \in (0,1), \\
\|\nabla_x u_\varepsilon(s)\|_{\mathrm{L}^\infty(\mathbb{R}^3_+)} &\lesssim \|\mathrm{D}^2_x u_\varepsilon(s)\|^{\beta_r}_{\mathrm{L}^r(\mathbb{R}^3_+)} \|u_\varepsilon(s)\|^{1-\beta_r}_{\mathrm{L}^2(\mathbb{R}^3_+)}, \quad r > 3, \quad \beta_r \in (0,1).
\end{aligned}$$

Thanks to (4.4.3), we can use the conditional Theorem 4.3.3 on the interval $[0,T]$, which means that for all $s \in [0,T]$

$$\|u_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{R}^3_+)} \lesssim \frac{\Psi\left(\|u^0_\varepsilon\|^2_{\mathrm{L}^1 \cap \mathrm{L}^2(\mathbb{R}^3_+)} + 1 \right)}{(1+s)^{\frac{3}{4}}}.$$

We then get

$$\begin{aligned}
&\int_{T_0}^T \int_{\mathbb{R}^3_+} |\mathrm{I}^\natural(t,x)|^r \,\mathrm{d}x \,\mathrm{d}t \\
&\lesssim \varepsilon^r \|(1 + |v|^{\frac{r}{r-1}})|v|^k f^0_\varepsilon\|^{r-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \| |v|^k f^0_\varepsilon \|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \\
&\qquad \times \int_0^T \|\mathrm{D}^2_x u_\varepsilon(s)\|^{r\alpha_r}_{\mathrm{L}^r(\mathbb{R}^3_+)} \|u_\varepsilon(s)\|^{r(1-\alpha_r)}_{\mathrm{L}^2(\mathbb{R}^3_+)} \,\mathrm{d}s \\
&\lesssim \varepsilon^r \|(1 + |v|^{\frac{r}{r-1}})|v|^k f^0_\varepsilon\|^{r-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \| |v|^k f^0_\varepsilon \|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \\
&\qquad \times \Psi\left(\|u^0_\varepsilon\|^2_{\mathrm{L}^1 \cap \mathrm{L}^2(\mathbb{R}^3_+)} + M \right)^{\frac{r(1-\alpha_r)}{2}} \int_0^T \|\mathrm{D}^2_x u_\varepsilon(s)\|^{r\alpha_r}_{\mathrm{L}^r(\mathbb{R}^3_+)} \frac{1}{(1+s)^{r(1-\alpha_r)\frac{3}{4}}} \,\mathrm{d}s \\
&\lesssim \varepsilon^r \|(1 + |v|^{\frac{r}{r-1}})|v|^k f^0_\varepsilon\|^{r-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \| |v|^k f^0_\varepsilon \|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \\
&\qquad \times \Psi\left(\|u^0_\varepsilon\|^2_{\mathrm{L}^1 \cap \mathrm{L}^2(\mathbb{R}^3_+)} + M \right)^{\frac{r(1-\alpha_r)}{2}} \|\mathrm{D}^2_x u_\varepsilon\|^{r\alpha_r}_{\mathrm{L}^r((0,T)\times\mathbb{R}^3_+)} \left(\int_0^T \frac{1}{(1+s)^{r\frac{3}{4}}} \,\mathrm{d}s \right)^{1-\alpha_r},
\end{aligned}$$

thanks to Hölder inequality. Since $r \geq 3 \geq \frac{4}{3}^+$, we have $r\frac{3}{4} > 1$ and the last integral in time is uniformly bounded in T therefore, by Young inequality, we have

$$\begin{aligned}&\int_{T_0}^{T}\int_{\mathbb{R}^3_+} |\mathrm{I}^\natural(t,x)|^r \,\mathrm{d}x\,\mathrm{d}t \\ &\lesssim \varepsilon^r \|(1+|v|^{\frac{r}{r-1}})|v|^k f^0_\varepsilon\|^{\frac{r-1}{1-\alpha_r}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \||v|^k f^0_\varepsilon\|^{\frac{1}{1-\alpha_r}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \Psi\left(\|u^0_\varepsilon\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac{r}{2}} \\ &\quad + \varepsilon^r \|\mathrm{D}^2_x u_\varepsilon\|^r_{\mathrm{L}^r((0,T)\times\mathbb{R}^3_+)}.\end{aligned}$$

The same procedure essentially leads to

$$\int_{T_0}^{T}\int_{\mathbb{R}^3_+} |\mathrm{II}^\natural(t,x)|^r \,\mathrm{d}x\,\mathrm{d}t \lesssim \varepsilon^r \||v|^k f^0_\varepsilon\|^{\frac{r}{1-\alpha_r}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \Psi\left(\|u^0_\varepsilon\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac{r}{2}} + \varepsilon^r \|\mathrm{D}^2_x u_\varepsilon\|^r_{\mathrm{L}^r((0,T)\times\mathbb{R}^3_+)}.$$

For the last term, we have

$$\begin{aligned}&\int_{T_0}^{T}\int_{\mathbb{R}^3_+} |\mathrm{III}^\natural(t,x)|^r \,\mathrm{d}x\,\mathrm{d}t \\ &\lesssim \varepsilon^r \||v|^k f^0_\varepsilon\|^r_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} (\Upsilon^0_\varepsilon)^r \int_0^T \|\mathrm{D}^2_x u_\varepsilon(s)\|^{r\beta_r}_{\mathrm{L}^r(\mathbb{R}^3_+)} \|u_\varepsilon(s)\|^{r(1-\beta_r)}_{\mathrm{L}^2(\mathbb{R}^3_+)}\,\mathrm{d}s \\ &\lesssim \varepsilon^r \||v|^k f^0_\varepsilon\|^r_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} (\Upsilon^0_\varepsilon)^r \\ &\quad \times \Psi\left(\|u^0_\varepsilon\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac{r(1-\beta_r)}{2}} \int_0^T \|\mathrm{D}^2_x u_\varepsilon(s)\|^{r\beta_r}_{\mathrm{L}^r(\mathbb{R}^3_+)} \frac{1}{(1+s)^{r(1-\beta_r)\frac{3}{4}}}\,\mathrm{d}s \\ &\lesssim \varepsilon^r \||v|^k f^0_\varepsilon\|^r_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} (\Upsilon^0_\varepsilon)^r \\ &\quad \times \Psi\left(\|u^0_\varepsilon\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac{r(1-\beta_r)}{2}} \|\mathrm{D}^2_x u_\varepsilon\|^{r\beta_r}_{\mathrm{L}^r((0,T)\times\mathbb{R}^3_+)} \left(\int_0^T \frac{1}{(1+s)^{r\frac{3}{4}}}\,\mathrm{d}s\right)^{1-\beta_r},\end{aligned}$$

as in the previous estimates. We finally obtain

$$\begin{aligned}&\int_{T_0}^{T}\int_{\mathbb{R}^3_+} |\mathrm{III}^\natural(t,x)|^r \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad\qquad \lesssim \varepsilon^r (\Upsilon^0_\varepsilon)^{\frac{p}{1-\beta_r}} \||v| f^0_\varepsilon\|^{\frac{r}{1-\beta_r}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3_+))} \Psi\left(\|u^0_\varepsilon\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^3_+)} + M\right)^{\frac{r}{2}} + \varepsilon^r \|\mathrm{D}^2_x u_\varepsilon\|^r_{\mathrm{L}^r((0,T)\times\mathbb{R}^3_+)}.\end{aligned}$$

We deduce the result by gathering all the terms together. □

Chapter 5

Well-posedness for thick spray equations

Based on the article [EHK23], prepublished and submitted in a journal, in collaboration with Daniel Han-Kwan.

5.1 Introduction and main results

5.1.1 System and notations

In this chapter, we are interested in the following coupling between fluid and particles:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v [f(u-v) - f\nabla_x p(\varrho)] = 0, \\ \partial_t(\alpha\varrho) + \mathrm{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u \otimes u) + \alpha\nabla_x p - \mathrm{div}_x(\tau[u]) = j_f - \rho_f u. \end{cases} \tag{TS}$$

This system describes the evolution of a cloud of particles (e.g. droplets or dust specks) in an underlying compressible fluid (e.g. a gas). Such a suspension is commonly referred to as a *spray* [Wil85]. More generally, the system (TS) belongs to the broad family of "multiphase flows" equations [Des10].

In this work, we study (TS) in the phase space $\in \mathbb{T}^d \times \mathbb{R}^d$, with $d \in \mathbb{N}\backslash\{0\}$. The first equation of (TS) is a kinetic equation of Vlasov-type on the particle distribution function $f(t,x,v) \in \mathbb{R}^+$ in the phase space (position-velocity), set for $t > 0$ and $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$. The other equations of (TS) are set for $t > 0$ and $x \in \mathbb{T}^d$ and are barotropic compressible Navier-Stokes equations on the fluid density $\varrho(t,x) \in \mathbb{R}^+$ and fluid velocity $u(t,x) \in \mathbb{R}^d$. Here, the function $\alpha(t,x) \in [0,1]$ is the volume fraction of the fluid.

Here, the system (TS) describes the so-called *thick sprays* and has been introduced and derived by O'Rourke in [O'R81]. Such coupling is appropriate for modeling two-phase mixtures where particles are small but occupy a non-negligible volume fraction of the whole suspension. The thick (or dense) spray regime is tipycally found in regions where droplets are injected in a carrying gas (see [Duk80, O'R81, Liu02]). The system (TS) has also been recognized as a set of equations linked to multilfuid systems, which are thoroughly described in [IH10]. Further details can be found in the overview provided in Section 5.1.4.

Let us detail and close the previous equations (with a normalization of all coefficients), explaining the meaning of the different terms which are involved.

- The kinetic moments (of order 0 and 1) of the distribution function f are defined as

$$\rho_f(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \quad \text{(Local density)},$$
$$j_f(t,x) := \int_{\mathbb{R}^d} f(t,x,v)v\,\mathrm{d}v, \quad \text{(Local current)}.$$

- The volume fraction $\alpha = \alpha(t,x) \in \mathbb{R}^+$ of the fluid is given by

$$\alpha(t,x) := 1 - \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v = 1 - \rho_f(t,x). \tag{5.1.1}$$

  In the *thick spray* regime, this quantity is not assumed to be close to 1 (this concerns suspensions for which typically, α is around the value 0.9), so that the volume fraction for the cloud of particles is not negligible compared to that of the fluid. It is in sharp contrast with the *thin spray* regime, that we shall recall below (see (VNS) in Section 5.1.4), where α is somehow directly set to 1 and thus does not explicitly appear in the system. Here, this quantity induces an extra coupling between both phases. We refer to [O'R81, Duk80, Des10] for comparisons between the thin and thick regimes.

- The pressure $p : \mathbb{R}^+ \to \mathbb{R}^+$ is a given $\mathscr{C}(\mathbb{R}^+) \cap \mathscr{C}^\infty(\mathbb{R}^+ \backslash \{0\})$ function such that, in the barotropic regime, the pressure term is a function $p = p(\varrho)$ depending only on the fluid density. One usually assumes that $p(0) = 0$ and $p'(\rho) > 0$. For reasons that will appear later, we shall also assume that p is such that

$$\rho \mapsto \rho p'(\rho) \text{ is nondecreasing on } \mathbb{R}^+. \tag{5.1.2}$$

  A common example is for instance $p(\varrho) = \varrho^\gamma$ for some $\gamma > 1$.

- The viscous stress tensor $\tau[u]$ is given by

$$\tau = 2\mu \mathrm{D}(u) + \lambda \mathrm{div}_x u \, \mathrm{I}_d,$$

  for some constants $\nu > 0$ and $\lambda \in \mathbb{R}$, and where $\mathrm{D}(u)$ stands for the deformation tensor defined as $\mathrm{D}(u) = (\nabla_x u + (\nabla_x u)^\intercal)/2$. In this chapter (see the possible generalization in Section 5.1.3), we choose the constants μ and λ so that

$$\mathrm{div}_x(\tau[u]) = \Delta_x u + \nabla_x \mathrm{div}_x u,$$

  but we emphasize that this choice has been made solely for simplicity, and that no special algebraic property arises in this case.

- The force $\Gamma = \Gamma(t,x,v) \in \mathbb{R}^d$ acting on the cloud of particles is defined by

$$\Gamma(t,x,v) = u(t,x) - v - \nabla_x[p(\varrho)](t,x).$$

  The first term $u(t,x) - v$, referred to as the drag force exerted by the fluid on the particles, is common to all fluid-kinetic couplings. Here, it is linear in the relative velocity between fluid and particles, and creates friction. The retroaction of this term in the momentum equation for the fluid is known as the *Brinkman force* and can be expressed as

$$-\int_{\mathbb{R}^d} (u - v) f \,\mathrm{d}v = j_f - \rho_f u.$$

  The second term in Γ, which is a pressure gradient from the fluid, is a specific feature of thick spray models (see also [Duk80, O'R81]) where the particles occupy a significant volume fraction of the two-phase mixture. The presence of this term is consistent with the fact that the system (TS) (or some of its variants) is formally linked to bifluid equations [DM10], where there is a common pressure gradient to both phases (see the overview below in Section 5.1.4). Note that the feedback of this term in the source term of the fluid momentum equation is

$$-\int_{\mathbb{R}^d} f(-\nabla_x[p(\varrho)])\,\mathrm{d}v = \nabla_x[p(\varrho)]\rho_f.$$

  Since $\alpha = 1 - \rho_f$, it explains the term $\alpha \nabla_x[p(\varrho)]$ in the left-hand side of the equation on u in (TS).

The unknowns of the problem are thus

$$f = f(t,x,v) \in \mathbb{R}^+, \quad \varrho = \varrho(t,x) \in \mathbb{R}^+, \quad u = u(t,x) \in \mathbb{R}^d,$$

and they are coupled through the drag term $u - v$, the pressure gradient $\nabla_x p(\varrho)$ and the volume fraction of the fluid $\alpha = 1 - \rho_f$. We finally prescribe initial conditions

$$f^{\text{in}} = f^{\text{in}}(x,v) \in \mathbb{R}^+, \quad \varrho^{\text{in}} = \varrho^{\text{in}}(x) \in \mathbb{R}^+, \quad u^{\text{in}} = u^{\text{in}}(x) \in \mathbb{R}^d.$$

We normalize the torus $\mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z})^d$ endowed with the normalized Lebesgue measure, so that $\mathrm{Leb}(\mathbb{T}^d) = 1$.

In this work, we investigate the *local well-posedness* of the thick spray equations (TS). The main difficulty comes from the fact that rough energy estimates on the transport and kinetic part of (TS) seem to result in a *loss of several derivatives*. As a matter of fact, suppose that we have a smooth solution (f, ϱ, u) with compact support to the system (TS). Since it is a coupling between a parabolic type equation on u and two transport equations on f and ϱ, the following observations can be made:

- a standard energy estimate for transport equations shows that a control of k derivatives of ϱ seems to require the control of $k+1$ derivatives of f. This is due to the coupling with the volume fraction α in the mass conservation equation;
- a standard energy estimate for transport(-kinetic) equations shows that a control of k derivatives of f seems to require the control of $k+1$ derivatives of ϱ. This comes from the pressure gradient in the force field of the Vlasov equation.

As a consequence, we obtain a control of k derivatives of f by $k+2$ derivatives of f (and similarly for ϱ) for any $k \in \mathbb{N}$, and so on. Rough estimates of transport type therefore seem to involve a formal loss of 2 derivatives on f or ϱ, which **makes the coupling singular**. Hence, standard techniques cannot be applied to obtain a solution to the system. In [BD06a], Baranger and Desvillettes conjectured anyway that the system is well-posed in Sobolev spaces.

In this chapter, we partly confirm this conjecture by showing that (TS) is locally well-posed in Sobolev spaces, when the initial data satisfy a *stability condition* of Penrose type. However, when this stability condition is violated, the system is actually ill-posed in the sense of Hadamard, see Section 1.6.3.2 of the Introduction, which means, loosely speaking, that it indeed displays losses of derivatives in this case.

The rest of the introduction is structured as follows. In Section 5.1.2, we introduce the Penrose stability condition for thick sprays and state our main result of local well-posedness for (TS). In Section 5.1.3, we describe several generalizations of the thick spray equations (TS), taking into account possible density-dependent drag or collisions in the kinetic equation, as well as non-barotropic Navier-Stokes equations. These variants will be treated in Sections 5.8–5.9–5.10. Section 5.1.4 is a general overview on fluid-kinetic systems, as well as on singular Vlasov equations. Finally, Section 5.1.5 provides a detailed outline of our method of proof.

5.1.2 Assumptions and main result

For $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$, we write $|\alpha| = \sum_{i=1}^d \alpha_i$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$. In this chapter, we will denote by H^k (or $\mathrm{H}^k(\mathbb{T}^d)$) the standard L^2 Sobolev spaces for functions depending on $x \in \mathbb{T}^d$. When it is necessary, we will denote $\mathrm{H}^k_{x,v}$ the same spaces for functions depending on $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$. To ease readibility, we shall sometimes abbreviate $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))$ and $\mathrm{L}^2(0,T;\mathrm{H}^k(\mathbb{T}^d))$ as $\mathrm{L}^2_T\mathrm{L}^2$ and $\mathrm{L}^2_T\mathrm{H}^k$. We will also use *weighted* $\mathrm{L}^2_{x,v}-$Sobolev spaces. They are defined as follows:

Definition 5.1.1. *For $k \in \mathbb{N}$ and $r \geq 0$ and $\mathrm{f} : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}^+$, we define the weighted (in velocity) Sobolev norms as*

$$\|\mathrm{f}\|_{\mathcal{H}^k_r} := \left(\sum_{|\alpha|+|\beta| \leq k} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \langle v \rangle^{2r} |\partial_x^\alpha \partial_v^\beta \mathrm{f}(x,v)|^2 \, \mathrm{d}x \, \mathrm{d}v \right)^{\frac{1}{2}},$$

where $\langle v \rangle = (1+|v|^2)^{\frac{1}{2}}$.

We now introduce the following Penrose function, which will be used to state the relevant corresponding Penrose stability condition.

Definition 5.1.2. *For any distribution function $\mathrm{f}(x,v)$ and density $\rho(x)$, we define the Penrose function*

$$\mathscr{P}_{\mathrm{f},\rho}(x,\gamma,\tau,k) := \frac{p'(\rho(x))\rho(x)}{1-\rho_{\mathrm{f}}(x)} \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot (\mathcal{F}_v \nabla_v \mathrm{f})(x,ks) \, \mathrm{d}s, \tag{5.1.3}$$

for $(x,\gamma,\tau,k) \in \mathbb{T}^d \times (0,+\infty) \times \mathbb{R} \in \mathbb{R}^d \backslash \{0\}$.

Definition 5.1.3. *We say that the couple $(\mathrm{f}(x,v),\rho(x))$ satisfies the c-**Penrose stability condition** (for the thick spray equations (TS)) if there exists $c > 0$ such that*

$$\forall x \in \mathbb{T}^d, \quad \inf_{(\gamma,\tau,k) \in (0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\}} |1 - \mathscr{P}_{\mathrm{f},\rho}(x,\gamma,\tau,k)| > c. \tag{$\mathbf{P}$}$$

When needed, we shall denote this condition by $(\mathbf{P})_c$.

As we shall explain later on, such a condition stems from the study of Vlasov equations in plasma physics. In the context of the Vlasov-Benney equation, a similar condition was the key to obtain local well-posedness in Sobolev spaces (and for its derivation in the quasineutral limit), see [HKR16]. We refer to Section 5.1.5 for more details.

Remark 5.1.4 (Sufficient conditions ensuring (**P**))**.** Let us describe a few different classes of profiles $(\mathrm{f}(x,v),\rho(x))$ which satisfy the Penrose stability condition (**P**). Several criteria on the shape of $\mathrm{f}(x,\cdot)$ for every $x \in \mathbb{T}^d$ indeed provide sufficient condition for (**P**) to hold (see for instance [MV11]). We shall assume that $\mathrm{f}(x,\cdot)$ is at least integrable and that the nonnegative prefactor in front of the integral in $\mathscr{P}_{\mathrm{f},\rho}$ is bounded from above on $\mathbb{T}^d$.

- First, any sufficiently small smooth profile f satisfies (**P**).
- When $d = 1$, the one bump profiles in velocity, that is to say profiles such that for all $x \in \mathbb{T}$, the function $v \mapsto \mathrm{f}(x,v)$ is increasing then decreasing, satisfy the Penrose condition.
- In any dimension $d \geq 1$, any profile such that for all $x \in \mathbb{T}^d$, $f(x,\cdot)$ is a radial non-increasing function in velocity, is Penrose stable. In particular, it includes the case of (local smooth) Maxwellians in velocity. If $d \geq 3$, any profile such that for all $x \in \mathbb{T}^d$, $f(x,\cdot)$ is a radial and strictly positive function in velocity, satisfies the condition.
- More generally, the following criterion on the marginals of f has been devised in [MV11] and ensures the Penrose stability: for all $x \in \mathbb{T}^d$,

$$\forall \omega_0 \in \mathbb{R}, \quad \forall k \in \mathbb{R}^d \backslash \{0\}, \quad \frac{1}{1+|k|^2} \frac{p'(\rho(x))\rho(x)}{1-\rho_{\mathrm{f}}(x)} \mathrm{p.v.} \int_{\mathbb{R}} \partial_y \mathrm{f}_{\frac{k}{|k|}}(x,y) \frac{1}{y+\omega_0} \, \mathrm{d}y < 1,$$

where *p.v.* stands for the principal value on $\mathbb{R}$ and where

$$\forall r \in \mathbb{R}, \quad \mathrm{f}_{\frac{k}{|k|}}(r) := \int_{\frac{k}{|k|}^\perp} \mathrm{f}\left(r\frac{k}{|k|} + w \right) \mathrm{d}w.$$

- Finally, any sufficiently small smooth perturbation of a Penrose stable profile will still satisfy (**P**). In particular, a slightly perturbed one-bump profile $f(x,\cdot)$ remains Penrose stable.

Our **main result** reads as follows.

Theorem 5.1.5. *There exist $m_0 > 0$ and $r_0 > 0$, depending only on the dimension, such that the following holds for all $m \geq m_0$ and $r \geq r_0$. Let*

$$f^{\mathrm{in}} \in \mathcal{H}^m_r, \quad \varrho^{\mathrm{in}} \in \mathrm{H}^{m+1}, \quad u^{\mathrm{in}} \in \mathrm{H}^m,$$

such that $(f^{\mathrm{in}}, \varrho^{\mathrm{in}})$ satisfies the c-Penrose stability condition $(\mathbf{P})_c$ for some $c > 0$ and

$$0 \leq f^{\mathrm{in}}, \quad \rho_{f^{\mathrm{in}}} < \Theta < 1, \quad 0 < \mu \leq \varrho^{\mathrm{in}}, \quad 0 < \underline{\theta} \leq (1-\rho_{f^{\mathrm{in}}})\varrho^{\mathrm{in}} \leq \overline{\theta},$$

for some constants $\Theta, \mu, \underline{\theta}, \overline{\theta}$. Then there exist $T > 0$ and a solution (f, ϱ, u) to (TS) with initial condition $(f^{\mathrm{in}}, \varrho^{\mathrm{in}}, u^{\mathrm{in}})$ such that

$$f \in \mathscr{C}\left([0,T]; \mathcal{H}^{m-1}_r\right), \quad \varrho \in \mathrm{L}^2(0,T;\mathrm{H}^m), \quad u \in \mathscr{C}\left([0,T];\mathrm{H}^m\right) \cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1}),$$

and with $(f(t), \varrho(t))$ satisfying the $c/2$-Penrose stability condition $(\mathbf{P})_{c/2}$ for all $t \in [0,T]$. In addition, this solution is unique in this regularity class.

In short, our result yields the local well-posedness for the thick spray equations, in the class of Penrose stable initial data. On the other hand, as already mentioned earlier, outside of this class, the system is ill-posed in the sense of Hadamard.

Remark 5.1.6. The uniqueness part in the previous statement must be understood as follows: if (f_1, ϱ_1, u_1) and (f_2, ϱ_2, u_2) are two solutions to (TS) on $[0,T]$ with the previous regularity and with the same initial condition $(f^{\mathrm{in}}, \varrho^{\mathrm{in}}, u^{\mathrm{in}})$ (satisfying (**P**)), then $(f_1, \varrho_1, u_1) = (f_2, \varrho_2, u_2)$ if $t \mapsto (f_1(t), \varrho_1(t))$ satisfies the Penrose stability condition (**P**) on $[0,T]$.

Remark 5.1.7. Let us point out the shift of one derivative in the regularity between f ans ϱ, which is reminiscent of the formal loss of derivatives that was evoked earlier.

Remark 5.1.8. In [BDD23], the authors consider the linearization of (TS) around radially non-increasing and homogeneous profile (for the kinetic part). This can be seen as a particular case of the Penrose stability condition (**P**). They obtain a stability estimate in L^2 around the solution generated by such particular data. However, it is not sufficient to provide a well-posedness result for the full non-linear equations. As a matter of fact, since the equations are quasilinear, one would need to prove the analogue of such stability estimates for all functions in a neighborhood of the aforementioned solution.

Remark 5.1.9. 1. It appears to be more natural (see in particular the proof of uniqueness in Section 5.7.3) to consider the optimal Penrose function

$$\mathcal{P}_{\mathrm{f},\rho}(x,\gamma,\tau,k) := \frac{p'(\rho(x))\rho(x)}{1-\rho_f(x)} \int_0^{+\infty} e^{-(\gamma+i\tau)s} ik \cdot (\mathcal{F}_v \nabla_v \mathrm{f})(x, ks)\, \mathrm{d}s,$$

instead of $\mathscr{P}_{\mathrm{f},\rho}$, as well as the related stability condition

$$\forall x \in \mathbb{T}^d, \quad \inf_{(\gamma,\tau,k)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\setminus\{0\}} |1 - \mathcal{P}_{\mathrm{f},\rho}(x,\gamma,\tau,k)| > c. \tag{Opt-P}$$

We refer to (**Opt-P**) as the *optimal Penrose stability condition.* The condition (**Opt-P**) is weaker than (**P**). Indeed, using a homogeneity argument combined with a continuity argument, it is possible to prove that if (**P**) holds for some $c > 0$ for (f, ρ), then (**Opt-P**) holds as well[1].

However, our strategy in this chapter will be based on a regularization of the force field in the Vlasov equation, and requires the stability condition (**P**). It is likely that, using the techniques of [CHKR23], one could assume (**Opt-P**) instead of (**P**) on $(f^{\mathrm{in}}, \varrho^{\mathrm{in}})$ to prove the well-posedness of thick spray equations. However, this would require substantial work.

2. The factor $\frac{1}{1+|k|^2}$ in the Penrose function $\mathscr{P}_{\mathrm{f},\rho}$ could appear as arbitrary. It is actually related to the explicit regularization procedure of the force field that will be made clearer later on. By a homogeneity argument, it is however possible to prove that the condition (**P**) is equivalent to

$$\forall x \in \mathbb{T}^d, \quad \inf_{(\gamma,\tau,k,\lambda)\in S^+\times(0,1]} |1 - \lambda \mathcal{P}_{\mathrm{f},\rho}(x,\gamma,\tau,k)| > c, \tag{5.1.4}$$

where

$$S^+ := \left\{(\gamma,\tau,k) \in (0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\} \mid \gamma^2+\tau^2+k^2 = 1\right\}.$$

This implies in particular that the factor $\frac{1}{1+|k|^2}$ could by instance be replaced by $\frac{1}{(1+|k|^2)^\alpha}$ for any $\alpha > 0$ in (5.1.3).

We refer to Remark 5.2.12 in Section 5.2.2 for the use of such reformulation of (**P**) in the regularization procedure.

Remark 5.1.10. In the Euler case for the fluid, that is for the same system on (f, ϱ, u) but **without** the term $-\Delta_x u - \nabla_x \mathrm{div}_x u$ in the equation for u, the question of well-posedness remains open.

5.1.3 Generalization to several variants

We are also interested in more complex versions of the system (TS) that take into account more physical effects. In this section, several such variants are presented. The main strategy of proof performed in this chapter for (TS) will be robust enough to handle such models: we will show how to obtain their local well-posedness in Sections 5.8–5.9–5.10.

5.1.3.1 Non-barotropic Navier-Stokes equations

If one wants to get rid of the barotropic-type assumption on the fluid, one has to consider its internal energy $\mathfrak{e} = \mathfrak{e}(t,x) \in \mathbb{R}^+$ as an additional unknown. This leads to the following system of equations

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + \mathrm{div}_v\Big[f(u-v) - f\nabla_x p(\varrho,\mathfrak{e})\Big] = 0, \\ \partial_t(\alpha\varrho) + \mathrm{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u\otimes u) + \alpha\nabla_x p(\varrho,\mathfrak{e}) - \Delta_x u - \nabla_x\mathrm{div}_x u = j_f - \rho_f u, \\ \partial_t(\alpha\varrho\mathfrak{e}) + \mathrm{div}_x(\alpha\varrho\mathfrak{e}u) + p(\varrho,\mathfrak{e})\left(\partial_t\alpha + \mathrm{div}_x(\alpha u)\right) = \displaystyle\int_{\mathbb{R}^d} |u-v|^2 f\,\mathrm{d}v, \\ \alpha = 1-\rho_f. \end{cases} \tag{5.1.5}$$

Here, $p : \mathbb{R}^+\times\mathbb{R}^+ \to \mathbb{R}$ is a given $\mathscr{C}(\mathbb{R}^+\times\mathbb{R}^+)\cap\mathscr{C}^\infty(\mathbb{R}^+\backslash\{0\}\times\mathbb{R}^+\backslash\{0\})$ function such that the pressure term is a function $p = p(\varrho,\mathfrak{e})$ depending on the fluid density and internal energy. For instance, a relation of the type $p(\varrho,\mathfrak{e}) = b\varrho\mathfrak{e}$ (for some $b > 0$) is a perfect gas pressure law.

The apparent loss of derivatives is still present in such model and occurs between f and $\mathfrak{e}$. In Section 5.8, we shall also justify how to build local in time solutions for (5.1.5) thanks to an adapted Penrose stability condition.

[1] We refer to the uniqueness part of the proof in Section 5.7.3 for more details.

5.1.3.2 Kinetic equation with collisions

In the thick spray regime, it is physically relevant to take into account collisions between droplets. A collision operator is therefore sometimes added in the kinetic equation, which turns into a Vlasov-Boltzmann type:

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v \Big[f(u-v) - f \nabla_x p(\varrho) \Big] = Q(f,f).$$

The quadratic operator $\mathcal{Q}(f,f) = \mathcal{Q}_\lambda(f,f)$ stands for some Boltzmann collision operator for (in)elastic hard-spheres. Here $\lambda \in (0,1]$ is given and is called the *restitution coeficcient* which is involved in the microscopic laws defining collisions between particles. We refer to Appendix 5.9 for some details about the precise definition of the collision operator $\mathcal{Q}_\lambda$ and some of the main basic features of inelastic collisions. Let us mention that the case $\lambda = 1$ corresponds to standard perfectly elastic collisions and that the inelastic case $\lambda \in (0,1)$ leads to a loss of kinetic energy along collisions (while mass and momentum are always conserved).

In Section 5.9, we will explain how to include a collision operator in the kinetic equation on f and still obtain an analogue of Theorem 5.1.5.

5.1.3.3 Density-dependent drag force

In many applications, the force $\Gamma = \Gamma(t,x,v) \in \mathbb{R}^d$ acting on the particle should actually present a density-dependent drag force, for instance of the form

$$\Gamma(t,x,v) = \varrho(t,x)(u(t,x)-v) - \nabla_x[p(\varrho)](t,x).$$

Compared to the one of (TS), this force displays an additional nonlinearity[2]. The Brinkman force in the Navier-Stokes equations also becomes

$$-\int_{\mathbb{R}^d} \varrho(u-v) f \,\mathrm{d}v = \varrho(j_f - \rho_f u).$$

Because of the modified term $\varrho(t,x) v \cdot \nabla_v f$ in the kinetic equation, this induces a potential growth in velocity which could become out of control. In Section 5.10, we will deal with the case of such density-dependent drag term, up to the additional assumption that the initial data f^{in} has a compact support in velocity.

5.1.3.4 Density-dependent viscosities

It is also possible to consider more general viscosity coefficients in the Navier-Stokes equations of (TS), that is replacing the differential operator $-\Delta_x u - \nabla_x \mathrm{div}_x u$ in the equation for u by

$$-\mathrm{div}_x \left(2\mu[\varrho] \mathrm{D}(u) + \lambda[\varrho] \mathrm{div}_x u \, \mathrm{I}_d \right),$$

for smooth non-negative coefficients $\mu, \lambda : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(0) = \lambda(0) = 0$. For the sake of simplicity, we restrict in this chapter to the case $\mu = 1$ and $\lambda = 0$. We claim however that our analysis applies *mutatis mutandis* to this more general situation. As a matter of fact, we will consider local in time strong solutions for which ϱ is non-vanishing.

[2]Note that a more physical model should deal with a nonlinear drag of the form $\mathrm{C}[\varrho, |u-v|](u-v)$, which is for the moment out of the scope of a rigorous mathematical analysis. However, our approach can for instance allow the treatment of a drag of the form $\varrho[(u-v) + \gamma(u-v)]$ where $\gamma \in \mathscr{C}^\infty(\mathbb{R}^d;\mathbb{R}^d)$ is such that $\gamma(0) = 0$ and $\gamma'(0) = 0$.

5.1.4 Overview on fluid-kinetic systems and related models

Let us provide an overview on couplings between fluid and kinetic equations, both from the modeling and the theoretical points of view. In particular, we want to highlight the differences between the main regimes for the description of sprays (namely, *thick* versus *thin*). We also review some existing works on singular Vlasov equations coming from plasma physics, our strategy for thick sprays being inspired by the study of such systems (see Section 5.1.5).

5.1.4.1 Fluid-kinetic description

We can trace back the introduction of fluid-kinetic couplings for the description of sprays involving a large number of particles to [Wil85, O'R81]. We also refer to [Des10] for a general overview on the description of multiphase flows, as well as to [Rei96]. Note that compared to an Eulerian-Eulerian description for both gas and droplets (where both phases are described at the macroscopic level, with (t, x) variables), a fluid-kinetic point of view seems to be well-suited for polydispersed flows (i.e. when the size of the droplets can vary). Indeed, no average is taken to compute the drag force for instance. Of course, many other physical effects such as coalescence, breakup, vaporization or chemical reactions can be included in the models (see e.g. [Lau02, Bar04]).

5.1.4.2 Thick spray case

Models for *thick sprays* have been explicitly introduced and formally derived in [Duk80] and in [O'R81, Chapter 2]. In particular, the pressure gradient acting on the dispersed phase is already present in [Duk80], while [O'R81, Chapter 2] also considers the additional contribution of collisions. The use of such complex models has then been pursued by O'Rourke's team in the Los Alamos National Laboratory, to develop the Kiva code [OA87, AOB89, OZS09]. From the numerical and modeling point of view, let us for also refer to the works [BDM14, BDGN12].

As explained above, the coupling between both phases makes the rigorous study of thick spray equations reputedly challenging. The mathematical study of such type of models is still in its infancy and only a few formal results are available.

In [DM10], the authors consider a formal hydrodynamic limit starting from a thick spray system of the type (5.1.5) (without fluid dissipation and with an additional energy variable for f) with an *inelastic collision operator*. The limit of Knudsen number tending to 0 allows to derive (at least formally) a two-fluid coupled system. The latter turns out to be a standard model of multiphase flow theory where the volume fraction is now an unknown (see the book [IH10]). It formally connects thick spray models to multifluid systems, where the presence of a common pressure term is standard. This somehow *a posteriori* explains the additional pressure gradient in the force field acting in the kinetic equation. A standard feature of this bifluid limiting system seems to be a lack of hyperbolicity [KSS03, NKDVLG05, HdM21], so that its behavior is *a priori* highly unstable. Note that preliminary computations performed in [Ram00] tend to indicate that adding some viscous term along some directions makes this type of system better-behaved.

A new understanding of thick sprays has been obtained in [BDD23] where the linear stability for (TS) and (5.1.5) is investigated. More precisely, the L^2 linear stability around a family of particular space-homogeneous profiles (for the kinetic phase) is obtained thanks to a suitable Lyapunov functional. The profiles in velocity are required to satisfy a property of monotonicity, this condition being a special example of the Penrose stability condition (**P**) that we shall impose on the initial condition (recall Remark 5.1.4 above).

Very recently, [FBD22] has proposed a new averaged version of thick spray models, where the pressure gradient $-\nabla_x p(\varrho)$ in the kinetic equation and the volume fraction α are regularized by including an extra convolution operator. Local existence in Sobolev spaces for this new version of the original thick spray model is obtained in the Euler case for the fluid, using tools from symmetrisable hyperbolic systems (see [BD06a, Mat10]).

5.1.4.3 Barotropic compressible Navier-Stokes equations

When $f = 0$ (and thus $\alpha = 1$), the system (TS) reduces to the standard compressible Navier-Stokes equations on (ϱ, u), in the barotropic-type regime. These equations have given rise to an abundant literature for more than half a century. Global weak solutions of finite energy have been built for the first time in [Lio98] for constant viscosity coefficients, a result extended in [FNP01] for more general pressure power laws. Another notion of weak solutions was also considered in Hoff [Hof87]. For more recent results allowing to include degenerate viscosities and more general pressure laws, see e.g. [BD03, BD06b, MV07, BJ18, VY16, BVY22].

In the framework of strong solutions (local in time or global with assumption on the data), we can for instance refer to [Ser59, Nas62] for classical ones, to [Sol80] for mild solutions, to [MN80, MN83] for solutions with high Sobolev regularity near equilibria, and to [HZ95] for the fine description of the time asymptotics of the system. Let us also mention the more recent works [Dan00, Dan01b, Dan01a, Dan05b, DFP20, CD10, DT22] for the study of the system in critical spaces.

5.1.4.4 Thin spray case: the Vlasov-Navier-Stokes system

Unlike the *thick spray* regime corresponding to (TS), there exists a rich literature on the so-called *thin spray* models. It corresponds to a regime where the particles volume fraction is small compared to that of the surrounding fluid. Here, the quantity α is set to 1 and does not appear in the system. The main term of the coupling which is retained is the drag force (that is $\Gamma(t, x, v) = u(t, x) - v$), and its feedback in the fluid equation.

An important fluid-kinetic model in this class is the so-called Vlasov-Navier-Stokes system, describing a monodispersed phase of small particles flowing in an ambient incompressible homogeneous viscous fluid, that takes the form

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v[f(u - v)] = 0, \\ \partial_t u + (u \cdot \nabla_x)u - \Delta_x u + \nabla_x p = j_f - \rho_f u, \\ \mathrm{div}_x\, u = 0. \end{cases} \tag{VNS}$$

This system has been for instance shown to provide a good description of medical aerosols in the upper part of the lung (see e.g. [BGLM15, BM21]).

From the mathematical point of view, many directions of research have been settled about (VNS) (and its variants) over the past twenty years. The Cauchy theory, adressing the existence of global weak solutions for (VNS) on a large class of domains in dimension 2 or 3, is by now well-developed (see e.g. [ABdMB97, Ham98, BDGM09]), and also allows for more complex physics in the model (see [BGM17, BMM20]). It mainly consists in obtaining a Leray weak solution for u and a renormalized weak solution (in the sense of DiPerna and Lions [DL89c]) for f, using a remarkable energy-dissipation identity that is satisfied by solutions to the system. In dimension 2, the uniqueness of such solutions has been shown in [HKMMM20].

More recently, several asymptotics of (VNS) have attracted a lot of attention. The question of the large-time dynamics for (VNS) has received several advances over the past few years. Roughly speaking, it is expected that the cloud of particles aligns its velocity on that of the fluid, that is

$$u(t) \underset{t\to+\infty}{\longrightarrow} v^\infty, \quad f(t) \underset{t\to+\infty}{\longrightarrow} \rho^\infty \otimes \delta_{v=v^\infty},$$

for some asymptotic velocity $v^\infty \in \mathbb{R}^3$ and profile $\rho^\infty(t, x)$.

The first complete answer justifying such a singular asymptotics has been obtained in [HKMM20] for Fujita-Kato type solutions, in the $3d$ torus case. In the whole space $\mathbb{R}^3$, this question is studied in [HK22] while the case of a $3d$ bounded domain (with absorption boundary conditions for f) is investigated in [EHKM21]. We also refer to [Ert21] for the half-space case, where the additional effect of a gravity force on the particles (combined with absorption at the boundary) leads to decay of the solutions to 0.

Another asymptotics is the so-called *hydrodynamic limit* starting from (VNS), related to high-friction regimes: further references can be found in the introduction of Chapter 4. Concerning the challenging open problem of the derivation of (VNS), we refer to Section 1.3.3 on the Introduction for more details.

5.1.4.5 Several variants of (VNS)

The Vlasov-Navier-Stokes system can also be considered with inhomogeneous or compressible Navier-Stokes equations [CK15, Cho17, CJ22b] and additional terms in kinetic equations [CK22, CJ22a].

Note that the case of compressible Euler equations for the fluid, coupled to a kinetic equation, has also been investigated. We refer to [BD06a] (for the thin spray case) and [Mat10] (for the so-called moderately thick spray case when collisions between particles are not neglected) where local in time strong solutions are built, thanks to ideas coming from hyperbolic systems.

5.1.4.6 Singular Vlasov equations

As we shall explain later on (see Section 5.1.5 on our strategy of proof), we shall take our inspiration from a different problem coming from plasma physics, which is the so-called *quasineutral limit* problem. More specifically, let us look at the dynamics of ions described by the following Vlasov-Poisson system

$$\begin{cases} \partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + E_\varepsilon\cdot\nabla_v f_\varepsilon = 0, \\ E_\varepsilon = -\nabla_x U_\varepsilon, \\ (\mathrm{Id}-\varepsilon^2\Delta_x)U_\varepsilon = \displaystyle\int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\,\mathrm{d}v - 1, \end{cases} \tag{VP$_\varepsilon$}$$

when $\varepsilon \ll 1$, corresponding to a small Debye length regime for the plasma. The issue at stake is the validity (or invalidity) of the formal limit $\varepsilon \to 0$, leading to

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x\rho_f\cdot\nabla_v f = 0, \\ \rho_f(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v. \end{cases} \tag{VB}$$

This system was named as the *Vlasov-(Dirac)-Benney* system by Bardos in [Bar13]. As directly seen on (VB), the force field in this Vlasov equation is one derivative less regular than the distribution function f itself, thus displaying an apparent loss of derivative.

The question of the justification of the quasineutral limit (from (VP$_\varepsilon$) to (VB)) and of the well-posedness of the limiting system (VB) has given rise to a wealth of literature for more than twenty years. Preliminary results have investigated the limit, up to to some defect measures [BG94, Gre95], and have been followed by a full justification in the analytic regime [Gre96] (see also [HKI17b, HKI17a]). We also refer to [Bre00, Mas01, HK11] for the case of singular monokinetic data leading to fluid equations. However, the quasineutral limit does not hold in general because instabilities for Vlasov-Poisson can take over (see [HKH15]).

In general, there also exist unstable homogeneous equilibria of (VB) around which the linearized equations have unbounded unstable spectrum (typically two bumps profile in velocity, leading to the so-called *two-stream* instability). Therefore (VB) may be ill-posed in the sense of Hadamard [BN12, HKN16, Bar20] in Sobolev spaces, even with arbitrary losses of derivatives and arbitrary small time.

A local theory for (VB) thus requires more assumptions on the initial data. For instance, a Cauchy-Kovalevskaya type theorem can be applied [JN11, BFJJ13, MV11] to show that there is local existence of analytic solutions for analytic initial data. In dimension $d = 1$, Sobolev initial data with a one bump profile in velocity (for all x) leads to local in time solution as shown in [BB13] (see also [BB15] for more properties).

In any dimension, the quasineutral limit and the well-posedness in Sobolev spaces of (VB) have been obtained in [HKR16] under a Penrose stability condition on the initial data f^{in}. In this work, the same type of condition as (**P**) was assumed, replacing the function $\mathscr{P}_{\mathrm{f},\rho}$ by $\mathscr{P}_{\mathrm{f}}^{\mathrm{VP}}$ defined as

$$\mathscr{P}_{\mathrm{f}}^{\mathrm{VP}}(x,\gamma,\tau,k) = \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot \left(\mathcal{F}_v \nabla_v \mathrm{f}\right)(x,ks)\,\mathrm{d}s.$$

Up to some prefactor coming from coupling for sprays, our assumption (**P**) in Theorem (5.1.5) is closely related to the one of [HKR16] and this work is the main inspiration of our analysis. We also refer to the recent work [Cha23] where the existence of local in time solutions for mildly singular Vlasov equations is shown (without assuming any stability condition).

Note that the rescaling $(t,x,v) \mapsto (t/\varepsilon, x/\varepsilon, v)$ in (VP_ε) leads to the same equation with $\varepsilon = 1$, hence connecting the quasineutral limit to an issue of large-time dynamics. The Penrose stability condition (**P**) appearing on a homogeneous profile $\mathrm{f}(v)$ is actually a necessary condition for its long-time stability in the Vlasov-Poisson equation (VP_ε) with $\varepsilon = 1$. This last issue is also linked to the Landau damping effect, which has been proven to hold in a small (Gevrey) neighborhood of such stable profile. In the torus, we refer to the breakthrough work [MV11], as well as to [BMM16, GNR21].

5.1.5 Strategy of the proof

We conclude this introduction by presenting a detailed outline of the proof. This will allow at the same time to explain the structure of the chapter. In order to ease readability and highlight the main features of the analysis, we voluntarily state our results without precise assumptions.

As explained above, and for the sake of clarity, we shall focus on (TS). This system indeed retains the main features and difficulties of this chapter. Our result and proof will be generalized to the more complete systems presented in Section 5.1.3 (see the Sections 5.8–5.9–5.10).

In the preliminary **Section 5.2**, we start by deriving several *a priori* energy estimates on the system (TS). We show in Proposition 5.2.3 that for all $t \in [0,T]$

$$\forall t \in [0,T], \quad \|\varrho(t)\|_{\mathrm{H}^m} \leq \|\varrho^{\text{in}}\|_{\mathrm{H}^m} \Phi\Big(T, \cdots, \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m+1})}, \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m+1}_r)}\Big), \tag{5.1.6}$$

where Φ is a continuous function which is increasing with respect to each of its arguments and where $\cdots$ involves lower order terms. On the other hand, we have for all $t \in [0,T]$

$$\|\rho_f(t)\|_{\mathrm{H}^m} + \|j_f(t)\|_{\mathrm{H}^m} \lesssim \|f(t)\|_{\mathcal{H}^m_r} \leq \|f^{\text{in}}\|^2_{\mathcal{H}^m_r} \Phi\Big(T, \cdots, \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)}, \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}\Big), \tag{5.1.7}$$

The estimates (5.1.6) and (5.1.7) thus yield a loss of two derivatives for the fluid density ϱ.This formally prevents the use of standard techniques to obtain a (local in time) solution. The main goal of the analysis is to show that these losses are only apparent when the initial condition $(f^{\text{in}}, \varrho^{\text{in}})$ satisfies the Penrose stability condition (**P**).

To this end, we first consider the following regularization of the system (see also the Remark 5.2.12), which includes a parameter $\varepsilon \in (0,1)$ which is bound to go to 0:

$$\begin{cases} \partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \mathrm{div}_v[f_\varepsilon E_\varepsilon - f_\varepsilon v] = 0, \\ \partial_t((1-\rho_{f_\varepsilon})\varrho_\varepsilon) + \mathrm{div}_x((1-\rho_{f_\varepsilon})\varrho_\varepsilon u_\varepsilon) = 0, \\ (1-\rho_{f_\varepsilon})\Big(\varrho_\varepsilon\big[\partial_t u_\varepsilon + (u_\varepsilon\cdot\nabla_x)u_\varepsilon\big] + \nabla_x p(\varrho_\varepsilon)\Big) - \Delta_x u_\varepsilon - \nabla_x \mathrm{div}_x\, u_\varepsilon = j_{f_\varepsilon} - \rho_{f_\varepsilon} u_\varepsilon, \\ f_{\varepsilon|t=0} = f^{\mathrm{in}}, \quad \varrho_{\varepsilon|t=0} = \varrho^{\mathrm{in}}, \quad u_{\varepsilon|t=0} = u^{\mathrm{in}}, \end{cases}$$

where

$$\rho_{f_\varepsilon}(t,x) = \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\,\mathrm{d}v, \quad j_{f_\varepsilon}(t,x) = \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)v\,\mathrm{d}v,$$
$$E_\varepsilon = -p'(\varrho_\varepsilon)\nabla_x\left[\mathrm{J}_\varepsilon\varrho_\varepsilon\right] + u_\varepsilon, \quad \mathrm{J}_\varepsilon = (\mathrm{Id} - \varepsilon^2\Delta_x)^{-1}.$$

When $\varepsilon > 0$, the regularized system can be seen as a non-singular coupling between compressible Navier-Stokes and the Vlasov equation; as a result, classical energy methods allow to build local in time solutions, away from vacuum (this is performed in Appendix 5.B). However, the point is to obtain uniform in ε estimates on some interval of time which has to be independent of ε. With this goal in mind, we set up a bootstrap argument that starts in the end of Section 5.2.

We introduce

$$\mathcal{N}_{m,r}(f_\varepsilon,\varrho_\varepsilon,u_\varepsilon,T) := \|f_\varepsilon\|_{\mathrm{L}^\infty\left(0,T;\mathcal{H}^{m-1}_r\right)} + \|\varrho_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} + \|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap\mathrm{L}^2(0,T;\mathrm{H}^{m+1})},$$

for $T > 0$ and we want to obtain a uniform (in ε) estimate for this quantity. This will pave the way for a compactness argument allowing to pass to the limit in the previous regularized system, when $\varepsilon \to 0$. Observe the shift of one derivative between the norm on f_ε and that on ϱ_ε. By (5.1.7), a control on $\|\varrho_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}$ and $\|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)}$ implies a control on $\|f_\varepsilon\|_{\mathrm{L}^\infty\left(0,T;\mathcal{H}^{m-1}_r\right)}$. Hence, the main challenge is to derive an estimate for $\|\varrho_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}$.

Our main observation is that, using the definition of α in the equation of conservation of mass, the fluid density satisfies a transport equation of the type

$$\partial_t\varrho_\varepsilon + u_\varepsilon\cdot\nabla_x\varrho_\varepsilon + \frac{\varrho_\varepsilon}{1-\rho_{f_\varepsilon}}\mathrm{div}_x\left[j_{f_\varepsilon} - \rho_{f_\varepsilon}u_\varepsilon\right] = \text{lower order terms}, \tag{5.1.8}$$

therefore ϱ_ε depends on f_ε only through its moments in velocity ρ_{f_ε} and j_{f_ε}. The goal of Sections 5.3-5.4-5.5 is thus to relate these two moments to the fluid density ϱ itself.

In **Section 5.3**, we initiate the study of the Vlasov equation satisfied by f_ε with a Lagrangian point of view. We study the characteristics curves for the kinetic dynamics with friction

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{X}^{s;t}(x,v) = \mathrm{V}^{s;t}(x,v), \qquad \mathrm{X}^{t;t}(x,v) = x, \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}^{s;t}(x,v) = -\mathrm{V}^{s;t}(x,v) + E_\varepsilon(s,\mathrm{X}^{s;t}(x,v),\mathrm{V}^{s;t}(x,v)), \qquad \mathrm{V}^{t;t}(x,v) = v, \end{cases} \tag{5.1.9}$$

stemming from the Vlasov equation in (5.1.5). The term $-v$ in the force field E_ε is responsible for the friction dynamics. To simplify its study, we want to straighten the total kinetic operator

$$\partial_t + v\cdot\nabla_x + \mathrm{div}_v((E_\varepsilon - v)\cdot)$$

into

$$\partial_t + v \cdot \nabla_x - v \cdot \nabla_v, \tag{5.1.10}$$

for short times. The operator in (5.1.10) corresponds to the free dynamics with friction.

More precisely, we prove (see Lemma 5.3.1) that for T small enough (independent of ε), $x \in \mathbb{T}^d$ and $s, t \in [0,T]$, there exists a diffeomorphism $\psi_{s,t}(x,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ satisfying for all $v \in \mathbb{R}^d$

$$\mathrm{X}^{s;t}\,(x, \psi_{s,t}(x,v)) = x + (1 - e^{t-s})v. \tag{5.1.11}$$

In addition, we provide several useful Sobolev estimates on ψ. We call this diffeomorphism the *straightening change of variable* in velocity.

The heart of the proof appears in the remaining sections. In **Section 5.4**, we study some smoothing averaging operators that will be crucial for the subsequent analysis of Section 5.5. In short, it will enable us to split all the quantities exhibiting a loss of derivatives into a leading term and a good remainder which will be controlled.

Let us consider the following kernel operator which has been considered in [HKR16]:

$$\mathrm{K}_G^{\mathrm{free}}[H](t,x) = \int_0^t \int_{\mathbb{R}^d} [\nabla_x H](s, x-(t-s)v) \cdot G(t,s,x,v)\, \mathrm{d}v\, \mathrm{d}s.$$

Despite the apparent loss of derivative, it is proved in [HKR16] that this operator is bounded in $\mathrm{L}^2_T \mathrm{L}^2_x$ as soon as the kernel G is sufficiently smooth and decaying in velocity, a result related to the classical averaging lemmas [GLPS88]. We refer to the introduction of Section 5.4 for more references and explanations about this aspect. We shall provide here extensions of this result to the natural averaging operator for the dynamics with friction (associated with (5.1.10)), namely

$$\mathrm{K}_G^{\mathrm{fric}}[H](t,x) = \int_0^t \int_{\mathbb{R}^d} [\nabla_x H](s, x+(1-e^{t-s})v) \cdot G(t,s,x,v)\, \mathrm{d}v\, \mathrm{d}s.$$

We shall see in Proposition 5.4.4 that $\mathrm{K}_G^{\mathrm{fric}}$ is also bounded in $\mathrm{L}^2_T \mathrm{L}^2_x$ under similar smoothness and decay assumptions for the kernel G. It was also observed in [HKR23] that when the kernel cancels out on the diagonal $s = t$, the operator $\mathrm{K}_G^{\mathrm{free}}$ becomes bounded from $\mathrm{L}^2_T \mathrm{L}^2_x$ to $\mathrm{L}^2_T \mathrm{H}^1_x$, i.e. we gain one extra derivative in x; the same holds as well for $\mathrm{K}_G^{\mathrm{fric}}$, see Proposition 5.4.5.

A key result (see Proposition 5.4.7) we prove is the fact that the *difference* between the two latter operators also allows to gain a derivative in x, namely $\mathrm{K}_G^{\mathrm{free}} - \mathrm{K}_G^{\mathrm{fric}}$ is bounded from $\mathrm{L}^2_T \mathrm{L}^2_x$ to $\mathrm{L}^2_T \mathrm{H}^1_x$. Propositions 5.4.4, 5.4.5 and 5.4.7 will be used at multiple times in this work.

Section 5.5 is dedicated to the proper analysis of the kinetic moments ρ_f and j_f. The main result provided in Proposition 5.5.1 is the fact that for all $|I| \leq m$, we can write

$$\begin{aligned}
\partial_x^I \rho_f(t,x) &= p'(\varrho(t,x)) \int_0^t \int_{\mathbb{R}^d} \nabla_x [\mathrm{J}_\varepsilon \partial_x^I \varrho](s, x-(t-s)v) \cdot \nabla_v f(t,x,v)\, \mathrm{d}v\, \mathrm{d}s + R, \\
\partial_x^I j_f(t,x) &= p'(\varrho(t,x)) \int_0^t \int_{\mathbb{R}^d} v \nabla_x [\mathrm{J}_\varepsilon \partial_x^I \varrho](s, x-(t-s)v) \cdot \nabla_v f(t,x,v)\, \mathrm{d}v\, \mathrm{d}s + R,
\end{aligned} \tag{5.1.12}$$

when R stands for a well-controlled remainder in $\mathrm{L}^2_T \mathrm{H}^1_x$. Combining with the continuity results for the averaging operator $\mathrm{K}^{\mathrm{free}}_{p'(\varrho)\nabla_v f}$, this proves that the loss of derivative for ρ_f and j_f in (5.1.7) was only apparent.

To obtain these identities, the first step is to derive a good equation satisfied by $\partial_x^I f$; to this end it is natural to apply the operator ∂_x^I to the Vlasov equation. We readily obtain

$$\partial_t \partial_x^I f_\varepsilon + v \cdot \nabla_x \partial_x^I f_\varepsilon + \mathrm{div}_v(\partial_x^I f_\varepsilon (E_\varepsilon - v)) + \mathrm{div}_v([\partial_x^I, E_\varepsilon] f_\varepsilon) = 0,$$

and we observe that the commutator involves

- the main order term
$$\mathrm{div}_v(\partial_x^I(E_\varepsilon)f_\varepsilon) \tag{5.1.13}$$
that will account for the leading term in the identities (5.1.12),
- low order terms that can be controlled,
- but also terms of the form

$$\begin{aligned} &\text{(I)} \qquad \partial_x E_\varepsilon \cdot \partial_x^J \nabla_v f_\varepsilon, \quad |J| = m-1,\\ &\text{(II)} \qquad \partial_x^2 E_\varepsilon \cdot \partial_x^J \nabla_v f_\varepsilon, \quad |J| = m-2. \end{aligned}$$

The terms of type (I) clearly cannot be considered as remainders since they involve m derivatives of f_ε, which we do not uniformly control. The terms of type (II) are not remainders either since we expect to plug in the identities (5.1.12) in the equation for ϱ, and this involves an extra derivative in x, thus also resulting in terms with m derivatives of f_ε. To overcome this difficulty, we argue as in [HKR16] and consider an augmented unknown $\mathcal{F} = (\partial_{x,v}^I f_\varepsilon)_{|I|=m-1,m}$ which satisfies a system of the form
$$\partial_t \mathcal{F} + v \cdot \nabla_x \mathcal{F} - v \cdot \nabla_v \mathcal{F} + \mathrm{div}_v(E_\varepsilon \mathcal{F}) + \mathcal{M}\mathcal{F} + \mathcal{L} = \mathcal{R},$$
where $\mathcal{M}$ is a bounded linear map, $\mathcal{L}$ stands for the terms like (5.1.13), and $\mathcal{R}$ is a well-controlled remainder. Note though that in [HKR16], only the terms of type (I) are relevant and the augmented unknown only involves derivatives of order m.

Controlling the averages in velocity of the whole family $(\partial_{x,v}^I f_\varepsilon)_{|I|=m-1,m}$ allows to recover derivatives of ρ_f and j_f in H^m. We finally rely on the Duhamel formula combined with an integration in velocity along the characteristics, on the straightening change of variables in velocity $\psi_{s,t}$ satisfying (5.1.11), and on the crucial gain of derivatives provided by the kernel operators $\mathrm{K}^{\mathrm{free}}_{\nabla_v f}$ and $\mathrm{K}^{\mathrm{fric}}_{\nabla_v f}$ to deduce (5.1.12).

We refer to this approach as a *semi-Lagrangian* one in the sense that we we first apply derivatives on the kinetic equation and then integrate along the characteristics to obtain equations bearing on moments.

Section 5.6 is then devoted to the obtention of an estimate for $\|\varrho_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}$. Taking derivatives in the transport equation (5.1.8) on ϱ and using (5.1.12), one can write an equation on $\partial_x^I \varrho$ for all $|I| = m$ under the form

$$\partial_t \partial_x^I \varrho + u \cdot \nabla_x \partial_x^I \varrho + \frac{\varrho}{1-\rho_f} \mathrm{div}_x \Big[\mathrm{K}^{\mathrm{free}}_{vp'(\varrho)\nabla_v f}(\mathrm{J}_\varepsilon \partial_x^I \varrho) - \mathrm{K}^{\mathrm{free}}_{p'(\varrho)\nabla_v f}(\mathrm{J}_\varepsilon \partial_x^I \varrho) u \Big] = \text{lower order terms.}$$

Based on this equation, and using some commutation properties relating the operators div_x and $\mathrm{K}^{\mathrm{free}}_{vp'(\varrho)\nabla_v f}$, it is then possible to prove (see Proposition 5.6.1) that for all $|I| = m$, the function $\partial_x^I \varrho$ satisfies

$$\left(\mathrm{Id} - \frac{\varrho}{1-\rho_f} \mathrm{K}^{\mathrm{free}}_G \circ \mathrm{J}_\varepsilon \right) \Big[\partial_t \partial_x^I \varrho + u \cdot \nabla_x \partial_x^I \varrho \Big] = \mathcal{R}, \tag{5.1.14}$$
$$G(t,x,v) = p'(\varrho(t,x)) \nabla_v f(t,x,v), \tag{5.1.15}$$

where $\mathcal{R}$ is a well-controlled remainder. The equality (5.1.14) has to be seen as a structural *factorization* of the equation on $\partial_x^I \varrho$, between the operators

$$\mathrm{Id} - \frac{\varrho}{1-\rho_f} \mathrm{K}^{\mathrm{free}}_G \circ \mathrm{J}_\varepsilon,$$

and

$$\partial_t + u \cdot \nabla_x.$$

This relation is fully based on the coupling with the kinetic part.

The main goal is then to derive some good $\mathrm{L}^2_T\mathrm{L}^2_x$ estimates on $\partial_x^I \varrho$. Again following [HKR16], the idea is to relate $\frac{\varrho}{1-\rho_f}\mathrm{K}_G^{\mathrm{free}} \circ \mathrm{J}_\varepsilon$ to a pseudodifferential operator and use pseudodifferential calculus to derive a suitable estimate. This is where the Penrose stability condition steps in and plays a crucial role: it will allow to obtain estimates without loss.

Compared to the analysis of [HKR16], the extra derivative due to the transport operator in (5.1.14) forces us to consider time-dependent symbols; this requires an extension on the whole line $\mathbb{R}$ of all functions, ensuring in the process that the Penrose stability condition still holds globally, see Subsection 5.6.3. For any $\gamma > 0$, there holds (see Lemma 5.6.14)

$$e^{-\gamma t}\mathrm{K}_G^{\mathrm{free}}\Big[e^{\gamma \bullet} H\Big](t,x) := \mathrm{Op}^\gamma(a_{f,\varrho})(H)(t,x), \quad \text{on} \quad (0,T)\times \mathbb{T}^d,$$

with

$$a_{f,\varrho}(t,x,\gamma,\tau,k) := p'(\varrho(t,x)) \int_0^{+\infty} e^{-(\gamma+i\tau)s} ik \cdot (\mathcal{F}_v \nabla_v f)(t,x,ks)\,\mathrm{d}s,$$

and thus (5.1.14) turns into the pseudodifferential equation

$$\left(\mathrm{Id} - \frac{\varrho}{1-\rho_f}\mathrm{Op}^\gamma(a_{f,\varrho}) \circ \mathrm{J}_\varepsilon\right)\Big[\partial_t \partial_x^I \varrho + u \cdot \nabla_x \partial_x^I \varrho\Big] = \mathcal{R}. \tag{5.1.16}$$

Here, Op^γ refers to a pseudodifferential quantization on $\mathbb{R}\times\mathbb{T}^d$ and with parameter $\gamma > 0$ (see Section 5.C in the Appendix for more details). By observing that

$$\frac{\varrho}{1-\rho_f} a_{f,\varrho} = \mathscr{P}_{f,\varrho},$$

where

$$\mathscr{P}_{f(t),\varrho(t)}(x,\gamma,\tau,k) := \frac{p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)} \int_0^{+\infty} e^{-(\gamma+i\tau)s}\frac{ik}{1+|k|^2} \cdot (\mathcal{F}_v \nabla_v f)(t,x,ks)\,\mathrm{d}s,$$

we discover that the Penrose stability condition

$$\forall t \in \mathbb{R}, \quad \inf_{(x,\gamma,\tau,k)} |1 - \mathscr{P}_{f(t),\varrho(t)}(x,\gamma,\tau,k)| > 0$$

thus asserts the *ellipticity* of the symbol involved in the equation (5.1.16). Assuming an $\mathrm{L}^2_T\mathrm{L}^2_x$ estimate on

$$H = \partial_t \partial_x^\alpha \varrho + u \cdot \nabla_x \partial_x^\alpha \varrho,$$

a standard hyperbolic energy estimate associated to the transport part $\partial_t + u\cdot\nabla_x$ thus leads to a suitable estimate on $\partial_x^\alpha \varrho$ (see Corollary 5.6.22). Roughly speaking, the Penrose stability condition shows that the equation (5.1.14) can be seen as a factorization between a *elliptic* part and an *hyperbolic* part.

Obtaining a control on H solution to the previous pseudodifferential equation (5.1.16) is then enough to conclude. To do so, we rely on a semiclassical (in ε) pseudodifferential calculus (with large parameter γ) whose aim is to invert the equation on H up to some small remainder[3]. The key

[3] This part of our analysis (with a large parameter) is reminiscent of the use of Lopatinskii determinant or Evans functions to obtain good estimates in hyperbolic boundary value problems or singular stable boundary layer problems (see e.g. [Mét01, MZ05, Rou04]).

is that one can consider the symbol $(1-\mathscr{P}_{f,\varrho})^{-1}$. This yields an $\mathrm{L}^2_T\mathrm{L}^2_x$ estimate for H in terms of the remainder $\mathcal{R}$ (see Corollary 5.6.20).

Section 5.7 is eventually dedicated to the conclusion of the proof, gathering all the previous steps and estimates of the bootstrap analysis. We obtain the desired uniform estimate for the quantity $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T)$, which is valid for some time $T>0$ independent of ε. The existence part of Theorem 5.1.5 is then easily deduced by a compactness argument on $[0,T]$. The uniqueness part requires an additional argument, in the same spirit as the strategy previously devised.

In **Section 5.8**, we show how one can easily adapt the strategy performed in this chapter to treat the case of a non-barotropic fluid with an additional equation on the internal energy for the fluid.

In **Section 5.9**, we describe how one can include an inelastic collision operator of Bolzmann type in the kinetic equation (see (5.9.1)). Note that our method follows an idea used in [Mat10], which allows to overcome the loss of weight in velocity from the collision operator *thanks to* the friction term in the original Vlasov equation.

In **Section 5.10**, we consider the case of a density-dependent drag force, for which one can also prove a local well-posedness result, with the limitation that the initial data f^{in} has a compact support in velocity.

We refer to the precise statements of the Section 5.8–5.9–5.10 for more details about the corresponding existence results.

Let us finally describe the content of the **Appendices**, gathered in the end of this chapter.

- In Appendix 5.A, we state several useful functional inequalities on commutators, products and composition on $\mathbb{T}^d$ and $\mathbb{T}^d\times\mathbb{R}^d$.
- In Appendix 5.B, we justify the main steps providing the existence of a local in time solution $(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon)$ to the regularized system (5.1.5) when $\varepsilon>0$ is fixed.
- In Appendix 5.C, we recall and give the main notions on pseudodifferential calculus (with parameter) that we shall need in this chapter.

In the rest of the chapter, we shall use the standard notation $A \lesssim B$ for $A \leq cB$ for some $c>0$ which is independent of A, B and of ε, but that may change from line to line. Furthermore, Λ will stand for a nonnegative continuous function which is independent of ε, nondecreasing with respect to each of its arguments, that may depend implicitly on the initial data and that may change from line to line. Finally, we denote by $[P,Q]=PQ-QP$ the commutator between two operators P and Q.

5.2 Preliminaries

In this section, we initiate the bootstrap strategy that will be used to prove Theorem 5.1.5.

Throughout this chapter, we will constantly use the following lemma (which is a straightforward consequence of the Cauchy-Schwarz inequality).

Lemma 5.2.1. *For all nonnegative measurable function* $\mathrm{f}(x,v)$ *and* $k\in\mathbb{N}$*, we have*

$$\forall \ell \in \mathbb{N}, \quad \forall r > \frac{d}{2}+k, \quad \left\| \int_{\mathbb{R}^d} |v|^k f(\cdot, v)\,\mathrm{d}v \right\|_{\mathrm{H}^\ell} \lesssim \|\mathrm{f}\|_{\mathcal{H}^\ell_r}.$$

In particular, we have

$$\forall \ell \in \mathbb{N}, \quad \forall r > \frac{d}{2}, \quad \|\rho_{\mathrm{f}}\|_{\mathrm{H}^\ell} \lesssim \|\mathrm{f}\|_{\mathcal{H}^\ell_r},$$

$$\forall \ell \in \mathbb{N}, \quad \forall r > \frac{d}{2}+1, \quad \|j_{\mathrm{f}}\|_{\mathrm{H}^\ell} \lesssim \|\mathrm{f}\|_{\mathcal{H}^\ell_r}.$$

5.2.1 Energy estimates

Our aim is to obtain some *a priori* estimates for smooth solutions to the system (TS). We first study the fluid density ϱ, which is shown to satisfy a hyperbolic-type equation.

Lemma 5.2.2. *Let $T > 0$, $c > 0$ and (f, ϱ, u) satisfying (TS) on $[0, T]$ with $1 - \rho_f \geq c$ on $[0, T]$. Defining the operator $\mathscr{L}^{u,f}$ as*

$$\mathscr{L}^{u,f} := \partial_t + u \cdot \nabla_x + B^{u,f},$$

where

$$B^{u,f} := \frac{1}{1-\rho_f}\mathrm{div}_x\,[F + u]\,\mathrm{Id}, \quad F(t,x) := (j_f - \rho_f u)(t,x),$$

the fluid density ϱ satisfies

$$\mathscr{L}^{u,f}\varrho = 0 \quad \textit{on } [0,T].$$

Proof. The transport equation on ϱ in (TS) can be rewritten as

$$(1-\rho_f)\,(\partial_t\varrho + u\cdot\nabla_x\varrho) + \varrho\Big(\partial_t(1-\rho_f) + u\cdot\nabla_x(1-\rho_f)\Big) + (1-\rho_f)\varrho\mathrm{div}_x\,u = 0.$$

Integrating the Vlasov equation in the v variable, we obtain the equation of conservation

$$\partial_t\rho_f + \mathrm{div}_x\,j_f = 0,$$

therefore $\partial_t(1-\rho_f) = \mathrm{div}_x\,j_f$ and we get

$$\begin{aligned}
0 &= \partial_t\varrho + u\cdot\nabla_x\varrho + \frac{\varrho}{1-\rho_f}\left[\mathrm{div}_x j_f - u\cdot\nabla_x\rho_f\right] + \varrho\mathrm{div}_x\,u \\
&= \partial_t\varrho + u\cdot\nabla_x\varrho + \frac{\varrho}{1-\rho_f}\mathrm{div}_x\left[j_f - \rho_f u\right] + \frac{\varrho}{1-\rho_f}\rho_f\mathrm{div}_x\,u + \varrho\mathrm{div}_x\,u \\
&= \partial_t\varrho + u\cdot\nabla_x\varrho + \frac{\varrho}{1-\rho_f}\mathrm{div}_x\left[j_f - \rho_f u\right] + \frac{\varrho}{1-\rho_f}\mathrm{div}_x\,u.
\end{aligned}$$

We recognize the expression of the Brinkman force $F := j_f - \rho_f u$ and therefore ϱ satisfies the equation

$$\partial_t\varrho + u\cdot\nabla_x\varrho + \frac{1}{1-\rho_f}\mathrm{div}_x\left[F+u\right]\varrho = 0,$$

which is the claimed result. □

We are now able to derive a Sobolev estimate bearing on the fluid density ϱ, in which we control ℓ derivatives of ϱ by $\ell+1$ derivatives of f and u.

Proposition 5.2.3. *For all $\ell, r > 1 + d/2$, $c > 0$ and $T > 0$ and all smooth functions (f, ϱ, u) such that $1 - \rho_f \geq c$ and*

$$\mathscr{L}^{u,f}\varrho = 0 \quad \textit{on } [0,T],$$

we have the estimate

$$\forall t\in[0,T], \quad \|\varrho(t)\|_{\mathrm{H}^\ell} \leq \|\varrho^{\mathrm{in}}\|_{\mathrm{H}^\ell} e^{C_T(u,f)T}\exp\left[Te^{C_T(u,f)T}\mathcal{Q}_\ell(T,u,f)\right],$$

where

$$\begin{aligned}
C_T(u,f) &= \|\mathrm{div}_x\,u\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^d)} + 2\|B^{u,f}\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^d)}, \\
\mathcal{Q}_\ell(T,u,f) &= \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^\ell)} + \left\|\frac{1}{1-\rho_f}\right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^\ell)}\Big(\|f\|^2_{\mathrm{L}^\infty(0,T;\mathcal{H}^{\ell+1}_r)} + \|u\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^{\ell+1})}\Big),
\end{aligned}$$

Remark 5.2.4. Note that for the same exponents ℓ and r, one has

$$C_T(u,f) \lesssim \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^\ell)} + \left\| \frac{1}{1-\rho_f} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \left(\|f\|^2_{\mathrm{L}^\infty(0,T;\mathcal{H}^{\ell+1}_r)} + \|u\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^{\ell+1})} \right).$$

Remark 5.2.5. Let us mention for any smooth solution (f,ϱ,u) to (TS), the function $(1-\rho_f)\varrho$ satisfies

$$\mathscr{L}^{u,0}[(1-\rho_f)\varrho] = 0.$$

As a consequence, the result of Proposition 5.2.3 holds for $(1-\rho_f)\varrho$ instead of ϱ, by considering $\mathcal{Q}_\ell(T,u,0)$ and $C_T(u,0)$.

Proof of Proposition 5.2.3. The proof is standard but for the sake of completeness and for highlighting the dependency in f and u, let us write it. First, suppose we have a smooth function h satisfying

$$\mathscr{L}^{u,f}(h) = S,$$

where S is a given smooth source term. Performing an L^2_x-energy estimate, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{\mathrm{L}^2} &= -2\langle u\cdot\nabla_x h, h\rangle_{\mathrm{L}^2} - 2\langle B^{u,f} h, h\rangle_{\mathrm{L}^2} + 2\langle S, h\rangle_{\mathrm{L}^2} \\ &= \int_{\mathbb{T}^d} (\mathrm{div}_x\, u - 2B^{u,f})|h|^2 \,\mathrm{d}x + 2\langle S, h\rangle_{\mathrm{L}^2}.\end{aligned}$$

By the Cauchy-Schwarz inequality, this yields for all $t \in (0,T)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h(t)\|^2_{\mathrm{L}^2} \leq C_T(u,f)\|h(t)\|^2_{\mathrm{L}^2} + 2\|S(t)\|_{\mathrm{L}^2}\|h(t)\|_{\mathrm{L}^2},$$

where $C_T(u,f) := \|\mathrm{div}_x\, u\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^d)} + 2\|B^{u,f}\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^d)}$. By Grönwall's inequality, we deduce

$$\|h(t)\|_{\mathrm{L}^2} \leq \|h(0)\|_{\mathrm{L}^2} e^{C_T(u,f)t/2} + \int_0^t e^{C_T(u,f)(t-\tau)/2}\|S(\tau)\|_{\mathrm{L}^2}\,\mathrm{d}\tau.$$

Now, let us assume that ϱ is such that $\mathscr{L}^{u,f}(\varrho) = 0$. Let $\beta \in \mathbb{N}^d$ such that $|\beta| \leq \ell$. Since

$$\mathscr{L}^{u,f}(\partial^\beta_x \varrho) = -\left[\partial^\beta_x, u\cdot\nabla_x + B^{u,f}\right]\varrho,$$

the first part of the proof with $h = \partial^\beta_x\varrho$ and $S = -\left[\partial^\beta_x, u\cdot\nabla_x + B^{u,f}\right]\varrho$ leads to

$$\|\partial^\beta_x\varrho(t)\|_{\mathrm{L}^2} \leq \|\partial^\beta_x\varrho(0)\|_{\mathrm{L}^2} e^{C_T(u,f)t/2} + \int_0^t e^{C_T(u,f)(t-\tau)/2}\left\|\left[\partial^\beta_x, u\cdot\nabla_x + B^{u,f}\right]\varrho(\tau)\right\|_{\mathrm{L}^2}\mathrm{d}\tau.$$

We can estimate the commutator $\left[\partial^\beta_x, u\cdot\nabla_x + B^{u,f}\right]\varrho = \left[\partial^\beta_x, u\cdot\right](\nabla_x\varrho) + \left[\partial^\beta_x, B^{u,f}\right](\varrho)$ thanks to Proposition 5.A.1. This gives

$$\begin{aligned}\sum_{0\leq|\beta|\leq\ell} \left\| \left[\partial^\beta_x, u\cdot\nabla_x + B^{u,f}\right]\varrho\right\|_{\mathrm{L}^2} &\leq \sum_{0\leq|\beta|\leq\ell} \left\|\left[\partial^\beta_x, u\cdot\right](\nabla_x\varrho)\right\|_{\mathrm{L}^2} + \sum_{0\leq|\beta|\leq\ell}\left\|\left[\partial^\beta_x, B^{u,f}\right](\varrho)\right\|_{\mathrm{L}^2} \\ &\lesssim \|\nabla_x u\|_{\mathrm{L}^\infty}\|\nabla_x\varrho\|_{\mathrm{H}^{\ell-1}} + \|u\|_{\mathrm{H}^\ell}\|\nabla_x\varrho\|_{\mathrm{L}^\infty} \\ &\qquad + \|\nabla_x B^{u,f}\|_{\mathrm{L}^\infty}\|\varrho\|_{\mathrm{H}^{\ell-1}} + \|B^{u,f}\|_{\mathrm{H}^\ell}\|\varrho\|_{\mathrm{L}^\infty} \\ &\lesssim \|u\|_{\mathrm{H}^\ell}\|\varrho\|_{\mathrm{H}^\ell} + \|B^{u,f}\|_{\mathrm{H}^\ell}\|\varrho\|_{\mathrm{H}^\ell},\end{aligned}$$

by Sobolev embedding, since $\ell > 1 + d/2$. By summing on β, we eventually get for all $t \in [0, T]$

$$\|\varrho(t)\|_{\mathrm{H}^\ell} \leq e^{C_T(u,f)t/2} \left(\|\varrho^{\mathrm{in}}\|_{\mathrm{H}^\ell} + \int_0^t \left[\|u(\tau)\|_{\mathrm{H}^\ell} + \|B(\tau)\|_{\mathrm{H}^\ell} \right] \|\varrho(\tau)\|_{\mathrm{H}^\ell} \, \mathrm{d}\tau \right).$$

Again by Grönwall's inequality, we get for all $t \in [0, T]$

$$\|\varrho(t)\|_{\mathrm{H}^\ell} \leq e^{C_T(u,f)T/2} \|\varrho^{\mathrm{in}}\|_{\mathrm{H}^\ell} \exp\left[T e^{C_T(u,f)T/2} \left(\|u\|_{\mathrm{L}^\infty(0,T,\mathrm{H}^\ell)} + \|B^{u,f}\|_{\mathrm{L}^\infty(0,T,\mathrm{H}^\ell)} \right) \right].$$

To conclude, we write for all $\tau \in (0, T)$

$$\begin{aligned} &\|u\|_{\mathrm{L}^\infty(0,T,\mathrm{H}^\ell)} + \|B^{u,f}\|_{\mathrm{L}^\infty(0,T,\mathrm{H}^\ell)} \\ &\qquad \leq \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^\ell)} + \left\| \frac{1}{1 - \rho_f} \right\|_{\mathrm{L}^\infty((0,T);\mathrm{H}^\ell)} \|\mathrm{div}_x \left[j_f - \rho_f u + u \right] \|_{\mathrm{L}^\infty(0,T;\mathrm{H}^\ell)}, \end{aligned}$$

and use

$$\|\mathrm{div}_x \left[j_f - \rho_f u + u \right] \|_{\mathrm{H}^\ell} \lesssim \|f\|_{\mathcal{H}_r^{\ell+1}} + \|f\|_{\mathcal{H}_r^{\ell+1}} \|u\|_{\mathrm{H}^{\ell+1}} + \|u\|_{\mathrm{H}^{\ell+1}},$$

via $\ell + 1 > d/2$ and Lemma 5.2.1 with $r > 1 + d/2$. This concludes the proof. □

Let us now start the study of the kinetic equation satisfied by f.

Definition 5.2.6. *For any vector field $u(t,x)$ and function $\varrho(t,x)$, we define the kinetic transport operator $\mathcal{T}^{u,\varrho}$ as*

$$\mathcal{T}^{u,\varrho} := \partial_t + v \cdot \nabla_x - v \cdot \nabla_v + E^{u,\varrho}(t,x) \cdot \nabla_v - d\mathrm{Id},$$

where

$$E^{u,\varrho}(t,x) := u(t,x) - p'(\varrho) \nabla_x \varrho(t,x).$$

By developing the divergence (in v) term in the kinetic equation, the Vlasov equation on f in (TS) can be rewritten

$$\mathcal{T}^{u,\varrho} f = 0.$$

We now state several standard useful estimates to handle the force field $E^{u,\varrho}$.

Lemma 5.2.7. *Let $T > 0$. If $\varrho \geq c > 0$ on $[0,T]$ for some given constant c, then the following hold:*

- *for all $k \geq 0$ and $t \in [0,T]$, we have*

$$\|p'(\varrho(t))\|_{\mathrm{H}^k} \leq \Lambda \left(\|\varrho(t)\|_{\mathrm{L}^\infty} \right) \|\varrho(t)\|_{\mathrm{H}^k}. \tag{5.2.1}$$

- *for all $k > 3 + d/2$ and $t \in [0,T]$, we have*

$$\|E^{u,\varrho}(t)\|_{\mathrm{H}^k} \lesssim \|u(t)\|_{\mathrm{H}^k} + \Lambda \left(\|\varrho(t)\|_{\mathrm{H}^{k-2}} \right) \|\varrho(t)\|_{\mathrm{H}^{k+1}}. \tag{5.2.2}$$

Proof. To prove (5.2.1), we rely on the paralinearization theorem of Bony applied to p' (see Proposition 5.A.3 and Remark 5.A.4 in the Appendix) thanks to the assumption on the pressure p and the lower bound on ϱ.

To prove (5.2.2), we only have to estimate the term $p'(\varrho)\nabla_x\varrho$. We use the following tame estimate for products (see Proposition 5.A.2 in the Appendix)

$$\begin{aligned}\|p'(\varrho(t))\nabla_x\varrho(t)\|_{\mathrm{H}^k} &\lesssim \|p'(\varrho(t))\|_{\mathrm{L}^\infty}\|\nabla_x\varrho(t)\|_{\mathrm{H}^k} + \|p'(\varrho(t))\|_{\mathrm{H}^k}\|\nabla_x\varrho(t)\|_{\mathrm{L}^\infty} \\ &\lesssim \|p'(\varrho(t))\|_{\mathrm{L}^\infty}\|\varrho(t)\|_{\mathrm{H}^{k+1}} + \|p'(\varrho(t))\|_{\mathrm{H}^k}\|\nabla_x\varrho(t)\|_{\mathrm{L}^\infty},\end{aligned}$$

therefore by Sobolev embedding, we have for $s_1 > d/2$ and $s_2 > 1 + d/2$,

$$\|E^{u,\varrho}(t)\|_{\mathrm{H}^k} \lesssim \|u(t)\|_{\mathrm{H}^k} + \|p'(\varrho(t))\|_{\mathrm{H}^{s_1}}\|\varrho(t)\|_{\mathrm{H}^{k+1}} + \|p'(\varrho(t))\|_{\mathrm{H}^k}\|\varrho(t)\|_{\mathrm{H}^{s_2}}.$$

With $s_1 = s_2 = k - 2$, this yields

$$\|E^{u,\varrho}(t)\|_{\mathrm{H}^k} \lesssim \|u(t)\|_{\mathrm{H}^k} + \|p'(\varrho(t))\|_{\mathrm{H}^{k-2}}\|\varrho(t)\|_{\mathrm{H}^{k+1}} + \|p'(\varrho(t))\|_{\mathrm{H}^k}\|\varrho(t)\|_{\mathrm{H}^{k-2}}.$$

In view of (5.2.1), there exists a continuous nondecreasing function $\mathrm{C}_{k,p'} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{aligned}\|E^{u,\varrho}(t)\|_{\mathrm{H}^k} \lesssim \|u(t)\|_{\mathrm{H}^k} &+ \mathrm{C}_{k,p'}\left(\|\varrho(t)\|_{\mathrm{L}^\infty}\right)\|\varrho(t)\|_{\mathrm{H}^{k-2}}\|\varrho(t)\|_{\mathrm{H}^{k+1}} \\ &+ \mathrm{C}_{k,p'}\left(\|\varrho(t)\|_{\mathrm{L}^\infty}\right)\|\varrho(t)\|_{\mathrm{H}^k}\|\varrho(t)\|_{\mathrm{H}^{k-2}},\end{aligned}$$

and finally by using Sobolev embedding (with $k - 2 > d/2$), we get

$$\|E^{u,\varrho}(t)\|_{\mathrm{H}^k} \lesssim \|u(t)\|_{\mathrm{H}^k} + \widetilde{\mathrm{C}}_{k,p'}\left(\|\varrho(t)\|_{\mathrm{H}^{k-2}}\right)\|\varrho(t)\|_{\mathrm{H}^{k+1}},$$

for another function $\widetilde{\mathrm{C}}_{k,p'}$ of the same type. This concludes the proof. □

Definition 5.2.8. *For $\beta \in \mathbb{N}^d$, we define $\widehat{\beta} \in \mathbb{N}^d$ and $\overline{\beta} \in (\mathbb{N} \cup \{-1\})^d$ as*

$$\begin{aligned}\widehat{\beta}^k &:= (\beta_1, \cdots, \beta_{k-1}, \beta_k + 1, \beta_{k+1}, \cdots, \beta_d), \quad k \in [\![1, d]\!], \\ \overline{\beta}^k &:= (\beta_1, \cdots, \beta_{k-1}, \beta_k - 1, \beta_{k+1}, \cdots, \beta_d), \quad k \in [\![1, d]\!].\end{aligned}$$

We have the following straightforward lemma of commutation for the kinetic equation.

Lemma 5.2.9. *For any $\alpha, \beta \in \mathbb{N}^d$ and for any smooth function $f(t, x, v)$, we have*

$$\left[\partial_x^\alpha\partial_v^\beta, \mathcal{T}^{u,\varrho}\right] f = \sum_{\substack{i=1 \\ \beta_i \neq 0}}^{d} \left(\partial_x^{\widehat{\alpha}^i}\partial_v^{\overline{\beta}^i} f - \partial_x^\alpha\partial_v^\beta f\right) + \left[\partial_x^\alpha\partial_v^\beta, E^{u,\varrho}(t,x)\cdot\nabla_v\right] f.$$

The Sobolev estimate for the kinetic equation goes as follows, showing that we can control m derivatives of f by $m + 1$ derivatives of ϱ and m derivatives of u.

Proposition 5.2.10. *For all $r \geq 0$, $m > 3 + d/2$, $c > 0$, there exists $C > 0$ such that for all $T > 0$ all smooth functions (f, ϱ, u) satisfying*

$$\mathcal{T}^{u,\varrho}(f) = 0 \quad \textit{on } [0, T],$$

and $\varrho \geq c$ on $[0, T]$, there holds, for all $t \in [0, T]$

$$\|f(t)\|^2_{\mathcal{H}^m_r} \leq \|f(0)\|^2_{\mathcal{H}^m_r}\exp\left[C\left((1 + \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)})T + \sqrt{T}\Lambda\left(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\right)\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}\right)\right].$$

Proof. By Lemma 5.2.9, we have

$$\mathcal{T}^{u,\varrho}(\partial_x^\alpha \partial_v^\beta f) = -\sum_{\substack{i=1\\ \beta_i\neq 0}}^{d} \left(\partial_x^{\widehat{\alpha}^i}\partial_v^{\overline{\beta}^i} f - \partial_x^\alpha \partial_v^\beta f\right) - \left[\partial_x^\alpha \partial_v^\beta, E^{u,\varrho}(t,x)\cdot\nabla_v\right] f,$$

for all $\alpha, \beta \in \mathbb{N}^d$. We take the scalar product of this equality with $(1+|v|^2)^r \partial_x^\alpha \partial_v^\beta f$, sum for all $|\alpha|+|\beta| \leq m$ and then integrate on $\mathbb{T}^d \times \mathbb{R}^d$. For the left-hand side, we observe that

$$\begin{aligned}
&\sum_{|\alpha|+|\beta|\leq m} \int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r \mathcal{T}^{u,\varrho}(\partial_x^\alpha \partial_v^\beta f)\partial_x^\alpha \partial_v^\beta f \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2_{\mathcal{H}^m_r} - d\|f(t)\|^2_{\mathcal{H}^m_r} \\
&\qquad + \sum_{|\alpha|+|\beta|\leq m} \int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r [v\cdot\nabla_x - v\cdot\nabla_v + E^{u,\varrho}(t,x)\cdot\nabla_v](\partial_x^\alpha \partial_v^\beta f)\partial_x^\alpha \partial_v^\beta f \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2_{\mathcal{H}^m_r} - d\|f(t)\|^2_{\mathcal{H}^m_r} + \sum_{|\alpha|+|\beta|\leq m} \int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r \mathrm{div}_x\left(v\frac{|\partial_x^\alpha \partial_v^\beta f|^2}{2}\right) \\
&\qquad + \sum_{|\alpha|+|\beta|\leq m} \int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r \mathrm{div}_v\left((E^{u,\varrho}-v)\frac{|\partial_x^\alpha \partial_v^\beta f|^2}{2}\right) + d\|f(t)\|^2_{\mathcal{H}^m_r} \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2_{\mathcal{H}^m_r} - \sum_{|\alpha|+|\beta|\leq m} \int_{\mathbb{T}^d\times\mathbb{R}^d} \nabla_v(1+|v|^2)^r\cdot(E^{u,\varrho}-v)\frac{|\partial_x^\alpha \partial_v^\beta f|^2}{2},
\end{aligned}$$

and that the last term satisfies

$$\sum_{|\alpha|+|\beta|\leq m} \int_{\mathbb{T}^d\times\mathbb{R}^d} \nabla_v(1+|v|^2)^r\cdot(E^{u,\varrho}-v)\frac{|\partial_x^\alpha \partial_v^\beta f|^2}{2} \leq (1+\|E^{u,\varrho}(t)\|_{\mathrm{L}^\infty})\|f(t)\|^2_{\mathcal{H}^m_r}.$$

We now look at the two terms of the right-hand side. For the first one, we have

$$-\sum_{|\alpha|+|\beta|\leq m} \int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r \sum_{\substack{i=1\\ \beta_i\neq 0}}^{d} \left(\partial_x^{\widehat{\alpha}^i}\partial_v^{\overline{\beta}^i} f - \partial_x^\alpha \partial_v^\beta f\right)\partial_x^\alpha \partial_v^\beta f \lesssim \|f(t)\|^2_{\mathcal{H}^m_r},$$

while for the second one, we write

$$\begin{aligned}
&\sum_{|\alpha|+|\beta|\leq m} \int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r \left[\partial_x^\alpha \partial_v^\beta, E^{u,\varrho}(t,x)\cdot\nabla_v\right] f \partial_x^\alpha \partial_v^\beta f \\
&\qquad\qquad \lesssim \|(1+|v|^2)^{r/2}\left[\partial_x^\alpha \partial_v^\beta, E^{u,\varrho}(t,x)\cdot\nabla_v\right] f\|_{\mathrm{L}^2_{x,v}}\|f(t)\|_{\mathcal{H}^m_r} \\
&\qquad\qquad \lesssim \|E^{u,\varrho}(t)\|_{\mathrm{H}^m}\|f(t)\|^2_{\mathcal{H}^m_r},
\end{aligned}$$

by the Cauchy-Schwarz inequality and the product law (5.A.3) in Sobolev spaces (since $m > 1+d$) of Lemma 5.A.6. All in all, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2_{\mathcal{H}^m_r} \lesssim (1+\|E^{u,\varrho}(t)\|_{\mathrm{L}^\infty}+\|E^{u,\varrho}(t)\|_{\mathrm{H}^m})\|f(t)\|^2_{\mathcal{H}^m_r} \lesssim (1+\|E^{u,\varrho}(t)\|_{\mathrm{H}^m})\|f(t)\|^2_{\mathcal{H}^m_r}, \quad (5.2.3)$$

if $m > d/2$. Invoking the estimate (5.2.2) of Lemma 5.2.7, and by integrating in time the inequality (5.2.3), we obtain

$$\|f(t)\|^2_{\mathcal{H}^m_r} \leq \|f(0)\|^2_{\mathcal{H}^m_r} + C\int_0^t \left(1+\|u(s)\|_{\mathrm{H}^m} + \Lambda\left(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\right)\|\varrho(s)\|_{\mathrm{H}^{m+1}}\right)\|f(s)\|^2_{\mathcal{H}^m_r}\,\mathrm{d}s,$$

for all $t \in [0,T)$ and for some constant $C > 0$. Using Cauchy-Schwarz and Grönwall's inequality, this implies for all $t \in [0,T)$

$$\|f(t)\|^2_{\mathcal{H}^m_r} \leq \|f(0)\|^2_{\mathcal{H}^m_r} \exp\left[C\left((1+\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)})T + \sqrt{T}\Lambda\left(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\right)\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}\right)\right],$$

and this concludes the proof. □

The estimates given by Proposition 5.2.3 and Proposition 5.2.10 show a possible loss of derivatives between f and ϱ. This constitutes the main obstacle of the analysis.

5.2.2 Regularization of the system and setup of the bootstrap

To (temporarily) bypass this problem, we introduce the following regularized version of the equations. Since the pressure gradient in the force field of the Vlasov equation seems to cause estimates with a loss of derivative, we smooth out this precise term. For all $\varepsilon > 0$, we consider

$$\begin{cases} \partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \mathrm{div}_v[f_\varepsilon(u_\varepsilon - v)] - p'(\varrho_\varepsilon)\nabla_x\left[(I-\varepsilon^2\Delta_x)^{-1}\varrho_\varepsilon\right]\cdot\nabla_v f_\varepsilon = 0, \\ \partial_t\varrho_\varepsilon + u_\varepsilon\cdot\nabla_x\varrho_\varepsilon + \dfrac{\varrho_\varepsilon}{1-\rho_{f_\varepsilon}}\mathrm{div}_x\left[j_{f_\varepsilon} - \rho_{f_\varepsilon}u_\varepsilon\right] = -\dfrac{\varrho_\varepsilon}{1-\rho_{f_\varepsilon}}\mathrm{div}_x u_\varepsilon, \\ \partial_t u_\varepsilon + (u_\varepsilon\cdot\nabla_x)u_\varepsilon + \dfrac{1}{\varrho_\varepsilon}\nabla_x p(\varrho_\varepsilon) - \dfrac{1}{\varrho_\varepsilon(1-\rho_{f_\varepsilon})}\left[\Delta_x + \nabla_x\mathrm{div}_x\right]u_\varepsilon = \dfrac{1}{\varrho_\varepsilon(1-\rho_{f_\varepsilon})}(j_{f_\varepsilon} - \rho_{f_\varepsilon}u_\varepsilon), \\ f_{\varepsilon|t=0} = f^{\mathrm{in}}, \quad \varrho_{\varepsilon|t=0} = \varrho^{\mathrm{in}}, \quad u_{\varepsilon|t=0} = u^{\mathrm{in}}, \end{cases} \tag{S_ε}$$

where

$$\rho_{f_\varepsilon}(t,x) := \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\,\mathrm{d}v, \quad j_{f_\varepsilon}(t,x) := \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)v\,\mathrm{d}v.$$

Let us highlight that we have used the rewriting of the transport equation on ϱ_ε based on Lemma 5.2.2.

Definition 5.2.11. *For all $\varepsilon > 0$, we define the regularized kinetic transport operator $\mathcal{T}^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}$ as*

$$\mathcal{T}^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon} := \partial_t + v\cdot\nabla_x - v\cdot\nabla_v + E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}(t,x)\cdot\nabla_v - d\mathrm{Id},$$

where

$$E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}(t,x) := u_\varepsilon(t,x) - p'(\varrho_\varepsilon)\nabla_x\left[\mathrm{J}_\varepsilon\varrho_\varepsilon\right](t,x), \quad \mathrm{J}_\varepsilon := (I-\varepsilon^2\Delta_x)^{-1}.$$

Remark 5.2.12. The regularization through the operator J_ε could appear as quite arbitrary. In view of the equivalent Penrose condition (5.1.4) appearing in Remark 5.1.9, it is actually possible to consider a more general Fourier multiplier

$$\mathcal{J}_\varepsilon = m(\varepsilon D),$$

associated to a smooth function $m : \mathbb{R}^d \to (0,1]$ such that

$$\forall k \in \mathbb{R}^d, \quad m(k) \leq \frac{C}{1+|k|^2}.$$

For all $\varepsilon > 0$, the Vlasov equation satisfied by f in (S_ε) can be recast as

$$\mathcal{T}^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon} f = 0.$$

Relying on the elliptic regularity provided by J_ε, we now have the following estimates for the regularized force field $E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}$.

Lemma 5.2.13. *Let $\varepsilon > 0$ and $T > 0$. If $\varrho \geq c > 0$ on $[0,T]$, then for all $k > 3 + d/2$ and $t \in [0,T]$*

$$\|E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}(t)\|_{\mathrm{H}^k} \lesssim \|u(t)\|_{\mathrm{H}^k} + \frac{1}{\varepsilon}\Lambda\left(\|\varrho(t)\|_{\mathrm{H}^{k-2}}\right)\|\varrho(t)\|_{\mathrm{H}^k}. \tag{5.2.4}$$

Thanks to the regularization, we can overcome the loss of derivative exhibited by the estimate of Proposition 5.2.10, up to some factor which is diverging when $\varepsilon \to 0$.

Proposition 5.2.14. *For all $r \geq 0$, $m > 3 + d/2$, $c > 0$, there exists $C > 0$ such that for all $\varepsilon > 0$, $T > 0$ and all smooth functions (f, ϱ, u) satisfying*

$$\mathcal{T}^{u,\varrho}_{\mathrm{reg},\varepsilon}(f) = 0 \quad \textit{on} \quad [0,T],$$

and $\varrho \geq c$ on $[0,T]$, there holds, for all $t \in [0,T]$

$$\|f(t)\|^2_{\mathcal{H}^m_r} \leq \|f(0)\|^2_{\mathcal{H}^m_r} \exp\left[C\left((1 + \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)})T + \frac{\sqrt{T}}{\varepsilon}\Lambda\left(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\right)\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}\right)\right].$$

Proof. The proof is the same as for Proposition 5.2.10, using (5.2.4) instead of (5.2.2) to conclude. □

We shall also need the following condition about the pointwise bounds for the densities.

Definition 5.2.15. *Let $T > 0$. For any nonnegative functions $f(t,x,v)$ and $\varrho(t,x)$ on $[0,T]$, we define the property*

$$\forall t \in [0,T], \quad \rho_f(t) \leq \frac{\Theta + 1}{2}, \quad \frac{\mu}{2} \leq \varrho(t), \quad \frac{\underline{\theta}}{2} \leq (1 - \rho_f(t))\varrho(t) \leq 2\overline{\theta}, \tag{$\mathrm{B}^{\mu,\theta}_{\Theta}(T)$}$$

where $\Theta, \mu, \underline{\theta}, \overline{\theta}$ are given in the statement of Theorem 5.1.5.

We will be able to propagate the condition ($\mathrm{B}^{\mu,\theta}_{\Theta}(T)$) thanks to the following lemmas, giving some rough pointwise control for the local particle density and the fluid density.

Lemma 5.2.16. *Assume that $\rho_{f^{\mathrm{in}}} < \Theta$. For any $\ell, r > 1 + d/2$, any smooth solution $f(t,x,v)$ to the Vlasov equation in (S_ε) satisfies for all $T > 0$*

$$\|\rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^d))} \leq \Theta + CT\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^\ell_r)},$$
$$\|1 - \rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^d))} \leq \Theta + CT\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^\ell_r)},$$

where Θ has been introduced in the statement of Theorem 5.1.5, and where $C = C_\ell > 0$ only depends on ℓ. Furthermore, if $\Theta + CT\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^\ell_r)} < 1$, then for all $t \in [0,T]$ and $x \in \mathbb{T}^d$

$$\frac{1}{1 - \rho_f(t,x)} \lesssim \frac{1}{1 - \Theta - CT\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^\ell_r)}},$$
$$\frac{1}{1 - \rho_f(t,x)} \gtrsim \frac{1}{\Theta + CT\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^\ell_r)}}.$$

Proof. Integrating the Vlasov equation with respect to velocity, one gets the conservation law $\partial_t \rho_f + \mathrm{div}_x\, j_f = 0$. We thus have

$$\rho_f(t) = \rho_{f^{\mathrm{in}}} + \int_0^t \mathrm{div}_x(j_f)(s)\,\mathrm{d}s,$$

therefore by using Sobolev embedding, we get for all $t \in [0,T]$

$$\|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \leq \|\rho_{f^{\mathrm{in}}}\|_{\mathrm{L}^\infty(\mathbb{T}^d)} + \int_0^t \|\mathrm{div}_x(j_f)(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s \leq \Theta + CT\|j_f\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^\ell(\mathbb{T}^d))},$$

if $\ell > 1 + d/2$, for some $C = C_\ell > 0$. We obtain the conclusion by using Lemma 5.2.1. The last estimates stated in the lemma directly follow. □

Lemma 5.2.17. *Assume that* $0 < \underline{\theta} \leq (1-\rho_{f^{\text{in}}})\varrho^{\text{in}} \leq \overline{\theta}$. *Let* $T > 0$. *For* $\ell > 1 + d/2$, *any smooth solution* $(f(t,x,v), \varrho(t,x), u(t,x))$ *to* (S_ε) *satisfies for all* $t \in [0,T]$

$$\underline{\theta} \exp\left(-T \left\|u\right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^\ell)}\right) \leq (1-\rho_f(t))\varrho(t) \leq \overline{\theta} \exp\left(T \left\|u\right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^\ell)}\right),$$

where $\underline{\theta}, \overline{\theta}$ *have been introduced in the statement of Theorem 5.1.5.*

Proof. The proof is a straighforward application of the method of characteristics applied to $(1-\rho_f)\varrho$, as the solution to the continuity equation

$$\partial_t((1-\rho_f)\varrho) + \mathrm{div}_x\left((1-\rho_f)\varrho u\right) = 0.$$

We obtain the conclusion in view of the assumption on $(1-\rho_{f^{\text{in}}})\varrho^{\text{in}}$. □

The following proposition then shows that, thanks to the regularization, we can actually build for all $\varepsilon > 0$ a local solution to the regularized system (S_ε).

Proposition 5.2.18. *There exist* $r_0 > 0$ *and* $m_0 > 0$ *such that the following holds. For all* $\varepsilon > 0$, *the system* (S_ε) *is locally well-posed in Sobolev spaces, that is if* $r > r_0$, $m > m_0$ *and if*

$$(f^{\text{in}}, \varrho^{\text{in}}, u^{\text{in}}) \in \mathcal{H}^m_r \times \mathrm{H}^m \times \mathrm{H}^m,$$

satisfies

$$0 \leq f^{\text{in}}, \quad \rho_{f^{\text{in}}} < \Theta < 1, \quad 0 < \mu \leq \varrho^{\text{in}}, \quad 0 < \underline{\theta} \leq (1-\rho_{f^{\text{in}}})\varrho^{\text{in}} \leq \overline{\theta},$$

for some fixed constants $\Theta, \mu, \underline{\theta}, \overline{\theta}$, *then there exist* $T = T(\varepsilon) > 0$ *and a unique solution* $(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon)$ *to the regularized system* (S_ε) *on* $(0, T(\varepsilon)]$ *such that*

$$(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon) \in \mathscr{C}(0,T;\mathcal{H}^m_r) \times \mathscr{C}(0,T;\mathrm{H}^m) \times \mathscr{C}(0,T;\mathrm{H}^m) \cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1}),$$

and starting at $(f^{\text{in}}, \varrho^{\text{in}}, u^{\text{in}})$. *Furthermore, the condition* ($\mathrm{B}^{\mu,\theta}_\Theta(T)$) *is satisfied by* $(f_\varepsilon, \varrho_\varepsilon)$ *with* $T = T(\varepsilon)$.

Proof. The proof is mainly based on the a priori estimates we have just derived, through a classical approximation procedure. Because of the regularization on the gradient of ϱ_ε in the Vlasov equation, the procedure is fairly standard. For the reader's convenience, we write the proof in Section 5.B of the Appendix. □

Here ever after and until the end of this chapter, we consider exponents $r > 0$ **and** $m > 0$ **which can be taken large enough. They will be chosen later on, in the end of the proof.**

We now introduce the following quantity, in view of the expected result of Theorem 5.1.5.

Definition 5.2.19. *For any functions* $f(t,x,v)$, $\varrho(t,x)$ *and* $u(t,x)$, *we set*

$$\mathcal{N}_{m,r}(f,\varrho,u,T) := \|f\|_{\mathrm{L}^\infty\left(0,T;\mathcal{H}^{m-1}_r\right)} + \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} + \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap\mathrm{L}^2(0,T;\mathrm{H}^{m+1})},$$

where $T > 0$.

The proof of Theorem 5.1.5 will rely on a bootstrap argument. Let $\varepsilon > 0$. From Proposition 5.2.18, consider the maximal time of existence T_ε^* to the system (S_ε). By definition, Proposition 5.2.18 ensures that

$$\forall T < T_\varepsilon^*, \quad \mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) < +\infty, \quad \text{and} \quad (\mathrm{B}_\Theta^{\mu,\theta}(T)) \quad \text{holds.}$$

So we can consider the time

$$T_\varepsilon = T_\varepsilon(R) := \sup\{T \in [0, T_\varepsilon^*[, \quad \mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) \leq R \quad \text{and} \quad (\mathrm{B}_\Theta^{\mu,\theta}(T)) \quad \text{holds}\},$$

where $R > 0$ will be chosen large enough and independent of ε. By continuity, we observe that $T_\varepsilon > 0$ if R is taken large enough and independent of ε. In particular, for all $t \in [0, T_\varepsilon]$, we have

$$0 < \frac{1-\Theta}{2} \leq 1 - \rho_f(t), \quad \frac{1}{1-\rho_f(t)} \leq \frac{2}{1-\Theta}. \tag{5.2.5}$$

Our main goal is to prove that R can be chosen large enough so that there exists $T(R) > 0$ independent of ε such that

$$\forall \varepsilon \in (0,1), \quad T(R) \leq T_\varepsilon(R).$$

Such a lower bound independent of ε will pave the way for a compactness argument when $\varepsilon \to 0$, leading to the existence of a solution for (TS) on $[0, T(R)]$. In what follows, we will work on the interval of time $[0, T_\varepsilon]$.

We have the following trichotomy:

- either $T_\varepsilon^* = +\infty$ and $T_\varepsilon = T_\varepsilon^*$, then there is nothing to do because $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) \leq R$ for all times $T > 0$;
- either $T_\varepsilon^* < +\infty$ and $T_\varepsilon = T_\varepsilon^*$, we shall see soon enough that this is impossible as this leads to a contradiction;
- else, $T_\varepsilon < T_\varepsilon^*$ and $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T_\varepsilon) = R$.

Let us show how to exclude the second case. We need the following lemma.

Lemma 5.2.20. *Let $\varepsilon > 0$. If $m > 3 + d/2$ and $r \geq 1 + d/2$, we have for all $T \in [0, T_\varepsilon)$*

$$\|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \lesssim \|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \leq \Lambda\left(T, R, \|\varrho^{\mathrm{in}}\|_{\mathrm{H}^{m-2}}\right).$$

Proof. From Proposition 5.2.3, we know that for all $T \in [0, T_\varepsilon)$ and $t \in [0, T]$

$$\|\varrho_\varepsilon(t)\|_{\mathrm{H}^{m-2}} \leq \|\varrho^{\mathrm{in}}\|_{\mathrm{H}^{m-2}} e^{C_{m-2}(T,u_\varepsilon,f_\varepsilon)T} \exp\left[T e^{C_{m-2}(T,u_\varepsilon,f_\varepsilon)T} \mathcal{Q}_{m-2}(T, u_\varepsilon, f_\varepsilon)\right],$$

provided that $m-2, r > 1 + d/2$, and where

$$C_{m-2}(T, u_\varepsilon, f_\varepsilon) \leq R + 2R^2 \left\|\frac{1}{1-\rho_{f_\varepsilon}}\right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)}, \ \mathcal{Q}_{m-2}(T, u_\varepsilon, f_\varepsilon) \leq R + 2R^2 \left\|\frac{1}{1-\rho_{f_\varepsilon}}\right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}.$$

because $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) \leq R$ for all $T \in [0, T_\varepsilon)$. By Sobolev embedding and the bound (5.2.5), this means that for $r > 1 + d/2$ and $m > 1 + d/2 + 2$, we have for all $T \in [0, T_\varepsilon)$

$$\|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \lesssim \|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \leq \Lambda\left(T, R, \|\varrho^{\mathrm{in}}\|_{\mathrm{H}^{m-2}}, \left\|\frac{1}{1-\rho_{f_\varepsilon}}\right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\right).$$

To conclude, we only have to understand the last term in the previous function Λ: by Lemma 5.A.5, there exists a continuous nonnegative nondecreasing function C_m such that

$$\left\| \frac{1}{1-\rho_{f_\varepsilon}} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \leq 1 + \mathrm{C}_m \left(\|\rho_{f_\varepsilon}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \right) \|\rho_{f_\varepsilon}\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \lesssim \Lambda(R),$$

thanks to Lemma 5.2.1 and the fact that $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) \leq R$. This concludes the proof. □

Remark 5.2.21. A careful inspection of the proof of Proposition 5.2.3 reveals that for all $k \leq m-2$ with $k > 3 + d/2$, $r > 1 + d/2$ and $T \in [0, T_\varepsilon)$

$$\|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)} \lesssim \Lambda(\|\varrho^{\mathrm{in}}\|_{\mathrm{H}^k}) + T\Lambda(T, R).$$

As a corollary, we can now exclude the second case written above: if $T_\varepsilon^* < +\infty$ and $T_\varepsilon = T_\varepsilon^*$, then according to Proposition 5.2.14 and Lemma 5.2.20,

$$\sup_{t \in [0, T_\varepsilon^*)} \|f_\varepsilon(t)\|^2_{\mathcal{H}^m_r} \leq \|f^{\mathrm{in}}\|^2_{\mathcal{H}^m_r} \exp\left[C \left((1+R)T_\varepsilon + \frac{\sqrt{T_\varepsilon}}{\varepsilon} \Lambda \left(T, R, \|\varrho^{\mathrm{in}}\|_{\mathrm{H}^{m-2}} \right) \right) \right] < +\infty.$$

The previous inequality means that the solution could be continued beyond T_ε^* which is impossible by maximality of T_ε^*. This case is thus impossible.

From now on, we assume that $T_\varepsilon < T_\varepsilon^*$ and $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T_\varepsilon) = R$. In view of our bootstrap stategy, we need to estimate $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T)$ for all $T < T_\varepsilon$.

In the end of this section, we will show that the first and third terms appearing in the quantity $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T_\varepsilon)$, that is $\|f_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)}$ and $\|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}$, can be handled by energy estimates. The main part of the upcoming analysis will be to provide a uniform control in ε for the term $\|\varrho\|_{\mathrm{L}^2(0,T;\mathcal{H}^m_r)}$.

In the following lemma, we give an estimate independent of ε for the term $\|f_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)}$ in $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T)$, for all $T < T_\varepsilon$.

Lemma 5.2.22. *For $m > 1 + d/2 + 2$ and $r > 1 + d/2$, the solution $(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon)$ to (S_ε) satisfies for all $T \in [0, T_\varepsilon)$*

$$\|f_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)} \leq \|f^{\mathrm{in}}\|_{\mathcal{H}^{m-1}_r} + T^{\frac{1}{4}}\Lambda(T, R).$$

Proof. Following the same steps leading to (5.2.3) in the proof of Proposition 5.2.10, we have for all $t \in [0, T_\varepsilon)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f_\varepsilon(t)\|^2_{\mathcal{H}^{m-1}_r} \lesssim (1 + \|E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}(t)\|_{\mathrm{H}^{m-1}})\|f_\varepsilon(t)\|^2_{\mathcal{H}^{m-1}_r},$$

since $m - 1 > d/2$, therefore

$$\|f_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)} \leq \|f^{\mathrm{in}}\|^2_{\mathcal{H}^{m-1}_r} + \|f_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)} \left(T + \int_0^T \|E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}(s)\|_{\mathrm{H}^{m-1}}\,\mathrm{d}s \right).$$

Using now the estimate (5.2.2) from Lemma 5.2.7, we get

$$\int_0^T \|\mathrm{E}^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}(s)\|_{\mathrm{H}^{m-1}}\,\mathrm{d}s \lesssim \sqrt{T}\|u_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})} + \sqrt{T}\Lambda \left(\|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \right) \|\varrho_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}.$$

We thus infer that for all $T \in [0, T_\varepsilon)$

$$\|f_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)} \leq \|f^{\mathrm{in}}\|^2_{\mathcal{H}^{m-1}_r} + C\|f_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)}\Bigg(T + \sqrt{T}\|u_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})} + \sqrt{T}\Lambda\left(\|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\right)\|\varrho_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}\Bigg),$$

where $C > 0$ is independent of ε. Since $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) \leq R$ for all $T \in [0, T_\varepsilon)$, this gives

$$\|f_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)} \leq \|f^{\mathrm{in}}\|^2_{\mathcal{H}^{m-1}_r} + CR^2\Bigg(T + \sqrt{T}R + \sqrt{T}\Lambda\left(\|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\right)R\Bigg).$$

To obtain a uniform bound in ε for the term $\Lambda\left(\|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\right)$, we use Lemma 5.2.20, leading to the conclusion of the lemma. □

We conclude this section with an estimate for the term $\|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1})}$ appearing in $\mathcal{N}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T)$. We will crucially rely on the smoothing provided by the differential operator $\Delta_x + \nabla_x \mathrm{div}_x$ from the Navier-Stokes equation on u_ε. Indeed, we have the following lemma.

Lemma 5.2.23. *The differential operator $-\Delta_x - \nabla_x \mathrm{div}_x$ is elliptic.*

Proof. The operator $-\Delta_x - \nabla_x \mathrm{div}_x$ is associated to the matrix Fourier multiplier $L(k) = |k|^2 \mathrm{I}_d + k \otimes k$ ($k \in \mathbb{Z}^d$). One can then prove (see e.g. [DT22]) that

$$2^{-3\frac{d}{2}}|k|^{2d} \leq |\det L(k)|,$$

which yields the desired ellipticity. □

We are now able to prove the following proposition.

Proposition 5.2.24. *For $m > 2 + d/2$ and $r \geq 0$, for all $\varepsilon > 0$, the solution $(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon)$ to (S_ε) satisfies for all $T \in [0, T_\varepsilon)$*

$$\|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} + \|u_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})} \lesssim \left(1 + T^{1/2}\Lambda(T,R)\right)\left(\|u^{\mathrm{in}}\|_{\mathrm{H}^m} + T^{1/2}\Lambda(T,R) + \Lambda(T,R)\|\varrho_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}\right).$$

Proof. First, we rewrite the equation on u_ε as

$$\partial_t u_\varepsilon - \sigma((1-\rho_{f_\varepsilon})\varrho_\varepsilon)\Big(\Delta_x + \nabla_x \mathrm{div}_x\Big)u_\varepsilon = F,$$

with

$$F := -(u_\varepsilon \cdot \nabla_x)u_\varepsilon - \nabla_x \pi(\varrho_\varepsilon) + \sigma((1-\rho_{f_\varepsilon})\varrho_\varepsilon)(j_{f_\varepsilon} - \rho_{f_\varepsilon} u_\varepsilon),$$

where $\sigma(s) := \frac{1}{s}$, $\pi(s) = \int_0^s \frac{p'(\tau)}{\tau}\,\mathrm{d}\tau$. We then apply ∂_x^β in the equation for $|\beta| \leq m-1$, which gives

$$\partial_t(\partial_x^\beta u_\varepsilon) - \sigma((1-\rho_{f_\varepsilon})\varrho_\varepsilon)\Big(\Delta_x + \nabla_x \mathrm{div}_x\Big)(\partial_x^\beta u_\varepsilon) = \partial_x^\beta F + \Big[\partial_x^\beta, \sigma((1-\rho_{f_\varepsilon})\varrho_\varepsilon)\Big]((\Delta_x + \nabla_x \mathrm{div}_x)u_\varepsilon),$$

and then multiply the equation with $-\Big(\Delta_x + \nabla_x \mathrm{div}_x\Big)(\partial_x^\beta u)$ so that, by integrating on $\mathbb{T}^d$ and with an integration by parts:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}\Big(|\nabla_x \partial_x^\beta u_\varepsilon|^2 + |\mathrm{div}_x \partial_x^\beta u_\varepsilon|^2\Big)\,\mathrm{d}x + \int_{\mathbb{T}^d}\sigma((1-\rho_{f_\varepsilon})\varrho_\varepsilon)|\Big(\Delta_x + \nabla_x \mathrm{div}_x\Big)(\partial_x^\beta u_\varepsilon)|^2\,\mathrm{d}x \\
&= -\int_{\mathbb{T}^d}\Big(\Delta_x + \nabla_x \mathrm{div}_x\Big)(\partial_x^\beta u_\varepsilon)\cdot \partial_x^\beta F\,\mathrm{d}x \\
&\quad -\int_{\mathbb{T}^d}\Big[\partial_x^\beta, \sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\Big]((\Delta_x + \nabla_x \mathrm{div}_x)u_\varepsilon)\cdot(\Delta_x + \nabla_x \mathrm{div}_x)(\partial_x^\beta u_\varepsilon)\,\mathrm{d}x.
\end{aligned}$$

Thanks to the Cauchy-Schwarz and Young inequalities, and after integration in time, we get for all $\eta > 0$ and $t \in (0,T)$

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{T}^d}|\nabla_x \partial_x^\beta u_\varepsilon(t)|^2\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^d}\left(\sigma((1-\rho_{f_\varepsilon})\varrho_\varepsilon) - \eta\right)|(\Delta_x + \nabla_x \mathrm{div}_x)\partial_x^\beta u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\le \frac{1}{2}\int_{\mathbb{T}^d}\Big(|\nabla_x \partial_x^\beta u^{\mathrm{in}}|^2 + |\mathrm{div}_x \partial_x^\beta u^{\mathrm{in}}|^2\Big)\,\mathrm{d}x \\
&\qquad + \frac{1}{2\eta}\int_0^t\int_{\mathbb{T}^d}|\partial_x^\beta F|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2\eta}\int_0^t\int_{\mathbb{T}^d}\Big|\Big[\partial_x^\beta, \sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\Big]((\Delta_x + \nabla_x \mathrm{div}_x)u_\varepsilon)\Big|^2\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Let us deal with the last term: by the commutator inequality from Proposition 5.A.1, we have

$$\begin{aligned}
&\int_{\mathbb{T}^d}\Big|\Big[\partial_x^\beta, \sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\Big]((\Delta_x + \nabla_x \mathrm{div}_x)u_\varepsilon)\Big|^2\,\mathrm{d}x \\
&\le M\Big(\|\nabla_x \sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\|_{\mathrm{L}^\infty}^2\,\|(\Delta_x + \nabla_x \mathrm{div}_x)u_\varepsilon\|_{\mathrm{H}^{m-2}}^2 \\
&\qquad\qquad + \|\sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\|_{\mathrm{H}^{m-1}}^2\,\|(\Delta_x + \nabla_x \mathrm{div}_x)u_\varepsilon\|_{\mathrm{L}^\infty}^2\Big) \\
&\le M\Big(\|\sigma'\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\nabla_x\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\|_{\mathrm{L}^\infty}^2\,\|u\|_{\mathrm{H}^m}^2 \\
&\qquad\qquad + \|\sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\|_{\mathrm{H}^{m-1}}^2\,\|(\Delta_x + \nabla_x \mathrm{div}_x)u_\varepsilon\|_{\mathrm{L}^\infty}^2\Big),
\end{aligned}$$

for some constant $M > 0$ independent of time. Combining the Sobolev embedding (with $m > 2 + d/2$) and Remark 5.A.4, we get

$$\begin{aligned}
&\frac{1}{2\eta}\int_0^t\int_{\mathbb{T}^d}\Big|\Big[\partial_x^\beta, \sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\Big]((\Delta_x + \nabla_x \mathrm{div}_x)u_\varepsilon)\Big|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad\qquad \le M\frac{1}{2\eta}\Lambda\Big(\|(1-\rho_{f_\varepsilon})\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})}\Big)\int_0^t\|u_\varepsilon(\tau)\|_{\mathrm{H}^m}^2\,\mathrm{d}\tau,
\end{aligned}$$

for another constant $M > 0$. Note that by Remark 5.2.5, we have

$$\|(1-\rho_{f_\varepsilon})\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})} \le \Lambda(T, R, \|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)}) \le \Lambda(T, R).$$

All in all, we get for all $\eta > 0$ and $t \in (0,T)$

$$\begin{aligned}
&\frac{1}{2}\Big\|\nabla_x \partial_x^\beta u_\varepsilon(t)\Big\|_{\mathrm{L}^2}^2 + \int_0^t\int_{\mathbb{T}^d}\left(\sigma((1-\rho_{f_\varepsilon})\varrho_\varepsilon) - \eta\right)|(\Delta_x + \nabla_x \mathrm{div}_x)\partial_x^\beta u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad\qquad \le \|u^{\mathrm{in}}\|_{\mathrm{H}^m}^2 + \frac{1}{\eta}\|F\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})}^2 + M\frac{\Lambda(T,R)}{2\eta}\int_0^t\|u_\varepsilon(\tau)\|_{\mathrm{H}^m}^2\,\mathrm{d}\tau.
\end{aligned}$$

Thanks to the condition ($\mathrm{B}^{\mu,\theta}_{\Theta}(T)$), we can choose $\eta = 1/4\overline{\theta}$ so that

$$\forall (t,x) \in [0,T] \times \mathbb{T}^d, \quad \frac{1}{4\overline{\theta}} < \sigma((1-\rho_{f_\varepsilon})\varrho_\varepsilon)(t,x) - \eta.$$

Summing for all $|\beta| = m$ and invoking the elliptic regularity for the operator $-\Delta_x - \nabla_x \mathrm{div}_x$ given by Lemma 5.2.23, we get for all $t \in (0,T)$

$$\begin{aligned}\|u_\varepsilon(t)\|^2_{\mathrm{H}^m} &\leq \|u_\varepsilon(t)\|^2_{\mathrm{H}^m} + \|u_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})} \\ &\lesssim \|u^{\mathrm{in}}\|^2_{\mathrm{H}^m} + \|F\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})} + \Lambda(T,R)\int_0^t \|u_\varepsilon(\tau)\|^2_{\mathrm{H}^m}\,\mathrm{d}\tau.\end{aligned}$$

By Grönwall's lemma, we deduce for all $t \in (0,T)$

$$\|u_\varepsilon(t)\|^2_{\mathrm{H}^m} \leq \Lambda(T,R)\left(\|u^{\mathrm{in}}\|^2_{\mathrm{H}^m} + \|F\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})}\right),$$

which then implies, by using again the previous inequality, that for all $t \in (0,T)$

$$\|u_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} + \|u_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})} \lesssim (1+T\Lambda(T,R))\left(\|u^{\mathrm{in}}\|^2_{\mathrm{H}^m} + \|F\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})}\right).$$

To conclude, let us now estimate the norm of the source term F. We have

$$\begin{aligned}\|F\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})} &\leq \int_0^T \Bigg(\|(u_\varepsilon\cdot\nabla_x)u_\varepsilon(\tau)\|^2_{\mathrm{H}^{m-1}} + \|\nabla_x\pi(\varrho_\varepsilon(\tau))\|^2_{\mathrm{H}^{m-1}} \\ &\qquad\qquad + \left\|\frac{1}{(1-\rho_{f_\varepsilon})\varrho_\varepsilon}(j_{f_\varepsilon} - \rho_{f_\varepsilon}u_\varepsilon)(\tau)\right\|^2_{\mathrm{H}^{m-1}}\Bigg)\,\mathrm{d}\tau \\ &\leq \int_0^T \left(\|u_\varepsilon(\tau)\|^2_{\mathrm{H}^{m-1}}\|u_\varepsilon(\tau)\|^2_{\mathrm{H}^m} + \|\pi(\varrho_\varepsilon(\tau))\|^2_{\mathrm{H}^m}\right)\mathrm{d}\tau \\ &\qquad\qquad + T\left\|\frac{1}{(1-\rho_{f_\varepsilon})\varrho_\varepsilon}(j_{f_\varepsilon} - \rho_{f_\varepsilon}u_\varepsilon)\right\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})}.\end{aligned}$$

In the rest of the proof, we shall make a constant use of the condition ($\mathrm{B}^{\mu,\theta}_{\Theta}(T)$). By Proposition 5.A.3 in the Appendix, we have

$$\|\pi(\varrho_\varepsilon)\|_{\mathrm{H}^m} \lesssim \Lambda(\|\varrho_\varepsilon\|_{\mathrm{L}^\infty})\|\varrho_\varepsilon\|_{\mathrm{H}^m},$$

from which we infer thanks to Sobolev embedding (taking $m > 2 + \frac{d}{2}$)

$$\begin{aligned}&\|F\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})} \\ &\lesssim T\,\|u_\varepsilon\|^4_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} + \Lambda(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)})\|\varrho_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^m)} + T\left\|\frac{1}{(1-\rho_{f_\varepsilon})\varrho_\varepsilon}(j_{f_\varepsilon} - \rho_{f_\varepsilon}u_\varepsilon)\right\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})} \\ &\lesssim TR^4 + \Lambda(\|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})})\|\varrho_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^m)} + T\left\|\frac{1}{(1-\rho_{f_\varepsilon})\varrho_\varepsilon}(j_{f_\varepsilon} - \rho_{f_\varepsilon}u_\varepsilon)\right\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})},\end{aligned}$$

since $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) \leq R$ for all $T \in [0, T_\varepsilon)$. The first term is then adressed thanks to Lemma 5.2.20 and it remains to estimate the last term. For this one, we have

$$\begin{aligned}\left\|\frac{1}{(1-\rho_{f_\varepsilon})\varrho_\varepsilon}(j_{f_\varepsilon} - \rho_{f_\varepsilon}u_\varepsilon)\right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})} &\leq \left\|\frac{1}{(1-\rho_{f_\varepsilon})\varrho_\varepsilon}\right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})}\|(j_{f_\varepsilon} - \rho_{f_\varepsilon}u_\varepsilon)\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})} \\ &\leq \Lambda(T,R)\left\|\frac{1}{(1-\rho_{f_\varepsilon})\varrho_\varepsilon}\right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})},\end{aligned}$$

thanks to Lemma 5.2.1 and $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) \leq R$ for all $T \in [0, T_\varepsilon)$. By Remark 5.A.4, we then have by Sobolev embedding

$$\begin{aligned}\left\| \frac{1}{(1-\rho_{f_\varepsilon})\varrho_\varepsilon}(j_f - \rho_f u) \right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})} &\leq \Lambda(T, R, \|(1-\rho_{f_\varepsilon})\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)})\|(1-\rho_{f_\varepsilon})\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})} \\ &\leq \Lambda(T, R, \|(1-\rho_{f_\varepsilon})\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})}) \\ &\leq \Lambda(T, R, \|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)}) \\ &\leq \Lambda(T, R),\end{aligned}$$

where we have also used Remark 5.2.5. This eventually concludes the proof. □

Remark 5.2.25. By looking at the previous proof, we have for $T \in [0, T_\varepsilon)$

$$\begin{aligned}\|u_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)} + \|u\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{k+1})} &\leq \|u^{\mathrm{in}}\|^2_{\mathrm{H}^k} + T\Lambda(T,R) + \Lambda(T,R)\|\varrho_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^k)} \\ &\leq \|u^{\mathrm{in}}\|^2_{\mathrm{H}^k} + T\Lambda(T,R) + T\Lambda(T,R)\|\varrho_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)},\end{aligned}$$

for all $k > 2 + d/2$ such that $k \leq m-2$, therefore by Remark 5.2.21, we obtain

$$\|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)} + \|u_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^{k+1})} \lesssim \left(1 + T^{1/2}\Lambda(T,R)\right)\left(\|u^{\mathrm{in}}\|_{\mathrm{H}^k} + \|\varrho^{\mathrm{in}}\|_{\mathrm{H}^k}\right) + T^{1/2}\Lambda(T,R).$$

So far, Lemma 5.2.22 and Proposition 5.2.24 show that it remains to control the second term in $\mathcal{N}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T)$, that is $\|\varrho_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}$, to perform a bootstrap argument. This will constitute the heart of our analysis and will be the purpose of the remaining sections.

5.3 Trajectories and straightening change of variable

In this short section, we study the trajectories associated to a Vlasov equation with friction and force field $\mathrm{F}(t,x)$. We show that for small times, their geometry can be simplified thanks to a straightening change of variable in velocity. Loosely speaking, this allows to boil down the dynamics to that associated with free-transport with friction. This procedure will be useful in Section 5.5.

Let $T > 0$. Given $\mathrm{F}(s,x) \in \mathbb{R}^d$ a given vector field defined on $[0,T] \times \mathbb{T}^d$ and satisfying

$$\mathrm{F} \in \mathrm{L}^2(0,T;\mathrm{W}^{1,\infty}(\mathbb{T}^d)),$$

we can consider, thanks to the Cauchy-Lipschitz theorem, the solution $s \mapsto (\mathrm{X}^{s;t}(x,v), \mathrm{V}^{s;t}(x,v)) \in \mathbb{T}^d \times \mathbb{R}^d$ of the following system of ODE:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{X}^{s;t}(x,v) = \mathrm{V}^{s;t}(x,v), \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}^{s;t}(x,v) = -\mathrm{V}^{s;t}(x,v) + \mathrm{F}(s, \mathrm{X}^{s;t}(x,v)), \\ \mathrm{X}^{t;t}(x,v) = x, \\ \mathrm{V}^{t;t}(x,v) = v. \end{cases}$$

Later on, we will apply this to $\mathrm{F} = E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}$, which has been defined in the beginning of Section 5.2.2 (see the end of the current section). Integrating the previous system of ODE, we have

$$\mathrm{X}^{s;t}(x,v) = x + (1-e^{t-s})v + \int_t^s (1-e^{\tau-s})\mathrm{F}\left(\tau, \mathrm{X}^{\tau;t}(x,v)\right)\,\mathrm{d}\tau, \tag{5.3.1}$$

$$\mathrm{V}^{s;t}(x,v) = e^{t-s}v + \int_t^s e^{\tau-s}\mathrm{F}\left(\tau, \mathrm{X}^{\tau;t}(x,v)\right)\,\mathrm{d}\tau. \tag{5.3.2}$$

Considering the full kinetic transport operator

$$\mathcal{T}_{\mathrm{F}} = \partial_t + v \cdot \nabla_x - v \cdot \nabla_v + \mathrm{F}(t,x) \cdot \nabla_v - d\mathrm{Id},$$

the method of characteristics shows that a smooth function $f(t,x,v)$ satisfying

$$\begin{cases} \mathcal{T}_{\mathrm{F}} f = 0, \\ f_{|t=0} = f^{\mathrm{in}}, \end{cases}$$

can be represented as

$$f(t,x,v) = e^{dt} f^{\mathrm{in}} \left(\mathrm{X}^{0;t}(x,v), \mathrm{V}^{0;t}(x,v) \right).$$

Note also that for all $t, s \in [0,T]$, the map

$$(x,v) \mapsto \left(\mathrm{X}^{s;t}(x,v), \mathrm{V}^{s;t}(x,v) \right)$$

is a diffeomorphism from $\mathbb{T}^d \times \mathbb{R}^d$ to itself, which Jacobian value is $e^{d(s-t)}$.

The main goal of this section is to prove that for short times, and modulo a *straightening change of variable* in velocity, it is possible to come down to the free dynamics with friction associated to the transport operator

$$\mathcal{T}^{\mathrm{fric}} = \partial_t + v \cdot \nabla_x - v \cdot \nabla_v - d\mathrm{Id}.$$

This corresponds to the previous system of ODE with $\mathrm{F} = 0$, and for which the solution $(\mathrm{X}^{s;t}_{\mathrm{fric}}, \mathrm{V}^{s;t}_{\mathrm{fric}})$ is

$$\mathrm{X}^{s;t}_{\mathrm{fric}} = x + (1 - e^{t-s})v, \quad \mathrm{V}^{s;t}_{\mathrm{fric}} = e^{t-s} v.$$

Namely, we have the following lemma.

Lemma 5.3.1. *Let $T > 0$ and $k \geq 1$. Let $\mathrm{F} \in \mathrm{L}^2(0,T; \mathrm{W}^{k,\infty}(\mathbb{T}^d))$ be a vector field such that*

$$\|\mathrm{F}\|_{\mathrm{L}^2(0,T;\mathrm{W}^{k,\infty}(\mathbb{T}^d))} \leq \Lambda(T, \mathrm{R}),$$

for some $\mathrm{R} > 0$. There exists $\overline{T}(\mathrm{R}) > 0$ such that for all $x \in \mathbb{T}^d$ and $s,t \in [0, \min(\overline{T}(\mathrm{R}), T)]$, there exists a diffeomorphism $\psi_{s,t}(x,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ satisfying for all $v \in \mathbb{R}^d$

$$\mathrm{X}^{s;t}\left(x, \psi_{s,t}(x,v)\right) = x + (1 - e^{t-s})v,$$

which furthermore verifies the estimates

$$\frac{1}{C} \leq \det\left(\mathrm{D}_v \psi_{s,t}(x,v)\right) \leq C, \tag{5.3.3}$$

$$\sup_{s,t \in [0,T]} \left\| \partial^{\beta}_{x,v} \left(\psi_{s,t}(x,v) - v \right) \right\|_{\mathrm{L}^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \leq \varphi(T) \Lambda(T, \mathrm{R}), \quad |\beta| \leq k, \tag{5.3.4}$$

$$\sup_{s,t \in [0,T]} \left\| \partial^{\beta}_{x,v} \partial_s \psi_{s,t}(x,v) \right\|_{\mathrm{L}^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \leq \varphi(T) \Lambda(T, \mathrm{R}), \quad |\beta| \leq k-1, \tag{5.3.5}$$

for some $C > 0$ and some nondecreasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ vanishing at 0.

Proof. We follow the approach of [HKNR21]. We observe that if we set

$$\widetilde{\mathrm{X}}^{s;t}(x,v) := \frac{1}{1-e^{t-s}}\left[\mathrm{X}^{s;t}(x,v) - x - (1-e^{t-s})v\right],$$

it comes down to prove that for s,t small enough, the mapping $\phi^{s,t,x} : v \mapsto v + \widetilde{\mathrm{X}}^{s;t}(x,v)$ is a small Lipschitz perturbation of the identity: denoting its inverse $\psi_{s,t}(x,\cdot)$, it will satisfy

$$v = \psi_{s,t}(x,v) + \widetilde{\mathrm{X}}^{s,t}(x,\psi_{s,t}(x,v)), \tag{5.3.6}$$

and the first conclusion of the lemma will follow. We introduce the remainder

$$\mathrm{Y}^{s;t}(x,v) = \mathrm{X}^{s;t}(x,v) - x - (1-e^{t-s})v,$$

which, in view of (5.3.1), satisfies for all $s,t \in [0,T]$

$$\begin{aligned}\mathrm{Y}^{s;t}(x,v) &= \int_s^t (e^{\tau-s}-1)\mathrm{F}\left(\tau, \mathrm{X}^{\tau;t}(x,v)\right)\,\mathrm{d}\tau \\ &= \int_s^t (e^{\tau-s}-1)\mathrm{F}\left(\tau, x + (1-e^{t-\tau})v + \mathrm{Y}^{\tau;t}(x,v)\right)\,\mathrm{d}\tau.\end{aligned} \tag{5.3.7}$$

We now have

$$\widetilde{\mathrm{X}}^{s;t}(x,v) = \frac{1}{e^{t-s}-1}\int_s^t (e^{\tau-s}-1)\mathrm{F}\left(\tau, x + (1-e^{t-\tau})v + \mathrm{Y}^{\tau;t}(x,v)\right)\,\mathrm{d}\tau. \tag{5.3.8}$$

Thus, estimates on Y and its derivatives obtained thanks to (5.3.7) shall provide estimates on $\widetilde{\mathrm{X}}$ and its derivatives via (5.3.8).

Let us assume that $s \le t$ (the case $s \ge t$ can be treated similarly). First, we have

$$\begin{aligned}\|\nabla_x \mathrm{Y}^{s;t}\|_{\mathrm{L}^\infty_{x,v}} &\le \int_s^t (e^{\tau-s}-1)\|\nabla_x \mathrm{F}\left(\tau, x + (1-e^{t-\tau})v + \mathrm{Y}^{\tau;t}(x,v)\right)\|_{\mathrm{L}^\infty_{x,v}}(1 + \|\nabla_x \mathrm{Y}^{\tau;t}\|_{\mathrm{L}^\infty_{x,v}})\,\mathrm{d}\tau \\ &\le \int_s^t (e^{\tau-s}-1)\|\nabla_x \mathrm{F}(\tau)\|_{\mathrm{L}^\infty_x}(1 + \|\nabla_x \mathrm{Y}^{\tau;t}\|_{\mathrm{L}^\infty_{x,v}})\,\mathrm{d}\tau \\ &\le (e^T-1)T^{1/2}\|\nabla_x \mathrm{F}\|_{\mathrm{L}^2(0,T;\mathrm{L}^\infty)}\left(1 + \sup_{\tau\le t}\|\nabla_x \mathrm{Y}^{\tau;t}\|_{\mathrm{L}^\infty_{x,v}}\right).\end{aligned}$$

By the assumption on the vector field F, we get

$$\|\nabla_x \mathrm{Y}^{s;t}\|_{\mathrm{L}^\infty_{x,v}} \le (e^T-1)T^{1/2}\Lambda(T,\mathrm{R})\left(1 + \sup_{\tau\le t}\|\nabla_x \mathrm{Y}^{\tau;t}\|_{\mathrm{L}^\infty_{x,v}}\right).$$

In the following of the proof, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ will stand for a generic continuous function, vanishing at 0, that may change from line to line. This yields

$$\|\nabla_x \mathrm{Y}^{s;t}\|_{\mathrm{L}^\infty_{x,v}} \le \sup_{\tau\le t}\|\nabla_x \mathrm{Y}^{\tau;t}\|_{\mathrm{L}^\infty_{x,v}} \le \frac{(e^T-1)T^{1/2}\Lambda(T,\mathrm{R})}{1-(e^T-1)T^{1/2}\Lambda(T,\mathrm{R})} \lesssim \varphi(T)\Lambda(T,\mathrm{R}), \tag{5.3.9}$$

for T small enough. In a similar way, we have

$$\begin{aligned}\|\nabla_v \mathrm{Y}^{s;t}\|_{\mathrm{L}^\infty_{x,v}} &\le \int_s^t (e^{\tau-s}-1)\|\nabla_x \mathrm{F}(\tau)\|_{\mathrm{L}^\infty_{x,v}}(1 - e^{t-\tau} + \|\nabla_v \mathrm{Y}^{\tau;t}\|_{\mathrm{L}^\infty_{x,v}})\,\mathrm{d}\tau \\ &\le (e^T-1)T^{1/2}\Lambda(T,R)\left(1 + \sup_{\tau\le t}\|\nabla_v \mathrm{Y}^{\tau;t}\|_{\mathrm{L}^\infty_{x,v}}\right),\end{aligned}$$

therefore

$$\|\nabla_v \mathrm{Y}^{s;t}\|_{\mathrm{L}^\infty_{x,v}} \lesssim \varphi(T)\Lambda(T,\mathrm{R}), \tag{5.3.10}$$

for T small enough. We then deduce the following estimates. There holds

$$\begin{aligned}\|\nabla_x \widetilde{\mathrm{X}}^{s;t}\|_{\mathrm{L}^\infty_{x,v}} &\leq \frac{1}{e^{t-s}-1}\int_s^t (e^{\tau-s}-1)\|\nabla_x \mathrm{F}(\tau)\|_{\mathrm{L}^\infty_{x,v}}(1+\|\nabla_x \mathrm{Y}^{\tau;t}\|_{\mathrm{L}^\infty_{x,v}})\,\mathrm{d}\tau \\ &\lesssim T^{1/2}\Lambda(T,R)(1+\varphi(T)\Lambda(T,\mathrm{R})),\end{aligned}$$

thanks to (5.3.9), as well as

$$\begin{aligned}\|\nabla_v \widetilde{\mathrm{X}}^{s;t}\|_{\mathrm{L}^\infty_{x,v}} &\leq \frac{1}{e^{t-s}-1}\int_s^t (e^{\tau-s}-1)\|\nabla_x \mathrm{F}(\tau)\|_{\mathrm{L}^\infty_{x,v}}(1-e^{\tau-t}+\|\nabla_v \mathrm{Y}^{\tau;t}\|_{\mathrm{L}^\infty_{x,v}})\,\mathrm{d}\tau \\ &\lesssim T^{1/2}\Lambda(T,\mathrm{R})(1+\varphi(T)\Lambda(T,\mathrm{R})),\end{aligned}$$

thanks to (5.3.10). We have proven that for T small enough, we have

$$\|\nabla_x \widetilde{\mathrm{X}}^{s;t}\|_{\mathrm{L}^\infty_{x,v}} + \|\nabla_v \widetilde{\mathrm{X}}^{s;t}\|_{\mathrm{L}^\infty_{x,v}} \leq \varphi(T)\Lambda(T,\mathrm{R}). \tag{5.3.11}$$

For T small enough, we therefore obtain the existence of the desired diffeomorphism $\psi_{s,t}(x,\cdot)$. We also have

$$0 < \left|\det\left(\nabla_v \psi^x_{s,t}(v)\right)\right| = \left|\det\left(\mathrm{Id} + \nabla_v \widetilde{\mathrm{X}}^{s,t}(x,\psi^x_{s,t}(v))\right)\right|^{-1}.$$

We thus infer the uniform bound (5.3.3) from the estimate (5.3.11) and the continuity of $M \mapsto |\det(M)|$, reducing $\overline{T}(\mathrm{R})$ if necessary.

Let us finally prove the estimates (5.3.4)–(5.3.5). For (5.3.4), we proceed by induction on the length of α. In view of (5.3.6), we obtain the result for $|\alpha| = 0$. For the case $|\alpha| = 1$, we differentiate the identity (5.3.6) and get, with $\nabla = \nabla_x$ or $\nabla = \nabla_v$

$$\nabla(v - \psi^x_{s,t}(v)) = \nabla\widetilde{\mathrm{X}}^{s;t}(x,\psi^x_{s,t}(v)) - \nabla\widetilde{\mathrm{X}}^{s;t}(x,\psi^x_{s,t}(v))\nabla(v-\psi^x_{s,t}(v)),$$

therefore thanks to (5.3.11) (reducing again $\overline{T}(\mathrm{R})$ if necessary), we have

$$\left\|\nabla\left(\psi^x_{s,t}(v) - v\right)\right\|_{\mathrm{L}^\infty_{x,v}} \leq \frac{\|\nabla\widetilde{\mathrm{X}}^{s;t}\|_{\mathrm{L}^\infty_{x,v}}}{1-\|\nabla\widetilde{\mathrm{X}}^{s;t}\|_{\mathrm{L}^\infty_{x,v}}} \leq \varphi(T)\Lambda(T,\mathrm{R}).$$

This yields the result for $|\alpha| = 1$. If $1 < |\alpha| \leq k$ and if the result holds for all $|\widetilde{\alpha}| < |\alpha|$, we apply $\partial^\alpha_{x,v}$ in (5.3.6) and use the Faà di Bruno's formula:

$$\partial^\alpha_{x,v}\left(\psi^x_{s,t}(v) - v\right) = \sum_{\mu,\nu} C_{\mu,\nu}\partial^\mu_{x,v}\widetilde{\mathrm{X}}^{s,t}(\mathrm{z}(x,v)) \prod_{\substack{1\leq|\beta|\leq|\alpha| \\ 1\leq j\leq 2d}} (\partial^\beta_{x,v}\mathrm{z}(x,v)_j)^{\nu_{\beta_j}}, \quad \mathrm{z}(x,v) := (x,\psi^x_{s,t}(v)),$$

where the sum is taken on (μ,ν) such that $1 \leq |\mu| \leq |\alpha|$ and $\nu_k \in \mathbb{N}\setminus\{0\}$ with

$$\forall 1 \leq j \leq 2d, \quad \sum_{1\leq|\beta|\leq|\alpha|} \nu_{\beta_j} = \mu_j, \quad \text{and} \quad \sum_{\substack{1\leq|\beta|\leq|\alpha| \\ 1\leq j\leq 2d}} \nu_{\beta_j}\beta = \alpha.$$

We proceed as in the case $|\alpha| = 1$. We isolate the terms with multi-indices μ such that $|\mu| = 1$ (giving associated $\nu_{\beta_j} = 1$ for all $1 \leq j \leq 2d$): the terms $\partial^\alpha_{x,v}\psi_{s,t}$ (given by $\nu_{\alpha_j} = 1$) in the product are treated as above, while derivatives of order strictly less than $|\alpha|$ are bounded thanks to the induction hypothesis. This procedure is allowed provided that uniform bounds (in time) of the same type for $\|\widetilde{\mathrm{X}}^{s,t}\|_{\mathrm{W}^{k,\infty}_{x,v}}$ ($k \leq |\alpha|$) hold true.

Such bounds are obtained by performing the same induction at the level of $\mathrm{Y}^{s;t}$ first (using the same principle as before with (5.3.7)) and then for $\widetilde{\mathrm{X}}^{s,t}$ (arguing as before with (5.3.8)).

Concerning the estimate (5.3.5), we have by (5.3.6)

$$\partial_s \psi^x_{s,t}(v) = -\partial_s \widetilde{\mathrm{X}}^{s,t}(x, \psi^x_{s,t}(v)) - \nabla_v \widetilde{\mathrm{X}}^{s,t}(x, \psi^x_{s,t}(v)) \partial_s \psi^x_{s,t}(v),$$

and by (5.3.8),

$$\partial_s \widetilde{\mathrm{X}}^{s;t}(x,v) = -\frac{e^{t-s}}{1-e^{t-s}} \widetilde{\mathrm{X}}^{s;t}(x,v) - \frac{1}{1-e^{t-s}} \int_s^t e^{\tau - s} \mathrm{F}\left(\tau, x + (1-e^{t-\tau})v + \mathrm{Y}^{\tau;t}(x,v)\right) \mathrm{d}\tau.$$

Since

$$\|\widetilde{\mathrm{X}}^{s;t}\|_{\mathrm{L}^\infty_{x,v}} \leq T\Lambda(T, \mathrm{R}),$$

and $\frac{Te^T}{e^T-1}$ is bounded in a neighborhood of 0, we obtain an estimate on $\|\partial_s \widetilde{\mathrm{X}}^{s;t}(x,v)\|_{\mathrm{L}^\infty_{x,v}}$ and then on $\|\partial_s \psi^x_{s,t}(v)\|_{\mathrm{L}^\infty_{x,v}}$ as before. Using the same induction procedure as for (5.3.4), we finally obtain (5.3.5). □

Remark 5.3.2. From the $\mathrm{W}^{k,\infty}_{x,v}$-bounds we have obtained on $\mathrm{Y}^{s,t}$ along the proof, and because

$$\mathrm{Y}^{s;t}(x,v) = \mathrm{X}^{s;t}(x,v) - x - (1-e^{t-s})v,$$

we can infer that

$$\sup_{s,t\in[0,T]} \left\| \partial^\beta_{x,v} \left(\mathrm{X}_{s,t}(x,v) - x - (1-e^{t-s})v \right) \right\|_{\mathrm{L}^\infty(\mathbb{T}^d_x \times \mathbb{R}^d_v)} \leq \varphi(T)\Lambda(T,R), \quad |\beta| \leq k. \tag{5.3.12}$$

Likewise, by considering

$$\mathrm{W}^{s,t}(x,v) = \mathrm{V}^{s,t}(x,v) - e^{t-s}v,$$

one can obtain the estimate

$$\sup_{s,t\in[0,T]} \left\| \partial^\beta_{x,v} \left(\mathrm{V}_{s,t}(x,v) - e^{t-s}v \right) \right\|_{\mathrm{L}^\infty(\mathbb{T}^d_x \times \mathbb{R}^d_v)} \leq \varphi(T)\Lambda(T,R), \quad |\beta| \leq k. \tag{5.3.13}$$

Let us conclude this section by showing that in Lemma 5.3.1, one can consider

$$(T, \mathrm{F}) = (T_\varepsilon, E^{u_\varepsilon, \varrho_\varepsilon}_{\mathrm{reg},\varepsilon}),$$

for a given $\varepsilon > 0$, where $(T_\varepsilon, E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon})$ have been defined in the set-up of the bootstrap argument of Section 5.2.2. Let us prove that the assumptions of Lemma 5.3.1 indeed hold with $k = \lfloor m-2-d/2 \rfloor$. By the estimate (5.2.2) from Lemma 5.2.7, we have for all $t \in (0, T_\varepsilon)$ and $\ell < m - 1 - d/2$

$$\begin{aligned}\left\| E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}(t) \right\|_{\mathrm{W}^{\ell,\infty}_x} &\lesssim \left\| E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}(t) \right\|_{H^{m-1}} \\ &\lesssim \|u_\varepsilon(t)\|_{\mathrm{H}^{m-1}} + \Lambda\left(\|\varrho_\varepsilon(t)\|_{\mathrm{H}^{m-2}}\right) \|\varrho_\varepsilon(t)\|_{\mathrm{H}^m}.\end{aligned}$$

Appealing to Lemma 5.2.20, and using $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T_\varepsilon) \leq R$ for all $T \leq T_\varepsilon$, we deduce

$$\|E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}\|_{\mathrm{L}^2(0,T;\mathrm{W}^{k,\infty}(\mathbb{T}^d))} \leq \Lambda(T,R),$$

for $k = \lfloor m - 2 - d/2 \rfloor$.

5.4 Averaging operators related to the dynamics with friction

For any smooth vector field $G(t,s,x,v) \in \mathbb{R}^d$, we consider the following integral operator $\mathrm{K}_G^{\mathrm{free}}$ acting on functions $H(t,x)$:

$$\mathrm{K}_G^{\mathrm{free}}[H](t,x) := \int_0^t \int_{\mathbb{R}^d} [\nabla_x H](s, x-(t-s)v) \cdot G(t,s,x,v)\,\mathrm{d}v\,\mathrm{d}s.$$

This operator, featuring an apparent loss of derivative in space, was systematically studied in [HKR16]. It was proven in [HKR16, Proposition 5.1 and Remark 5.1] that this loss is only apparent, provided that the kernel is sufficiently smooth and decaying in velocity. The statement goes as follows.

Proposition 5.4.1. *Let $T>0$. If $p>1+d$ and $\sigma>d/2$ then for all $H \in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))$*

$$\left\| \mathrm{K}_G^{\mathrm{free}}[H] \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \lesssim \sup_{0\leq s,t\leq T} \|G(t,s)\|_{\mathcal{H}^p_\sigma} \|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.$$

As already noted in [HKR16], this smoothing estimate is reminiscent of (but different from) the so-called kinetic averaging lemmas. Namely, Proposition 5.4.1 provides the gain of one full derivative.

Averaging lemmas are well-known to provide powerful regularity and compactness results in the study of kinetic equations. Loosely speaking, moments in velocity of the solutions appear to gain some regularity compared to the solutions themselves, which are just transported along the flow of the equation. We refer to [GPS85, Ago84, GLPS88] for the introduction of the averaging lemmas, and to [DL89b, PS98, GSR02, JV04, ASR11, AM14, JLT22, AL21] for several extensions of such results.

A thorough comparison between standard kinetic averaging lemmas and the estimate from Proposition 5.4.1 can be found in [HK19]. We finally refer to [HKNR18] for the use of Proposition 5.4.1 for a sligthly different purpose, as well as to [Cha23] for an extension of this proposition.

In this section, we prove crucial smoothing estimates adaptated to kinetic equations with friction, in the spirit of Proposition 5.4.1. First, we define the corresponding integral operator.

Definition 5.4.2. *For any smooth vector field $G(t,s,x,v)$, we define the following integral operators acting on functions $H(t,x)$ by*

$$\mathrm{K}_G^{\mathrm{fric}}[H](t,x) := \int_0^t \int_{\mathbb{R}^d} [\nabla_x H](s, x+(1-e^{t-s})v) \cdot G(t,s,x,v)\,\mathrm{d}v\,\mathrm{d}s,$$

where $(t,x) \in \mathbb{R}^+ \times \mathbb{T}^d$.

Assuming that the kernel G is sufficiently smooth and decaying in velocity, we will prove several continuity and regularization estimates for $\mathrm{K}_G^{\mathrm{free}}$ and $\mathrm{K}_G^{\mathrm{fric}}$ (see Propositions 5.4.4–5.4.5–5.4.7 below).

In what follows, $\mathcal{F}_{x,v}$ will refer to the Fourier transform on $\mathbb{T}^d \times \mathbb{R}^d$ defined as

$$\mathcal{F}_{x,v} h(k,\xi) = \int_{\mathbb{T}^d\times\mathbb{R}^d} e^{-i(k\cdot x+v\cdot\xi)} h(x,v)\,\mathrm{d}x\,\mathrm{d}v, \quad (k,\xi) \in \mathbb{Z}^d \times \mathbb{R}^d.$$

Our first result is the following.

Proposition 5.4.3. *There exists $C>0$ such that the followings holds. Suppose that $G_{[q]}(t,s,x,v)$ is a kernel of the form*

$$G_{[q]}(t,s,x,v) = (t-s)^q \mathcal{G}(t,s,x,v),$$

with $q \in \mathbb{N}$. *For every* $T > 0$ *satisfying*

$$\|\mathcal{G}\|_{T,s_1,s_2} := \sup_{0\leq t\leq T}\left(\sum_{m\in\mathbb{Z}^d}\sup_{0\leq s\leq t}\sup_{\xi\in\mathbb{R}^d}\left\{(1+|m|)^{s_2}(1+|\xi|)^{s_1}|(\mathcal{F}_{x,v}\mathcal{G})(t,s,m,\xi)|\right\}^2\right)^{\frac{1}{2}} < +\infty,$$

for $s_1 > 1+2q$ *and* $s_2 > d/2+2q$,

$$\left\|\nabla_x^q \mathrm{K}^{\mathrm{free}}_{G_{[q]}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} + \left\|\nabla_x^q \mathrm{K}^{\mathrm{fric}}_{G_{[q]}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq C\|\mathcal{G}\|_{T,s_1,s_2}\|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.$$

Proof. We only perform the proof in the case of $\mathrm{K}^{\mathrm{fric}}_{G_{[q]}}$ (the proof is similar for $\mathrm{K}^{\mathrm{free}}_{G_{[q]}}$). Writing for all $t \geq 0$

$$H(t,x) := \sum_{k\in\mathbb{Z}^d}\widehat{H}_k(t)e^{ik\cdot x} \quad \text{in } \mathrm{L}^2(\mathbb{T}^d),$$

we have

$$\begin{aligned}
\mathrm{K}^{\mathrm{fric}}_{G_{[q]}}[H](t,x) &= \int_0^t \sum_{k\in\mathbb{Z}^d}\widehat{H}_k(s)e^{ik\cdot x}(ik)\cdot\int_{\mathbb{R}^d} e^{-ik\cdot(e^{t-s}-1)v}G_{[q]}(t,s,x,v)\,\mathrm{d}v\,\mathrm{d}s\\
&= \int_0^t \sum_{k\in\mathbb{Z}^d}\widehat{H}_k(s)e^{ik\cdot x}(ik)\cdot(\mathcal{F}_v G_{[q]})(t,s,x,k(e^{t-s}-1))\,\mathrm{d}s.
\end{aligned}$$

We now expand $G_{[q]}$ in Fourier series along the x variable so that

$$\begin{aligned}
\mathrm{K}^{\mathrm{fric}}_{G_{[q]}}[H](t,x) &= \sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} e^{i(k+\ell)\cdot x}\int_0^t \widehat{H}_k(s)(ik)\cdot(\mathcal{F}_{x,v}G_{[q]})(t,s,\ell,k(e^{t-s}-1))\,\mathrm{d}s\\
&= \sum_{\ell'\in\mathbb{Z}^d} e^{i\ell'\cdot x}\left\{\sum_{k\in\mathbb{Z}^d}\int_0^t \widehat{H}_k(s)(ik)\cdot(\mathcal{F}_{x,v}G_{[q]})(t,s,\ell'-k,k(e^{t-s}-1))\,\mathrm{d}s\right\},
\end{aligned}$$

and then

$$\begin{aligned}
&\nabla_x^q \mathrm{K}^{\mathrm{free}}_{(t-s)^q\mathcal{G}}[H](t,x)\\
&\qquad= \sum_{\ell\in\mathbb{Z}^d} e^{i\ell\cdot x}\left\{(i\ell)^q\sum_{k\in\mathbb{Z}^d}\int_0^t (t-s)^q\widehat{H}_k(s)(ik)\cdot(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))\,\mathrm{d}s\right\}.
\end{aligned}$$

By Parseval equality and the Cauchy-Schwarz inequality (in frequency and time), we get

$$\begin{aligned}
&\|\nabla_x^q \mathrm{K}^{\mathrm{fric}}_{(t-s)\mathcal{G}}[F](t)\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\\
&= \sum_{\ell\in\mathbb{Z}^d}|\ell|^{2q}\left|\sum_{k\in\mathbb{Z}^d}\int_0^t (t-s)^q\widehat{H}_k(s)(ik)\cdot(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))\,\mathrm{d}s\right|^2\\
&\leq \sum_{\ell\in\mathbb{Z}^d}|\ell|^{2q}\left(\sum_{k\in\mathbb{Z}^d}\int_0^t (t-s)^{2q}|\widehat{H}_k(s)|^2|k\cdot(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))|\,\mathrm{d}s\right)\\
&\quad\times\left(\sum_{k\in\mathbb{Z}^d}\int_0^t |k\cdot(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))|\mathrm{d}s\right).
\end{aligned}$$

Integrating in time yields

$$\begin{aligned}
&\|\nabla_x^q \mathrm{K}^{\mathrm{fric}}_{(t-s)\mathcal{G}}[F]\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\times\mathbb{T}^d))} \\
&\leq \sum_{\ell\in\mathbb{Z}^d} |\ell|^{2q} \int_0^T \int_0^t \sum_{k\in\mathbb{Z}^d} (t-s)^{2q} |\widehat{H}_k(s)|^2 |k\cdot(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))|\,\mathrm{d}s\mathrm{d}t \\
&\quad \times \sup_{\ell\in\mathbb{Z}^d} \sup_{t\in(0,T)} \sum_{k\in\mathbb{Z}^d} \int_0^t |k\cdot(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))|\mathrm{d}s \\
&= (\mathrm{I}) \times (\mathrm{II}).
\end{aligned}$$

A first step is to note that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d} \int_0^t |k\cdot(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))|\mathrm{d}s \\
&\qquad \leq \sum_{k\in\mathbb{Z}^d} \sup_{\substack{s\in(0,t)\\ \xi\in\mathbb{R}^d}} (1+|\xi|)^\alpha |\mathcal{F}_{x,v}\mathcal{G}(t,s,\ell-k,\xi))| \int_0^t \frac{|k|}{(1+|k|(e^{t-s}-1))^\alpha}\mathrm{d}s \\
&\qquad \leq \sum_{k\in\mathbb{Z}^d} \sup_{\substack{s\in(0,t)\\ \xi\in\mathbb{R}^d}} (1+|\xi|)^\alpha |\mathcal{F}_{x,v}\mathcal{G}(t,s,\ell-k,\xi))| \int_0^t \frac{|k|}{(1+|k|(t-s))^\alpha}\mathrm{d}s,
\end{aligned}$$

and that the change of variable $\tau = |k|(t-s)$ in the last integral yields

$$\int_0^t \frac{|k|}{(1+|k|(t-s))^\alpha}\mathrm{d}s \leq \int_0^{+\infty} \frac{\mathrm{d}\tau}{(1+\tau)^\alpha} < +\infty,$$

provided that $\alpha > 1$. With this observation, we can treat the term (II) and obtain by the Cauchy-Schwarz inequality in k

$$(\mathrm{II}) \leq \sup_{\ell\in\mathbb{Z}^d} \sup_{t\in(0,T)} \sum_{k\in\mathbb{Z}^d} \sup_{\substack{s\in(0,t)\\ \xi\in\mathbb{R}^d}} (1+|\xi|)^{s_1} |\mathcal{F}_{x,v}\mathcal{G}(t,s,\ell-k,\xi))| \leq \|\mathcal{G}\|_{T,s_1,s_2},$$

for $s_1 > 1$ and $s_2 > d/2$. For the term (I), we use Fubini-Tonelli theorem and get

$$\begin{aligned}
(\mathrm{I}) &= \sum_{\ell\in\mathbb{Z}^d} |\ell|^{2q} \int_0^T \int_0^t \sum_{k\in\mathbb{Z}^d} (t-s)^{2q} |\widehat{H}_k(s)|^2 |k| |(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))|\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^T \sum_{k\in\mathbb{Z}^d} |\widehat{H}_k(s)|^2 \int_s^T \sum_{\ell\in\mathbb{Z}^d} (t-s)^{2q} |\ell|^{2q} |k| |(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))|\,\mathrm{d}t\,\mathrm{d}s \\
&\leq \|H\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \sup_{k\in\mathbb{Z}^d} \sup_{0\leq s\leq T} \int_s^T \sum_{\ell\in\mathbb{Z}^d} (t-s)^{2q} |\ell|^{2q} |k| |(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))|\,\mathrm{d}t.
\end{aligned}$$

Note that the last expression can be taken into account for $k \in \dot{\mathbb{Z}}^d$ only (indeed, the term corresponding to $k=0$ vanishes in (I)). We then have

$$\sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\leq s\leq T}\int_s^T (t-s)^{2q}|k|\sum_{\ell\in\mathbb{Z}^d}|\ell|^{2q}|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))|\,\mathrm{d}t$$
$$\leq \sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\leq s\leq T}\int_s^T \frac{(t-s)^{2q}|k|}{(1+|k|(e^{t-s}-1))^{\alpha_1}}$$
$$\times\sum_{\ell\in\mathbb{Z}^d}|\ell|^{2q}(1+|k|(e^{t-s}-1))^{\alpha_1})|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))|\,\mathrm{d}t$$
$$\leq \sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\leq s\leq T}\int_s^T \frac{(t-s)^{2q}|k|}{(1+|k|(e^{t-s}-1))^{\alpha_1}}\,\mathrm{d}t$$
$$\times\sup_{0\leq s\leq T}\sup_{s\leq t\leq T}\sum_{\ell\in\mathbb{Z}^d}|\ell|^{2q}\sup_{\xi}(1+|\xi|^{\alpha_1})|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,\xi)|$$
$$\leq \sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\leq s\leq T}\int_s^T \frac{(t-s)^{2q}|k|}{(1+|k|(t-s))^{\alpha_1}}\,\mathrm{d}t$$
$$\times\sup_{0\leq s\leq T}\sup_{s\leq t\leq T}\sum_{\ell\in\mathbb{Z}^d}|\ell+k|^{2q}\sup_{\xi}(1+|\xi|^{\alpha_1})|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell,\xi)|,$$

therefore

$$\sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\leq s\leq T}\int_s^T (t-s)^{2q}|k|\sum_{\ell\in\mathbb{Z}^d}|\ell|^{2q}|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))|\,\mathrm{d}t$$
$$\leq \sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\leq s\leq T}\int_s^T \frac{(t-s)^{2q}|k|}{(1+|k|(t-s))^{\alpha_1}}\,\mathrm{d}t$$
$$\times\sup_{0\leq s\leq T}\sup_{s\leq t\leq T}\sum_{\ell\in\mathbb{Z}^d}|\ell|^{2q}\sup_{\xi}(1+|\xi|^{\alpha_1})|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell,\xi)|$$
$$+\sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\leq s\leq T}\int_s^T \frac{(t-s)^{2q}|k|^{1+2q}}{(1+|k|(t-s))^{\alpha_1}}\,\mathrm{d}t$$
$$\times\sup_{0\leq s\leq T}\sup_{s\leq t\leq T}\sum_{\ell\in\mathbb{Z}^d}\sup_{\xi}(1+|\xi|^{\alpha_1})|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell,\xi)|$$
$$=\mathrm{S}_1+\mathrm{S}_2.$$

Let us treat these two terms separately.

- For S_1, we have by the Cauchy-Schwarz inequality

$$\mathrm{S}_1\leq \sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\leq s\leq T}\int_s^T \frac{(t-s)^{2q}|k|}{(1+|k|(t-s))^{\alpha_1}}\,\mathrm{d}t$$
$$\times\sup_{0\leq s\leq T}\sup_{s\leq t\leq T}\left(\sum_{\ell\in\mathbb{Z}^d}\sup_{\xi}\left\{(1+|\ell|^{2q+\alpha_2})(1+|\xi|^{\alpha_1})|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell,\xi)|\right\}^2\right)^{\frac{1}{2}},$$

if $\alpha_2>d/2$. For the integral term, we write for $\alpha_1>3$.

$$\sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\leq s\leq T}\int_s^T \frac{(t-s)^{2q}|k|}{(1+|k|(t-s))^{\alpha_1}}\,\mathrm{d}t=\sup_{k\in\dot{\mathbb{Z}}^d}\frac{1}{|k|^{2q}}\sup_{0\leq s\leq T}\int_0^{|k|(T-s)}\frac{\tau^{2q}}{(1+\tau)^{\alpha_1}}\,\mathrm{d}\tau$$
$$\leq \sup_{k\in\dot{\mathbb{Z}}^d}\frac{1}{|k|^{2q}}\int_0^{+\infty}\frac{\tau^{2q}}{(1+\tau)^{\alpha_1}}\,\mathrm{d}\tau,$$

which is a finite constant independent of k and T (since $\alpha_1 > 1+2q$) therefore

$$\mathrm{S}_1 \lesssim \sup_{0\le t\le T}\left(\sum_{\ell\in\mathbb{Z}^d}\sup_{0\le s\le t}\sup_{\xi\in\mathbb{R}^d}\left\{(1+|\ell|^{2q+\alpha_2})(1+|\xi|^{\alpha_1})|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell,\xi)|\right\}^2\right)^{\frac{1}{2}}.$$

- For S_2 , we have for $\alpha_2 > d/2$

$$\mathrm{S}_2 \le \sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\le s\le T}\int_s^T \frac{(t-s)^{2q}|k|^{1+2q}}{(1+|k|(t-s))^{\alpha_1}}\,\mathrm{d}t$$
$$\times \sup_{0\le s\le T}\sup_{s\le t\le T}\left(\sum_{\ell\in\mathbb{Z}^d}\sup_{\xi}\left\{(1+|m|^{\alpha_2})(1+|\xi|^{\alpha_1})|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell,\xi)|\right\}^2\right)^{\frac{1}{2}}.$$

The integral term now reads for $\alpha_1 > 3$

$$\sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\le s\le T}\int_s^T \frac{(t-s)^{2q}|k|^{1+2q}}{(1+|k|(t-s))^{\alpha_1}}\,\mathrm{d}t = \sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\le s\le T}\int_0^{|k|(T-s)} \frac{\tau^{2q}}{(1+\tau)^{\alpha_1}}\,\mathrm{d}\tau \le \int_0^{+\infty}\frac{\tau^{2q}}{(1+\tau)^{\alpha_1}}\,\mathrm{d}\tau,$$

therefore

$$\mathrm{S}_1 \lesssim \sup_{0\le t\le T}\left(\sum_{\ell\in\mathbb{Z}^d}\sup_{0\le s\le t}\sup_{\xi\in\mathbb{R}^d}\left\{(1+|\ell|^{\alpha_2})(1+|\xi|^{\alpha_1})|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell,\xi)|\right\}^2\right)^{\frac{1}{2}}.$$

We have thus proven that

$$(\mathrm{I}) \lesssim \|H\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}\left(\|\mathcal{G}\|_{T,s_1,s_2} + \|\mathcal{G}\|_{T,s_1,s_2+2q}\right),$$

for $s_1 > 1+2q$ and $s_2 > d/2+2q$.All in all, we get

$$\|\nabla_x^q \mathrm{K}^{\mathrm{fric}}_{G_{[q]}}[H]\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \le C\|\mathcal{G}\|_{T,s_1,s_2}\|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))},$$

which ends the proof. □

We then deduce the two following propositions. The first one is a direct consequence of Proposition 5.4.3 with $q=0$, and states the continuity of $\mathrm{K}^{\mathrm{free}}_G$ and $\mathrm{K}^{\mathrm{fric}}_G$ on $\mathrm{L}^2_T\mathrm{L}^2_x$.

Proposition 5.4.4. *There exists $C>0$ such that for every $T>0$, if $p>1+d$ and $\sigma>d/2$ then for all $H\in\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))$*

$$\left\|\mathrm{K}^{\mathrm{free}}_G[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} + \left\|\mathrm{K}^{\mathrm{fric}}_G[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \le C\sup_{0\le s,t\le T}\|G(t,s)\|_{\mathcal{H}^p_\sigma}\|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.$$

Proof. By Proposition 5.4.3 with $q=0$, we have

$$\left\|\mathrm{K}^{\mathrm{free}}_G[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} + \left\|\mathrm{K}^{\mathrm{fric}}_G[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \le C\|G\|_{T,s_1,s_2}\|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.$$

for any $s_1 > 1$ and $s_2 > d/2$. Now appealing to [HKR16, Remark 3], one can prove that for all $p > 1+d$ and $\sigma > d/2$, there exist $s_2 > d/2$ and $s_1 > 1$ such that

$$\|G\|_{T,s_1,s_2} \lesssim \sup_{0\le s,t\le T}\|G(t,s)\|_{\mathcal{H}^p_\sigma},$$

hence the result. □

When the kernel G vanishes along the diagonal in time $\{t = s\}$, Proposition 5.4.3 with $q = 1$ leads to the following additional regularizing effect of the operators K_G (as already observed in [HKR23]). Loosely speaking, the operators $\mathrm{K}_G^{\mathrm{free}}$ and $\mathrm{K}_G^{\mathrm{fric}}$ are bounded from $\mathrm{L}^2_T\mathrm{L}^2_x$ to $\mathrm{L}^2_T\dot{\mathrm{H}}^1_x$ in this case.

Proposition 5.4.5. *There exists $C > 0$ such that if the kernel G satisfies*

$$G(t, t, x, v) = 0,$$

the following holds. Let $T > 0$. If $p > 7 + d$ and $\sigma > d/2$ then for all $H \in \mathrm{L}^2(0, T; \mathrm{L}^2(\mathbb{T}^d))$

$$\left\|\mathrm{K}_G^{\mathrm{free}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^d))} + \left\|\mathrm{K}_G^{\mathrm{fric}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^d))} \le C(1 + T) \sup_{0\le s,t\le T} \|\partial_s G(t, s)\|_{\mathcal{H}^p_\sigma} \|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.$$

Proof. Since $G(t, t, x, v) = 0$, the Taylor's formula shows that

$$G(t, s, x, v) = (t - s)\widetilde{G}(t, s, x, v), \quad \widetilde{G}(t, s, x, v) := -\int_0^1 \partial_s G(t, t + \tau(s - t), x, v)\, \mathrm{d}\tau.$$

By Proposition 5.4.3 with $q = 1$, we get for $s_1 > 3$ and $s_2 > d/2 + 2$

$$\|\nabla_x \mathrm{K}_G^{\mathrm{free}}[H]\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} + \|\nabla_x \mathrm{K}_G^{\mathrm{fric}}[H]\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \le C\|\widetilde{G}\|_{T,s_1,s_2}\|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.$$

To conclude, we observe that

$$\begin{aligned}
&\{(1 + |m|)^{s_2}(1 + |\xi|)^{s_1}|(\mathcal{F}_{x,v}\widetilde{G})(t, s, m, \xi)|\}^2 \\
&\qquad \le \left((1 + |m|)^{s_2}(1 + |\xi|)^{s_1} \int_0^1 |\mathcal{F}_{x,v}(\partial_s G)(t, t + \tau(s - t), m, \xi)|\, \mathrm{d}\tau\right)^2 \\
&\qquad \le \int_0^1 \{(1 + |m|)^{s_2}(1 + |\xi|)^{s_1}|\mathcal{F}_{x,v}(\partial_s G)(t, t + \tau(s - t), m, \xi)|\}^2\, \mathrm{d}\tau,
\end{aligned}$$

therefore

$$\|\nabla_x \mathrm{K}_G^{\mathrm{free}}[H]\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} + \|\nabla_x \mathrm{K}_G^{\mathrm{fric}}[H]\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \le C\|\partial_s G\|_{T,s_1,s_2}\|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.$$

Now appealing to [HKR16, Remark 3], one can prove that for all $p > 2\ell + s + 1 + d$ and $\sigma > d/2$ (with $\ell, s \in \mathbb{R}^+$), there exist $s_2 > \ell + d/2$ and $s_1 > s + 1$ such that

$$\|G\|_{T,s_1,s_2} \lesssim \sup_{0\le s,t\le T} \|G(t, s)\|_{\mathcal{H}^p_\sigma}.$$

Since $G(t, t, x, v) = 0$, we also have

$$\|G\|_{T,s_1,s_2} \lesssim T \sup_{0\le s,t\le T} \|\partial_s G(t, s)\|_{\mathcal{H}^p_\sigma}.$$

By Proposition 5.4.4, and taking $\ell = s = 2$, we end up with the desired conclusion. □

Remark 5.4.6. A variant of Proposition 5.4.5 holds in the following form: there exists $C > 0$ such that for $p > 7 + d$ and $\sigma > d/2$, and

$$G(t, t, x, v) = 0,$$

we have for all $H \in \mathrm{L}^2(0, T; \dot{\mathrm{H}}^{-1}(\mathbb{T}^d))$

$$\left\|\mathrm{K}_G^{\mathrm{free}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} + \left\|\mathrm{K}_G^{\mathrm{fric}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \le C(1 + T) \sup_{0\le s,t\le T} \|\partial_s G(t, s)\|_{\mathcal{H}^p_\sigma} \|H\|_{\mathrm{L}^2(0,T;\dot{\mathrm{H}}^{-1}(\mathbb{T}^d))}.$$

We don't detail the proof, which follows the same lines as the one of Proposition 5.4.5.

We finally investigate the smoothing properties of the difference operator $\mathrm{K}_G^{\mathrm{free}} - \mathrm{K}_G^{\mathrm{fric}}$. A somewhat surprising result is the fact that this operator gains one additional derivative. This is the content of the following proposition.

Proposition 5.4.7. *There exists $C > 0$ such that for every $T > 0$, if $p > 8 + d$ and $\sigma > 1 + d/2$ then for all $H \in \mathrm{L}^2(0, T; \mathrm{L}^2(\mathbb{T}^d))$,*

$$\left\|\mathrm{K}_G^{\mathrm{free}}[H] - \mathrm{K}_G^{\mathrm{fric}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^d))} \leq C\varphi(T) \sup_{0\leq s,t\leq T} \|G(t,s)\|_{\mathcal{H}^p_\sigma} \|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))},$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nondecreasing function.

Proof. Following the computations performed in the proof of Proposition 5.4.5, we have

$$\mathrm{K}_G^{\mathrm{free}}[F](t,x) - \mathrm{K}_G^{\mathrm{fric}}[F](t,x) = \sum_{\ell\in\mathbb{Z}^d} e^{i\ell\cdot x}\Bigg\{ \sum_{k\in\mathbb{Z}^d} \int_0^t \widehat{F}_k(s)(ik) \cdot \big[(\mathcal{F}_{x,v}G)(t,s,\ell-k,k(t-s)) \\ -(\mathcal{F}_{x,v}G)(t,s,\ell-k,k(e^{t-s}-1))\big]\, \mathrm{d}s\Bigg\},$$

therefore, if we set

$$\Theta(t,s,\ell,k) := (\mathcal{F}_{x,v}G)(t,s,\ell-k,k(t-s)) - (\mathcal{F}_{x,v}G)(t,s,\ell-k,k(e^{t-s}-1)),$$

we get

$$\|\nabla_x \left(\mathrm{K}_G^{\mathrm{free}}[H] - \mathrm{K}_G^{\mathrm{fric}}[H]\right)(t)\|^2_{\mathrm{L}^2(\mathbb{T}^d)} \leq \sum_{\ell\in\mathbb{Z}^d} |\ell|^2 \left(\sum_{k\in\mathbb{Z}^d}\int_0^t |\widehat{H}_k(s)||k||\Theta(t,s,\ell,k)|\,\mathrm{d}s\right)^2.$$

We also have, by setting $\mathcal{G}^{t,s}_{\ell-k} = (\mathcal{F}_{x,v}G)(t,s,\ell-k,\bullet)$

$$\begin{aligned}|\Theta(t,s,\ell,k)| &= |\mathcal{G}^{t,s}_{\ell-k}(k(e^{t-s}-1)) - \mathcal{G}^{t,s}_{\ell-k}(k(t-s))| \\ &= |\mathcal{G}^{t,s}_{\ell-k}(k(t-s) + k(t-s)^2\varphi(t-s)) - \mathcal{G}^{t,s}_{\ell-k}(k(t-s))| \\ &\leq \sup_{\theta\in[0,1]} |\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}\left(\xi^{t,s}_\theta(k(t-s))\right)||k|(t-s)^2\varphi(t-s),\end{aligned}$$

where $\varphi(z) = \sum_{i\geq 0} \frac{z^i}{(i+2)!}$ and $\xi^{t,s}_\theta(z) = z + \theta z(t-s)\varphi(t-s)$. By continuity, there exists $\theta^\star \in [0,1]$ (which may depend on all the other variables) such that

$$|\Theta(t,s,\ell,k)| \leq |\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}\left(\xi^{t,s}_{\theta^\star}(k(t-s))\right)||k|(t-s)^2\varphi(t-s).$$

This yields

$$\begin{aligned}&\|\nabla_x \left(\mathrm{K}_G^{\mathrm{free}}[H] - \mathrm{K}_G^{\mathrm{fric}}[H]\right)(t)\|^2_{\mathrm{L}^2(\mathbb{T}^d)} \\ &\leq \sum_{\ell\in\mathbb{Z}^d} |\ell|^2 \left(\sum_{k\in\mathbb{Z}^d}\int_0^t |\widehat{H}_k(s)||k|^2(t-s)^2\varphi(t-s)|\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}\left(\xi^{t,s}_{\theta^\star}(k(t-s))\right)|\,\mathrm{d}s\right)^2 \\ &\leq \sum_{\ell\in\mathbb{Z}^d} |\ell|^2 \left(\sum_{k\in\mathbb{Z}^d}\int_0^t |\widehat{H}_k(s)|^2|k|^2(t-s)^3\varphi(t-s)^2|\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}\left((\xi^{t,s}_{\theta^\star}(k(t-s))\right)|\,\mathrm{d}s\right) \\ &\quad\times \left(\sum_{k\in\mathbb{Z}^d}\int_0^t |k|^2(t-s)|\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}\left(\xi^{t,s}_{\theta^\star}(k(t-s))\right)|\,\mathrm{d}s\right),\end{aligned}$$

thanks to the Cauchy-Schwarz inequality. As in the proof of Proposition 5.4.5, we obtain by integrating in time that

$$\begin{aligned}
&\|\nabla_x \left(\mathrm{K}_G^{\mathrm{free}}[H] - \mathrm{K}_G^{\mathrm{fric}}[H]\right)\|^2_{\mathrm{L}^2((0,T)\times\mathbb{T}^d)} \\
&\quad\leq \sum_{\ell\in\mathbb{Z}^d} |\ell|^2 \int_0^T \int_0^t \sum_{k\in\mathbb{Z}^d} |\widehat{H}_k(s)|^2 |k|^2 (t-s)^3 \varphi(t-s)^2 |\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}\left(\xi^{t,s}_{\theta^\star}(k(t-s))\right)| \,\mathrm{d}s\,\mathrm{d}t \\
&\qquad\qquad \times \sup_{\ell\in\mathbb{Z}^d} \sup_{t\in(0,T)} \sum_{k\in\mathbb{Z}^d} \int_0^t |k|^2 (t-s) |\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}\left(\xi^{t,s}_{\theta^\star}(k(t-s))\right)| \mathrm{d}s \\
&\quad= (\mathrm{A}) \times (\mathrm{B}).
\end{aligned}$$

First, we note that for all $\theta \in [0,1]$, $k \in \mathbb{Z}^d$ and $0 \leq s \leq t$

$$\begin{aligned}
|\xi^{t,s}_\theta(k(t-s))| &= |k(t-s) + \theta k(t-s)^2 \varphi(t-s)| \\
&= |k|(t-s)\left[1 + \theta(t-s)\varphi(t-s)\right] \\
&\geq |k|(t-s).
\end{aligned}$$

For (B), we thus have

$$\begin{aligned}
(\mathrm{B}) &\leq \sup_{\ell\in\mathbb{Z}^d} \sup_{t\in(0,T)} \sum_{k\in\mathbb{Z}^d} \int_0^t (1 + |\xi^{t,s}_{\theta^\star}(k(t-s))|)^{\beta_1} |\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}\left(\xi^{t,s}_{\theta^\star}(k(t-s))\right)| \frac{|k|^2(t-s)}{(1+|\xi^{t,s}_{\theta^\star}(k(t-s))|)^{\beta_1}} \mathrm{d}s \\
&\leq \sup_{\ell\in\mathbb{Z}^d} \sup_{t\in(0,T)} \sum_{k\in\mathbb{Z}^d} \sup_{s,\xi} \{(1+|\xi|)^{\beta_1} |\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}(\xi)|\} \int_0^t \frac{|k|^2(t-s)}{(1+|k|(t-s))^{\beta_1}} \mathrm{d}s.
\end{aligned}$$

Since

$$\int_0^t \frac{|k|^2(t-s)}{(1+|k|(t-s))^{\beta_1}} \mathrm{d}s = \int_0^{|k|t} \frac{\tau}{(1+\tau)^{\beta_1}} \mathrm{d}\tau \leq \int_0^{+\infty} \frac{\tau}{(1+\tau)^{\beta_1}} \mathrm{d}\tau < +\infty,$$

if $\beta_1 > 2$, we get

$$(\mathrm{B}) \lesssim \sup_{\ell\in\mathbb{Z}^d} \sup_{t\in(0,T)} \sum_{k\in\mathbb{Z}^d} \sup_{s,\xi} \{(1+|\xi|)^{\beta_1} |\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}(\xi)|\}.$$

By choosing $\beta_2 > d/2$ and by the Cauchy-Schwarz inequality, this yields

$$(\mathrm{B}) \lesssim \sup_{t\in(0,T)} \left(\sum_{k\in\mathbb{Z}^d} \sup_{s,\xi} \{(1+|k|)^{\beta_2} (1+|\xi|)^{\beta_1} |\nabla_\xi \mathcal{G}^{t,s}_k(\xi)|\}^2 \right)^{\frac{1}{2}}.$$

Let us estimate the other term (A). By the Fubini-Tonelli theorem, we have

$$\begin{aligned}
(\mathrm{A}) &= \sum_{\ell\in\mathbb{Z}^d} |\ell|^2 \int_0^T \int_0^t \sum_{k\in\mathbb{Z}^d} |\widehat{H}_k(s)|^2 |k|^2 (t-s)^3 \varphi(t-s)^2 |\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}\left(\xi^{t,s}_{\theta^\star}(k(t-s))\right)| \,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^T \sum_{k\in\mathbb{Z}^d} |\widehat{H}_k(s)|^2 \int_s^T \sum_{\ell\in\mathbb{Z}^d} |\ell|^2 |k|^2 (t-s)^3 \varphi(t-s)^2 |\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}\left(\xi^{t,s}_{\theta^\star}(k(t-s))\right)| \,\mathrm{d}t\,\mathrm{d}s \\
&\leq \|H\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \sup_{k\in\mathbb{Z}^d} \sup_{0\leq s\leq T} \int_s^T \sum_{\ell\in\mathbb{Z}^d} |\ell|^2 |k|^2 (t-s)^3 \varphi(t-s)^2 |\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}\left(\xi^{t,s}_{\theta^\star}(k(t-s))\right)| \,\mathrm{d}t.
\end{aligned}$$

As in the proof of Proposition 5.4.3, we have

$$
\begin{aligned}
&\sup_{k\in\mathbb{Z}^d}\sup_{0\le s\le T}\int_s^T |k|^2(t-s)^3\varphi(t-s)^2\sum_{\ell\in\mathbb{Z}^d}|\ell|^2|\nabla_\xi\mathcal{G}^{t,s}_{\ell-k}\left(\xi^{t,s}_{\theta^\star}(k(t-s))\right)|\,\mathrm{d}t\\
&\le \sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\le s\le T}\int_s^T\frac{|k|^2(t-s)^3\varphi(t-s)^2}{(1+|k|(t-s))^{\alpha_1}}\,\mathrm{d}t\\
&\qquad\qquad\times\sup_{0\le s\le T}\sup_{s\le t\le T}\sum_{\ell\in\mathbb{Z}^d}|\ell|^2\sup_\xi(1+|\xi|^{\alpha_1})|\nabla_\xi\mathcal{G}^{t,s}_\ell(\xi)|\\
&\quad+\sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\le s\le T}\int_s^T\frac{|k|^4(t-s)^3\varphi(t-s)^2}{(1+|k|(t-s))^{\alpha_1}}\,\mathrm{d}t\\
&\qquad\qquad\times\sup_{0\le s\le T}\sup_{s\le t\le T}\sum_{\ell\in\mathbb{Z}^d}\sup_\xi(1+|\xi|^{\alpha_1})|\nabla_\xi\mathcal{G}^{t,s}_\ell(\xi)|\\
&=\mathrm{T}_1+\mathrm{T}_2.
\end{aligned}
$$

We treat these two terms in a separate way.

- In T_1, the integral term can be bounded via

$$
\begin{aligned}
\sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\le s\le T}\int_s^T\frac{|k|^2(t-s)^3\varphi(t-s)^2}{(1+|k|(t-s))^{\alpha_1}}\,\mathrm{d}t&\le\sup_{k\in\dot{\mathbb{Z}}^d}\frac{1}{|k|^2}\sup_{0\le s\le T}\varphi(T-s)^2\int_0^{+\infty}\frac{\tau^3}{(1+\tau)^{\alpha_1}}\,\mathrm{d}\tau.\\
&\le\varphi(T)^2\sup_{k\in\dot{\mathbb{Z}}^d}\frac{1}{|k|^2}\int_0^{+\infty}\frac{\tau^3}{(1+\tau)^{\alpha_1}}\,\mathrm{d}\tau\\
&\lesssim\varphi(T)^2,
\end{aligned}
$$

provided that $\alpha_1>4$. This implies that for any $\alpha_2>d/2$

$$
\mathrm{T}_1\lesssim\varphi(T)^2\sup_{0\le s\le T}\sup_{s\le t\le T}\left(\sum_{m\in\mathbb{Z}^d}\sup_\xi\left\{(1+|m|^{2+\alpha_2})(1+|\xi|^{\alpha_1})|\nabla_\xi\mathcal{G}^{t,s}_m(\xi)|\right\}^2\right)^{\frac12}.
$$

- In T_2, the integral term can be bounded in a similar way via

$$
\begin{aligned}
\sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\le s\le T}\int_s^T\frac{|k|^4(t-s)^3\varphi(t-s)^2}{(1+|k|(t-s))^{\alpha_1}}\,\mathrm{d}t&\le\sup_{0\le s\le T}\varphi(T-s)^2\int_0^{+\infty}\frac{\tau^3}{(1+\tau)^{\alpha_1}}\,\mathrm{d}\tau.\\
&\le\varphi(T)^2\int_0^{+\infty}\frac{\tau^3}{(1+\tau)^{\alpha_1}}\,\mathrm{d}\tau\\
&\lesssim\varphi(T)^2,
\end{aligned}
$$

provided that $\alpha_1>4$. This implies that for any $\alpha_2>d/2$

$$
\mathrm{T}_2\lesssim\varphi(T)\sup_{0\le s\le T}\sup_{s\le t\le T}\left(\sum_{m\in\mathbb{Z}^d}\sup_\xi\left\{(1+|m|^{\alpha_2})(1+|\xi|^{\alpha_1})|\nabla_\xi\mathcal{G}^{t,s}_m(\xi)|\right\}^2\right)^{\frac12}.
$$

All in all, we get for $\alpha_1 > 4$ and $\alpha_2 > 2 + d/2$

$$\begin{aligned}
(\mathrm{A}) &\lesssim \varphi(T)^2 \|H\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \\
&\qquad \times \sup_{0\leq s\leq T} \sup_{s\leq t\leq T} \left(\sum_{m\in\mathbb{Z}^d} \sup_{\xi} \left\{ (1+|m|^{\alpha_2})(1+|\xi|^{\alpha_1}) |\nabla_\xi \mathcal{G}^{t,s}_m(\xi)| \right\}^2 \right)^{\frac{1}{2}} \\
&\lesssim \varphi(T)^2 \|H\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \\
&\qquad \times \sup_{0\leq t\leq T} \left(\sum_{m\in\mathbb{Z}^d} \sup_{0\leq s\leq t} \sup_{\xi} \left\{ (1+|m|^{\alpha_2})(1+|\xi|^{\alpha_1}) |\nabla_\xi \mathcal{G}^{t,s}_m(\xi)| \right\}^2 \right)^{\frac{1}{2}},
\end{aligned}$$

We have thus proven that for $s_1 > 4$ and $s_2 > 2 + d/2$

$$\left\| \nabla_x \left(\mathrm{K}^{\mathrm{free}}_G[H] - \mathrm{K}^{\mathrm{fric}}_G[H] \right) \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \lesssim \varphi(T) \, \|v \otimes G\|_{T,s_1,s_2} \, \|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))},$$

where we have used the semi-norm $\| \cdot \|_{T,s_1,s_2}$ from Proposition 5.4.3. We can now conclude as in the proof of Proposition 5.4.5. We observe that for all $p > 2\ell + s + 1 + d$ and $\sigma > 1 + d/2$ (with $\ell, s \in \mathbb{R}^+$), there exist $s_1 > s + 1$ and $s_2 > \ell + d/2$ such that

$$\|v \otimes G\|_{T,s_1,s_2} \lesssim \sup_{0\leq s,t\leq T} \|G(t,s)\|_{\mathcal{H}^p_\sigma}.$$

By taking $\ell = 2$ ans $s = 3$, and by finally using Proposition 5.4.3, we reach the desired conclusion. □

5.5 Analysis of the kinetic moments

Following the bootstrap procedure initiated in Section 5.2.2, we want to control $\|\varrho_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}$ uniformly in ε and for $T < T_\varepsilon$. In view of the transport equation bearing on ϱ_ε (see Lemma 5.2.2), we will relate the kinetic moments ρ_{f_ε} and j_{f_ε} to the fluid density ϱ_ε itself.

In this section, to ease readability, we drop out the subscripts ε when we refer to the solution. Recall that Λ will always stand for a nonnegative continuous function which is independent of ε, nondecreasing with respect to each of its argument, that may depend implicitly on the initial data and that may change from line to line.

For all $T \in [0, T_\varepsilon)$ small enough, the goal of this section is thus to prove the following result.

Proposition 5.5.1. *Let $T \in (0, \min\left(T_\varepsilon(R), \overline{T}(R)\right)$. For all $|I| \leq m$, we have for any $t \in (0,T)$,*

$$\begin{aligned}
\partial^I_x \rho_f(t,x) &= p'(\varrho(t,x)) \int_0^t \int_{\mathbb{R}^d} \nabla_x [\mathrm{J}_\varepsilon \partial^I_x \varrho](s, x-(t-s)v) \cdot \nabla_v f(t,x,v) \, \mathrm{d}v \, \mathrm{d}s + R^I[\rho_f](t,x), \\
\partial^I_x j_f(t,x) &= p'(\varrho(t,x)) \int_0^t \int_{\mathbb{R}^d} v \nabla_x [\mathrm{J}_\varepsilon \partial^I_x \varrho](s, x-(t-s)v) \cdot \nabla_v f(t,x,v) \, \mathrm{d}v \, \mathrm{d}s + R^I[j_f](t,x),
\end{aligned}$$

where the remainders $R^I[\rho_f]$ and $R^I[j_f]$ satisfy

$$\left\| R^I[\rho_f] \right\|_{\mathrm{L}^2(0,T,\mathrm{H}^1_x)} \leq \Lambda(T,R), \quad \left\| R^I[j_f] \right\|_{\mathrm{L}^2(0,T,\mathrm{H}^1_x)} \leq \Lambda(T,R).$$

We recall the definition of the time $T_\varepsilon(R)$ from our bootstrap procedure settled in Section 5.2.2, as well as the definition of the time $\overline{T}(R)$ from Lemma 5.3.1 in Section 5.3. Note that it is independent of ε. In the rest of this section, we will always implicitly consider times $T > 0$ such that

$$T < \min\left(T_\varepsilon(R), \overline{T}(R)\right).$$

From Proposition 5.5.1, we can immediately infer the following corollary.

Corollary 5.5.2. *For $m > 2 + d$, $\sigma > 1 + d/2$ and $|I| \leq m$, we have*

$$\|\partial_x^I \rho_f\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R),$$
$$\|\partial_x^I j_f\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R).$$

Proof. By Proposition 5.5.1, we can write

$$\partial_x^I \rho_f = p'(\varrho)\mathrm{K}_G^{\mathrm{free}}[\mathrm{J}_\varepsilon \partial_x^I \varrho] + R^I[\rho_f],$$

with $G(t,x,v) = \nabla_v f(t,x,v)$ and $\left\|R^I[\rho_f]\right\|_{\mathrm{L}^2(0,T,\mathrm{H}^1_x)} \leq \Lambda(T,R)$. Since the kernel G satisfies for $p > 1 + d$

$$\sup_{0\leq t\leq T} \|G\|_{\mathcal{H}^p_\sigma} \leq \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_\sigma)} \leq \Lambda(T,R),$$

we can use the estimate from Proposition 5.4.1 to get

$$\begin{aligned}\|\partial_x^I \rho_f\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} &\lesssim \|p'(\varrho)\|_{\mathrm{L}^2(0,T;\mathrm{L}^\infty)}\Lambda(T,R)\|\partial_x^I \varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} + \|R^I[\rho_f]\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \\ &\lesssim \mathrm{C}(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})})\Lambda(T,R) + \Lambda(T,R) \\ &\leq \Lambda(T,R),\end{aligned}$$

by Sobolev embedding, Proposition 5.A.3 and Lemma 5.2.20. The same argument applies for $\|\partial_x^I j_f\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}$. □

Our strategy to prove Proposition 5.5.1 goes as follows:

- first, we take derivatives in the Vlasov equation to obtain a system of coupled kinetic equations satisfied by the augmented unknown $(\partial_x^I \partial_v^J f_\varepsilon)_{|I|+|J|=m-1,m}$;
- next, we study the average in velocity of $\mathcal{F}$ by relying on Duhamel formula and the Lagrangian point of view of Section 5.3. We isolate the leading terms and prove estimates for the remainders, using crucially the techniques developed in Sections 5.3 and 5.4.

5.5.1 The integro-differential system for derivatives of moments

5.5.1.1 Applying derivatives

We start with the following algebraic lemma, where we apply $\partial_x^I \partial_v^J$ to the Vlasov equation. Let us recall the notation $\widehat{\alpha}^k$ and $\overline{\alpha}^k$ for shifted indices (see Definition 5.2.8).

Lemma 5.5.3. *For any $I = (i_1, \cdots, i_d), J = (j_1, \cdots, , j_d) \in \mathbb{N}^d$ such that $|I| + |J| \in \{m-1, m\}$ and for any smooth function $f(t,x,v)$, we have*

$$\left[\partial_x^I \partial_v^J, \mathcal{T}^{u,\varrho}_{\mathrm{reg},\varepsilon}\right] f = \partial_x^I \partial_v^J E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v f + \mathcal{M}^{I,J}\mathcal{F} + \mathcal{R}_1^{I,J} + \mathcal{R}_0^{I,J},$$

where

$$\mathcal{F} := \left(\partial_x^I \partial_v^J f\right)_{\substack{I,J\in\mathbb{N}^d,\\ |I|+|J|\in\{m-1,m\}}},$$

$$\mathcal{M}^{I,J}\mathcal{F} := \sum_{p=1}^d \mathbf{1}_{j_p\neq 0}\left(\partial_x^{\widehat{I}^p}\partial_v^{\overline{J}^p} f - \partial_x^I\partial_v^J f\right) + \mathbf{1}_{|I|>2}\sum_{\substack{0<\alpha<I\\ |\alpha|\in\{1,2\}}}\binom{I}{\alpha}\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\partial_x^{I-\alpha}\partial_v^J f$$
$$+ \mathbf{1}_{|I|=1,2}\partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\partial_v^J\nabla_v f,$$

$$\mathcal{R}_1^{I,J} := \mathbf{1}_{\substack{|I|>2\\ |J|\neq 0}}\partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\partial_v^J f + \mathbf{1}_{|I|>1}\sum_{\substack{0<\alpha<I\\ |\alpha|=m-1}}\binom{I}{\alpha}\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\partial_x^{I-\alpha}\partial_v^J f,$$

where

$$\left\|\mathcal{R}_1^{I,J}\right\|_{\mathrm{L}^2(0,T;\mathcal{H}^0_r)} \leq \Lambda(T,R), \tag{5.5.1}$$

and where $\mathcal{R}_0^{I,J}$ *is a remainder satisfying,*

$$\left\|\mathcal{R}_0^{I,J}\right\|_{\mathrm{L}^2(0,T;\mathcal{H}^1_r)} \leq \Lambda(T,R). \tag{5.5.2}$$

Proof. Using Lemma 5.2.9, we have

$$\partial_x^I\partial_v^J(\mathcal{T}^{u,\varrho}_{\mathrm{reg},\varepsilon} f) = \mathcal{T}^{u,\varrho}_{\mathrm{reg},\varepsilon}(\partial_x^I\partial_v^J f) + \sum_{p=1}^d \mathbf{1}_{j_p\neq 0}\left(\partial_x^{\widehat{I}^p}\partial_v^{\overline{J}^p} f - \partial_x^I\partial_v^J f\right) + \left[\partial_x^I\partial_v^J, E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\right] f.$$

Since the force $E^{u,\varrho}_{\mathrm{reg},\varepsilon}(t,x)$ does not depend on v, we expand the commutator as

$$\left[\partial_x^I\partial_v^J, E(t,x)\cdot\nabla_v\right] f = \mathbf{1}_{I\neq 0}\partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\partial_v^J\nabla_v f + \mathbf{1}_{|I|>1}\sum_{0<\alpha<I}\binom{I}{\alpha}\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\partial_x^{I-\alpha}\partial_v^J\nabla_v f.$$

Note that if $J\neq 0$ then $|I|\leq m-1$, and if $|I|=1,2$ then $J\neq 0$. The terms that cannot be considered as remainders in the last sum are those such that $|I|-|\alpha|+|J|+1\in\{m-1,m\}$, that is for $|\alpha|\in\{1,2\}$. We thus have

$$\left[\partial_x^I\partial_v^J, E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\right] f = \mathbf{1}_{\substack{|I|\neq 0\\ J=0}}\partial_x^I\partial_v^J E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v f + \mathbf{1}_{|I|=1,2}\partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\partial_v^J\nabla_v f$$
$$+ \mathbf{1}_{|I|>2}\sum_{\substack{0<\alpha<I\\ |\alpha|\in\{1,2\}}}\binom{I}{\alpha}\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\partial_x^{I-\alpha}\partial_v^J f$$
$$+ \mathcal{R}_1^{I,J} + \mathcal{R}_0^{I,J},$$

where

$$\mathcal{R}_1^{I,J} := \mathbf{1}_{\substack{|I|>2\\ |J|\neq 0}}\partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\partial_v^J\nabla_v f + \mathbf{1}_{|I|>1}\sum_{\substack{0<\alpha<I\\ |\alpha|=m-1}}\binom{I}{\alpha}\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\partial_x^{I-\alpha}\partial_v^J f,$$

$$\mathcal{R}_0^{I,J} := \mathbf{1}_{|I|>1}\sum_{\substack{0<\alpha<I\\ 3\leq|\alpha|\leq m-2}}\binom{I}{\alpha}\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\partial_x^{I-\alpha}\partial_v^J f.$$

Let us estimate the remainder $\mathcal{R}_0^{I,J}$ in $\mathcal{H}_r^1$: setting $\chi(v) = (1+|v|^2)^{r/2}$, we have

$$
\begin{aligned}
\|\mathcal{R}_0^{I,J}\|_{\mathcal{H}_r^1} \lesssim & \sum_{\substack{0<\alpha<I \\ 3\leq|\alpha|\leq m-2}} \|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha}\partial_v^J f\|_{\mathrm{L}^2_{x,v}} \\
& + \sum_{\substack{0<\alpha<I \\ 3\leq|\alpha|\leq m-2}} \sum_{k=1}^d \Big(\|\chi \partial_x^{\widehat{\alpha}^k} E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha}\partial_v^J f\|_{\mathrm{L}^2_{x,v}} + \|\chi \partial_x^{\alpha} E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \partial_{x_k}\nabla_v \partial_x^{I-\alpha}\partial_v^J f\|_{\mathrm{L}^2_{x,v}} \\
& \qquad + \|\chi \partial_x^{\alpha} E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \partial_{v_k}\nabla_v \partial_x^{I-\alpha}\partial_v^J f\|_{\mathrm{L}^2_{x,v}}\Big).
\end{aligned}
$$

- If $\frac{m+1}{2} \leq |\alpha| \leq m-2$, then for all k

$$
\begin{aligned}
\|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha}\partial_v^J f\|^2_{\mathrm{L}^2_{x,v}} &+ \|\chi \partial_x^{\widehat{\alpha}^k} E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha}\partial_v^J f\|^2_{\mathrm{L}^2_{x,v}} \\
&\leq \Big(\|\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|^2_{\mathrm{L}^2} + \|\partial_x^{\widehat{\alpha}^k} E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|^2_{\mathrm{L}^2}\Big) \int_{\mathbb{R}^d} \chi(v)^2 \|\nabla_v \partial_x^{I-\alpha}\partial_v^J f\|^2_{\mathrm{L}^\infty}\,\mathrm{d}v \\
&\lesssim \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|^2_{\mathrm{H}^{m-1}} \int_{\mathbb{R}^d} \chi(v)^2 \|\nabla_v \partial_v^J f\|^2_{\mathrm{H}^\sigma}\,\mathrm{d}v,
\end{aligned}
$$

provided that $\sigma > |I-\alpha| + d/2$. Since $|J| + 1 + |I| - |\alpha| + d/2 \leq m+1-|\alpha| + d/2 \leq \frac{m+1+d}{2}$ and since $m > 4+d$, we can find such a σ so that

$$
\|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha}\partial_v^J f\|_{\mathrm{L}^2_{x,v}} + \|\chi \partial_x^{\widehat{\alpha}^k} E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha}\partial_v^J f\|_{\mathrm{L}^2_{x,v}} \lesssim \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|_{\mathrm{H}^{m-1}} \|f\|_{\mathcal{H}_r^{m-1}}.
$$

Likewise, we have for all k

$$
\|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \partial_{x_k}\nabla_v \partial_x^{I-\alpha}\partial_v^J f\|^2_{\mathrm{L}^2_{x,v}} \leq \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|^2_{\mathrm{H}^{m-1}} \int_{\mathbb{R}^d} \chi(v)^2 \|\nabla_v \partial_v^J f\|^2_{\mathrm{H}^\sigma}\,\mathrm{d}v,
$$

provided that $\sigma > 1 + |I-\alpha| + d/2$. Since $|J| + 1 + 1 + |I| - |\alpha| + d/2 \leq m+2-|\alpha| + d/2 \leq \frac{m+3+d}{2}$ and since $m > 6+d$, there exists such a σ so that

$$
\|\chi \partial_x^\alpha E \cdot \partial_{x_k}\nabla_v \partial_x^{I-\alpha}\partial_v^J f\|_{\mathrm{L}^2_{x,v}} \lesssim \|E\|_{\mathrm{H}^{m-1}} \|f\|_{\mathcal{H}_r^{m-1}}.
$$

The same procedure can be applied for the terms $\|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \partial_{v_k}\nabla_v \partial_x^{I-\alpha}\partial_v^J f\|_{\mathrm{L}^2_{x,v}}$.

- If $3 \leq |\alpha| < \frac{m+1}{2}$, then

$$
\begin{aligned}
\|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha}\partial_v^J f\|_{\mathrm{L}^2_{x,v}} &+ \|\chi \partial_x^{\widehat{\alpha}^k} E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha}\partial_v^J f\|_{\mathrm{L}^2_{x,v}} \\
&\leq \Big(\|\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|_{\mathrm{L}^\infty} + \|\partial_x^{\widehat{\alpha}^k} E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|_{\mathrm{L}^\infty}\Big) \|\chi \nabla_v \partial_x^{I-\alpha}\partial_v^J f\|_{\mathrm{L}^2_{x,v}} \\
&\lesssim \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|_{\mathrm{H}^\sigma} \|f\|_{\mathcal{H}_r^{m-1}},
\end{aligned}
$$

provided that $\sigma > 1 + |\alpha| + d/2$. Since $1 + |\alpha| + d/2 \leq \frac{m+3+d}{2}$ and $m > 5+d$, we can find such a σ so that

$$
\|\chi \partial_x^{\widehat{\alpha}^k} E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha}\partial_v^J f\|_{\mathrm{L}^2_{x,v}} \lesssim \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|_{\mathrm{H}^{m-1}} \|f\|_{\mathcal{H}_r^{m-1}}.
$$

Likewise, we have for all k

$$
\|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \partial_{x_k}\nabla_v \partial_x^{I-\alpha}\partial_v^J f\|^2_{\mathrm{L}^2_{x,v}} \leq \|\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|_{\mathrm{L}^\infty} \|\chi \partial_x \nabla_v \partial_x^{I-\alpha}\partial_v^J f\|_{\mathrm{L}^2_{x,v}} \lesssim \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|_{\mathrm{H}^\sigma} \|f\|_{\mathcal{H}_r^{m-1}},
$$

provided that $\sigma > |I-\alpha| + d/2$. Since $|\alpha| + d/2 \leq \frac{m+2+d}{2}$ and $m > 4+d$, there exists such a σ so that

$$\|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x \partial_x^{I-\alpha} \partial_v^J f\|_{\mathrm{L}^2_{x,v}} \lesssim \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|_{\mathrm{H}^{m-1}} \|f\|_{\mathcal{H}^{m-1}_r}.$$

The same procedure can be applied for the terms $\|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \partial_{v_k} \nabla_v \partial_x^{I-\alpha} \partial_v^J f\|_{\mathrm{L}^2_{x,v}}$.

All in all, we have proven that for all $t \in [0,T]$

$$\|\mathcal{R}_0^{I,J}(t)\|_{\mathcal{H}^1_r} \lesssim \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}(t)\|_{\mathrm{H}^{m-1}} \|f(t)\|_{\mathcal{H}^{m-1}_r} \leq \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)} \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}(t)\|_{\mathrm{H}^{m-1}} \leq R\|E(t)\|_{\mathrm{H}^{m-1}}.$$

By the estimate (5.2.2) from Lemma 5.2.7 and by Lemma 5.2.20, we finally have

$$\left\|\mathcal{R}_0^{I,J}\right\|_{\mathrm{L}^2(0,T;\mathcal{H}^1_r)} \leq \Lambda(T,R).$$

With the same exact arguments, we easily obtain the fact that

$$\left\|\mathcal{R}_1^{I,J}\right\|_{\mathrm{L}^2(0,T;\mathcal{H}^0_r)} \leq \Lambda(T,R),$$

and this concludes the proof.

□

Remark 5.5.4. We will actually obtain an improved $\mathrm{L}^2(0,T;\mathcal{H}^1_r)$ estimate for the term $\mathcal{R}_1^{I,J}$ (or, more preciely, related terms) in the end of the current section.

We can see $\mathcal{M}^{I,J}\mathcal{F}$ appearing in Lemma 5.5.3 as a linear combination of $\mathcal{F}^{K,L} = \partial_x^K \partial_v^L f$. More precisely, we can write for all (I,J),

$$\mathcal{M}^{I,J}\mathcal{F} = \sum_{K,L} \mathcal{M}_{(I,J),(K,L)} \mathcal{F}^{K,L},$$

$$\mathcal{M}_{(I,J),(K,L)} := \sum_{p=1}^d \mathbf{1}_{j_p \neq 0} (\mathbf{1}_{(K,L)=(\widehat{I}^p, \overline{J}^p)} - \mathbf{1}_{(K,L)=(I,J)}) + \mathbf{1}_{|I|=1,2} \sum_{p=1}^d \mathbf{1}_{(K,L)=(0,\widehat{J}^p)} \partial_x^I (E^{u,\varrho}_{\mathrm{reg},\varepsilon})_p$$
$$+ \sum_{p=1}^d \sum_{\substack{0<\alpha<I \\ |\alpha| \in \{1,2\}}} \binom{I}{\alpha} \mathbf{1}_{(K,L)=(I-\alpha,\widehat{J}^p)} \partial_x^\alpha (E^{u,\varrho}_{\mathrm{reg},\varepsilon})_p.$$

Let us observe that the coefficient involved in the operator $\mathcal{M}$ involve only 0, 1 or 2 derivatives of the force field $E^{u,\varrho}_{\mathrm{reg},\varepsilon}(t,x)$, but nothing coming from f.

We finally introduce the following additional notations which will allow us to reformulate Lemma 5.5.3 in a compact way.

Definition 5.5.5. *We consider the following quantities:*

1. *$\mathcal{R}_0$ and $\mathcal{R}_1$ are the vectors defined by*

$$\mathcal{R}_0 = \left(\mathcal{R}_0^{I,J}\right)_{\substack{I,J \in \mathbb{N}^d, \\ |I|+|J| \in \{m-1,m\}}}, \quad \mathcal{R}_1 = \left(\mathcal{R}_0^{I,J}\right)_{\substack{I,J \in \mathbb{N}^d, \\ |I|+|J| \in \{m-1,m\}}};$$

2. *$\mathcal{M}$ is the linear map defined by*

$$\mathcal{M} = \left(\mathcal{M}_{(I,J),(K,L)}\right)_{\substack{(I,J),(K,L) \\ |I|+|J|,|K|+|L| \in \{m-1,m\}}};$$

3. *$\mathcal{L}$ is the vector defined by*

$$\mathcal{L} = \left(\partial_x^I \partial_v^J E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v f\right)_{\substack{I,J \in \mathbb{N}^d, \\ |I|+|J| \in \{m-1,m\}}}.$$

5.5.1.2 The semi-Lagrangian approach

If f satisfies the Vlasov equation (in a strong sense), we have $\partial_x^I \partial_v^J (\mathcal{T}^{u,\varrho}_{\mathrm{reg},\varepsilon} f) = 0$ for any I, J. We can use Lemma 5.5.3 to obtain the following coupled system of equations satisfied by the family $\mathcal{F} = (\partial_x^I \partial_v^J f)_{I,J}$:

$$\mathcal{T}^{u,\varrho}_{\mathrm{reg},\varepsilon} \mathcal{F} + \mathcal{M}\mathcal{F} + \mathcal{L} = -\mathcal{R}_0 - \mathcal{R}_1. \tag{5.5.3}$$

For any function $g(t,x,v)$, we set

$$\widetilde{g}(t,x,v) = g(t, \mathrm{X}^{t;0}(x,v), \mathrm{V}^{t;0}(x,v)),$$

where

$$s \mapsto \mathrm{Z}^{s;t}(x,v) = \Big(\mathrm{X}^{s;t}(x,v), \mathrm{V}^{s;t}(x,v)\Big)$$

is the solution to

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{X}^{s;t}(x,v) = \mathrm{V}^{s;t}(x,v), \quad \mathrm{X}^{t;t}(x,v) = x \in \mathbb{T}^d, \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}^{s;t}(x,v) = -\mathrm{V}^{s;t}(x,v) + E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, \mathrm{X}^{s;t}(x,v)), \quad \mathrm{V}^{t;t}(x,v) = v \in \mathbb{R}^d. \end{cases} \tag{5.5.4}$$

After the composition by $(t,x,v) \mapsto (t, \mathrm{X}^{t;0}(x,v), \mathrm{V}^{t;0}(x,v))$, we thus obtain by the method of characteristics

$$\partial_t \widetilde{\mathcal{F}} + \widetilde{\mathcal{M}}\widetilde{\mathcal{F}} + \widetilde{\mathcal{L}} = d\widetilde{\mathcal{F}} - \widetilde{\mathcal{R}}_0 - \widetilde{\mathcal{R}}_1. \tag{5.5.5}$$

To deal with the coupling matrix $\mathcal{M}$, we introduce the following object.

Definition 5.5.6. *For all (x,v) and $s,t \geq 0$, we define the resolvant operator $\mathfrak{N}^{s,t}(x,v)$ as the solution $s \mapsto \mathfrak{N}^{s,t}(x,v)$ of*

$$\begin{cases} \partial_s \mathfrak{N}^{s;t} + \Big[\mathcal{M} \circ \mathrm{Z}^{s;0} - d\mathrm{Id}\Big]\mathfrak{N}^{s;t} = 0, \\ \mathfrak{N}^{t;t} = \mathrm{Id}. \end{cases} \tag{5.5.6}$$

The resolvant is well-defined thanks to the Cauchy-Lipschitz theorem. We also have

$$\mathfrak{N}^{s;t}(x,v) = e^{d(s-t)}\mathrm{N}^{s;t}(x,v),$$

where

$$\begin{cases} \partial_s \mathrm{N}^{s;t} + \mathcal{M} \circ \mathrm{Z}^{s;0}\mathrm{N}^{s;t} = 0, \\ \mathrm{N}^{t;t} = \mathrm{Id}. \end{cases} \tag{5.5.7}$$

For the upcoming analysis, we need the following bounds on the resolvant.

Lemma 5.5.7. *For all $0 \leq k < m - 3 - d/2$, we have*

$$\sup_{0 \leq s,t \leq T} \|\mathrm{N}^{s;t}\|_{\mathrm{W}^{k,\infty}_{x,v}} + \sup_{0 \leq s,t \leq T} \|\partial_s \mathrm{N}^{s;t}\|_{\mathrm{W}^{k,\infty}_{x,v}} + \sup_{0 \leq s,t \leq T} \|\partial_t \mathrm{N}^{s;t}\|_{\mathrm{W}^{k,\infty}_{x,v}} \leq \Lambda(T,R).$$

Proof. We have

$$\mathrm{N}^{s;t} = \mathrm{Id} - \int_t^s [\mathcal{M} \circ \mathrm{Z}^{\tau;0}]\mathrm{N}^{\tau;t}\,\mathrm{d}\tau.$$

By the definition of coefficients of the matrix $\mathcal{M}$, we also have

$$\|\mathcal{M}\|_{\mathrm{L}^2(0,T;\mathrm{L}^\infty)} \lesssim \sup_{|\alpha| \leq 2} \|\partial_x^\alpha E\|_{\mathrm{L}^2(0,T;\mathrm{L}^\infty)} \leq \Lambda(T,R),$$

thanks to the estimate (5.2.2) from Lemma 5.2.7 and by Sobolev embedding (with $m-1 > 2+d/2$). Grönwall's lemma leads to the conclusion. □

We first obtain the following decomposition for the kinetic moments of the vector $\mathcal{F}$.

Lemma 5.5.8. *We have*

$$\int_{\mathbb{R}^d} \mathcal{F}(t,x,v)\,\mathrm{d}v = \mathcal{I}^0_{in}(t,x) + \mathcal{I}^0_{\mathcal{R}_0}(t,x) + \mathcal{I}^0_{\mathcal{R}_1}(t,x) \\ - \int_0^t e^{d(t-s)} \int_{\mathbb{R}^d} \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))\mathcal{L}(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s,$$

and

$$\int_{\mathbb{R}^d} v\otimes\mathcal{F}(t,x,v)\,\mathrm{d}v = \mathcal{I}^1_{in}(t,x) + \mathcal{I}^1_{\mathcal{R}_0}(t,x) + \mathcal{I}^1_{\mathcal{R}_1}(t,x) \\ - \int_0^t e^{d(t-s)} \int_{\mathbb{R}^d} v\otimes\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))\mathcal{L}(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s,$$

where

$$\begin{aligned}
\mathcal{I}^0_{in}(t,x) &:= e^{dt}\int_{\mathbb{R}^d} \mathrm{N}^{t;0}(\mathrm{Z}^{0;t}(x,v))\mathcal{F}_{|t=0}(\mathrm{Z}^{0;t}(x,v))\,\mathrm{d}v,\\
\mathcal{I}^0_{\mathcal{R}_j}(t,x) &:= -\int_0^t e^{d(t-s)}\int_{\mathbb{R}^d} \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))\mathcal{R}_j(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s, \quad j=0,1,\\
\mathcal{I}^1_{in}(t,x) &:= e^{dt}\int_{\mathbb{R}^d} v\otimes\mathrm{N}^{t;0}(\mathrm{Z}^{0;t}(x,v))\mathcal{F}_{|t=0}(\mathrm{Z}^{0;t}(x,v))\,\mathrm{d}v,\\
\mathcal{I}^1_{\mathcal{R}_j}(t,x) &:= -\int_0^t e^{d(t-s)}\int_{\mathbb{R}^d} v\otimes\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))\mathcal{R}_j(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s; \quad j=0,1.
\end{aligned}$$

Proof. We only explain the case of the moment of order 0, the other being similar. Starting from the equation (5.5.5) and using the resolvant operators $\mathfrak{N}$ and N defined in (5.5.6) and (5.5.7), we have

$$\begin{aligned}
\widetilde{\mathcal{F}}(t) &= \mathfrak{N}^{t,0}\widetilde{\mathcal{F}}_{|t=0} - \int_0^t \mathfrak{N}^{t,s}\left[\widetilde{\mathcal{L}}(s)+\widetilde{\mathcal{R}}_0(s)+\widetilde{\mathcal{R}}_1(s)\right]\,\mathrm{d}s\\
&= e^{dt}\mathrm{N}^{t;0}\widetilde{\mathcal{F}}_{|t=0} - \int_0^t e^{d(t-s)}\mathrm{N}^{t;s}\left[\widetilde{\mathcal{L}}(s)+\widetilde{\mathcal{R}}_0(s)+\widetilde{\mathcal{R}}_1(s)\right]\,\mathrm{d}s.
\end{aligned}$$

After a composition by the map $(t,x,v)\mapsto(t,\mathrm{X}^{0;t}(x,v),\mathrm{V}^{0;t}(x,v))$, we obtain

$$\mathcal{F}(t) = e^{dt}\left[\mathrm{N}^{t;0}\circ\mathrm{Z}^{0;t}\right]\mathcal{F}_{|t=0}\circ\mathrm{Z}^{0;t} - \int_0^t e^{d(t-s)}\left[\mathrm{N}^{t;s}\circ\mathrm{Z}^{0;t}\right]\Big(\mathcal{R}_0(s,\mathrm{Z}^{s;t})+\mathcal{R}_1(s,\mathrm{Z}^{s;t})\Big)\,\mathrm{d}s \\ - \int_0^t e^{d(t-s)}\left[\mathrm{N}^{t;s}\circ\mathrm{Z}^{0;t}\right]\mathcal{L}(s,\mathrm{Z}^{s;t})\,\mathrm{d}s.$$

We reach the desired conclusion by integrating in velocity.

□

5.5.2 First remainders

Let us show straightaway that some of the previous terms can be considered as remainders.

Lemma 5.5.9. *We have*

$$\|\mathcal{I}^0_{in}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)}\| + \|\mathcal{I}^1_{in}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R)\|f^{\mathrm{in}}\|_{\mathcal{H}^{m+1}_r}, \tag{5.5.8}$$

$$\|\mathcal{I}^0_{\mathcal{R}_0}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} + \|\mathcal{I}^1_{\mathcal{R}_0}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R), \tag{5.5.9}$$

$$\|\mathcal{I}^0_{\mathcal{R}_1}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} + \|\mathcal{I}^1_{\mathcal{R}_1}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R). \tag{5.5.10}$$

Proof. We shall first estimate the term $\mathcal{I}_{in}^0$. We have

$$|\mathcal{I}_{in}^0(t,x)| \lesssim e^{dT} \sup_{0\leq t\leq T} \|\mathrm{N}^{t;s}\|_{\mathrm{L}^\infty_{x,v}} \int_{\mathbb{R}^d} |\mathcal{F}_0(\mathrm{Z}^{0;t}(x,v))| \,\mathrm{d}v \lesssim \Lambda(T,R) \sum_{I,J} \int_{\mathbb{R}^d} |\partial_x^I \partial_v^J f^{\mathrm{in}}(\mathrm{Z}^{0;t}(x,v))| \,\mathrm{d}v,$$

thanks to Lemma 5.5.7 with $k=0$. Using the generalized Minkowski inequality and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\|\mathcal{I}_{in}^0\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} &\lesssim \Lambda(T,R) \sum_{I,J} \left\| \int_{\mathbb{R}^d} \|\partial_x^I \partial_v^J f^{\mathrm{in}}(\mathrm{Z}^{0;t}(\cdot,v))\|_{\mathrm{L}^2} \,\mathrm{d}v \right\|_{\mathrm{L}^2(0,T)} \\ &\lesssim \Lambda(T,R) \sum_{I,J} \left\| \left(\int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r |\partial_x^I \partial_v^J f^{\mathrm{in}}(\mathrm{Z}^{0;t}(x,v))|^2 \,\mathrm{d}x\,\mathrm{d}v \right)^{1/2} \right\|_{\mathrm{L}^2(0,T)},\end{aligned}$$

since $2r > d$. We then perform the change of variable $(x,v)\mapsto \mathrm{Z}^{0;t}(x,v)$ which yields

$$\begin{aligned}&\int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r |\partial_x^I \partial_v^J f^{\mathrm{in}}(\mathrm{Z}^{0;t}(x,v))|^2 \,\mathrm{d}x\,\mathrm{d}v \\ &\qquad\qquad \lesssim \Lambda(T,R) \int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|\mathrm{V}^{t;0}(x,v)|^2)^r |\partial_x^I \partial_v^J f^{\mathrm{in}}(x,v)|^2 \,\mathrm{d}x\,\mathrm{d}v.\end{aligned}$$

Since

$$\mathrm{V}^{t;0}(x,v) = e^{-t}v + \int_0^t e^{\tau-t} E^{u,\varrho}_{\mathrm{reg},\varepsilon}(\tau, \mathrm{X}^{\tau;t}(x,v)) \,\mathrm{d}\tau,$$

we have by Sobolev embedding

$$|\mathrm{V}^{t;0}(x,v)|^2 \leq |v|^2 + \left| \int_0^t \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}(\tau)\|_{\mathrm{L}^\infty} \,\mathrm{d}\tau \right|^2 \lesssim |v|^2 + T\|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})}.$$

By the estimate (5.2.2) from Lemma 5.2.7, we get

$$|\mathrm{V}^{t;0}(x,v)|^2 \leq |v|^2 \Lambda(T,R),$$

and then

$$\begin{aligned}\int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|\mathrm{V}^{t;0}(x,v)|^2)^r |\partial_x^I \partial_v^J f^{\mathrm{in}}(x,v)|^2 \,\mathrm{d}x\,\mathrm{d}v &\lesssim \Lambda(T,R) \int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r |\partial_x^I \partial_v^J f^{\mathrm{in}}(x,v)|^2 \,\mathrm{d}x\,\mathrm{d}v \\ &\lesssim \Lambda(T,R) \|f^{\mathrm{in}}\|^2_{\mathcal{H}^m_r}.\end{aligned}$$

This implies

$$\|\mathcal{I}_{in}^0\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R) \|f^{\mathrm{in}}\|_{\mathcal{H}^m_r}.$$

Likewise, using $2r > d+1$, we have

$$\|\mathcal{I}_{in}^1\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \lesssim \Lambda(T,R) \|f^{\mathrm{in}}\|_{\mathcal{H}^m_r}.$$

We also have

$$\left[\mathcal{I}_{in}^0\right]_{(I,J)}(t,x) = e^{dt} \int_{\mathbb{R}^d} \sum_{(K,L)} \mathrm{N}^{t;0}_{(I,J),(K,L)}(\mathrm{Z}^{0;t}(x,v)) \left[\mathcal{F}_{|t=0}\right]_{(K,L)} (\mathrm{Z}^{0;t}(x,v)) \,\mathrm{d}v,$$

and

$$\nabla_x \left[\mathcal{I}^0_{in}\right]_{(I,J)}(t,x) = e^{dt}\int_{\mathbb{R}^d}\sum_{(K,L)} \nabla_x \mathrm{Z}^{0;t}(x,v)\nabla_x \mathrm{N}^{t;0}_{(I,J),(K,L)}(\mathrm{Z}^{0;t}(x,v))\left[\mathcal{F}_{|t=0}\right]_{(K,L)}(\mathrm{Z}^{0;t}(x,v))\,\mathrm{d}v$$
$$+ e^{dt}\int_{\mathbb{R}^d}\sum_{(K,L)} \mathrm{N}^{t;0}_{(I,J),(K,L)}(\mathrm{Z}^{0;t}(x,v))\nabla_x \mathrm{Z}^{0;t}(x,v)\nabla_x\left[\mathcal{F}_{|t=0}\right]_{(K,L)}(\mathrm{Z}^{0;t}(x,v))\,\mathrm{d}v.$$

The same procedure as before, using Lemma 5.5.7 with $k=1$ and the pointwise bounds (5.3.12)-(5.3.13) from Remark 5.3.2, gives

$$\|\nabla_x \mathcal{I}^0_{in}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} + \|\nabla_x \mathcal{I}^1_{in}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \lesssim \Lambda(T,R)\|f^{\mathrm{in}}\|_{\mathcal{H}^{m+1}_r}.$$

We now estimate the terms $\mathcal{I}^0_{\mathcal{R}_0}$ and $\mathcal{I}^1_{\mathcal{R}_0}$. We use the same arguments as before, with an additional Cauchy-Schwarz inequality in time leading to

$$\begin{aligned}\|\mathcal{I}^0_{\mathcal{R}_0}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} &\leq \Lambda(T,R)\sum_{I,J}\left\|\int_0^t\int_{\mathbb{R}^d}\left\|\mathcal{R}^{I,J}_0(s,\mathrm{Z}^{s;t}(\cdot,v))\right\|_{\mathrm{L}^2}\,\mathrm{d}s\,\mathrm{d}v\right\|_{\mathrm{L}^2(0,T)}\\
&\leq \Lambda(T,R)\left\|\int_0^t \|\mathcal{R}_0(s)\|_{\mathcal{H}^0_r}\,\mathrm{d}s\right\|_{\mathrm{L}^2(0,T)}\\
&\leq \Lambda(T,R)T\,\|\mathcal{R}_0\|_{\mathrm{L}^2(0,T;\mathcal{H}^0_r)}\\
&\leq \Lambda(T,R),\end{aligned}$$

thanks to (5.5.2) in Lemma 5.5.3. We obtain the same result for $\mathcal{I}^1_{\mathcal{R}_0}$. Using again (5.5.2) for the first order derivative, we also have

$$\|\mathcal{I}_{\mathcal{R}_0}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} + \|\mathcal{I}^1_{\mathcal{R}_0}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \lesssim \Lambda(T,R).$$

For the estimate in $\mathrm{L}^2(0,T;\mathrm{L}^2)$ of the last term $\mathcal{I}^0_{\mathcal{R}_1}$, we end up with the conclusion thanks to the inequality (5.5.1) in Lemma 5.5.3, combined with the same arguments as before. □

Note that the previous lemma does not give any control on $\|\nabla_x \mathcal{I}^0_{\mathcal{R}_1}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}+\|\nabla_x \mathcal{I}^1_{\mathcal{R}_1}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}$. The treatment of these terms requires additional arguments that we will develop in the next subsections. For now, we merely state the result and postpone the proof to the end of Subsection 5.5.4.

Lemma 5.5.10. *We have*

$$\|\mathcal{I}^0_{\mathcal{R}_1}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} + \|\mathcal{I}^1_{\mathcal{R}_1}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R). \tag{5.5.11}$$

5.5.3 The leading terms and the conclusion

In this section, we focus on the following two terms:

$$\mathfrak{L}^0(t,x) := -\int_0^t e^{d(t-s)}\int_{\mathbb{R}^d}\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))\mathcal{L}(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s,$$

and

$$\mathfrak{L}^1(t,x) := -\int_0^t e^{d(t-s)}\int_{\mathbb{R}^d}v\otimes\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))\mathcal{L}(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s.$$

The goal is to prove that $\mathfrak{L}^0$ and $\mathfrak{L}^1$ can be decomposed as a sum of a leading term and a remainder in $\mathrm{L}^2_T\mathrm{H}^1$. This will imply the result stated in Proposition 5.5.1. Since the treament of $\mathfrak{L}^1$ is similar, we focus on $\mathfrak{L}^0$.

First, we have the following decomposition, which introduces several remainder terms that we shall estimate later on. Recall the definition of the straightening diffeomorphism $\psi_{s,t}(x,v)$ from Lemma 5.3.1.

Lemma 5.5.11. *For $|I| \leq m$, we have*

$$[\mathfrak{L}^0]_{(I,0)}(t,x) = -\int_0^t \int_{\mathbb{R}^d} \partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x-(t-s)v)\cdot \nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s + \mathscr{R}^{\mathrm{Diff}}_I(t,x) + \mathscr{R}^{\mathrm{Duha}}_{I,1}(t,x) + \mathscr{R}^{\mathrm{Duha}}_{I,2}(t,x),$$

with

$$\mathscr{R}^{\mathrm{Diff}}_I(t,x) := -\int_0^t \int_{\mathbb{R}^d} \Big[\partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x+(1-e^{t-s})v) - \partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x-(t-s)v)\Big]\cdot \nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s,$$

$$\mathscr{R}^{\mathrm{Duha}}_{I,1}(t,x) := -\sum_K \int_0^t \int_{\mathbb{R}^d} \partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x+(1-e^{t-s})v)\cdot \mathrm{H}^{K,I}(s,t,x,v)\,\mathrm{d}v\,\mathrm{d}s,$$

$$\mathscr{R}^{\mathrm{Duha}}_{I,2}(t,x) := -\sum_K \int_0^t \int_{\mathbb{R}^d} \partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x+(1-e^{t-s}v))\cdot \mathfrak{H}^{K,I}(t,s,x,v)\,\mathrm{d}v\,\mathrm{d}s,$$

where $\mathrm{H}^{K,I}$ and $\mathfrak{H}^{K,I}$ are vector fields defined by

$$\mathrm{H}^{K,I}(t,s,x,v) := \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)}\mathrm{J}^{s,t}(x,v)\nabla_v f(t,x,\psi_{s,t}(x,v)) - \nabla_v f(t,x,v), \tag{5.5.12}$$

and

$$\mathfrak{H}^{K,I}(t,s,x,v) := \int_s^t e^{d(t-\tau)}\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)} \left(\nabla_x f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v))) - \nabla_v f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v)))\right)\mathrm{J}^{s,t}(x,v)\,\mathrm{d}\tau, \tag{5.5.13}$$

with

$$\mathrm{J}^{s,t}(x,w) = |\det\left(\mathrm{D}_w\psi_{s,t}(x,w)\right)|.$$

Proof. Let $T \in \left(0, \min\Big(T_\varepsilon(R), \overline{T}(R)\Big)\right.$. We have for $|I| \leq m$

$$\begin{aligned}
&[\mathfrak{L}^0]_{(I,0)}(t,x)\\
&= -\sum_{(K,L)} \int_0^t e^{d(t-s)} \int_{\mathbb{R}^d} \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))_{(I,0),(K,L)}\partial_x^K\partial_v^L E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s,\mathrm{X}^{s;t}(x,v))\cdot\nabla_v f(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s\\
&= -\sum_{K} \int_0^t e^{d(t-s)} \int_{\mathbb{R}^d} \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))_{(I,0),(K,0)}\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s,\mathrm{X}^{s;t}(x,v))\cdot\nabla_v f(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s,
\end{aligned} \tag{5.5.14}$$

since $E^{u,\varrho}_{\mathrm{reg},\varepsilon}$ does not depend on the v variable. By Lemma 5.2.9, we know that

$$\mathcal{T}^{u,\varrho}_{\mathrm{reg},\varepsilon}(\nabla_v f) = \nabla_v f - \nabla_x f.$$

Invoking Duhamel formula, we get

$$\nabla_v f(t,x,v) = e^{d(t-s)}\nabla_v f(s,\mathrm{Z}^{s;t}(x,v)) + \int_s^t e^{d(t-\tau)}\left(\nabla_v f(\tau,\mathrm{Z}^{\tau;t}(x,v)) - \nabla_x f(\tau,\mathrm{Z}^{\tau;t}(x,v))\right)\mathrm{d}\tau,$$

and therefore

$$e^{d(t-s)}\nabla_v f(s,\mathrm{Z}^{s;t}(x,v)) = \nabla_v f(t,x,v) + \int_s^t e^{d(t-\tau)}\left(\nabla_x f(\tau,\mathrm{Z}^{\tau;t}(x,v)) - \nabla_v f(\tau,\mathrm{Z}^{\tau;t}(x,v))\right)\mathrm{d}\tau.$$

Inserting this expression in (5.5.14) yields

$$[\mathfrak{L}^0]_{(I,0)}(t,x) = \mathscr{L}_I^1(t,x) + \mathscr{L}_I^2(t,x), \tag{5.5.15}$$

where

$$\begin{aligned}
\mathscr{L}_I^1(t,x) &:= -\sum_K \int_0^t \int_{\mathbb{R}^d} \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))_{(I,0),(K,0)}\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s,\mathrm{X}^{s;t}(x,v))\cdot\nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s,\\
\mathscr{L}_I^2(t,x) &:= -\sum_K \int_0^t \int_{\mathbb{R}^d}\int_s^t e^{d(t-\tau)}\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))_{(I,0),(K,0)}\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s,\mathrm{X}^{s;t}(x,v))\\
&\qquad\qquad\cdot\left(\nabla_x f(\tau,\mathrm{Z}^{\tau;t}(x,v)) - \nabla_v f(\tau,\mathrm{Z}^{\tau;t}(x,v))\right)\mathrm{d}\tau\,\mathrm{d}v\,\mathrm{d}s.
\end{aligned}$$

Let us transform these two expressions in order to make the terms $\mathscr{R}_I^1(t,x)$, $\mathscr{R}_I^2(t,x)$ and $\mathscr{R}_I^3(t,x)$ appear.

• We first focus on $\mathscr{L}_I^1$, which will produce the leading term in the result. Using the change of variable $v=\psi_{s,t}(x,w)$ coming from Lemma 5.3.1, we have (since $t\leq \overline{T}(R)$)

$$\begin{aligned}
\mathscr{L}_I^1(t,x) = -\sum_K \int_0^t \int_{\mathbb{R}^d} &\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)}\\
&\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s,x+(1-e^{t-s})v)\cdot\nabla_v f(t,x,\psi_{s,t}(x,v))\mathrm{J}^{s,t}(x,v)\,\mathrm{d}v\,\mathrm{d}s,
\end{aligned}$$

where $\mathrm{J}^{s,t}(x,w) = |\det(\mathrm{D}_w\psi_{s,t}(x,w))|$. We obtain

$$\begin{aligned}
\mathscr{L}_I^1(t,x) = &-\sum_K \int_0^t \int_{\mathbb{R}^d} \partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s,x+(1-e^{t-s})v)\cdot\nabla_v f(t,x,\psi_{t,t}(x,v))\\
&\qquad\qquad \mathrm{N}^{t;t}(\mathrm{Z}^{0;t}(x,\psi_{t,t}(x,v)))_{(I,0),(K,0)}\mathrm{J}^{t,t}(x,v)\,\mathrm{d}v\,\mathrm{d}s\\
&-\sum_K \int_0^t \int_{\mathbb{R}^d} \partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s,x+(1-e^{t-s})v)\cdot \mathrm{H}^{K,I}(t,s,x,v)\,\mathrm{d}v\,\mathrm{d}s,
\end{aligned}$$

with

$$\begin{aligned}
\mathrm{H}^{K,I}(t,s,x,v) := &\,\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)}\mathrm{J}^{s,t}(x,v)\nabla_v f(t,x,\psi_{s,t}(x,v))\\
&-\mathrm{N}^{t;t}(\mathrm{Z}^{0;t}(x,\psi_{t,t}(x,v)))_{(I,0),(K,0)}\mathrm{J}^{t,t}(x,v)\nabla_v f(t,x,v).
\end{aligned}$$

Now observe that since $\mathrm{N}^{t;t}=\mathrm{Id}$ and $\psi_{t,t}=\mathrm{Id}$, we have

$$\nabla_v f(t,x,\psi_{t,t}(x,v))\mathrm{N}^{t;t}(\mathrm{Z}^{0;t}(x,\psi_{t,t}(x,v)))_{(I,0),(K,0)}\mathrm{J}^{t,t}(x,v) = \mathbf{1}_{I=K}\nabla_v f(t,x,v),$$

therefore

$$\begin{aligned}\mathscr{L}_I^1(t,x) &= -\int_0^t \int_{\mathbb{R}^d} \partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x+(1-e^{t-s})v)\cdot \nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s \\ &\quad - \sum_K \int_0^t \int_{\mathbb{R}^d} \partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x+(1-e^{t-s})v)\cdot \mathrm{H}^{K,I}(s,t,x,v)\,\mathrm{d}v\,\mathrm{d}s \\ &= -\int_0^t \int_{\mathbb{R}^d} \partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x-(t-s)v)\cdot \nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s + \mathscr{R}_I^{\mathrm{Diff}}(t,x) + \mathscr{R}_{I,1}^{\mathrm{Duha}}(t,x).\end{aligned}$$

• To deal with the term $\mathscr{L}_I^2$, we apply again the change of variable $v=\psi_{s,t}(x,w)$ from Lemma 5.3.1 and get (since $T \leq \overline{T}(R)$)

$$\mathscr{L}_I^2(t,x) = -\sum_K \int_0^t \int_{\mathbb{R}^d} \partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x+(1-e^{t-s}v))\cdot \mathfrak{H}^{K,I}(t,s,x,v)\,\mathrm{d}v\,\mathrm{d}s,$$

where

$$\begin{aligned}\mathfrak{H}^{K,I}(t,s,x,v) := \int_s^t e^{d(t-\tau)} \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)} \\ (\nabla_x f(\tau, \mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v))) - \nabla_v f(\tau, \mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v))))\,\mathrm{J}^{s,t}(x,v)\,\mathrm{d}\tau,\end{aligned}$$

which means that $\mathscr{L}_I^2(t,x) = \mathscr{R}_{I,2}^{\mathrm{Duha}}(t,x)$. Combining the previous decompositions eventually yields the conclusion.

□

We now have the following lemma, which is the continuation of Lemma 5.5.11, in which we express $[\mathfrak{L}^0]_{(I,0)}$ as a sum of a leading term and a remainder which is controlled in $\mathrm{L}^2_T\mathrm{H}^1$. The proof, which is rather lengthy and technical, is based on the smoothing estimates of Section 5.4. We refer to Subsection 5.5.4 where the proof is postponed.

Lemma 5.5.12. *We have for all $|I| = m$,*

$$[\mathfrak{L}^0]_{(I,0)}(t,x) = p'(\varrho(t,x)) \int_0^t \int_{\mathbb{R}^d} \nabla_x[\mathrm{J}_\varepsilon \partial_x^I \varrho](s, x-(t-s)v)\cdot \nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s + \mathscr{R}_I(t,x),$$

with

$$\|\mathscr{R}_I\|_{\mathrm{L}^2(0,T,\mathrm{H}^1)} \leq \Lambda(T,R).$$

We can finally proceed with the proof of Proposition 5.5.1.

Proof of Proposition 5.5.1. We only treat the case of $\partial_x^I \rho_f$, the one of $\partial_x^I j_f$ being similar. First invoking Lemma 5.5.8, we have

$$\partial_x^I \rho_f = [\mathcal{I}^0_{in}]_{(I,0)} + [\mathcal{I}^0_{\mathcal{R}_0}]_{(I,0)} + [\mathcal{I}^0_{\mathcal{R}_1}]_{(I,0)} + [\mathfrak{L}^0]_{(I,0)}.$$

Thanks to Lemmas 5.5.9–5.5.10–5.5.12, we infer

$$\begin{aligned}\partial_x^I \rho_f(t,x) &= p'(\varrho(t,x)) \int_0^t \int_{\mathbb{R}^d} \nabla_x[\mathrm{J}_\varepsilon \partial_x^I \varrho](s, x-(t-s)v)\cdot \nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s \\ &\quad + [\mathcal{I}^0_0]_{(I,0)}(t,x) + [\mathcal{I}^0_{\mathcal{R}_0}]_{(I,0)}(t,x) + [\mathcal{I}^0_{\mathcal{R}_1}]_{(I,0)}(t,x) + \mathscr{R}_I(t,x),\end{aligned}$$

where the previous remainders are estimated as

$$\begin{aligned}\|\mathcal{I}^0_{in}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)}\| &\leq \Lambda(T,R)\|f^{\mathrm{in}}\|_{\mathcal{H}^{m+1}_r}, \\ \|\mathcal{I}^0_{\mathcal{R}_0}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} + \|\mathcal{I}^0_{\mathcal{R}_1}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} &\leq \Lambda(T,R), \\ \|\mathscr{R}_I\|_{\mathrm{L}^2(0,T,\mathrm{H}^1)} &\leq \Lambda(T,R).\end{aligned}$$

This concludes the proof. □

5.5.4 Estimates of the last remainders

In this subsection, we mainly aim at giving a proof for Lemma 5.5.12 and Lemma 5.5.10, that we have previously stated. We shall rely on the crucial smoothing estimates derived in Section 5.4 to treat the different remainders. Broadly speaking, there are three types of terms requiring a gain of regularity:

- **Type I**: terms that will be treated thanks to the continuity estimate of Proposition 5.4.1, as in [HKR16], and of Proposition 5.4.4. It will be used for the terms $\nabla_x \mathcal{I}^0_{\mathcal{R}_1}$ and $\nabla_x \mathcal{I}^1_{\mathcal{R}_1}$ in the proof of Lemma 5.5.10.
- **Type II**: terms involving a kernel vanishing on the diagonal in time. They will be treated by the regularization estimate of Proposition 5.4.5. Such terms will appear in the proof of Lemma 5.5.12, as well as for the remainders $\mathscr{R}_I^{\mathrm{Diff}}, \mathscr{R}_{I,1}^{\mathrm{Duha}}$ and $\mathscr{R}_{I,2}^{\mathrm{Duha}}$.
- **Type III**: terms involving the difference between the integral operators $\mathrm{K}^{\mathrm{free}}$ and $\mathrm{K}^{\mathrm{fric}}$ (see Section 5.4. They will be handled thanks to the regularization estimate of Proposition 5.4.7. They will also appear in the treatment of the different remainders.

Let us mention that the previous gains require to control a fixed number of derivatives of the kernels that are involved (see Section 5.4).

Recall also the expression

$$E^{u,\varrho}_{\mathrm{reg},\varepsilon}(t,x) = u(t,x) - p'(\varrho)\nabla_x \left[\mathrm{J}_\varepsilon \varrho\right](t,x),$$

as well as Definition 5.2.8 for shifted indices. In order to check that the assumptions of the smoothing estimates of Section 5.4 are satisfied, it is convenient to have the following result.

Lemma 5.5.13. *For any $K \in \mathbb{N}^d$ such that $|K| > 0$, we have*

$$\begin{aligned}\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon} = -p'(\varrho)\nabla_x[\partial_x^K \mathrm{J}_\varepsilon \varrho] + \partial_x^K u - \sum_{\ell=0}^{\lfloor \frac{|K|-1}{2} \rfloor} \sum_{\substack{i=1,\cdots,d \\ \beta \in \mathbb{B}_K(i,\ell)}} \binom{K}{\widehat{\beta}^i} \nabla_x(\partial_x^{\overline{K}^i - \beta} \mathrm{J}_\varepsilon \varrho)\nabla_x(\partial_x^\beta p'(\varrho)) \cdot e_i \\ - \sum_{\ell=\lfloor \frac{|K|-1}{2} \rfloor +1}^{|K|-1} \sum_{\substack{i=1,\cdots,d \\ \beta \in \mathbb{B}_K(i,\ell)}} \binom{K}{\widehat{\beta}^i} \nabla_x(\partial_x^\beta p'(\varrho)) \cdot e_i \nabla_x(\partial_x^{\overline{K}^i - \beta} \mathrm{J}_\varepsilon \varrho),\end{aligned}$$

where

$$\mathbb{B}_K(i,\ell) := \left\{\beta \in \mathbb{N}^d \mid |\beta| = \ell, \quad 0 < \widehat{\beta}^i \leq K\right\}, \quad i = 1,\cdots,d, \quad \ell = 0,\cdots,|K|-1.$$

Proof. The proof directly follows from Leibniz formula, which provides

$$\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon} = -p'(\varrho)\nabla_x \partial_x^K \varrho + \partial_x^K u - \sum_{\ell=0}^{|K|-1} \sum_{\substack{i=1,\cdots,d \\ \beta \in \mathbb{B}_K(i,\ell)}} \binom{K}{\widehat{\beta}^i} \nabla_x(\partial_x^\beta p'(\varrho)) \cdot e_i \nabla_x(\partial_x^{\overline{K}^i - \beta} \mathrm{J}_\varepsilon \varrho),$$

and from which we infer the conclusion. □

In the next lemma, we show how to obtain the leading term of Lemma 5.5.12 (up to some good remainder) from the integral term of Lemma 5.5.11.

Lemma 5.5.14. *We have*

$$
\begin{aligned}
&-\int_0^t \int_{\mathbb{R}^d} \partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x-(t-s)v)\cdot \nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s \\
&\qquad = p'(\varrho(t,x)) \int_0^t \int_{\mathbb{R}^d} \nabla_x[\mathrm{J}_\varepsilon \partial_x^I \varrho](s,x-(t-s)v)\cdot \nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s + \widetilde{\mathscr{R}}_I(t,x),
\end{aligned}
$$

where the remainder $\widetilde{\mathscr{R}}_I$ *satisfies*

$$
\left\| \widetilde{\mathscr{R}}_I \right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R).
$$

Proof. Let us introduce the following vector fields

$$
G_1(s,t,x,v) = [p'(\varrho(s,x-(t-s)v) - p'(\varrho(t,x))]\nabla_v f(t,x,v),
$$

and

$$
\begin{aligned}
G_{3,i}^{\beta}(s,t,x,v) &:= \Big(\nabla_x(\partial_x^\beta p'(\varrho))(s,x-(t-s)v)\cdot e_i\Big)\nabla_v f(t,x,v), \\
G_{4,i}^{K,\beta}(s,t,x,v) &:= \Big(\nabla_x(\partial_x^{\overline{K}^i-\beta}\mathrm{J}_\varepsilon \varrho)(s,x-(t-s)v)\cdot \nabla_v f(t,x,v)\Big)e_i,
\end{aligned}
$$

for

$$
\beta \in \mathbb{B}_K(i,\ell) = \left\{ \beta \in \mathbb{N}^d \mid |\beta| = \ell, \;\; 0 < \widehat{\beta}^i \leq K \right\}, \quad i = 1,\cdots,d, \quad \ell = 0,\cdots,|K|-1.
$$

Thanks to Lemma 5.5.13, we can write

$$
\begin{aligned}
&-\int_0^t \int_{\mathbb{R}^d} \partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x-(t-s)v)\cdot \nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s \\
&= p'(\varrho(t,x)) \int_0^t \int_{\mathbb{R}^d} \nabla_x[\mathrm{J}_\varepsilon \partial_x^I \varrho](s,x-(t-s)v)\cdot \nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s \\
&\qquad\qquad + \mathbf{S_1}(t,x) + \mathbf{S_2}(t,x) + \mathbf{S_3}(t,x) + \mathbf{S_4}(t,x),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{S_1}(t,x) &:= \mathrm{K}^{\mathrm{free}}_{G_1}[\partial_x^I \mathrm{J}_\varepsilon \varrho], \\
\mathbf{S_2}(t,x) &:= -\int_0^t \int_{\mathbb{R}^d} \partial_x^I u(s,x-(t-s)v)\cdot \nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s, \\
\mathbf{S_3}(t,x) &:= \sum_{\ell=0}^{\lfloor \frac{|I|-1}{2} \rfloor} \sum_{\substack{i=1,\cdots,d \\ \beta \in \mathbb{B}_I(i,\ell)}} \binom{I}{\widehat{\beta}^i} \mathrm{K}^{\mathrm{free}}_{G_{3,i}^{\beta}}\left[\partial_x^{\overline{I}^i-\beta}\mathrm{J}_\varepsilon \varrho\right], \\
\mathbf{S_4}(t,x) &:= \sum_{\ell=\lfloor \frac{|I|-1}{2} \rfloor+1}^{|I|-1} \sum_{\substack{i=1,\cdots,d \\ \beta \in \mathbb{B}_I(i,\ell)}} \binom{I}{\widehat{\beta}^i} \mathrm{K}^{\mathrm{free}}_{G_{4,i}^{\overline{I}^i,\beta}}\left[\partial_x^{\beta} p'(\varrho)\right].
\end{aligned}
$$

The treatment of the term $\mathbf{S_2}$ will follow from a straightforward estimate. The terms $\mathbf{S_3}$ and $\mathbf{S_4}$ are terms of **Type I** and we will use the continuity estimates provided by Proposition 5.4.1. The term $\mathbf{S_1}$, which already contains I derivatives of ϱ, is a term of **Type II** (since $G_1(t,t,x,v) = 0$) and we will rely on the regularization estimate of Proposition 5.4.5.

• **Estimate of $\mathbf{S_1}$**: Since $G_1(t,t,x,v)=0$, we use Proposition 5.4.5 to get

$$\begin{aligned}\|\mathbf{S_1}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} &\lesssim (1+T)\sup_{0\leq s,t\leq T}\|\partial_s G_1(s,t)\|_{\mathcal{H}^\ell_\sigma}\|\partial_x^I\varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\\ &\lesssim (1+T)\sup_{0\leq s,t\leq T}\|\partial_s G_1(s,t)\|_{\mathcal{H}^\ell_\sigma}\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)},\end{aligned}$$

for $\ell>7+d$ and $\sigma>d/2$. A direct computation gives

$$\partial_s G_1(s,t,x,v)=p''(\varrho)\left[\partial_s\varrho+v\cdot\nabla_x\varrho\right](s,x-(t-s)v)\nabla_v f(t,x,v).$$

We thus have for all $0\leq s,t\leq T$

$$\begin{aligned}\|\partial_s G_1(s,t,x,v)\|^2_{\mathcal{H}^\ell_\sigma} &\lesssim C_T\sum_{|\mu|+|\nu|\leq\ell}\sum_{\gamma=0}^{\mu+\nu}\left(\|\partial_x^\gamma(p''(\varrho)\partial_s\varrho(s))\|^2_{\mathrm{L}^\infty}+\|\partial_x^\gamma(p''(\varrho)\nabla_x\varrho(s))\|^2_{\mathrm{L}^\infty}\right)\\ &\qquad\qquad\times\int_{\mathbb{T}^d\times\mathbb{R}^d}\langle v\rangle^{2\sigma+2}|\partial_{x,v}^{\mu+\nu-\gamma}\nabla_v f(t,x,v)|^2\,\mathrm{d}x\,\mathrm{d}v\\ &\lesssim C_T\left(\|p''(\varrho)\partial_s\varrho(s)\|^2_{\mathrm{H}^k}+\|p''(\varrho)\nabla_x\varrho(s)\|^2_{\mathrm{H}^k}\right)\|f(t)\|^2_{\mathcal{H}^{m-1}_{\sigma+1}}\\ &\lesssim C_T\|p''(\varrho(s))\|^2_{\mathrm{H}^k}\left(\|\partial_s\varrho(s)\|^2_{\mathrm{H}^k}+\|\nabla_x\varrho(s)\|^2_{\mathrm{H}^k}\right)\|f(t)\|^2_{\mathcal{H}^{m-1}_{\sigma+1}},\end{aligned}$$

for $k>\frac{d}{2}+\ell>\frac{d}{2}+1+d$ and $m-1>\ell>1+d$. Using the equation satisfied by ϱ, we get

$$\begin{aligned}\|p''(\varrho(s))\|_{\mathrm{H}^k}&\leq\Lambda\left(\|\varrho(s)\|_{\mathrm{H}^k}\right)\|\varrho(s)\|_{\mathrm{H}^k},\\ \|\partial_s\varrho(s)\|_{\mathrm{H}^k}+\|\nabla_x\varrho(s)\|_{\mathrm{H}^k}&\lesssim\|u\|_{\mathrm{H}^k}\|\nabla_x\varrho\|_{\mathrm{H}^k}+\left\|\frac{\varrho}{1-\rho_f}\right\|_{\mathrm{H}^k}\|\mathrm{div}_x\left(j_f-\rho_f u+u\right)\|_{\mathrm{H}^k}+\|\varrho\|_{\mathrm{H}^{k+1}},\end{aligned}$$

thanks to the algebra property of H^k. Taking $k+1<m-3$ and using the bootstrap assumption combined with (the proof) of Lemma 5.2.20, we obtain

$$\sup_{0\leq s,t\leq T}\|\partial_s G_1(s,t)\|_{\mathcal{H}^\ell_\sigma}\lesssim\Lambda(T,R).$$

• **Estimate of $\mathbf{S_2}$**: Using the generalized Minkowski inequality followed by the Cauchy-Schwarz inequality, we have for $r>d/2$

$$\begin{aligned}\|\mathbf{S_2}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} &\leq\left\|\int_0^t\left(\int_{\mathbb{T}^d\times\mathbb{R}^d}(1+|v|^2)^r|\partial_x^I u(s,x-(t-s)v)|^2|\nabla_v f(t,x,v)|^2\,\mathrm{d}x\,\mathrm{d}v\right)^{1/2}\mathrm{d}s\right\|_{\mathrm{L}^2(0,T)}\\ &\leq\left\|\int_0^t\left(\int_{\mathbb{T}^d\times\mathbb{R}^d}(1+|v|^2)^r|\partial_x^I u(s,y)|^2|\nabla_v f(t,y+(t-s)v,v)|^2\,\mathrm{d}y\,\mathrm{d}v\right)^{1/2}\mathrm{d}s\right\|_{\mathrm{L}^2(0,T)}\\ &\leq\sup_{t\in[0,T]}\left(\int_{\mathbb{R}^d}(1+|v|^2)^r\|\nabla_v f(t,\cdot,v)\|^2_{\mathrm{L}^\infty}\,\mathrm{d}v\right)^{1/2}\left\|\int_0^t\|\partial_x^I u(s)\|_{\mathrm{L}^2}\mathrm{d}s\right\|_{\mathrm{L}^2(0,T)}\\ &\lesssim\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)}\|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)},\end{aligned}$$

by Sobolev embedding in the last line, since $m-1>1+d/2$. This yields

$$\|\mathbf{S_2}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\lesssim\Lambda(T,R).$$

Likewise, we have since $m-1 > 2 + d/2$

$$\begin{aligned}\|\nabla_x \mathbf{S_2}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} &\lesssim \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)}\|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} + \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)}\|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})} \\ &\leq \Lambda(T,R),\end{aligned}$$

which gives the conclusion.

• **Estimate of $\mathbf{S_3}$ and $\mathbf{S_4}$**: for all $0 \leq |\beta| \leq \lfloor \frac{|I|-1}{2} \rfloor$ and $i = 1, \cdots, d$, we use Proposition 5.4.1 (after taking one derivative in space) to get

$$\left\| \mathrm{K}^{\mathrm{free}}_{G^{\beta}_{3,i}} \left[\partial_x^{\overline{I}^i - \beta} \mathrm{J}_\varepsilon \varrho \right] \right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \lesssim \sup_{0\leq s,t\leq T} \|G^\beta_{3,i}(t,s)\|_{\mathcal{H}^{\ell+1}_\sigma} \left\| \partial_x^{\overline{I}^i-\beta} \varrho \right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)},$$

for $\ell > 1 + d$ and $\sigma > d/2$ therefore

$$\left\| \mathrm{K}^{\mathrm{free}}_{G^{\beta}_{3,i}} \left[\partial_x^{\overline{I}^i - \beta} \mathrm{J}_\varepsilon \varrho \right] \right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \lesssim \sup_{0\leq s,t\leq T} \|G^\beta_{3,i}(t,s)\|_{\mathcal{H}^{\ell+1}_\sigma} \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}.$$

We have for all $0 \leq s,t \leq T$

$$\begin{aligned}&\|G^\beta_{3,i}(t,s)\|^2_{\mathcal{H}^{\ell+1}_\sigma} \\ &\leq C_T \sum_{|\mu|+|\nu|\leq \ell+1} \sum_{\gamma=0}^{\mu+\nu} \|\partial_x^\gamma \nabla_x (\partial_x^\beta p'(\varrho))(s)\|^2_{\mathrm{L}^\infty} \int_{\mathbb{T}^d\times\mathbb{R}^d} \langle v\rangle^{2\sigma} |\partial^{\mu+\nu-\gamma}_{x,v} \nabla_v f(t,x,v)|^2 \,\mathrm{d}x\,\mathrm{d}v \\ &\lesssim C_T \|p'(\varrho)\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)} \|f(t)\|^2_{\mathcal{H}^{m-1}_\sigma},\end{aligned}$$

provided that $m-1 \geq \ell+2$ and $k > \frac{d}{2}+|\gamma|+1+|\beta|$. Since $\ell > 1+d$ and $\frac{d}{2}+|\gamma|+1+|\beta| \leq \frac{d}{2}+\ell+2+\frac{m-1}{2}$, a condition such as $3d+9 < m$ ensures that

$$\|G^\beta_{3,i}(t,s)\|_{\mathcal{H}^{\ell+1}_\sigma} \lesssim C_T \|p'(\varrho)\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_\sigma)} \leq \Lambda(T,R).$$

thanks to (5.2.1) from Lemma 5.2.7, Sobolev embedding, and Lemma 5.2.20. We thus obtain

$$\left\| \mathrm{K}^{\mathrm{free}}_{G^{\beta}_{3,i}} \left[\partial_x^{\overline{I}^i - \beta} \mathrm{J}_\varepsilon \varrho \right] \right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R).$$

We proceed in the same way for $\mathbf{S_4}$: for all $\lfloor \frac{|I|-1}{2} \rfloor + 1 \leq |\beta| \leq |I|-1$ and $i = 1, \cdots, d$, we take one derivative in space and use Proposition 5.4.1 to write for $\ell > 1+d$ and $\sigma > d/2$

$$\left\| \mathrm{K}^{\mathrm{free}}_{G^{\overline{I}^i,\beta}_{4,i}} \left[\partial_x^\beta p'(\varrho) \right] \right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \sup_{0\leq s,t\leq T} \|G^{\overline{I}^i,\beta}_{4,i}(t,s)\|_{\mathcal{H}^{\ell+1}_\sigma} \|p'(\varrho)\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}.$$

The kernel $G^{\overline{I}^i,\beta}_{4,i}$ is estimated as before: using Leibniz rule, we have for all $0 \leq s,t \leq T$

$$\|G^{\overline{I}^i,\beta}_{4,i}(t,s)\|^2_{\mathcal{H}^{\ell+1}_\sigma} \lesssim C_T \|\varrho\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)} \|f(t)\|^2_{\mathcal{H}^{m-1}_\sigma},$$

provided that $m-1 > 2+\ell$ and $k > \frac{d}{2} + |\gamma| + 1 + |\overline{I}^i| - |\beta|$. Since $\ell > 1+d$ and

$$\frac{d}{2} + |\gamma| + 1 + |\overline{I}^i| - |\beta| \leq \frac{d + 2\ell + m + 1}{2},$$

a condition such as $3d+9 < m$ ensures that

$$\|G^{\overline{I}^i,\beta}_{4,i}(t,s)\|_{\mathcal{H}^{\ell+1}_\sigma} \lesssim C_T \|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_\sigma)}.$$

We thus obtain by Sobolev embedding, the bound (5.2.1) in Lemma 5.2.7 and Lemma 5.2.20.

$$\left\| \mathrm{K}^{\mathrm{free}}_{G^{\overline{I}^i,\beta}_{4,i}} \left[\partial_x^\beta p'(\varrho)\right] \right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R).$$

□

The remaining task is to show that the remainder terms $\mathscr{R}_I^{\mathrm{Diff}}, \mathscr{R}_{I,1}^{\mathrm{Duha}}$ and $\mathscr{R}_{I,2}^{\mathrm{Duha}}$ introduced in Lemma 5.5.11 are well-controlled in $\mathrm{L}^2(0,T;\mathrm{H}^1)$. For the first one, we have the following lemma.

Lemma 5.5.15. *We have*

$$\left\| \mathscr{R}_I^{\mathrm{Diff}} \right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R).$$

Proof. In view of Lemma 5.5.13, let us write for $0 \leq |\mu| \leq \lfloor \frac{|K|-1}{2} \rfloor$ and $\lfloor \frac{|K|-1}{2} \rfloor + 1 \leq |\nu| \leq |K|-1$

$$\begin{aligned}
&(\nabla_x[\partial_x^{\overline{K}^i-\mu}\mathrm{J}_\varepsilon\varrho]\nabla_x[\partial_x^\mu p'(\varrho)]\cdot e_i + \nabla_x[\partial_x^\mu p'(\varrho)]\cdot e_i \nabla_x[\partial_x^{\overline{K}^i-\nu}\mathrm{J}_\varepsilon\varrho])(s,x-(t-s)v)\cdot\nabla_v f(t,x,v) \\
&-(\nabla_x[\partial_x^{\overline{K}^i-\mu}\mathrm{J}_\varepsilon\varrho]\nabla_x[\partial_x^\mu p'(\varrho)]\cdot e_i + \nabla_x[\partial_x^\mu p'(\varrho)]\cdot e_i \nabla_x[\partial_x^{\overline{K}^i-\nu}\mathrm{J}_\varepsilon\varrho])(s,x+(1-e^{t-s})v)\cdot\nabla_v f(t,x,v) \\
&= \Big(\nabla_x[\partial_x^{\overline{K}^i-\mu}\mathrm{J}_\varepsilon\varrho](s,x-(t-s)v) - \nabla_x[\partial_x^{\overline{K}^i-\mu}\mathrm{J}_\varepsilon\varrho](s,x+(1-e^{t-s})v)\Big)\cdot G_{8,\mu,i}(s,t,x,v) \\
&\quad + \nabla_x[\partial_x^{\overline{K}^i-\mu}\mathrm{J}_\varepsilon\varrho](s,x+(1-e^{t-s})v)\cdot G_{9,\mu,i}(s,t,x,v) \\
&\quad + \nabla_x[\partial_x^\nu p'(\varrho)](s,x-(t-s)v)\cdot G^K_{10,\nu,i}(s,t,x,v) \\
&\quad + \Big(\nabla_x[\partial_x^\nu p'(\varrho)](s,x-(t-s)v) - \nabla_x[\partial_x^\nu p'(\varrho)](s,x+(1-e^{t-s})v)\Big)\cdot G^K_{11,\nu,i}(s,t,x,v).
\end{aligned}$$

where

$$\begin{aligned}
G_{8,\mu,i}(s,t,x,v) &:= \Big(\nabla_x[\partial_x^\mu p'(\varrho)](s,x-(t-s)v)\cdot e_i \nabla_v f(t,x,v)\Big), \\
G_{9,\mu,i}(s,t,x,v) &:= \Big((\nabla_x[\partial_x^\mu p'(\varrho)]\cdot e_i - \nabla_x[\partial_x^\mu p'(\varrho)](s,x+(1-e^{t-s})v)\cdot e_i)\nabla_v f(t,x,v)\Big), \\
G^K_{10,\nu,i}(s,t,x,v) &:= \Big((\nabla_x[\partial_x^{\overline{K}^i-\nu}\mathrm{J}_\varepsilon\varrho](s,x-(t-s)v) \\
&\qquad\qquad - \nabla_x[\partial_x^{\overline{K}^i-\nu}\mathrm{J}_\varepsilon\varrho](s,x+(1-e^{t-s})v))\cdot\nabla_v f(t,x,v)e_i\Big), \\
G^K_{11,\nu,i}(s,t,x,v) &:= \Big(\nabla_x[\partial_x^{\overline{K}^i-\nu}\mathrm{J}_\varepsilon\varrho](s,x+(1-e^{t-s})v)\cdot\nabla_x f(t,x,v)e_i\Big).
\end{aligned}$$

By Lemma 5.5.13, we can thus rewrite

$$\mathscr{R}_I^{\mathrm{Diff}} = \mathbf{S_5} + \mathbf{S_6} + \mathbf{S_7} + \mathbf{S_8} + \mathbf{S_9} + \mathbf{S_{10}} + \mathbf{S_{11}},$$

where

$$\mathbf{S_5}(t,x) := -\int_0^t \int_{\mathbb{R}^d} \left[\partial_x^I u(s,x+(1-e^{t-s})v) - \partial_x^I u(s,x-(t-s)v)\right]\cdot\nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s,$$

and

$$\mathbf{S_6} := \mathrm{K}^{\mathrm{free}}_{G_6}[\partial_x^I \mathrm{J}_\varepsilon \varrho] - \mathrm{K}^{\mathrm{fric}}_{G_6}[\partial_x^I \mathrm{J}_\varepsilon \varrho], \quad \ddot{o}G_6(t,s,x,v) := p'(\varrho(s, x-(t-s)v))\nabla_v f(t,x,v),$$

$$\mathbf{S_7} := \mathrm{K}^{\mathrm{fric}}_{G_7}[\partial_x^I \mathrm{J}_\varepsilon \varrho],\ G_7(t,s,x,v) := \Big[(p'(\varrho)(s, x-(t-s)v) - p'(\varrho)(s, x+(1-e^{t-s})v))\nabla_v f(t,x,v)\Big],$$

$$\mathbf{S_8} := \sum_{\ell=0}^{\lfloor \frac{|I|-1}{2} \rfloor} \sum_{\substack{i=1,\cdots,d \\ \beta \in \mathbb{B}_I(i,\ell)}} \binom{I}{\widehat{\beta}^i} \left(\mathrm{K}^{\mathrm{free}}_{G_{8,\beta,i}}[\partial_x^{\overline{I}^i-\beta} \mathrm{J}_\varepsilon \varrho] - \mathrm{K}^{\mathrm{fric}}_{G_{8,\beta,i}}[\partial_x^{\overline{I}^i-\beta} \mathrm{J}_\varepsilon \varrho] \right),$$

$$\mathbf{S_9} := \sum_{\ell=0}^{\lfloor \frac{|I|-1}{2} \rfloor} \sum_{\substack{i=1,\cdots,d \\ \beta \in \mathbb{B}_I(i,\ell)}} \binom{I}{\widehat{\beta}^i} \mathrm{K}^{\mathrm{fric}}_{G_{9,\beta,i}} \left[\partial_x^{\overline{I}^i-\beta} \mathrm{J}_\varepsilon \varrho\right],$$

$$\mathbf{S_{10}} := \sum_{\ell=\lfloor \frac{|I|-1}{2} \rfloor+1}^{|I|-1} \sum_{\substack{i=1,\cdots,d \\ \beta \in \mathbb{B}_I(i,\ell)}} \binom{I}{\widehat{\beta}^i} \mathrm{K}^{\mathrm{free}}_{G^I_{10,\beta,i}} \left[\partial_x^{\beta} p'(\varrho)\right],$$

$$\mathbf{S_{11}} := \sum_{\ell=\lfloor \frac{|I|-1}{2} \rfloor+1}^{|I|-1} \sum_{\substack{i=1,\cdots,d \\ \beta \in \mathbb{B}_I(i,\ell)}} \binom{I}{\widehat{\beta}^i} \left(\mathrm{K}^{\mathrm{free}}_{G^I_{11,\beta,i}}[\partial_x^{\beta} p'(\varrho)] - \mathrm{K}^{\mathrm{fric}}_{G^I_{11,\beta,i}}[\partial_x^{\beta} p'(\varrho)] \right).$$

Let us explain how to estimate each of these terms. The term $\mathbf{S_5}$ will be estimated by a direct proof. All the other terms actually require the use of the smoothing estimates coming either from Proposition 5.4.5 or Proposition 5.4.7:

- for $\mathbf{S_6}, \mathbf{S_8}$ and $\mathbf{S_{11}}$, we have the difference of the operators $\mathrm{K}^{\mathrm{free}}$ and $\mathrm{K}^{\mathrm{fric}}$ which appears: these terms are therefore of **Type III** and we will use Proposition 5.4.7;
- since the kernels G_7, G_9 and G_{10} vanish on the diagonal $\{s=t\}$, the terms $\mathbf{S_7}, \mathbf{S_9}$ and $\mathbf{S_{10}}$ are of **Type II** and we will appeal to Proposition 5.4.5.

Let us now turn to the estimates.

- **Estimate of $\mathbf{S_5}$**: the argument is the same as for $\mathbf{S_2}$ above. We obtain

$$\|\mathbf{S_5}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \lesssim \Lambda(T,R).$$

- **Estimate of $\mathbf{S_6}, \mathbf{S_8}$ and $\mathbf{S_{11}}$**: by Proposition 5.4.7, we have for all $\ell > 8+d$ and $\sigma > 1+d/2$

$$\begin{aligned}\|\mathbf{S_6}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^d))} &\lesssim \sup_{0\leq s,t\leq T} \|G_6(t,s)\|_{\mathcal{H}^\ell_\sigma} \|\partial_x^I \varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \\ &\leq \Lambda(T,R),\end{aligned}$$

thanks to the bound (5.2.1) in Lemma 5.2.7, Lemma 5.2.20 and provided that we can take $m-2 \geq \frac{d}{2}+8+d$.

Likewise, for $\mathbf{S_8}$, we use Proposition 5.4.7 to get

$$\|\mathbf{S_8}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \lesssim \sup_{\substack{0\leq s,t\leq T \\ 0\leq|\beta|\leq\lfloor\frac{|I|-1}{2}\rfloor \\ i=1,\cdots,d}} \|G_{8,\beta,i}(s,t)\|_{\mathcal{H}^\ell_\sigma} \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})},$$

for all $\ell > 8+d$ and $\sigma > 1+d/2$. As in the treatment of $\mathbf{S_3}$ above, we deduce

$$\|\mathbf{S_8}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \lesssim C_T \|p'(\varrho)\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_\sigma)} \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})} \leq \Lambda(T,R),$$

since $m > 3d+21$. We argue in the same way for $\mathbf{S_{11}}$ (see the treatment of $\mathbf{S_4}$ above) and obtain

$$\|\mathbf{S_{11}}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R).$$

• **Estimate of $\mathbf{S_7}, \mathbf{S_9}$ and $\mathbf{S_{10}}$:** we proceed exactly as in the estimate of $\mathbf{S}_1$ above, since $G_7(t,t,x,v) = 0$. Here, we have

$$\partial_s G_7(s,t,x,v) = \Big[p''(\varrho)\,[\partial_s\varrho + v\cdot\nabla_x\varrho]\,(s,x-(t-s)v) \\ - p''(\varrho)\left[\partial_s\varrho + e^{t-s}v\cdot\nabla_x\varrho\right](s,x+(1-e^{t-s})v)\Big]\nabla_v f(t,x,v).$$

Using triangle inequality with Proposition 5.4.5, we end up with

$$\|\mathbf{S_7}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R).$$

For $\mathbf{S_9}$, using $G_{9,\beta,i}(t,t,x,v) = 0$ for $0 \leq |\beta| \leq \lfloor \frac{|I|-1}{2} \rfloor$, we also have by Proposition 5.4.5

$$\left\| \mathrm{K}^{\mathrm{fric}}_{G_{9,\beta,i}}\left[\partial_x^{\overline{I}^i-\beta}\mathrm{J}_\varepsilon\varrho\right]\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \lesssim (1+T)\sup_{0\leq s,t\leq T}\|\partial_s G_{9,\beta,i}(s,t)\|_{\mathcal{H}^\ell_\sigma}\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})},$$

for all $\ell > 7+d$ and $\sigma > d/2$. We then proceed as in the estimate of $\mathbf{S}_1$ to deal with the time derivative, combined with what we have done for the estimate of $\mathbf{S}_3$ (since $m > 3d+19$ for instance) and get

$$\|\mathbf{S_9}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R)\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_\sigma)}\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})} \leq \Lambda(T,R).$$

Finally, we use the same exact arguments as before for $\mathbf{S_{10}}$ (see the estimate of $\mathbf{S}_4$ above for instance) to get

$$\|\mathbf{S_{10}}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R)\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})} \leq \Lambda(T,R).$$

□

The second term $\mathscr{R}^{\mathrm{Duha}}_{I,1}$ from Lemma 5.5.11 is estimated thanks to the following lemma.

Lemma 5.5.16. *We have*

$$\left\|\mathscr{R}^{\mathrm{Duha}}_{I,1}\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R).$$

Proof. Let us introduce the following vector fields

$$G^{K,I}_{14,i,\beta}(s,t,x,v) := \Big(\nabla_x(\partial_x^\beta p'(\varrho))(s,x+(1-e^{t-s}v))\cdot e_i\Big)\mathrm{H}^{K,I}(s,t,x,v),$$

$$G^{K,I}_{15,i,\beta}(s,t,x,v) := \Big(\nabla_x(\partial_x^{\overline{K}^i-\beta}\mathrm{J}_\varepsilon\varrho)(s,x+(1-e^{t-s}v))\cdot\mathrm{H}^{K,I}(s,t,x,v)\Big)e_i,$$

for

$$\beta \in \mathbb{B}_K(i,\ell) = \left\{\beta\in\mathbb{N}^d \mid |\beta| = \ell,\quad 0 < \widehat{\beta}^i \leq K\right\},\quad i=1,\cdots,d,\quad \ell = 0,\cdots,|K|-1,$$

and where we recall the expression of the kernel H defined in (5.5.12) by

$$\mathrm{H}^{K,I}(t,s,x,v) := \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)}\mathrm{J}^{s,t}(x,v)\nabla_v f(t,x,\psi_{s,t}(x,v)) - \nabla_v f(t,x,v).$$

By Lemma 5.5.13, we now decompose $\mathscr{R}_{I,1}^{\mathrm{Duha}}$ as

$$\mathscr{R}_{I,1}^{\mathrm{Duha}} = \mathbf{S_{12}} + \mathbf{S_{13}} + \mathbf{S_{14}} + \mathbf{S_{15}},$$

where

$$\mathbf{S_{12}}(t,x) := -\sum_K \int_0^t \int_{\mathbb{R}^d} \partial_x^K u(s, x+(1-e^{t-s})v) \cdot \mathrm{H}^{K,I}(t,s,x,v)\,\mathrm{d}v\,\mathrm{d}s,$$

and

$$\mathbf{S_{13}} := \sum_K \mathrm{K}^{\mathrm{fric}}_{G_{13}}\left[\partial_x^K \mathrm{J}_\varepsilon \varrho\right], \qquad G_{13}(t,s,x,v) := p'(\varrho)(s,x+(1-e^{t-s})v)\mathrm{H}^{K,I}(t,s,x,v),$$

$$\mathbf{S_{14}} := \sum_K \sum_{\ell=0}^{\lfloor \frac{|K|-1}{2} \rfloor} \sum_{\substack{i=1,\cdots,d \\ \beta\in\mathbb{B}_K(i,\ell)}} \binom{I}{\widehat{\beta}^i} \mathrm{K}^{\mathrm{free}}_{G^{K,I}_{14,i,\beta}}\left[\partial_x^{\overline{K}^i-\beta}\mathrm{J}_\varepsilon\varrho\right],$$

$$\mathbf{S_{15}} := \sum_K \sum_{\ell=\lfloor \frac{|K|-1}{2} \rfloor+1}^{|K|-1} \sum_{\substack{i=1,\cdots,d \\ \beta\in\mathbb{B}_K(i,\ell)}} \binom{K}{\widehat{\beta}^i} \mathrm{K}^{\mathrm{free}}_{G^{K,I}_{15,i,\beta}}\left[\partial_x^{\beta}p'(\varrho)\right].$$

Let us estimate each of theses terms. Note that $\mathrm{H}^{K,I}(t,t,x,v)=0$ so that the kernels appearing in $\mathbf{S_{13}}, \mathbf{S_{14}}$ and $\mathbf{S_{15}}$ vanish in the diagonal in time: these terms are therefore of **Type II** and we will use the regularization property from Proposition 5.4.5 to handle them.

- **Estimate of $\mathbf{S_{12}}$:** We proceed as in the estimate for $\mathbf{S}_2$ above and first get for $k > 1+\frac{d}{2}$

$$\|J_8\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \lesssim \sum_K \left\|\mathrm{H}^{K,I}\right\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_r)} \|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}.$$

Now observe that for s,t, we have

$$\begin{aligned}
&\left\|\mathrm{H}^{K,I}(s,t)\right\|_{\mathcal{H}^k_r} \\
&\lesssim \left\|J^{s,t}\right\|_{\mathrm{W}^{k,\infty}_{x,v}} \left\|\mathrm{N}^{s;t}(\mathrm{Z}^{0;t}(\cdot,\psi_{s,t}))\nabla_v f(t,\cdot,\psi_{s,t})\right\|_{\mathcal{H}^k_r} + \|f(t)\|_{\mathcal{H}^{k+1}_r} \\
&\lesssim \left\|J^{s,t}\right\|_{\mathrm{W}^{k,\infty}_{x,v}} \left(1+\left\|\mathrm{Z}^{0;t}\right\|_{\mathrm{W}^{k,\infty}_{x,v}}\right)\left(1+\|\psi_{s,t}\|_{\dot{\mathrm{W}}^{k,\infty}_{x,v}}\right)\left\|\mathrm{N}^{t;s}\right\|_{\mathrm{W}^{k,\infty}_{x,v}} \sum_{|\gamma|\leq k}\left\|\partial^\gamma_{x,v}(\nabla_v f)(t,\cdot,\psi_{s,t})\right\|_{\mathcal{H}^0_r} \\
&\quad + \|f(t)\|_{\mathcal{H}^{k+1}_r} \\
&\lesssim \Lambda(T,R)\sum_{|\gamma|\leq k}\left\|\partial^\gamma_{x,v}(\nabla_v f)(t,\cdot,\psi_{s,t})\right\|_{\mathcal{H}^0_r} + \|f(t)\|_{\mathcal{H}^{k+1}_r},
\end{aligned}$$

thanks to the estimate (5.3.5) of Lemma 5.3.1, Remark 5.3.2 and Lemma 5.5.7, and since $m > 4+d$. To handle the last sum, we write

$$\begin{aligned}
&\int_{\mathbb{T}^d\times\mathbb{R}^d} |\partial^\gamma_{x,v}\nabla_v f(t,x,\psi_{s,t}(x,v))|^2(1+|v|^2)^r\,\mathrm{d}x\,\mathrm{d}v \\
&\qquad\leq \int_{\mathbb{T}^d\times\mathbb{R}^d} |\partial^\gamma_{x,v}\nabla_v f(t,x,\psi_{s,t}(x,v))|^2(1+|v-\psi_{s,t}(x,v)|^2+|\psi_{s,t}(x,v)|^2)^r\,\mathrm{d}x\,\mathrm{d}v,
\end{aligned}$$

and use the change of variable $v\mapsto w=\psi_{s,t}(x,v)$ from Lemma 5.3.1, combined with the bounds (5.3.4) and (5.3.3) to get (choosing k such that $k+1\leq m-1$)

$$\left\|\mathrm{H}^{K,I}(s,t)\right\|_{\mathcal{H}^k_r} \leq \Lambda(T,R),$$

Taking a supremum in time we obtain

$$\|\mathbf{S_{12}}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R).$$

which yields the result.

• **Estimate of $\mathbf{S_{13}}, \mathbf{S_{14}}$ and $\mathbf{S_{15}}$**: Since $\mathrm{H}^{K,I}(t,t,x,v)=0$, we observe that $G_{13}(t,t,x,v)=0$. By Proposition 5.4.5, we therefore have for $\ell > 7+d$ and $\sigma > d/2$

$$\begin{aligned}
\|\mathbf{S_{13}}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} &\lesssim (1+T)\sum_K \sup_{0\leq s,t\leq T} \|\partial_s\left[p'(\varrho(s,x+(1-e^{t-s})v))\mathrm{H}^{K,I}(s,t)\right]\|_{\mathcal{H}^\ell_\sigma}\|\partial_x^K\varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\\
&\lesssim C_T\sum_K \sup_{0\leq s,t\leq T}\|p''(\varrho)\partial_s\varrho(s,x+(1-e^{t-s})v))\mathrm{H}^{K,I}(s,t)\|_{\mathcal{H}^\ell_\sigma}\|\partial_x^K\varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\\
&\quad + C_T\sum_K \sup_{0\leq s,t\leq T}\|p''(\varrho)\nabla_x\varrho(s,x+(1-e^{t-s})v))\mathrm{H}^{K,I}(s,t)\|_{\mathcal{H}^\ell_{\sigma+1}}\|\partial_x^K\varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\\
&\quad + C_T\sum_K \sup_{0\leq s,t\leq T}\|p'(\varrho(s,x+(1-e^{t-s})v))\partial_s\mathrm{H}^{K,I}(s,t)\|_{\mathcal{H}^\ell_\sigma}\|\partial_x^K\varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}.
\end{aligned}$$

The two first terms can be handled by similar arguments to the ones used for $\mathbf{S}_1$ and $\mathbf{S_{12}}$, a fixed number of derivatives being involved. For the last one, we proceed as for the other terms, combined with the arguments used for $\mathbf{S}_{12}$, and write for all t,s

$$\begin{aligned}
&\|p'(\varrho(s,x+(1-e^{t-s})v))\partial_s\mathrm{H}^{K,I}(s,t)\|_{\mathcal{H}^\ell_\sigma}\\
&\leq C_T\|p'(\varrho)\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\|\partial_s\mathrm{H}^{K,I}(s,t)\|_{\mathcal{H}^\ell_\sigma}\\
&\leq \Lambda(T,R)\Big(\|J^{s,t}\|_{\mathrm{W}^{\ell,\infty}_{x,v}}\|\nabla_v f(t)\|_{\mathcal{H}^\ell_\sigma}+\|\partial_s J^{s,t}\|_{\mathrm{W}^{\ell,\infty}_{x,v}}\|\nabla_v f(t,\cdot,\psi_{s,t})\|_{\mathcal{H}^\ell_\sigma}+\|\nabla_v f(t)\|_{\mathcal{H}^{\ell+1}_\sigma}\Big)\\
&\leq \Lambda(T,R),
\end{aligned}$$

since $m > 3d/2+11$. This yields

$$\|\mathbf{S_{13}}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R).$$

Next, for $\mathbf{S_{14}}$, we use the fact that $G^{K,I}_{14,i,\beta}(t,t,x,v)=0$ for $0\leq|\beta|\leq\lfloor\frac{|K|-1}{2}\rfloor$ and $i=1,\cdots,d$ so that by Proposition 5.4.5, we have

$$\left\|\mathrm{K}_{G^{K,I}_{14,i,\beta}}\left[\partial_x^{\overline{K}^i-\beta}\mathrm{J}_\varepsilon\varrho\right]\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \lesssim (1+T)\sup_{0\leq t,s\leq T}\left\|\partial_s G^{K,I}_{14,i,\beta}(s,t)\right\|_{\mathcal{H}^\ell_\sigma}\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)},$$

for $\ell > 7+d$ and $\sigma > d/2$. Next, we have for all t,s

$$\begin{aligned}
\left\|\partial_s G^{K,I}_{14,i,\beta}(s,t)\right\|_{\mathcal{H}^\ell_\sigma} \lesssim{}& \left\|\partial_s[\nabla_x(\partial_x^\beta p'(\varrho))(s,x+(1-e^{t-s})v)]\mathrm{H}^{K,I}(s,t)\right\|_{\mathcal{H}^\ell_\sigma}\\
&+\left\|\nabla_x(\partial_x^\beta p'(\varrho))(s,x+(1-e^{t-s})v)\partial_s\mathrm{H}^{K,I}(s,t)\right\|_{\mathcal{H}^\ell_\sigma}.
\end{aligned}$$

The first term can be handled by arguments similar to the ones used for $\mathbf{S_9}$, $\mathbf{S}_1$ and for $\mathbf{S_{12}}$. The second can be adressed by the same procedure where one relies on the arguments used for $\mathbf{S_{13}}$. Likewise, we obtain

$$\|\mathbf{S_{14}}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R).$$

□

We eventually estimate the third term $\mathscr{R}^{\mathrm{Duha}}_{I,2}$ from Lemma 5.5.11.

Lemma 5.5.17. *We have*

$$\left\|\mathscr{R}_{I,2}^{\mathrm{Duha}}\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R).$$

Proof. We proceed as before by introducing the vector fields

$$G_{18,i,\beta}^{K,I}(t,s,x,v) := \Big(\nabla_x(\partial_x^{\beta} p'(\varrho))(s,x+(1-e^{t-s}v))\cdot e_i\Big)\,\mathfrak{H}^{K,I}(s,t,x,v),$$

$$G_{19,i,\beta}^{K,I}(t,s,x,v) := \Big(\nabla_x(\partial_x^{\overline{K}^i-\beta}\mathrm{J}_\varepsilon\varrho)(s,x+(1-e^{t-s}v))\cdot\mathfrak{H}^{K,I}(s,t,x,v)\Big)e_i,$$

where

$$\beta\in\mathbb{B}_K(i,\ell)=\left\{\beta\in\mathbb{N}^d \mid |\beta|=\ell,\quad 0<\widehat{\beta}^i\leq K\right\},\quad i=1,\cdots,d,\quad \ell=0,\cdots,|K|-1.$$

Let us also recall the expression of the kernel $\mathfrak{H}$ defined in (5.5.13) by

$$\mathfrak{H}^{K,I}(t,s,x,v) := \int_s^t e^{d(t-\tau)}\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)}$$
$$\left(\nabla_x f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v)))-\nabla_v f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v)))\right)\mathrm{J}^{s,t}(x,v)\,\mathrm{d}\tau.$$

Thanks to Lemma 5.5.13, we can write

$$\mathscr{R}_{I,2}^{\mathrm{Duha}}=\mathbf{S}_{16}+\mathbf{S}_{17}+\mathbf{S}_{18}+\mathbf{S}_{19},$$

where

$$\mathbf{S}_{16}(t,x) := -\sum_K\int_0^t\int_{\mathbb{R}^d}\partial_x^K u(s,x+(1-e^{t-s}v))\cdot\mathfrak{H}^{K,I}(t,s,x,v)\,\mathrm{d}v\,\mathrm{d}s,$$

and

$$\mathbf{S}_{17} := \mathrm{K}^{\mathrm{fric}}[\partial_x^K\mathrm{J}_\varepsilon\varrho],\quad G_{17}(t,s,x,v) := p'(\varrho(s,x+(1-e^{t-s})v))\mathfrak{H}^{K,I}(t,s,x,v),$$

$$\mathbf{S}_{18} := \sum_K\sum_{\ell=0}^{\lfloor\frac{|K|-1}{2}\rfloor}\sum_{\substack{i=1,\cdots,d\\ \beta\in\mathbb{B}_K(i,\ell)}}\binom{K}{\widehat{\beta}^i}\mathrm{K}_{G_{18,i,\beta}^{K,I}}\left[\partial_x^{\overline{K}^i-\beta}\mathrm{J}_\varepsilon\varrho\right],$$

$$\mathbf{S}_{19} := \sum_K\sum_{\ell=\lfloor\frac{|K|-1}{2}\rfloor+1}^{|K|-1}\sum_{\substack{i=1,\cdots,d\\ \beta\in\mathbb{B}_K(i,\ell)}}\binom{K}{\widehat{\beta}^i}\mathrm{K}_{G_{19,i,\beta}^{K,I}}\left[\partial_x^{\beta}p'(\varrho)\right].$$

Let us estimate these terms, as we have done previously. Let us observe that $\mathfrak{H}^{K,I}(t,t,x,v)=0$ therefore the terms $\mathbf{S}_{17},\mathbf{S}_{18}$ and $\mathbf{S}_{19}$ are of **Type II** and we can rely on Proposition 5.4.5.

• **Estimate of $\mathbf{S}_{16}$**: We mainly proceed as for $\mathbf{S}_{12}$, the kernel being changed from H to $\mathfrak{H}$. Hence, we only have to give an estimate for $\left\|\mathfrak{H}^{K,I}(s,t)\right\|_{\mathcal{H}_r^k}$ (for k and r large enough). As in the estimate of $\mathbf{S}_{12}$, we use $\mathrm{W}_{x,v}^{k,\infty}$ bounds on $\psi_{s,t},\mathrm{Z}^{s;t}$ and $\mathrm{N}^{s;t}$ from Lemmas 5.3.1, Remark 5.3.2 and Lemma 5.5.7 to write

$$\begin{aligned}
&\left\|\mathfrak{H}^{K,I}(s,t)\right\|_{\mathcal{H}_r^k}^2\\
&\lesssim\Lambda(T,R)\left\|J^{s,t}\right\|_{\mathrm{W}_{x,v}^{k,\infty}}^2\left(1+\left\|\mathrm{N}^{s;t}\right\|_{\mathrm{W}_{x,v}^{k,\infty}}^2\right)\left\|\int_s^t\left[\nabla_x f(\tau,\mathrm{Z}^{\tau;t}(\cdot,\psi_{s,t}))-\nabla_v f(\tau,\mathrm{Z}^{\tau;t}(\cdot,\psi_{s,t}))\right]\mathrm{d}\tau\right\|_{\mathcal{H}_r^k}^2\\
&\lesssim\Lambda(T,R)\sum_{|\gamma|\leq k}\int_{\mathbb{T}^d\times\mathbb{R}^d}\langle v\rangle^{2r}\left|\int_s^t\left[\partial_{x,v}^{\gamma}(\nabla_x f)(\tau,\mathrm{Z}^{\tau;t}(\cdot,\psi_{s,t}))-\partial_{x,v}^{\gamma}(\nabla_v f)(\tau,\mathrm{Z}^{\tau;t}(\cdot,\psi_{s,t}))\right]\mathrm{d}\tau\right|^2\mathrm{d}x\,\mathrm{d}v.
\end{aligned}$$

By the the generalized Minkowski inequality, the last expression is bounded by

$$\sum_{|\gamma|\leq k}\left(\int_s^t\left(\int_{\mathbb{T}^d\times\mathbb{R}^d}\langle v\rangle^{2r}|\partial^\gamma_{x,v}(\nabla_x f)(\tau,\mathrm{Z}^{\tau;t}(\cdot,\psi_{s,t}))-\partial^\gamma_{x,v}(\nabla_v f)(\tau,\mathrm{Z}^{\tau;t}(\cdot,\psi_{s,t}))|^2\,\mathrm{d}x\,\mathrm{d}v\right)^{1/2}\mathrm{d}\tau\right)^2$$
$$\leq \Lambda(T,R)\sum_{|\gamma|\leq k}\left(\int_s^t\left(\int_{\mathbb{T}^d\times\mathbb{R}^d}\langle v\rangle^{2r}|\partial^\gamma_{x,v}(\nabla_x f)(\tau,x,v)-\partial^\gamma_{x,v}(\nabla_v f)(\tau,x,v)|^2\,\mathrm{d}x\,\mathrm{d}v\right)^{1/2}\mathrm{d}\tau\right)^2,$$

where we have performed the change of variable $v\mapsto w=\psi_{s,t}(x,v)$ from Lemma 5.3.1 followed by $(x,w)\mapsto \mathrm{Z}^{0;t}(x,w)$. Here, we have used the bounds on the Jacobian from (5.3.3), as well as the one on $|v-\psi_{s,t}(x,v)|$ via (5.3.4). It follows that

$$\left\|\mathfrak{H}^{K,I}(s,t)\right\|^2_{\mathcal{H}^k_r}\lesssim \Lambda(T,R)|t-s|^2\sup_{0\leq\tau\leq T}\left\{\|\nabla_x f(\tau)\|^2_{\mathcal{H}^k_r}+\|\nabla_v f(\tau)\|^2_{\mathcal{H}^k_r}\right\},$$

and therefore

$$\|\mathbf{S_{16}}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)}\leq\Lambda(T,R).$$

• **Estimate of $\mathbf{S}_{17},\mathbf{S}_{18}$ and $\mathbf{S}_{19}$**: We mainly proceed as for $\mathbf{S}_{13},\mathbf{S}_{14}$ and $\mathbf{S}_{15}$, using Proposition 5.4.5. As before, the kernel has just been changed from H to $\mathfrak{H}$. Hence, we only have to provide an estimate for $\|\partial_s\mathfrak{H}^{K,I}(s,t)\|_{\mathcal{H}^k_r}$ (for k and r large enough). We have

$$\begin{aligned}
\partial_s\mathfrak{H}^{K,I}(s,t,x,v)=\ &\partial_s J^{s,t}\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)}\\
&\quad\int_s^t e^{d(t-\tau)}\left[\nabla_x f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v)))-\nabla_v f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v)))\right]\mathrm{d}\tau\\
&+J^{s,t}\partial_s\left\{\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)}\right\}\\
&\quad\int_s^t e^{d(t-\tau)}\left[\nabla_x f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v)))-\nabla_v f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v)))\right]\mathrm{d}\tau\\
&+J^{s,t}\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)}\\
&\quad\int_s^t e^{d(t-\tau)}\partial_s\left[\nabla_x f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v)))-\nabla_v f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v)))\right]\mathrm{d}\tau\\
&-e^{d(t-s)}\mathrm{J}^{s,t}\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)}\\
&\qquad\left[\nabla_x f(s,\mathrm{Z}^{s;t}(x,\psi_{s,t}(x,v)))-\nabla_v f(s,\mathrm{Z}^{s;t}(x,\psi_{s,t}(x,v)))\right].
\end{aligned}$$

Using the same $\mathrm{W}^{k,\infty}_{x,v}$ bounds on $J^{s,t}$, $\mathrm{N}^{s;t}$ and $\dot{\mathrm{W}}^{k,\infty}_{x,v}$ bounds on $\psi_{s,t}$ as before, as well as on their time derivatives, we obtain

$$\|\partial_s\mathfrak{H}^{K,I}(s,t)\|_{\mathcal{H}^k_r}\leq\Lambda(T,R).$$

This eventually yields

$$\|\mathbf{S_{17}}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)}+\|\mathbf{S_{18}}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)}+\|\mathbf{S_{19}}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)}\leq\Lambda(T,R).$$

This ends the proof. □

We end this section by eventually giving a proof of Lemma 5.5.10. Let us mention that it only requires the use of the continuity estimate coming from Proposition 5.4.4 and not the ones of Proposition 5.4.5-5.4.7.

Proof of Lemma 5.5.10. In view of the estimate (5.5.10) of Lemma 5.5.9, we only have to prove that

$$\|\nabla_x \mathcal{I}^0_{\mathcal{R}_1}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} + \|\nabla_x \mathcal{I}^1_{\mathcal{R}_1}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R).$$

We only write the proof for $\nabla_x \mathcal{I}^0_{\mathcal{R}_1}$. Let us recall that

$$\mathcal{I}^0_{\mathcal{R}_1}(t,x) = -\int_0^t e^{d(t-s)} \int_{\mathbb{R}^d} \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))\mathcal{R}_1(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s$$

where

$$\mathcal{R}_1 = \left(\mathcal{R}_1^{K,L}\right)_{|K|+|L|\in\{m-1,m\}},$$

with

$$\mathcal{R}_1^{K,L} = \mathbf{1}_{\substack{|K|>2\\ L\neq 0}} \partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\partial_v^L f + \mathbf{1}_{|K|>1}\sum_{\substack{0<\alpha<K\\ |\alpha|=m-1}} \binom{L}{\alpha}\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\partial_x^{L-\alpha}\partial_v^K f.$$

We have for all (I,J)

$$\left[\mathcal{I}^0_{\mathcal{R}_1}\right]_{(I,J)}(t,x) = -\int_0^t e^{d(t-s)}\int_{\mathbb{R}^d}\sum_{(K,L)} \mathrm{N}^{t;s}_{(I,J),(K,L)}(\mathrm{Z}^{0;t}(x,v))\left[\mathcal{R}_1\right]_{(K,L)}(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s,$$

therefore

$$\begin{aligned}
&\nabla_x\left[\mathcal{I}^0_{\mathcal{R}_1}\right]_{(I,J)}(t,x)\\
&= -\int_0^t e^{d(t-s)}\int_{\mathbb{R}^d}\sum_{(K,L)}\nabla_x \mathrm{Z}^{0;t}(x,v)\nabla_x\mathrm{N}^{t;s}_{(I,J),(K,L)}(\mathrm{Z}^{0;t}(x,v))\left[\mathcal{R}_1\right]_{(K,L)}(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s\\
&\quad-\int_0^t e^{d(t-s)}\int_{\mathbb{R}^d}\sum_{(K,L)}\mathrm{N}^{t;s}_{(I,J),(K,L)}(\mathrm{Z}^{0;t}(x,v))\nabla_x\mathrm{Z}^{0;t}(x,v)\nabla_x\left[\mathcal{R}_1\right]_{(K,L)}(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s.
\end{aligned}$$

- For the first term, there is no derivative on $\mathcal{R}_1$. We can proceed exactly as for $\mathcal{R}_0$ in Lemma 5.5.9, relying on the estimate (5.5.10).
- The second term is more involved, since $\nabla_x\left[\mathcal{R}_1\right]_{(K,L)}$ contains several types of terms:
  - some are of the form
  $$\mathbf{1}_{\substack{|K|>2\\ L\neq 0}}\nabla_x[\nabla_v\partial_v^L f]\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}.$$
  They involve at most m derivatives of ϱ and at most $2+(m-3)=m-1$ derivatives of f. We can bound this term in $\mathrm{L}^2(0,T;\mathcal{H}^0_r)$, following the argument used for $\mathcal{R}_0$ in the proof of Lemma 5.5.3. Its contribution to $\nabla_x\mathcal{I}^0_{\mathcal{R}_1}$ is handled as in the proof of Lemma 5.5.9 (for the term $\nabla_x\mathcal{I}^0_{\mathcal{R}_0}$);
  - some are of the form
  $$\nabla_x[\nabla_v\partial_x^{L-\alpha}\partial_v^K f]\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon},$$
  with $|K|>1$ and $|\alpha|=m-1$. They involve at most m derivatives of ϱ and at most 3 derivatives of f. We can also bound this term in $\mathrm{L}^2(0,T;\mathcal{H}^0_r)$, following the argument used for $\mathcal{R}_0$ in the proof of Lemma 5.5.3. As before, we can follow the proof of Lemma 5.5.9 to control its contribution to $\nabla_x\mathcal{I}^0_{\mathcal{R}_1}$;

– some are of the form

$$\mathbf{1}_{\substack{|K|>2\\ L\neq 0}}\nabla_x[\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}]\nabla_v\partial_v^L f,$$

and of the form

$$\nabla_x[\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}]\nabla_v\partial_x^{L-\alpha}\partial_v^K f,$$

with $|K|>1$ and $|\alpha|=m-1$. These terms involve at most $m+1$ derivatives of ϱ and cannot be directly treated as the previous ones. We need to rely on the smoothing estimates from Section 5.4 (by making some terms of **Type I** appear).

We then focus on the two last types of terms, and we have to deal with

$$\mathbb{W}_1^{K,L}(t,x) := -\int_0^t e^{d(t-s)}\int_{\mathbb{R}^d} \mathrm{N}^{t;s}_{(I,J),(K,L)}(\mathrm{Z}^{0;t}(x,v)) \\ \nabla_x \mathrm{Z}^{0;t}(x,v)\mathbf{1}_{\substack{|K|>2\\ L\neq 0}}\nabla_x[\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}]\nabla_v\partial_v^L f(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s,$$

and

$$\mathbb{W}_2^{K,L}(t,x) := -\int_0^t e^{d(t-s)}\int_{\mathbb{R}^d} \mathrm{N}^{t;s}_{(I,J),(K,L)}(\mathrm{Z}^{0;t}(x,v)) \\ \nabla_x \mathrm{Z}^{0;t}(x,v)\mathbf{1}_{|K|>1}\sum_{\substack{0<\alpha<K\\ |\alpha|=m-1}}\binom{L}{\alpha}\nabla_x[\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}]\nabla_v\partial_x^{L-\alpha}\partial_v^K f(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s,$$

for $|K|+|L|\leq m$. Let us turn to the estimates of these terms in $\mathrm{L}^2(0,T;\mathrm{L}^2)$.

• **Estimate of** $\mathbb{W}_1^{K,L}$ **when** $\frac{m-2}{2}\leq |L| < m-2$: since $|K|+|L|\leq m$, this implies that $|K|\leq 1+m/2$. As in the proof of Lemma 5.5.9, we have by Cauchy-Schwarz inequality

$$\left\|\mathbb{W}_1^{K,L}\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R)\left\|\nabla_x\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\partial_v^L f\right\|_{\mathrm{L}^2(0,T;\mathcal{H}^0_r)},$$

and, by setting $\chi(v)=(1+|v|^2)^{r/2}$, we have

$$\|\chi\nabla_x\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\partial_v^L f\|_{\mathrm{L}^2_{x,v}} \lesssim \|\nabla_x\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|_{\mathrm{L}^\infty_x}\|\chi\nabla_v\partial_v^L f\|_{\mathrm{L}^2_{x,v}} \lesssim \|E\|_{\mathrm{H}^k}\|f\|_{\mathcal{H}^{m-1}_r},$$

with $k>\frac{d}{2}+1+|K|$. Since $d+6\leq m$, we can choose k such that

$$\left\|\mathbb{W}_1^{K,L}\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R)\|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})}\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)} \leq \Lambda(T,R),$$

and conclude as in the proof of Lemma 5.5.3. We obtain in that case

$$\left\|\mathbb{W}_1^{K,L}\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R).$$

• **Estimate of** $\mathbb{W}_1^{K,L}$ **when** $1\leq |L|\leq \frac{m-2}{2}-1$: in that case, we only have $|K|\leq m-1$. We shall rely on the smoothing estimate of Proposition 5.4.4 (to treat terms of **Type I**) and to recover the loss of the extra derivative on ϱ. To do so, we first write

$$|\mathbb{W}_1^{K,L}(t,x)| \\ \leq \Lambda(T,R)\left\|\nabla_x\mathrm{Z}^{0;t}\right\|_{\mathrm{L}^\infty_{x,v}}\sup_{0\leq s\leq T}\left\|\mathrm{N}^{t;s}\right\|_{\mathrm{L}^\infty_{x,v}}\mathbf{1}_{\substack{|K|>2\\ L\neq 0}}\left|\int_0^t\int_{\mathbb{R}^d}\nabla_x[\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}]\nabla_v\partial_v^L f(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s\right|,$$

and we perform the change of variable $v \mapsto \psi_{s,t}(x,w)$ from Lemma 5.3.1 in the last integral (since $t \leq \overline{T}(R)$) to get

$$
\begin{aligned}
&|\mathbb{W}_1^{K,L}(t,x)| \\
&\leq \Lambda(T,R)\mathbf{1}_{\substack{|K|>2 \\ L\neq 0}} \left| \int_0^t \int_{\mathbb{R}^d} \nabla_x[\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}](s, x+(1-e^{t-s})w)\nabla_v \partial_v^L f(s, \mathrm{Z}^{s;t}(x,\psi_{s,t}(x,w)))\,\mathrm{d}w\,\mathrm{d}s \right|,
\end{aligned}
$$

thanks to the bounds (5.3.3) of Lemma 5.3.1, Remark 5.3.2 and Lemma 5.5.7.

Relying on the decomposition of Lemma 5.5.13, we can make the operator $\mathrm{K}^{\mathrm{fric}}$ appear, hence dealing with terms of **Type I**, and use Proposition 5.4.4 (combined with the bounds on $\mathrm{Z}^{s;t}$ and $\psi_{s,t}$) to estimate the last integral in $\mathrm{L}^2(0,T;\mathrm{L}^2)$. The proof follows the same lines as the ones of Lemma 5.5.14. Note that Proposition 5.4.4 only requires an estimate of $\nabla_v \partial_v^L f$ in $\mathrm{L}^\infty(0,T;\mathcal{H}^s_r)$ with $s > 1+d$. Since $m > 2d+2$, we obtain in that case

$$
\left\| \mathbb{W}_1^{K,L} \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R).
$$

• **Estimate of** $\mathbb{W}_2^{K,L}$: to treat this term, we observe that it only involves at most 2 derivatives of f and the gradient of $m-1$ derivatives of the force field $E^{u,\varrho}_{\mathrm{reg},\varepsilon}$. Hence, we can exactly perform the same as previously for $\mathbb{W}_1^{K,L}$. We likewise obtain

$$
\left\| \mathbb{W}_2^{K,L} \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R).
$$

This concludes the proof of Lemma 5.5.10. □

5.6 Analysis of the fluid density

We pursue our goal which is to obtain an uniform control on $\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}$. In the previous section, we have related the kinetic moments ρ_f and j_f to the fluid density ϱ, up to some well controlled remainders. In this section, we build on these relations to further analyze ϱ.

We start by taking derivatives in the transport equation satisfied by ϱ. By using the key Proposition 5.5.1 (which was precisely the main outcome of Section 5.5), we obtain a factorization of the equation on the derivatives between

- a purely hyperbolic part, which is the transport operator $\partial_t + u \cdot \nabla_x$;
- an integro-differential operator part.

This is where the crucial Penrose condition (**P**) steps in and allows to justify that this last operator is actually elliptic in space-time and therefore can provide $\mathrm{L}^2_T\mathrm{L}^2_x$ estimates without loss. This relies on a semiclassical pseudodifferential analysis, in the spirit of [HKR16].

In this section, we will use the notation M_{in}, which stands for a positive constant depending only on the initial data.

5.6.1 Equation on the derivatives of the fluid density

For $T \in [0, \min\left(T_\varepsilon(R), \overline{T}(R)\right)$, the aim of this section is to prove the following proposition.

Proposition 5.6.1. *Setting $h = \partial_x^\alpha \varrho$ for $|\alpha| \leq m$, one has*

$$
\left(\mathrm{Id} - \frac{\varrho}{1-\rho_f}\mathrm{K}_G^{\mathrm{free}} \circ \mathrm{J}_\varepsilon\right)\left[\partial_t h + u\cdot\nabla_x h\right] = \mathcal{R}, \quad t \in (0,T), \tag{5.6.1}
$$

with $G(t,x,v) = p'(\varrho(t,x))\nabla_v f(t,x,v)$ *and*

$$\|\mathcal{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T, R, \|h(0)\|_{\mathrm{H}^1(\mathbb{T}^d)}).$$

We start with a commutation result when one takes derivatives in the equation on ϱ.

Lemma 5.6.2. *For all* $|\alpha| \leq m$, $\partial_x^\alpha \varrho$ *satisfies the equation*

$$\partial_t(\partial_x^\alpha \varrho) + u \cdot \nabla_x(\partial_x^\alpha \varrho) + \frac{\varrho}{1-\rho_f}\mathrm{div}_x \left[j_{\partial_x^\alpha f} - \rho_{\partial_x^\alpha f} u\right] = R^\alpha,$$

with

$$\|R^\alpha\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T, R).$$

Proof. Recall that by Lemma 5.2.2, the transport equation on ϱ reads as

$$\partial_t \varrho + u \cdot \nabla_x \varrho + \frac{\varrho}{1-\rho_f}\mathrm{div}_x \left[j_f - \rho_f u\right] = S, \quad S = -\frac{\varrho}{1-\rho_f}\mathrm{div}_x\, u.$$

We get, for all $\alpha \in \mathbb{N}^d$,

$$\partial_t(\partial_x^\alpha \varrho) + u \cdot \nabla_x(\partial_x^\alpha \varrho) + [\partial_x^\alpha, u\cdot\nabla_x]\varrho + \frac{\varrho}{1-\rho_f}\mathrm{div}_x\partial_x^\alpha\left(j_f - \rho_f u\right) + \left[\partial_x^\alpha, \frac{\varrho\,\mathrm{div}_x}{1-\rho_f}\right](j_f - \rho_f u) = \partial_x^\alpha S,$$

and therefore $\partial_x^\alpha \varrho$ satisfies the equation

$$\partial_t(\partial_x^\alpha \varrho) + u \cdot \nabla_x(\partial_x^\alpha \varrho) + \frac{\varrho}{1-\rho_f}\mathrm{div}_x \left[j_{\partial_x^\alpha f} - \rho_{\partial_x^\alpha f} u\right] = R^\alpha,$$

where R^α is a remainder defined by

$$\begin{aligned} R^\alpha &:= \partial_x^\alpha S - [\partial_x^\alpha, u\cdot\nabla_x]\varrho - \left[\partial_x^\alpha, \frac{\varrho\,\mathrm{div}_x}{1-\rho_f}\right](j_f - \rho_f u) + \frac{\varrho}{1-\rho_f}\mathrm{div}_x\left([\partial_x^\alpha, u]\rho_f\right) \\ &:= \partial_x^\alpha S + \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3. \end{aligned}$$

Let us estimate each of these terms in $\mathrm{L}^2(0,T;\mathrm{L}^2)$, for all $|\alpha| \leq m$.

- **Estimate of** $\partial_x^\alpha S$: we use the tame estimate from Proposition 5.A.2 to write

$$\|\partial_x^\alpha S\|_{\mathrm{L}^2} \lesssim \left\|\frac{\varrho}{1-\rho_f}\right\|_{\mathrm{L}^\infty} \|\mathrm{div}_x\, u\|_{\mathrm{H}^m} + \left\|\frac{\varrho}{1-\rho_f}\right\|_{\mathrm{H}^m} \|\mathrm{div}_x\, u\|_{\mathrm{L}^\infty} = \mathfrak{S}_1 + \mathfrak{S}_2.$$

Since $m-2 > d/2$, we have by Lemma 5.2.20 and (5.2.5)

$$\|\mathfrak{S}_1\|_{\mathrm{L}^2(0,T)} \lesssim \left\|\frac{1}{1-\rho_f}\right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})} \leq \Lambda\left(T,R\right).$$

For $\mathfrak{S}_2$, we combine the tame estimate from Proposition 5.A.2 with Lemma 5.A.5 which provides

$$\begin{aligned} \|\mathfrak{S}_2\|_{\mathrm{L}^2(0,T)} &\lesssim \|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} + \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} \left\|\frac{1}{1-\rho_f}\right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} \\ &\quad + \|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \left(\|\rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)}\right) \|\rho_f\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} \\ &\lesssim \Lambda(T,R) + \Lambda(T,R)\, \|\rho_f\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}, \end{aligned}$$

since $m-2>d/2$ and again thanks to Lemma 5.2.20 and (5.2.5). We obtain

$$\|\partial_x^\alpha S\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R),$$

by using Corollary 5.5.2.

- **Estimate of $\mathcal{C}_1$:** we have $-\mathcal{C}_1 = [\partial_x^\alpha, u\cdot](\nabla_x \varrho)$ therefore the commutator estimate from Proposition 5.A.1 yields

$$\|\mathcal{C}_1\|_{\mathrm{L}^2} \lesssim \|\nabla_x u\|_{\mathrm{L}^\infty} \|\nabla_x \varrho\|_{\mathrm{H}^{m-1}} + \|u\|_{\mathrm{H}^m} \|\nabla_x \varrho\|_{\mathrm{L}^\infty} \lesssim \|u\|_{\mathrm{H}^m} \|\varrho\|_{\mathrm{H}^m},$$

since $m>1+d/2$. We then obtain by Corollary 5.5.2

$$\|\mathcal{C}_1\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \lesssim \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} \leq \Lambda(T,R).$$

- **Estimate of $\mathcal{C}_2$:** we have

$$\mathcal{C}_2 = \left[\partial_x^\alpha, \frac{\varrho}{1-\rho_f}\right] (\mathrm{div}_x(j_f - \rho_f u)).$$

Applying the commutator estimate from Proposition 5.A.1, we get

$$\|\mathcal{C}_2\|_{\mathrm{L}^2} \lesssim \left\| \nabla_x \frac{\varrho}{1-\rho_f} \right\|_{\mathrm{L}^\infty} \|j_f - \rho_f u\|_{\mathrm{H}^m} + \left\| \frac{\varrho}{1-\rho_f} \right\|_{\mathrm{H}^m} \|\mathrm{div}_x(j_f - \rho_f u)\|_{\mathrm{L}^\infty} = \mathcal{C}_{2,1} + \mathcal{C}_{2,2}.$$

For $\mathcal{C}_{2,1}$, we infer from Sobolev embedding (since $m-2>1+d/2$)

$$\begin{aligned}
\left\| \nabla_x \frac{\varrho}{1-\rho_f} \right\|_{\mathrm{L}^\infty} &\leq \left\| \frac{1}{1-\rho_f} \right\|_{\mathrm{L}^\infty} \|\nabla_x \varrho\|_{\mathrm{L}^\infty} + \left\| \frac{\varrho}{(1-\rho_f)^2} \right\|_{\mathrm{L}^\infty} \|\nabla_x \rho_f\|_{\mathrm{L}^\infty} \\
&\lesssim \left\| \frac{1}{1-\rho_f} \right\|_{\mathrm{L}^\infty} \|\varrho\|_{\mathrm{H}^{m-2}} + \left\| \frac{1}{(1-\rho_f)} \right\|_{\mathrm{L}^\infty}^2 \|\rho\|_{\mathrm{H}^{m-2}} \|\nabla_x \rho_f\|_{\mathrm{L}^\infty} \\
&\leq \Lambda(T,R),
\end{aligned}$$

thanks to Lemma 5.2.20 and Corollary 5.5.2, therefore the tame estimate coming from Proposition 5.A.2 entails

$$\begin{aligned}
\|\mathcal{C}_{2,1}\|_{\mathrm{L}^2(0,T)} \lesssim \Lambda(T,R) \Big(&\|j_f\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} + \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}_r^{m-1})} \|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} \\
&+ \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} \|\rho_f\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} \Big),
\end{aligned}$$

since $m-1>d/2$. Invoking Lemma 5.2.1 and Corollary 5.5.2, we get

$$\|\mathcal{C}_{2,1}\|_{\mathrm{L}^2(0,T)} \leq \Lambda(T,R).$$

For $\mathcal{C}_{2,2}$, we observe that since $m-1>1+d/2$

$$\|\mathrm{div}_x(j_f - \rho_f u)\|_{\mathrm{L}^\infty} \lesssim \|j_f\|_{\mathrm{H}^{m-1}} + \|\rho_f\|_{\mathrm{H}^{m-1}} \|u\|_{\mathrm{H}^{m-1}},$$

therefore we obtain

$$\|\mathcal{C}_{2,2}\|_{\mathrm{L}^2(0,T)} \leq \Lambda(T,R),$$

by using what we have done for $\mathfrak{S}_2$ above.

- **Estimate of $\mathcal{C}_3$:** we have

$$\mathcal{C}_3 = \frac{\varrho}{1-\rho_f}\Big([\partial_x^\alpha, \mathrm{div}_x\, u\cdot](\rho_f) + [\partial_x^\alpha, u\cdot](\nabla_x \rho_f)\Big) = \frac{\varrho}{1-\rho_f}\left(\mathcal{C}_{3,1} + \mathcal{C}_{3,2}\right).$$

Applying Proposition 5.A.1, we get for $m+1 > 2 + d/2$

$$\|\mathcal{C}_{3,1}\|_{\mathrm{L}^2} \lesssim \|\nabla_x \mathrm{div}_x\, u\|_{\mathrm{L}^\infty}\|\rho_f\|_{\mathrm{H}^{m-1}} + \|\mathrm{div}_x\, u\|_{\mathrm{H}^m}\|\rho_f\|_{\mathrm{L}^\infty} \lesssim \|u\|_{\mathrm{H}^{m+1}}\|\rho_f\|_{\mathrm{H}^{m-1}},$$

$$\|\mathcal{C}_{3,2}\|_{\mathrm{L}^2} \lesssim \|\nabla_x u\|_{\mathrm{L}^\infty}\|\nabla_x \rho_f\|_{\mathrm{H}^{m-1}} + \|u\|_{\mathrm{H}^m}\|\nabla_x\rho_f\|_{\mathrm{L}^\infty} \lesssim \|u\|_{\mathrm{H}^m}\|\rho_f\|_{\mathrm{H}^m} + \|u\|_{\mathrm{H}^m}\|\rho_f\|_{\mathrm{H}^{m-1}}.$$

Taking the L^2-norm in time and using Lemma 5.2.1, we have for $\mathcal{C}_{3,1}$

$$\|\mathcal{C}_{3,1}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \lesssim \|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}\|\rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})} \leq \Lambda(T,R),$$

while for $\mathcal{C}_{3,2}$, we have

$$\begin{aligned}\|\mathcal{C}_{3,2}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} &\lesssim \|u\|_{\mathrm{L}^\infty(0,T,\mathrm{H}^m)}\|\rho_f\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} + \|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}\|\rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})}\\ &\leq \Lambda(T,R),\end{aligned}$$

thanks to Corollary 5.5.2. □

Let us now transform the equation on the derivatives of ϱ obtained in Lemma 5.6.2. To ease readibility let us momentaneously set

$$\mathrm{K}_{1,G}^{\mathrm{free}}[F](t,x) := \int_0^t \int_{\mathbb{R}^d} v[\nabla_x F](s, x-(t-s)v)\cdot G(t,s,x,v)\,\mathrm{d}v\,\mathrm{d}s. \tag{5.6.2}$$

Lemma 5.6.3. *For $h = \partial_x^\alpha \varrho$ with $|\alpha| \leq m$, one has*

$$\partial_t h + u\cdot\nabla_x h + \frac{\varrho}{1-\rho_f}\mathrm{div}_x\Big[\mathrm{K}_{1,G}^{\mathrm{free}}(\mathrm{J}_\varepsilon h) - \mathrm{K}_G^{\mathrm{free}}(\mathrm{J}_\varepsilon h)u\Big] = \mathcal{R},$$

with

$$G(t,x,v) := p'(\varrho(t,x))\nabla_v f(t,x,v),$$

and where the remainder $\mathcal{R}$ satisfies

$$\|\mathcal{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T,R).$$

Proof. By Lemma 5.6.2, we have

$$\partial_t h + u\cdot\nabla_x h + \frac{\varrho}{1-\rho_f}\mathrm{div}_x\left[j_{\partial_x^\alpha f} - \rho_{\partial_x^\alpha f}u\right] = R^\alpha,$$

with

$$\|R^\alpha\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R).$$

Thanks to Proposition 5.5.1, we can write

$$\rho_{\partial_x^\alpha f} = \mathrm{K}_G^{\mathrm{free}}(\mathrm{J}_\varepsilon \partial_x^\alpha \varrho) + \mathrm{R}^\alpha[\rho_f], \quad j_{\partial_x^\alpha f} = \mathrm{K}_{1,G}^{\mathrm{free}}(\mathrm{J}_\varepsilon \partial_x^\alpha\varrho) + \mathrm{R}^\alpha[j_f],$$

with $G(t,x,v) := p'(\varrho(t,x))\nabla_v f(t,x,v)$ and where

$$\|\mathrm{R}^\alpha[\rho_f]\|_{\mathrm{L}^2(0,T,\mathrm{H}^1)} \leq \Lambda(T,R), \quad \|\mathrm{R}^\alpha[j_f]\|_{\mathrm{L}^2(0,T,\mathrm{H}^1)} \leq \Lambda(T,R).$$

We obtain

$$\partial_t h + u\cdot\nabla_x h + \frac{\varrho}{1-\rho_f}\mathrm{div}_x\Big[\mathrm{K}_{1,G}^{\mathrm{free}}(\mathrm{J}_\varepsilon h) - \mathrm{K}_G^{\mathrm{free}}(\mathrm{J}_\varepsilon h)u\Big] = R^\alpha - \mathrm{div}_x\Big[\mathrm{R}^\alpha[j_f] - \mathrm{R}^\alpha[\rho_f]u\Big].$$

Thanks to the aforementioned estimates in $\mathrm{L}^2_T\mathrm{H}^1_x$, we obtain the desired estimate on the remainder. □

Remark 5.6.4. In the sequel, we shall rely on the following estimate: for all $\ell > 0$ and $\sigma > 0$ such that $\ell < m - d/2 - 2$, we have

$$\sup_{0 \leq t \leq T} \|G(t)\|_{\mathcal{H}^\ell_\sigma} \leq \Lambda(T, R), \tag{5.6.3}$$

where we recall that

$$G(t, x, v) = p'(\varrho(t, x))\nabla_v f(t, x, v).$$

Indeed, we have

$$\begin{aligned}\|p'(\varrho(t))\nabla_v f(t)\|^2_{\mathcal{H}^\ell_\sigma} &\lesssim \sum_{|\mu|+|\nu|\leq \ell} \sum_{\gamma=0}^{\mu+\nu} \|\partial_x^\gamma (p'(\varrho(t)))\|^2_{\mathrm{L}^\infty} \int_{\mathbb{T}^d \times \mathbb{R}^d} \langle v \rangle^{2\sigma} |\partial_{x,v}^{\mu+\nu-\gamma} \nabla_v f(t, x, v)|^2 \, \mathrm{d}x \, \mathrm{d}v \\ &\lesssim \|p'(\varrho(t))\|^2_{\mathrm{H}^k} \|f(t)\|^2_{\mathcal{H}^{m-1}_\sigma},\end{aligned}$$

if $m - 1 \geq \ell$ and $k > \frac{d}{2} + \ell$. Invoking Lemma 5.2.20 by choosing also $k \leq m - 2$, we obtain the estimate (5.6.3).

Our goal is now to understand the term

$$\mathrm{div}_x \Big[\mathrm{K}^{\mathrm{free}}_{1,G}(\mathrm{J}_\varepsilon h) - \mathrm{K}^{\mathrm{free}}_G(\mathrm{J}_\varepsilon h) u\Big].$$

We will show that up to good remainders, it is related to the transport part $\partial_t h + u \cdot \nabla_x h$ appearing in the equation of Lemma 5.6.3. This is the object of the following Lemmas 5.6.5–5.6.6 which are crucial commutation results.

Lemma 5.6.5. *For all $|\alpha| \leq m$ and $h = \partial_x^\alpha \varrho$, there holds*

$$\mathrm{div}_x(\mathrm{K}^{\mathrm{free}}_G[\mathrm{J}_\varepsilon h] u) = \mathrm{K}^{\mathrm{free}}_G[\mathrm{J}_\varepsilon (u \cdot \nabla_x h)] + \mathcal{R},$$

with

$$\|\mathcal{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T, R).$$

Proof. First, let us prove that for all smooth function $\mathfrak{h}(t, x)$, we have

$$\mathrm{div}_x(\mathrm{K}^{\mathrm{free}}_G[\mathfrak{h}] u) = \mathrm{K}^{\mathrm{free}}_G[(u \cdot \nabla_x \mathfrak{h})] + \mathcal{R}, \tag{5.6.4}$$

with

$$\|\mathcal{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T, R, \|\mathfrak{h}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}). \tag{5.6.5}$$

We have

$$\begin{aligned}\mathrm{div}_x(u\mathrm{K}^{\mathrm{free}}_G[\mathfrak{h}]) &= u \cdot \nabla_x \mathrm{K}^{\mathrm{free}}_G[\mathfrak{h}] + (\mathrm{div}_x u)\mathrm{K}^{\mathrm{free}}_G[\mathfrak{h}] \\ &= \mathrm{K}^{\mathrm{free}}_G[(u \cdot \nabla_x \mathfrak{h})] + (\mathrm{div}_x u)\mathrm{K}^{\mathrm{free}}_G[\mathfrak{h}] + [u \cdot \nabla_x, \mathrm{K}^{\mathrm{free}}_G][\mathfrak{h}].\end{aligned}$$

Using the notation $\partial_i = \partial_{x_i}$, we have

$$[u \cdot \nabla_x, \mathrm{K}^{\mathrm{free}}_G][\mathfrak{h}](t, x) = \sum_{i=1}^d \int_{\mathbb{R}^d} u_i(t, x)\partial_i(\mathrm{K}^{\mathrm{free}}_G[\mathfrak{h}])(t, x) - \sum_{i=1}^d \int_{\mathbb{R}^d} \mathrm{K}^{\mathrm{free}}_G[(u_i \partial_i \mathfrak{h})](t, x),$$

and then

$$\begin{aligned}
[u\cdot\nabla_x, \mathrm{K}_G^{\mathrm{free}}][\mathfrak{h}](t,x) &= \sum_{i=1}^d \int_{\mathbb{R}^d} u_i(t,x) \int_0^t \int_{\mathbb{R}^d} \nabla_x(\partial_i\mathfrak{h})(s,x-(t-s)v)\cdot G(t,x,v)\,\mathrm{d}v\,\mathrm{d}s \\
&\quad + \sum_{i=1}^d \int_{\mathbb{R}^d} u_i(t,x) \int_0^t \int_{\mathbb{R}^d} \nabla_x\mathfrak{h}(s,x-(t-s)v)\cdot \partial_i G(t,x,v)\,\mathrm{d}v\,\mathrm{d}s \\
&\quad - \sum_{i=1}^d \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \partial_i\mathfrak{h}(s,x-(t-s)v)\nabla_x u_i(s,x-(t-s)v)\cdot G(t,x,v)\,\mathrm{d}v\,\mathrm{d}s \\
&\quad - \sum_{i=1}^d \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} u_i(s,x-(t-s)v)\nabla_x(\partial_i\mathfrak{h})(s,x-(t-s)v)\cdot G(t,x,v)\,\mathrm{d}v\,\mathrm{d}s \\
&= \mathrm{K}^{\mathrm{free}}_{(u\cdot\nabla_x)G}[\mathfrak{h}](t,x) - \mathrm{K}^{\mathrm{free}}_{(G\cdot\nabla_x)\widetilde{u}}[\mathfrak{h}](t,x) \\
&\quad + \sum_{i=1}^d \int_{\mathbb{R}^d} \nabla_x(\partial_i\mathfrak{h})(s,x-(t-s)v)\cdot\Big((u_i(t,x)-\widetilde{u}_i(s,t,x,v))G(t,x,v)\Big)\,\mathrm{d}v\,\mathrm{d}s,
\end{aligned}$$

where we have set $\widetilde{u}(s,t,x,v) = u(s,x-(t-s)v)$. We thus get

$$\mathrm{div}_x(u\mathrm{K}_G^{\mathrm{free}}[\mathfrak{h}]) = \mathrm{K}_G^{\mathrm{free}}[(u\cdot\nabla_x\mathfrak{h})] + (\mathrm{div}_x u)\mathrm{K}_G^{\mathrm{free}}[\mathfrak{h}] + \mathrm{K}^{\mathrm{free}}_{(u\cdot\nabla_x)G}[\mathfrak{h}] - \mathrm{K}^{\mathrm{free}}_{(G\cdot\nabla_x)\widetilde{u}}[\mathfrak{h}] + \sum_{i=1}^d \mathrm{K}^{\mathrm{free}}_{(u_i-\widetilde{u}_i)G}[\partial_i\mathfrak{h}],$$

which gives a decomposition as (5.6.4). In the sequel, we shall constantly use the estimate (5.6.3) of Remark 5.6.4, that is, for all $0<p<m-d/2-2$ and $\sigma>0$

$$\sup_{0\le t\le T}\|G(t)\|_{\mathcal{H}^p_\sigma} \le \Lambda(T,R). \tag{5.6.6}$$

Let us estimate the different terms of the previous decomposition in order to prove (5.6.5). For the first one, we use the smoothing estimate of Proposition 5.4.1 to directly get for $\ell > 1+d$ and $\sigma > d/2$

$$\left\|(\mathrm{div}_x u)\mathrm{K}_G^{\mathrm{free}}[\mathfrak{h}]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \le \Lambda(T,R)\sup_{0\le t\le T}\|G(t)\|_{\mathcal{H}^\ell_\sigma}\|\mathfrak{h}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \le \Lambda(T,R)\|\mathfrak{h}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)},$$

since $m>3+3d/2$. For the second and third ones, Proposition 5.4.1 yields

$$\begin{aligned}
&\left\|\mathrm{K}^{\mathrm{free}}_{(u\cdot\nabla_x)G}[\mathfrak{h}]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} + \left\|\mathrm{K}^{\mathrm{free}}_{(G\cdot\nabla_x)\widetilde{u}}[\mathfrak{h}]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \\
&\qquad \lesssim \left(\sup_{0\le s,t\le T}\|(u\cdot\nabla_x)G(t,s)\|_{\mathcal{H}^p_\sigma}. + \sup_{0\le s,t\le T}\|(G\cdot\nabla_x)\widetilde{u}(t,s)\|_{\mathcal{H}^p_\sigma}\right)\|\mathfrak{h}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)},
\end{aligned}$$

if $\sigma>d/2$ and $p>1+d$. With the same arguments as in Remark 5.6.4 to estimate the terms inside the parentheses, we get (since $m>4+3d/2$)

$$\left\|\mathrm{K}^{\mathrm{free}}_{(u\cdot\nabla_x)G}[\mathfrak{h}]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} + \left\|\mathrm{K}^{\mathrm{free}}_{(G\cdot\nabla_x)\widetilde{u}}[\mathfrak{h}]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \lesssim \Lambda(T,R)\|\mathfrak{h}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}.$$

For the last term, we observe that the kernel $(u_i(t)-\widetilde{u}_i(t,s,x,v))G(t,x,v)$ vanishes at $s=t$, therefore Remark 5.4.6 implies

$$\begin{aligned}
\left\|\mathrm{K}^{\mathrm{free}}_{(u_i-\widetilde{u}_i)G}[\partial_i\mathfrak{h}]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} &\lesssim \Lambda(T)\sup_{0\le s,t\le T}\|\partial_s(u_i-\widetilde{u}_i)G(t,s)\|_{\mathcal{H}^p_\sigma}\|\mathfrak{h}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \\
&= \Lambda(T)\sup_{0\le s,t\le T}\|\partial_s\widetilde{u}_i(t,s)\|_{\mathrm{H}^k}\sup_{0\le t\le T}\|G(t)\|_{\mathcal{H}^p_\sigma}\|\mathfrak{h}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)},
\end{aligned}$$

for $p > 7 + d/2$, $k > p + d/2$ and $\sigma > d/2$. Using the equation on u, we obtain (since $m > 9 + d$)

$$\left\| \mathrm{K}^{\mathrm{free}}_{(u_i - \widetilde{u}_i)G}[\partial_i \mathfrak{h}] \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R) \left\| \mathfrak{h} \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} .$$

All in all, this yields the claimed estimate (5.6.5).

Now applying (5.6.4) and (5.6.5) with $\mathfrak{h} = \mathrm{J}_\varepsilon h$, we get

$$\mathrm{div}_x(u\mathrm{K}^{\mathrm{free}}_G[\mathrm{J}_\varepsilon h]) = \mathrm{K}^{\mathrm{free}}_G[(u \cdot \nabla_x \mathrm{J}_\varepsilon h)] + \mathcal{R},$$

with

$$\|\mathcal{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T, R, \|\mathrm{J}_\varepsilon h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}) \leq \Lambda(T, R, \|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}).$$

Finally observe that

$$\mathrm{K}^{\mathrm{free}}_G[(u \cdot \nabla_x \mathrm{J}_\varepsilon h)] = \mathrm{K}^{\mathrm{free}}_G[\mathrm{J}_\varepsilon(u \cdot \nabla_x h)] + \mathrm{K}^{\mathrm{free}}_G[[u \cdot \nabla_x, \mathrm{J}_\varepsilon]h].$$

Relying once again on Proposition 5.4.1, we thus have

$$\left\| \mathrm{K}^{\mathrm{free}}_G[[u \cdot \nabla_x, \mathrm{J}_\varepsilon]h] \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T,R) \left\| [u \cdot \nabla_x, \mathrm{J}_\varepsilon]h] \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} .$$

Invoking (a variant of the proof of) [BGS07, Theorem C.14] about the commutator between a differential operator of order 1 and a regularizing operator, we get

$$\left\| [u \cdot \nabla_x, \mathrm{J}_\varepsilon]h] \right\|_{\mathrm{L}^2(\mathbb{T}^d)} \lesssim \|u\|_{\mathrm{W}^{1,\infty}(\mathbb{T}^d)} \|h\|_{\mathrm{L}^2(\mathbb{T}^d)} ,$$

where this estimate is independent of ε. We obtain

$$\left\| \mathrm{K}^{\mathrm{free}}_G[[u \cdot \nabla_x, \mathrm{J}_\varepsilon]h] \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T,R),$$

which concludes the proof.

□

Lemma 5.6.6. *For all $|\alpha| \leq m$ and $h = \partial^\alpha_x \varrho$, and with $G(s,t,x,v) = p'(\varrho(t,x))\nabla_v f(t,x,v)$, there holds*

$$\mathrm{div}_x \mathrm{K}^{\mathrm{free}}_{1,G}[\mathrm{J}_\varepsilon h] = -\mathrm{K}^{\mathrm{free}}_G[\mathrm{J}_\varepsilon \partial_s h] + \mathcal{R},$$

with

$$\|\mathcal{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T, R, \|h(0)\|_{\mathrm{H}^1(\mathbb{T}^d)}).$$

Proof. Recall the definition (5.6.2) for $\mathrm{K}^{\mathrm{free}}_{1,G}$. We first write

$$\mathrm{div}_x \mathrm{K}^{\mathrm{free}}_{1,G}[\mathrm{J}_\varepsilon h] = p'(\varrho(t,x)) \int_0^t \int_{\mathbb{R}^d} \nabla_x \mathrm{div}_x \, [v \mathrm{J}_\varepsilon h](s, x-(t-s)v)] \cdot \nabla_v f(t,x,v) \, \mathrm{d}v \, \mathrm{d}s + \mathcal{R},$$

with

$$\begin{aligned} \mathcal{R}(t,x) &:= \int_{\mathbb{R}^d} \nabla_x [\mathrm{J}_\varepsilon h](s, x-(t-s)v) \cdot \left[(v \cdot \nabla_x)(p'(\varrho(t,x))\nabla_v f(t,x,v)) \right] \mathrm{d}v \, \mathrm{d}s \\ &= \mathrm{K}^{\mathrm{free}}_{(v\cdot\nabla_x)(p'(\varrho)\nabla_v f)}[\mathrm{J}_\varepsilon h]. \end{aligned}$$

Thanks to Proposition 5.4.1, we thus have for $p > 1 + d$ and $\sigma > d/2$

$$\begin{aligned} \|\mathcal{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} &\lesssim \|\mathrm{J}_\varepsilon h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \sup_{0 \leq s,t \leq T} \|(v \cdot \nabla_x p'(\varrho))\nabla_v f(t,s)\|_{\mathcal{H}^p_\sigma} \\ &\leq \Lambda(T,R). \end{aligned}$$

Now observe the identity

$$\partial_s\Big[\mathrm{J}_\varepsilon h(s,x-(t-s)v)\Big] = (\partial_s \mathrm{J}_\varepsilon h)(s,x-(t-s)v) + \mathrm{div}_x(v\mathrm{J}_\varepsilon h)(s,x-(t-s)v).$$

Since

$$\int_{\mathbb{R}^d} \nabla_x \mathrm{J}_\varepsilon h(t,x)\cdot\nabla_v f(t,x,v)\,\mathrm{d}v = 0,$$

we have

$$\int_0^t\int_{\mathbb{R}^d} \partial_s\Big[\nabla_x\mathrm{J}_\varepsilon h(s,x-(t-s)v)\Big]\cdot\nabla_v f(t,x,v)\,\mathrm{d}v\mathrm{d}s = -\int_{\mathbb{R}^d}\nabla_x\mathrm{J}_\varepsilon h(0,x-tv)\cdot\nabla_v f(t,x,v)\,\mathrm{d}v.$$

Using the generalized Minkowski inequality and the Sobolev embedding, the last term is estimated in $\mathrm{L}^2(0,T,\mathrm{L}^2)$ by

$$\|\nabla_x\mathrm{J}_\varepsilon h(0)\|_{\mathrm{L}^2(\mathbb{T}^d)}\left\|\int_{\mathbb{R}^d}\sup_{x\in\mathbb{T}^d}|\nabla_v f(\cdot,x,v)|\,\mathrm{d}v\right\|_{\mathrm{L}^2(0,T)} \leq \Lambda(T,R)\|h(0)\|_{\mathrm{H}^1(\mathbb{T}^d)}.$$

We deduce that we can write

$$\mathrm{div}_x\mathrm{K}^{\mathrm{free}}_{1,G}[\mathrm{J}_\varepsilon h] = -\mathrm{K}^{\mathrm{free}}_G[\partial_s\mathrm{J}_\varepsilon h] + \mathcal{R},$$

with

$$\|\mathcal{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T,R,\|h(0)\|_{\mathrm{H}^1(\mathbb{T}^d)}).$$

□

Combining the results of Lemmas 5.6.3–5.6.5–5.6.6 leads to the proof of Proposition 5.6.1.

5.6.2 Propagation of the Penrose condition for short times

We now show how to propagate the Penrose stability condition (**P**) for short times. This will allow to study the operator

$$\mathrm{Id} - \frac{\varrho}{1-\rho_f}\mathrm{K}^{\mathrm{free}}_G\circ\mathrm{J}_\varepsilon$$

in the next sections, the outset being its ellipticity.

First, we shall need several estimates on the time derivatives of the solutions. We have the following basic lemma.

Lemma 5.6.7. *Let $s\geq 0$ and $\sigma\geq 0$. For all $T\in(0,T_\varepsilon)$, the following holds.*

- *If $s>d$ and $s+1>d/2$, we have*

$$\|\partial_t f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^s_\sigma)} \lesssim \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{s+1}_{\sigma+1})} + \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{s+1}_\sigma)}\left(\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} + \Lambda\left(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{s+1})}\right)\right).$$

- *If $s>d/2$ and $\sigma>1+d/2$, we have*

$$\|\partial_t\rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \lesssim \|\partial_t\rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} \lesssim \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{1+s}_\sigma)}.$$

- *If $s>d/2$ and $\sigma>1+d/2$, we have*

$$\begin{aligned}\|\partial_t\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} &\lesssim \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)}\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{1+s})} \\ &\quad + \left\|\frac{1}{1-\rho_f}\right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)}\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)}\Big(\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{1+s}_\sigma)} + \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{1+s})} \\ &\quad + \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{1+s}_\sigma)}\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{1+s})}\Big).\end{aligned}$$

Proof. • Using the Vlasov equation satisfied by f, we get

$$\|\partial_t f\|_{\mathcal{H}^s_\sigma} \lesssim \|f\|_{\mathcal{H}^s_\sigma} + \|v\cdot\nabla_x f\|_{\mathcal{H}^s_\sigma} + \|v\cdot\nabla_v f\|_{\mathcal{H}^s_\sigma} + \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v f\|_{\mathcal{H}^s_\sigma} \lesssim \|f\|_{\mathcal{H}^{s+1}_{\sigma+1}} + \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v f\|_{\mathcal{H}^s_\sigma}.$$

Next, combining the estimate (5.A.2) of Lemma 5.A.6 and the estimate (5.2.2) of Lemma 5.2.7, we have for $s>d$ such that $s>3+d/2$

$$\|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v f\|_{\mathcal{H}^s_\sigma} \lesssim \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|_{\mathrm{H}^s}\|f\|_{\mathcal{H}^{s+1}_\sigma} \lesssim \|f\|_{\mathcal{H}^{s+1}_\sigma}\left(\|u(t)\|_{\mathrm{H}^s} + \Lambda(\|\varrho(t)\|_{\mathrm{H}^{s+1}})\right),$$

hence providing the first estimate.

• To estimate $\partial_t\rho_f$, we use the fact that $\partial_t\rho_f = -\mathrm{div}_x j_f$ so that we have by Sobolev embedding

$$\|\partial_t\rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \lesssim \|\partial_t\rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} \lesssim \|j_f\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{1+s})} \lesssim \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{1+s}_\sigma)},$$

thanks to Lemma 5.2.1. This gives the second estimate.

• To estimate $\partial_t\varrho$, we use the equation

$$\partial_t\varrho = -u\cdot\nabla_x\varrho - \frac{1}{1-\rho_f}\mathrm{div}_x(j_f-\rho_f u+u)\varrho,$$

from which we infer

$$\begin{aligned}\|\partial_t\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} &\lesssim \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)}\|\nabla_x\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} \\ &\quad + \left\|\frac{1}{1-\rho_f}\right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)}\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)}\|j_f-\rho_f u+u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{1+s})}.\end{aligned}$$

We conclude by using Lemma 5.2.1. □

The propagation of the Penrose condition (**P**) for short times is obtained in the following lemma.

Lemma 5.6.8. *There exists $\widetilde{T}_0(R)>0$ independent of ε such that the following holds: if $(f^{\mathrm{in}},\varrho^{\mathrm{in}})$ satisfies the c-Penrose stability condition $(\mathbf{P})_c$ for some $c>0$ then $(f(t),\varrho(t))$ satisfies the $\frac{c}{2}$-Penrose stability condition $(\mathbf{P})_{c/2}$ for all $t\in\left[0,\min\left(\widetilde{T}_0(R),T^\varepsilon\right)\right]$.*

Proof. Let $T<T^\varepsilon$ and recall the Definition 5.1.3 of the Penrose function $\mathscr{P}$. We start writing for all $t\in[0,T]$

$$\begin{aligned}1-\mathscr{P}_{f(t),\varrho(t)}(x,\gamma,\tau,k) &= 1-\frac{p'(\varrho(t,x))\rho(t,x)}{1-\rho_f(t,x)}\int_0^{+\infty} e^{-(\gamma+i\tau)s}\frac{ik}{1+|k|^2}\cdot\left(\mathcal{F}_v\nabla_v f\right)(t,x,ks)\,\mathrm{d}s \\ &= 1-\mathscr{P}_{f^{\mathrm{in}},\varrho^{\mathrm{in}}}(x,\gamma,\tau,k)+\mathbf{A}_0(t,x,\gamma,\tau,k)+\mathbf{B}_0(t,x,\gamma,\tau,k),\end{aligned}$$

where

$$\begin{aligned}&\mathbf{A}_0(t,x,\gamma,\tau,k)\\ &\quad := -\frac{p'(\varrho^{\mathrm{in}}(x))\varrho^{\mathrm{in}}(x)}{1-\rho_{f^{\mathrm{in}}}(x)}\left(\int_0^{+\infty} e^{-(\gamma+i\tau)s}\frac{ik}{1+|k|^2}\cdot\left[\left(\mathcal{F}_v\nabla_v f\right)(t,x,ks)-\left(\mathcal{F}_v\nabla_v f^{\mathrm{in}}\right)(x,ks)\right]\mathrm{d}s\right),\end{aligned}$$

and

$$\begin{aligned}&\mathbf{B}_0(t,x,\gamma,\tau,k)\\ &\quad := \left(\frac{p'(\varrho^{\mathrm{in}}(x))\varrho^{\mathrm{in}}(x)}{1-\rho_{f^{\mathrm{in}}}(x)}-\frac{p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)}\right)\int_0^{+\infty} e^{-(\gamma+i\tau)s}\frac{ik}{1+|k|^2}\cdot\left(\mathcal{F}_v\nabla_v f\right)(t,x,ks)\,\mathrm{d}s.\end{aligned}$$

• **Estimate of $\mathbf{A}_0$**: using Taylor's formula, we have

$$
\begin{aligned}
|\mathbf{A}_0(t,x,\gamma,\tau,k)| &\leq \left\| \frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}} \right\|_{\mathrm{L}^\infty} \int_0^{+\infty} |k| \int_0^T |(\mathcal{F}_v \nabla_v \partial_t f)(\theta,x,ks)| \, \mathrm{d}\theta \, \mathrm{d}s \\
&\leq \left\| \frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}} \right\|_{\mathrm{L}^\infty} \int_0^T \int_0^{+\infty} \left| (\mathcal{F}_v \nabla_v \partial_t f)\left(\theta, x, \frac{k}{|k|}s\right) \right| \mathrm{d}s \, \mathrm{d}\theta \\
&\lesssim \left\| \frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}} \right\|_{\mathrm{L}^\infty} \int_0^T \sup_{s\in\mathbb{R}^+} (1+s^2) \left| (\mathcal{F}_v \nabla_v \partial_t f)\left(\theta, x, \frac{k}{|k|}s\right) \right| \mathrm{d}\theta,
\end{aligned}
$$

therefore we get for $\sigma > d/2$

$$
\begin{aligned}
|\mathbf{A}_0(t,x,\gamma,\tau,k)| &\lesssim \left\| \frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}} \right\|_{\mathrm{L}^\infty} \int_0^T \sup_{s\in\mathbb{R}^+} \sum_{|\beta|\leq 2} \left| \left(\mathcal{F}_v \partial_v^\beta \nabla_v \partial_t f\right)\left(\theta, x, \frac{k}{|k|}s\right) \right| \mathrm{d}\theta \\
&\lesssim \left\| \frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}} \right\|_{\mathrm{L}^\infty} \int_0^T \left(\sum_{|\beta|\leq 2} \int_{\mathbb{R}^d} (1+|v|^2)^\sigma \left| \partial_v^\beta \nabla_v \partial_t f(\theta,x,v) \right|^2 \mathrm{d}v \right)^{\frac{1}{2}} \mathrm{d}\theta \\
&\lesssim \left\| \frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}} \right\|_{\mathrm{L}^\infty} T \|\partial_t f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{3+s}_\sigma)},
\end{aligned}
$$

for all $s > d/2$, thanks to the Sobolev embedding. Invoking Lemma 5.6.7 (taking $3+s > d$ and $3+s+1 > d/2$), we have

$$
\|\partial_t f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{3+s}_\sigma)} \leq \Lambda(T,R),
$$

thanks to Lemma 5.2.20, choosing $s+6 \leq m$. Therefore there exists a universal constant $C>0$ such that

$$
|\mathbf{A}_0(t,x,\gamma,\tau,k)| \leq C \left\| \frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}} \right\|_{\mathrm{L}^\infty} T\Lambda(T,R).
$$

• **Estimate of $\mathbf{B}_0$**: Likewise, we have for all $s > d/2$ such that $3+s < m-1$

$$
\begin{aligned}
|\mathbf{B}_0(t,x,\gamma,\tau,k)| &\lesssim \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{3+s}_\sigma)} \left| \frac{p'(\varrho(t,x))\rho(t,x)}{1-\varrho_f(t,x)} - \frac{p'(\varrho^{\mathrm{in}}(x))\varrho^{\mathrm{in}}(x)}{1-\rho_{f^{\mathrm{in}}}(x)} \right| \\
&\leq R \int_0^T \left| \partial_t \left\{ \frac{p'(\varrho(\theta,x))\varrho(\theta,x)}{1-\rho_f(\theta,x)} \right\} \right| \mathrm{d}\theta \\
&\leq RT \left\| \partial_t \left\{ \frac{p'(\varrho)\varrho}{1-\rho_f} \right\} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)},
\end{aligned}
$$

Now observe that

$$
\partial_t \left\{ \frac{p'(\varrho)\varrho}{1-\rho_f} \right\} = \frac{1}{1-\rho_f} \left(\varrho \partial_t \varrho \, p''(\varrho) + p'(\varrho)\partial_t \varrho \right) + p'(\varrho)\varrho \frac{\partial_t \rho_f}{(1-\rho_f)^2},
$$

therefore by the Sobolev embedding, we get for all $s > d/2$

$$
\begin{aligned}
\left\| \partial_t \left\{ \frac{p'(\varrho)\varrho}{1-\rho_f} \right\} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} &\leq \left\| \frac{1}{1-\rho_f} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \|\partial_t \varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \\
&\qquad \times \left(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} \|p''(\varrho)\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} + \|p'(\varrho)\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} \right) \\
&\quad + \left\| \frac{1}{1-\rho_f} \right\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \|p'(\varrho)\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} \|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} \|\partial_t \rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)}.
\end{aligned}
$$

Using Lemma 5.6.7 with Proposition 5.2.20, we obtain as before

$$|\mathbf{B}_0(t,x,\gamma,\tau,k)| \leq CT\Lambda\left(T,R\right).$$

All in all, we have for all $t \in [0,T]$

$$|\mathbf{A}_0(t,x,\gamma,\tau,k)| + |\mathbf{B}_0(t,x,\gamma,\tau,k)| \leq C\left(\left\|\frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}}\right\|_{\mathrm{L}^\infty} + 1\right) T\Lambda\left(T,R\right).$$

Consequently, there exists $\eta_0 > 0$ independent of ε (and depending on c) such that for $\widetilde{T}_0 = \widetilde{T}_0(R)$ satisfying

$$C\left(\left\|\frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}}\right\|_{\mathrm{L}^\infty} + 1\right) \widetilde{T}_0\Lambda\left(\widetilde{T}_0,R\right) \leq \eta_0,$$

the $\frac{c}{2}$-Penrose stability condition $(\mathbf{P})_{c/2}$ holds true for $(f(t),\varrho(t))$ whenever $t \in \left[0,\left(\widetilde{T}_0(R),T^\varepsilon\right)\right]$, provided that $(f^{\mathrm{in}},\varrho^{\mathrm{in}})$ satisfies the c-Penrose stability condition $(\mathbf{P})_c$.

□

5.6.3 Extension of the solution

In this section, our goal is to construct a suitable extension in time of the solution $(f_\varepsilon,\varrho_\varepsilon,u_\varepsilon)$ on the whole line $\mathbb{R}$. This technical step is required in view of the subsequent pseudodifferential analysis (in time-space) of Sections 5.6.4–5.6.5 – the symbols being dependent on our solutions. A main issue is to obtain an extension still satisfying the Penrose stability condition (for all times): we refer to the later Proposition 5.6.17.

The parameter R being fixed, there exists by continuity a nonnegative time $\widehat{T}(R) \leq 1$ (independent of ε) such that

$$\forall T \in [0,\widehat{T}(R)], \quad T\Lambda(T,R) \leq T^{1/2}\Lambda(T,R) \leq T^{1/4}\Lambda(T,R) \leq 1.$$

Define $T_\varepsilon^\star$ (depending on R) as

$$T_\varepsilon^\star := \min\left(T_\varepsilon(R),\overline{T}(R),\widetilde{T}_0(R),\widehat{T}(R)\right). \tag{5.6.7}$$

In particular, the Penrose stability condition holds on $[0,T_\varepsilon^\star]$ thanks to Lemma 5.6.8.

Consider two nonnegative nonincreasing cutoffs $\chi,\underline{\chi} \in \mathscr{C}^\infty(\mathbb{R})$ such that

$$\forall t \in \mathbb{R}, \quad \chi(t) = \begin{cases} 1, & t \leq 0, \\ 0, & t \geq 1, \end{cases} \quad \underline{\chi}(t) = \begin{cases} 1, & t \leq 0, \\ 1/2, & t \geq 1. \end{cases}$$

We set for $\delta > 0$ to be fixed later, $\chi_\delta(t) := \chi(t/\delta)$.

Given a solution (f,ϱ,u) to the system (S_ε), we consider its extension $(\widetilde{f},\widetilde{\varrho},\widetilde{u})$ as follows. Given $(N_u,N_f,N_\varrho) \in \mathbb{N}^3$ to be determined later on (by the number of derivatives we will use), we define:

• **Extension in time for** u: we set

$$\widetilde{u}(t) = \begin{cases} u(t), & t \in [0,T_\varepsilon^\star], \\ \chi(t-T_\varepsilon^\star)\displaystyle\sum_{k=0}^{N_u} \partial_t^k u(T_\varepsilon^\star)\frac{(t-T_\varepsilon^\star)^k}{k!}, & t \geq T_\varepsilon^\star, \\ \chi(-t)\displaystyle\sum_{k=0}^{N_u} \partial_t^k u(0)\frac{t^k}{k!}, & t \leq 0. \end{cases}$$

In particular, the extension is 0 after $t = T_\varepsilon^\star + 1$ and before $t = -1$.

• **Extension in time for** f: we set

$$\widetilde{f}(t) = \begin{cases} f(t), & t \in [0, T_\varepsilon^\star], \\ \chi_\delta(t - T_\varepsilon^\star) f(T_\varepsilon^\star) + \chi_\delta(t - T_\varepsilon^\star) \displaystyle\sum_{k=1}^{N_f} \partial_t^k f(T_\varepsilon^\star) \frac{(t - T_\varepsilon^\star)^k}{k!}, & t \geq T_\varepsilon^\star, \\ \chi_\delta(-t) f^{\mathrm{in}} + \chi_\delta(-t) \displaystyle\sum_{k=1}^{N_f} \partial_t^k f(0) \frac{t^k}{k!}, & t \leq 0. \end{cases}$$

In particular, the extension is 0 after $t = T_\varepsilon^\star + \delta$ and before $t = -\delta$.

• **Extension in time for** ϱ: we set

$$\widetilde{\varrho}(t) = \begin{cases} \varrho(t), & t \in [0, T_\varepsilon^\star], \\ \underline{\chi}(t - T_\varepsilon^\star) \varrho(T_\varepsilon^\star) + \chi_\delta(t - T_\varepsilon^\star) \displaystyle\sum_{k=1}^{N_\varrho} \partial_t^k \varrho(T_\varepsilon^\star) \frac{(t - T_\varepsilon^\star)^k}{k!}, & t \geq T_\varepsilon^\star, \\ \underline{\chi}(-t) \varrho^{\mathrm{in}} + \chi_\delta(-t) \displaystyle\sum_{k=1}^{N_\varrho} \partial_t^k \varrho(0) \frac{t^k}{k!}, & t \leq 0. \end{cases}$$

In particular, the extension is constant in time equal to $\varrho(T_\varepsilon^\star)/2$ after $t = T_\varepsilon^\star + 1$ and equal to $\varrho^{\mathrm{in}}/2$ before $t = -1$

The bounds from above and below ($\mathrm{B}_\Theta^{\mu,\theta}(T)$) on ϱ and ρ_f are still valid for $\widetilde{\varrho}$ and $\rho_{\widetilde{f}}$, provided that we choose the parameter δ small enough.

Remark 5.6.9. Since $T_\varepsilon^\star \leq 1$, we observe that the Penrose function $\mathscr{P}_{\widetilde{f}(t), \widetilde{\varrho}(t)}$ has a compact support in time included in $[0, 2]$.

Here ever after, we drop out the tilde notation and we shall always consider the extension of our solutions. Let us conclude this section by explaining how we will deal with such an extension:

- Replacing the former solution (defined on $[0, T_\varepsilon^\star]$) by its extension (f, ϱ, u) on $\mathbb{R}$, we observe that (f, ϱ, u) satisfies (S_ε) with the addition of a new source term S^{new} in the r.h.s which has a support included in $\mathbb{R} \backslash [0, T_\varepsilon^\star]$.

- The results of Section 5.5 and Subsection 5.6.1 remain true on $[0, T_\varepsilon^\star)$.

We also refer to Proposition 5.6.17 below where we will prove that the extension (f, ϱ, u) satisfies a Penrose stability condition for all times (the proof requiring some technical estimates from the upcoming Section 5.6.4).

5.6.4 Bounds on the symbols

The aim of this section is twofold:

- obtain some bounds in terms of the initial data for some symbol seminorms of the Penrose function introduced in (5.1.3) (depending on the extension (f, ϱ));

- propagate the Penrose stability condition (**P**) on the whole line in time for the extension (f, ϱ, u).

These two ingredients are required to obtain crucial elliptic estimates in Section 5.6.5.

Before stating the next lemma, consider the symbol seminorms (5.C.1)–(5.C.2)–(5.C.3) introduced in the Section 5.C in the Appendix: for any $\mathrm{M} \geq 0$ and for any symbol $a(t,x,\eta)$ with $\eta = (\gamma, \tau, k) \in (0,+\infty) \times \mathbb{R} \times \mathbb{R}^d \backslash \{0\}$, we set

$$\omega[a] := \sup_{\substack{\alpha\in\mathbb{N}^d \\ \alpha_i \in\{0,1\}}} \|(1+t)\partial_x^\alpha a\|_{\mathrm{L}^\infty_{t,x,\eta}} + \sup_{\substack{\alpha\in\mathbb{N}^d \\ \alpha_i \in\{0,1\}}} \|(1+t)\partial_t\partial_x^\alpha a\|_{\mathrm{L}^\infty_{t,x,\eta}},$$

$$\Omega[a] := \sup_{\substack{\alpha\in\mathbb{N}^d \\ \alpha_i \in\{0,1\}}} \left\{ \||\eta|\partial_x^\alpha \nabla_{\tau,k} a\|_{\mathrm{L}^\infty_{t,x,\eta}} + \||\eta|\partial_x^\alpha \nabla_{\tau,k}\partial_t a\|_{\mathrm{L}^\infty_{t,x,\eta}} \right\}$$
$$+ \sup_{\substack{\alpha\in\mathbb{N}^d \\ \alpha_i \in\{0,1\}}} \left\{ \||\eta|\partial_x^\alpha \partial_\tau \nabla_{\tau,k} a\|_{\mathrm{L}^\infty_{t,x,\eta}} + \||\eta|\partial_x^\alpha \partial_\tau\nabla_{\tau,k}\partial_t a\|_{\mathrm{L}^\infty_{t,x,\eta}} \right\},$$

$$\Xi[a]_{\mathrm{M}} := \sup_{\substack{1\leq|\alpha|\leq 1+\mathrm{M} \\ \beta=0,1,2,3,4}} \left\|\partial_x^\alpha \partial_t^\beta a\right\|_{\mathrm{L}^\infty_{t,x,\eta}}.$$

Lemma 5.6.10. *For $(t,x,\gamma,\tau,k) \in \mathbb{R}\times\mathbb{T}^d\times(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\}$, set*

$$a_f(t,x,\gamma,\tau,k) := \int_0^{+\infty} e^{-(\gamma+i\tau)s} ik\cdot (\mathcal{F}_v\nabla_v f)(t,x,ks)\,\mathrm{d}s. \tag{5.6.8}$$

The symbol a_f is (positively) homogeneous of degree zero in (γ,τ,k) in the sense that

$$\forall (t,x), \quad \forall\eta = (\gamma,\tau,k), \quad \forall\lambda>0 \quad a_f(t,x,\lambda\eta) = a_f(t,x,\eta).$$

Furthermore, for any $A>0$ and $r>d/2+4$, we have

$$\omega[a_f] \lesssim \sup_{i=0,1} \|(1+t)\partial_t^i f\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{H}^\ell_r)}, \quad 3+\frac{3d}{2}<\ell, \tag{5.6.9}$$

$$\Xi[a_f]_M \lesssim \sup_{i=0,1,2,3,4} \|\partial_t^i f\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{H}^\ell_r)}, \quad 4+M+\frac{d}{2}<\ell, \tag{5.6.10}$$

$$\Omega[a_f] \lesssim \sup_{i=0,1} \|\partial_t^i f\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{H}^\ell_r)}, \quad 7+\frac{3d}{2}<\ell. \tag{5.6.11}$$

Proof. The homogeneity is obtained by performing the change of variable $s=\frac{s'}{\lambda}$ (for $\lambda>0$) in the integrals in s defining $a_f(t,x,\eta)$. In what follows, we will rely on the estimate

$$|\partial_t^\delta\partial_x^\alpha\partial_\xi^\beta \mathcal{F}_v\nabla_v f(t,x,\xi)| \lesssim \frac{1}{1+|\xi|^q}\left(\int_{\mathbb{R}^d}(1+|v|^2)^{\sigma+|\beta|}|\nabla_v\partial_t^\delta\partial_x^\alpha(I-\Delta_v)^{\frac{q}{2}} f(t,x,v)|^2\,\mathrm{d}v\right)^{\frac12}, \tag{5.6.12}$$

which is valid for all $\sigma>d/2$, $q>0$ and any $(\delta,\alpha,\beta)\in\mathbb{N}\times\mathbb{N}^d\times\mathbb{N}^d$.

Consequently, for any $\alpha\in\mathbb{N}^d$ with $\alpha_i=0,1$, we apply (5.6.12) with $\delta=0,1$, $q=2$ and $\beta=0$, and obtain for $\chi(v)=(1+|v|^2)^{\frac{\sigma}{2}}$

$$\begin{aligned}|\partial_t^\delta\partial_x^\alpha a_f(t,x,\eta)| &\lesssim \left(\int_{\mathbb{R}^d}(1+|v|^2)^{\sigma}|\nabla_v\partial_t^\delta\partial_x^\alpha(I-\Delta_v) f(t,x,v)|^2\,\mathrm{d}v\right)^{\frac12}\int_0^{+\infty}\frac{|k|}{1+s^2|k|^2}\,\mathrm{d}s \\ &\lesssim \left\|\chi\partial_t^\delta\partial_x^\alpha f(t,x)\right\|_{\mathrm{H}^3_v(\mathbb{R}^d)}\int_0^{+\infty}\frac{1}{1+s^2}\,\mathrm{d}s \\ &\lesssim \|\partial_t^\delta f(t)\|_{\mathcal{H}_\sigma^{3+|\alpha|+\frac{d}{2}+\kappa}},\end{aligned}$$

for all $\kappa > 0$, thanks to the Sobolev embedding. We then deduce

$$\omega[a_f] \lesssim \|(1+t)f\|_{\mathrm{L}^\infty\left(\mathbb{R};\mathcal{H}_\sigma^{3+d+\frac{d}{2}+\kappa}\right)} + \|(1+t)\partial_t f\|_{\mathrm{L}^\infty\left(\mathbb{R};\mathcal{H}_\sigma^{3+d+\frac{d}{2}+\kappa}\right)},$$

and hence the claimed inequality (5.6.9). The inequality (5.6.10) can be obtained in the same way.

Let us now turn to the proof of (5.6.11). First, observe by the homogeneity of a_f in $\eta = (\gamma, \tau, k)$, it is enough to estimates the quantities $\left\|\partial_x^\alpha \nabla_{\tau,k}\partial_t^\delta a_f\right\|_{\mathrm{L}^\infty_{t,x}\mathrm{L}^\infty_\eta(S^+)}$ and $\left\|\partial_x^\alpha \partial_\tau \nabla_{\tau,k}\partial_t^\delta a\right\|_{\mathrm{L}^\infty_{t,x}\mathrm{L}^\infty_\eta(S^+)}$ with $\delta = 0,1$, $\alpha \in \mathbb{N}^d$ with $\alpha_i = 0,1$ and where

$$S^+ := \left\{\widetilde{\eta} = (\widetilde{\gamma}, \widetilde{\tau}, \widetilde{k}) \in (0,+\infty) \times \mathbb{R} \times \mathbb{R}^d \backslash\{0\} \mid \widetilde{\gamma}^2 + \widetilde{\tau}^2 + \widetilde{k}^2 = 1\right\}.$$

We thus need to estimate the following symbols

$$I_1^{\alpha,\delta}(t,x,\widetilde{\eta}) = \int_0^{+\infty} e^{-(\widetilde{\gamma}+i\widetilde{\tau})s} n \cdot \left(\partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f\right)(t,x,\widetilde{k}s)\,\mathrm{d}s, \quad n \in \mathbb{R}^d, \quad |n| = 1,$$
$$I_2^{\alpha,\delta}(t,x,\widetilde{\eta}) = \int_0^{+\infty} e^{-(\widetilde{\gamma}+i\widetilde{\tau})s} s\widetilde{k} \cdot \left(\partial_\xi^\beta \partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f\right)(t,x,\widetilde{k}s)\,\mathrm{d}s, \quad |\beta| \in \{0,1\}.$$

and

$$J_{1,q_1}^{\alpha,\delta}(t,x,\widetilde{\eta}) = \int_0^{+\infty} e^{-(\widetilde{\gamma}+i\widetilde{\tau})s} s^{q_1} n \cdot \left(\partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f\right)(t,x,\widetilde{k}s)\,\mathrm{d}s, \quad n \in \mathbb{R}^d, \quad |n| = 1, \quad q_1 \in \{1,2\},$$
$$J_{2,q_2}^{\alpha,\delta}(t,x,\widetilde{\eta}) = \int_0^{+\infty} e^{-(\widetilde{\gamma}+i\widetilde{\tau})s} s^{q_2} \widetilde{k} \cdot \left(\partial_\xi \partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f\right)(t,x,\widetilde{k}s)\,\mathrm{d}s, \quad q_2 \in \{2,3\},$$

for $\widetilde{\eta} \in S^+$.

We focus on the terms $J_{1,q_1}^{\alpha,\delta}$ and $J_{2,q_2}^{\alpha,\delta}$ by following [HKR16, Lemma 16], the treatment of the symbols $I_1^{\alpha,\delta}$ and $I_2^{\alpha,\delta}$ being similar and involving fewer derivatives.

If $|\widetilde{k}| \geq 1/2$, invoking (5.6.12) with $q = q_1 + 3$ and $\beta = 0$ yields as before

$$\begin{aligned}|J_{1,q_1}^{\alpha,\delta}(t,x,\widetilde{\eta})| &\lesssim \left(\int_{\mathbb{R}^d} (1+|v|^2)^\sigma |\nabla_v \partial_t^\delta \partial_x^\alpha (I-\Delta_v)^{\frac{q_1+3}{2}} f(t,x,v)|^2 \,\mathrm{d}v\right)^{\frac{1}{2}} \int_0^{+\infty} \frac{s^{q_1}}{1+s^{q_1+3}|\widetilde{k}|^{q_1+3}}\,\mathrm{d}s \\ &\lesssim \|\partial_t^\delta f(t)\|_{\mathcal{H}_\sigma^{4+q_1+|\alpha|+\frac{d}{2}+}} \int_0^{+\infty} \frac{s^{q_1}}{1+s^{q_1+3}}\,\mathrm{d}s,\end{aligned}$$

since $|\widetilde{k}|$ is bounded from below. We thus obtain a uniform estimate in this case. Otherwise, if $|\widetilde{k}| \leq 1/2$, then $\widetilde{\gamma}^2 + \widetilde{\tau}^2 \geq 3/4$ and we can therefore rely on the exponential to integrate by parts in s in the integral defining $J_{1,q_1}^{\alpha,\delta}(t,x,\widetilde{\eta})$. If $q_1 = 1$, we first get

$$\begin{aligned}J_{1,1}^{\alpha,\delta}(t,x,\widetilde{\eta}) &= \frac{1}{\widetilde{\gamma}+i\widetilde{\tau}} \int_0^{+\infty} e^{-(\widetilde{\gamma}+i\widetilde{\tau})s} n \cdot \left(\partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f\right)(t,x,\widetilde{k}s)\,\mathrm{d}s \\ &\quad + \frac{1}{\widetilde{\gamma}+i\widetilde{\tau}} \int_0^{+\infty} e^{-(\widetilde{\gamma}+i\widetilde{\tau})s} s n \cdot \left(\mathrm{D}_\xi \partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f\right)(t,x,\widetilde{k}s)\widetilde{k}\,\mathrm{d}s,\end{aligned}$$

and integrating by parts once again yields the estimate (since $\widetilde{\gamma}^2 + \widetilde{\tau}^2 \geq 3/4$)

$$\begin{aligned}|J_{1,1}^{\alpha,\delta}(t,x,\widetilde{\eta})| &\lesssim |\partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f(t,x,0)| + \int_0^{+\infty} |\widetilde{k}||\mathrm{D}_\xi \partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f(t,x,\widetilde{k}s)|\,\mathrm{d}s \\ &\quad + \int_0^{+\infty} s|\widetilde{k}|^2 |\mathrm{D}_\xi^2 \partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f(t,x,\widetilde{k}s)|\,\mathrm{d}s.\end{aligned}$$

Using (5.6.12) with $q = 2$, $|\beta| = 0, 1$, or $q = 3$, $|\beta| = 2$ now provides for all $\kappa > 0$

$$
\begin{aligned}
|J^{\alpha,\delta}_{1,1}(t,x,\widetilde{\eta})| &\lesssim \|\partial_t^\delta f(t)\|_{\mathcal{H}_\sigma^{4+|\alpha|+\frac{d}{2}+\kappa}} \left(1 + \int_0^{+\infty} \frac{|\widetilde{k}|}{1+|\widetilde{k}|^2 s^2}\,\mathrm{d}s + \int_0^{+\infty} \frac{|\widetilde{k}|^2 s}{1+|\widetilde{k}|^3 s^3}\,\mathrm{d}s\right) \\
&\lesssim \|\partial_t^\delta f(t)\|_{\mathcal{H}_{\sigma+2}^{4+|\alpha|+\frac{d}{2}+\kappa}} \left(1 + \int_0^{+\infty} \frac{1}{1+s^2}\,\mathrm{d}s + \int_0^{+\infty} \frac{s}{1+s^3}\,\mathrm{d}s\right),
\end{aligned}
$$

and hence the uniform estimate for this symbol. If $q_1 = 2$, we use the same strategy with additional integration by parts and (5.6.12) with $q = 2$, $|\beta| = 0, 1$, or $q = 3$, $|\beta| = 2$, or $q = 4$, $|\beta| = 3$ to get

$$
\begin{aligned}
&|J^{\alpha,\delta}_{1,2}(t,x,\widetilde{\eta})| \\
&\lesssim |J^{\alpha,\delta}_{1,1}(t,x,\widetilde{\eta})| + \int_0^{+\infty} |\widetilde{k}||\mathrm{D}_\xi \partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f(t,x,\widetilde{k}s)|\,\mathrm{d}s + \int_0^{+\infty} s|\widetilde{k}|^2 |\mathrm{D}_\xi^2 \partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f(t,x,\widetilde{k}s)|\,\mathrm{d}s \\
&\quad + \int_0^{+\infty} s^2 |\widetilde{k}|^3 |\mathrm{D}_\xi^3 \partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f(t,x,\widetilde{k}s)|\,\mathrm{d}s. \\
&\lesssim \|\partial_t^\delta f(t)\|_{\mathcal{H}_{\sigma+3}^{5+|\alpha|+\frac{d}{2}+}} \left(1 + \int_0^{+\infty} \frac{|\widetilde{k}|}{1+|\widetilde{k}|^2 s^2}\,\mathrm{d}s + \int_0^{+\infty} \frac{|\widetilde{k}|^2 s}{1+|\widetilde{k}|^3 s^3}\,\mathrm{d}s + \int_0^{+\infty} \frac{|\widetilde{k}|^3 s^2}{1+|\widetilde{k}|^4 s^4}\,\mathrm{d}s\right) \\
&\lesssim \|\partial_t^\delta f(t)\|_{\mathcal{H}_{\sigma+3}^{5+|\alpha|+\frac{d}{2}+}} \left(1 + \int_0^{+\infty} \frac{1}{1+s^2}\,\mathrm{d}s + \int_0^{+\infty} \frac{s}{1+s^3}\,\mathrm{d}s + \int_0^{+\infty} \frac{s^2}{1+s^4}\,\mathrm{d}s\right).
\end{aligned}
$$

Gathering the two cases together, we deduce the estimate

$$
\left\| J^{\alpha,\delta}_{1,q_1} \right\|_{\mathrm{L}^\infty_{t,x}\mathrm{L}^\infty_\eta(S^+)} \lesssim \|\partial_t^\delta f\|_{\mathrm{L}^\infty\left(\mathbb{R};\mathcal{H}_{\sigma+3}^{6+|\alpha|+\frac{d}{2}+\kappa}\right)},
$$

for all $\kappa > 0$. Likewise, we can apply these arguments to $J^{\alpha,\delta}_{2,q_2}$, by invoking again (5.6.12) with $q = q_2 + 3$, $\beta = 1$ if $|\widetilde{k}| \leq 1/2$, or with at most $q = 5$, $\beta = 4$ if $|\widetilde{k}| \geq 1/2$. All in all, we obtain

$$
\left\| J^{\alpha,\delta}_{2,q_1} \right\|_{\mathrm{L}^\infty_{t,x}\mathrm{L}^\infty_\eta(S^+)} \lesssim \|\partial_t^\delta f\|_{\mathrm{L}^\infty\left(\mathbb{R};\mathcal{H}_{\sigma+4}^{7+|\alpha|+\frac{d}{2}+\kappa}\right)}.
$$

We have eventually proven the estimate (5.6.11). □

In view of Lemma 5.6.10, it will be useful to have the following estimates on f.

Lemma 5.6.11. *For $k > d$ and $\sigma > 0$, we have*

$$
\sup_{i=0,1} \left\|(1+t)\partial_t^i f\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{H}^k_\sigma)} \leq \Lambda(1+M_{\mathrm{in}}), \quad \sigma+4<r, \quad k+3<m, \tag{5.6.13}
$$

$$
\sup_{i=0,1,2,3,4} \left\|\partial_t^i f\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{H}^k_\sigma)} \leq \Lambda(1+M_{\mathrm{in}}), \quad \sigma+4<r, \quad k+6<m. \tag{5.6.14}
$$

Proof. Thanks to our choice of extension for (f, ϱ, u), and picking $N_f = 4, N_\varrho = 3$ and $N_u = 3$, it is sufficient to study the estimates on $[0, T]$ with $T \in [0, T_\varepsilon^\star]$. Here, we shall constantly use Remarks 5.2.21 –5.2.25. Taking all the exponents k large enough in the following, we can always assume that the Sobolev spaces that we use are algebras.

We proceed inductively, relying on the equations satisfied by f, ϱ and u. For $i = 0$, we directly use Lemma 5.2.22 with $k \leq m-1$ to get

$$
\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_\sigma)} \leq \|f^{\mathrm{in}}\|_{\mathcal{H}^{m-1}_\sigma} + T^{\frac{1}{4}}\Lambda(T,R) \leq M_{\mathrm{in}} + 1.
$$

For $i=1$, we use Lemma 5.6.7 to get for $k>d$ and $k+1>d/2$

$$\|\partial_t f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_\sigma)} \lesssim \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{k+1}_{\sigma+1})} + \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{k+1}_\sigma)} \left(\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)} + \Lambda\left(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{k+1})}\right)\right),$$

therefore using Remarks 5.2.21–5.2.25 and the previous estimate on f, we deduce

$$\|\partial_t f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_\sigma)} \leq \Lambda(1+M_{\mathrm{in}}),$$

if $k \leq m-3$. For $i=2$, we take one derivative in time in the Vlasov equation and get

$$\begin{aligned}\|\partial_t^2 f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_\sigma)} &\lesssim \|\partial_t f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{k+1}_{\sigma+1})} \\ &\quad + \|E^{\varrho,u}_{\mathrm{reg},\varepsilon}\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)}\|\partial_t f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{k+1}_\sigma)} + \|\partial_t E^{\varrho,u}_{\mathrm{reg},\varepsilon}\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)}\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{k+1}_\sigma)} \\ &\leq \Lambda(1+M_{\mathrm{in}}) + \Lambda(1+M_{\mathrm{in}})\|\partial_t E^{\varrho,u}_{\mathrm{reg},\varepsilon}\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k)},\end{aligned}$$

if we take $k \leq m-4$. Since

$$\partial_t E^{\varrho,u}_{\mathrm{reg},\varepsilon} = \partial_t u - \partial_t \varrho p''(\varrho)\mathrm{J}_\varepsilon \nabla_x \varrho - p'(\varrho)\mathrm{J}_\varepsilon \nabla_x \partial_t \varrho,$$

it is enough to estimate $\|\partial_t u\|_{\mathrm{H}^k}$ and $\|\partial_t \varrho\|_{\mathrm{H}^{k+1}}$. Thanks to the equation on u and on ϱ, we easily get the fact that this involves ρ_f, j_f, ϱ and u in $\mathrm{L}^\infty(0,T;\mathrm{H}^{k+2})$. Using Lemma 5.2.22 and 5.2.21–5.2.25 with $k \leq m-4$ shows that

$$\|\partial_t^2 f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_\sigma)} \lesssim \Lambda(1+M_{\mathrm{in}}).$$

Now for $i=3$, we observe that $\|\partial_t^3 f\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)}$ is estimated by, at most, $\|\partial_t^2 f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{k+1}_{\sigma+1})}$ and $\|\partial_t^2 E^{\varrho,u}_{\mathrm{reg},\varepsilon}\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_\sigma)}$, thus requiring to estimate at most $\|\partial_t^2 u\|_{\mathrm{H}^k}$ and $\|\partial_t^2 \varrho\|_{\mathrm{H}^{k+1}}$. Using again the equation on u and ϱ, this now involves u in $\mathrm{L}^\infty(0,T;\mathrm{H}^{k+4})$ and ρ_f, j_f, ϱ in $\mathrm{L}^\infty(0,T;\mathrm{H}^{k+3})$. Using Lemma 5.2.22 and Remarks 5.2.21–5.2.25 with $k \leq m-5$ shows that

$$\|\partial_t^3 f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_\sigma)} \leq \Lambda(1+M_{\mathrm{in}}).$$

The same can be done for $i=4$, implying for $k \leq m-6$

$$\|\partial_t^4 f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_\sigma)} \leq \Lambda(1+M_{\mathrm{in}}).$$

This allows us to conclude the proof. □

We are now in position to prove the following result on the Penrose symbol $\mathscr{P}_{f,\varrho}$, which is, as we recall here:

$$\mathscr{P}_{f,\varrho}(t,x,\gamma,\tau,\eta) = \frac{p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)} \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2}\cdot\left(\mathcal{F}_v \nabla_v f\right)(t,x,\eta s)\,\mathrm{d}s,$$

and is defined for all $t \in \mathbb{R}$. The symbol seminorms (5.C.1)–(5.C.2)–(5.C.3) of $\mathscr{P}_{f,\varrho}$ are first estimated as follows.

Lemma 5.6.12. *For any $k>d/2$ and $\mathrm{M} \geq 0$, we have*

$$\omega[\mathscr{P}_{f,\varrho}] \leq \Lambda\left(\sup_{i=0,1}\left\|\partial_t^i \varrho\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})}, \sup_{i=0,1}\left\|\partial_t^i \rho_f\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})}\right)\omega[a_f], \tag{5.6.15}$$

$$\Xi[\mathscr{P}_{f,\varrho}]_{\mathrm{M}} \leq \Lambda\left(\sup_{i=0,1,2,3,4}\left\|\partial_t^i \varrho\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{1+M+k})}, \sup_{i=0,1,2,3,4}\left\|\partial_t^i \rho_f\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{1+\mathrm{M}+k})}\right)\Xi[a_f]_{\mathrm{M}}, \tag{5.6.16}$$

$$\Omega[\mathscr{P}_{f,\varrho}] \leq \Lambda\left(\sup_{i=0,1}\left\|\partial_t^i \varrho\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})}, \sup_{i=0,1}\left\|\partial_t^i \rho_f\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})}\right)\Omega[a_f]. \tag{5.6.17}$$

Proof. Since the symbol $\mathscr{P}_{f,\varrho}$ depends only on (γ,τ,k) through a_f, we only prove the estimate (5.6.15), the treatment of (5.6.16) and (5.6.17) being similar.

First, we write the symbol $\mathscr{P}_{f,\varrho}$ as

$$\mathscr{P}_{f,\varrho}(t,x,\gamma,\tau,k) = \mathfrak{m}(\varrho(t,x),\rho_f(t,x))\frac{1}{1+|k|^2}a_f(t,x,\gamma,\tau,k),$$

where a_f has been defined in (5.6.8) and where

$$\mathfrak{m}(\varrho(t,x),\rho_f(t,x)) := \frac{p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)}.$$

We have

$$\begin{aligned}&\omega[\mathscr{P}_{f,\varrho}]\\ &\lesssim \sup_{\substack{\alpha\in\times\mathbb{N}^d\\ \alpha_i\in\{0,1\}}} \|(1+t)\partial_x^\alpha\left(\mathfrak{m}(\varrho,\rho_f)a_f\right)\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{T}^d;\mathrm{L}^\infty_\eta)} + \sup_{\substack{\alpha\in\times\mathbb{N}^d\\ \alpha_i\in\{0,1\}}} \|(1+t)\partial_x^\alpha\left(\mathfrak{m}(\varrho,\rho_f)\partial_t a_f\right)\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{T}^d;\mathrm{L}^\infty_\eta)}\\ &\quad + \sup_{\substack{\alpha\in\times\mathbb{N}^d\\ \alpha_i\in\{0,1\}}} \|(1+t)\partial_x^\alpha\left(\partial_1\mathfrak{m}(\varrho,\rho_f)\partial_t\varrho + \partial_2\mathfrak{m}(\varrho,\rho_f)\partial_t\rho_f)a_f\right)\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{T}^d;\mathrm{L}^\infty_\eta)}.\end{aligned}$$

Using Leibniz rule, we get for all $\alpha\in\mathbb{N}^d$ such that $\alpha_i\in\{0,1\}$

$$(1+t)\left|\partial_x^\alpha\left(\mathfrak{m}(\varrho,\rho_f)a_f\right)\right| \lesssim \sum_{\beta\leq\alpha}\left|\partial_x^\beta\left(\mathfrak{m}(\varrho,\rho_f)\right)\right|\left|\partial_x^{\alpha-\beta}a_f\right| \leq \omega[a_f]\sum_{\beta\leq\alpha}\left|\partial_x^\beta\left(\mathfrak{m}(\varrho,\rho_f)\right)\right|,$$

and one can observe that for all $k>\frac{d}{2}$

$$\sum_{\beta\leq\alpha}\left|\partial_x^\beta\left(\mathfrak{m}(\varrho,\rho_f)\right)\right| \lesssim \|\mathfrak{m}(\varrho,\rho_f)\|_{\mathrm{H}^{d+k}} \leq \Lambda(\|\varrho\|_{\mathrm{L}^\infty},\|\rho_f\|_{\mathrm{L}^\infty})\|\varrho\|_{\mathrm{H}^{d+k}}\|\rho_f\|_{\mathrm{H}^{d+k}},$$

where we have used Proposition 5.A.3 in the Appendix and the fact that H^{k+d} is an algebra. Doing the same with the term involving $\partial_t a_f$, we obtain

$$\begin{aligned}&\sup_{\substack{\alpha\in\times\mathbb{N}^d\\ \alpha_i\in\{0,1\}}} \|(1+t)\partial_x^\alpha\left(\mathfrak{m}(\varrho,\rho_f)a_f\right)\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{T}^d;\mathrm{L}^\infty_\eta)} + \sup_{\substack{\alpha\in\times\mathbb{N}^d\\ \alpha_i\in\{0,1\}}} \|(1+t)\partial_x^\alpha\left(\mathfrak{m}(\varrho,\rho_f)\partial_t a_f\right)\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{T}^d;\mathrm{L}^\infty_\eta)}\\ &\qquad\leq \Lambda\left(\|\varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^\infty)},\|\rho_f\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^\infty)}\right)\|\varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})}\|\rho_f\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})}\,\omega[a_f].\end{aligned}$$

Likewise, the same kind of computations, that we do not detail, show that for all $k>\frac{d}{2}$

$$\begin{aligned}&\sup_{\substack{\alpha\in\times\mathbb{N}^d\\ \alpha_i\in\{0,1\}}} \|\partial_x^\alpha\left([\partial_1 G(\varrho,\rho_f)\partial_t\varrho + \partial_2 G(\varrho,\rho_f)]\partial_t\rho_f)a_f\right)\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{T}^d;\mathrm{L}^\infty_\eta)}\\ &\leq \Lambda\left(\|\varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^\infty)},\|\rho_f\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^\infty)}\right)\\ &\qquad\times\Lambda\left(\|\varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})},\|\partial_t\varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})},\|\rho_f\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})},\|\partial_t\rho_f\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})}\right)\omega[a_f].\end{aligned}$$

This concludes the proof of the estimate (5.6.15) of the lemma, thanks to Sobolev embedding. □

We finally obtain the following result, yielding some control on the seminorms of the Penrose function in terms of the initial data only.

Corollary 5.6.13. *There hold*

$$\omega[\mathscr{P}_{f,\varrho}] \lesssim \Lambda(1+M_{\text{in}}), \tag{5.6.18}$$

$$\Xi[\mathscr{P}_{f,\varrho}]_{\mathrm{M}} \lesssim \Lambda(1+M_{\text{in}}), \quad 2\mathrm{M} < m-11-d/2, \tag{5.6.19}$$

$$\Omega\,[\mathscr{P}_{f,\varrho}] \lesssim \Lambda(1+M_{\text{in}}). \tag{5.6.20}$$

Proof. We combine the estimates (5.6.9)–(5.6.10)–(5.6.11) of Lemma 5.6.10 with (5.6.15)–(5.6.16)–(5.6.17) of Lemma 5.6.12. We first get for $3+3d/2 \leq \ell < m-3$

$$\begin{aligned}\omega[\mathscr{P}_{f,\varrho}] &\leq \Lambda\left(\sup_{i=0,1}\left\|\partial_t^i\varrho\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+\ell})}, \sup_{i=0,1}\left\|\partial_t^i\rho_f\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+\ell})}\right)\sup_{i=0,1}\|(1+t)\partial_t^i f\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{H}^\ell_r)} \\ &\leq \Lambda\left(\sup_{i=0,1}\left\|\partial_t^i\varrho\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+\ell})}, \sup_{i=0,1}\left\|\partial_t^i f\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{H}^{d+\ell}_\sigma)}\right)\Lambda(1+M_{\text{in}}),\end{aligned}$$

for $\sigma > d/2$. Hence, by using the equation on ϱ with Remarks 5.2.21 and Lemma (5.2.22), and by taking $\ell < m-3-d$, we can use (5.6.13) from Lemma 5.6.11 to obtain (5.6.18). Next, we have for $4+\mathrm{M}+d/2 < \ell < m-6$

$$\begin{aligned}\Xi[\mathscr{P}_{f,\varrho}]_{\mathrm{M}} &\leq \Lambda\left(\sup_{i=0,1,2,3,4}\left\|\partial_t^i\varrho\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{1+\mathrm{M}+\ell})}, \sup_{i=0,1,2,3,4}\left\|\partial_t^i\rho_f\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{1+\mathrm{M}+\ell})}\right)\sup_{i=0,1,2,3,4}\|\partial_t^i f\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{H}^\ell_r)} \\ &\leq \Lambda\left(\sup_{i=0,1,2,3,4}\left\|\partial_t^i\varrho\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{1+\mathrm{M}+\ell})}, \sup_{i=0,1,2,3,4}\left\|\partial_t^i f\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{H}^{1+\mathrm{M}+\ell}_\sigma)}\right)\Lambda(1+M_{\text{in}}),\end{aligned}$$

for $\sigma > d/2$. Using (5.6.14) from Lemma 5.6.11, the equation on ϱ, Remark 5.2.21 and Lemma (5.2.22) with $\ell+1+\mathrm{M} < m-6$, we deduce (5.6.19). Finally, we also have for $7+3d/2 < \ell < m-3-d$

$$\Omega[\mathscr{P}_{f,\varrho}] \leq \Lambda\left(\sup_{i=0,1}\left\|\partial_t^i\varrho\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+\ell})}, \sup_{i=0,1}\left\|\partial_t^i f\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{H}^{d+\ell}_\sigma)}\right)\Lambda(1+M_{\text{in}}),$$

for $\sigma > d/2$ and as before, we obtain (5.6.20). □

5.6.5 Elliptic estimates through pseudodifferential analysis

Let $T \in (0, T_\varepsilon^\star)$. In view of the equation (5.6.1) on $h = \partial_x^\alpha \varrho$ obtained in Proposition 5.6.1, we initiate the study of the equation

$$\left(\mathrm{Id} - \frac{\varrho}{1-\rho_f}\mathrm{K}_G^{\text{free}} \circ \mathrm{J}_\varepsilon\right)\left[\widetilde{H}\right] = \widetilde{\mathcal{R}}, \quad 0 \leq t \leq T, \tag{5.6.21}$$

where $\widetilde{\mathcal{R}}$ is a given source term defined on $(0,T)$. Given a solution $\widetilde{H}$ to this equation, we want to derive an $\mathrm{L}^2(0,T;\mathrm{L}^2)$ estimate of $\widetilde{H}$ in terms of $\widetilde{R}$. This will be possible thanks to the Penrose stability condition satisfied by $(f(t),\varrho(t))$ (see the forthcoming Proposition 5.6.17).

Note that the operator involved in the equation (5.6.21) depends on ϱ and f, which are defined for all times.

Following [HKR16], we would like to link the operator $\mathrm{Id} - \frac{\varrho}{1-\rho_f}\mathrm{K}_G^{\text{free}} \circ \mathrm{J}_\varepsilon$ which appears in (5.6.21) to a pseudodifferential operator of order 0 in time-space. We will use the following notation for the Fourier transform in time-space, and we also refer to Section 5.C in the Appendix:

$$\forall (\tau,k) \in \mathbb{R}\times\mathbb{Z}^d, \quad \mathcal{F}_{t,x}g(\tau,k) = \int_{\mathbb{R}\times\mathbb{T}^d} e^{-i(\tau t + k\cdot x)} g(t,x)\,\mathrm{d}t\,\mathrm{d}x.$$

For symbols of the form $a(t,x,\gamma,\tau,k)$ on $[0,T]\times\mathbb{T}^d\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\}$ and in the Schwartz class, we rely on the quantification

$$\mathrm{Op}^\gamma(a)(h)(t,x)=\frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{Z}^d}e^{i(\tau t+k\cdot x)}a(t,x,\gamma,\tau,k)\mathcal{F}_{t,x}h(\tau,k)\,\mathrm{d}\tau\,\mathrm{d}k.$$

The measure on $\mathbb{Z}^d$ is the discrete measure. Note that the symbols we shall use are defined on $\mathbb{R}\times\mathbb{T}^d$ in the physical space, thanks to the extension procedure from Section 5.6.3. Note also that we shall handle symbols defined in the whole space $\mathbb{R}^d$ for the k variable, even if we only use them for $k\in\mathbb{Z}^d$ in the formula.

For $\gamma>0$ (which will be chosen large enough in the end, but always independently of ε), we set

$$\widetilde{H}(t,x):=e^{\gamma t}H(t,x),\quad \widetilde{\mathcal{R}}(t,x):=e^{\gamma t}\mathcal{R}(t,x).$$

We can rewrite (5.6.21) as

$$H(t,x)-\frac{\varrho}{1-\rho_f}e^{-\gamma t}\mathrm{K}_G^{\mathrm{free}}\Big[e^{\gamma\bullet}\mathrm{J}_\varepsilon H\Big](t,x)=\mathcal{R}(t,x),\quad 0\le t\le T. \tag{5.6.22}$$

First, let us extend the solution H by zero for times $t<0$ and let us choose any smooth and compactly extension in time of H after time T. We still call this solution H. The following lemma now provides the link between the integro-differential equation (5.6.22) and a pseudodifferential equation.

Lemma 5.6.14. *We have*

$$e^{-\gamma t}\mathrm{K}_G^{\mathrm{free}}\Big[e^{\gamma\bullet}H\Big](t,x)=p'(\varrho(t,x))\mathrm{Op}^\gamma(a_f)(H)(t,x),\quad \text{on}\ \ \mathbb{R}\times\mathbb{T}^d,$$

where a_f is defined in (5.6.8). In particular, the equation (5.6.22) on H reads

$$H-\frac{p'(\varrho)\varrho}{1-\rho_f}\mathrm{Op}^\gamma(a_f)(\mathrm{J}_\varepsilon H)=S,\quad \text{on}\ \ \mathbb{R}\times\mathbb{T}^d, \tag{5.6.23}$$

where $S(t,x)$ is a source term defined on $\mathbb{R}\times\mathbb{T}^d$ such that $S_{|(-\infty;0)}=0$ and $S_{|[0,T]}=\mathcal{R}$.

Proof. We write

$$H(t,x)=\frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{Z}^d}e^{i(\tau t+k\cdot x)}\mathcal{F}_{t,x}H(\tau,k)\,\mathrm{d}\tau\,\mathrm{d}k,$$

therefore

$$\begin{aligned}&e^{-\gamma t}\mathrm{K}_G^{\mathrm{free}}(e^{\gamma\bullet}H)(t,x)\\&=\frac{p'(\varrho(t,x))}{(2\pi)^{d+1}}\int_{-\infty}^t\int_{\mathbb{R}^d}\int_{\mathbb{R}\times\mathbb{Z}^d}e^{-\gamma(t-s)}e^{i(\tau s+k\cdot(x-(t-s)v)}ik\cdot\nabla_v f(t,x,v)\mathcal{F}_{t,x}H(\tau,k)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}\tau\,\mathrm{d}k,\end{aligned}$$

because H is 0 on negative times. We use the Fubini theorem (which holds since $\gamma>0$) and get

$$\begin{aligned}e^{-\gamma t}\mathrm{K}_G^{\mathrm{free}}(e^{\gamma\bullet}h)(t,x)=\frac{p'(\varrho(t,x))}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{Z}^d}e^{i(\tau t+k\cdot x)}\\\left(\int_{-\infty}^t e^{-(\gamma+i\tau)(t-s)}ik\cdot\int_{\mathbb{R}^d}e^{-ik\cdot v(t-s)}\nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s\right)\mathcal{F}_{t,x}H(\tau,k)\,\mathrm{d}\tau\,\mathrm{d}k,\end{aligned}$$

therefore, setting $s' = t - s$

$$\begin{aligned}
&e^{-\gamma t}\mathrm{K}_G^{\text{free}}(e^{\gamma\bullet}h)(t,x) \\
&= \frac{p'(\varrho(t,x))}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{Z}^d} e^{i(\tau t + k\cdot x)}\left(\int_0^{+\infty} e^{-(\gamma+i\tau)s} ik\cdot\int_{\mathbb{R}^d} e^{-ik\cdot vs}\nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s\right)\mathcal{F}_{t,x}H(\tau,k)\,\mathrm{d}\tau\,\mathrm{d}k \\
&= \frac{p'(\varrho(t,x))}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{Z}^d} e^{i(\tau t + k\cdot x)}\left(\int_0^{+\infty} e^{-(\gamma+i\tau)s} ik\cdot\left(\mathcal{F}_v\nabla_v f\right)(t,x,ks)\,\mathrm{d}s\right)\mathcal{F}_{t,x}H(\tau,k)\,\mathrm{d}\tau\,\mathrm{d}kw \\
&= p'(\varrho(t,x,))\mathrm{Op}(a_f)(h)(t,x).
\end{aligned}$$

This provides the desired equality. The fact that the source term S in the final pseudodifferential equation (5.6.23) satisfies $S_{|(-\infty;0)} = 0$ comes from the equation and the fact that H is zero for negative times. □

Having in mind a semiclassical approach (see also Apppendix 5.C), we introduce the following quantization.

Definition 5.6.15. *For any symbol $b(t,x,\gamma,\tau,k)$ on $R\times\mathbb{T}^d\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\}$, we set for $\varepsilon\in(0,1)$*

$$\begin{aligned}
b^\varepsilon(t,x,\gamma,\tau,k) &:= b(t,x,\varepsilon\gamma,\varepsilon\tau,\varepsilon k), \\
\mathrm{Op}^{\gamma,\varepsilon}(b) &:= \mathrm{Op}^\gamma(b^\varepsilon).
\end{aligned}$$

Lemma 5.6.16. *We have*

$$p'(\varrho)\mathrm{Op}^\gamma(a_{f,\varrho})(\mathrm{J}_\varepsilon H) = \mathrm{Op}^{\gamma,\varepsilon}(\mathfrak{a}_{f,\varrho})(H), \tag{5.6.24}$$

where

$$\mathfrak{a}_f(t,x,\eta) := \frac{1}{1+|k|^2}p'(\varrho(t,x))a_f(t,x,\eta).$$

Proof. We have the exact composition formula

$$\begin{aligned}
p'(\varrho)\mathrm{Op}^\gamma(a_f)(\mathrm{J}_\varepsilon H) &= \mathrm{Op}^\gamma(\widetilde{a}_{\varepsilon,f,\varrho})(H), \\
\widetilde{a}_{\varepsilon,f,\varrho}(t,x,\eta) &:= p'(\varrho(t,x))a_f(t,x,\eta)\frac{1}{1+|\varepsilon k|^2},
\end{aligned}$$

since we are composing on the right by a Fourier multiplier. Since $a_{f,\varrho}$ is homogeneous of degree 0 in the variable η (see Proposition 5.6.10), we have

$$\widetilde{a}_{\varepsilon,f}(t,x,\eta) = p'(\varrho(t,x))a_f(t,x,\varepsilon\eta)\frac{1}{1+|\varepsilon k|^2} = \mathfrak{a}_{f,\varrho}(t,x,\varepsilon\eta),$$

and the conclusion follows. □

In the following proposition, we show that we can choose an extension of (f,ϱ,u) as in Subsection 5.6.3 and such that it satisfies a Penrose condition for all times. By definition of $T_\varepsilon^\star$, we know that

$$\forall t\in[0,T_\varepsilon^\star],\quad \inf_{(x,\gamma,\tau,k)}|1-\mathscr{P}_{f(t),\varrho(t)}(x,\gamma,\tau,k)|\geq c_0/2,$$

where we recall the expression of the Penrose symbol:

$$\mathscr{P}_{f(t),\varrho(t)}(x,\gamma,\tau,k) = \frac{p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)}\int_0^{+\infty} e^{-(\gamma+i\tau)s}\frac{ik}{1+|k|^2}\cdot\left(\mathcal{F}_v\nabla_v f\right)(t,x,ks)\,\mathrm{d}s.$$

We want this condition to be true for the extension of (f,ϱ). We have the following result, which requires the technical assumption (5.1.2) on the pressure.

Proposition 5.6.17. *There exists $\delta^\star = \delta(c_0, M_{\mathrm{in}}) > 0$ small enough such that, considering the extension of f and ϱ with respect to $T_\varepsilon^\star$ and $\delta^\star$, we have*

$$\forall t \in \mathbb{R}, \quad \inf_{(x,\gamma,\tau,k)} |1 - \mathscr{P}_{f(t),\varrho(t)}(x,\gamma,\tau,k)| \geq c_0/4.$$

Proof. We only treat the case of $t \in (T_\varepsilon^\star, +\infty)$, the case of negative times being identical. Let us set

$$\chi_\delta^\star(t) := \chi_\delta(t - T_\varepsilon^\star), \quad \underline{\chi}^\star(t) := \underline{\chi}(t - T_\varepsilon^\star),$$

where we use the notations of Section 5.6.3. By definition of the extension for f, we have

$$\begin{aligned}
&1 - \mathscr{P}_{f(t),\varrho(t)} \\
&= 1 - \frac{\chi_\delta^\star(t) p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)} \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot (\mathcal{F}_v \nabla_v f)(T_\varepsilon^\star, x, ks)\,\mathrm{d}s \\
&\quad - \frac{\chi_\delta^\star(t) p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)} \sum_{k=1}^{N_f} \frac{(t-T_\varepsilon^\star)^k}{k!} \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot \left(\mathcal{F}_v \nabla_v \partial_t^k f\right)(T_\varepsilon^\star, x, ks)\,\mathrm{d}s.
\end{aligned} \tag{5.6.25}$$

We next proceed to the following decompositions: we have

$$\begin{aligned}
\frac{1}{1-\rho_f(t)} &= \frac{1}{1 - \chi_\delta^\star(t)\rho_f(T_\varepsilon^\star) - \underline{\chi}^\star(t)\sum_{k=1}^{N_f} \partial_t^k \rho_f(T_\varepsilon^\star)\frac{(t-T_\varepsilon^\star)^k}{k!}} \\
&= \frac{1}{1-\chi_\delta^\star(t)\rho_f(T_\varepsilon^\star)} + \frac{1}{1-\chi_\delta^\star(t)\rho_f(T_\varepsilon^\star)} \frac{Q^\star(t)}{1-Q^\star(t)},
\end{aligned}$$

where

$$Q^\star(t) := \frac{\chi_\delta^\star(t)\sum_{k=1}^{N_f} \partial_t^k \rho_f(T_\varepsilon^\star)\frac{(t-T_\varepsilon^\star)^k}{k!}}{1-\chi_\delta^\star(t)\rho_f(T_\varepsilon^\star)},$$

and by writing

$$\varrho(t) = \underline{\chi}^\star(t)\varrho(T_\varepsilon^\star) + R^\star(t), \quad R^\star(t) := \chi_\delta(t - T_\varepsilon^\star)\sum_{k=1}^{N_\varrho} \varrho^{(k)}(T_\varepsilon^\star)\frac{(t-T_\varepsilon^\star)^k}{k!},$$

we also have

$$p'(\varrho(t))\varrho(t) = p'\left(\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star)\right)\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star) + S^\star(t),$$

where

$$S^\star(t) := \left[p'\left(\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star) + R^\star(t)\right) - p'\left(\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star)\right)\right]\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star) + p'\left(\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star) + R^\star(t)\right)R^\star(t).$$

Note that

$$S^\star(t) = \underset{t\to T_\varepsilon^\star}{\mathcal{O}}(t - T_\varepsilon^\star).$$

Now, the equality (5.6.25) turns into

$$\begin{aligned}
&1 - \mathscr{P}_{f(t),\varrho(t)} \\
&= 1 - \frac{\chi_\delta^\star(t) p'\left(\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star)\right)\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star)}{1-\chi_\delta^\star(t)\rho_f(T_\varepsilon^\star)} \frac{1}{1+|k|^2} \int_0^{+\infty} e^{-(\gamma+i\tau)s} ik \cdot (\mathcal{F}_v \nabla_v f)(T_\varepsilon^\star, x, ks)\,\mathrm{d}s \\
&\quad + \mathfrak{E}^\star(t),
\end{aligned}$$

with a remainder

$$
\begin{aligned}
&\mathfrak{E}^\star(t) \\
&:= -\chi^\star_\delta(t)\Bigg(\frac{p'\left(\underline{\chi}^\star(t)\varrho(T^\star_\varepsilon)\right)\underline{\chi}^\star(t)\varrho(T^\star_\varepsilon)}{1-\chi^\star_\delta(t)\rho_f(T^\star_\varepsilon)}\frac{Q^\star(t)}{1-Q^\star(t)}\int_0^{+\infty} e^{-(\gamma+i\tau)s}\frac{ik}{1+|k|^2}\cdot\left(\mathcal{F}_v\nabla_v f\right)(T^\star_\varepsilon,x,ks)\,\mathrm{d}s \\
&\quad + \frac{S^\star(t)}{1-\chi^\star_\delta(t)\rho_f(T^\star_\varepsilon)}\int_0^{+\infty} e^{-(\gamma+i\tau)s}\frac{ik}{1+|k|^2}\cdot\left(\mathcal{F}_v\nabla_v f\right)(T^\star_\varepsilon,x,ks)\,\mathrm{d}s \\
&\quad + \frac{S^\star(t)}{1-\chi^\star_\delta(t)\rho_f(T^\star_\varepsilon)}\frac{Q^\star(t)}{1-Q^\star(t)}\int_0^{+\infty} e^{-(\gamma+i\tau)s}\frac{ik}{1+|k|^2}\cdot\left(\mathcal{F}_v\nabla_v f\right)(T^\star_\varepsilon,x,ks)\,\mathrm{d}s \\
&\quad + \frac{p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)}\sum_{k=1}^{N_f}\frac{(t-T^\star_\varepsilon)^k}{k!}\int_0^{+\infty} e^{-(\gamma+i\tau)s}\frac{ik}{1+|k|^2}\cdot\left(\mathcal{F}_v\nabla_v\partial_t^k f\right)(T^\star_\varepsilon,x,ks)\,\mathrm{d}s\Bigg).
\end{aligned}
$$

We claim that an homogeneity argument shows that

$$
\begin{aligned}
&\inf_{(x,\gamma,\tau,k)}\left|1-\frac{\chi^\star_\delta(t)p'\left(\underline{\chi}^\star(t)\varrho(T^\star_\varepsilon,x)\right)\underline{\chi}^\star(t)\varrho(T^\star_\varepsilon,x)}{1-\chi^\star_\delta(t)\rho_f(T^\star_\varepsilon,x)}\frac{1}{1+|k|^2}\int_0^{+\infty} e^{-(\gamma+i\tau)s}ik\cdot\left(\mathcal{F}_v\nabla_v f\right)(T^\star_\varepsilon,x,ks)\,\mathrm{d}s\right| \\
&\geq \inf_{(x,\gamma,\tau,k)}\left|1-\frac{p'\left(\varrho(T^\star_\varepsilon,x)\right)\varrho(T^\star_\varepsilon,x)}{1-\rho_f(T^\star_\varepsilon,x)}\frac{1}{1+|k|^2}\int_0^{+\infty} e^{-(\gamma+i\tau)s}ik\cdot\left(\mathcal{F}_v\nabla_v f\right)(T^\star_\varepsilon,x,ks)\,\mathrm{d}s\right| \geq c_0/2.
\end{aligned}
$$

Indeed, we know from Lemma 5.6.10 that for all $x\in\mathbb{T}^d$, the function

$$
(\gamma,\tau,k)\mapsto\int_0^{+\infty} e^{-(\gamma+i\tau)s}ik\cdot\left(\mathcal{F}_v\nabla_v f\right)(T^\star_\varepsilon,x,ks)\,\mathrm{d}s
$$

is homogeneous of degree 0, and since by Assumption (5.1.2) and by construction we have

$$
0\leq\frac{\chi^\star_\delta(t)p'\left(\underline{\chi}^\star(t)\varrho(T^\star_\varepsilon,x)\right)\underline{\chi}^\star(t)\varrho(T^\star_\varepsilon,x)}{1-\chi^\star_\delta(t)\rho_f(T^\star_\varepsilon,x)}\leq\frac{p'\left(\varrho(T^\star_\varepsilon,x)\right)\varrho(T^\star_\varepsilon,x)}{1-\rho_f(T^\star_\varepsilon,x)},
$$

we can rely on Remark 5.1.9 (see (5.1.4)) to obtain the previous claim. By writing

$$
\inf_{(x,\gamma,\tau,k)}|1-\mathscr{P}_{f(t),\varrho(t)}(x,\gamma,\tau,k)|\geq\frac{c_0}{2}-\sup_{(x,\gamma,\tau,k)}|\mathfrak{E}^\star(t,x,\gamma,\tau,k)|
$$

thanks to the triangular inequality, it remains to prove that a suitable choice of δ can lead to

$$
\forall t\in[T^\star_\varepsilon,+\infty),\quad\sup_{(x,\gamma,\tau,k)}|\mathfrak{E}^\star(t,x,\gamma,\tau,k)|\leq\frac{c_0}{4}.
$$

First note that the remainder $\mathfrak{E}^\star$ has a factor $\chi^\star_\delta(t)$ in factor of all the terms in its expression, and is therefore compactly supported in time, with support in $(T^\star_\varepsilon,T^\star_\varepsilon+\delta)$. Relying on the bounds for f and ϱ depending only on M_{in} (see Remarks 5.2.21 and Lemma 5.2.22), we can proceed as in the proof of the estimates (5.6.18)–(5.6.19)–(5.6.20) and show that

$$
|\mathfrak{E}^\star(t)|\leq\Lambda(M_{\mathrm{in}})\chi^\star_\delta(t)|t-T^\star_\varepsilon|\leq\Lambda(M_{\mathrm{in}})\delta.
$$

This procedure is allowed since the extension and the remainder $\mathfrak{E}^\star$ only involve a finite number of derivatives in time of ϱ and f at $t=T^\star_\varepsilon$. Choosing δ small enough is now sufficient to conclude. □

We are now in position to provide the key $\mathrm{L}^2(\mathbb{R},\mathrm{L}^2)$ estimates for a solution to the equation (5.6.23). We will actually consider a slightly different pseudodifferential equation, which is

$$\mathcal{H} - \frac{p'(\varrho)\varrho}{1-\rho_f}\mathrm{Op}^{\gamma}(a_f)(\mathrm{J}_\varepsilon \mathcal{H}) = \mathcal{S}, \quad \text{on} \ \ \mathbb{R}\times\mathbb{T}^d, \tag{5.6.26}$$

where $\mathcal{S}=S$ on $[0,T]$ and where $\mathcal{S}$ is zero outisde $[0,T]$. The main part of our analysis will provide some estimates $\mathrm{L}^2(\mathbb{R};\mathrm{L}^2)$ on the solution $\mathcal{H}$ of the equation (5.6.26), and we will show subsequently that it will provide an $\mathrm{L}^2(0,T,\mathrm{L}^2)$ on H, solution to the original equation (5.6.22). This will be based on a causality principle for the pseudodifferential equation (5.6.22) (see Lemma 5.6.19 below).

Proposition 5.6.18. *Assume that $\mathcal{H}$ is a solution of the equation (5.6.26) on $\mathbb{R}\times\mathbb{T}^d$. There exists $\Lambda(M_{\mathrm{in}})>0$ such that for any $\gamma\geq\Lambda(M_{\mathrm{in}})$, we have*

$$\|\mathcal{H}\|_{\mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)} \leq \Lambda(M_{\mathrm{in}})\|\mathcal{S}\|_{\mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)}.$$

In particular, we have

$$\|\mathcal{H}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^d)} \leq \Lambda(M_{\mathrm{in}})\|\mathcal{S}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^d)}.$$

Proof. Thanks to (5.6.24) and Lemma 5.6.14, the equation (5.6.26) can be rewritten as

$$\left(\mathrm{Id} - \frac{\varrho}{1-\rho_f}\mathrm{Op}^{\gamma,\varepsilon}(\mathfrak{a}_{f,\varrho})\right)(\mathcal{H}) = \mathcal{S},$$

where $\mathfrak{a}_{f,\varrho}$ has been defined in (5.6.24). Now observe that, recalling the definition (5.1.3) of the Penrose symbol $\mathscr{P}_{f,\varrho}$, we have

$$\frac{\varrho}{1-\rho_f}\mathfrak{a}_{f,\varrho} = \mathscr{P}_{f,\varrho},$$

therefore H satisfies

$$\mathrm{Op}^{\gamma,\varepsilon}(1-\mathscr{P}_{f,\varrho})(\mathcal{H}) = \mathcal{S}. \tag{5.6.27}$$

Relying on Proposition 5.6.17 on the Penrose condition satisfied by the (extension) of $(f(t),\varrho(t))$ on $\mathbb{R}$, we can consider

$$c_{f,\varrho} := \frac{1}{1-\mathscr{P}_{f,\varrho}}.$$

Note that the symbol $c_{f,\varrho}-1$ vanishes outside a compact set in time and hence, in view of the estimates (5.6.18)–(5.6.19)–(5.6.20) of Corollary 5.6.13 on the symbol $\mathscr{P}_{f,\varrho}$ and the Faà di Bruno's formula, we get

$$\omega[c^{\varepsilon}_{f,\varrho}-1] + \Omega\left[c^{\varepsilon}_{f,\varrho}-1\right] \leq \Lambda(1+M_{\mathrm{in}}). \tag{5.6.28}$$

Applying $\mathrm{Op}^{\gamma,\varepsilon}(c_{f,\varrho})$ to the previous equation (5.6.27) yields

$$\begin{aligned}\mathcal{H} &= \mathcal{S} + \mathrm{Op}^{\gamma,\varepsilon}(c_{f,\varrho}-1)(\mathcal{S}) + \Big[\mathrm{Op}^{\gamma,\varepsilon}((c_{f,\varrho}-1)(1-\mathscr{P}_{f,\varrho})) - \mathrm{Op}^{\gamma,\varepsilon}(c_{f,\varrho}-1)\,\mathrm{Op}^{\gamma,\varepsilon}(1-\mathscr{P}_{f,\varrho})\Big](\mathcal{H}) \\ &= \mathcal{S} + \mathrm{Op}^{\gamma,\varepsilon}(c_{f,\varrho}-1)(\mathcal{S}) - \Big[\mathrm{Op}^{\gamma,\varepsilon}((c_{f,\varrho}-1)\mathscr{P}_{f,\varrho}) - \mathrm{Op}^{\gamma,\varepsilon}(c_{f,\varrho}-1)\,\mathrm{Op}^{\gamma,\varepsilon}(\mathscr{P}_{f,\varrho})\Big](\mathcal{H}).\end{aligned}$$

Using the L^2 continuity property from Theorem 5.C.2 and the commutation estimates from Proposition 5.C.3 in the Appendix, we get for all $\gamma > 0$ and $\mathrm{M} > 1 + 2d$

$$\|\mathcal{H}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} \leq (1 + C\omega[c_{f,\varrho} - 1]) \|\mathcal{S}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} + \frac{C}{\gamma}\Omega[c^{\varepsilon}_{f,\varrho} - 1]\Xi[\mathscr{P}^{\varepsilon}_{f,\varrho}]_{\mathrm{M}} \|\mathcal{H}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)},$$

for some constant $C > 0$ depending only on the dimension. By homogeneity of the previous seminorms with respect to the semiclassical quantification in ε (in particular the fact that $\Omega[c^{\varepsilon}_{f,\varrho} - 1] \leq \Omega[c_{f,\varrho} - 1]$ for $\varepsilon \leq 1$) , we infer

$$\|\mathcal{H}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} \leq (1 + C\omega[c_{f,\varrho} - 1]) \|\mathcal{S}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} + \frac{C}{\gamma}\Omega[c_{f,\varrho} - 1]\Xi[\mathscr{P}_{f,\varrho}]_{\mathrm{M}} \|\mathcal{H}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)}.$$

Thanks to (5.6.28) and the estimate (5.6.19) of Corollary 5.6.13, we get

$$\|\mathcal{H}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} \leq C\Lambda(1 + M_{\mathrm{in}}) \|\mathcal{S}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} + \frac{C}{\gamma}\Lambda(1 + M_{\mathrm{in}}) \|\mathcal{H}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)}.$$

Taking γ large enough with respect to M_{in} allows to perform an absorption argument: we get the existence of some $\Lambda(M_{\mathrm{in}}) > 0$ such that for any $\gamma \geq \Lambda(M_{\mathrm{in}})$, we have

$$\|\mathcal{H}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} \lesssim \Lambda(M_{\mathrm{in}})\|\mathcal{S}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)},$$

which was the desired conclusion. The last inequality which was stated comes from the fact that $\mathcal{S}$ is zero outside $[0, T]$. □

Let us briefly explain how to obtain a solution $\mathcal{H}$ of the equation (5.6.26) with source term $\mathcal{S}$. The main idea is to use the estimate derived in Proposition 5.6.18. Indeed, setting

$$A_\gamma := \mathrm{Op}^\gamma(1 - \mathscr{P}_{f,\varrho}), \quad B_\gamma := \mathrm{Op}^\gamma\left(\frac{1}{1 - \mathscr{P}_{f,\varrho}}\right),$$

one can repeat the proof of Proposition 5.6.18 to show that

$$B_\gamma A_\gamma = \mathrm{Id} + \frac{1}{\gamma}C_\gamma,$$

where C_γ is a bounded operator on $L^2(\mathbb{R}\times\mathbb{T}^d)$, whose norm is uniformly bounded independently of γ. Hence, for γ large enough, we obtain the fact that $B_\gamma A_\gamma$ is invertible on $L^2(\mathbb{R}\times\mathbb{T}^d)$, and so A_γ has a left-inverse. By the same argument for $A_\gamma B_\gamma$, we obtain the fact that A_γ is invertible on $L^2(\mathbb{R}\times\mathbb{T}^d)$, for γ large enough. This leads to the existence of a solution to the equation (5.6.26).

Let us now show how to relate the solutions of the equations (5.6.23) and (5.6.26) on $[0, T]$. This comes from the following causality principle.

Lemma 5.6.19. *Let $T > 0$. Consider the solution H to the equation (5.6.23) and $\mathcal{H}$ a solution to the equation (5.6.26) on $\mathbb{R}\times\mathbb{T}^d$. Then*

$$H|_{[0,T]} = \mathcal{H}|_{[0,T]}.$$

Proof. First introduce for $\gamma > 0$

$$\widetilde{H}(t, x) := e^{\gamma t}H(t, x), \quad \widetilde{S}(t, x) := e^{\gamma t}S(t, x),$$

and

$$\widetilde{\mathcal{H}}(t,x) := e^{\gamma t} H(t,x), \quad \widetilde{\mathcal{S}}(t,x) := e^{\gamma t}\mathcal{S}(t,x).$$

In view of the equation (5.6.23) and (5.6.26), we have by linearity

$$e^{-\gamma t}(\widetilde{H} - \widetilde{\mathcal{H}}) - \frac{p'(\varrho)\varrho}{1-\rho_f}\mathrm{Op}^\gamma(a_f)\left(\mathrm{J}_\varepsilon e^{-\gamma \cdot}(\widetilde{H} - \widetilde{\mathcal{H}})\right) = e^{-\gamma t}(\widetilde{S} - \widetilde{\mathcal{S}}), \quad \text{on } \mathbb{R}\times\mathbb{T}^d.$$

Note that the source term of this equation is zero on $(-\infty;T]$, by definition of S and $\mathcal{S}$ which satisfy

$$S|_{(-\infty,T]} = \mathcal{S}|_{(-\infty,T]}.$$

By Proposition 5.6.18, there exists $\Lambda(M_{\mathrm{in}})$ such that for all $\gamma \geq \Lambda(M_{\mathrm{in}})$

$$\begin{aligned}\int_0^T e^{-2\gamma t}\left\|\widetilde{H}(t) - \widetilde{\mathcal{H}}(t)\right\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\,\mathrm{d}t &\leq \|e^{-\gamma \cdot}(\widetilde{H} - \widetilde{\mathcal{H}})\|^2_{\mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)} \\ &\leq \Lambda(M_{\mathrm{in}})^2 \int_T^{+\infty} e^{-2\gamma t}\left\|\widetilde{S}(t) - \widetilde{\mathcal{S}}(t)\right\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\,\mathrm{d}t,\end{aligned}$$

for some $\gamma_0 > 0$ and $C_0 \geq 0$. We thus infer

$$\int_0^T \left\|\widetilde{H}(t) - \widetilde{\mathcal{H}}(t)\right\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\,\mathrm{d}t \leq \Lambda(M_{\mathrm{in}})^2 \int_T^{+\infty} e^{2\gamma(T-t)}\left\|\widetilde{S}(t) - \widetilde{\mathcal{S}}(t)\right\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\,\mathrm{d}t.$$

By letting $\gamma \to +\infty$, we deduce that $\widetilde{H}|_{[0,T]} = \widetilde{\mathcal{H}}|_{[0,T]}$, hence the result. ☐

As a consequence, we can eventually obtain an estimate for the equation (5.6.21) on $(0,T)\times\mathbb{T}^d$.

Corollary 5.6.20. *Consider $\widetilde{H}$ the solution to (5.6.21) on $(0,T)\times\mathbb{T}^d$. We have*

$$\left\|\widetilde{H}\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T, M_{\mathrm{in}})\left\|\widetilde{\mathcal{R}}\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.$$

Proof. First recall that defining

$$\widetilde{H}(t,x) := e^{\gamma t} H(t,x), \quad \widetilde{\mathcal{R}}(t,x) := e^{\gamma t}\mathcal{R}(t,x),$$

and extending H by zero for $t < 0$, it is solution of (5.6.23) on $\mathbb{R}\times\mathbb{T}^d$ with a source term S satisfying $S_{|(-\infty;0)} = 0$ and $S_{|[0,T]} = \mathcal{R}$. By Lemma 5.6.19 and Proposition 5.6.18, we thus get for all $\gamma \geq \Lambda(M_{\mathrm{in}})$

$$\int_0^T e^{-2\gamma t}\left\|\widetilde{H}(t)\right\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\,\mathrm{d}t \leq \Lambda(M_{\mathrm{in}})^2 \int_0^T e^{-2\gamma t}\left\|\widetilde{\mathcal{R}}(t)\right\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\,\mathrm{d}t,$$

therefore taking $\gamma = \Lambda(M_{\mathrm{in}})$ provides

$$\left\|\widetilde{H}\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq e^{\Lambda(M_{\mathrm{in}})T}\Lambda(M_{\mathrm{in}})\left\|\widetilde{\mathcal{R}}\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))},$$

which concludes the proof. ☐

5.6.6 Final hyperbolic estimates

To conclude this section, it remains to perform an energy estimate on the hyperbolic part

$$(\partial_t + u\cdot\nabla_x)(h).$$

Let us observe that by Remark 5.2.25, we have

$$\|\mathrm{div}_x\, u\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \lesssim \left(1+T^{1/2}\Lambda(T,R)\right) M_{\mathrm{in}} + T^{1/2}\Lambda(T,R),$$

by Sobolev embedding (since $m > 1 + d/2$), therefore for all $t \in (0, T_\varepsilon^\star)$, there holds

$$\|\mathrm{div}_x\, u\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty)} \leq 1 + 2M_{\mathrm{in}}. \tag{5.6.29}$$

We can therefore state the following lemma.

Lemma 5.6.21. *Let* $T \in \left(0, \min\left(T_\varepsilon(R), \overline{T}(R), \widetilde{T}_0(R), \widehat{T}(R)\right)\right)$. *Assume that* h *is a solution of the equation*

$$(\partial_t + u\cdot\nabla_x)(h) = \widetilde{H}, \quad t\in[0,T],$$

There holds

$$\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq e^{(1+M_{\mathrm{in}})T} T^{\frac12}\left(\|h(0)\|_{\mathrm{L}^2} + T^{\frac12}\left\|\widetilde{H}\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}\right).$$

Proof. The proof is standard; the hyperbolic nature of the equation provides pointwise in time L^2 bounds thanks to an energy estimate (in the same spirit as the proof of Proposition 5.2.3). For all $t\in[0,T]$, we have

$$\|h(t)\|_{\mathrm{L}^2} \leq e^{\|\mathrm{div}_x\, u\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty)}t}\|h(0)\|_{\mathrm{L}^2} + \int_0^t e^{\|\mathrm{div}_x\, u\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty)}(t-\tau)}\|\widetilde{H}(\tau)\|_{\mathrm{L}^2}\,\mathrm{d}\tau.$$

By (5.6.29), we get by the Cauchy-Schwarz inequality in time that for all $t\in[0,T]$

$$\|h(t)\|_{\mathrm{L}^2} \leq e^{(1+M_{\mathrm{in}})T}\left(\|h(0)\|_{\mathrm{L}^2} + T^{\frac12}\int_0^T \|\widetilde{H}(\tau)\|^2_{\mathrm{L}^2}\,\mathrm{d}\tau\right),$$

hence the conclusion, eventually taking the L^2 norm in time on $(0,T)$ in the previous inequality. □

Gathering the results of Lemma 5.6.2, Proposition 5.6.1, Corollary 5.6.20 and Lemma 5.6.21, we directly infer the following statement.

Corollary 5.6.22. *For all* $|\alpha| \leq m$, *for all* $T \in \left(0, \min\left(T_\varepsilon(R), \overline{T}(R), \widetilde{T}_0(R), \widehat{T}(R)\right)\right)$, *we have the estimate*

$$\|\partial_x^\alpha \varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq T^{\frac12}\Lambda(M_{\mathrm{in}}, T, R).$$

5.7 End of the proof

5.7.1 Conclusion of the bootstrap

Let us conclude the bootstrap argument. We choose

$$T \in \left(0, \min\left(T_\varepsilon(R), \overline{T}(R), \widetilde{T}_0(R), \widehat{T}(R)\right)\right).$$

We can consider the following explicit quantity, which appears in all the estimates from Section 5.6:

$$M_{\text{in}} := \left\|f^{\text{in}}\right\|_{\mathcal{H}^m_r} + \left\|\varrho^{\text{in}}\right\|_{\mathrm{H}^{m+1}} + \left\|u^{\text{in}}\right\|_{\mathrm{H}^m}.$$

We now use Corollary 5.6.22 to get

$$\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} \leq T^{1/2}\Lambda(M_{\text{in}}, T, R).$$

We also invoke the energy estimate of Lemma 5.2.22 on f that yields

$$\|f\|_{\mathrm{L}^\infty\left(0,T;\mathcal{H}^{m-1}_r\right)} \leq M_{\text{in}} + T^{\frac{1}{4}}\Lambda\left(T, R\right),$$

and the energy estimate of Proposition 5.2.24 on u giving

$$\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap\mathrm{L}^2(0,T;\mathrm{H}^{m+1})} \leq M_{\text{in}} + T^{1/2}\Lambda(M_{\text{in}}, T, R) + \Lambda(T, R)\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)},$$

and therefore, using the previous estimate on ϱ, there holds

$$\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap\mathrm{L}^2(0,T;\mathrm{H}^{m+1})} \leq M_{\text{in}} + T^{1/2}\Lambda(M_{\text{in}}, T, R).$$

Combining all these estimates together, we finally obtain the key estimate

$$\mathcal{N}_{m,r}(f, \varrho, u, T) \leq C\left(M_{\text{in}} + T^{1/4}\Lambda(T, R) + T^{1/2}\Lambda(M_{\text{in}}, T, R)\right),$$

for some universal constant $C > 0$. Note that the previous r.h.s is independent of ε. Next, we choose R large enough so that

$$CM_{\text{in}} < \frac{1}{2}R.$$

Now, with R being fixed, we use the continuity at 0 of the function

$$s \mapsto s^{1/4}\Lambda(s, R) + s^{1/2}\Lambda(M_{\text{in}}, s, R)$$

to find some time $T^\sharp > 0$ independent of ε with

$$T^\sharp \in \left(0, \min\left(\overline{T}(R), \widetilde{T}_0(R), \widehat{T}(R)\right)\right),$$

and such that for every $T \in [0, T^\sharp]$

$$C\left(T^{1/4}\Lambda(M_{\text{in}}, T, R) + T^{1/2}\Lambda(M_{\text{in}}, T, R)\right) < \frac{1}{2}R.$$

We deduce that for all $T \in [0, \min(T^\sharp, T_\varepsilon(R))]$, we have $\mathcal{N}_{m,r}(f, \varrho, u, T) < R$. In addition, thanks to Lemmas 5.2.16–5.2.17, we easily get the fact that the condition ($\mathrm{B}^{\mu,\theta}_{\Theta}(T)$) is satisfied, up to reducing $T^\sharp$ so that $\Theta + T^\sharp R < 1$ and

$$T^\sharp \in \left(0, \frac{1}{R}\min\left(\frac{1-\Theta}{2}, \ln(2), \ln\left(\frac{2\theta}{\mu}\right)\right)\right).$$

Since we were assuming that $\mathcal{N}_{m,r}(f, \varrho, u, T_\varepsilon(R)) = R$, this shows that we must have $T_\varepsilon > T^\sharp$.

In conclusion, we have found $R > 0$ and $T > 0$ such that for all $\varepsilon > 0$

$$\mathcal{N}_{m,r}(f, \varrho, u, T) \leq R.$$

5.7.2 Existence of a solution

Let us first focus on the existence part of Theorem 5.1.5. We will rely on a standard compactness argument, that we briefly detail. In view of Section 5.7.1, there exist $T > 0$, $R >$ and

$$(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon) \in \mathscr{C}([0,T];\mathcal{H}^m_r) \times \mathscr{C}([0,T];\mathrm{H}^m) \times \mathscr{C}([0,T];\mathrm{H}^m) \cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1})$$

a solution to (S_ε) with initial data $(f^{\mathrm{in}}, \varrho^{\mathrm{in}}, u^{\mathrm{in}})$ such that

$$\sup_{\varepsilon\in(0,1]} \mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) \leq R.$$

Hence, we deduce that $(f_\varepsilon)_\varepsilon$ is bounded in $\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)$, $(\varrho_\varepsilon)_\varepsilon$ is bounded in $\mathrm{L}^2(0,T;\mathrm{H}^m)$ and $(u_\varepsilon)_\varepsilon$ is bounded in $\mathrm{L}^\infty(0,T;\mathrm{H}^m)$ and in $\mathrm{L}^2(0,T;\mathrm{H}^{m+1})$. From Lemma 5.2.20, we also know that $(\varrho_\varepsilon)_\varepsilon$ is bounded in $\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})$. We deduce that $(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon)$ has a weak(-$\star$) limit (f, ϱ, u) (up to some extraction) in the previous spaces.

Furthermore, using the equations satisfied by f_ε, ϱ_ε and u_ε, we observe that $(\partial_t f_\varepsilon)_{\varepsilon\leq 1}$ is bounded in $\mathrm{L}^\infty(0,T;\mathcal{H}^{m-3}_{r-1})$, $(\partial_t \varrho_\varepsilon)_{\varepsilon\leq 1}$ is bounded in $\mathrm{L}^\infty(0,T;\mathrm{H}^{m-3})$ and $(\partial_t u_\varepsilon)_{\varepsilon\leq 1}$ is bounded in $\mathrm{L}^\infty(0,T;\mathrm{H}^{m-3})$. Invoking the Aubin-Lions-Simon lemma (see e.g. [BF12, Theorem II.5.16]), we deduce that, up to some extraction, we have

$$f_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} f \ \text{ in } \mathscr{C}([0,T];\mathcal{H}^{m-2}_{r-1}), \quad \varrho_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \varrho \ \text{ in } \mathscr{C}([0,T];\mathrm{H}^{m-3}), \quad u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u \ \text{ in } \mathscr{C}([0,T];\mathrm{H}^{m-1}).$$

These strong convergences allow us to pass to the limit in (S_ε) and to obtain the fact that (f, ϱ, u) is a solution to the thick spray equations (TS) on $[0,T]$, and that (f, ϱ) satisfies a Penrose stability condition (**P**) on the same interval of time.

It remains to prove the fact that $f \in \mathscr{C}([0,T];\mathcal{H}^{m-1}_r)$ and $u \in \mathscr{C}([0,T];\mathrm{H}^m)$, as we only have $f \in \mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)$ and $u \in \mathrm{L}^\infty(0,T;\mathrm{H}^m)$ for the moment. Since $f \in \mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r) \cap \mathscr{C}_w([0,T];\mathcal{H}^{m-2}_{r-1})$ and $u \in \mathrm{L}^\infty(0,T;\mathrm{H}^m) \cap \mathscr{C}_w([0,T];\mathrm{H}^{m-1})$, we know (see e.g. [BF12, Lemma II.5.9]) that $f \in \mathscr{C}_w([0,T];\mathcal{H}^{m-1}_r)$ and $u \in \mathscr{C}_w([0,T];\mathrm{H}^m)$. It is now sufficient to prove that $t \mapsto \|f(t)\|_{\mathcal{H}^{m-1}_r}$ and $t \mapsto \|u(t)\|_{\mathrm{H}^m}$ are continuous functions on $[0,T]$ to conclude. Coming back to the energy estimates of Subsection 5.2.2, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2_{\mathcal{H}^{m-1}_r} < +\infty, \quad \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{\mathrm{H}^m} < +\infty.$$

Since $t \mapsto \|f(t)\|^2_{\mathcal{H}^{m-1}_r}$ and $t \mapsto \|u(t)\|^2_{\mathrm{H}^m}$ are integrable (because $f \in \mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)$ and $u \in \mathrm{L}^\infty(0,T;\mathrm{H}^m)$), we obtain the fact that $t \mapsto \|f(t)\|^2_{\mathcal{H}^{m-1}_r}$ and $t \mapsto \|u(t)\|^2_{\mathrm{H}^m}$ belongs to $\mathrm{W}^{1,1}(0,T)$, hence the continuity in time of these quantities. This finally yields the desired continuity for f and u and concludes the proof.

5.7.3 Uniqueness of the solution

Let us turn to the uniqueness part of Theorem 5.1.5. Let (f_1, ϱ_1, u_1) and (f_2, ϱ_2, u_2) be two solutions of (TS) belonging to

$$\mathrm{L}^\infty(0,\boldsymbol{T};\mathcal{H}^{m-1}_r) \times \mathrm{L}^2(0,\boldsymbol{T};\mathrm{H}^m) \times \mathrm{L}^\infty(0,\boldsymbol{T};\mathrm{H}^m) \cap \mathrm{L}^2(0,\boldsymbol{T};\mathrm{H}^{m+1}),$$

for some $\boldsymbol{T} > 0$, with the same initial condition $(f^{\mathrm{in}}, \varrho^{\mathrm{in}}, u^{\mathrm{in}})$ and such that $t \mapsto (f_1(t), \varrho_1(t))$ satisfies the Penrose stability condition (**P**) on $[0,\boldsymbol{T}]$. Let us set

$$f := f_1 - f_2, \quad \alpha_i := 1 - \rho_{f_i}, \quad \alpha := \alpha_1 - \alpha_2, \quad \varrho := \varrho_1 - \varrho_2, \quad u := u_1 - u_2,$$

and

$$R := \max_{i=1,2}\left(\|f_i\|_{\mathrm{L}^\infty(0,\boldsymbol{T};\mathcal{H}^{m-1}_r)} + \|\varrho_i\|_{\mathrm{L}^2(0,\boldsymbol{T};\mathrm{H}^m)} + \|u_i\|_{\mathrm{L}^\infty(0,\boldsymbol{T};\mathrm{H}^m)\cap\mathrm{L}^2(0,\boldsymbol{T};\mathrm{H}^{m+1}),}\right).$$

Note that R depends on $\boldsymbol{T}$.

• **Step 1**: let us show that $\varrho_1 = \varrho_2$, at least on a small interval of time. The key is to obtain $\mathrm{L}^2(0,T;\mathrm{L}^2)$ estimates for ϱ for some $T = T(R) \leq \boldsymbol{T}$. To this end, we write the equation satisfied by ϱ as

$$\begin{aligned}
&\alpha_1\left(\partial_t\varrho + u_1\cdot\nabla_x\varrho\right) + \varrho_1\left(\partial_t\alpha + u_1\cdot\nabla_x\alpha\right)\\
&= -\Bigg(\partial_t\varrho_2\alpha + \partial_t\alpha_2\varrho + [\varrho u_1 + \varrho_2 u]\cdot\nabla_x\alpha_2 + [\alpha u_1 + \alpha_2 u]\cdot\nabla_x\varrho_2\\
&\qquad\qquad + \alpha\rho_1\mathrm{div}_x u_1 + \alpha_2\varrho\mathrm{div}_x u_1 + \alpha_2\varrho_2\mathrm{div}_x u\Bigg),
\end{aligned}$$

and since $\partial_t\alpha_i = \mathrm{div}_x\, j_{f_i}$, we get

$$\partial_t\varrho + u_1\cdot\nabla_x\varrho + \frac{\varrho_1}{1-\rho_{f_1}}\mathrm{div}_x(j_f - \rho_f u_1) = S_{1,2}(f,\varrho,u), \tag{5.7.1}$$

where

$$\begin{aligned}
S_{1,2}(f,\varrho,u) := -\frac{1}{1-\rho_{f_1}}\Bigg(&-\partial_t\varrho_2\rho_f + \partial_t\alpha_2\varrho + [\varrho u_1 + \varrho_2 u]\cdot\nabla_x\alpha_2 + [-\rho_f u_1 + \alpha_2 u]\cdot\nabla_x\varrho_2\\
&- \rho_f\rho_1\mathrm{div}_x u_1 + \alpha_2\varrho\mathrm{div}_x u_1 + \alpha_2\varrho_2\mathrm{div}_x u - \varrho_2\rho_f\mathrm{div}_x\, u_1\Bigg).
\end{aligned}\tag{5.7.2}$$

The r.h.s $S_{1,2}(f,\varrho,u)$ can be seen as a source term whose $\mathrm{L}^2(0,T;\mathrm{L}^2)$ norm will be estimated by that of ϱ, without any loss of derivative in (f,ϱ,u). Note that we have rewritten the equation as above in order to perform the right pseudodifferential factorization. It is of course reminiscent of what we have done in Section 5.6. Next, the equations on the differences f and u read

$$\begin{cases}
\partial_t f + v\cdot\nabla_x f + \mathrm{div}_v\big[(u_2 - v)f - \nabla_x p(\varrho_2)f\big] + \Big(u - \nabla_x p(\varrho_1) + \nabla_x p(\varrho_2)\Big)\cdot\nabla_v f_1 = 0,\\[2mm]
\partial_t u + (u_2\cdot\nabla_x)u + (u\cdot\nabla_x)u_1 + \dfrac{1}{\varrho_1}\nabla_x p(\varrho_1) - \dfrac{1}{\varrho_2}\nabla_x p(\varrho_2)\\
\qquad - \dfrac{1}{\alpha_1\varrho_1}\Big(\Delta_x + \nabla_x\mathrm{div}_x\Big)u - \left(\dfrac{1}{\alpha_1\varrho_1} - \dfrac{1}{\alpha_2\varrho_2}\right)\Big(\Delta_x + \nabla_x\mathrm{div}_x\Big)u_2 = j_f - \rho_f u_1 - \rho_{f_2}u,
\end{cases}\tag{5.7.3}$$

and in particular

$$\begin{aligned}
&\partial_t f + v\cdot\nabla_x f + \mathrm{div}_v\big[(u_2 - v)f - \nabla_x p(\varrho_2)f\big]\\
&\qquad\qquad = p'(\varrho_1)\nabla_x\varrho\cdot\nabla_v f_1 + (p'(\varrho_1) - p'(\varrho_2))\nabla_x\varrho_2\cdot\nabla_v f_1 - u\cdot\nabla_x f_1.
\end{aligned}$$

Again, the last two terms in the r.h.s should be seen as some source terms, without any loss of derivative in (f,ϱ,u). One can show that

$$\|(p'(\varrho_1) - p'\varrho_2))\nabla_x\varrho_2\cdot\nabla_v f_1\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} + \|u\cdot\nabla_x f_1\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R)\,\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}.$$

The proof of the estimate for the second term will be similar to the one for the term $S[\varrho]$ we will treat below.

Arguing as in Sections 5.3–5.4–5.5, but for a force field

$$\mathfrak{F}(t,x) := u_2(t,x) - \nabla_x p(\varrho_2(t,x)),$$

instead of $E^{\varrho,u}_{\mathrm{reg},\varepsilon}$, one can also prove that for $T(R)$ small enough, we have for all $t \in [0, T(R)]$

$$\rho_f(t,x) = p'(\varrho_1(t,x)) \int_0^t \int_{\mathbb{R}^d} \nabla_x \varrho(s, x-(t-s)v) \cdot \nabla_v f_1(t,x,v) \, \mathrm{d}s \, \mathrm{d}v + \mathcal{R}[\rho_f](t,x),$$
$$j_f(t,x) = p'(\varrho_1(t,x)) \int_0^t \int_{\mathbb{R}^d} v \nabla_x \varrho(s, x-(t-s)v) \cdot \nabla_v f_1(t,x,v) \, \mathrm{d}s \, \mathrm{d}v + \mathcal{R}[j_f](t,x),$$

where

$$\|\mathcal{R}[\rho_f]\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} + \|\mathcal{R}[j_f]\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R) \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}.$$

Here, we have also used the fact that $f_{|t=0} = 0$. Injecting these expressions in equation (5.7.1) on ϱ, we get as in Section 5.6

$$\partial_t \varrho + u_1 \cdot \nabla_x \varrho + \frac{\varrho_1}{1-\rho_{f_1}} \mathrm{div}_x \left[\mathrm{K}^{\mathrm{free}}_{1,G_1}(\varrho) - \mathrm{K}^{\mathrm{free}}_{G_1}(\varrho) u_1 \right] = S[\varrho],$$

with

$$G_1(t,x,v) := p'(\varrho_1(t,x)) \nabla_v f_1(t,x,v),$$

and

$$S[\varrho] = -\frac{\varrho_1}{1-\rho_{f_1}} \mathrm{div}_x \left(\mathcal{R}[j_f] - \mathcal{R}[\rho_f] u_1 \right) + S_{1,2}(f, \varrho, u),$$

(the operator $\mathrm{K}^{\mathrm{free}}_{1,G_1}$ being defined in (5.6.2)) and then

$$\left(\mathrm{Id} - \frac{\varrho_1}{1-\rho_{f_1}} \mathrm{K}^{\mathrm{free}}_{G_1} \right) \left[\partial_t \varrho + u_1 \cdot \nabla_x \varrho \right] = S[\varrho], \quad t \in (0,T). \tag{5.7.4}$$

It is straightforward to prove that the source $S[\varrho]$ satisfies

$$\|S[\varrho]\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R) \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}.$$

The main issue comes from obtaining the same estimate for the term $S_{1,2}(u,f)$ defined in (5.7.2). We observe that it is made of a sum of three types of terms, of the following form:

- some terms where ϱ is in factor, so that we directly obtain the estimate;
- some terms where ρ_f is in factor: we can rely on the previous decomposition of ρ_f as $\rho_f = \mathrm{K}^{\mathrm{free}}_{G_1}[\varrho] + \mathcal{R}[\rho_f]$ and an $\mathrm{L}^2(0,T;\mathrm{L}^2)$ estimate is then provided by the smoothing estimate of Proposition 5.4.4;
- some terms where u or $\mathrm{div}_x\, u$ is in factor: to estimate them, we perfom the same kind of energy estimate as in Subsection 5.B of the Appendix, following the steps leading to the equality (5.B.4), which gives after integration in time (since $u_{|t=0} = 0$)

$$\begin{aligned} &\|u(t)\|^2_{\mathrm{L}^2} + \int_0^t \|\mathrm{div}_x\, u(s)\|^2_{\mathrm{L}^2} \, \mathrm{d}s \\ &\lesssim \int_0^t \|u(s)\|^2_{\mathrm{L}^2} \, \mathrm{d}s + \int_0^t \left(\|\rho_f(s)\|^2_{\mathrm{L}^2} + \|j_f(s)\|^2_{\mathrm{L}^2} + \|(\alpha_1 \varrho_1(s) - \alpha_2 \varrho_2(s)\|^2_{\mathrm{L}^2} \right) \mathrm{d}s \\ &\lesssim \int_0^t \|u(s)\|^2_{\mathrm{L}^2} \, \mathrm{d}s + \Lambda(T,R) \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}. \end{aligned}$$

Here, we have also used the smoothing estimate from Proposition 5.4.4 combined with the previous decomposition of ρ_f and j_f. We obtain the desired control on u after an application of Grönwall's lemma.

As in Subsection 5.6.5, we then study the equation

$$\left(\mathrm{Id} - \frac{\varrho_1}{1-\rho_{f_1}}\mathrm{K}^{\mathrm{free}}_{G_1}\right)\left[\widetilde{H}\right] = \widetilde{\mathcal{R}}, \quad 0 \leq t \leq T,$$

where $\widetilde{\mathcal{R}}$ is a given source term and we want to derive an $\mathrm{L}^2(0,T;\mathrm{L}^2)$ estimate on the solution $\widetilde{H}$. After applying the same extension procedure for the coefficients (depending on (f_1, ϱ_1, u_1)) in the equation as in Section 5.6.3, and by setting $\widetilde{H}(t,x) := e^{\gamma t}H(t,x)$ and $\widetilde{\mathcal{R}}(t,x) = e^{\gamma t}\mathcal{R}(t,x)$ for $\gamma > 0$ (with the same continuation by zero outside $[0,T]$ as in Subsection 5.6.5), we are led to the study of the pseudodifferential equation

$$\mathrm{Op}^\gamma(1-\mathcal{P}_{f_1,\varrho_1})(H) = \mathcal{R},$$

where

$$\mathcal{P}_{f_1(t),\varrho_1(t)}(x,\gamma,\tau,k) := \frac{p'(\varrho_1(x))\varrho_1(x)}{1-\rho_{f_1}(x)}\int_0^{+\infty} e^{-(\gamma+i\tau)s} ik\cdot(\mathcal{F}_v\nabla_v f_1)(t,x,ks)\,\mathrm{d}s.$$

Observe that there is no factor $(1+|k|^2)^{-1}$ in the definition of this symbol, because there is no regularization operator in the equation. Note also that the estimates (5.6.15)–(5.6.16)–(5.6.17) holds true for $\mathcal{P}_{f_1,\varrho_1}$, in view of the regularity of (f_1,ϱ_1). Likewise, as in Corollary 5.6.20, we have

$$\left\|\widetilde{H}\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T,R)\left\|\widetilde{\mathcal{R}}\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))},$$

provided that one can apply $\mathrm{Op}^\gamma\left(\frac{1}{1-\mathcal{P}_{f_1,\varrho_1}}\right)$ to the previous pseudodifferential equation and take γ large enough in order to invert it up to a small remainder. To do so, we prove that if the condition (**P**)

$$\forall t\in\mathbb{R},\quad \forall x\in\mathbb{T}^d,\quad \inf_{(\gamma,\tau,k)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\setminus\{0\}}|1-\mathscr{P}_{f_1(t),\varrho_1(t)}(x,\gamma,\tau,k)|>c, \tag{\textbf{P}}$$

holds for some $c>0$, then the condition

$$\forall t\in\mathbb{R},\quad \forall x\in\mathbb{T}^d,\quad \inf_{(\gamma,\tau,k)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\setminus\{0\}}|1-\mathcal{P}_{f_1(t),\varrho_1(t)}(x,\gamma,\tau,k)|>c, \tag{\textbf{Opt-P}}$$

holds as well. Here, we implicitly consider the extension in time of f_1 and ϱ_1, as it was done in Section 5.6.3. To this end, we rely on a homogeneity argument, as in [HKR16]: we define the function

$$\widetilde{\mathscr{P}}_{f_1,\varrho_1}(x,\widetilde{\gamma},\widetilde{\tau},\widetilde{k},\sigma) := \mathscr{P}_{f_1,\varrho_1}(x,\sigma\widetilde{\gamma},\sigma\widetilde{\tau},\sigma\widetilde{k}),\quad x\in\mathbb{T}^d,\ (\widetilde{\gamma},\widetilde{\tau},\widetilde{k})\in S^+,\ \sigma>0,$$

where

$$S^+ := \left\{(\widetilde{\gamma},\widetilde{\tau},\widetilde{k})\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\setminus\{0\}\mid \widetilde{\gamma}^2+\widetilde{\tau}^2+\widetilde{k}^2=1\right\}.$$

Since f_1 is regular enough, one can prove that $\widetilde{\mathscr{P}}_{f_1,\varrho_1}$ can be extended as a continuous function on $\mathbb{T}^d\times S^+\times[0,+\infty)$ and we obtain

$$\forall t\in\mathbb{R},\quad \forall x\in\mathbb{T}^d,\quad \inf_{(\widetilde{\gamma},\widetilde{\tau},\widetilde{k},\sigma)\in S^+\times[0,+\infty)}|1-\widetilde{\mathscr{P}}_{f_1(t),\varrho_1(t)}(x,\widetilde{\gamma},\widetilde{\tau},\widetilde{k},\sigma)|>c.$$

In view of the homogeneity of degree 0 of the symbol $a_{f,\varrho}$ with respect to the variable (γ,τ,η) (see Lemma 5.6.16), we also have

$$\widetilde{\mathscr{P}}_{f_1,\varrho_1}(x,\widetilde{\gamma},\widetilde{\tau},\widetilde{k},\sigma)=\frac{1}{1+\sigma^2|\widetilde{k}|^2}\frac{p'(\varrho_1(x))\varrho_1(x)}{1-\rho_{f_1}(x)}\int_0^{+\infty}e^{-(\widetilde{\gamma}+i\widetilde{\tau})s}i\widetilde{k}\cdot(\mathcal{F}_v\nabla_v f_1)(t,x,\widetilde{k}s)\,\mathrm{d}s,$$

hence

$$\widetilde{\mathscr{P}}_{f_1(t),\varrho_1(t)}(x,\widetilde{\gamma},\widetilde{\tau},\widetilde{k},0)=\mathcal{P}_{f_1(t),\varrho_1(t)}(x,\widetilde{\gamma},\widetilde{\tau},\widetilde{k}),$$

from which we infer that (**Opt-P**) holds on S^+. Again, the homogeneity of degree 0 of the quantity $\mathscr{P}_{f_1(t),\varrho_1(t)}(x,\gamma,\tau,k)$ with respect to the variable (γ,τ,k) implies that (**Opt-P**) holds.

Next, we come up with the transport equation on ϱ

$$\partial_t\varrho+u_1\cdot\nabla_x\varrho=\widetilde{H},$$

where $\varrho_{|t=0}=0$ and where the source term $\widetilde{H}$ has been shown to satisfied

$$\left\|\widetilde{H}\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}\leq\Lambda(T,R)\left\|S[\varrho]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}\leq\Lambda(T,R)\left\|\varrho\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.$$

Performing an L^2 energy estimate gives for all $t\in[0,T]$ (since $\varrho_{|t=0}=0$)

$$\|\varrho(t)\|_{\mathrm{L}^2}\leq\Lambda(T,R)\int_0^t\|\widetilde{H}(\tau)\|_{\mathrm{L}^2}\,\mathrm{d}\tau\leq T^{\frac{1}{2}}\Lambda(T,R)\left\|\varrho\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))},$$

by the Cauchy-Schwarz inequality. We end up with

$$\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}\leq T\Lambda(T,R)\left\|\varrho\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.$$

We deduce the fact that there exists a small enough $T=T(R)>0$ which depends only on R such that

$$\forall t\in[0,T(R)],\quad\varrho(t)=0.$$

• **Step 2**: let us now prove that $f_1=f_2$ and $u_1=u_2$ on $[0,T(R)]$. It is in fact a direct consequence of $\varrho=\varrho_1-\varrho_2=0$ on $[0,T(R)]$. Indeed, the previous step has shown that for all $t\in[0,T(R)]$

$$\|u(t)\|_{\mathrm{L}^2}^2\lesssim\int_0^t\|u(s)\|_{\mathrm{L}^2}^2\,\mathrm{d}s+\Lambda(t,R)\left\|\varrho\right\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\lesssim\int_0^t\|u(s)\|_{\mathrm{L}^2}^2\,\mathrm{d}s,$$

therefore we directly have $u=0$ on $[0,T(R)]$. The equations on (5.7.3) on f now turn into

$$\begin{cases}\partial_t f+v\cdot\nabla_x f-v\cdot\nabla_v f+(u_2-\nabla_x p(\varrho_2))\cdot\nabla_v f=df,\\ f_{|t=0}=0.\end{cases}$$

The method of characteristics thus shows that $f=0$ on $[0,T(R)]$.

In conclusion, we have obtained $(f,\varrho,u)=(0,0,0)$ on $[0,T(R)]$. We eventually observe that we can repeat this procedure starting from $t=T(R)$ instead of $t=0$. As a matter of fact, $f_1(T(R),\cdot)$ still satisfies the Penrose stability condition. Since the time $T(R)$ only depends on R, we obtain $\varrho=0$ on $[0,2T(R)]$ and then $(f,u)=(0,0)$ on $[0,2T(R)]$. After a finite number of steps, this yields $(f,\varrho,u)=(0,0,0)$ on $[0,\boldsymbol{T}]$. This finally concludes the proof of the uniqueness part of the statement.

5.8 Generalization to the non-barotropic case

In this section, we show how to handle the case of the full Navier-Stokes system for non-barotropic fluids, where we consider the additional internal energy $\mathfrak{e} \in \mathbb{R}^+$ for the fluid and where the the pressure depends on ϱ and $\mathfrak{e}$. As explained in the introduction, the system which is at stake is the following:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v \Big[f(u-v) - f \nabla_x p(\varrho, \mathfrak{e}) \Big] = 0, \\ \partial_t(\alpha\varrho) + \mathrm{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u \otimes u) + \alpha \nabla_x p(\varrho, \mathfrak{e}) - \Delta_x u - \nabla_x \mathrm{div}_x u = j_f - \rho_f u, \\ \partial_t(\alpha\varrho\mathfrak{e}) + \mathrm{div}_x(\alpha\varrho\mathfrak{e}u) + p(\varrho,\mathfrak{e})\left(\partial_t \alpha + \mathrm{div}_x(\alpha u)\right) = \displaystyle\int_{\mathbb{R}^d} |u-v|^2 f \, \mathrm{d}v, \\ \alpha = 1 - \rho_f. \end{cases} \tag{5.8.1}$$

Here, we will assume[4] that the pressure law is given as

$$p(\varrho, \mathfrak{e}) = \pi(\varrho\mathfrak{e}),$$

for some given function $\pi : \mathbb{R}^+ \to \mathbb{R}$ such that $\pi \in \mathscr{C}(\mathbb{R}^+) \cap \mathscr{C}^\infty(\mathbb{R}^+ \backslash \{0\})$. For instance, the relation $p(\varrho, \mathfrak{e}) = b\varrho\mathfrak{e}$ (with $b > 0$) is a perfect gas pressure law. Similarly to the technical hypothesis (5.1.2), we shall assume that

$$y \mapsto \pi'(y)(y + \pi(y)) \quad \text{is nondecreasing on} \quad \mathbb{R}.$$

Setting

$$\vartheta := \varrho\mathfrak{e},$$

the system in $(f, \varrho, u, \vartheta)$ can be rewritten as

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v \left[f(u - v - \nabla_x \pi(\vartheta)) \right] = 0, \\ \partial_t(\alpha\varrho) + \mathrm{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u \otimes u) + \alpha \nabla_x \pi(\vartheta) - \Delta_x u - \nabla_x \mathrm{div}_x u = j_f - \rho_f u, \\ \partial_t(\alpha\vartheta) + \mathrm{div}_x(\alpha\vartheta u) + \pi(\vartheta)\left(\partial_t \alpha + \mathrm{div}_x(\alpha u)\right) = \displaystyle\int_{\mathbb{R}^d} |v-u|^2 f \, \mathrm{d}v, \\ \alpha = 1 - \rho_f. \end{cases} \tag{TS$_e$}$$

As seen below, it is significant to define the following Penrose symbol of a function $(f(x,v), \vartheta(x))$ as

$$\mathbf{P}^{\mathrm{energy}}_{f,\vartheta}(x,\gamma,\tau,\eta) := \frac{\pi'(\vartheta(x))\left[\vartheta(x) + \pi(\vartheta(x))\right]}{1-\rho_f(x)} \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot \left(\mathcal{F}_v \nabla_v f\right)(x, s\eta) \, \mathrm{d}s. \tag{5.8.2}$$

We can now introduce the following Penrose condition, adapted to (TS$_e$), which is that there exists $c > 0$ such that

$$\forall x \in \mathbb{T}^d, \quad \inf_{(\gamma,\tau,\eta) \in (0,+\infty)\times\mathbb{R}\times\mathbb{R}^d \backslash \{0\}} |1 - \mathbf{P}^{\mathrm{energy}}_{f,\vartheta}(x,\gamma,\tau,\eta)| > c, \tag{$\mathbf{P}^{\mathrm{energy}}$}$$

Our main result for the system (TS$_e$) reads as follows.

[4] It is likely that more general pressure $p(\varrho, \mathfrak{e})$ could be treated by our method.

Theorem 5.8.1. *There exist $m_0 > 0$ and $r_0 > 0$, depending only on the dimension, such that the following holds for all $m \geq m_0$ and $r \geq r_0$. Let*

$$f^{\mathrm{in}} \in \mathcal{H}^m_r, \quad \varrho^{\mathrm{in}} \in \mathrm{H}^{m-1}, \quad u^{\mathrm{in}} \in \mathrm{H}^m, \quad \vartheta^{\mathrm{in}} \in \mathrm{H}^{m+1},$$

such that $(f^{\mathrm{in}}, \vartheta^{\mathrm{in}})$ satisfies the c-Penrose stability condition $(\mathbf{P}^{\mathrm{energy}})_c$ (with $c > 0$) and

$$0 \leq f^{\mathrm{in}}, \quad \rho_{f^{\mathrm{in}}} < \Upsilon < 1, \quad 0 < \mu \leq \varrho^{\mathrm{in}}, \vartheta^{\mathrm{in}} \quad 0 < \underline{\nu} \leq (1-\rho_{f^{\mathrm{in}}})\varrho^{\mathrm{in}}, (1-\rho_{f^{\mathrm{in}}})\vartheta^{\mathrm{in}} \leq \overline{\nu},$$

for some fixed constants $\Upsilon, \mu, \underline{\nu}, \overline{\nu}$. Then there exist $T > 0$ and a solution $(f, \varrho, u, \vartheta)$ to (TS$_e$) with initial condition $(f^{\mathrm{in}}, \varrho^{\mathrm{in}}, u^{\mathrm{in}}, \vartheta^{\mathrm{in}})$ such that

$$f \in \mathscr{C}\left([0,T]; \mathcal{H}^{m-1}_r\right), \quad \varrho \in \mathscr{C}([0,T]; \mathrm{H}^{m-1}), \quad u \in \mathscr{C}\left([0,T]; \mathrm{H}^m\right) \cap \mathrm{L}^2(0,T; \mathrm{H}^{m+1}), \quad \vartheta \in \mathrm{L}^2(0,T;\mathrm{H}^m),$$

and with $(f(t), \vartheta(t))$ satisfying the $c/2$-Penrose stability condition $(\mathbf{P}^{\mathrm{energy}})_{c/2}$ for all $t \in [0,T]$. In addition, this solution is unique in this class.

Let us explain the main strategy for the proof of Theorem 5.8.1. First, we observe that the equation on ϑ can be rewritten as

$$\partial_t \vartheta + u \cdot \nabla_x \vartheta + \frac{\vartheta + \pi(\vartheta)}{1-\rho_f} \mathrm{div}(j_f - \rho_f u) = -\frac{\vartheta + \pi(\vartheta)}{1-\rho_f} \mathrm{div}_x\, u + \frac{1}{1-\rho_f} \int_{\mathbb{R}^d} |v-u|^2 f \, \mathrm{d}v. \tag{5.8.3}$$

Appart from the last term, this equation has exactly the same structure as the one for ϱ in Lemma 5.2.2. Hence, the following estimate holds

$$\|\vartheta(t)\|_{\mathrm{H}^m} \leq \|\varrho^{\mathrm{in}}\|_{\mathrm{H}^m} \Phi\Big(T, \cdots, \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m+1})}, \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m+1}_r)}\Big),$$

and features the same loss of derivative as for ϱ in (TS). Note also that the equation on ϱ can be directly solved once f and u are given. As in Section 5.2.2, we consider the regularization $-\pi'(\vartheta)\mathrm{J}_\varepsilon \nabla_x \vartheta$ of the term $-\nabla_x \pi(\vartheta)$ in the kinetic equation of (TS$_e$) and introduce the quantity

$$\begin{aligned}\mathbf{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, \vartheta_\varepsilon, T) := \|f_\varepsilon\|_{\mathrm{L}^\infty\left(0,T;\mathcal{H}^{m-1}_r\right)} + \|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})} \\ + \|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1})} + \|\vartheta_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}.\end{aligned}$$

Following the bootstrap procedure we have set for the case of (TS), we mainly want to control the quantity $\|\vartheta_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}$.

Using (5.8.3) (and dropping the dependency in ε) it is possible to obtain the following equation on $h = \partial_x^\alpha \vartheta$ with $|\alpha| \leq m$ (see Proposition 5.6.1): one has

$$\left(\mathrm{Id} - \frac{\vartheta + \pi(\vartheta)}{1-\rho_f} \mathrm{K}^{\mathrm{free}}_G \circ \mathrm{J}_\varepsilon\right)\Big[\partial_t h + u \cdot \nabla_x h\Big] = \mathcal{R}, \quad t \in (0,T),$$

with $G(t,x,v) = \pi'(\vartheta(t,x))\nabla_v f(t,x,v)$ and

$$\|\mathcal{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T, R, \|h(0)\|_{\mathrm{H}^1(\mathbb{T}^d)}).$$

Following the arguments of Section 5.6, we are thus led to the study of the following pseudodifferential equation

$$H - \frac{\pi'(\vartheta)(\vartheta + \pi(\vartheta))}{1-\rho_f} \mathrm{Op}^\gamma(a_f)(\mathrm{J}_\varepsilon H) = \mathcal{R}, \quad \text{on} \quad (0,T) \times \mathbb{T}^d,$$

where a_f is defined in (5.6.8), that is

$$\mathrm{Op}^{\gamma,\varepsilon}(1-\mathbf{P}^{\text{energy}}_{f,\vartheta})(H)=\mathcal{R}.$$

In particular, it explains the introduction of the Penrose symbol (5.8.2) above, which allows to invert the previous equation on H, up to a small remainder.

Some additional arguments have also to be given to treat the last term in (5.8.3). Since

$$\int_{\mathbb{R}^d}|v-u|^2 f\,\mathrm{d}v=\int_{\mathbb{R}^d}|v|^2 f\,\mathrm{d}v+|u|^2\rho_f-2u\cdot j_f,$$

we have to include the treatment of the second order moment in velocity

$$m_2 f(t,x):=\int_{\mathbb{R}^d}|v|^2 f(t,x,v)\,\mathrm{d}v$$

in the analysis of Section 5.5. In addition to Proposition 5.5.1, we have

Proposition 5.8.2. *For all $|I|\leq m$, we have for any $t\in(0,T)$*

$$\partial_x^I m_2 f(t,x)=p'(\varrho(t,x))\int_0^t\int_{\mathbb{R}^d}|v|^2\nabla_x[\mathrm{J}_\varepsilon\partial_x^I\varrho](s,x-(t-s)v)\cdot\nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s+R^I[m_2 f](t,x),$$

where the remainder $R^I[m_2 f]$ satisfies

$$\left\|R^I[M_2 f]\right\|_{\mathrm{L}^2(0,T,\mathrm{H}^1_x)}\leq\Lambda(T,R).$$

In particular, we have

$$\left\|\partial_x^I\int_{\mathbb{R}^d}|v-u|^2 f\,\mathrm{d}v\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\leq\Lambda(T,R).$$

The fairly straightforward adaptation of the analysis of Section 5.5 is left to the reader.

5.9 Generalization to the inelastic Boltzmann case

In this section, we consider the case where one takes into account inelastic collisions between particles. This corresponds to the following Vlasov-Boltzmann equation in the coupling (the other equations on (ϱ,u) being unchanged):

$$\begin{cases}\partial_t f+v\cdot\nabla_x f+\mathrm{div}_v\left[f(u-v)-f\nabla_x p(\varrho)\right]=\mathcal{Q}_\lambda(f,f),\\ \partial_t(\alpha\varrho)+\mathrm{div}_x(\alpha\varrho u)=0,\\ \partial_t(\alpha\varrho u)+\mathrm{div}_x(\alpha\varrho u\otimes u)+\alpha\nabla_x p-\Delta_x u-\nabla_x\mathrm{div}_x\,u=j_f-\rho_f u,\end{cases}\tag{TS-Coll}$$

where $\mathcal{Q}_\lambda(f,f)$ stands for a quadratic collision operator of Boltzmann-type, in a inelastic hard-spheres regime. Here, the fixed parameter $\lambda\in(0,1)$ corresponds to the so-called *restitution coefficient*: if $'v$ and $'v_\star$ denote the velocities of two particles before collision, their respective velocities v and $v_\star$ after collision are given by

$$\begin{cases}v={}'v-\dfrac{1+\lambda}{2}('u\cdot n)n,\\ v_\star={}'v_\star+\dfrac{1+\lambda}{2}('u\cdot n)n,\end{cases}$$

where $'u := 'v - 'v_\star$ is the relative pre-collision velocity and $n \in \mathbb{S}^{d-1}$ is a unit vector that points from the particle center with velocity v to the particle center with velocity $v_\star$ at the impact. Note that $\lambda = 1$ corresponds to the standard elastic case.

In this representation, given two distribution functions $f = f(v)$ and $g = g(v)$, we can consider the following expression for Boltzmann collision operator, as a difference of a gain and loss term

$$\mathcal{Q}_\lambda(f, g) = \mathcal{Q}_\lambda^+(f, g) - \mathcal{Q}_\lambda^-(f, g),$$

where, setting $u = v - v_\star$ and $\widehat{u} = u/|u|$, we define

$$\begin{aligned} \mathcal{Q}_\lambda^+(f, g)(v) &= \frac{1}{\lambda^2} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |u \cdot n| b(\widehat{u} \cdot n) f('v) g('v_\star) \, \mathrm{d}v_\star \mathrm{d}n, \\ \mathcal{Q}_\lambda^-(f, g)(v) &= f(v) \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |u \cdot n| b(\widehat{u} \cdot n) g(v_\star) \, \mathrm{d}v_\star \mathrm{d}n. \end{aligned} \tag{5.9.1}$$

Here, the function $b \in \mathrm{L}^1([-1, 1])$ is a given angular collision kernel. For the sake of simplicity, we consider $b \equiv 1$. Note that in the (truly) inelastic case $\lambda \in (0, 1)$, we have

$$|v|^2 + |v_\star|^2 = |'v|^2 + |'v_\star|^2 - \frac{1 - \lambda^2}{2} ('u \cdot n)^2,$$

thus inducing a loss of kinetic energy at each collision, while mass and momentum are conserved. We refer to [Vil06] (see also the introduction of [ALT20]) for more details on this model coming from the theory of *granular media* and which describes a cloud of macroscopic particles of size larger than that usually described by the usual Boltzmann equation with elastic collisions (for so-called molecular gases). To include dissipative effects, inelastic collisions are thus considered. Note that the presence of such a collision operator (with a large parameter ε^{-1} in front of it) formally leads to a biphasic fluid model when starting from (TS-Coll)) (see [DM10]). We also refer to [O'R81, Bar04, DC09] for its use in the study of sprays.

Our main result (which also includes the elastic case $\lambda = 1$) reads as follows.

Theorem 5.9.1. *Let $\lambda \in (0, 1]$. There exist $m_0 > 0$, depending only on the dimension, such that the following holds for all $m \geq m_0$. Let*

$$e^{|v|^2} f^{\text{in}} \in \mathcal{H}_0^m, \quad \varrho^{\text{in}} \in \mathrm{H}^{m+1}, \quad u^{\text{in}} \in \mathrm{H}^m,$$

such that $(f^{\text{in}}, \varrho^{\text{in}})$ satisfies the c-Penrose stability condition $(\mathbf{P})_c$ (with $c > 0$) and

$$0 \leq f^{\text{in}} \quad \rho_{f^{\text{in}}} < \Theta < 1, \quad 0 < \mu \leq \varrho^{\text{in}}, \quad 0 < \underline{\theta} \leq (1 - \rho_{f^{\text{in}}}) \varrho^{\text{in}} \leq \overline{\theta},$$

for some fixed constants $\Theta, \mu, \underline{\theta}, \overline{\theta}$. Then there exist $T > 0$ and a solution (f, ϱ, u) to (TS-Coll) with initial condition $(f^{\text{in}}, \varrho^{\text{in}}, u^{\text{in}})$ such that

$$e^{|v|^2} f \in \mathscr{C}\left([0, T]; \mathcal{H}_0^{m-1}\right), \quad \varrho \in \mathrm{L}^2(0, T; \mathrm{H}^m), \quad u \in \mathscr{C}([0, T]; \mathrm{H}^m) \cap \mathrm{L}^2(0, T; \mathrm{H}^{m+1}),$$

and with $(f(t), \varrho(t))$ satisfying the $c/2$-Penrose stability condition $(\mathbf{P})_{c/2}$ for all $t \in [0, T]$. In addition, this solution is unique in this class.

As seen below, the friction term in the kinetic equation comes in handy in order to treat some of the new terms due to the collision operator. This was remarked already in [Mat10].

Let us present the main changes that must be considered in our strategy of proof and that are due to the collision operator. It mainly concerns:

- the energy estimates for f from Section 5.2;
- the integro-differential system for the derivatives of f from Section 5.5.1.

The rest of Section 5.2 and of Section 5.5 then remains unchanged, as well as Sections 5.6–5.7.

5.9.1 New energy estimates for the kinetic part

Let us focus on the energy estimates for the new kinetic equation

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v[f(u-v) - f\nabla_x p(\varrho) f] = \mathcal{Q}_\lambda(f,f). \tag{5.9.2}$$

Following [Mat10], we first define $g(t,x,v) = e^{|v|^2} f(t,x,v)$ with f solving (5.9.2). Setting

$$E^{u,\varrho} = u - \nabla_x p(\varrho),$$

it implies that g satisfies the following modified Vlasov-Boltzmann equation

$$\partial_t g + v \cdot \nabla_x g + \mathrm{div}_v[(E^{u,\varrho} - v)g] - 2v \cdot (E^{u,\varrho} - v) g = \Gamma_\lambda[g,g], \tag{5.9.3}$$

where for all functions $h_1(v), h_2(v)$, the operator $\Gamma_\lambda[g,g] = \Gamma_\lambda^+[g,g] - \Gamma_\lambda^-[g,g]$ is defined via

$$\Gamma_\lambda^+[h_1,h_2](v) := \frac{1}{\lambda^2}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} e^{-|v_\star|^2 - \frac{1-\lambda^2}{2}(('v-'v_\star)\cdot n)^2} |u\cdot n| h_1('v) h_2('v_\star)\, \mathrm{d}v_\star \mathrm{d}n,$$

$$\Gamma_\lambda^-[h_1,h_2](v) := h_2(v)\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} e^{-|v_\star|^2} |u\cdot n| h_1(v_\star)\, \mathrm{d}v_\star \mathrm{d}n.$$

Note that the additional term $2v\cdot(E^{u,\varrho} - v)g$ comes from the friction term in (5.9.2). Using

$$'u = 'v_\star - 'v = v_\star - v - (1+\lambda)('u\cdot n)n = v_\star - v + (1+\lambda)\lambda(u\cdot n)n,$$

we observe that for all $\lambda \in (0,1)$, there exists a constant $c(\lambda) > 0$ (and $c(1) = 0$) such that

$$\Gamma_\lambda^+[h_1,h_2] := \frac{1}{\lambda^2}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} e^{-|v_\star|^2 - c(\lambda)((v-v_\star)\cdot n)^2} |(v-v_\star)\cdot n| h_1('v) h_2('v_\star)\, \mathrm{d}v_\star \mathrm{d}n.$$

The exponential inside the integral is roughly behaving as $e^{-|v_\star|^2 - c(\lambda)|v-v_\star|^2}$. We have the following bilinear estimates on the previous collision operators, where some loss of weights in velocity classically shows up.

Lemma 5.9.2. *There exists $s = s(d) > 0$ large enough such that for all $\sigma \geq 0$, we have for any smooth nonnegative function $g = g(x,v)$*

$$\sum_{|\alpha|+|\beta|\leq s} \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} \langle v\rangle^{2\sigma} \partial_x^\alpha \partial_v^\beta [\Gamma_\lambda(g,g)] \partial_x^\alpha \partial_v^\beta g \,\mathrm{d}x\,\mathrm{d}v \lesssim \|g\|^2_{\mathcal{H}^s_\sigma} \|g\|_{\mathcal{H}^s_{\sigma+1}}.$$

Proof. We refer to [Mat10, Lemma 2.3] combined to the (proof of) [ALT20, Appendix A]. □

The key estimate allowing to recover the previous loss of weight then comes from the following lemma bearing on the extra term $2v\cdot(E^{u,\varrho} - v)g$.

Lemma 5.9.3. *Let $s \geq 0$ and $\sigma > 0$. For any smooth nonnegative function $g = g(x,v)$ and $\delta \in (0,1)$, we have*

$$\sum_{|\alpha|+|\beta|\leq s} \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} \langle v\rangle^{2\sigma} \partial_x^\alpha \partial_v^\beta \left[2v\cdot(E^{u,\varrho} - v)g\right] \partial_x^\alpha \partial_v^\beta g \,\mathrm{d}x\,\mathrm{d}v$$
$$\lesssim -(1-\delta)\|g\|^2_{\mathcal{H}^s_{\sigma+1}} + \left(1 + \frac{1}{\delta}\|E^{u,\varrho}\|^2_{\mathrm{H}^s}\right)\|g\|^2_{\mathcal{H}^s_\sigma}.$$

Proof. We refer to [Mat10, Lemma 2.7]. □

Let us show how one can now obtain an *a priori* energy estimate for a solution g to (5.9.3), that reads as

$$\mathcal{T}^{u,\varrho}(g) - 2v \cdot (E^{u,\varrho} - v)g = \Gamma_\lambda(g,g),$$

where

$$\mathcal{T}^{u,\varrho} = \partial_t + v \cdot \nabla_x - v \cdot \nabla_v + E^{u,\varrho}(t,x) \cdot \nabla_v - d\mathrm{Id}.$$

The result is the following.

Lemma 5.9.4. *For all $r \geq 0$, $m > 3 + d/2$, $c > 0$ and all $T > 0$, for all smooth functions (f, ϱ, u) satisfying*

$$\partial_t g + v \cdot \nabla_x g - v \cdot \nabla_v g + E^{u,\varrho}(t,x) \cdot \nabla_v - 2v \cdot (E^{u,\varrho} - v)g - dg = \Gamma_\lambda(g,g) \quad \text{on} \quad [0,T],$$

and $\varrho \geq c$ on $[0,T]$, there holds, for all $t \in [0,T]$

$$\begin{aligned}\mathcal{E}_{m,\sigma}[g(t)] \leq \|g(0)\|^2_{\mathcal{H}^m_\sigma} \exp\Big[C\Big((1 + \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} + \|\mathcal{E}_{m,\sigma}[g]\|_{\mathrm{L}^\infty(0,T)})T \\ + \sqrt{T}\Lambda\left(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\right)\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}\Big)\Big],\end{aligned} \tag{5.9.4}$$

for some universal constant $C > 0$ and where

$$\mathcal{E}_{m,\sigma}[g(t)] := \|g(t)\|^2_{\mathcal{H}^m_\sigma} + \frac{1}{4}\int_0^t \|g(\tau)\|^2_{\mathcal{H}^m_{\sigma+1}}\,\mathrm{d}\tau.$$

Proof. By Lemma 5.2.9, we first have

$$\begin{aligned}\mathcal{T}^{u,\varrho}(\partial^\alpha_x\partial^\beta_v g) = -\sum_{\substack{i=1\\ \beta_i\neq 0}}^{d}\left(\partial_x^{\widehat{\alpha}^i}\partial_v^{\overline{\beta}^i} g - \partial^\alpha_x\partial^\beta_v g\right) - \left[\partial^\alpha_x\partial^\beta_v, E^{u,\varrho}(t,x)\cdot\nabla_v\right]g \\ + \partial^\alpha_x\partial^\beta_v\left[2v\cdot(E^{u,\varrho} - v)g\right] + \partial^\alpha_x\partial^\beta_v\left[\Gamma_\lambda(f,f)\right].\end{aligned}$$

The analogue of inequality (5.2.3) from Proposition 5.2.10 is, thanks to Lemmas 5.9.2–5.9.3, for all $\delta \in (0,1)$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|g(t)\|^2_{\mathcal{H}^m_\sigma} + (1-\delta)\|g(t)\|^2_{\mathcal{H}^m_{\sigma+1}} \\ &\lesssim \left(1 + \|E^{u,\varrho}(t)\|_{\mathrm{H}^m} + \frac{1}{\delta}\|E^{u,\varrho}(t)\|^2_{\mathrm{H}^m}\right)\|g(t)\|^2_{\mathcal{H}^m_\sigma} + \|g(t)\|^2_{\mathcal{H}^m_\sigma}\|g(t)\|_{\mathcal{H}^m_{\sigma+1}} \\ &\lesssim \left(1 + \|E^{u,\varrho}(t)\|_{\mathrm{H}^m} + \frac{1}{\delta}\|E^{u,\varrho}(t)\|^2_{\mathrm{H}^m}\right)\|g(t)\|^2_{\mathcal{H}^m_\sigma} \\ &\quad + \frac{2}{1-\delta}\|g(t)\|^4_{\mathcal{H}^m_\sigma} + \frac{1-\delta}{2}\|g(t)\|^2_{\mathcal{H}^m_{\sigma+1}},\end{aligned}$$

therefore after absorbing the last term with $\delta = 1/2$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{m,\sigma}[g(t)] \lesssim \left(1 + \|E^{u,\varrho}(t)\|^2_{\mathrm{H}^m} + \mathcal{E}_{m,\sigma}[g(t)]\right)\mathcal{E}_{m,\sigma}[g(t)].$$

Concluding as in the proof of Proposition 5.2.10, we finally obtain the result. □

The bootstrap argument from Section 5.2.2 is then performed with the quantity

$$\mathscr{N}_{m,r}(g,\varrho,u,T) := \|\mathcal{E}_{m-1,\sigma}[g]\|_{\mathrm{L}^\infty(0,T)} + \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} + \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap\mathrm{L}^2(0,T;\mathrm{H}^{m+1})},$$

instead of $\mathcal{N}_{m,r}(f,\varrho,u,T)$, by considering the same regularization $\mathrm{J}_\varepsilon\nabla_x\varrho$ in the equation (5.9.3). Taking into account (5.9.4) and the quantity $\mathscr{N}_{m,r}(g,\varrho,u,T)$, the content of Section 5.2 can be modified accordingly. Concerning the local in time existence for the regularized system from Section 5.B, we just modify the kinetic part of the scheme of approximation:

$$\begin{cases} \partial_t g^{n+1} + v\cdot\nabla_x g^{n+1} + \mathrm{div}_v\left[(E^n(t,x)-v)g^{n+1}\right] - 2v\cdot(E^{u,\varrho}_{\mathrm{reg},\varepsilon}-v)g^{n+1} \\ \qquad\qquad\qquad\qquad = \Gamma^+_\lambda[g^n,g^n] - \Gamma^-_\lambda[g^n,g^{n+1}], \\ f^{n+1}{}_{|t=0} = f^{in}. \end{cases}$$

where $\varepsilon>0$ is given. We refer to [Mat10] for more details.

5.9.2 Equation on the augmented variable $\mathcal{F}$

Let us highlight the main changes that arise in the end of Subsection 5.5.1, more precisely in Definition 5.5.5. Recall that the goal is to consider a new unknown $\mathcal{F} = \left(\partial_x^K\partial_v^K f\right)_{|K|+|L|\in\{m-1,m\}}$. Here, one has to consider a new coupling matrix $\mathbb{M} = \left(\mathbb{M}_{(I,J),(K,L)}\right)$ which takes into account the collision operator $\mathcal{Q}$, defined by

$$\mathbb{M}_{(I,J),(K,L)} := \mathcal{M}_{(I,J),(K,L)} + \mathcal{M}^{\mathcal{Q}}_{(I,J),(K,L)},$$

with $|I|+|J|, |K|+|L| \in \{m-1,m\}$ and where

- $\mathcal{M}_{(I,J),(K,L)} \in \mathbb{R}$ stands for the former terms of the coupling matrix already appearing in Definition 5.5.5;
- $\mathcal{M}^{\mathcal{Q}}_{(I,J),(K,L)}$ is an new operator term coming from the collision operator and defined by the relation

$$\begin{aligned} \partial_x^I\partial_v^J\mathcal{Q}_\lambda(f,f) &= \sum_{\substack{0\le\alpha\le I\\0\le\beta\le J}} \binom{I}{\alpha}\binom{J}{\beta}\mathcal{Q}_\lambda(\partial_x^\alpha\partial_v^\beta f, \partial_x^{I-\alpha}\partial_v^{J-\beta}f) \\ &= \sum_{|K|+|L|\in\{m-1,m\}} \mathcal{M}^{\mathcal{Q}}_{(I,J),(K,L)}\left[\partial_x^K\partial_v^L f\right], \end{aligned}$$

  that is

$$\mathcal{M}^{\mathcal{Q}}_{(I,J),(K,L)}(\bullet) := \sum_{\substack{0\le\alpha\le I\\0\le\beta\le J\\|\alpha|+|\beta|\le1}} \binom{I}{\alpha}\binom{J}{\beta}\mathbf{1}_{\substack{K=I-\alpha\\L=J-\beta}}\mathcal{Q}_\lambda\left(\partial_x^\alpha\partial_v^\beta f,\bullet\right).$$

  Hence, $\mathcal{M}^{\mathcal{Q}}$ is a matrix with operator coefficients, acting on $\mathcal{F} = \left(\partial_x^K\partial_v^K f\right)_{|K|+|L|\in\{m-1,m\}}$.

Using the same notations as in Section 5.5.1, we obtain the following equation satisfied by $\mathcal{F} = \left(\partial_x^K\partial_v^K f\right)_{|K|+|L|\in\{m-1,m\}}$ (see (5.5.3))

$$\mathcal{T}^{u,\varrho}_{\mathrm{reg},\varepsilon}\mathcal{F} + \mathbb{M}\mathcal{F} + \mathcal{L} = -\mathcal{R}_0 - \mathcal{R}_1.$$

After the composition by $(t,x,v) \mapsto (t, \mathrm{X}^{t;0}(x,v), \mathrm{V}^{t;0}(x,v))$ where $\mathrm{Z} = (\mathrm{X},\mathrm{V})$ is the solution to (5.5.4), there holds the equation

$$\partial_t \widetilde{\mathcal{F}} + \widetilde{\mathbb{M}}\widetilde{\mathcal{F}} + \widetilde{\mathcal{L}} = d\widetilde{\mathcal{F}} - \widetilde{\mathcal{R}}_0 - \widetilde{\mathcal{R}}_1,$$

with $\widetilde{g}(t,x,v) = g(t, \mathrm{X}^{t;0}(x,v), \mathrm{V}^{t;0}(x,v))$. We can still consider the resolvant associated to the previous operator $\mathbb{M} - d\mathrm{Id}$, that is the solution $s \mapsto \mathfrak{N}^{s,t}(x,v)$ of

$$\begin{cases} \partial_s \mathfrak{N}^{s;t} + \left[\mathbb{M} \circ \mathrm{Z}^{s;0} - d\mathrm{Id}\right] \mathfrak{N}^{s;t} = 0, \\ \qquad\qquad\qquad\qquad \mathfrak{N}^{t;t} = \mathrm{Id}, \end{cases}$$

whose existence and uniqueness is still provided by the Cauchy-Lipschitz theorem, Hence, this shows that the contribution of the collision operator $\mathcal{Q}$ can be handled by the modified operator $\mathbb{M}$. The strategy of proof which is performed in the rest of Section 5.5 applies *mutatis mutandis*.

5.10 Generalization to the density-dependent drag case

In this section, we show how one can deal with the case of density-dependent drag in the force acting on the particles, that is, with the notation of the introduction, when the force in the kinetic equation is

$$\Gamma(t,x,v) = \varrho(t,x)(u(t,x) - v) - \nabla_x[p(\varrho)](t,x).$$

This additional factor also appears in the feedback of particles on the fluid, that is in the source term in the Navier-Stokes equations. We are thus led to consider the following system:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x p \cdot \nabla_v f + \mathrm{div}_v[f\varrho(u-v) - f\nabla_x p(\varrho)] = 0, \\ \partial_t(\alpha\varrho) + \mathrm{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u \otimes u) + \alpha\nabla_x p - \Delta_x u - \nabla_x \mathrm{div}_x\, u = \varrho(j_f - \rho_f u), \\ \alpha = 1 - \rho_f. \end{cases} \tag{5.10.1}$$

Our main result of local well-posedness is the following.

Theorem 5.10.1. *Consider the same assumptions of Theorem 5.1.5. Assume also that f^{in} is compactly supported in velocity. Then the conclusion of Theorem 5.1.5 holds for the density-dependent drag case of System* (5.10.1).

To fix notations, let us assume that

$$\mathrm{supp}\, f^{\mathrm{in}} \subset \mathbb{T}^d \times \mathrm{B}(0, M^{\mathrm{in}}), \quad M^{\mathrm{in}} > 0. \tag{5.10.2}$$

In what follows, we shall only present the main modifications which have to be added to the strategy used in this chapter.

5.10.1 Modification of the energy estimates and of the bootstrap argument

Our first goal is to adapt the energy estimates of Section 5.2 and the bootstrap procedure to the density-dependent drag case. The main change comes from the estimate for f from Proposition 5.2.10.

Indeed, let us set

$$\mathcal{T}^{u,\varrho}_{\mathrm{drag}} = \partial_t + v\cdot\nabla_x - \varrho v\cdot\nabla_v + E^{u,\varrho}(t,x)\cdot\nabla_v - d\varrho\mathrm{Id},$$

where $E^{u,\varrho}_{\mathrm{drag}} := \varrho u - \nabla_x p(\varrho)$. Observe that because of the term $\varrho v\cdot\nabla_v$ (coming from friction), a growth in velocity can occur in the analysis, that is if f is controlled in $\mathcal{H}^m_r$, then this term would *a priori* require a control in $\mathcal{H}^m_{r+1}$. This explains the additional assumption of compact support in velocity in the next proposition. We mention though that the use of exponential-weighted norms in velocity could relax this assumption (see the work [Asa86] on the Vlasov-Maxwell equations, and also [CJ22a] in the context of fluid-kinetic equations).

Proposition 5.10.2. *For all $r\geq 0$, $m>3+d/2$, $c>0$ and all $T>0$, for all smooth functions (f,ϱ,u) with f having a compact support in velocity:*

$$\forall t\in[0,T],\quad \operatorname{supp} f(t)\subset\mathbb{T}^d\times\mathrm{B}(0,M(t)),$$

for some $M\in\mathrm{L}^\infty(0,T)$, satisfying

$$\mathcal{T}^{u,\varrho}_{\mathrm{drag}}(f)=0\quad\text{on } [0,T],$$

and $\varrho\geq c$ on $[0,T]$, the following holds. For all $t\in[0,T]$, we have

$$\|f(t)\|^2_{\mathcal{H}^m_r}\leq\|f(0)\|^2_{\mathcal{H}^m_r}\exp\Big[C(1+\|M\|_{\mathrm{L}^\infty(0,T)})\Big(T+\sqrt{T}\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)}\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} + \sqrt{T}\Lambda\left(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\right)\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}\Big)\Big],$$

Proof. Let us suppose that there exists $M\in\mathrm{L}^\infty(0,T)$ sucht that

$$\forall t\in[0,T],\quad \operatorname{supp} f(t)\subset\mathbb{T}^d\times\mathrm{B}(0,M(t)).$$

Since $\mathcal{T}^{u,\varrho}_{\mathrm{drag}}(f)=0$, we have by Lemma 5.2.9

$$\mathcal{T}^{u,\varrho}_{\mathrm{drag}}(\partial^\alpha_x\partial^\beta_v f) = -\sum_{\substack{i=1\\ \beta_i\neq 0}}^d \partial_x^{\widehat{\alpha}^i}\partial_v^{\widehat{\beta}^i}f + \Big[\partial^\alpha_x\partial^\beta_v,\varrho(t,x)v\cdot\nabla_v\Big] - \Big[\partial^\alpha_x\partial^\beta_v, E^{u,\varrho}_{\mathrm{drag}}(t,x)\cdot\nabla_v\Big]f + d\Big[\partial^\alpha_x\partial^\beta_v,\varrho\mathrm{Id}\Big]f,$$

for all $\alpha,\beta\in\mathbb{N}^d$. We now take the scalar product of this equality with $(1+|v|^2)^r\partial^\alpha_x\partial^\beta_v f$, sum for all $|\alpha|+|\beta|\leq m$ and then integrate on $\mathbb{T}^d\times\mathbb{R}^d$. For the left-hand side, we have as in Proposition 5.2.10

$$\sum_{|\alpha|+|\beta|\leq m}\int_{\mathbb{T}^d\times\mathbb{R}^d}(1+|v|^2)^r\mathcal{T}^{u,\varrho}_{\mathrm{drag}}(\partial^\alpha_x\partial^\beta_v f)\partial^\alpha_x\partial^\beta_v f = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2_{\mathcal{H}^m_r} - \sum_{|\alpha|+|\beta|\leq m}\int_{\mathbb{T}^d\times\mathbb{R}^d}\nabla_v(1+|v|^2)^r\cdot(E^{u,\varrho}_{\mathrm{drag}}-\varrho v)\frac{|\partial^\alpha_x\partial^\beta_v f|^2}{2},$$

the last term satisfying

$$\sum_{|\alpha|+|\beta|\leq m}\int_{\mathbb{T}^d\times\mathbb{R}^d}\nabla_v(1+|v|^2)^r\cdot(E^{u,\varrho}_{\mathrm{drag}}-\varrho v)\frac{|\partial^\alpha_x\partial^\beta_v f|^2}{2}\leq(\|\varrho(t)\|_{\mathrm{L}^\infty}+\|E^{u,\varrho}_{\mathrm{drag}}(t)\|_{\mathrm{L}^\infty})\|f(t)\|^2_{\mathcal{H}^m_r}.$$

We now look at the four terms of the right-hand side. For the first, third and fourth ones, we proceed as in the proof of Proposition 5.2.10 (with a variant of (5.A.3)) and get

$$\sum_{|\alpha|+|\beta|\leq m}\int_{\mathbb{T}^d\times\mathbb{R}^d}(1+|v|^2)^r\Bigg(-\sum_{\substack{i=1\\ \beta_i\neq 0}}^{d}\partial_x^{\widehat{\alpha}^i}\partial_v^{\widehat{\beta}^i}f+\Big[\partial_x^{\alpha}\partial_v^{\beta},E^{u,\varrho}_{\mathrm{drag}}(t,x)\cdot\nabla_v+d\varrho\mathrm{Id}\Big]\,f\Bigg)\partial_x^{\alpha}\partial_v^{\beta}f$$
$$\lesssim(1+\|E^{u,\varrho}_{\mathrm{drag}}(t)\|_{\mathrm{H}^m}+\|\varrho(t)\|_{\mathrm{H}^m})\|f(t)\|^2_{\mathcal{H}^m_r}.$$

The treatment of the third term requires the use of the compact support in velocity of f. Invoking the inequality (5.A.4), we have

$$\sum_{|\alpha|+|\beta|\leq m}\int_{\mathbb{T}^d\times\mathbb{R}^d}(1+|v|^2)^r\left[\partial_x^{\alpha}\partial_v^{\beta},\varrho(t,x)v\cdot\nabla_v\right]\partial_x^{\alpha}\partial_v^{\beta}f\lesssim(1+M(t))\|\varrho(t)\|_{\mathrm{H}^m}\|f(t)\|^2_{\mathcal{H}^m_r}.$$

All in all, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2_{\mathcal{H}^m_r}\lesssim(1+M(t))(1+\|\varrho(t)\|_{\mathrm{H}^m}+\|E^{u,\varrho}(t)\|_{\mathrm{H}^m})\|f(t)\|^2_{\mathcal{H}^m_r}, \tag{5.10.3}$$

if $m>d/2$. As in the proof of Proposition 5.2.10, an by Sobolev embedding, we have

$$\|E^{u,\varrho}(t)\|_{\mathrm{H}^m}\lesssim\|\varrho(t)\|_{\mathrm{H}^m}\|u(t)\|_{\mathrm{H}^m}+\Lambda\left(\|\varrho(t)\|_{\mathrm{H}^{m-2}}\right)\|\varrho(t)\|_{\mathrm{H}^{m+1}}.$$

By integrating in time the inequality (5.10.3), we get

$$\begin{aligned}\|f(t)\|^2_{\mathcal{H}^m_r}&\leq\|f(0)\|^2_{\mathcal{H}^m_r}\\&+C\int_0^t(1+M(s))\Big(1+\|\varrho(s)\|_{\mathrm{H}^m}\|u(s)\|_{\mathrm{H}^m}+\Lambda\Big(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\Big)\|\varrho(s)\|_{\mathrm{H}^{m+1}}\Big)\|f(s)\|^2_{\mathcal{H}^m_r}\,\mathrm{d}s,\end{aligned}$$

for all $t\in[0,T)$ and for some constant $C>0$ independent of ε. Using the Cauchy-Schwarz inequality and the Grönwall's inequality, this implies for all $t\in[0,T)$

$$\begin{aligned}\|f(t)\|^2_{\mathcal{H}^m_r}\leq\|f(0)\|^2_{\mathcal{H}^m_r}\exp\Big[&C(1+\|M\|_{\mathrm{L}^\infty(0,T)})\Big(T+\sqrt{T}\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)})\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)})\\&+\sqrt{T}\Lambda\Big(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\Big)\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}\Big)\Big],\end{aligned}$$

and this concludes the proof. □

The proof of the other estimates from Section 5.2 is mainly unchanged and details are left to the reader. We need to adapt the bootstrap procedure, by taking into account the need of a compact support in velocity. We thus define the following modified condition.

Definition 5.10.3. *Let $T>0$. For any nonnegative functions $f(t,x,v)$ and $\varrho(t,x)$ on $[0,T]$, we define the property*

$$\forall t\in[0,T],\quad\begin{cases}\rho_f(t)\leq\dfrac{\Theta+1}{2},\quad\dfrac{\mu}{2}\leq\varrho(t),\quad\dfrac{\underline{\theta}}{2}\leq(1-\rho_f(t))\varrho(t)\leq 2\overline{\theta},\\ \qquad\qquad\qquad\qquad\mathrm{supp}\,f(t)\subset\mathbb{T}^d\times\mathrm{B}(0,1+M^{\mathrm{in}}),\end{cases}\tag{$\mathrm{B}^{\mu,\theta}_{\Theta,M^{\mathrm{in}}}(T)$}$$

where $\Theta,\mu,\underline{\theta},\overline{\theta}$ are given in the statement of Theorem 5.1.5 and where M^{in} has been introduced in (5.10.2).

If $T^*_\varepsilon>0$ is the maximal time of existence to the system (S_ε) (with density-dependent drag term), we introduce the following time for all $\varepsilon>0$:

$$T_\varepsilon=T_\varepsilon(R):=\sup\left\{T\in[0,T^*_\varepsilon[,\ \ \mathcal{N}_{m,r}(f_\varepsilon,\varrho_\varepsilon,u_\varepsilon,T)\leq R\ \text{ and }\ (\mathrm{B}^{\mu,\theta}_{\Theta,M^{\mathrm{in}}}(T))\ \text{ holds}\right\},\tag{5.10.4}$$

where $R>0$ has to be chosen large enough and independent of ε. Here, $\mathcal{N}_{m,r}(f_\varepsilon,\varrho_\varepsilon,u_\varepsilon,T)$ is exactly the same quantity as in Definition 5.2.19.

5.10.2 Modification in the straightening change of variable

As a matter of fact, the main difference in the analysis appears in the part related to characteristics, namely Section 5.3. Our purpose here is to explain how to modify the arguments of Section 5.3 about the straightening change of variable in velocity, that is Lemma 5.3.1. It turns out that in the density-dependent drag case, obtaining a suitable diffeomorphism $\psi_x^{s;t}$ is not as straightforward as in Lemma 5.3.1. Indeed, again because of the term $-\varrho v \cdot \nabla_v f$, there could be a growth in velocity in the dynamics which prevents our proof of Lemma 5.3.1, which is based on a perturbative approach, to directly hold. This is where the assumption compact support in velocity appears to be crucial; we do not know whether it is possible to replace it here by an exponential moment assumption.

With the notations of Section 5.3, we will actually directly straighten the total kinetic operator

$$\mathcal{T}_{\mathrm{drag,F}} = \partial_t + v \cdot \nabla_x - \varrho(t,x) v \cdot \nabla_v + \mathrm{F}(t,x) \cdot \nabla_v - d\mathrm{Id},$$

into the free-transport operator

$$\mathcal{T}^{\mathrm{free}} = \partial_t + v \cdot \nabla_x.$$

For $(x,v) \in \mathbb{T}^d \times \mathrm{B}(0, 1+\mathrm{M}^{\mathrm{in}})$ and $t \in [0,T]$, we consider the solution $s \mapsto (\mathrm{X}^{s;t}_{\mathrm{drag}}, \mathrm{V}^{s;t}_{\mathrm{drag}})(x,v)$ to the following system of differential equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} \mathrm{X}^{s;t}_{\mathrm{drag}} = \mathrm{V}^{s;t}_{\mathrm{drag}}, \quad \mathrm{X}^{t;t}_{\mathrm{drag}}(x,v) = x, \\ \dfrac{\mathrm{d}}{\mathrm{d}s} \mathrm{V}^{s;t}_{\mathrm{drag}} = -\varrho(s, \mathrm{X}^{s;t}_{\mathrm{drag}}) \mathrm{V}^{s;t}_{\mathrm{drag}} + \mathrm{F}(s, \mathrm{X}^{s;t}_{\mathrm{drag}}), \quad \mathrm{V}^{t;t}_{\mathrm{drag}}(x,v) = v. \end{cases}$$

Lemma 5.10.4. *Let $T > 0$ and $k \geq 1$. Let $\mathrm{F} \in \mathrm{L}^2(0,T; \mathrm{W}^{k,\infty}(\mathbb{T}^d))$ be a vector field such that*

$$\|\mathrm{F}\|_{\mathrm{L}^2(0,T;\mathrm{W}^{k,\infty}(\mathbb{T}^d))} \leq \Lambda(T, \mathrm{R}),$$

for some $\mathrm{R} > 0$. There exists $\overline{T}(\mathrm{R}) > 0$ such that for all $x \in \mathbb{T}^d$ and $s,t \in [0, \min(\overline{T}(\mathrm{R}), T)]$, there exists a diffeomorphism $\psi_{s,t}(x, \cdot) : \mathrm{B}(0, 1+\mathrm{M}^{\mathrm{in}}) \to \mathrm{B}(0, 1+\mathrm{M}^{\mathrm{in}})$ satisfying for all $v \in \mathrm{B}(0, 1+\mathrm{M}^{\mathrm{in}})$

$$\mathrm{X}^{s;t}_{\mathrm{drag}}(x, \psi_{s,t}(x,v)) = x - (t-s)v,$$

which furthermore verifies the estimates

$$\begin{aligned} \frac{1}{C} \leq \det\left(\mathrm{D}_v \psi_{s,t}(x,v)\right) &\leq C, \\ \sup_{s,t \in [0,T]} \left\| \partial^{\alpha}_{x,v} \left(\psi_{s,t}(x,v) - v\right) \right\|_{\mathrm{L}^\infty(\mathbb{T}^d \times \mathbb{R}^d)} &\leq \varphi(T) \Lambda(T, \mathrm{R}), \quad |\alpha| \leq k, \\ \sup_{s,t \in [0,T]} \left\| \partial^{\beta}_{x,v} \partial_s \psi_{s,t}(x,v) \right\|_{\mathrm{L}^\infty(\mathbb{T}^d \times \mathbb{R}^d)} &\leq \varphi(T) \Lambda(T, \mathrm{R}), \quad |\beta| \leq k-1, \end{aligned}$$

for some $C > 0$ and some nondecreasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ vanishing at 0.

Proof. Let us sketch the proof. Dropping the (x,v) dependency in the trajectories, we have

$$\mathrm{V}^{s;t}_{\mathrm{drag}} = \exp\left(\int_s^t \varrho(\tau, \mathrm{X}^{\tau;t}_{\mathrm{drag}}) \, \mathrm{d}\tau \right) v - \int_s^t \exp\left(\int_s^\tau \varrho(\tau', \mathrm{X}^{\tau';t}_{\mathrm{drag}}) \, \mathrm{d}\tau' \right) \mathrm{F}(\tau, \mathrm{X}^{\tau;t}_{\mathrm{drag}}) \, \mathrm{d}\tau,$$

from which we deduce

$$\begin{aligned}
\mathrm{X}^{s,t}_{\mathrm{drag}} &= x - \left(\int_s^t \exp\left(\int_{s'}^t \varrho(\tau, \mathrm{X}^{\tau;t}_{\mathrm{drag}})\,\mathrm{d}\tau\right)\mathrm{d}s'\right) v \\
&\qquad + \int_s^t \int_{s'}^t \exp\left(\int_{s'}^{\tau} \varrho(\tau', \mathrm{X}^{\tau';t}_{\mathrm{drag}})\,\mathrm{d}\tau'\right) \mathrm{F}(\tau, \mathrm{X}^{\tau;t}_{\mathrm{drag}})\,\mathrm{d}\tau\,\mathrm{d}s' \\
&= x - (t-s)\Bigg[v + \left(\frac{1}{t-s}\int_s^t \exp\left(\int_{s'}^t \varrho(\tau, \mathrm{X}^{\tau;t}_{\mathrm{drag}})\,\mathrm{d}\tau\right)\mathrm{d}s' - 1\right) v \\
&\qquad - \frac{1}{t-s}\int_s^t \int_{s'}^t \exp\left(\int_{s'}^{\tau} \varrho(\tau', \mathrm{X}^{\tau';t}_{\mathrm{drag}})\,\mathrm{d}\tau'\right) \mathrm{F}(\tau, \mathrm{X}^{\tau;t}_{\mathrm{drag}})\,\mathrm{d}\tau\,\mathrm{d}s' \Bigg].
\end{aligned}$$

Taking one derivative in velocity, we observe that because of the term

$$\left(\frac{1}{t-s}\int_s^t \exp\left(\int_{s'}^t \varrho(\tau, \mathrm{X}^{\tau;t}_{\mathrm{drag}})\,\mathrm{d}\tau\right)\mathrm{d}s' - 1\right) v,$$

we obtain a term where the derivative is not falling on the v factor. Hence, the latter is *a priori* unbounded, inducing a potential linear growth in velocity of the derivative. But since we restrict to $v \in \mathrm{B}(0, 1 + \mathrm{M}^{\mathrm{in}})$, we can however deduce a rough bound on $\nabla_v \mathrm{X}^{s;t}_{\mathrm{drag}}$ by Grönwall's inequality, which takes the form

$$|\nabla_v \mathrm{X}^{s;t}_{\mathrm{drag}}| \lesssim \exp(T\Lambda(T,R)(|v|+1))T\Lambda(T,R) \lesssim \exp(T\Lambda(T,R)(2+M^{\mathrm{in}}))T\Lambda(T,R).$$

We can now check that for T small enough, for all $0 \leq s, t \leq T$, the map

$$\begin{aligned}
v \mapsto v &+ \left(\frac{1}{t-s}\int_s^t \exp\left(\int_{s'}^t \varrho(\tau, \mathrm{X}^{\tau;t}_{\mathrm{drag}})\,\mathrm{d}\tau\right)\mathrm{d}s' - 1\right) v \\
&\qquad - \frac{1}{t-s}\int_s^t \int_{s'}^t \exp\left(\int_{s'}^{\tau} \varrho(\tau', \mathrm{X}^{\tau';t}_{\mathrm{drag}})\,\mathrm{d}\tau'\right) E^{u,\varrho}_{\mathrm{drag}}(\tau, \mathrm{X}^{\tau;t}_{\mathrm{drag}})\,\mathrm{d}\tau\,\mathrm{d}s'
\end{aligned}$$

is a $\mathscr{C}^1$ diffeomorphism from $\mathrm{B}(0, 1+\mathrm{M}^{\mathrm{in}})$ onto its image, since it is a small Lipschitz perturbation of the identity map. Details are left to the reader. □

As a result, for small times, we can directly come down to the case of free-transport case.

5.10.3 Modifications in remainder terms and conclusion of the bootstrap

To conclude, let us point out the last main modifications which have to be made to conclude the bootstrap argument.

In Section 5.5, one shall be careful when handling the remainder terms $\mathscr{R}_I^{\mathrm{Diff}}, \mathscr{R}_{I,1}^{\mathrm{Duha}}, \mathscr{R}_{I,2}^{\mathrm{Duha}}, \mathscr{R}_I$ in $\mathrm{L}^2(0,T;\mathrm{H}^1)$ because they now involve some terms with at most $m+1$ derivatives on ϱu stemming from the force field $E^{u,\varrho}_{\mathrm{drag}}$. It was somehow harmless in the linear-drag case because the corresponding terms involved only u, which has an additional regularity provided by the Navier-Stokes equation, namely a control in $\mathrm{L}^\infty(0,T;\mathrm{H}^m) \cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1})$ (see the terms $\mathbf{S_2}$, $\mathbf{S_5}$, $\mathbf{S_{12}}$, $\mathbf{S_{16}}$). Since we only have a control of ϱ in $\mathrm{L}^2(0,T;\mathrm{H}^m)$, this requires a additional argument.

The main idea is to rely on the same type of decomposition as in Lemma 5.5.13 combined with the use of the smoothing estimates from Section 5.4. The expression $\partial_x^K(\varrho u)$ ($K \in \mathbb{N}^d$) will involve terms or sum of terms

- of the form
$$\varrho\partial_x^K u \quad \text{or} \quad \partial_x^{\overline{K}^i-\beta} u \nabla_x(\partial_x^\beta \varrho)\cdot e_i,$$
with $i = 1, \cdots, d$ and $|\beta| = 0, \cdots, |\lfloor \frac{|K|-1}{2} \rfloor$. They are treated using L^∞ bounds on ϱ (with Sobolev embeddings) and L^2 bounds on u;

- of the form
$$\nabla_x(\partial_x^\beta \varrho)\cdot e_i \partial_x^{\overline{K}^i-\beta} u,$$
with $i = 1, \cdots, d$ and $|\beta| = \lfloor \frac{|K|-1}{2} \rfloor + 1, \cdots, |K|-1$ These terms are adressed thanks to Proposition 5.4.4. We refer to Section 5.5 for the same kind of procedure for the estimates on remainders via the smoothing estimates of Section 5.4.

Let us eventually briefly conclude by showing how the bootstrap procedure ends, when one considers the condition ($\mathrm{B}^{\mu,\theta}_{\Theta,M^{\mathrm{in}}}(T)$). Namely, we have to show how to control the compact support in velocity for f, for short times. We rely on the Lagrangian structure of the equation which implies a finite propagation in time of the support of f in velocity, namely

$$\forall t>0, \quad \operatorname{supp} f(t) \subset \mathbb{T}^d \times \mathrm{V}^{t;0}_{\mathrm{drag}}(\operatorname{supp} f^{\mathrm{in}}).$$

Since

$$\mathrm{V}^{t;0}_{\mathrm{drag}} = \exp\left(-\int_0^t \varrho(\tau, \mathrm{X}^{\tau;0}_{\mathrm{drag}})\,\mathrm{d}\tau\right) v + \int_0^t \exp\left(-\int_\tau^t \varrho(\tau', \mathrm{X}^{\tau';0}_{\mathrm{drag}})\,\mathrm{d}\tau'\right) E^{u,\varrho}_{\mathrm{drag}}(\tau, \mathrm{X}^{\tau;0}_{\mathrm{drag}})\,\mathrm{d}\tau,$$

the same kind of estimates on the trajectories as those performed in the proof of Lemma 5.10.4 give for all $(x,v) \in \mathbb{T}^d \times \mathrm{B}_v(0, M^{\mathrm{in}})$

$$|\mathrm{V}^{t;0}_{\mathrm{drag}}| \leq |v| + T\Lambda(T,R) \leq M^{\mathrm{in}} + T\Lambda(T,R).$$

As a consequence, choosing T small enough (once R is given) is sufficient to control the size of the support of $f(t)$ for short times.

Appendix

5.A Useful (para-)differential inequalities on $\mathbb{T}^d_x$ and $\mathbb{T}^d_x \times \mathbb{R}^d_v$

We recall and state several classic inequalities of (para-)differential type. First, we have the following tame estimate for commutators (see [MB02, Lemma 3.4]).

Proposition 5.A.1. *Let $s \geq 1$. There exists $C_s > 0$ such that for any functions $g, E \in \mathrm{H}^s \cap \mathrm{L}^\infty$, we have*

$$\forall |\alpha| \leq s, \quad \|[\partial_x^\alpha, E]g\|_{\mathrm{L}^2} \leq C_s \left(\|\nabla_x E\|_{\mathrm{L}^\infty}\|g\|_{\mathrm{H}^{s-1}} + \|E\|_{\mathrm{H}^s}\|g\|_{\mathrm{L}^\infty}\right).$$

The following result is about tame estimates in Sobolev spaces (see e.g. [BCD11, Corollary 2.86]).

Proposition 5.A.2. *Let $s > 0$. There exists $C_s > 0$ such that for all $w_1, w_2 \in \mathrm{H}^s \cap \mathrm{L}^\infty$, we have*

$$\|w_1 w_2\|_{\mathrm{H}^s_x} \leq C_s \left(\|w_1\|_{\mathrm{L}^\infty}\|w_2\|_{\mathrm{H}^s} + \|w_1\|_{\mathrm{H}^s}\|w_2\|_{\mathrm{L}^\infty}\right).$$

We finally have the following result of Bony about Sobolev continuity of the composition by a smooth function (see e.g. [BCD11, Corollary 2.87] or [Dan05a, Proposition 1.4.8] for a more precise version)

Proposition 5.A.3. *Let I be an open interval of $\mathbb{R}$ containing 0. Let $s > 0$ and σ be the smallest integer such that $\sigma > s$. Let $F \in \mathrm{W}^{\sigma+1,\infty}(I;\mathbb{R})$ such that $F(0) = 0$. If $w \in \mathrm{H}^s$ has value in $J \Subset I$, then there exists $C_s > 0$ such that*

$$\|F(w)\|_{\mathrm{H}^s} \leq C_s(1 + \|w\|_{\mathrm{L}^\infty})^\sigma \|F'\|_{\mathrm{W}^{\sigma,\infty}(I)} \|w\|_{\mathrm{H}^s}.$$

Remark 5.A.4. Note that if $0 \notin I$, and $w \in \mathrm{H}^s$ with value in $J \Subset I$, we can extend F outside I by a smooth extension such that $F(0) = 0$, the previous proposition being still valid. Indeed, by Faà di Bruno's formula, we observe that $\|F(w)\|_{\mathrm{H}^s}$ only involves F through its derivatives evaluated at w.

Lemma 5.A.5. *For all $k \in \mathbb{N}$ and $g : \mathbb{T}^d \to \mathbb{R}^+$ such that $0 < c \leq g \leq C < 1$, we have*

$$\left\| \frac{1}{1-g} \right\|_{\mathrm{H}^k} \leq 1 + \mathrm{C}_k \left(\|g\|_{\mathrm{L}^\infty}\right) \|g\|_{\mathrm{H}^k},$$

for some non-decreasing continuous functions $\mathrm{C}_k : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$.

Proof. We rely on Proposition 5.A.3 by writing

$$\left\| \frac{1}{1-g} \right\|_{\mathrm{H}^k} = \|F(g) - F(0)\|_{\mathrm{H}^k} + 1,$$

and this directly concludes the proof. □

Let us finally state several product and commutator laws using weighted-Sobolev norms.

Lemma 5.A.6. *Let $s \geq 0$. Consider a smooth nonnegative function $\chi = \chi(v)$ such that $|\partial^\gamma \chi| \lesssim_\gamma \chi$ for all $\gamma \in \mathbb{N}^d$ such that $|\gamma| \leq s$.*

- *For any functions $f = f(x,v)$, $g = g(x,v)$ and $k \geq s/2$, we have*

$$\|\chi f g\|_{\mathrm{H}^k_{x,v}} \lesssim \|f\|_{W^{k,\infty}_{x,v}} \|\chi g\|_{\mathrm{H}^s_{x,v}} + \|g\|_{W^{k,\infty}_{x,v}} \|\chi f\|_{\mathrm{H}^s_{x,v}}. \tag{5.A.1}$$

- *For any functions $a = a(x)$, $F = F(x,v)$ and $s_0 > d$, we have*

$$\|\chi a F\|_{\mathrm{H}^s_{x,v}} \lesssim \|a\|_{\mathrm{H}^{s_0}_x} \|\chi F\|_{\mathrm{H}^s_{x,v}} + \|a\|_{\mathrm{H}^s_x} \|\chi F\|_{\mathrm{H}^s_{x,v}}. \tag{5.A.2}$$

- *For any vector field $E = E(x)$ and any function $f = f(x,v)$, there holds for all $s_0 > 1 + d$ and all $\alpha, \beta \in \mathbb{N}^d$ satisfying $|\alpha| + |\beta| = s \geq 1$*

$$\left\| \chi \left[\partial_x^\alpha \partial_v^\beta, E(x) \cdot \nabla_v\right] f \right\|_{\mathrm{L}^2_{x,v}} \lesssim \|E\|_{\mathrm{H}^{s_0}_x} \|\chi f\|_{\mathrm{H}^s_{x,v}} + \|E\|_{\mathrm{H}^s_x} \|\chi f\|_{\mathrm{H}^s_{x,v}}. \tag{5.A.3}$$

- *For any functions $a = a(x)$, $f = f(x,v)$ such that f has a compact suport in velocity, there holds for all $s_0 > 1 + d$ and all $\alpha, \beta \in \mathbb{N}^d$ satisfying $|\alpha| + |\beta| = s \geq 1$*

$$\left\| \chi \left[\partial_x^\alpha \partial_v^\beta, a(x) v \cdot \nabla_v\right] f \right\|_{\mathrm{L}^2_{x,v}} \lesssim (1 + M_f) \|a\|_{\mathrm{H}^{s_0}_x} \|\chi f\|_{\mathrm{H}^s_{x,v}} + \|a\|_{\mathrm{H}^s_x} \|\chi f\|_{\mathrm{H}^s_{x,v}}, \tag{5.A.4}$$

  where

$$\operatorname{supp} f \subset \mathbb{T}^d \times \mathrm{B}(0, M_f), \quad \mathrm{M}_f \in (0, +\infty).$$

Proof. We refer to [HKR16, Lemma 1, Section 3] for the proof the (5.A.1)–(5.A.2)–(5.A.3). Let us briefly sketch the proof of (5.A.4). By expanding the commutator, we have to estimate terms of the form

$$I_{\gamma,\mu} = \left\| \chi \partial_x^\gamma a \partial_v^\mu (v_i) \partial_{v_i} \partial_x^{\alpha-\gamma} \partial_v^{\beta-\mu} f \right\|_{\mathrm{L}^2_{x,v}}$$

for some $1 \leq i \leq d$ and with $(\gamma,\mu) \neq (0,0)$, $\gamma \leq \alpha$, $\mu \leq \beta$ and $|\mu| < 2$. If $\gamma \neq 0$ and $\mu \neq 0$, then $I_{\gamma,\mu} = 0$ or $\partial_v^\mu(v_i) = 1$ and we can conclude as for (5.A.3). If $\gamma = 0$ and $\mu > 0$ then $\partial_v^\mu(v_i) = 0$ or 1 and we conclude by Sobolev embedding in x since $I_{\gamma,\mu} \leq \|a\|_{\mathrm{L}^\infty_x} \|\chi f\|_{\mathrm{H}^s_{x,v}}$. Lastly, if $\gamma > 0$ and $\mu = 0$, we rely on the compact support in velocity for f to get

$$I_{\gamma,\mu} \leq M_f \left\| \chi \partial_x^\gamma a \partial_{v_i} \partial_x^{\alpha-\gamma} \partial_v^\beta f \right\|_{\mathrm{L}^2_{x,v}},$$

and we end the proof as for (5.A.3). □

5.B Local well-posedness for (S_ε): proof of Proposition 5.2.18

Recall that we want to find a solution $(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon)$ (with $\varepsilon > 0$) defined on some interval $[0, T_\varepsilon^*)$ to the system

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon - p'(\varrho_\varepsilon) \nabla_x \left[(I - \varepsilon^2 \Delta_x)^{-1} \varrho_\varepsilon \right] \cdot \nabla_v f_\varepsilon + \mathrm{div}_v [f_\varepsilon (u_\varepsilon - v)] = 0, \\ \partial_t (\alpha_\varepsilon \varrho_\varepsilon) + \mathrm{div}_x (\alpha_\varepsilon \varrho_\varepsilon u_\varepsilon) = 0, \\ \partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla_x) u_\varepsilon + \dfrac{1}{\varrho_\varepsilon} \nabla_x p(\varrho_\varepsilon) - \dfrac{1}{\varrho_\varepsilon (1 - \rho_{f_\varepsilon})} \Big(\Delta_x + \nabla_x \mathrm{div}_x \Big) u_\varepsilon = \dfrac{1}{\varrho_\varepsilon (1 - \rho_{f_\varepsilon})} (j_{f_\varepsilon} - \rho_{f_\varepsilon} u_\varepsilon), \\ f_{\varepsilon|t=0} = f^{\mathrm{in}}, \quad \varrho_{\varepsilon|t=0} = \varrho^{\mathrm{in}}, \quad u_{\varepsilon|t=0} = u^{\mathrm{in}}, \end{cases} \qquad (\widetilde{S}_\varepsilon)$$

where

$$\rho_{f_\varepsilon}(t,x) := \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\, \mathrm{d}v, \quad j_{f_\varepsilon}(t,x) := \int_{\mathbb{R}^d} f_\varepsilon(t,x,v) v \, \mathrm{d}v, \quad \alpha_\varepsilon(t,x) = 1 - \rho_{f_\varepsilon}(t,x).$$

To do so, we rely on a standard iterative scheme. We will derive two types of estimates on the inductive solutions to that scheme:

- first, a *uniform bound in high regularity* which allows to obtain, through a weak compactness argument, a weak converging (sub)sequence in spaces with high regularity. This will be possible if we consider a small enough time of existence.
- next, *contraction estimates in low regularity* which aim at proving that this sequence of solutions is a Cauchy sequence in spaces with lower regularity, thus strongly converging.

This will be enough in order the pass to the limit in the iterative scheme, obtaining a solution with the high order regularity. Note that the first step is required to prove the second one. Uniqueness will classicaly follow from the exact same computations that have been performed in order to derive the contraction estimates.

We fix $\varepsilon > 0$ and we now drop the dependency in ε (we can take $\varepsilon = 1$ for instance): we set

$$f^0 = f^{in}, \quad \varrho^0 = \varrho^{in}, \quad u^0 = u^{in},$$

and for all $n \in \mathbb{N}$, a triplet (f^n, u^n, ϱ^n) and a time $T_n > 0$ being given with

$$(f^n, \varrho^n, u^n) \in \mathscr{C}(0,T_n;\mathcal{H}^m_r) \times \mathscr{C}(0,T_n;\mathrm{H}^m) \times \mathscr{C}(0,T_n;\mathrm{H}^m) \cap \mathrm{L}^2(0,T_n;\mathrm{H}^{m+1}),$$

we consider the system

$$\begin{cases} \partial_t f^{n+1} + v \cdot \nabla_x f^{n+1} + \mathrm{div}_v \left[f^{n+1}(E^n(t,x) - v) \right] = 0, \\ \partial_t \mathfrak{m}^{n+1} + \mathrm{div}_x(\mathfrak{m}^{n+1} u^n) = 0, \\ \partial_t u^{n+1} - \dfrac{1}{\mathfrak{m}^n}\Big(\Delta_x + \nabla_x \mathrm{div}_x\Big) u^{n+1} = -(u^n \cdot \nabla_x) u^n - \nabla_x \pi(\varrho^n) + \dfrac{1}{\mathfrak{m}^n}(j_{f^n} - \rho_{f^n} u^n), \\ \varrho^{n+1} = \dfrac{1}{1-\rho_{f^n}} \mathfrak{m}^{n+1}, \\ E^n := u^n - p'(\varrho^n) \nabla_x \left[(I - \varepsilon^2 \Delta_x)^{-1} \varrho^n \right], \\ f^{n+1}{}_{|t=0} = f^{\mathrm{in}}, \quad \mathfrak{m}^{n+1}{}_{|t=0} = \alpha^{\mathrm{in}} \varrho^{\mathrm{in}}, \quad u^{n+1}{}_{|t=0} = u^{\mathrm{in}}, \end{cases} \tag{$\widetilde{S}^{n+1}$}$$

with

$$\pi'(x) := \frac{p'(x)}{x}, \quad \pi(0) = 0,$$

and where

$$\rho_{f^n}(t,x) := \int_{\mathbb{R}^d} f^n(t,x,v)\,\mathrm{d}v, \quad j_{f^n}(t,x) := \int_{\mathbb{R}^d} f^n(t,x,v) v\,\mathrm{d}v.$$

We also set

$$R_0 := 100 \left(\|f^{\mathrm{in}}\|_{\mathcal{H}^m_r} + \|\mathfrak{m}^{\mathrm{in}}\|_{\mathrm{H}^m} + \|u^{\mathrm{in}}\|_{\mathrm{H}^m} \right).$$

Step 1: construction of $(f^{n+1}, \varrho^{n+1}, u^{n+1})$ and uniform estimates. In what follows, we construct the next iteration of the scheme which is a solution to ($\widetilde{S}^{n+1}$), thus proving that our inductive scheme is well-defined. We derive at the same time some uniform estimates in high regularity on the sequence of solutions.

Let $n \in \mathbb{N}$. By induction, we assume that there exists $T > 0$ (depending on ε) such that for all $k = 0, \cdots, n$

$$\|f^k\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^m_r)} + \|\mathfrak{m}^k\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} + \|u^k\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m) \cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1})} < R_0,$$

the functions f_k and ϱ_k are nonnegative, and for $t \in [0,T]$

$$\forall t \in [0,T], \quad \rho_{f_k}(t) \leq \frac{\Theta + 1}{2}, \quad \frac{\mu}{2} \leq \varrho_k(t), \quad \frac{\underline{\theta}}{2} \leq \mathfrak{m}_k(t) \leq 2\overline{\theta}, \tag{5.B.1}$$

where $\Theta, \mu, \underline{\theta}, \overline{\theta}$ are the constants given in Proposition 5.2.18. In particular, we have

$$1 - \rho_{f_k}(t) \geq \frac{1 - \Theta}{2} > 0.$$

Note also that via Sobolev embedding, Lemma 5.A.5 and Lemma 5.2.1, we have

$$\begin{aligned} \|\varrho^n\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} &\leq \left\| \frac{1}{1 - \rho_{f^{n-1}}} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} \|\mathfrak{m}^n\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} \\ &\leq \Big(1 + \Lambda(\|\rho_{f^{n-1}}\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)}\Big) \|\rho_{f^{n-1}}\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} \|\mathfrak{m}^n\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^m)} \\ &\leq \Big(1 + \Lambda(\|f^{n-1}\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^m_r)}\Big) \|f^{n-1}\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^m_r)} \|\mathfrak{m}^n\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} \\ &\leq \Lambda(R_0). \end{aligned}$$

In what follows, we rely on the a priori estimates of Section 5.2.

• We can obtain a unique nonnegative solution $f^{n+1} \in \mathscr{C}(0,T;\mathcal{H}^m_r)$ to the Vlasov equation by the method of characteristics. Since there is the regularization due to J_ε, we can invoke Proposition 5.2.14 so that we have for all $t \in [0,T]$

$$\begin{aligned}\|f^{n+1}(t)\|_{\mathcal{H}^m_r} &\leq \|f^{\mathrm{in}}\|_{\mathcal{H}^m_r} \exp\left[C\left((1+\|u^n\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^m)})t + \frac{t}{\varepsilon}\Lambda\left(\|\varrho^n\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^{m-2})}\right)\|\varrho^n\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^m)}\right)\right] \\ &\leq \frac{R_0}{100}\exp\left[C\left((1+R_0)t + \frac{t}{\varepsilon}\Lambda(R_0)\right)\right].\end{aligned}$$

We then define $T_{[1]} = T_{[1]}(R_0) > 0$ with $T_{[1]}(R_0) < T$ (depending on ε) such that

$$\frac{R_0}{100}\exp\left[C\left((1+R_0)T_{[1]} + \frac{\sqrt{T_{[1]}}}{\varepsilon}\Lambda(R_0)\right)\right] < \frac{R_0}{3}.$$

• Since $u^n \in \mathscr{C}(0,T;\mathrm{H}^{m+1})$ is given, we can construct a nonnegative solution $\mathfrak{m}^{n+1}$ to the second equation in ($\widetilde{S}^{n+1}$), relying on standard arguments for continuity equations. We obtain a unique solution $\mathfrak{m}^{n+1} \in \mathscr{C}^1(0,T;\mathrm{H}^m)$, with for all $t \in [0,T]$

$$\|\mathfrak{m}^{n+1}(t)\|_{\mathrm{H}^m} \leq e^{\|\mathrm{div}_x u^n\|_{\mathrm{L}^\infty((0,t)\times\mathbb{T}^d)}t/2}\left(\|\mathfrak{m}^{in}\|_{\mathrm{H}^m} + \int_0^t \|u^n(\tau)\|_{\mathrm{H}^{m+1}}\|\mathfrak{m}^{n+1}(\tau)\|_{\mathrm{H}^m}\,\mathrm{d}\tau\right),$$

thanks to Proposition 5.2.3. Assuming that $e^{R_0 t/2} \leq 2$, we infer by Grönwall's lemma

$$\begin{aligned}\|\mathfrak{m}^{n+1}(t)\|_{\mathrm{H}^m} &\leq 2\|\mathfrak{m}^{in}\|_{\mathrm{H}^m}\exp\left(2\int_0^t \|u^n(\tau)\|_{\mathrm{H}^{m+1}}\,\mathrm{d}\tau\right) \\ &\leq 2\|\mathfrak{m}^{in}\|_{\mathrm{H}^m}\exp\left(2\sqrt{t}\|u^n\|_{\mathrm{L}^2(0,t;\mathrm{H}^{m+1})}\right),\end{aligned}$$

therefore for such times t, we have

$$\forall t \in [0,T], \quad \|\mathfrak{m}^{n+1}(t)\|_{\mathrm{H}^m} \leq 2\frac{R_0}{100}e^{2\sqrt{t}R_0}.$$

We then define $T_{[2]} = T_{[2]}(R_0) > 0$ such that $e^{R_0 T_{[2]}/2} \leq 2$ and such that

$$2\frac{R_0}{100}e^{2\sqrt{T_{[2]}}R_0} < \frac{R_0}{3}.$$

• We define u^{n+1} as the unique solution of the parabolic equation

$$\partial_t w - \frac{1}{\mathfrak{m}^n}\Big(\Delta_x + \nabla_x \mathrm{div}_x\Big)w = -(u^n\cdot\nabla_x)u^n - \nabla_x\pi(\varrho^n) + \frac{1}{\mathfrak{m}^n}(j_{f^n} - \rho_{f^n}u^n),$$

starting from u^{in}, which satisfies

$$\begin{aligned}&\|u^{n+1}\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^m)} + \|u^{n+1}\|_{\mathrm{L}^2(0,t;\mathrm{H}^{m+1})} \\ &\leq (1+\sqrt{t}\Lambda(t,R_0))\Bigg(\|u^{in}\|_{\mathrm{H}^m} + \sqrt{t}\Lambda(\|\varrho^n\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^m)})\|\varrho^n\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^m)} + \sqrt{t}\,\|u^n\|^2_{\mathrm{L}^\infty(0,t;\mathrm{H}^m)} \\ &\qquad + \sqrt{t}\left\|\frac{1}{\mathfrak{m}^n}(j_{f^n} - \rho_{f^n}u^n)\right\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^{m-1})}\Bigg).\end{aligned}$$

Here, we have applied the same argument as for the proof of Proposition 5.2.24, making the term $\|\varrho^n\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^m)}$ appear to get a factor $\sqrt{t}$. The second term in the parenthesis is controlled by $\Lambda(R_0)$ while for the third term, we can use the same of kind of arguments (with Remark 5.A.4) to get

$$\begin{aligned}\Big\|\frac{1}{\mathfrak{m}^n}(j_{f^n}-\rho_{f^n}u^n)\Big\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^{m-1})}\\ &\leq \Big\|\frac{1}{\mathfrak{m}^n}\Big\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^{m-1})}\Big(\|j_{f^n}\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^{m-1})}+\|\rho_{f^n}\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^{m-1})}\|u^n\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^{m-1})}\Big)\\ &\leq \Lambda(R_0).\end{aligned}$$

We eventually obtain

$$\|u^{n+1}\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^m)}+\|u^{n+1}\|_{\mathrm{L}^2(0,t;\mathrm{H}^{m+1})}\leq(1+\sqrt{t}\Lambda(t,R_0))\frac{R_0}{100}+\sqrt{t}\Lambda(R_0),$$

We then define $T_{[3]}=T_{[3]}(R_0)>0$ with $T_{[3]}(R_0)<T$ such that

$$\Big(1+\sqrt{T_{[3]}}\Lambda(T_{[3]},R_0)\Big)\frac{R_0}{100}+\sqrt{T_{[3]}}\Lambda(R_0)<\frac{R_0}{3}.$$

• Lastly, we can rely on Lemmas 5.2.16–5.2.17 to find a time $T_{[4]}=T_{[4]}(R_0)>0$ such that the condition (5.B.1) is satisfied for $(f^{n+1},\mathfrak{m}^{n+1},u^{n+1})$ on the interval $[0,\min(T_{[4]},T))$.

All in all, we define

$$T(R_0):=\min\Big(T_{[1]}(R_0),T_{[2]}(R_0),T_{[3]}(R_0),T_{[4]}(R_0)\Big)<T,$$

which may depends on ε but which is independent of n. An induction procedure based on the three previous estimate shows that one can obtain a time $T^{[\varepsilon]}>0$ and a sequence $(f^n,\mathfrak{m}^n,u^n)_{n\in\mathbb{N}}$ satisfying for all $n\in\mathbb{N}$

$$(f^n,\mathfrak{m}^n,u^n)\in\mathscr{C}(0,T^{[\varepsilon]};\mathcal{H}^m_r)\times\mathscr{C}(0,T^{[\varepsilon]};\mathrm{H}^m)\times\mathscr{C}(0,T^{[\varepsilon]};\mathrm{H}^m)\cap\mathrm{L}^2(0,T;\mathrm{H}^{m+1}),$$

with (5.B.1) and the uniform estimate

$$\|f^n\|_{\mathrm{L}^\infty(0,T^{[\varepsilon]};\mathcal{H}^m_r)}+\|\mathfrak{m}^n\|_{\mathrm{L}^\infty(0,T^{[\varepsilon]};\mathrm{H}^m)}+\|u^n\|_{\mathrm{L}^\infty(0,T^{[\varepsilon]};\mathrm{H}^m)\cap\mathrm{L}^2(0,T^{[\varepsilon]};\mathrm{H}^{m+1})}<R_0. \tag{5.B.2}$$

Step 2: contraction estimates in $\mathrm{L}^\infty_T\mathcal{H}^0_r\times\mathrm{L}^\infty_T\mathrm{L}^2\times\mathrm{L}^\infty_T\mathrm{L}^2\cap\mathrm{L}^2_T\mathrm{H}^1$. For $n\in\mathbb{N}\backslash\{0\}$, we set

$$g^n:=f^{n+1}-f^n,\quad \mathfrak{M}^n=\mathfrak{m}^{n+1}-\mathfrak{m}^n,\quad w^n:=u^{n+1}-u^n,$$

which satisfy the system of equations:

$$\begin{cases}\partial_t g^n+v\cdot\nabla_x g^n+\mathrm{div}_v\left[g^n(E^n(t,x)-v)\right]+\Big(E^n-E^{n-1}\Big)\cdot\nabla_v f^n=0,\\ \partial_t\mathfrak{M}^n+\mathrm{div}_x(\mathfrak{M}^n u^n)=-\mathrm{div}_x(\mathfrak{m}^n w^{n-1}),\\ \partial_t w^n-\dfrac{1}{\mathfrak{m}^{n+1}}\Big(\Delta_x+\nabla_x\mathrm{div}_x\Big)w^n=\mathfrak{S}^n,\\ \varrho^{n+1}:=\dfrac{1}{1-\rho_{f^n}}\mathfrak{m}^{n+1},\\ E^n:=u^n-p'(\varrho^n)\nabla_x\left[(I-\varepsilon^2\Delta_x)^{-1}\varrho^n\right],\\ g^n{}_{|t=0}=0,\quad \mathfrak{M}^n{}_{|t=0}=0,\quad w^n{}_{|t=0}=0,\end{cases} \tag{5.B.3}$$

and where

$$\mathfrak{S}^n := \left(\frac{1}{\mathfrak{m}^{n+1}} - \frac{1}{\mathfrak{m}^n}\right)\Big(\Delta_x + \nabla_x \mathrm{div}_x\Big) u^n - (w^{n-1}\cdot\nabla_x)u^n - (u^{n-1}\cdot\nabla_x)w^{n-1}$$
$$- \nabla_x\left[\pi(\varrho^{n-1}) - \pi(\varrho^n)\right] + \frac{1}{\mathfrak{m}^n}(j_{f^n} - \rho_{f^n}u^n) - \frac{1}{\mathfrak{m}^{n-1}}(j_{f^{n-1}} - \rho_{f^{n-1}}u^{n-1}).$$

Let us derive some L^2 estimates on $(g^n, \mathfrak{M}^n, w^n)$. They will be satisfied on $[0,\widetilde{T}]$ for some $\widetilde{T}^{[\varepsilon]} < T^{[\varepsilon]}$.

• We perform $\mathrm{L}^2_{x,v}$-weighted estimates in the first equation on g^n and as before (see (5.2.3)), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|g^n(t)\|^2_{\mathcal{H}^0_r} \lesssim (1 + \|E^n(t)\|_{\mathrm{H}^m})\|g^n(t)\|^2_{\mathcal{H}^0_r} + \left\|E^n(t) - E^{n-1}(t)\right\|_{\mathrm{L}^2}\|f^n(t)\|_{\mathcal{H}^m_r}\|g^n(t)\|_{\mathcal{H}^0_r}.$$

For $t\in(0,T)$, we can infer

$$\begin{aligned}\|g^n(t)\|_{\mathcal{H}^0_r} &\lesssim \int_0^t (1 + \|E^n(s)\|_{\mathrm{H}^m})\|g^n(s)\|_{\mathcal{H}^0_r}\,\mathrm{d}s + \int_0^t \left\|E^n(s) - E^{n-1}(s)\right\|_{\mathrm{L}^2}\|f^n(s)\|_{\mathcal{H}^m_r}\,\mathrm{d}s \\ &\leq \sqrt{t}\Big(\left(\sqrt{t} + \|E^n\|_{\mathrm{L}^2(0,t;\mathrm{H}^m)}\right)\|g^n\|_{\mathrm{L}^\infty(0,t;\mathcal{H}^0_r)} + R_0\left\|E^n - E^{n-1}\right\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\Big) \\ &\lesssim \sqrt{t}\Big(\left(\sqrt{t} + \frac{\sqrt{t}}{\varepsilon}R_0\right)\|g^n\|_{\mathrm{L}^\infty(0,t;\mathcal{H}^0_r)} + R_0\left\|E^n - E^{n-1}\right\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\Big).\end{aligned}$$

Choosing $\widetilde{T}^{[\varepsilon]} < T^{[\varepsilon]}$ small enough independent of n, we can absorb the first term in the parenthesis in the left-hand side so that for all $t\in(0,\widetilde{T}^{[\varepsilon]})$

$$\|g^n(t)\|_{\mathcal{H}^0_r} \lesssim \sqrt{t}R_0\left\|E^n - E^{n-1}\right\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}.$$

Now observe that by Sobolev embedding and Proposition 5.A.3, we have

$$\begin{aligned}\|E^n - E^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)} &\leq \|u^n - u^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)} + \left\|p'(\varrho^n)\nabla_x\left[\mathrm{J}_\varepsilon\varrho^n\right] - p'(\varrho^{n-1})\nabla_x\left[\mathrm{J}_\varepsilon\varrho^{n-1}\right]\right\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)} \\ &\leq \|u^n - u^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)} + \|p'(\varrho^n)\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty)}\left\|\nabla_x\left[\mathrm{J}_\varepsilon(\varrho^n - \varrho^{n-1}\right]\right\|_{\mathrm{L}^2(0,t\mathrm{L}^2)} \\ &\quad + \|p'(\varrho^n) - p'(\varrho^{n-1})\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\left\|\nabla_x\left[\mathrm{J}_\varepsilon\varrho^{n-1}\right]\right\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty)} \\ &\lesssim \|u^n - u^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)} + \Lambda(R_0)\frac{1}{\varepsilon}\left\|\varrho^n - \varrho^{n-1}\right\|_{\mathrm{L}^2(0,t\mathrm{L}^2)} \\ &\quad + \Lambda(t,R_0)\frac{1}{\varepsilon}\|p'(\varrho^n) - p'(\varrho^{n-1})\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)} \\ &\lesssim \|u^n - u^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)} + \Lambda(t,R_0)\frac{1}{\varepsilon}\|\mathfrak{m}^n - \mathfrak{m}^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}.\end{aligned}$$

This yields for all $t\in(0,\widetilde{T}^{[\varepsilon]})$

$$\|g^n\|_{\mathrm{L}^\infty(0,t;\mathcal{H}^0_r)} \lesssim \sqrt{t}\left(R_0\|w^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)} + \Lambda(t,R_0)\frac{1}{\varepsilon}\|\mathfrak{M}^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\right).$$

• An L^2 energy estimate on the second equation on $\mathfrak{M}^n$ also leads to

$$\begin{aligned}
&\|\mathfrak{M}^n(t)\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^2)}\\
&\leq e^{\|\mathrm{div}_x u_n\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty)}t}\int_0^t \left\|\mathrm{div}_x\, \mathfrak{m}^n w^{n-1}(\tau)\right\|_{\mathrm{L}^2} \mathrm{d}\tau\\
&\leq e^{R_0 t}\left(\int_0^t \left\|w^{n-1}\cdot\nabla_x \mathfrak{m}^n(\tau)\right\|_{\mathrm{L}^2}\mathrm{d}\tau + \int_0^t \left\|\mathfrak{m}^n \mathrm{div}_x\, w^{n-1}(\tau)\right\|_{\mathrm{L}^2}\mathrm{d}\tau\right)\\
&\leq \sqrt{t}e^{R_0 t}\left(\|\nabla_x \mathfrak{m}^n\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty)}\left\|w^{n-1}\right\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)} + \|\mathfrak{m}^n\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty)}\left\|\mathrm{div}_x w^{n-1}\right\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\right)\\
&\lesssim \sqrt{t}\Lambda(t,R_0)\left\|w^{n-1}\right\|_{\mathrm{L}^2(0,t;\mathrm{H}^1)},
\end{aligned}$$

for all $t\in(0,\widetilde{T}^{[\varepsilon]})$.

• Performing an $\mathrm{L}^\infty_T\mathrm{L}^2\cap\mathrm{L}^2_T\mathrm{H}^1$ estimate by multiplying by w_n in the parabolic equation satisfied by w^n yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^n\|^2_{\mathrm{L}^2} + \int_{\mathbb{T}^d}\frac{1}{\mathfrak{m}^{n+1}}\left(|\nabla_x w^n|^2 + |\mathrm{div}_x w^n|^2\right)\\
&\qquad = \sum_{k=1}^5\int_{\mathbb{T}^d}\mathfrak{S}^n_k\cdot w^n - \sum_{i=1}^d\left\{\int_{\mathbb{T}^d}\nabla_x w^n_i\cdot\nabla_x\left(\frac{1}{\mathfrak{m}^{n+1}}\right)w^n_i - \int_{\mathbb{T}^d}\mathrm{div}_x\, w^n\partial_i\left(\frac{1}{\mathfrak{m}^{n+1}}\right)w^n_i\right\},
\end{aligned}$$

with

$$\begin{aligned}
&\mathfrak{S}^n_1 := \left(\frac{1}{\mathfrak{m}^{n+1}}-\frac{1}{\mathfrak{m}^n}\right)\Big(\Delta_x+\nabla_x\mathrm{div}_x\Big)u^n,\quad \mathfrak{S}^n_2 := -(w^{n-1}\cdot\nabla_x)u^n,\quad \mathfrak{S}^n_3 := -(u^{n-1}\cdot\nabla_x)w^{n-1},\\
&\qquad \mathfrak{S}^n_4 := -\nabla_x\left[\pi(\varrho^{n-1})-\pi(\varrho^n)\right],\quad \mathfrak{S}^n_5 := \frac{1}{\mathfrak{m}^n}(j_{f^n}-\rho_{f^n}u^n) - \frac{1}{\mathfrak{m}^{n-1}}(j_{f^{n-1}}-\rho_{f^{n-1}}u^{n-1}).
\end{aligned}$$

For all $\eta>0$, we obtain by Young inequality

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^n\|^2_{\mathrm{L}^2} + \int_{\mathbb{T}^d}\left(\frac{1}{\mathfrak{m}^{n+1}} - \eta\left\|\nabla_x\left(\frac{1}{\mathfrak{m}^{n+1}}\right)\right\|^2_{\mathrm{L}^\infty}\right)\left(|\nabla_x w^n|^2 + |\mathrm{div}_x w^n|^2\right)\\
&\qquad\qquad \leq \sum_{k=1}^5\int_{\mathbb{T}^d}\mathfrak{S}^n_k\cdot w^n + \frac{1}{\eta}\int_{\mathbb{T}^d}|w^n|^2.
\end{aligned}$$

Let us focus on the source terms $\mathfrak{S}^n_k$. We have by Sobolev embedding

$$\begin{aligned}
\int_{\mathbb{T}^d}\mathfrak{S}^n_1\cdot w^n &\lesssim \|(\Delta_x+\nabla_x\mathrm{div}_x)u^n\|^2_{\mathrm{L}^\infty}\left\|\frac{1}{\mathfrak{m}^{n+1}}-\frac{1}{\mathfrak{m}^n}\right\|^2_{\mathrm{L}^2} + \int_{\mathbb{T}^d}|w^n|^2\\
&\lesssim R_0^2\left\|\frac{1}{\mathfrak{m}^{n+1}\mathfrak{m}^n}\right\|^2_{\mathrm{L}^\infty}\left\|\mathfrak{m}^{n+1}-\mathfrak{m}^n\right\|^2_{\mathrm{L}^2} + \int_{\mathbb{T}^d}|w^n|^2\\
&\lesssim \Lambda(R_0)\|\mathfrak{M}^n\|^2_{\mathrm{L}^2} + \|w^n\|^2_{\mathrm{L}^2}.
\end{aligned}$$

We next have

$$\begin{aligned}
\int_{\mathbb{T}^d}\mathfrak{S}^n_2\cdot w^n \lesssim \int_{\mathbb{T}^d}|\mathfrak{S}^n_2|^2 + \int_{\mathbb{T}^d}|w^n|^2 &\leq \left\|\nabla_x u^{n-1}\right\|^2_{\mathrm{L}^\infty}\left\|w^{n-1}\right\|^2_{\mathrm{L}^2} + \int_{\mathbb{T}^d}|w^n|^2\\
&\leq R_0^2\left\|w^{n-1}\right\|^2_{\mathrm{L}^2} + \|w^n\|^2_{\mathrm{L}^2},
\end{aligned}$$

again by Sobolev embedding while for all $\delta_1 > 0$, we have by integration by parts and Sobolev embedding

$$\int_{\mathbb{T}^d} \mathfrak{S}_3^n \cdot w^n \lesssim \left\|\nabla_x u^{n-1}\right\|_{\mathrm{L}^\infty}^2 \left\|w^{n-1}\right\|_{\mathrm{L}^2}^2 + \|w^n\|_{\mathrm{L}^2}^2 + \frac{1}{\delta_1}\left\|u^{n-1}\right\|_{\mathrm{L}^\infty}^2 \left\|w^{n-1}\right\|_{\mathrm{L}^2}^2 + \delta_1 \int_{\mathbb{T}^d} |\nabla_x w^n|^2$$
$$\lesssim \Lambda_{\delta_1}(R_0)\left\|w^{n-1}\right\|_{\mathrm{L}^2} + \|w^n\|_{\mathrm{L}^2}^2 + \delta_1 \int_{\mathbb{T}^d} |\nabla_x w^n|^2.$$

We also have for all $\delta_2 > 0$

$$\int_{\mathbb{T}^d} \mathfrak{S}_4^n \cdot w^n \lesssim \frac{1}{\delta_2}\int_{\mathbb{T}^d} |\left[\pi(\varrho^{n-1}) - \pi(\varrho^n)\right]|^2 + \delta_2 \int_{\mathbb{T}^d} |\mathrm{div}_x\, w^n|^2$$
$$\lesssim \Lambda_{\delta_2}(R_0)\left\|\varrho^{n-1} - \varrho^n\right\|_{\mathrm{L}^2}^2 + \delta_2 \int_{\mathbb{T}^d} |\mathrm{div}_x\, w^n|^2.$$

For $\mathfrak{S}_5^n$, we write

$$\mathfrak{S}_5^n = \left(\frac{1}{\mathfrak{m}^n} - \frac{1}{\mathfrak{m}^{n-1}}\right)(j_{f^n} - \rho_{f^n} u^n) + \frac{1}{\mathfrak{m}^{n-1}}(j_{f^n} - j_{f^{n-1}})$$
$$+ \frac{1}{\mathfrak{m}^n}\left(\rho_{f^{n-1}} - \rho_{f^n}\right) u^{n-1} + \frac{1}{\mathfrak{m}^{n-1}}\rho_{f^n}(u^{n-1} - u^n),$$

therefore by Sobolev embedding and Lemma 5.2.1

$$\int_{\mathbb{T}^d} \mathfrak{S}_5^n \cdot w^n \lesssim \|j_{f^n} - \rho_{f^n} u^n\|_{\mathrm{L}^\infty}^2 \left\|\frac{1}{\mathfrak{m}^n} - \frac{1}{\mathfrak{m}^{n-1}}\right\|_{\mathrm{L}^2}^2 + \left\|\frac{1}{\mathfrak{m}^n}\right\|_{\mathrm{L}^\infty}^2 \left\|j_{f^n} - j_{f^{n-1}}\right\|_{\mathrm{L}^2}^2$$
$$+ \left\|\frac{1}{\mathfrak{m}^n}\right\|_{\mathrm{L}^\infty}^2 \left\|u^{n-1}\right\|_{\mathrm{L}^\infty}^2 \left\|\rho_{f^n} - \rho_{f^{n-1}}\right\|_{\mathrm{L}^2}^2 + \left\|\frac{1}{\mathfrak{m}^{n-1}}\right\|_{\mathrm{L}^\infty}^2 \|\rho_{f^n}\|_{\mathrm{L}^\infty}^2 \left\|u^{n-1} - u^n\right\|_{\mathrm{L}^2}^2$$
$$+ \int_{\mathbb{T}^d} |w^n|^2$$
$$\lesssim \Lambda(R_0)\left(\left\|\mathfrak{m}^n - \mathfrak{m}^{n-1}\right\|_{\mathrm{L}^2}^2 + \left\|f^n - f^{n-1}\right\|_{\mathcal{H}_r^0}^2 + \left\|u^{n-1} - u^n\right\|_{\mathrm{L}^2}^2\right) + \|w^n\|_{\mathrm{L}^2}^2.$$

We finally obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|w^n(t)\|_{\mathrm{L}^2}^2 + \int_{\mathbb{T}^d}\left(\frac{1}{\mathfrak{m}^{n+1}} - \eta R_0^2 - \delta_1 - \delta_2\right)\left(|\nabla_x w^n|^2 + |\mathrm{div}_x w^n|^2\right) \\
&\leq \Lambda_{\delta_{1,2},\eta}(R_0)\left(\left\|\mathfrak{M}^{n-1}(t)\right\|_{\mathrm{L}^2}^2 + \left\|g^{n-1}(t)\right\|_{\mathcal{H}_r^0}^2 + \left\|w^{n-1}(t)\right\|_{\mathrm{L}^2}^2\right) + \Lambda(R_0)\|\mathfrak{M}^n(t)\|_{\mathrm{L}^2}^2 \\
&\quad + \Lambda_\eta \|w^n(t)\|_{\mathrm{L}^2}^2.
\end{aligned} \tag{5.B.4}$$

for some constant $\Lambda_\eta > 0$. A good choice of η, δ_1 and δ_2 with respect to $\inf \frac{1}{\mathfrak{m}^{n+1}}$ combined with Grönwall's lemma shows that

$$\|w^n(t)\|_{\mathrm{L}^2}^2 \lesssim e^{\Lambda_\eta t}\int_0^t \left(\left\|\mathfrak{M}^{n-1}(\tau)\right\|_{\mathrm{L}^2}^2 + \left\|g^{n-1}(\tau)\right\|_{\mathcal{H}_r^0}^2 + \left\|w^{n-1}(\tau)\right\|_{\mathrm{L}^2}^2 + \|\mathfrak{M}^n(\tau)\|_{\mathrm{L}^2}^2\right)\mathrm{d}\tau.$$

Reducing $\widetilde{T}^{[\varepsilon]}$ such that $\widetilde{T}^{[\varepsilon]} \leq 1/\Lambda_\eta$, we can integrate in time the previous differential inequality (5.B.4) to obtain for all $t \in (0, \widetilde{T}^{[\varepsilon]})$

$$\|w^n(t)\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^2)}^2 + \|w^n(t)\|_{\mathrm{L}^2(0,t;\mathrm{H}^1)}^2$$
$$\lesssim \Lambda(R_0)t\left(\left\|\mathfrak{M}^{n-1}\right\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^2)}^2 + \left\|g^{n-1}\right\|_{\mathrm{L}^\infty(0,t;\mathcal{H}_r^0)}^2 + \left\|w^{n-1}\right\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^2)}^2\right) + \Lambda(R_0)t\|\mathfrak{M}^n\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^2)}^2.$$

Combining the three previous points, we discover the following inequality valid for all $t \in (0, \widetilde{T}^{[\varepsilon]})$

$$\begin{aligned}
&\|g^n(t)\|_{\mathrm{L}^\infty(0,t;\mathcal{H}^0_r)} + \|\mathfrak{M}^n(t)\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^2)} + \|w^n(t)\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^2)\cap \mathrm{L}^2(0,t;\mathrm{H}^1)} \\
&\lesssim \sqrt{t}\left(R_0\sqrt{t}\|w^{n-1}\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^2)} + \Lambda(t,R_0)\frac{1}{\varepsilon}\|\mathfrak{M}^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\right) + \sqrt{t}\Lambda(t,R_0)\left\|w^{n-1}\right\|_{\mathrm{L}^2(0,t;\mathrm{H}^1)} \\
&\quad + \Lambda(R_0)t\left(\left\|\mathfrak{M}^{n-1}\right\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^2)} + \left\|g^{n-1}\right\|_{\mathrm{L}^\infty(0,t;\mathcal{H}^0_r)} + \left\|w^{n-1}\right\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^2)}\right) + \Lambda(R_0)t\,\|\mathfrak{M}^n\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^2)}\,.
\end{aligned}$$

Reducing again $\widetilde{T}^{[\varepsilon]}$, it means that that for all $n \in \mathbb{N}\backslash\{0\}$

$$\begin{aligned}
&\|g^n\|_{\mathrm{L}^\infty(0,\widetilde{T}^{[\varepsilon]};\mathcal{H}^0_r)} + \|\mathfrak{M}^n\|_{\mathrm{L}^\infty(0,\widetilde{T}^{[\varepsilon]};\mathrm{L}^2)} + \|w^n\|_{\mathrm{L}^\infty(0,\widetilde{T}^{[\varepsilon]};\mathrm{L}^2)\cap\mathrm{L}^2(0,\widetilde{T}^{[\varepsilon]};\mathrm{H}^1)} \\
&\quad \le \frac{1}{2}\left(\|g^{n-1}\|_{\mathrm{L}^\infty(0,\widetilde{T}^{[\varepsilon]};\mathcal{H}^0_r)} + \left\|\mathfrak{M}^{n-1}\right\|_{\mathrm{L}^\infty(0,\widetilde{T}^{[\varepsilon]};\mathrm{L}^2)} + \|w^{n-1}\|_{\mathrm{L}^\infty(0,\widetilde{T}^{[\varepsilon]};\mathrm{L}^2)\cap\mathrm{L}^2(0,\widetilde{T}^{[\varepsilon]};\mathrm{H}^1)}\right). \qquad (5.B.5)
\end{aligned}$$

Note that $\widetilde{T}^{[\varepsilon]}$ has been chosen independent of $n \in \mathbb{N}\backslash\{0\}$.

Step 3: obtaining a unique solution to (S_ε). Combining the uniform bound (5.B.2) with the contraction estimate (5.B.5), we deduce that the sequence $(f_n, \mathfrak{m}_n, u_n)$ is weakly($-\star$) compact in the space

$$\mathrm{L}^\infty(0,\widetilde{T}^{[\varepsilon]};\mathcal{H}^m_r) \times \mathrm{L}^\infty(0,\widetilde{T}^{[\varepsilon]};\mathrm{H}^m) \times \mathrm{L}^\infty(0,\widetilde{T}^{[\varepsilon]};\mathrm{H}^m) \cap \mathrm{L}^2(0,\widetilde{T}^{[\varepsilon]};\mathrm{H}^{m+1}),$$

and is a Cauchy sequence in the space

$$\mathrm{L}^\infty(0,\widetilde{T}^{[\varepsilon]};\mathcal{H}^0_r) \times \mathrm{L}^\infty(0,\widetilde{T}^{[\varepsilon]};\mathrm{L}^2) \times \mathrm{L}^\infty(0,\widetilde{T}^{[\varepsilon]};\mathrm{L}^2) \cap \mathrm{L}^2(0,\widetilde{T}^{[\varepsilon]};\mathrm{H}^1).$$

This shows that the whole sequence converges weakly($-\star$) in the first space. The limit (f, m, u) belongs in particular to the first space with high regularity. Using weak-strong convergence principles then allows to prove that the limit is a solution to the system (S_ε) on $(0,\widetilde{T}^{[\varepsilon]})$ in the sense of distribution. We don't detail this part of the proof. Using the equation and the time derivative of the solution, one can shows that the solution actually belongs to

$$\mathrm{X}^m_{\widetilde{T}^{[\varepsilon]}} := \mathscr{C}(0,\widetilde{T}^{[\varepsilon]};\mathcal{H}^m_r) \times \mathscr{C}(0,\widetilde{T}^{[\varepsilon]};\mathrm{H}^m) \times \mathscr{C}(0,\widetilde{T}^{[\varepsilon]};\mathrm{H}^m) \cap \mathrm{L}^2(0,\widetilde{T}^{[\varepsilon]};\mathrm{H}^{m+1}).$$

The uniqueness of the Cauchy problem for (S_ε) in the former space is eventually obtained by mimicking the contraction estimates of the Step 2. In fact, if $(f_1, \mathfrak{m}_1, u_1)$ and $(f_2, \mathfrak{m}_2, u_2)$ are two solutions in $\mathrm{X}^m_{\widetilde{T}^{[\varepsilon]}}$ starting at the same initial condition, performing the same computations prove that there exists $\overline{T}^{[\varepsilon]}$ (depending on $\|f_{1,2}, \mathfrak{m}_{1,2}, u_{1,2}\|_{\mathrm{X}^m_{\widetilde{T}^{[\varepsilon]}}}$) and with $\overline{T}^{[\varepsilon]} < \widetilde{T}^{[\varepsilon]}$ such that for all $t \in [0, \overline{T}^{[\varepsilon]}]$, we have

$$\begin{aligned}
&\|(f_1 - f_2)(t)\|^2_{\mathcal{H}^0_r} + \|(\mathfrak{m}_1 - \mathfrak{m}_2)(t)\|^2_{\mathrm{L}^2} + \|(u_1 - u_2)(t)\|^2_{\mathrm{L}^2} \\
&\qquad\qquad \le \frac{1}{2}\left(\|(f_1 - f_2)(t)\|^2_{\mathcal{H}^0_r} + \|(\mathfrak{m}_1 - \mathfrak{m}_2)(t)\|^2_{\mathrm{L}^2} + \|(u_1 - u_2)(t)\|^2_{\mathrm{L}^2}\right),
\end{aligned}$$

from which we directly infer $(f_1, \mathfrak{m}_1, u_1) = (f_2, \mathfrak{m}_2, u_2)$ on $[0, \overline{T}^{[\varepsilon]}]$. Repeating this procedure starting from $\overline{T}^{[\varepsilon]} < T'_\varepsilon$, we obtain $(f_1, \mathfrak{m}_1, u_1) = (f_2, \mathfrak{m}_2, u_2)$ on $[0, 2\overline{T}^{[\varepsilon]}]$. After a finite number of steps, we eventually obtain uniqueness on $[0, \widetilde{T}^{[\varepsilon]}]$.

5.C Tools from pseudodifferential calculus with a large parameter on $\mathbb{R} \times \mathbb{T}^d$

In this section, we collect several results on pseudodifferential calculus that we shall need in this article. We refer to [AG07, Mét01] for a more general and complete approach. Here, our framework is adapted to the physical space $\mathbb{R} \times \mathbb{T}^d$.

For any symbol of the form $a(t, x, \gamma, \tau, k)$ defined on $\mathbb{R} \times \mathbb{T}^d \times (0, +\infty) \times \mathbb{R} \times \mathbb{R}^d \backslash \{0\}$, we use the quantization

$$\mathrm{Op}^\gamma(a)(h)(t, x) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}\times\mathbb{Z}^d} e^{i(\tau t + k\cdot x)} a(t, x, \gamma, \tau, k) \mathcal{F}_{t,x} h(\tau, k) \, \mathrm{d}\tau \, \mathrm{d}k.$$

Here, we use the discrete measure on $\mathbb{Z}^d$. Above, the variable $\gamma > 0$ should be seen as a parameter. The Fourier transform of any function $h(t, x)$ defined on $\mathbb{R} \times \mathbb{T}^d$ is denoted by

$$\mathcal{F}_{t,x} h(\tau, k) = \int_{\mathbb{R}\times\mathbb{Z}^d} e^{-i(\tau t + k\cdot x)} h(t, x) \, \mathrm{d}t \, \mathrm{d}x,$$

while the Fourier transform of any symbol $a(t, x, \gamma, \tau, k)$ defined on $\mathbb{R} \times \mathbb{T}^d \times (0, +\infty) \times \mathbb{R} \times \mathbb{R}^d \backslash \{0\}$ is written as

$$\mathcal{F}_{t,x} a(\theta, \ell, \gamma, \tau, k) = \int_{\mathbb{R}\times\mathbb{Z}^d} e^{-i(\theta t + \ell\cdot x)} a(t, x, \gamma, \tau, k) \, \mathrm{d}t \, \mathrm{d}x.$$

For the sake of readibility, we introduce the notation

$$\eta := (\gamma, \tau, k) \in (0, +\infty) \times \mathbb{R} \times \mathbb{R}^d \backslash \{0\},$$
$$|\eta| := \left(\gamma^2 + \tau^2 + |k|^2\right)^{\frac{1}{2}},$$

and we will use the notation $\mathrm{L}^\infty_{t,x,\eta}$ to denote $\mathrm{L}^\infty(\mathbb{R} \times \mathbb{T}^d \times (0, +\infty) \times \mathbb{R} \times \mathbb{R}^d \backslash \{0\})$.

We state L^2 continuity results of Calderón-Vaillancourt-type: namely, we shall ask for L^∞ bounds in all the variables for the symbols of the operators (see [CV71, Hwa87]). We first introduce the following family of seminorms for our symbols.

Notation 5.C.1. *For any* $\mathrm{M} \geq 0$ *and for any symbol* $a(t, x, \eta)$ *with* $\eta = (\gamma, \tau, k)$*, we set*

$$\omega[a] := \sup_{\substack{\alpha\in\mathbb{N}^d \\ \alpha_i\in\{0,1\}}} \|(1+t)\partial_x^\alpha a\|_{\mathrm{L}^\infty_{t,x,\eta}} + \sup_{\substack{\alpha\in\mathbb{N}^d \\ \alpha_i\in\{0,1\}}} \|(1+t)\partial_t \partial_x^\alpha a\|_{\mathrm{L}^\infty_{t,x,\eta}}, \tag{5.C.1}$$

$$\begin{aligned}\Omega[a] := &\sup_{\substack{\alpha\in\mathbb{N}^d \\ \alpha_i\in\{0,1\}}} \left\{ \||\eta|\partial_x^\alpha \nabla_{\tau,k} a\|_{\mathrm{L}^\infty_{t,x,\eta}} + \||\eta|\partial_x^\alpha \nabla_{\tau,k} \partial_t a\|_{\mathrm{L}^\infty_{t,x,\eta}} \right\} \\ &+ \sup_{\substack{\alpha\in\mathbb{N}^d \\ \alpha_i\in\{0,1\}}} \left\{ \||\eta|\partial_x^\alpha \partial_\tau \nabla_{\tau,k} a\|_{\mathrm{L}^\infty_{t,x,\eta}} + \||\eta|\partial_x^\alpha \partial_\tau \nabla_{\tau,k} \partial_t a\|_{\mathrm{L}^\infty_{t,x,\eta}} \right\},\end{aligned} \tag{5.C.2}$$

$$\Xi[a]_{\mathrm{M}} := \sup_{\substack{1\leq|\alpha|\leq 1+\mathrm{M} \\ \beta=0,1,2,3,4}} \left\| \partial_x^\alpha \partial_t^\beta a \right\|_{\mathrm{L}^\infty_{t,x,\eta}}. \tag{5.C.3}$$

The following result states the L^2 continuity of pseudodifferential operator with symbol having a finite seminorm $\omega[\cdot]$. We refer to [Hwa87, Theorem 1] for a proof, which adaptation to the physical space $\mathbb{R} \times \mathbb{T}^d$ is not difficult[5].

[5] More precisely, one can introduce a weight $(1 + t)^{-2}$ in [Hwa87, proof of Theorem 1, eq (2.5)] to get some integrability in time. This turns out to be sufficient to consider the seminorm $\omega[\cdot]$ in our statement.

Theorem 5.C.2. *There exists $C_d > 0$ such that if a is a symbol satisfying $\omega[a] < \infty$ then the following holds: for every $\gamma > 0$, we have*

$$\forall h \in \mathrm{L}^2(\mathbb{R} \times \mathbb{T}^d), \quad \|\mathrm{Op}^\gamma(a)(h)\|_{\mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)} \leq C_d \omega[a] \|h\|_{\mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)},$$

Next, we have the following symbolic calculus result, with the use of the parameter $\gamma > 0$. Note that taking γ large can obviously be useful in view of an absorption argument. Here, we have to assume that one symbol has a compact support in time because the time variable $t \in \mathbb{R}$ in unbounded.

Proposition 5.C.3. *There exists $C_d > 0$ and continuous nonnegative and nondecreasing function Λ such that for any symbols a, b satisfying*

$$\Omega[a], \Xi[b]_{\mathrm{M}} < \infty, \quad \mathrm{M} > 1 + 2d,$$
$$\nabla_x b \text{ has compact support in time},$$

the following holds: for all $\gamma > 0$ and all $h \in \mathrm{L}^2(\mathbb{R} \times \mathbb{T}^d)$, we have

$$\|\mathrm{Op}^\gamma(a)\mathrm{Op}^\gamma(b)(h) - \mathrm{Op}^\gamma(ab)(h)\|_{\mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)} \leq \frac{C_d}{\gamma}\Lambda(|\mathrm{supp}_t \nabla_x b|)\Omega[a]\Xi[b]_{\mathrm{M}}\|h\|_{\mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)}.$$

Proof. A standard formula on pseudodifferential operators first shows that

$$\mathrm{Op}^\gamma(a)\mathrm{Op}^\gamma(b) = \mathrm{Op}^\gamma(c),$$

with

$$\begin{aligned} &c(t, x, \gamma, \tau, k) \\ &= \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}\times\mathbb{T}^d} \int_{\mathbb{R}\times\mathbb{Z}^d} e^{i(\tau'-\tau)(t-t')} e^{i(k'-k)\cdot(x-x')} a(t, x, \gamma, \tau', k') b(t', x', \gamma, \tau, k) \, \mathrm{d}t' \mathrm{d}x' \mathrm{d}\tau' \mathrm{d}k' \\ &= \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}\times\mathbb{Z}^d} e^{i(\tau' t + k'\cdot x)} a(t, x, \gamma, \tau + \tau', k + k') \mathcal{F}_{t,x} b(\tau', k', \gamma, \tau, k) \, \mathrm{d}\tau' \mathrm{d}k'. \end{aligned}$$

Therefore for $\eta = (\gamma, \tau, k) \in (0, +\infty) \times \mathbb{R} \times \mathbb{R}^d \backslash \{0\}$, we have

$$\begin{aligned} c(t, x, \eta) - a(t, x, \eta) b(t, x, \eta) &= \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}\times\mathbb{Z}^d} e^{i(\tau' t + k'\cdot x)} \mathcal{F}_{t,x} b(\tau', k', \eta) \\ &\qquad \left\{ \int_0^1 \nabla_{\tau,k} a(t, x, \gamma, \tau + s\tau', k + sk') \cdot (\tau', k') \, \mathrm{d}s \right\} \mathrm{d}\tau' \mathrm{d}k' \\ &=: \frac{1}{\gamma} \mathrm{m}(t, x, \eta), \end{aligned}$$

where

$$\mathrm{m}(t, x, \eta) := \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}\times\mathbb{Z}^d} e^{i(\tau' t + k'\cdot x)} \mathcal{F}_{t,x} b(\tau', k', \eta) \mathcal{J}(a)(t, x, \eta, \tau', k') \, \mathrm{d}\tau' \mathrm{d}k',$$

with

$$\mathcal{J}(a)(t, x, \eta, \tau', k') := \int_0^1 \gamma \nabla_{\tau,k} a(t, x, \gamma, \tau + s\tau', k + sk') \cdot (\tau', k') \, \mathrm{d}s.$$

Since $\mathrm{Op}^\gamma(a)\mathrm{Op}^\gamma(b)-\mathrm{Op}^\gamma(ab) = \frac{1}{\gamma}\mathrm{Op}^\gamma(\mathrm{m})$, and in view of the continuity property stated in Theorem 5.C.2, it remains to estimate the seminorm $\omega[\mathrm{m}]$. For all $\alpha \in \mathbb{N}^d$ such that $\alpha_i \in \{0,1\}$, we now have

$$(1+t)\mathrm{m}(t,x,\eta) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}\times\mathbb{Z}^d} (1-i\partial_{\tau'})(e^{i\tau' t})e^{ik'\cdot x}\mathcal{F}_{t,x}b(\tau',k',\eta)\mathcal{J}(a)(t,x,\eta,\tau',k')\mathrm{d}\tau'\mathrm{d}k',$$

and

$$\begin{aligned}(1+t)\partial_t\mathrm{m}(t,x,\eta) &= \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}\times\mathbb{Z}^d} i\tau'(1-i\partial_{\tau'})(e^{i\tau' t})e^{ik'\cdot x}\mathcal{F}_{t,x}b(\tau',k',\eta)\mathcal{J}(a)(t,x,\eta,\tau',k')\mathrm{d}\tau'\mathrm{d}k' \\ &\quad + \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}\times\mathbb{Z}^d} (1-i\partial_{\tau'})(e^{i\tau' t})e^{ik'\cdot x}\mathcal{F}_{t,x}b(\tau',k',\eta)\partial_t\left(\mathcal{J}(a)\right)(t,x,\eta,\tau',k')\mathrm{d}\tau'\mathrm{d}k'.\end{aligned}$$

Tedious but standard computations then show that for $\mathrm{M} > d$, we have

$$\begin{aligned}&\omega[\mathrm{m}] \\ &\lesssim \Omega[a] \sup_{\substack{\alpha\in\mathbb{N}^d \\ \alpha_i\in\{0,1\}}} \left\|(1+|\tau'|^2)|k'|^{1+|\alpha|}\mathcal{F}_{t,x}\left[(1+t)b\right]\right\|_{\mathrm{L}^1(\mathbb{R}_{\tau'}\times\mathbb{Z}^d_{k'};\mathrm{L}^\infty_\eta)} \\ &\lesssim \Omega[a]\left(\sup_{1\le|\alpha|\le 1+\mathrm{M}} \left\|\mathcal{F}_{t,x}\left[(1+t)\partial_x^\alpha b\right]\right\|_{\mathrm{L}^1(\mathbb{R}\times\mathbb{Z}^d;\mathrm{L}^\infty_\eta)} + \sup_{\substack{1\le|\alpha|\le 1+\mathrm{M}\\ \beta=1,2}} \left\|\mathcal{F}_{t,x}\left[(1+t)\partial_x^\alpha\partial_t^\beta b\right]\right\|_{\mathrm{L}^1(\mathbb{R}\times\mathbb{Z}^d;\mathrm{L}^\infty_\eta)}\right).\end{aligned}$$

As a consequence, we obtain for $\mathrm{M} > 1+2d$

$$\begin{aligned}\omega[\mathrm{m}] &\lesssim \Omega[a]\left(\sup_{\substack{1\le|\alpha|\le 1+\mathrm{M}\\ \beta=0,1,2}} \left\|\mathcal{F}_{t,x}\left[(1+t)\partial_x^\alpha\partial_t^\beta b\right]\right\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{Z}^d;\mathrm{L}^\infty_\eta)}\right. \\ &\qquad\qquad\left. + \sup_{\substack{1\le|\alpha|\le 1+\mathrm{M}\\ \beta=0,1,2,3,4}} \left\|\mathcal{F}_{t,x}\left[(1+t)\partial_x^\alpha\partial_t^\beta b\right]\right\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{Z}^d;\mathrm{L}^\infty_\eta)}\right) \\ &\lesssim \Omega[a] \sup_{\substack{1\le|\alpha|\le 1+\mathrm{M}\\ \beta=0,1,2,3,4}} \left\|(1+t)\partial_x^\alpha\partial_t^\beta b\right\|_{\mathrm{L}^1(\mathbb{R}\times\mathbb{T}^d;\mathrm{L}^\infty_\eta)},\end{aligned}$$

and therefore

$$\omega[\mathrm{m}] \lesssim \Lambda(|\mathrm{supp}_t\,\nabla_x b|)\Omega[a]\Xi[b]_\mathrm{M}.$$

From Theorem 5.C.2, we have for all $h \in \mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)$

$$\|\mathrm{Op}^\gamma(\mathrm{m})h\|_{\mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)} \le C_d\omega[\mathrm{m}]\|h\|_{\mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)} \lesssim C_d\Lambda(|\mathrm{supp}_t\,\nabla_x b|)\Omega[a]\Xi[b]_\mathrm{M}\|h\|_{\mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)}.$$

We obtain the conclusion since $\mathrm{Op}^\gamma(a)\mathrm{Op}^\gamma(b) - \mathrm{Op}^\gamma(ab) = \frac{1}{\gamma}\mathrm{Op}^\gamma(\mathrm{m})$. □

Bibliography

[ABdMB97] O. Anoshchenko and A. Boutet de Monvel-Berthier. The existence of the global generalized solution of the system of equations describing suspension motion. *Math. Methods Appl. Sci.*, 20(6):495–519, 1997.

[AC93] L. Arkeryd and C. Cercignani. A global existence theorem for the initial-boundary-value problem for the Boltzmann equation when the boundaries are not isothermal. *Arch. Ration. Mech. Anal.*, 125(3):271–287, 1993.

[ACM05] L. Ambrosio, G. Crippa, and S. Maniglia. Traces and fine properties of a BD class of vector fields and applications. *Ann. Fac. Sci. Toulouse, Math. (6)*, 14(4):527–541, 2005.

[AG07] S. Alinhac and P. Gérard. *Pseudo-differential operators and the Nash–Moser theorem. Transl. from the French by Stephen S. Wilson*, volume 82 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2007.

[Ago84] V.I. Agoshkov. Spaces of functions with differential-difference characteristics and smoothness of solutions of the transport equation. *Sov. Math., Dokl.*, 29:662–666, 1984.

[AH07] H. Abidi and T. Hmidi. On the global well-posedness for Boussinesq system. *J. Differ. Equations*, 233(1):199–220, 2007.

[Aiz78] M. Aizenman. On vector fields as generators of flows: a counterexample to nelson's conjecture. *Ann. Math. (2)*, 107(2):287–296, 1978.

[AL21] D. Arsénio and N. Lerner. An energy method for averaging lemmas. *Pure Appl. Anal.*, 3(2):319–362, 2021.

[All91] G. Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes I. Abstract framework, a volume distribution of holes. *Arch. Ration. Mech. Anal.*, 113:209–259, 1991.

[ALT20] R. Alonso, B. Lods, and I. Tristani. Fluid dynamic limit of Boltzmann equation for granular hard–spheres in a nearly elastic regime. *arXiv preprint arXiv:2008.05173*, 2020.

[AM14] D. Arsénio and N. Masmoudi. A new approach to velocity averaging lemmas in Besov spaces. *J. Math. Pures Appl. (9)*, 101(4):495–551, 2014.

[Amb04] L. Ambrosio. Transport equation and Cauchy problem for BV vector fields. *Invent. Math.*, 158(2):227–260, 2004.

[Amb17] L. Ambrosio. Well posedness of ODE's and continuity equations with nonsmooth vector fields, and applications. *Revista Matemática Complutense*, 30(3):427–450, 2017.

[AOB89] A. Amsden, P.J. O'Rourke, and T.D. Butler. KIVA-II: A computer program for chemically reactive flows with sprays. Technical report, Los Alamos National Lab.(LANL), Los Alamos, NM (United States), 1989.

[Ars75] A.A. Arsenev. Existence in the large of a weak solution to the Vlasov system of equations. *Zhurnal Vychislitelnoi Matematiki i Matematicheskoi Fiziki*, 15:136–147, 1975.

[Ars12] D. Arsénio. From Boltzmann's equation to the incompressible Navier–Stokes–Fourier system with long-range interactions. *Arch. Ration. Mech. Anal.*, 206:367–488, 2012.

[Asa86] K. Asano. On local solutions of the initial value problem for the Vlasov-Maxwell equation. *Commun. Math. Phys.*, 106:551–568, 1986.

[ASR11] D. Arsénio and L. Saint-Raymond. Compactness in kinetic transport equations and hypoellipticity. *J. Funct. Anal.*, 261(10):3044–3098, 2011.

[Bar70] C. Bardos. Problèmes aux limites pour les équations aux dérivées partielles du premier ordre à coefficients réels; théorèmes d'approximation; application à l'équation de transport. *Annales scientifiques de l'École normale supérieure*, 3(2):185–233, 1970.

[Bar04] C. Baranger. Modelling of oscillations, breakup and collisions for droplets: the establishment of kernels for the TAB model. *Math. Models Methods Appl. Sci.*, 14(05):775–794, 2004.

[Bar13] C. Bardos. About a variant of the $1d$ Vlasov equation, dubbed "Vlasov-Dirac-Benney equation". *Sémin. Laurent Schwartz, EDP Appl.*, 15:ex, 2012-2013.

[Bar20] A. Baradat. Nonlinear instability in Vlasov type equations around rough velocity profiles. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 37(3):489–547, 2020.

[BB13] C. Bardos and N. Besse. The Cauchy problem for the Vlasov-Dirac-Benney equation and related issues in fluid mechanics and semi-classical limits. *Kinet. Relat. Models*, 6(4):893–917, 2013.

[BB15] C. Bardos and N. Besse. Hamiltonian structure, fluid representation and stability for the Vlasov-Dirac-Benney equation. In *Hamiltonian partial differential equations and applications*, volume 75 of *Fields Inst. Commun.*, pages 1–30. Fields Inst. Res. Math. Sci., Toronto, ON, 2015.

[BBC16] A. Bohun, F. Bouchut, and G. Crippa. Lagrangian solutions to the Vlasov-Poisson system with L^1 density. *J. Differ. Equations*, 260(4):3576–3597, 2016.

[BCD11] H. Bahouri, J-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343. Springer, 2011.

[BD85] C. Bardos and P. Degond. Global existence for the Vlasov-Poisson equation in 3 space variables with small initial data. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 2(2):101–118, 1985.

[BD03] D. Bresch and B. Desjardins. Existence of global weak solutions for a 2d viscous shallow water equations and convergence to the quasi-geostrophic model. *Commun. Math. Phys.*, 238(1-2):211–223, 2003.

[BD06a] C. Baranger and L. Desvillettes. Coupling Euler and Vlasov equations in the context of sprays: the local-in-time, classical solutions. *J. Hyperbolic Differ. Equ.*, 3(01):1–26, 2006.

[BD06b] D. Bresch and B. Desjardins. On the construction of approximate solutions for the 2d viscous shallow water model and for compressible Navier-Stokes models. *J. Math. Pures Appl. (9)*, 86(4):362–368, 2006.

[BD07] D. Bresch and B. Desjardins. On the existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids. *J. Math. Pures Appl. (9)*, 87(1):57–90, 2007.

[BDC20] N. Briggs, G. Dall'Olmo, and H. Claustre. Major role of particle fragmentation in regulating biological sequestration of CO2 by the oceans. *Science*, 367(6479):791–793, 2020.

[BDD23] C. Buet, B. Després, and L. Desvillettes. Linear stability of thick sprays equations. *J. Stat. Phys.*, 190(3):53, 2023.

[BDGM09] L. Boudin, L. Desvillettes, C. Grandmont, and A. Moussa. Global existence of solutions for the coupled Vlasov and Navier-Stokes equations. *Differential Integral Equations*, 22(11-12):1247–1271, 2009.

[BDGN12] S. Benjelloun, L. Desvillettes, J.M. Ghidaglia, and K. Nielsen. Modeling and simulation of thick sprays through coupling of a finite volume Euler equation solver and a particle method for a disperse phase. *Note Mat.*, 32(1):63–85, 2012.

[BDGR17] E. Bernard, L. Desvillettes, F. Golse, and V. Ricci. A derivation of the Vlasov-Navier-Stokes model for aerosol flows from kinetic theory. *Commun. Math. Sci.*, 15(6):1703–1741, 2017.

[BDGR18] E. Bernard, L. Desvillettes, F. Golse, and V. Ricci. A derivation of the Vlasov-Stokes system for aerosol flows from the kinetic theory of binary gas mixtures. *Kinet. Relat. Models*, 11(1):43–69, 2018.

[BDM14] S. Benjelloun, L. Desvillettes, and A Moussa. Existence theory for the kinetic-fluid coupling when small droplets are treated as part of the fluid. *J. Hyperbolic Differ. Equ.*, 11(01):109–133, 2014.

[Bed22] J. Bedrossian. A brief introduction to the mathematics of Landau damping. *arXiv preprint arXiv:2211.13707*, 2022.

[Ber20] A. Bernou. *Long-time behavior of kinetic equations with boundary effects.* PhD thesis, Sorbonne université, 2020.

[BF12] F. Boyer and P. Fabrie. *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models*, volume 183. Springer Science & Business Media, 2012.

[BF23] A. Blaustein and F. Filbet. Concentration phenomena in Fitzhugh–Nagumo equations: A mesoscopic approach. *SIAM J. Math. Anal.*, 55(1):367–404, 2023.

[BFJJ13] M. Bossy, L. Fontbona, P-E. Jabin, and J-F Jabir. Local existence of analytical solutions to an incompressible Lagrangian stochastic model in a periodic domain. *Commun. Partial Differ. Equations*, 38(7-9):1141–1182, 2013.

[BG91] C. Bardos and D. Golse, F.and Levermore. Fluid dynamic limits of kinetic equations. I. Formal derivations. *J. Stat. Phys.*, 63(1):323–344, 1991.

[BG94] Y. Brenier and E. Grenier. Limite singulière du système de Vlasov-Poisson dans le régime de quasi neutralité: le cas indépendant du temps. *C. R. Acad. Sci. Paris Sér. I Math.*, 318(2):121–124, 1994.

[BGG$^+$20] L. Boudin, C. Grandmont, B. Grec, S. Martin, A. Mecherbet, and F. Noël. Fluid-kinetic modelling for respiratory aerosols with variable size and temperature. *ESAIM: Proceedings and Surveys*, 67:100–119, 2020.

[BGL93] C. Bardos, F. Golse, and D. Levermore. Fluid dynamic limits of kinetic equations. II. Convergence proofs for the Boltzmann equation. *Commun. Pure Appl. Math.*, 46(5):667–753, 1993.

[BGLM15] L. Boudin, C. Grandmont, A. Lorz, and A. Moussa. Modelling and numerics for respiratory aerosols. *Commun. Comput. Phys.*, 18(3):723–756, 2015.

[BGM17] L. Boudin, C. Grandmont, and A. Moussa. Global existence of solutions to the incompressible Navier-Stokes-Vlasov equations in a time-dependent domain. *J. Differ. Equations*, 262:1317–1340, 2017.

[BGS07] S. Benzoni-Gavage and D. Serre. *Multi-dimensional hyperbolic partial differential equations. First-order systems and applications.* Oxford Math. Monogr. Oxford: Oxford University Press, 2007.

[BGSRS22] T. Bodineau, I. Gallagher, L. Saint-Raymond, and S. Simonella. Dynamics of dilute gases: a statistical approach. *arXiv preprint arXiv:2201.10149*, 2022.

[BH77] W. Braun and K. Hepp. The Vlasov dynamics and its fluctuations in the 1/n limit of interacting classical particles. *Commun. Math. Phys.*, 56(2):101–113, 1977.

[BJ18] D. Bresch and P-E. Jabin. Global existence of weak solutions for compressible Navier-Stokes equations: thermodynamically unstable pressure and anisotropic viscous stress tensor. *Ann. Math. (2)*, 188(2):577–684, 2018.

[BJW19] D. Bresch, P-E. Jabin, and Z. Wang. Modulated free energy and mean field limit. *Séminaire Laurent Schwartz—EDP et applications*, pages 1–22, 2019.

[BLR92] C Bardos, G. Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.*, 30(5):1024–1065, 1992.

[BM88] W. Borchers and T. Miyakawa. L2 decay for the Navier-Stokes flow in halfspaces. *Math. Ann*, 282(1):139–155, 1988.

[BM90] W. Borchers and T. Miyakawa. Algebraic L2 decay for Navier-Stokes flows in exterior domains. *Acta Math.*, 165(1):189–227, 1990.

[BM17] L. Brandolese and C. Mouzouni. A short proof of the large time energy growth for the Boussinesq system. *J. Nonlinear Sci.*, 27(5):1589–1608, 2017.

[BM21] L. Boudin and D. Michel. Three-dimensional numerical study of a fluid-kinetic model for respiratory aerosols with variable size and temperature. *J. Comput. Theor. Transp.*, 50(5):507–527, 2021.

[BMAM19] M. Briant, S. Merino-Aceituno, and C. Mouhot. From Boltzmann to incompressible Navier-Stokes in Sobolev spaces with polynomial weight. *Anal. Appl., Singap.*, 17(1):85–116, 2019.

[BMM16] J. Bedrossian, N. Masmoudi, and C. Mouhot. Landau damping: paraproducts and Gevrey regularity. *Ann. PDE*, 2(1):Art. 4, 71, 2016.

[BMM20] L. Boudin, D. Michel, and A. Moussa. Global existence of weak solutions to the incompressible Vlasov-Navier-Stokes system coupled to convection-diffusion equations. *Math. Models Methods Appl. Sci.*, 2020.

[BN12] C. Bardos and A. Nouri. A Vlasov equation with Dirac potential used in fusion plasmas. *J. Math. Phys.*, 53(11):115621, 16, 2012.

[Bol72] L. Boltzmann. *Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen.* Sitzungberichte der Kaiserlichen Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Classe 66, 275–370, 1872.

[Boy05] F. Boyer. Trace theorems and spatial continuity properties for the solutions of the transport equation. *Differ. Integral Equ.*, 18(8):891–934, 2005.

[Bre89] Y. Brenier. A Vlasov-Poisson formulation of the Euler equations for perfect incompressible fluids. *Rapport de recherche INRIA*, 1989.

[Bre00] Y. Brenier. Convergence of the Vlasov-Poisson system to the incompressible Euler equations. *Comm. Partial Differential Equations*, 25(3-4):737–754, 2000.

[Bri49] H.C. Brinkman. A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles. *Flow, Turbulence and Combustion*, 1:27–34, 1949.

[Bri15] M. Briant. From the Boltzmann equation to the incompressible Navier–Stokes equations on the torus: a quantitative error estimate. *J. Differ. Equations*, 259(11):6072–6141, 2015.

[BS12] L. Brandolese and M. Schonbek. Large time decay and growth for solutions of a viscous Boussinesq system. *Trans. Am. Math. Soc.*, 364(10):5057–5090, 2012.

[BS18] L. Brandolese and M. Schonbek. Large time behavior of the Navier-Stokes flow. In *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids. Part I. Incompressible fluids. Unsteady viscous Newtonian fluids.* Springer International Publishing, 2018.

[BVY22] D. Bresch, A.. Vasseur, and C. Yu. Global existence of entropy-weak solutions to the compressible Navier-Stokes equations with non-linear density dependent viscosities. *J. Eur. Math. Soc. (JEMS)*, 24(5):1791–1837, 2022.

[CC91] M. Cannone and C. Cercignani. A trace theorem in kinetic theory. *Appl. Math. Lett.*, 4(6):63–67, 1991.

[CC20] J. Carrillo and Y-P. Choi. Quantitative error estimates for the large friction limit of Vlasov equation with nonlocal forces. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 37(4):925–954, 2020.

[CCM18] S. Caprino, G. Cavallaro, and C. Marchioro. The Vlasov-Poisson equation in $\mathbb{R}^3$ with infinite charge and velocities. *J. Hyperbolic Differ. Equ.*, 15(3):407–442, 2018.

[CD10] F. Charve and R. Danchin. A global existence result for the compressible Navier–Stokes equations in the critical Lp framework. *Arch. Ration. Mech. Anal.*, 198(1):233–271, 2010.

[CDGG06] J-Y. Chemin, B. Desjardins, I. Gallagher, and E. Grenier. *Mathematical geophysics: An introduction to rotating fluids and the Navier-Stokes equations*, volume 32. Oxford University Press on Demand, 2006.

[CDL08] G. Crippa and C. De Lellis. Estimates and regularity results for the DiPerna-Lions flow. *J. Reine Angew. Math.*, 616:15–46, 2008.

[CDS14] G. Crippa, C. Donadello, and L. Spinolo. Initial–boundary value problems for continuity equations with BV coefficients. *J. Math. Pures Appl. (9)*, 102(1):79–98, 2014.

[Ces84] M. Cessenat. Théoremes de trace Lp pour des espaces de fonctions de la neutronique. *C. R. Acad. Sci., Paris, Sér. I*, 299(16):831–834, 1984.

[CF88] P. Constantin and C. Foias. *Navier-Stokes Equations.* University of Chicago Press, 1988.

[CFF19] J. Crevat, G. Faye, and F. Filbet. Rigorous derivation of the nonlocal reaction-diffusion FitzHugh–Nagumo system. *SIAM J. Math. Anal.*, 51(1):346–373, 2019.

[CG06] J. A Carrillo and T. Goudon. Stability and asymptotic analysis of a fluid-particle interaction model. *Commun. Partial. Differ. Equ.*, 31(9):1349–1379, 2006.

[CGL08] J. Carrillo, T. Goudon, and P. Lafitte. Simulation of fluid and particles flows: Asymptotic preserving schemes for bubbling and flowing regimes. *J. Comput. Phys.*, 227(16):7929–7951, 2008.

[CH20] K. Carrapatoso and M. Hillairet. On the derivation of a Stokes–Brinkman problem from Stokes equations around a random array of moving spheres. *Commun. Math. Phys.*, 373(1):265–325, 2020.

[Cha06] D. Chae. Global regularity for the 2D Boussinesq equations with partial viscosity terms. *Adv. Math.*, 203(2):497–513, 2006.

[Cha23] T. Chaub. Local well-posedness for a class of singular Vlasov equations. *Kinet. Relat. Models*, 16(2):187–206, 2023.

[Che90] J-Y. Chemin. Dynamique des gaz à masse totale finie. *Asymptotic Anal.*, 3(3):215–220, 1990.

[Che92] J-Y. Chemin. Remarques sur l'existence globale pour le système de Navier–Stokes incompressible. *SIAM J. Math. Anal.*, 23(1):20–28, 1992.

[CHKR23] K. Carrapatoso, D. Han-Kwan, and F. Rousset. Wellposedness of singular Vlasov equations under optimal stability conditions (working title). In preparation, 2023.

[Cho17] Y-P. Choi. Finite-time blow-up phenomena of Vlasov/Navier–Stokes equations and related systems. *J. Math. Pures Appl. (9)*, 108(6):991–1021, 2017.

[CIP13] C. Cercignani, R. Illner, and M. Pulvirenti. *The mathematical theory of dilute gases*, volume 106. Springer Science & Business Media, 2013.

[CJ22a] Y-P. Choi and J. Jung. Local well-posedness for the compressible Navier-Stokes-BGK model in Sobolev spaces with exponential weight. *arXiv preprint arXiv:2209.14729*, 2022.

[CJ22b] Y-P. Choi and J. Jung. On regular solutions and singularity formation for Vlasov/Navier-Stokes equations with degenerate viscosities and vacuum. *Kinet. Relat. Models*, 15(5):843–891, 2022.

[CK15] Y-P. Choi and B. Kwon. Global well-posedness and large-time behavior for the inhomogeneous Vlasov–Navier–Stokes equations. *Nonlinearity*, 28(9):3309, 2015.

[CK22] K. Choi, Y-P.and Kang and J-M. Kim, H.K.and Kim. Temporal decays and asymptotic behaviors for a Vlasov equation with a flocking term coupled to incompressible fluid flow. *Nonlinear Anal., Real World Appl.*, 63:46, 2022. Id/No 103410.

[CM12] P. Cherrier and A. Milani. *Linear And Quasi-Linear Evolution Equations In Hilbert Spaces*. American Mathematical Society Providence, 2012.

[CMP94] M. Cannone, Y. Meyer, and F. Planchon. Solutions auto-similaires des équations de Navier-Stokes. *Séminaire Équations aux dérivées partielles (Polytechnique) dit aussi" Séminaire Goulaouic-Schwartz"*, pages 1–10, 1994.

[CMX21] D. Chae, Q. Miao, and L. Xue. Global regularity of non-diffusive temperature fronts for the 2d viscous Boussinesq system. *arXiv preprint arXiv:2110.06442*, 2021.

[Cob23] D. Cobb. On the well-posedness of a fractional Stokes-Transport system. *arXiv preprint arXiv:2301.10511*, 2023.

[CP83] R. Caflisch and G. Papanicolaou. Dynamic theory of suspensions with Brownian effects. *SIAM J. Appl. Math.*, 43(4):885–906, 1983.

[Cre20] J. Crevat. Asymptotic limit of a spatially-extended mean-field FitzHugh–Nagumo model. *Math. Models Methods Appl. Sci.*, 30(5):957–990, 2020.

[CV71] A. Calderón and R. Vaillancourt. On the boundedness of pseudo-differential operators. *J. Math. Soc. Japan*, 23(2):374–378, 1971.

[CWY20] H. Cui, W. Wang, and L. Yao. Asymptotic analysis for 1D compressible Navier-Stokes-Vlasov equations. *Commun. Pure Appl. Anal.*, 19(5):2737, 2020.

[Dan85] F. Dannan. Integral inequalities of Gronwall-Bellman-Bihari type and asymptotic behavior of certain second order nonlinear differential equations. *J. Math. Anal. Appl.*, 108(1):151–164, 1985.

[Dan00] R. Danchin. Global existence in critical spaces for compressible Navier-Stokes equations. *Invent. Math.*, 141(3):579–614, 2000.

[Dan01a] R. Danchin. Global existence in critical spaces for flows of compressible viscous and heat-conductive gases. *Arch. Ration. Mech. Anal.*, 160(1):1–39, 2001.

[Dan01b] R. Danchin. Local theory in critical spaces for compressible viscous and heat-conductive gases. *Commun. Partial Differ. Equations*, 26(7-8):1183–1233, 2001.

[Dan05a] R. Danchin. Fourier analysis methods for PDE's. *https://perso.math.u-pem.fr/danchin.raphael/cours/courschine.pdf*, 2005.

[Dan05b] R. Danchin. On the uniqueness in critical spaces for compressible Navier-Stokes equations. *Nonlinear Differ. Equ. Appl.*, 12(1):111–128, 2005.

[DC09] S. De Chaisemartin. *Modèles Eulériens et simulation numérique de la dispersion turbulente de brouillards qui s'évaporent.* PhD thesis, PhD thesis, Ecole Centrale Paris, 2009.

[DDE12] F. Demengel, G. Demengel, and R. Erné. *Functional spaces for the theory of elliptic partial differential equations.* Springer, 2012.

[DE88] M. Doi and S. Edwards. *The theory of polymer dynamics*, volume 73. oxford university press, 1988.

[Dep02] N. Depauw. Non-unicité du transport par un champ de vecteurs presque BV. *Séminaire Équations aux dérivées partielles (Polytechnique) dit aussi" Séminaire Goulaouic-Schwartz"*, pages 1–9, 2002.

[Des10] L. Desvillettes. Some aspects of the modeling at different scales of multiphase flows. *Comput. Methods Appl. Mech. Eng.*, 199(21-22):1265–1267, 2010.

[DFP20] R. Danchin, F. Fanelli, and M. Paicu. A well-posedness result for viscous compressible fluids with only bounded density. *Anal. PDE*, 13(1):275–316, 2020.

[DGL23] A-L. Dalibard, J. Guillod, and A. Leblond. Long-time behavior of the stokes-transport system in a channel. *arXiv preprint arXiv:2306.00780*, 2023.

[DGR08] L. Desvillettes, F. Golse, and V. Ricci. The mean-field limit for solid particles in a Navier-Stokes flow. *J. Stat. Phys.*, 131(5):941–967, 2008.

[DL88] R. DiPerna and P-L. Lions. Solutions globales d'équations du type Vlasov-Poisson. *C. R. Acad. Sci., Paris, Sér. I*, 307(12):655–658, 1988.

[DL89a] R. DiPerna and P-L. Lions. Global weak solutions of Vlasov-Maxwell systems. *Commun. Pure Appl. Math.*, 42(6):729–757, 1989.

[DL89b] R. DiPerna and P-L. Lions. Global weak solutions of Vlasov-Maxwell systems. *Commun. Pure Appl. Math.*, 42(6):729–757, 1989.

[DL89c] R. DiPerna and P-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.

[DL06] C. De Lellis. *Notes on hyperbolic systems of conservation laws and transport equations.* University of Zurich, Institute of Mathematics, 2006.

[DM10] L. Desvillettes and L. Mathiaud. Some aspects of the asymptotics leading from gas-particles equations towards multiphase flows equations. *J. Stat. Phys.*, 141(1):120–141, 2010.

[DM19] R. Danchin and P. Mucha. The incompressible Navier-Stokes equations in vacuum. *Commun. Pure Appl. Math.*, 72(7):1351–1385, 2019.

[DMLM95] G. Dal Maso, P. Lefloch, and F. Murat. Definition and weak stability of nonconservative products. *J. Math. Pures Appl. (9)*, 74(6):483–548, 1995.

[Dob79] R.L. Dobrushin. Vlasov equations. *Funktsional. Anal. i Prilozhen*, 13(2):48—-58, 1979.

[DP08] R. Danchin and M. Paicu. Les théorèmes de Leray et de Fujita-Kato pour le système de Boussinesq partiellement visqueux. *Bull. Soc. Math. Fr.*, 136(2):261–309, 2008.

[DS20] M. Duerinckx and S. Serfaty. Mean field limit for coulomb-type flows. *Duke Mathematical Journal*, 169(15):2887–2935, 2020.

[DS21] L. Dong and Y. Sun. On asymptotic stability of the 3d Boussinesq equations without thermal conduction. *arXiv preprint arXiv:2107.10082*, 2021.

[DT22] R. Danchin and P. Tolksdorf. Critical regularity issues for the compressible Navier–Stokes system in bounded domains. *Math. Ann.*, pages 1–57, 2022.

[Dud18] R.M. Dudley. *Real Analysis And Probability.* CRC Press, 2018.

[Duf05] G. Dufour. *Modélisation multi-fluide eulérienne pour les écoulements diphasiques à inclusions dispersées.* PhD thesis, PhD thesis, Université Paul Sabatier Toulouse III, 2005.

[Duk80] J.K. Dukowicz. A particle-fluid numerical model for liquid sprays. *Journal of computational Physics*, 35(2):229–253, 1980.

[DWZZ18] C. Doering, J. Wu, K. Zhao, and X. Zheng. Long time behavior of the two-dimensional Boussinesq equations without buoyancy diffusion. *Physica D*, 376:144–159, 2018.

[DZ17] R. Danchin and X. Zhang. Global persistence of geometrical structures for the Boussinesq equation with no diffusion. *Commun. Partial Differ. Equations*, 42(1):68–99, 2017.

[EHK23] L. Ertzbischoff and D. Han-Kwan. On well-posedness for thick spray equations. *arXiv preprint arXiv:2303.09467*, 2023.

[EHKM21] L. Ertzbischoff, D. Han-Kwan, and A. Moussa. Concentration versus absorption for the Vlasov–Navier–Stokes system on bounded domains. *Nonlinearity*, 34(10):6843, 2021.

[Ert21] L. Ertzbischoff. Decay and absorption for the Vlasov-Navier-Stokes system with gravity in a half-space. *arXiv preprint arXiv:2107.02200*, 2021.

[Ert22] L. Ertzbischoff. Global derivation of a Boussinesq-Navier-Stokes type system from fluid-kinetic equations. *arXiv preprint arXiv:2202.08181*, 2022.

[FBD22] V. Fournet, C.. Buet, and B. Després. Local-in-time existence of strong solutions to an averaged thick sprays model. *HAL preprint hal-03881187*, 2022.

[FK64] H. Fujita and T. Kato. On the Navier-Stokes initial value problem, i. *Arch. Ration. Mech. Anal.*, 16:269–315, 1964.

[FK19] A. Figalli and M-J. Kang. A rigorous derivation from the kinetic Cucker-Smale model to the pressureless Euler system with nonlocal alignment. *Anal. PDE*, 12(3):843–866, 2019.

[FKS05] R. Farwig, H. Kozono, and H. Sohr. An Lq-approach to Stokes and Navier-Stokes equations in general domains. *Acta Math.*, 195(1):21–53, 2005.

[FLR19] F. Flandoli, M. Leocata, and C. Ricci. The Vlasov-Navier-Stokes equations as a mean field limit. *Discrete Contin. Dyn. Syst. Ser. B*, 24(8):3741–3753, 2019.

[FLR21] F. Flandoli, M. Leocata, and C. Ricci. The Navier-Stokes-Vlasov-Fokker-Planck system as a scaling limit of particles in a fluid. *J. Math. Fluid Mech.*, 23(2):Paper No. 40, 39, 2021.

[FMRT01] C. Foias, O. Manley, R. Rosa, and R. Temam. *Navier-Stokes Equations And Turbulence*, volume 83. Cambridge University Press, 2001.

[FNN16] E. Feireisl, Y. Namlyeyeva, and Š. Nečasová. Homogenization of the evolutionary Navier–Stokes system. *Manuscripta Mathematica*, 149:251–274, 2016.

[FNP01] E. Feireisl, A. Novotný, and H. Petzeltová. On the existence of globally defined weak solutions to the Navier-Stokes equations. *J. Math. Fluid Mech.*, 3(4):358–392, 2001.

[FS12] E. Feireisl and M. Schonbek. On the Oberbeck–Boussinesq approximation on unbounded domains. *Nonlinear Partial Differential Equations: The Abel Symposium 2010*, pages 131–168, 2012.

[Gal11] G. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations: Steady-state problems.* Springer Science & Business Media, 2011.

[Ger22] P. Gervais. On the convergence from Boltzmann to Navier-Stokes-Fourier for general initial data. *arXiv preprint arXiv:2201.02825*, 2022.

[GGBS22] F. Gancedo, R. Granero-Belinchón, and E. Salguero. Long time interface dynamics for gravity Stokes flow. *arXiv preprint arXiv:2211.03437*, 2022.

[GGJ20] F. Gancedo and E. García-Juárez. Regularity results for viscous 3d Boussinesq temperature fronts. *Commun. Math. Phys.*, 376:1705–1736, 2020.

[GH19] A. Giunti and R. Höfer. Homogenisation for the Stokes equations in randomly perforated domains under almost minimal assumptions on the size of the holes. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 36(7):1829–1868, 2019.

[GHKM18] O. Glass, D. Han-Kwan, and A. Moussa. The Vlasov–Navier–Stokes system in a 2d pipe: Existence and stability of regular equilibria. *Arch. Ration. Mech. Anal.*, 230(2):593–639, 2018.

[GHMZ10] T. Goudon, L. He, A. Moussa, and P. Zhang. The Navier–Stokes–Vlasov–Fokker–Planck system near equilibrium. *SIAM J. Math. Anal.*, 42(5):2177–2202, 2010.

[GI22] H. Grayer II. Dynamics of density patches in infinite prandtl number convection. *arXiv preprint arXiv:2207.09738*, 2022.

[GJV04a] T. Goudon, P-E. Jabin, and A. Vasseur. Hydrodynamic limit for the Vlasov-Navier-Stokes equations. Part I: Light particles regime. *Indiana Univ. Math. J.*, pages 1495–1515, 2004.

[GJV04b] T. Goudon, P-E. Jabin, and A. Vasseur. Hydrodynamic limit for the Vlasov-Navier-Stokes equations. Part II: Fine particles regime. *Indiana Univ. Math. J.*, pages 1517–1536, 2004.

[Gla96] R. Glassey. *The Cauchy problem in kinetic theory.* SIAM, 1996.

[Gli65] J. Glimm. Solutions in the large for nonlinear hyperbolic systems of equations. *Commun. Pure Appl. Math.*, 18(4):697–715, 1965.

[GLPS88] F. Golse, P-L. Lions, B. Perthame, and R. Sentis. Regularity of the moments of the solution of a transport equation. *J. Funct. Anal.*, 76(1):110–125, 1988.

[GM11] E. Guazzelli and J. Morris. *A physical introduction to suspension dynamics*, volume 45. Cambridge University Press, 2011.

[GNR21] E. Grenier, T. Nguyen, and I. Rodnianski. Landau damping for analytic and Gevrey data. *Math. Res. Lett.*, 28(6):1679–1702, 2021.

[Gol05] F. Golse. The Boltzmann equation and its hydrodynamic limits. In *Handbook of differential equations: Evolutionary equations. Vol. II*, pages 159–301. Amsterdam: Elsevier/North-Holland, 2005.

[Gol16] F. Golse. On the dynamics of large particle systems in the mean field limit. *Macroscopic and large scale phenomena: coarse graining, mean field limits and ergodicity*, pages 1–144, 2016.

[Gou01] T. Goudon. Asymptotic problems for a kinetic model of two-phase flow. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 131(6):1371–1384, 2001.

[GP04] T. Goudon and F. Poupaud. On the modeling of the transport of particles in turbulent flows. *ESAIM MATH. Model. Numer. Anal*, 38(4):673–690, 2004.

[GPS85] F. Golse, B. Perthame, and R. Sentis. Un résultat de compacité pour les équations de transport et application au calcul de la limite de la valeur propre principale d'un opérateur de transport. *C. R. Acad. Sci., Paris, Sér. I*, 301:341–344, 1985.

[Gre95] E. Grenier. Defect measures of the Vlasov-Poisson system in the quasineutral regime. *Comm. Partial Differential Equations*, 20(7-8):1189–1215, 1995.

[Gre96] E. Grenier. Oscillations in quasineutral plasmas. *Comm. Partial Differential Equations*, 21(3-4):363–394, 1996.

[GS87] R. Glassey and W. Strauss. Absence of shocks in an initially dilute collisionless plasma. *Commun. Math. Phys.*, 113(2):191–208, 1987.

[GS91] Y. Giga and H. Sohr. Abstract Lp estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains. *J. Funct. Anal.*, 102(1):72–94, 1991.

[GSR02] F. Golse and L. Saint-Raymond. Velocity averaging in L^1 for the transport equation. *C. R., Math., Acad. Sci. Paris*, 334(7):557–562, 2002.

[GSR04] F. Golse and L. Saint-Raymond. The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels. *Invent. Math.*, 155(1):81, 2004.

[GSR09] F. Golse and L. Saint-Raymond. The incompressible Navier–Stokes limit of the Boltzmann equation for hard cutoff potentials. *J. Math. Pures Appl. (9)*, 91(5):508–552, 2009.

[GSRT13] I. Gallagher, L. Saint-Raymond, and B. Texier. *From Newton to Boltzmann: hard spheres and short-range potentials.* Zur. Lect. Adv. Math. Zürich: European Mathematical Society (EMS), 2013.

[GT20] I. Gallagher and I. Tristani. On the convergence of smooth solutions from Boltzmann to Navier–Stokes. *Ann. Henri Lebesgue*, 3:561–614, 2020.

[GV14] D. Gérard-Varet. Phénomène d'amortissement dans les équations d'Euler. *Séminaire Bourbaki*, page 67, 2014.

[GWMP21] P. Gao, H. Wakeford, S. Moran, and V. Parmentier. Aerosols in exoplanet atmospheres. *J. Geophys. Res.*, 126, 2021.

[Haa06] M. Haase. The functional calculus for sectorial operators. In *The Functional Calculus for Sectorial Operators*, pages 19–60. Springer, 2006.

[Ham92] L. Hamdache. Initial-boundary value problems for the Boltzmann equation: global existence of weak solutions. *Arch. Ration. Mech. Anal.*, 119(4):309–353, 1992.

[Ham98] K. Hamdache. Global existence and large time behaviour of solutions for the Vlasov-Stokes equations. *Japan J. Ind. Appl. Math.*, 15(1):51, 1998.

[HdM21] E. Heulhard de Montigny. *Thermo-hydro-dynamic consistency and stiffness in general compressible multiphase flows.* PhD thesis, Université Paris-Saclay, 2021.

[Hei99] A. Heintz. Initial boundary value problems in irregular domains for nonlinear kinetic equations of Boltzmann type. *Transp. Theory Stat. Phys.*, 28(2):105–134, 1999.

[Hey80] J.G. Heywood. The Navier-Stokes equations: on the existence, regularity and decay of solutions. *Indiana Univ. Math. J.*, 29(5):639–681, 1980.

[Hey88] J.G. Heywood. Epochs of regularity for weak solutions of the Navier-Stokes equations in unbounded domains. *Tohoku Mathematical Journal, Second Series*, 40(2):293–313, 1988.

[Hey90] J.G. Heywood. Open problems in the theory of the Navier-Stokes equations for viscous incompressible flow. In *The Navier-Stokes Equations Theory and Numerical Methods*, pages 1–22. Springer, 1990.

[HH84] E. Horst and R. Hunze. Weak solutions of the initial value problem for the unmodified non-linear Vlasov equation. *Math. Methods Appl. Sci.*, 6:262–279, 1984.

[Hil18] M. Hillairet. On the homogenization of the Stokes problem in a perforated domain. *Arch. Ration. Mech. Anal.*, 230(3):1179–1228, 2018.

[Hil21] M. Hillairet. Derivation of the Stokes–Brinkman problem and extension to the Darcy regime. *J. Elliptic Parabol. Equ.*, pages 1–20, 2021.

[HJ07] M. Hauray and P-E. Jabin. N-particles approximation of the Vlasov equations with singular potential. *Arch. Ration. Mech. Anal.*, 183(3):489–524, 2007.

[HJ15] M. Hauray and P-E. Jabin. Particle approximation of Vlasov equations with singular forces: propagation of chaos. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(4):891–940, 2015.

[HJ20] R. Höfer and J. Jansen. Convergence rates and fluctuations for the Stokes-Brinkman equations as homogenization limit in perforated domains. *arXiv preprint arXiv:2004.04111*, 2020.

[HK07] T. Hmidi and S. Keraani. On the global well-posedness of the two-dimensional Boussinesq system with a zero diffusivity. *Adv. Differ. Equ.*, 12(4):461–480, 2007.

[HK11] D. Han-Kwan. Quasineutral limit of the Vlasov-Poisson system with massless electrons. *Commun. Partial Differ. Equations*, 36(7-9):1385–1425, 2011.

[HK17] D. Han-Kwan. Stabilité, limites singulières et conditions de contrôle géométrique en théorie cinétique. *Habilitation à diriger les recherches*, 2017.

[HK19] D. Han-Kwan. On propagation of higher space regularity for nonlinear Vlasov equations. *Anal. PDE*, 12(1):189–244, 2019.

[HK22] D. Han-Kwan. Large-time behavior of small-data solutions to the Vlasov-Navier-Stokes system on the whole space. *Probab. Math. Phys.*, 3(1):35–67, 2022.

[HKH15] D. Han-Kwan and M. Hauray. Stability issues in the quasineutral limit of the one-dimensional Vlasov-Poisson equation. *Comm. Math. Phys.*, 334(2):1101–1152, 2015.

[HKI17a] D. Han-Kwan and M. Iacobelli. Quasineutral limit for Vlasov–Poisson via Wasserstein stability estimates in higher dimension. *J. Differ. Equations*, 263(1):1–25, 2017.

[HKI17b] D. Han-Kwan and M. Iacobelli. The quasineutral limit of the Vlasov–Poisson equation in Wasserstein metric. *Commun. Math. Sci.*, 15(2):481 – 509, 2017.

[HKMM20] D. Han-Kwan, A. Moussa, and I. Moyano. Large time behavior of the Vlasov-Navier-Stokes system on the torus. *Arch. Ration. Mech. Anal.*, 236(3):1273–1323, 2020.

[HKMMM20] D. Han-Kwan, É. Miot, A. Moussa, and Iván Moyano. Uniqueness of the solution to the 2d Vlasov-Navier-Stokes system. *Rev. Mat. Iberoam.*, 36(1):37–60, 2020.

[HKMar] D. Han-Kwan and D. Michel. On hydrodynamic limits of the Vlasov-Navier-Stokes system. *Mem. Amer. Math. Soc.*, To appear.

[HKN16] D. Han-Kwan and T. Nguyen. Ill-posedness of the hydrostatic Euler and singular Vlasov equations. *Arch. Ration. Mech. Anal.*, 221(3):1317–1344, 2016.

[HKNR18] D. Han-Kwan, T. Nguyen, and F. Rousset. Long time estimates for the Vlasov-Maxwell system in the non-relativistic limit. *Commun. Math. Phys.*, 363(2):389–434, 2018.

[HKNR21] D. Han-Kwan, T. Nguyen, and F. Rousset. Asymptotic stability of equilibria for screened Vlasov-Poisson systems via pointwise dispersive estimates. *Ann. PDE*, 7(2):37, 2021. Id/No 18.

[HKR16] D. Han-Kwan and F. Rousset. Quasineutral limit for Vlasov-Poisson with Penrose stable data. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(6):1445–1495, 2016.

[HKR23] D. Han-Kwan and F. Rousset. From Vlasov-Poisson to the kinetic incompressible Euler equation (working title). In preparation, 2023.

[HL05] T. Hou and C. Li. Global well-posedness of the viscous Boussinesq equations. *Discrete Contin. Dyn. Syst.*, 12(1):1–12, 2005.

[HLP88] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988. Reprint of the 1952 edition.

[HMS19] M. Hillairet, A. Moussa, and F. Sueur. On the effect of polydispersity and rotation on the Brinkman force induced by a cloud of particles on a viscous incompressible flow. *Kinet. Relat. Models*, 12, 2019.

[Hof87] D. Hoff. Global existence for 1d, compressible, isentropic Navier-Stokes equations with large initial data. *Trans. Am. Math. Soc.*, 303:169–181, 1987.

[Höf18] F. Höfer. Sedimentation of inertialess particles in Stokes flows. *Commun. Math. Phys.*, 360(1):55–101, 2018.

[Höf20] R. Höfer. Sedimentation of particle suspensions in Stokes flows. *Universitäts-und Landesbibliothek Bonn*, 2020.

[Hop51] E. Hopf. Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. *Mathematische Nachrichten*, 4(1-6):213–231, 1951.

[HR10] T. Hmidi and F. Rousset. Global well-posedness for the Navier-Stokes-Boussinesq system with axisymmetric data. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 27(5):1227–1246, 2010.

[HS14] P. Han and M. Schonbek. Large time decay properties of solutions to a viscous Boussinesq system in a half space. *Adv. Differ. Equ.*, 19(1-2):87–132, 2014.

[HS21] R Höfer and R. Schubert. The influence of Einstein's effective viscosity on sedimentation at very small particle volume fraction. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 38(6):1897–1927, 2021.

[HS22] R. Höfer and R. Schubert. Sedimentation of particles with very small inertia in Stokes flows i: convergence to the transport-Stokes equations. *arXiv preprint arXiv:2302.04637*, 2022.

[Hwa87] I-L. Hwang. The L2-boundedness of pseudodifferential operators. *Trans. Am. Math. Soc.*, 302(1):55–76, 1987.

[HZ95] D. Hoff and K. Zumbrun. Multi-dimensional diffusion waves for the Navier-Stokes equations of compressible flow. *Indiana Univ. Math. J.*, 44(2):603–676, 1995.

[Hö18] R. Höfer. The inertialess limit of particle sedimentation modeled by the Vlasov-Stokes equations. *SIAM J. Math. Anal.*, 50(5):5446–5476, 2018.

[IH10] M. Ishii and T. Hibiki. *Thermo-fluid dynamics of two-phase flow.* Springer Science & Business Media, 2010.

[Jab00a] P. Jabin. Macroscopic limit of Vlasov type equations with friction. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 17(5):651–672, 2000.

[Jab00b] P-E. Jabin. Large time concentrations for solutions to kinetic equations with energy dissipation. *Commun. Partial Differ. Equations*, 25(3-4):541–557, 2000.

[Jab02] P-E. Jabin. Various levels of models for aerosols. *Math. Models Methods Appl. Sci.*, 12(7):903–919, 2002.

[Jab14] P-E. Jabin. A review of the mean field limits for Vlasov equations. *Kinetic & Related Models*, 7(4):661, 2014.

[Jea15] J. Jeans. On the theory of star-streaming and the structure of the universe. *Monthly Notices of the Royal Astronomical Society, Vol. 76, p. 70-84*, 76:70–84, 1915.

[JLT22] P-E. Jabin, H-Y. Lin, and E. Tadmor. Commutator method for averaging lemmas. *Anal. PDE*, 15(6):1561–1584, 2022.

[JN11] P-E. Jabin and A. Nouri. Analytic solutions to a strongly nonlinear Vlasov equation. *C. R., Math., Acad. Sci. Paris*, 349(9-10):541–546, 2011.

[JO04] P-E Jabin and F. Otto. Identification of the dilute regime in particle sedimentation. *Commun. Math. Phys.*, 250:415–432, 2004.

[JV04] P-E. Jabin and L. Vega. A real space method for averaging lemmas. *J. Math. Pures Appl. (9)*, 83(11):1309–1351, 2004.

[Kat84] T. Kato. Strong Lp-solutions of the Navier-Stokes equation in Rm, with applications to weak solutions. *Mathematische Zeitschrift*, 187:471–480, 1984.

[Koc90] D. Koch. Kinetic theory for a monodisperse gas–solid suspension. *Physics of Fluids A: Fluid Dynamics*, 2(10):1711–1723, 1990.

[KSS03] B.L. Keyfitz, R. Sanders, and M. Sever. Lack of hyperbolicity in the two-fluid model for two-phase incompressible flow. *Discrete Contin. Dyn. Syst., Ser. B*, 3(4):541–564, 2003.

[KT01] H. Koch and D. Tataru. Well-posedness for the Navier–Stokes equations. *Adv. Math.*, 157(1):22–35, 2001.

[KW20] I. Kukavica and W. Wang. Long time behavior of solutions to the 2d Boussinesq equations with zero diffusivity. *J. Dyn. Differ. Equations*, 32(4):2061–2077, 2020.

[Lad59] O.A. Ladyzhenskaya. Solution ”in the large” of the nonstationary boundary value problem for the Navier-Stokes system with two space variables. *Commun. Pure Appl. Math.*, 12:427–433, 1959.

[Lau02] F. Laurent. *Modélisation mathématique et numérique de la combustion de brouillards de gouttes poly- dispersées.* PhD thesis, Université Claude Bernard, Lyon, (in French), 2002.

[Leb22] A. Leblond. Well-posedness of the Stokes-transport system in bounded domains and in the infinite strip. *J. Math. Pures Appl. (9)*, 158:120–143, 2022.

[Ler34] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.*, 63(1):193–248, 1934.

[Lio98] P-L. Lions. *Mathematical topics in fluid mechanics. Vol. 2: Compressible models*, volume 10 of *Oxf. Lect. Ser. Math. Appl.* Oxford: Clarendon Press, 1998.

[Liu02] H. Liu. Science and engineering of droplets: fundamentals and applications. *Appl. Mech. Rev.*, 55(1):B16–B17, 2002.

[Loe06] G. Loeper. Uniqueness of the solution to the Vlasov–Poisson system with bounded density. *J. Math. Pures Appl. (9)*, 86(1):68–79, 2006.

[LP91] P-L. Lions and B. Perthame. Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. *Invent. Math.*, 105(1):415–430, 1991.

[LP17] D. Lazarovici and P. Pickl. A mean field limit for the Vlasov–Poisson system. *Arch. Ration. Mech. Anal.*, 225:1201–1231, 2017.

[LS78] O.A. Ladyzhenskaya and V.A. Solonnikov. Unique solvability of an initial-and boundary-value problem for viscous incompressible nonhomogeneous fluids. *Journal of Soviet Mathematics*, 9(5):697–749, 1978.

[Lun18] A. Lunardi. *Interpolation theory*, volume 16. Springer, 2018.

[Maj03] A. Majda. *Introduction to PDEs and waves for the atmosphere and ocean*, volume 9. Providence, RI: American Mathematical Society (AMS); New York, NY: Courant Institute of Mathematical Sciences, 2003.

[Maj12] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*, volume 53. Springer Science & Business Media, 2012.

[Mas84] K. Masuda. Weak solutions of Navier-Stokes equations. *Tôhoku Math. J. (2)*, 36(4):623–646, 1984.

[Mas01] M. Masmoudi. From Vlasov-Poisson system to the incompressible Euler system. *Commun. Partial Differ. Equations*, 26(9-10):1913–1928, 2001.

[Mat06] J. Mathiaud. *Etude de systemes de type gaz-particules.* PhD thesis, Cachan, Ecole normale supérieure, 2006.

[Mat10] J. Mathiaud. Local smooth solutions of a thin spray model with collisions. *Math. Models Methods Appl. Sci.*, 20(02):191–221, 2010.

[MB02] A. Majda and A. Bertozzi. *Vorticity and incompressible flow.* Camb. Texts Appl. Math. Cambridge: Cambridge University Press, 2002.

[Mec19] A. Mecherbet. Sedimentation of particles in Stokes flow. *Kinet. Relat. Models*, 12, 2019.

[Mec20] A. Mecherbet. On the sedimentation of a droplet in Stokes flow. *To appear in Comm Math Sci*, 2020.

[Mét01] G. Métivier. Stability of multidimensional shocks. In *Advances in the theory of shock waves*, pages 25–103. Boston, MA: Birkhäuser, 2001.

[Mic21] D. Michel. *Analyse mathématique et asymptotique de modèles couplés fluide-cinétique issus de la mécanique des fluides et des sciences du vivant.* PhD thesis, Sorbonne Université, 2021.

[Mis00a] S. Mischler. On the initial boundary value problem for the Vlasov-Poisson-Boltzmann system. *Commun. Math. Phys.*, 210(2):447–466, 2000.

[Mis00b] S. Mischler. On the trace problem for solutions of the Vlasov equation: The trace problem for solutions. *Commun. Partial Differ. Equations*, 25(7-8):1415–1443, 2000.

[Mis10] S. Mischler. Kinetic equations with Maxwell boundary conditions. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(5):719–760, 2010.

[MN80] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.*, 20(1):67–104, 1980.

[MN83] A. Matsumura and T. Nishida. Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids. *Commun. Math. Phys.*, 89(4):445–464, 1983.

[Mou09] A. Moussa. *Étude mathématique et numérique du transport d'aérosols dans le poumon humain.* PhD thesis, École normale supérieure de Cachan-ENS Cachan, 2009.

[Moy16] I. Moyano. Local null-controllability of the 2-d Vlasov-Navier-Stokes system. *arXiv preprint arXiv:1607.05578*, 2016.

[MPO03] T.A. Mather, D.M. Pyle, and C. Oppenheimer. Tropospheric volcanic aerosol. *Geophysical Monograph-American Geophysical Union*, 139:189–212, 2003.

[MS88] T. Miyakawa and H. Sohr. On energy inequality, smoothness and large time behavior in L2 for weak solutions of the Navier-Stokes equations in exterior domains. *Mathematische Zeitschrift*, 199(4):455–478, 1988.

[MS13] A. Moussa and F. Sueur. On a Vlasov–Euler system for 2D sprays with gyroscopic effects. *Asymptotic Anal.*, 81(1):53–91, 2013.

[MSHZ20] N. Masmoudi, B. Said-Houari, and W. Zhao. Stability of Couette flow for 2d Boussinesq system without thermal diffusivity. *arXiv preprint arXiv:2010.01612*, 2020.

[MV01] M. Massot and P. Villedieu. Modélisation multi-fluide eulérienne pour la simulation de brouillards denses polydispersés. *C. R. Acad. Sci., Paris, Sér. I*, 332(9):869–874, 2001.

[MV07] A. Mellet and A. Vasseur. On the barotropic compressible Navier-Stokes equations. *Commun. Partial Differ. Equations*, 32(3):431–452, 2007.

[MV08] A Mellet and A Vasseur. Asymptotic analysis for a Vlasov-Fokker-Planck/compressible Navier-Stokes system of equations. *Commun. Math. Phys.*, 281(3):573–596, 2008.

[MV11] C. Mouhot and C. Villani. On Landau damping. *Acta Math.*, 207(1):29–201, 2011.

[MV15] C. Mouhot and C. Villani. Kinetic theory. *Princeton Companion to Applied Mathematics, NJ Higham ed*, 2015.

[MZ05] G. Métivier and K. Zumbrun. *Large viscous boundary layers for noncharacteristic nonlinear hyperbolic problems*, volume 826 of *Mem. Am. Math. Soc.* Providence, RI: American Mathematical Society (AMS), 2005.

[Nas62] J. Nash. Le problème de Cauchy pour les équations différentielles d'un fluide général. *Bull. Soc. Math. Fr.*, 90:487–497, 1962.

[NKDVLG05] M. Ndjinga, A. Kumbaro, F. De Vuyst, and P. Laurent-Gengoux. Influence of interfacial forces on the hyperbolicity of the two-fluid model. In *5th International Symposium on Multiphase Flow, Heat Mass Transfer and Energy Conversion*, 2005.

[NW74] H. Neunzert and J. Wick. Die Approximation der Lösung von Integro-Differentialgleichungen durch endliche Punktmengen. *Numer. Behandlg. nichtlin. Integrodiffer.-u. Differ.-Glchg., Vortr. Tag. Oberwolfach 1973, Lect. Notes Math. 395, 275-290 (1974).*, 1974.

[OA87] P.J. O'Rourke and A. Amsden. The TAB method for numerical calculation of spray droplet breakup. Technical report, Los Alamos National Lab.(LANL), Los Alamos, NM (United States), 1987.

[O'R81] P.J. O'Rourke. Collective drop effects on vaporizing liquid sprays. Technical report, Los Alamos National Lab., NM (USA), 1981.

[OZS09] P.J. O'Rourke, P.P. Zhao, and D. Snider. A model for collisional exchange in gas/liquid/solid fluidized beds. *Chem. Eng. Sci*, 64(8):1784–1797, 2009.

[Pen60] O. Penrose. Electrostatic instabilities of a uniform non-Maxwellian plasma. *The Physics of Fluids*, 3(2):258–265, 1960.

[PS98] B. Perthame and P. E. Souganidis. A limiting case for velocity averaging. *Ann. Sci. Éc. Norm. Supér. (4)*, 31(4):591–598, 1998.

[Ram00] D. Ramos. *Quelques résultats mathématiques et simulations numériques d'écoulements régis par des modèles bifluides.* PhD thesis, ENS Cachan, (in French), 2000.

[Rei96] R.D. Reitz. Computer modeling of sprays. *Spray Technology Short Course, Pittsburgh, PA*, 1996.

[Ric17] V. Ricci. Derivation of models for thin sprays from a multiphase Boltzmann model. In *From Particle Systems to Partial Differential Equations: PSPDE IV, Braga, Portugal, December 2015*, pages 285–308. Springer, 2017.

[Rou04] F. Rousset. Stability of large Ekman boundary layers in rotating fluids. *Arch. Ration. Mech. Anal.*, 172(2):213–245, 2004.

[RRS16] J.C. Robinson, J.L. Rodrigo, and W. Sadowski. *The three-dimensional Navier-Stokes equations: Classical theory*, volume 157. Cambridge university press, 2016.

[Sal98] R. Salmon. *Lectures on geophysical fluid dynamics.* Oxford University Press, 1998.

[Sch86] M. Schonbek. Large time behaviour of solutions to the Navier-Stokes equations. *Commun. Partial Differ. Equations*, 11(7):733–763, 1986.

[Ser59] J. Serrin. On the uniqueness of compressible fluid motions. *Arch. Ration. Mech. Anal.*, 3:271–288, 1959.

[Sir10] W. Sirignano. *Fluid dynamics and transport of droplets and sprays.* Cambridge university press, 2010.

[Soh12] H. Sohr. *The Navier-Stokes equations: An elementary functional analytic approach.* Springer Science & Business Media, 2012.

[Sol80] V.A. Solonnikov. Solvability of the initial-boundary-value problem for the equations of motion of a viscous compressible fluid. *Journal of Soviet Mathematics*, 14(2):1120–1133, 1980.

[SR03] L. Saint-Raymond. Convergence of solutions to the Boltzmann equation in the incompressible Euler limit. *Arch. Ration. Mech. Anal.*, 166:47–80, 2003.

[SR09a] L. Saint-Raymond. *Hydrodynamic limits of the Boltzmann equation*, volume 1971 of *Lect. Notes Math.* Berlin: Springer, 2009.

[SR09b] L. Saint-Raymond. Hydrodynamic limits: some improvements of the relative entropy method. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 26(3):705–744, 2009.

[Sto50] G.G. Stokes. On the effect of internal friction of fluids on the motion of pendulums. *Trans. Camb. phi1. S0c*, 9(8):106, 1850.

[SY20] Y. Su and L. Yao. Hydrodynamic limit for the inhomogeneous incompressible Navier-Stokes/Vlasov-Fokker-Planck equations. *J. Differ. Equations*, 269(2):1079–1116, 2020.

[TWZZ20] L. Tao, J. Wu, K. Zhao, and X. Zheng. Stability near hydrostatic equilibrium to the 2d Boussinesq equations without thermal diffusion. *Arch. Ration. Mech. Anal.*, 237(2):585–630, 2020.

[Uka67] S. Ukai. Eigenvalues of the neutron transport operator for a homogeneous finite moderator. *J. Math. Anal. Appl.*, 18(2):297–314, 1967.

[Val17] G. Vallis. *Atmospheric and oceanic fluid dynamics.* Cambridge University Press, 2017.

[Vil02a] C. Villani. Limites hydrodynamiques de l'équation de Boltzmann. *Astérisque, SMF*, 282:365–405, 2002.

[Vil02b] C. Villani. A review of mathematical topics in collisional kinetic theory. In *Handbook of mathematical fluid dynamics. Vol. 1*, pages 71–305. Amsterdam: Elsevier, 2002.

[Vil06] C. Villani. Mathematics of granular materials. *J. Stat. Phys.*, 124(2-4):781–822, 2006.

[Vil21] C. Villani. *Topics in optimal transportation*, volume 58. American Mathematical Soc., 2021.

[Vla38] A.A. Vlasov. The vibrational properties of an electron gas. *J. Exp. Theor. Phys (In Russian)*, 8(3):291–318, 1938.

[VM20] F. Veron and L. Mieussens. An Eulerian model for sea spray transport and evaporation. *J. Fluid Mech.*, 897:A6, 2020.

[VY16] A. Vasseur and C. Yu. Existence of global weak solutions for 3d degenerate compressible Navier-Stokes equations. *Invent. Math.*, 206(3):935–974, 2016.

[Wie87] M. Wiegner. Decay results for weak solutions of the Navier-Stokes equations on Rn. *J. London Math. Soc.*, 2(2):303–313, 1987.

[Wil85] F.A. Williams. *Combustion theory.* Benjamin Cummings, second edition, 1985.

[WST86] R. West, D. Strobel, and M. Tomasko. Clouds, aerosols, and photochemistry in the Jovian atmosphere. *Icarus*, 65(2-3):161–217, 1986.

ECOLE DOCTORALE DE MATHEMATIQUES HADAMARD

Titre : Analyse mathématique de quelques systèmes d'équations de type fluide-cinétique

Mots clés : EDP, équations cinétiques, mécanique des fluides, dynamique en temps long, limite hydrodynamique, couplage singulier

Résumé : Cette thèse est consacrée à l'analyse mathématique de systèmes de type fluide-cinétique, qui décrivent l'évolution d'une suspension de particules au sein d'un fluide ambiant. Le point de vue adopté est celui de la théorie cinétique pour la phase dispersée et celui de la mécanique des fluides pour la phase continue.
Les Chapitres 2 et 3 sont dédiés à l'étude du comportement en temps long pour les équations de Vlasov-Navier-Stokes dans un domaine, avec condition d'absorption au bord pour les particules. Au Chapitre 2, nous analysons la compétition entre concentration en vitesse et absorption dans un domaine borné. Nous démontrons que la fonction de distribution des particules possède un comportement monocinétique en vitesse, et exhibons une grande variété de scénarios pour le profil asymptotique spatial. Au Chapitre 3, nous nous plaçons dans le cas du demi-espace en prenant en compte l'action de la force de gravité sur les particules. Nous montrons que les effets d'absorption au bord, combinés à la gravité, permettent d'obtenir la stabilité de la solution triviale pour ce système. Notre obtenons une famille d'estimations de décroissance en temps pour tous les moments en vitesse de la fonction de distribution, grâce à l'introduction d'une condition de contrôle géométrique appropriée.
Dans le Chapitre 4, nous utilisons les idées précédentes pour étudier une limite hydrodynamique des équations de Vlasov-Navier-Stokes avec gravité, dans un régime haute-friction. Nous obtenons ainsi la dérivation globale en temps d'un système de type Boussinesq-Navier-Stokes.
Finalement, le Chapitre 5 est consacré à l'étude mathématique d'un système des sprays épais, qui est un couplage singulier entre une équation cinétique et les équations des fluides compressibles. Dans le cas d'un fluide visqueux, nous démontrons l'existence et l'unicité d'une solution à régularité Sobolev, localement en temps, pour des données initiales satisfaisant un critère de stabilité à la Penrose. Il s'agit de la première construction rigoureuse de solution pour ce type de système, inspirée de travaux récents sur les équations de Vlasov singulières.

Title : Mathematical analysis of some fluid-kinetic systems of equations

Keywords : PDE, kinetic equations, fluid mechanics, long-time dynamics, hydrodynamic limit, singular coupling

Abstract : This thesis delves into the mathematical analysis of fluid-kinetic systems, which describe the evolution of a suspension of particles in an ambient fluid. In this framework, a kinetic equation is coupled with the standard equations of fluid mechanics.
The focus of Chapters 2 and 3 is the long-time behaviour of the Vlasov-Navier-Stokes equations in a domain, with absorption boundary condition for the particles. In Chapter 2, we analyse the competition between concentration in velocity and absorption in a bounded domain. We show that the particle distribution function has a monokinetic behaviour in velocity, and exhibit a wide variety of scenarios for the spatial asymptotic profile. In Chapter 3, we consider the half-space case, taking into account the action of the gravity force on the particles. The stability of the trivial solution for the system is explored and proven by combining both the absorption at the boundary and the gravity effects. Our approach is based on time-decay estimates for all moments in velocity of the distribution function, obtained by introducing an appropriate geometric control condition.
Chapter 4 builds on the previous ideas to study a hydrodynamic limit of the Vlasov-Navier-Stokes equations with gravity, in a high-friction regime. We obtain the global in time derivation of a Boussinesq-Navier-Stokes type system.
Chapter 5 is dedicated to the mathematical study of a thick spray system, which is a singular coupling between a kinetic equation and the compressible fluid equations. In the case of a viscous fluid, we prove the local in time strong well-posedness of the equations with Sobolev regularity, for initial data satisfying a Penrose stability condition. This is the first rigorous construction of solutions to this type of system, in the spirit of some recent works on singular Vlasov equations.

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