Notes of Everything

In fact, not even everything in algebraic geometry

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To Alexandre Grothendieck

Introduction

These notes are meant to be my point of view to modern algebraic geometry. As usual, the reader should ask why I think there's room for another such text if there exist already lots of great books about scheme theory such as [10], [15], and [36]. The *raison d'être* of these notes are three simple guiding principles.

- **Principle 1.** Since there's only a brief amount of time for a student to learn the foundations of algebraic geometry before doing research, I think it is unreasonable to learn any part of it twice. In these notes, I take a *hypermodern* approach: whenever it is sensible, I will always try my best to motivate the subjects in their modern form. For example, we'll study extensively the geometry of spectra, as topological spaces, in the part about commutative algebra, and we'll learn about how we can understand manifolds using sheaf-theoretical language. In this way, the reader will hopefully find schemes intuitive when they arrive in due time. Similarly, the part about homological algebra uses derived categories right from the start.

- **Principle 2.** In my experience, one of the main difficulties most students face when studying basic algebraic geometry is the need to learn commutative algebra, homological algebra and sheaf theory all at the same time. Possibly the only reference which tries to explain in detail all those topics is Vakil's *The Rising Sea* [36]. Nevertheless, I believe in the pedagogical advantage of learning each of these subjects in due time. This is the reason why these notes are divided in multiple parts.

- **Principle 3.** Algebraic geometry has deep relations with most other fields in mathematics. In most books, these interactions are at most briefly outlined. While filling those gaps may be trivial for the experienced mathematician, my experience shows that they aren't for the student. This is the reason why we'll, for example, study sheaves (and their cohomology) in a setting which accommodates not only schemes but also manifolds and complex analytic spaces. Similarly, the relation between number fields and function fields is illustrated throughout the entirety of these notes.

If you notice any errors / typos or have any suggestions, I would be very grateful to hear about them. Please email me at gabriel.ribeiro@polytechnique.edu.

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Part I.

Commutative Algebra

1. The spectrum of a ring

The overarching goal of this part (and of a large portion of these notes) is to show that commutative rings are very geometrical objects. Not only is the geometrical intuition extraordinarily fruitful, but so are the geometrical methods that we'll employ.

Since every book has to start somewhere, we assume some basic knowledge of abstract algebra and category theory. Also, as we're going to use manifolds as our prime example of geometrical object, a passing acquaintance of differential geometry is only going to be useful. The books [23], [32], and the first seven chapters of [1] provide more than enough preparation.

1.1. Maximal ideals are points of a space

Let M be a (smooth) compact manifold. We begin this chapter investigating how much geometrical information about M we can recover from its ring $A := C^\infty(M)$ of smooth functions. A first observation is that points of M give rise to ideals of A. Indeed, if $x \in M$, the subset

$$\mathfrak{m}_x := \{f \in A \mid f(x) = 0\}$$

is the kernel of the evaluation map

$$\begin{aligned} \mathrm{ev}_x : A &\to \mathbb{R} \\ f &\mapsto f(x). \end{aligned}$$

Since ev_x is clearly surjective, $\mathfrak{m}_x$ is actually a maximal ideal of A. Even more, all its maximal ideals are of this form!

Proposition 1.1.1 Let M be a compact manifold. The map $x \mapsto \mathfrak{m}_x$ defines a bijection $M \to \operatorname{Specm} A$, where $\operatorname{Specm} A$ is the set of maximal ideals of A.

Proof. We begin by proving that our map is surjective. Given that every proper ideal is contained in a maximal ideal, it suffices to prove that if some ideal I is not contained in any $\mathfrak{m}_x$, then $I = A$. If that's the case, for all $x \in M$ there's a function $f_x \in I$ such that $f(x) \neq 0$. In other words, the sets

$$D(f_x) := \{y \in M \mid f_x(y) \neq 0\}$$

form an open cover of M. By compactness, there is a finite set of functions $f_1, \dots, f_n \in I$ such that, for all $y \in M$, we have $f_i(y) \neq 0$ for at least some i. It follows that

$$f := \sum_{i=1}^{n} f_i^2 \in I$$

is nowhere vanishing on M and so is a unit; proving that $I = A$.

The injectivity follows readily from the smooth Urysohn lemma. Indeed, if $x, y \in M$ are different points, there's a function $f \in A$ such that $f \in \mathfrak{m}_x$ and $f \notin \mathfrak{m}_y$. In particular, $\mathfrak{m}_x \neq \mathfrak{m}_y$. □

Naturally, we would love for this to be a homeomorphism. Inspired by open sets that appeared in the proof above, we define

$$D(f) := \{\mathfrak{m} \in \operatorname{Specm} A \mid f \notin \mathfrak{m}\}$$

for $f \in A$, and we endow $\operatorname{Specm} A$ with the topology generated by these sets. Observe that $\mathfrak{m}_x \in D(f)$ if and only if $f(x) \neq 0$.

Corollary 1.1.2 With the topology defined above, the bijection $M \to \operatorname{Specm} A$ of the preceding proposition is a homeomorphism.

Proof. Denote our map $M \to \operatorname{Specm} A$ by Φ. By the discussion above, we have that

$$\Phi^{-1}(D(f)) = \{x \in M \mid \mathfrak{m}_x \in D(f)\} = \{x \in M \mid f(x) \neq 0\}$$

is an open subset of M. In particular, Φ is continuous. Recall that a continuous bijection between a compact and a Hausdorff space is necessarily a homeomorphism, so it suffices to prove that $\operatorname{Specm} A$ is Hausdorff. In other words, given two different points $x, y \in M$, we need to find disjoint neighborhoods of $\mathfrak{m}_x$ and $\mathfrak{m}_y$.

Let U and V be disjoint neighborhoods of x and y in M. By the smooth Urysohn lemma, there exists two functions $f, g \in A$ satisfying $f^{-1}(0) = M \setminus U$ and $g^{-1}(0) = M \setminus V$. It follows that $D(f)$ and $D(g)$ are disjoint neighborhoods of $\mathfrak{m}_x$ and $\mathfrak{m}_y$, respectively. □

What about the manifold structure of M? For that, we need to understand how to think about smooth maps $M \to N$ in terms of $\mathbb{R}$-algebra morphisms $C^\infty(N) \to C^\infty(M)$. Firstly, a smooth map $f : M \to N$ induces a morphism of $\mathbb{R}$-algebras given by

$$\begin{aligned} f^* : C^\infty(N) &\to C^\infty(M) \\ g &\mapsto g \circ f. \end{aligned}$$

We can also associate a smooth map to a morphism between the respective $\mathbb{R}$-algebras!

Proposition 1.1.3 Let $\varphi : C^\infty(N) \to C^\infty(M)$ be a morphism of $\mathbb{R}$-algebras. Then $\mathfrak{m} \mapsto \varphi^{-1}(\mathfrak{m})$ defines a continuous map $\operatorname{Specm}\varphi : \operatorname{Specm} C^\infty(M) \to \operatorname{Specm} C^\infty(N)$. Moreover, under the bijections of proposition 1.1.1, this map is smooth.

It's not *true that the preimage of a maximal ideal by a morphism of rings is always maximal. This need not be true even for morphisms of algebras over a field.*

Proof. Let $\mathfrak{m} \in \operatorname{Specm} C^\infty(M)$. For some $x \in M$, we have that $\mathfrak{m} = \ker \mathrm{ev}_x$ and so $\varphi^{-1}(\mathfrak{m}) = \ker(\mathrm{ev}_x \circ \varphi)$. We conclude that, in order for $\varphi^{-1}(\mathfrak{m})$ to be a maximal ideal, it suffices for $\mathrm{ev}_x \circ \varphi : C^\infty(N) \to \mathbb{R}$ to be surjective. But this follows from the fact that $\mathrm{ev}_x \circ \varphi$ is a composition of two morphisms of $\mathbb{R}$-algebras.

The continuity of $\operatorname{Specm}\varphi$ amounts to the fact that $(\operatorname{Specm}\varphi)^{-1}(D(g)) = D(\varphi(g))$ for all $g \in C^\infty(N)$. Indeed, a maximal ideal $\mathfrak{m}$ is an element of the left-hand side if $\varphi^{-1}(\mathfrak{m}) \in D(g)$. This means that $g \notin \varphi^{-1}(\mathfrak{m})$, which is equivalent to $\varphi(g) \notin \mathfrak{m}$. And this happens precisely when $\mathfrak{m}$ is an element of the right-hand side.

Now, consider the continuous map $f : M \to N$ defined as the composition

$$M \cong \operatorname{Specm} C^\infty(M) \xrightarrow{\operatorname{Specm}\varphi} \operatorname{Specm} C^\infty(N) \cong N.$$

This map sends $x \in M$ to the unique $y \in N$ which satisfies $\ker \mathrm{ev}_y = \mathfrak{m}_y = \varphi^{-1}(\mathfrak{m}_x) = \ker(\mathrm{ev}_x \circ \varphi)$. If K is this common kernel, both ev_y and $\mathrm{ev}_x \circ \varphi$ induce isomorphisms

$$A/K \xrightarrow{\sim} \mathbb{R}.$$

Since there's only one such isomorphism, this implies that $\mathrm{ev}_y = \mathrm{ev}_x \circ \varphi$.

In order to prove that f is smooth, we recall the following criterion. [32, Proposition 6.16] Let $f : M \to N$ be a continuous map between manifolds. If $g \circ f \in C^\infty(M)$ for every $g \in C^\infty(N)$, then f is smooth. It suffices then to prove that $g \circ f = \varphi(g) \in C^\infty(M)$.

If $g \in C^\infty(N)$, the function $f^*(g) = g \circ f \in C^\infty(M)$ sends a point $x \in M$ to $g(y)$, where y is the unique point of N satisfying $\mathrm{ev}_y = \mathrm{ev}_x \circ \varphi$. Applying g to this equality we obtain $g(y) = \varphi(g)(x)$, yielding our claim. ☐

Even more is true. All smooth maps $M \to N$ are of the form $\operatorname{Specm}\varphi$, for some morphism of $\mathbb{R}$-algebras $\varphi : C^\infty(N) \to C^\infty(M)$.

Corollary 1.1.4 Let CMan be the category of compact manifolds. The contravariant functor CMan $\to$ $\mathbb{R}$-Alg, given by $M \mapsto C^\infty(M)$, is fully faithful. That is, if M and N are compact manifolds, the function

$$\begin{gathered} \operatorname{Hom}_{\mathsf{Man}}(M, N) \to \operatorname{Hom}_{\mathbb{R}\text{-}\mathsf{Alg}}(C^\infty(N), C^\infty(M)) \\ f \mapsto f^* \end{gathered}$$

is bijective.

Proof. Let $\varphi : C^\infty(N) \to C^\infty(M)$ be a morphism of $\mathbb{R}$-algebras. The preceding proposition associates to φ a smooth map $M \to N$, which we'll denote by φ^*. We affirm that $\varphi \mapsto \varphi^*$ yields an inverse to $f \mapsto f^*$. Indeed, in the proof of the aforementioned proposition, we saw that $(\varphi^*)^* = \varphi$. Conversely, if $f : M \to N$ is any smooth map, the map $(f^*)^*$ sends $x \in M$ to the unique point $y \in N$ satisfying $\mathrm{ev}_y = \mathrm{ev}_x \circ f^*$. But $y = f(x)$ is one such point. This finishes the proof. ☐

We remark that this corollary implies that a compact manifold M is completely, even with its manifold structure, determined by the ring $A = C^\infty(M)$. Indeed, A determines entirely the functor $\mathrm{Hom}_{\mathsf{Man}}(M, -)$ which, by the Yoneda lemma, suffices to determine M itself.

1.2. Localization

Proposition 1.2.1 Let U be an open subset of a manifold M. Let $A = C^\infty(M)$ and $S = \{f \in A \mid f(x) \neq 0 \; \forall x \in U\}$. Then $S^{-1}A \cong C^\infty(U)$ and the canonical morphism $A \to S^{-1}A$ coincides with the restriction from $C^\infty(M) \to C^\infty(U)$.

Proof. ☐

Proposition 1.2.2 Let x be a point of a manifold M. Let $A = C^\infty(M)$ and $S = \{f \in A \mid f(x) \neq 0\} = A \setminus \mathfrak{m}_x$. Then $S^{-1}A \cong C^\infty_x(M)$, the ring of germs of functions at x, and the canonical morphism $A \to S^{-1}A$ coincides with the map which sends a global function to its equivalence class.

Proof. ☐

Proposition 1.2.3 espaço tangente é $\mathfrak{m}/\mathfrak{m}^2$.

Proof. ☐

Definition 1.2.1 — Regular ideal.

Proposition 1.2.4 Seja N um fechado de M. Então N é uma subvariedade fechada se e somente se I_N é um ideal regular de A.

Proof. ☐

1.3. The tangent space

1.4. Modules are vector bundles

Definition 1.4.1 — Locally free module. Let A be a ring and M be a A-module. We say that M is *locally free* if there exist $f_1, \dots, f_n \in A$ generating the unit ideal such that, for each i, M_{f_i} is a free A_{f_i}-module.

Proposition 1.4.1 — Serre-Swan.

1.5. The prime spectrum

Proposition 1.5.1 Let I be an ideal of a ring A. Then the projection map $\pi : A \to A/I$ induces the following bijections:

$$\operatorname{Spec} A/I = \{\mathfrak{p} \in \operatorname{Spec} A \mid I \subset \mathfrak{p}\} \quad \text{and} \quad \operatorname{Specm} A/I = \{\mathfrak{m} \in \operatorname{Specm} A \mid I \subset \mathfrak{m}\}.$$

Proof. Both results follow from the fact that

$$\frac{A/I}{J/I} \cong \frac{A}{J}$$

for every ideal J which contains I. In order to prove this isomorphism, notice that since $I \subset J$, we obtain a morphism $f : A/I \to A/J$ (which is defined explicitly by $a + I \mapsto a + J$) by the universal property of quotients. Its kernel is given by

$$\ker f = \{a + I \mid a + J = J\} = \{a + I \mid a \in J\} = \frac{J}{I}.$$

The result now follows from the fact that f is surjective. □

A recurrent motif in these notes will be the fact that rings are fundamentally geometric objects. In fact, we'll endow the spectrum of a ring with a topology, and we'll see the elements of A as *functions* on $\operatorname{Spec} A$. The value of a function $f \in A$ at a point $\mathfrak{p} \in \operatorname{Spec} A$ is defined to be $f \pmod{\mathfrak{p}}$. This notion is weird: these functions have images in different rings. Even worse: their images don't determine the function. Nevertheless, we shall see that this is one of the most fruitful ideas in all of mathematics.

If these functions of $\operatorname{Spec} A$ are to be continuous, we should have that their zero-set is closed. In other words, if $S \subset A$ is a collection of functions, we hope that their *vanishing set*

$$V(S) := \{\mathfrak{p} \in \operatorname{Spec} A \mid f(\mathfrak{p}) = 0 \text{ for all } \mathfrak{p} \in S\} = \{\mathfrak{p} \in \operatorname{Spec} A \mid S \subset \mathfrak{p}\}$$

is closed. So, in order to define a topology on $\operatorname{Spec} A$ we impose this and nothing more. But first we need to check that these sets form a topology. In order to simplify our life, we observe that if (S) is the ideal generated by a subset $S \subset A$, then $V((S)) = V(S)$. This allows us to just consider ideals. Moreover, if f is an element of A, we denote by $V(f)$ the set $V((f))$.

Lemma 1.5.2 Let A be a ring. Then,

(a) $V(0) = \operatorname{Spec} A$ and $V(1) = \varnothing$;

(b) $V(I) \cup V(J) = V(IJ)$ for every pair of ideals $I, J \subset A$;

(c) $\bigcap_i V(I_i) = V(\sum_i I_i)$ for every family $\{I_i\}$ of ideals of A.

Proof. The first item is immediate. The second item follows from the fact that $IJ \subset \mathfrak{p}$ implies $I \in \mathfrak{p}$ or $J \in \mathfrak{p}$. The last one is just the fact that ideals are subgroups and so if $I_i \subset \mathfrak{p}$ for every i, then $\sum_i I_i \subset \mathfrak{p}$. □

By this lemma, the sets of the form $V(I)$, where I runs through the ideals of A, form the closed sets of a topology on $\operatorname{Spec} A$.

Definition 1.5.1 — Zariski topology. Let A be a ring. The topology on $\operatorname{Spec} A$ whose closed sets are of the form $V(S)$, for some subset $S \subset A$, is called the *Zariski topology*.

We now see some properties of the vanishing set. First of all, it is clear that it is *inclusion-reversing*: if $S_1 \subset S_2$, then $V(S_2) \subset V(S_1)$. Also, since $(I \cap J)^2 \subset IJ \subset I \cap J$, this implies that $V(I \cap J) = V(IJ)$. Finally, we have that $V(\sqrt{I}) = V(I)$.

If $f \in A$, the complement of $V(f)$ in $\operatorname{Spec} A$ is so important that it deserves a name.

Definition 1.5.2 Let A be a ring and $f \in A$. Then the subset $D(f)$ of $\operatorname{Spec} A$ defined by

$$D(f) := \{\mathfrak{p} \in \operatorname{Spec} A \mid f(\mathfrak{p}) \neq 0\} = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$$

is called a *distinguished open set*.

These subsets are important for the following reason.

Proposition 1.5.3 The distinguished open sets form a basis of the Zariski topology on $\operatorname{Spec} A$.

Proof. This follows from the fact that the complement of $V(S)$ is $\bigcup_{f \in S} D(f)$. □

This implies some important properties of the spectrum $\operatorname{Spec} A$ as a topological space.

Corollary 1.5.4 Let $\varphi : A \to B$ be a morphism of rings. Then $\operatorname{Spec} \varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is a continuous map.

Proof. We affirm that $(\operatorname{Spec} \varphi)^{-1}(D(f)) = D(\varphi(f))$. In fact $\mathfrak{p} \in (\operatorname{Spec} \varphi)^{-1}(D(f))$ means that $\varphi^{-1}(\mathfrak{p}) \in D(f)$. Unraveling the definition of the distinguished open set we see that this happens precisely when $h \notin \varphi^{-1}(\mathfrak{p}) \iff \varphi(h) \notin \mathfrak{p}$. But this is the very definition of $\mathfrak{p} \in D(\varphi(f))$. □

The preceding corollary implies that Spec is a functor from the category of commutative rings to the category of topological spaces. We'll see eventually that by endowing Spec with a sheaf of rings we get an equivalence of categories.

Corollary 1.5.5 If $\mathfrak{p} \in \operatorname{Spec} A$, then $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. In particular, $\mathfrak{m} \in \operatorname{Spec} A$ is a closed point if and only if $\mathfrak{m}$ is a maximal ideal. Moreover, if A is an integral domain, the ideal (0) is dense in $\operatorname{Spec} A$. We say that it is the *generic point* of $\operatorname{Spec} A$.

Proof. The first part follows from the fact that

$$\overline{\{\mathfrak{p}\}} = \bigcap_{\mathfrak{p} \in V(I)} V(I) = V\left(\sum_{I \subset \mathfrak{p}} I\right) = V(\mathfrak{p}).$$

The other two parts are clear. □

We see $\operatorname{Specm} A$ as the subset of $\operatorname{Spec} A$ composed of the closed points. The maximal spectrum inherits the subspace topology as usual. Also, in general we say that a point $\mathfrak{p} \in \operatorname{Spec} A$ is a generic point for a closed subset X if $X = \overline{\{\mathfrak{p}\}}$.

As we just observed, in a multitude of cases the spectrum of a ring is not Hausdorff. (Since we may have points which are not closed.) We say that $\operatorname{Spec} A$ is *quasi-compact* in the following corollary since we reserve the word *compact* to Hausdorff spaces.

Corollary 1.5.6 The topological space $\operatorname{Spec} A$ is quasi-compact.

Proof. Since the distinguished open sets form a basis of the Zariski topology, it suffices to consider a covering of the form $\operatorname{Spec} A = \bigcup_i D(f_i)$. This is equivalent to the fact that

$$V\left(\sum_i (f_i)\right) = \bigcap_i V(f_i) = \varnothing.$$

In particular, $\sum_i (f_i) = (1)$ for any other ideal is contained in a maximal (thus prime) ideal. We conclude that 1 is a finite linear combination of the f_i:

$$1 = a_1 f_{i_1} + \ldots + a_n f_{i_n}.$$

Finally, the same argument implies that $\operatorname{Spec} A = \bigcup_{k=1}^{n} D(f_{i_k})$. □

Finally, we see that the bijection given by proposition 1.5.1 is actually a homeomorphism.

Corollary 1.5.7 If I is an ideal of A, then the morphism

$$\operatorname{Spec} \pi : \operatorname{Spec} A/I \to V(I) \subset \operatorname{Spec} A,$$

induced by the projection $\pi : A \to A/I$, is an homeomorphism.

Proof. The corollary 1.5.4 says that $\operatorname{Spec} \pi$ is a continuous map and the proposition **??** implies that it is a bijection onto its image $V(I)$. It remains just to prove that it is a closed map. But this also follows from proposition **??** since the image of $V(J/I)$ by $\operatorname{Spec} \pi$ is just $V(J)$. □

In particular, since $V(0) = V(\sqrt{(0)})$, we have that $\operatorname{Spec} A$ and $\operatorname{Spec} A/\sqrt{(0)}$ are homeomorphic. This shows that the spectrum, as a topological space, is not enough to determine the ring. We'll soon see that the situation changes once we endow the spectrum with a sheaf of rings.

We can verify that the ideals of a product ring $A_1 \times A_2$ are precisely those of the form $I_1 \times I_2$, where $I_1 \subset A_1$ and $I_2 \subset A_2$ are ideals. Furthermore, since $(A_1 \times A_2)/(I_1 \times I_2) \cong (A_1/I_1) \times (A_2/I_2)$, the ideal $I_1 \times I_2$ is prime (resp. maximal) if and only if one of the I_i is the entire ring and the other is prime (resp. maximal). This implies the following result.

Proposition 1.5.8 If A_1 and A_2 are rings, then $\operatorname{Spec} A_1 \times A_2$ is homeomorphic to $\operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2$.

Proof. We have morphisms (induced by the projections from the product)

$$\begin{aligned} \operatorname{Spec} A_1 &\to \operatorname{Spec} A_1 \times A_2 \\ \mathfrak{p} &\mapsto \mathfrak{p} \times A_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \operatorname{Spec} A_2 &\to \operatorname{Spec} A_1 \times A_2 \\ \mathfrak{p} &\mapsto A_1 \times \mathfrak{p}, \end{aligned}$$

which are continuous by the corollary 1.5.4 and well-defined by the previous discussion. By the universal property of coproducts, we get a morphism

$$\operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2 \to \operatorname{Spec} A_1 \times A_2,$$

that is a continuous bijection between quasi-compact spaces. We conclude that it is a homeomorphism. ☐

In fact, this is the only way in which the spectrum can lack connectedness. This can be seen by doing messy calculations but it will be clear once we have a structure sheaf on $\operatorname{Spec} A$.[1]

We finish this section by calculating some spectra.

Proposition 1.5.9 Let I be a nonzero ideal of a principal ideal domain A. Then I is prime if and only if it is maximal.

Proof. Let's suppose that $I = (a)$ is prime. Now, if $I \subset J$ for some ideal $J = (b)$, then $a = bc$ for some $c \in A$. But then $b \in I$ or $c \in I$ by primality of I. If $b \in I$, then $J = (b) \subset I$, which means that $J = I$. Else, if $c \in I$, then $c = da$ for some $d \in A$ and so

$$a = bc = bda.$$

Since I is nonzero, $a \neq 0$ and so $bd = 1$. It follows that b is a unit, and hence $J = A$. ☐

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[1] For those who know it, if $X = \operatorname{Spec} A = U \cup V$, with $U \cap V = \varnothing$, then the sheaf axioms imply that $A = \mathscr{O}_X(X) = \mathscr{O}_X(U) \times \mathscr{O}_X(V)$.

■ **Example 1.5.1 — Principal ideal domains.** Let A be a PID. Since A is, in particular, a UFD, prime and irreducible elements are one and the same. It follows that

$$\operatorname{Spec} A = \{(0)\} \cup \{(a) \mid a \in A \text{ is irreducible}\}.$$

This implies that

$$\begin{aligned}
\operatorname{Spec} \mathbb{Z} &= \{(0)\} \cup \{(p) \mid p \text{ is prime}\} \\
\operatorname{Spec} \overline{k}[x] &= \{(0)\} \cup \{(x-a) \mid a \in \overline{k}\} \\
\operatorname{Spec} \mathbb{R}[x] &= \{(0)\} \cup \{(x-a) \mid a \in \mathbb{R}\} \cup \{(x^2+ax+b) \mid a^2-4b<0\} \\
\operatorname{Spec} \mathbb{Z}[i] &= \{(0)\} \cup \{(1+i)\} \\
&\quad \cup \{(p) \mid p \text{ is prime number with } p \equiv 3 \pmod 4\} \\
&\quad \cup \left\{ (a \pm bi) \;\middle|\; \begin{array}{c} p \text{ is a prime number with} \\ p \equiv 1 \pmod 4 \text{ and } a^2+b^2=p \end{array} \right\}.
\end{aligned}$$

By proposition 1.5.9, the maximal spectrum of a PID is exactly the same thing minus $\{(0)\}$. Also, the proposition 1.5.1 implies that

$$\begin{aligned}
\operatorname{Spec} \mathbb{Z}/(n) &= \{(\overline{p}) \mid p \text{ is prime factor of } n\} \\
\operatorname{Spec} \overline{k}[x]/(f) &= \{(\overline{x-a}) \mid a \in \overline{k},\ f(a)=0\}
\end{aligned}$$

for $n \in \mathbb{Z}$ and $f \in \overline{k}[x]$ nonzero. ■

1.6. Examples

2. Localization

Just as the quotient by an ideal I reduces the amount of prime ideals of A to just those that contain I, localization will reduce the amount of prime ideals of A to just those that are contained in a given prime $\mathfrak{p}$. This simplification is arguably one of the fundamental tools in commutative algebra.

On a more geometric standpoint, we saw before that the closed subsets of $\operatorname{Spec} A$ can be identified to the spectrum of some A-algebra. While we can't do the same for every open subset of $\operatorname{Spec} A$, in this chapter we'll define a A-algebra A_f whose spectrum $\operatorname{Spec} A_f$ is homeomorphic to the distinguished open set $D(f) \subset \operatorname{Spec} A$.

2.1. Definition and universal property

The definition of the localization is very similar to that of the ring of fractions of an integral domain.

Definition 2.1.1 — Localization. A multiplicative subset S of a ring A is a subset closed under multiplication containing 1. We define the *localization* $S^{-1}A$ of A at S as the quotient of $A \times S$ by the equivalence relation defined by $(a_1, s_1) \sim (a_1, s_2)$ if and only if there is $s \in S$ such that

$$s(s_2 a_1 - s_1 a_2) = 0.$$

We'll denote the pair (a, s) as a/s and define ring operations on $S^{-1}A$ in the natural way:

$$\frac{a_1}{s_1} + \frac{a_2}{s_2} := \frac{s_2 a_1 + s_1 a_2}{s_1 s_2} \qquad \text{and} \qquad \frac{a_1}{s_1} \cdot \frac{a_2}{s_2} := \frac{a_1 a_2}{s_1 s_2}.$$

It is easily seen that these operations are independent of any choices. We have a natural morphism of rings $A \mapsto S^{-1}A$ given by $a \mapsto a/1$.

The condition is not simply $s_2 a_1 - s_1 a_2 = 0$ to assure transitivity when A is not an integral domain. We also observe that the localization map $A \mapsto S^{-1}A$ is not necessarily injective. This is the case precisely when S has no zero-divisors.

We can also localize modules (in particular, ideals) and algebras in precisely the same way. If S is a multiplicative subset of a ring A and M is a A-module, the localization $S^{-1}M$ is the $S^{-1}A$-module whose elements are fractions m/s with $m \in M$

and $s \in S$ such that

$$\frac{m_1}{s_1} = \frac{m_2}{s_2} \in S^{-1}M \iff \text{there is } s \in S \text{ such that } s(s_2m_1 - s_1m_2) = 0 \in M.$$

The module operations are defined as

$$\frac{m_1}{s_1} + \frac{m_2}{s_2} := \frac{s_2m_1 + s_1m_2}{s_1s_2} \qquad \text{and} \qquad \frac{a}{s_1} \cdot \frac{m}{s_2} := \frac{am}{s_1s_2}.$$

As before, these operations are independent of any choices. The localization of an algebra is defined similarly.

There are two kinds of localization that we'll see all the time. We describe them in the next examples.

■ **Example 2.1.1** If $\mathfrak{p}$ is a prime ideal of a ring A, then $A \setminus \mathfrak{p}$ is a multiplicative subset. (In fact, this is equivalent to $\mathfrak{p}$ being prime.) We denote the localization of A at this multiplicative subset by $A_{\mathfrak{p}}$. Similarly, the localization of a A-module M at this multiplicative subset is denoted by $M_{\mathfrak{p}}$. ■

■ **Example 2.1.2** If f is an element of a ring A, then $\{1, f, f^2, \dots\}$ is a multiplicative subset. We denote the localization of A at this multiplicative subset by A_f. Similarly, the localization of a A-module M at this multiplicative subset is denoted by M_f. ■

We emphasize that if f is a prime element of a ring, then A_f and $A_{(f)}$ are *not* the same ring.

As usual, the localization satisfies a universal property. Just as in the case of the quotient ring, this universal property characterizes morphisms getting out of $S^{-1}A$.

Proposition 2.1.1 — Universal property of the localization. Let S be a multiplicative subset of a ring A. Then, if $\varphi : A \to B$ is a morphism of rings such that $\varphi(S) \subset B^\times$, then there exists a unique morphism $\tilde{\varphi} : S^{-1}A \to B$ such that the diagram

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr] & & B \\ & S^{-1}A \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

commutes. In other words, $S^{-1}A$ is initial among the A-algebras B such that every element of S is sent to a unit in B.

Proof. Since $\varphi(s)$ is always a unit, $a/s \mapsto \varphi(a)\varphi(s)^{-1}$ defines a morphism of rings $S^{-1}A \to B$ which makes the diagram commute. It is clear that this is well-defined. The unicity follows from the fact that this is an initial object. ☐

It is clear that the localization of modules satisfies a similar universal property.

As we saw, the localization map need not be injective. Nevertheless, the same proof that we used before to show that $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism shows that the localization map is always an epimorphism.

Proposition 2.1.2 Let S be a multiplicative subset of a ring A. Then the localization map $A \mapsto S^{-1}A$ is an epimorphism.

Proof. Let A' be a ring and consider two parallel morphisms $\varphi_1, \varphi_2 : S^{-1}A \to A'$:

$$\begin{tikzcd} A \arrow[r] & S^{-1}A \arrow[r, shift left, "\varphi_1"] \arrow[r, shift right, "\varphi_2"'] & A'. \end{tikzcd}$$

If this diagram commutes, then $\varphi_1(a/1) = \varphi_2(a/1)$ for every $a \in A$. In particular, this holds for every $s \in S$. Then,

$$\varphi_1\left(\frac{a}{s}\right) = \varphi_1\left(\frac{a}{1}\right)\varphi_1\left(\frac{s}{1}\right)^{-1} = \varphi_2\left(\frac{a}{1}\right)\varphi_2\left(\frac{s}{1}\right)^{-1} = \varphi_2\left(\frac{a}{s}\right)$$

for every $a/s \in S^{-1}A$. It follows that $A \mapsto S^{-1}A$ is an epimorphism. □

We end this section by an easy but useful result.

Proposition 2.1.3 Let S be a multiplicative subset of a ring A. Then, $S^{-1}A = 0$ if and only if $0 \in S$.

Proof. We have that $S^{-1}A = 0$ if and only if $0/1 = 1/1$ in $S^{-1}A$. That is, if and only if there exists $s \in S$ such that $s(1 \cdot 0 - 1 \cdot 1) = 0$ in A. This means precisely that $0 \in S$. □

2.2. The localization functor

Let S be a multiplicative subset of a ring A. We already know to associate a $S^{-1}A$-module $S^{-1}M$ to each A-module M. This construction is actually functorial since to each morphism $\varphi : M \to N$ of A-modules we have an induced morphism of $S^{-1}A$-modules

$$\begin{aligned} S^{-1}\varphi : S^{-1}M &\to S^{-1}N \\ \frac{m}{s} &\mapsto \frac{\varphi(m)}{s}. \end{aligned}$$

As usual, it is clear that this definition is independent of representatives. A notable fact is that this functor is exact.

Just before the proof of this result, we ought to make a simple remark. A functor F is exact if it sends short exact sequences to short exact sequences. It is clear that, in order to check if a functor is exact, it suffices to show that it sends an exact sequence of the form

$$M \xrightarrow{\varphi} N \xrightarrow{\psi} P$$

in a exact sequence of the form

$$F(M) \xrightarrow{F(\varphi)} F(N) \xrightarrow{F(\psi)} F(P).$$

What is not so evident is the converse, which is also true. Suppose that F is exact, and let

$$M \xrightarrow{\varphi} N \xrightarrow{\psi} P$$

be an exact sequence. Now, behold the following diagram.

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & & 0 \\
 & & & \operatorname{im}\varphi \arrow[ur] \arrow[dr, hook] & & & & P/\operatorname{im}\psi \arrow[ur] & \\
 & & M \arrow[ur, two heads, "\varphi"] \arrow[rr, "\varphi"] & & N \arrow[rr, "\psi"] \arrow[dr, two heads, "\psi"'] & & P \arrow[ur, two heads] & & \\
 & \ker\varphi \arrow[ur, hook] & & & & \operatorname{im}\psi \arrow[ur, hook] \arrow[dr] & & & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

The diagonal arrows form short exact sequences and the triangles commute. The image of this diagram by F has short exact sequences in the diagonals since F is exact. We can then diagram chase to prove that the middle terms are exact as well. The same argument holds with sequences of any size.

Theorem 2.2.1 — Localization is exact. Let S be a multiplicative subset of a ring A and

$$M \xrightarrow{\varphi} N \xrightarrow{\psi} P$$

be a exact sequence of A-modules. Then,

$$S^{-1}M \xrightarrow{S^{-1}\varphi} S^{-1}N \xrightarrow{S^{-1}\psi} S^{-1}P$$

is an exact sequence of $S^{-1}A$-modules.

Proof. Notice that $\operatorname{im} S^{-1}\varphi \subset \ker S^{-1}\psi$ since $S^{-1}\psi \circ S^{-1}\varphi = S^{-1}(\psi \circ \varphi) = 0$. In order to show the reverse inclusion, let $n \in N$ and $s \in S$ such that $n/s \in \ker S^{-1}\psi$. This implies that

$$\frac{\psi(n)}{s} = \frac{0}{1} \in S^{-1}P.$$

In other words, that there exists $t \in S$ such that $\psi(tn) = t\psi(n) = 0$. Since $\ker \psi = \operatorname{im} \varphi$, there exists $m \in M$ such that $\varphi(m) = tn$. Then,

$$S^{-1}\varphi\left(\frac{m}{ts}\right) = \frac{\varphi(m)}{ts} = \frac{n}{s},$$

which implies that $n/s \in \operatorname{im} S^{-1}\varphi$. ☐

This theorem has numerous important corollaries, which we now describe.

Corollary 2.2.2 Let S be a multiplicative subset of a ring A and $\varphi : M \to N$ a morphism of A-modules. Then,

$$\ker S^{-1}\varphi = S^{-1}\ker\varphi, \quad \operatorname{coker} S^{-1}\varphi = S^{-1}\operatorname{coker}\varphi, \quad \operatorname{im} S^{-1}\varphi = S^{-1}\operatorname{im}\varphi.$$

In particular, φ is injective or surjective if and only if $S^{-1}\varphi$ is.

Proof. We only prove that $\ker S^{-1}\varphi = S^{-1}\ker\varphi$ as the others are similar. Localizing the exact sequence of A-modules

$$\begin{tikzcd} 0 \arrow[r] & \ker\varphi \arrow[r, hook] & M \arrow[r, "\varphi"] & N \end{tikzcd}$$

we get that

$$\begin{tikzcd} 0 \arrow[r] & S^{-1}\ker\varphi \arrow[r, hook] & S^{-1}M \arrow[r, "S^{-1}\varphi"] & S^{-1}N \end{tikzcd}$$

is an exact sequence of $S^{-1}A$-modules. In particular, $\ker S^{-1}\varphi = S^{-1}\ker\varphi$. ☐

Likewise, localizations commute with quotients.

Corollary 2.2.3 If N is a submodule of M, then $S^{-1}(M/N) = (S^{-1}M)/(S^{-1}N)$.

Proof. Since the following sequence is exact

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, hook] & N \arrow[r, two heads] & M/N \arrow[r] & 0 \end{tikzcd}$$

the exactness of localization implies that

$$\begin{tikzcd} 0 \arrow[r] & S^{-1}M \arrow[r, hook] & S^{-1}N \arrow[r, two heads] & S^{-1}(M/N) \arrow[r] & 0 \end{tikzcd}$$

is exact, which implies our result. ☐

If $M = A$ and I is an ideal of A, we get that $S^{-1}(A/I)$ and $(S^{-1}A)/(S^{-1}I)$ are isomorphic as A-modules. The next proposition shows that they are isomorphic as rings. Its proof is a well-played game of universal properties.

Proposition 2.2.4 Let I be an ideal and S be a multiplicative subset of a ring A. Then $S^{-1}I$ is an ideal of $S^{-1}A$ and

$$\overline{S}^{-1}(A/I) = (S^{-1}A)/(S^{-1}I),$$

where $\overline{S}$ is the image of S in A/I.

Proof. Its clear that $S^{-1}I$ is an ideal of $S^{-1}A$. We define

$$\begin{aligned} \varphi : S^{-1}A &\to \overline{S}^{-1}(A/I) \\ a/s &\mapsto \overline{a}/\overline{s}, \end{aligned}$$

where $\overline{a}$ and $\overline{s}$ are the images of a and s in A/I. This morphism is well-defined by the universal property of localization and $S^{-1}I$ is contained in its kernel. The universal property of quotients then gives a morphism

$$\begin{aligned} \tilde{\varphi} : (S^{-1}A)/(S^{-1}I) &\to \overline{S}^{-1}(A/I) \\ \overline{a/s} &\mapsto \overline{a}/\overline{s}. \end{aligned}$$

On the other hand, the composition $A \to S^{-1}A \to (S^{-1}A)/(S^{-1}I)$ induces a morphism

$$\begin{aligned} \psi : A/I &\to (S^{-1}A)/(S^{-1}I) \\ \overline{a} &\mapsto \overline{a/1}. \end{aligned}$$

The image of $\overline{S}$ is invertible and so the universal property of localization gives a morphism

$$\begin{aligned} \tilde{\psi} : \overline{S}^{-1}(A/I) &\to (S^{-1}A)/(S^{-1}I) \\ \overline{a} &\mapsto \overline{a/1}, \end{aligned}$$

which is clearly the inverse of $\tilde{\varphi}$. It follows that they are both isomorphisms. ☐

In a similar fashion, we leave to the reader the task of proving that if $\{M_i\}$ is a (possibly infinite) collection of A-modules and if M_1, M_2 are submodules of a A-module M, then

$$S^{-1}\left(\bigoplus_i M_i\right) = \bigoplus_i S^{-1}M_i \qquad \text{and} \qquad S^{-1}(M_1 \cap M_2) = S^{-1}M_1 \cap S^{-1}M_2.$$

Nevertheless, localization does *not* commute with arbitrary products. Moreover, if I and J are ideals of A, then

$$S^{-1}(IJ) = (S^{-1}I)(S^{-1}J) \qquad \text{and} \qquad \sqrt{S^{-1}I} = S^{-1}\sqrt{I}.$$

In particular, the nilradical of $S^{-1}A$ is $S^{-1}\sqrt{(0)}$.

In some cases, we also have a sort of converse of the theorem 2.2.1.

Theorem 2.2.5 — Local-global principle. Let A be a ring and M be a A-module. Then the following are equivalent:

(a) $M = 0$;

(b) $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} A$;

(c) $M_{\mathfrak{m}} = 0$ for all $\mathfrak{m} \in \operatorname{Specm} A$.

Proof. It is clear that $(a) \implies (b) \implies (c)$, so we prove $(c) \implies (a)$. If $m \in M \setminus \{0\}$, then the *annihilator* ideal

$$\operatorname{Ann}(m) := \{a \in A \mid am = 0\}$$

doesn't contain $1 \in A$ and so is contained in a maximal ideal $\mathfrak{m} \in \operatorname{Specm} A$. But the hypothesis that $M_{\mathfrak{m}} = 0$ implies that $m/1 = 0/1$ and so there exists $s \in A \setminus \mathfrak{m}$ such that $sm = 0$. This contradicts the fact that $\operatorname{Ann}(m) \subset \mathfrak{m}$. ☐

We will often use the local-global principle in the following form.

Corollary 2.2.6 Let $\varphi : M \to N$ be a morphism of A-modules. Then the following are equivalent:

(a) $\varphi : M \to N$ is injective (resp. surjective);

(b) $\varphi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is injective (resp. surjective) for all $\mathfrak{p} \in \operatorname{Spec} A$;

(c) $\varphi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective (resp. surjective) for all $\mathfrak{m} \in \operatorname{Specm} A$.

Proof. This follows from corollary 2.2.2 and the preceding theorem applied to the A-modules $\ker \varphi$ and $\operatorname{coker} \varphi$. ☐

We finish this section considering how submodules behave in localization.

Proposition 2.2.7 Let N' be a submodule of $S^{-1}M$. Then, $N' = S^{-1}N$ for some submodule N of M.

In fact, if $\rho : M \to S^{-1}M$ is the localization map, we can take $N = \rho^{-1}(N')$.

Proof. We show that $S^{-1}N \subset N'$. In fact, if $n/s \in S^{-1}N$, then $n/s = (1/s)\rho(n) \in N'$ since N' is a $S^{-1}A$-module. As for the reverse inclusion, if $n'/s \in N'$, then $\rho(n') = (s/1) \cdot (n'/s) \in N'$ by the same reason. It follows that $n' \in N$ and so $n'/s \in S^{-1}N$. ☐

In particular, the ideals of $S^{-1}A$ are all of the form $S^{-1}I$ for some ideal $I \subset A$.

2.3. The spectrum of the localized ring

Just as with quotients, we have an order-preserving bijection between $\operatorname{Spec} S^{-1}A$ and a subset of $\operatorname{Spec} A$.

Proposition 2.3.1 Let S be a multiplicative subset of a ring A. Then the localization map $A \to S^{-1}A$ induces an order-preserving bijection

$$\{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \varnothing\} \to \operatorname{Spec} S^{-1}A$$

given by $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$. In other words, the natural order-preserving map

$$\operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$$

is injective and has image $\{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \varnothing\}$.

Proof. Let $\rho : A \to S^{-1}A$ be the localization map. As we saw in the proposition 2.2.7, $\operatorname{Spec} \rho$ is injective.[1] It suffices then to show that its image is $\{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \varnothing\}$. Now, if $\mathfrak{p} \in \operatorname{Spec} A$, the proposition 2.2.4 implies that

$$(S^{-1}A)/(S^{-1}\mathfrak{p}) = \overline{S}^{-1}(A/\mathfrak{p}).$$

Since $A/\mathfrak{p}$ is an integral domain, $\overline{S}^{-1}(A/\mathfrak{p})$ is either contained in a field, in which case $S^{-1}\mathfrak{p}$ is prime, or is zero. The latter case happens precisely when $\overline{S}$ contains 0. That is, when $\mathfrak{p} \cap S \neq \varnothing$. ☐

With this, we can describe the spectra of the two main flavors of localization.

Corollary 2.3.2 If $\mathfrak{p} \in \operatorname{Spec} A$, we have a bijection

$$\begin{aligned} \{\mathfrak{q} \in \operatorname{Spec} A \mid \mathfrak{q} \subset \mathfrak{p}\} &\to \operatorname{Spec} A_{\mathfrak{p}} \\ \mathfrak{q} &\mapsto \mathfrak{q}A_{\mathfrak{p}}. \end{aligned}$$

Proof. This follows directly from the previous proposition. ☐

Corollary 2.3.3 If $f \in A$, we have a homeomorphism

$$\begin{aligned} D(f) &\to \operatorname{Spec} A_f \\ \mathfrak{p} &\mapsto \mathfrak{p}_f = \mathfrak{p}A_f. \end{aligned}$$

Proof. Let $\rho : A \to A_f$ be the localization map. By the previous proposition, this $\operatorname{Spec} \rho : \operatorname{Spec} A_f \to D(f)$ is a bijection. Since it is automatically continuous, it suffices

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[1] The equation $\mathfrak{p} = S^{-1}(\rho^{-1}(\mathfrak{p}))$ implies that $\operatorname{Spec} \rho$ has a left-inverse and so is injective.

to show that it is open. Now, let $D(a/f^n) \in \operatorname{Spec} A_f$ be a distinguished open set. Since $a/1$ and a/f^n differ by a unit, $D(a/f^n) = D(a/1)$. By the proof of corollary 1.5.4 and the fact that $\operatorname{Spec} \rho$ is injective, we have that the image of $D(a/1)$ in $\operatorname{Spec} A$ is precisely $D(a)$. This implies our result. ☐

This establishes what we promised in the introduction of this chapter. If $\mathfrak{p} \in \operatorname{Spec} A$, we have that

$$\begin{aligned} \operatorname{Spec} A/\mathfrak{p} &= \{\mathfrak{q} \in \operatorname{Spec} A \mid \mathfrak{q} \supset \mathfrak{p}\} \\ \operatorname{Spec} A_{\mathfrak{p}} &= \{\mathfrak{q} \in \operatorname{Spec} A \mid \mathfrak{q} \subset \mathfrak{p}\}. \end{aligned}$$

In this way, combining localizations and quotients, we can "filter" any set of primes that we with to study. We'll now rephrase this observation to obtain a very useful result. But for that we need a definition.

Definition 2.3.1 — Dimension and height. Let A be a ring. The *Krull dimension* of A, denoted $\dim A$, is the size n of the biggest chain of prime ideals (numbered from 0 to n)

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_n.$$

If there are arbitrarily big chains of prime ideals in A, we say that $\dim A = \infty$. If $\mathfrak{p} \in \operatorname{Spec} A$, we say that its *height* is $\operatorname{ht} \mathfrak{p} := \dim A_{\mathfrak{p}}$. In other words, the height of a prime ideal $\mathfrak{p} \subset A$ is the size n of the biggest chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$$

which are contained in $\mathfrak{p}$.

Observe that the size of a chain of prime ideals is not *the number of prime ideals, but the number of inclusions!*

We gather a couple of examples.

■ **Example 2.3.1** If k is a field, then $\dim k = 0$ since (0) is a maximal chain of primes. As we saw in example 1.5.1, if A is a PID that is not a field, then $\dim A = 1$. In particular, $\dim \mathbb{Z} = \dim k[x] = 1$. In that same example, we saw that $\dim \mathbb{Z}/(n) = \dim \overline{k}[x]/(f) = 0$ for $n \in \mathbb{Z}$ and $f \in \overline{k}[x]$ nonzero.

In general, the dimension of $k[x_1, \dots, x_n]$ is n, as expected. But we'll have to wait awhile to prove it. ■

As promised, our preceding discussion yields the following corollary.

Corollary 2.3.4 Let A be a ring and $\mathfrak{p} \in \operatorname{Spec} A$. Then,

$$\dim A \geqslant \operatorname{ht} \mathfrak{p} + \dim A/\mathfrak{p}.$$

Proof. Concatenating the biggest chains of prime ideals in $A_\mathfrak{p}$ and in $A/\mathfrak{p}$ we obtain a chain of prime ideals in A. The biggest ought to be at least as big as this one. ☐

2.4. Radical ideals and spectra

In this section, we introduce a function $I(\cdot)$ which takes subsets of $\operatorname{Spec} A$ to ideals of A. It is, in some sense, the inverse of the vanishing set function. As we shall see, their interplay will yield lots of algebraically and geometrically important results.

Definition 2.4.1 Let A be a ring and S be a subset of $\operatorname{Spec} A$. We define $I(S)$ to be the set of functions vanishing on S. In other words,

$$I(S) = \bigcap_{\mathfrak{p} \in S} \mathfrak{p} \subset A.$$

It is clear that $I(S)$ is an ideal of A, that $I(\cdot)$ is inclusion-reversing and that $I(\overline{S}) = I(S)$.

An ideal $J \subset A$ is said to be *radical* if $\sqrt{J} = J$. A important fact is that $I(S)$ is always radical. In fact, if $f \in \sqrt{I(S)}$, then f^n vanishes on S for some $n > 0$. This implies that f vanishes on S and so $f \in I(S)$. The fact that $I(S) \subset \sqrt{I(S)}$ is clear. As a matter of fact, the nilradical of a ring can be written using this function. Behold a clever proof!

Proposition 2.4.1 Let A be a ring. Then,

$$I(\operatorname{Spec} A) = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} = \sqrt{(0)}.$$

Proof. Since 0 is in every prime ideal, it is clear that $\sqrt{(0)}$ is contained in the intersection of all the primes. Conversely, if f is in the intersection of all primes, then A_f has no prime ideals. This implies that A_f is the zero-ring and so $0 = 1$ in A_f. In other words, there exists $n > 0$ such that $f^n = 0$. ☐

We now begin to see that $I(\cdot)$ is some sort of inverse of $V(\cdot)$.

Corollary 2.4.2 If J is an ideal of A, then

$$I(V(J)) = \bigcap_{J \subset \mathfrak{p}} \mathfrak{p} = \sqrt{J}.$$

Proof. Since an element $\overline{f} \in A/J$ is nilpotent if and only if $f \in \sqrt{J}$, it follows that the nilradical of A/J is $\sqrt{J}/J$. We then apply the preceding proposition to A/J:

$$I(\operatorname{Spec} A/J) = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A/J} \mathfrak{p} = \frac{\sqrt{J}}{J}.$$

The result now follows from the correspondence between the ideals of A/J and of A and the fact that $\operatorname{Spec} A/J = V(J)$ under this correspondence. ☐

Continuing on the idea that $I(\cdot)$ is some sort of inverse of $V(\cdot)$, we get the result below.

Proposition 2.4.3 Let S be a subset of $\operatorname{Spec} A$. Then $V(I(S)) = \overline{S}$.

Proof. This follows direct from the fact that

$$\overline{S} = \bigcap_{S \subset V(J)} V(J) = V\left(\sum_{J \subset I(S)} J \right) = V(I(S)),$$

where we observe that $S \subset V(J)$ means that $J \subset \mathfrak{p}$ for every $\mathfrak{p} \in S$. Thus $J \subset I(S)$. ☐

We finally obtain our desired result.

Theorem 2.4.4 Let A be a ring. Then we have a one-on-one, inclusion-reversing, correspondence

$$\left\{ \begin{array}{c} \text{Radical} \\ \text{ideals of } A \end{array} \right\} \underset{I(\cdot)}{\overset{V(\cdot)}{\rightleftarrows}} \left\{ \begin{array}{c} \text{Closed subsets} \\ \text{of } \operatorname{Spec} A \end{array} \right\}.$$

Proof. This is just a reinterpretation of the two previous results. ☐

We have a couple of corollaries of this result.

Corollary 2.4.5 Let A be a ring, J_1, J_2 be ideals of A and $f, g \in A$. Then,

$$V(J_1) \subset V(J_2) \iff \sqrt{J_2} \subset \sqrt{J_1} \iff J_2 \subset \sqrt{J_1}$$

and

$$D(f) \subset D(g) \iff \sqrt{(f)} \subset \sqrt{(g)} \iff f \in \sqrt{(g)}.$$

Proof. If $V(J_1) \subset V(J_2)$, we have that

$$\sqrt{J_2} = I(V(J_2)) \subset I(V(J_1)) = \sqrt{J_1}.$$

For the converse, $\sqrt{J_2} \subset \sqrt{J_1}$ implies that

$$V(J_1) = V(\sqrt{J_1}) \subset V(\sqrt{J_2}) = V(J_2)$$

since V reverses inclusions. Also, if $J_2 \subset \sqrt{J_1}$, then

$$V(J_1) = V(\sqrt{J_1}) \subset V(J_2).$$

The first part now implies that $\sqrt{J_2} \subset \sqrt{J_1}$. The converse is obvious as $J_2 \subset \sqrt{J_2}$. As for the results involving the distinguished open sets, they follow from what we already proved applied to $J_1 = (g)$ and $J_2 = (f)$. ☐

In the rest of this section, we'll see that we can restrict the correspondence of theorem 2.4.4 to obtain two other interesting correspondences. For that we need a topological definition.

Definition 2.4.2 — Irreducible space. A topological space is said to be *irreducible* if it is nonempty, and it is not the union of two proper closed subsets. In other words, a nonempty topological space X is irreducible if whenever $X = Y \cup Z$ with Y and Z closed in X, we have that $Y = X$ or $Z = X$.

Moreover, an *irreducible component* of a topological space is a maximal subspace that is irreducible for the induced topology.

These topological spaces have numerous peculiar properties. For example, in an irreducible space X, every nonempty open set is dense. In fact, if there were a nonempty open set which was not dense, then we could take its complement as Y and its closure as Z, contradicting the definition of irreducibility.

We can prove that if $T \subset X$ is irreducible, then so is its closure in X. In particular, irreducible components are closed. Also, it is clear that any irreducible space is connected (as the intersection of two nonempty open subsets is necessarily not empty). This implies that the irreducible components are included in the connected components.

Since the irreducible components of a Hausdorff space are the singletons (if a irreducible component had two points, then no open set would separate them), irreducible spaces are ubiquitous in algebraic geometry but not very popular outside of it.

We present our main example and a counter-example.

■ **Example 2.4.1** Let A be an integral domain. Then $\operatorname{Spec} A$ is irreducible. In fact, if $\operatorname{Spec} A = V(I) \cup V(J)$, then (0) ought to be in one of those vanishing sets. If $(0) \in V(I)$ we have that $I \subset (0)$ which implies that $V(I) = \operatorname{Spec} A$. Similarly, $(0) \in V(J)$ implies that $V(J) = \operatorname{Spec} A$. ■

■ **Example 2.4.2** Let k be a field and consider the ring $A = k[x,y]/(xy)$. If $\mathfrak{p} \in \operatorname{Spec} A$, then $\overline{xy} = 0 \in \mathfrak{p}$. In other words, $\overline{x} \in \mathfrak{p}$ or $\overline{y} \in \mathfrak{p}$. This means that

$$\operatorname{Spec} A = V(\overline{x}) \cup V(\overline{y}) = \operatorname{Spec} \frac{k[x,y]}{(x)} \cup \operatorname{Spec} \frac{k[x,y]}{(y)}.$$

We conclude that $\operatorname{Spec} A$ is reducible as it is the union of two proper closed subsets. As each of those closed subsets is homeomorphic to $\operatorname{Spec} k[x]$, they are the irreducible components of $\operatorname{Spec} A$. ■

In general, since $\operatorname{Spec} A = \operatorname{Spec} A/\sqrt{(0)}$ we have no hope that irreducibility of the spectrum implies that the ring is an integral domain. However, we have the next best thing.

Lemma 2.4.6 If $\operatorname{Spec} A$ is irreducible and A is reduced, then A is an integral domain.

Proof. Suppose that we have $fg = 0$ with f and g nonzero. Notice that corollary 2.4.5 implies that $D(f)$ is empty if and only if f is nilpotent, which is not the case here since $f \neq 0$ and A is reduced. Similarly, $D(g) \neq \varnothing$ but $D(f) \cap D(g) = D(fg) = D(0) = \varnothing$. This contradicts the fact that $\operatorname{Spec} A$ is irreducible, since in an irreducible space the intersection of two nonempty subsets is not empty. ☐

Corollary 2.4.7 Let A be a ring. Then we have a one-on-one, inclusion-reversing, correspondence

$$\left\{\begin{array}{c}\text{Prime}\\ \text{ideals of } A\end{array}\right\} \underset{I(\cdot)}{\overset{V(\cdot)}{\rightleftarrows}} \left\{\begin{array}{c}\text{Irreducible closed}\\ \text{subsets of } \operatorname{Spec} A\end{array}\right\}.$$

In other words, we have bijection between points of $\operatorname{Spec} A$ and irreducible closed subsets which is given by $\mathfrak{p} \mapsto \overline{\{\mathfrak{p}\}}$. This shows that each irreducible closed subset has precisely one generic point.

Proof. We have to prove that $V(J)$, where J is radical, is irreducible if and only if J is a prime ideal. Since $V(J)$ is homeomorphic to $\operatorname{Spec} A/J$, which is reduced by the assumption that J is radical, this follows from the preceding lemma. ☐

This corollary allows us to define dimension in a more general setting.

Definition 2.4.3 — Dimension of a topological space. Let X be a topological space. The *dimension* of X, denoted $\dim X$, is the size n of the biggest chain of irreducible closed subsets (numbered from 0 to n)

$$Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq \cdots \subsetneq Z_n.$$

If there are arbitrarily big chains of irreducible closed subsets in X, we say that $\dim X = \infty$.

From the inclusion-reversing correspondence between prime ideals of A and irreducible closed subsets of $\operatorname{Spec} A$, it follows that $\dim A = \dim \operatorname{Spec} A$.

It's possible to restrict our correspondence yet one more time to obtain another nice result. We need another definition for that.

Definition 2.4.4 — Minimal primes. A prime ideal is a *minimal prime* if it is minimal with respect to inclusion.

For example, in an integral domain (0) is a minimal prime. Also, we know that the nilradical of a ring is the intersection of all its prime ideals. It is clear from the definition that we can take only the minimal ones in this intersection.

We get another correspondence!

Corollary 2.4.8 Let A be a ring. Then we have a one-on-one, inclusion-reversing, correspondence

$$\left\{\begin{array}{c}\text{Minimal prime}\\ \text{ideals of } A\end{array}\right\} \underset{I(\cdot)}{\overset{V(\cdot)}{\rightleftarrows}} \left\{\begin{array}{c}\text{Irreducible com-}\\ \text{ponents of } \operatorname{Spec} A\end{array}\right\}.$$

Proof. Since this correspondence is inclusion-reversing, maximal elements in one side become minimal elements in the other. The result now follows from the fact that the irreducible components are the maximal subspaces that are irreducible. □

An important fact about minimal primes, proved by Emmy Noether, is that a Noetherian ring only has a finite number of minimal prime ideals. As we shall see, this is, surprisingly, a geometric result.

Definition 2.4.5 — Noetherian space. A topological space X is said to be *Noetherian* if it satisfies the descending chain condition for closed subsets. That is, for any sequence

$$Z_1 \supset Z_2 \supset Z_3 \supset \cdots$$

of closed sets in X, there's an integer n such that $Z_m = Z_n$ for all $m \geqslant n$.

Similarly to the case of Noetherian rings, a topological space X is Noetherian if and only if every nonempty set of closed subsets of X has a minimal element relative to inclusion.

This nomenclature can be explained by the fact that the spectrum of a Noetherian ring A is Noetherian. In fact, if

$$Z_1 \supset Z_2 \supset Z_3 \supset \cdots$$

is a sequence of closed subsets of $\operatorname{Spec} A$, then

$$I(Z_1) \subset I(Z_2) \subset I(Z_3) \subset \cdots$$

is an ascending chain of ideals of A. Since A is Noetherian, this sequence stabilizes. But, since the Z_i are closed, $V(I(Z_i)) = Z_i$ and so our original sequence also stabilizes.

Nevertheless, there are non-Noetherian rings with Noetherian spectrum.

■ **Example 2.4.3** Let k be a field and consider

$$A = \frac{k[x_1, x_2, \ldots]}{(x_1^2, x_2^2, \ldots)}.$$

Since each variable is nilpotent, every prime must contain $I = (x_1, x_2, \ldots)$. But $A/I = k$ and so this ideal is maximal. It follows that $\operatorname{Spec} A$ has only one point, thus is trivially a Noetherian space. But I is not finitely generated, hence A is not Noetherian. ■

The most important property of Noetherian topological spaces is the following.

Proposition 2.4.9 Let X be a Noetherian topological space. Then every nonempty closed subset Z can be expressed uniquely as a finite union

$$Z = Z_1 \cup \cdots \cup Z_n$$

of irreducible closed subsets, none contained in any other.

Proof. Let S be the set of all nonempty closed subsets of X which cannot be expressed as a finite union of irreducible closed subsets. We'll show that S is empty. Lets suppose it's nonempty and let $Z \in S$ be a minimal element. Since Z is not irreducible, we write $Z = Z_1 \cup Z_2$, where Z_1 and Z_2 are proper closed subsets of Z. By minimality, both Z_1 and Z_2 can be written as a finite union of irreducible closed subsets, thus so can Z. This contradicts the minimality of Z. We conclude that every nonempty closed subset of X can be written as a finite union of irreducible closed subsets. By throwing away a few, if necessary, we can suppose that none are contained in any other.

We now show uniqueness. Suppose that

$$Z = Z_1 \cup \cdots \cup Z_n = Z_1' \cup \cdots \cup Z_m'$$

are two such representations. Since $Z_1' \subset Z_1 \cup \cdots \cup Z_n$, we have that

$$Z_1' = (Z_1 \cap Z_1') \cup \cdots \cup (Z_n \cap Z_1').$$

But Z_1' is irreducible, so one of these factors is Z_1' itself. Without loss of generality we suppose that it is $Z_1 \cap Z_1'$. Thus $Z_1' \subset Z_1$. Similarly, $Z_1 \subset Z_r'$ for some r. As $Z_1' \subset Z_1 \subset Z_r'$, and Z_1' is contained in no other Z_i', we must have $r = 1$ and $Z_1' = Z_1$. The result follows. □

Last but not least, we prove Noether's result as promised.

Corollary 2.4.10 — Noether. Let A be a Noetherian ring. Then A has a finite number of minimal prime ideals.

Proof. Since A is Noetherian, $\operatorname{Spec} A$ is a Noetherian topological space and so has a finite number of irreducible components. As these irreducible components are in bijection with the minimal primes, the result follows. □

2.5. Local rings and Nakayama's lemma

Let $\mathfrak{p}$ be a prime ideal of a ring A. By the order-preserving bijection between the prime ideals of $A_{\mathfrak{p}}$ and the prime ideals of A which are contained in $\mathfrak{p}$, it follows that $A_{\mathfrak{p}}$ has precisely one maximal ideal. Namely, $\mathfrak{p}A_{\mathfrak{p}}$. These kind of rings which only have one maximal ideal are ubiquitous in algebraic geometry and in number theory, so we'll study them now.

Definition 2.5.1 A ring A is said to be *local* if it has a single maximal ideal $\mathfrak{m}$. In this case, the field $A/\mathfrak{m}$ is said to be the *residue field of* A. Moreover, if A and B are local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, respectively, we say that a morphism $\varphi : A \to B$ is *local* if $\varphi(\mathfrak{m}) \subset \mathfrak{n}$.

Observe that $\operatorname{Spec} \varphi(\mathfrak{n}) = \varphi^{-1}(\mathfrak{n})$ is a prime ideal of A and so is contained in $\mathfrak{m}$. If φ is local, then $\mathfrak{m} \subset \varphi^{-1}(\mathfrak{n})$ and so $\operatorname{Spec} \varphi(\mathfrak{n}) = \mathfrak{m}$. This is an equivalent way to define local morphisms. We also notice that a local morphism induces a morphism between the residue fields

$$\begin{aligned} A/\mathfrak{m} &\to B/\mathfrak{n} \\ a \bmod \mathfrak{m} &\mapsto \varphi(a) \bmod \mathfrak{n}. \end{aligned}$$

In other words, $A/\mathfrak{m} \hookrightarrow B/\mathfrak{n}$ is a field extension.

The following is a useful criterion to identifying local rings.

Proposition 2.5.1 A ring A is local if and only if the set $A \setminus A^\times$ is an ideal, in which case it is the unique maximal ideal.

Proof. If A is a local ring with maximal ideal $\mathfrak{m}$, corollary **??** says precisely that $\mathfrak{m} = A \setminus A^\times$. Conversely, suppose that $A \setminus A^\times$ is an ideal of A. Every proper ideal of A does not contain any units and so is contained in $A \setminus A^\times$. This shows that $A \setminus A^\times$ is the unique maximal ideal of A. □

Also, given a ring A and a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, there are two natural associated fields: the residue field of $A_\mathfrak{p}$ and the fraction field of $A/\mathfrak{p}$. It's reassuring to know that they are one and the same.

Proposition 2.5.2 Let A be a ring and $\mathfrak{p} \in \operatorname{Spec} A$. Then the residue field of $A_\mathfrak{p}$ coincides with the fraction field of $A/\mathfrak{p}$.

Proof. Since $\mathfrak{p}A_\mathfrak{p} = S^{-1}\mathfrak{p}$, where $S = A \setminus \mathfrak{p}$, we have that

$$\frac{A_\mathfrak{p}}{\mathfrak{p}A_\mathfrak{p}} = \frac{S^{-1}A}{S^{-1}\mathfrak{p}} = \overline{S}(A/\mathfrak{p}),$$

where $\overline{S}$ is the image of $A \setminus \mathfrak{p}$ in $A/\mathfrak{p}$. In other words, $\overline{S}$ is the set of nonzero elements of $A/\mathfrak{p}$, which implies that $\overline{S}(A/\mathfrak{p}) = \operatorname{Frac}(A/\mathfrak{p})$. □

Definition 2.5.2 Let A be a ring and $\mathfrak{p} \in \operatorname{Spec} A$. The *residue field of* A *at* $\mathfrak{p}$ is $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = \operatorname{Frac}(A/\mathfrak{p})$. We denote it by $\kappa(\mathfrak{p})$.

We now present some fundamental examples of local rings, along with their main properties.

■ **Example 2.5.1** Every field k is a local ring. Moreover, for every $n > 0$, the ring $A = k[x]/(x^n)$ is local. In fact, a maximal ideal of A would be a maximal ideal I of $k[x]$ containing (x^n). Since $(x)^n \subset I$ and I is prime, it follows that $(x) \subset I$. But (x) is a maximal ideal of $k[x]$. It follows that $I = (x)$ is the only maximal ideal of A. ■

■ **Example 2.5.2 — Formal power series.** Let A be a ring. The define the ring of formal power series $A[\![x]\!]$ by the limit

$$A[\![x]\!] := \varprojlim \frac{A[x]}{(x^n)},$$

indexed by the natural numbers as usually ordered, with morphisms $A[x]/(x^{n+m}) \to A[x]/(x^n)$ given by the natural projections. In other words,

$$A[\![x]\!] = \left\{ (f_n) \in \prod_{n=1}^{\infty} \frac{A[x]}{(x^n)} \;\middle|\; f_m \equiv f_n \bmod x^m \text{ for all } n \geqslant m \right\}.$$

The elements of $A[\![x]\!]$ are of the form

$$(a_0 \bmod x, a_0 + a_1 x \bmod x^2, a_0 + a_1 x + a_2 x^2 \bmod x^3, \dots)$$

and so are usually denoted by $a_0 + a_1 x + a_2 x^2 + \cdots$. We'll now study some of the properties of this important ring.

First of all, of A is an integral domain, then so is $A[\![x]\!]$. In fact, given two nonzero elements $\sum a_n x^n$ and $\sum b_n x^n$ in $A[\![x]\!]$, let i, j be the smallest indices so that $a_i \neq 0$ and $b_j \neq 0$. Then the coefficient of x^{i+j} in the product $(\sum a_n x^n)(\sum b_n x^n)$ is $a_i b_j$, which is nonzero if A is an integral domain.

Also, the group of units of $A[\![x]\!]$ is precisely

$$A[\![x]\!]^\times = \{a_0 + a_1 x + a_2 x^2 + \cdots \in A[\![x]\!] \mid a_0 \in A^\times\}.$$

In fact, an element $\sum a_n x^n \in A[\![x]\!]$ is a unit if and only if there are b_i's such that

$$\begin{aligned} 1 &= (a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots) \\ &= a_0 b_0 + (a_1 b_0 + a_0 b_1) x + (a_2 b_0 + a_1 b_1 + a_0 b_2) x^2 + \cdots . \end{aligned}$$

From the first term, we can see that if $\sum a_n x^n$ is a unit, then $a_0 \in A^\times$. Conversely, if $a_0 \in A^\times$, then we can recursively solve this equation: we let $b_0 = a_0^{-1}$, $b_1 = -a_0^{-1} a_1 b_0$, and so on.

Iterating the construction of this ring, we obtain the ring of formal power series in multiple variables $A[\![x_1, \dots, x_n]\!]$ which is defined by $A[\![x_1]\!] \cdots [\![x_n]\!]$. The two previous properties generalize in the same way to this ring: if A is an integral domain, then so is $A[\![x_1,\dots,x_n]\!]$ and the group $A[\![x_1,\dots,x_n]\!]^\times$ is constituted by those power series which have units in the first coefficient.

We now specialize to the case $A = k$, where k is a field. In this case it is clear that $k[\![x]\!] \setminus k[\![x]\!]^\times = (x)$ and so $k[\![x]\!]$ is a local ring with maximal ideal (x). Similarly, $k[\![x_1,\dots,x_n]\!]$ is a local ring with maximal ideal $(x_1,\dots,x_n)$.

A final important property of $k[\![x]\!]$ is the fact that it is a principal ideal domain. For that we observe that every nonzero element $f \in k[\![x]\!]$ can be written in a unique way as

$$f = x^n \times \underbrace{(a_n + a_{n+1}x + a_{n+1}x^n + \cdots)}_{\text{a unit}}, \quad \text{where } a_n \neq 0.$$

This is the factorization of f into irreducible elements. Now, let I be a nonzero ideal of $k[\![x]\!]$ and let n be the smallest positive integer such that $x^n \in I$. Clearly $(x^n) \subset I$. We'll show the reverse inclusion. If $f \in I$, then $f = ux^m$, where u is a unit and $m \geqslant n$. This implies that f is a multiple of x^n and so $f \in (x^n)$. In other words, $k[\![x]\!]$ is a PID and every ideal is of the form (x^n) for some n. ■

An arithmetical counterpart of the ring of formal power series is the ring of p-adic integers.

■ **Example 2.5.3 — p-adic integers.** Let $p \in \mathbb{Z}$ be a prime number. We define the ring of p-adic integers $\mathbb{Z}_p$ by the limit

$$\mathbb{Z}_p := \varprojlim \frac{\mathbb{Z}}{(p^n)} = \left\{ (a_n) \in \prod_{n=1}^{\infty} \frac{\mathbb{Z}}{(p^n)} \;\middle|\; a_m \equiv a_n \bmod p^m \text{ for all } n \geqslant m \right\}.$$

We note that this is *not* the localization of $\mathbb{Z}$ at the element $p \in \mathbb{Z}$. The context usually makes clear what ring we're talking about. This ring $\mathbb{Z}_p$ has a natural morphism (as it should, since $\mathbb{Z}$ is initial in Ring)

$$\begin{aligned} \mathbb{Z} &\to \mathbb{Z}_p \\ a &\mapsto (a \bmod p^n)_{n\geqslant 1}. \end{aligned}$$

This ring is injective since the only integer that is divisible by arbitrarily large powers of p is 0. We can then think of $\mathbb{Z}$ as the subring of $\mathbb{Z}_p$ which contains only finitely nonzero terms.

By the same proofs as before, the units in $\mathbb{Z}_p$ are the elements with nonzero first component and $\mathbb{Z}_p$ is a PID whose ideals are of the form (p^n) for some n. In particular $\mathbb{Z}_p$ is a local ring with maximal ideal (p). ■

We now present the main result of this section. Its significance lies within its corollaries, which allows us to study some properties of finitely generated modules over local rings as if they were vector spaces over the residue field.

Theorem 2.5.3 — Nakayama's lemma. Let M be a finitely generated module over A and let I be an ideal of A. If $IM = M$ then there exists $i \in I$ such that $im = m$ for all $m \in M$.

Proof. Let $m_1, \dots, m_n$ be generators for M. By the hypothesis, there are $a_{ij} \in I$ such that

$$\begin{cases} m_1 = a_{11}m_1 + a_{12}m_2 + \cdots + a_{1n}m_n \\ m_2 = a_{21}m_1 + a_{22}m_2 + \cdots + a_{2n}m_n \\ \quad \vdots \\ m_n = a_{n1}m_1 + a_{n2}m_2 + \cdots + a_{nn}m_n. \end{cases}$$

In other words, if $A = (a_{ij})$ and I is the $n \times n$ identity matrix, we have that

$$(I - A) \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Multiplying by the adjoint of $I - A$ we get that $\det(I - A)m_i = 0$ for all i. Since the m_i generate M, it follows that $\det(I - A)m = 0$ for all $m \in M$. But the fact that $a_{ij} \in I$ implies that $\det(I - A) \equiv \det(I) \equiv 1 \mod I$ and so the determinant of $I - A$ is $1 - i$ for some $i \in I$. The result follows. ☐

The idea of this proof is usually called the *determinant trick*. We'll use it to prove other results in the following chapters. The following corollaries are so related to Nakayama's lemma that they are usually called by the same name.

Corollary 2.5.4 Let M be a finitely generated A-module and $\varphi : M \to M$ a surjective endomorphism. Then φ is an isomorphism.

Proof. Consider M as a $A[x]$-module via the action given by $f \cdot m := \varphi(f)(m)$ for all $f \in A[x]$. Now, since φ is surjective, we have that $(x)M = M$ and so there exists $f \in (x)$ such that $\varphi(f)(m) = m$ for all $m \in M$. In particular, this equation shows that if $m \in \ker \varphi$ then $m = 0$. Thus, φ is injective. ☐

Corollary 2.5.5 Let A be a local ring, I an ideal of A and M be a finitely generated A-module. Then $IM = M$ implies that $M = 0$.

Proof. Nakayama's lemma implies the existence of $i \in I$ such that $(1 - i)m = 0$ for all $m \in M$. If $1 - i$ weren't a unit, it would be in the maximal ideal of A and then so would 1, which is absurd. It follows that $1 - i$ is a unit and then $M = 0$. ☐

> **Corollary 2.5.6** Let A be a local ring, I an ideal of A and M be a finitely generated A-module. If N is a submodule of M such that $M = N + IM$, then $M = N$.

Proof. We apply the previous corollary to the A-module M/N. □

The next two corollaries make precise our claim that Nakayama's lemma allows us to get information about finitely generated modules over local rings by working with vector spaces.

> **Corollary 2.5.7** Let A be a local ring, I an ideal of A and M, N be A-modules where M is finitely generated. Also, let $\varphi : N \to M$ be a morphism of A-modules. Then the induced morphism
> $$\tilde{\varphi} : \frac{N}{IN} \to \frac{M}{IM}$$
> is surjective if and only if φ is.

Proof. Observe that $\tilde{\varphi}$ is surjective if and only if $M = \varphi(N) + IM$. The result now follows from the previous corollary. □

> **Corollary 2.5.8** Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Also, let M be a finitely generated A-module. Then $m_1, \ldots, m_n \in M$ generate M if and only if $\overline{m_1}, \ldots, \overline{m_n}$ generate $M/\mathfrak{m}M$ as a k-vector space. In particular, any minimal set of generators of M has exactly $\dim_k M/\mathfrak{m}M$ elements.

Proof. Observe that $m_1, \ldots, m_n$ generate M if and only if the morphism

$$\begin{aligned} A^n &\to M \\ (a_1, \ldots, a_n) &\mapsto a_1 m_1 + \ldots + a_n m_m \end{aligned}$$

is surjective. By the previous corollary, this happens if and only if the induced morphism

$$\begin{aligned} k^n &\to M/\mathfrak{m}M \\ (\overline{a_1}, \ldots, \overline{a_n}) &\mapsto \overline{a_1 m_1} + \ldots + \overline{a_n m_m} \end{aligned}$$

is surjective. This, in turn, happens if and only if $\overline{m_1}, \ldots, \overline{m_n}$ generate $M/\mathfrak{m}M$ as a k-vector space. □

This last result doesn't work without the assumption that M is finitely generated! In particular, if we want to show that M is finitely generated, it does not *suffice to show that $M/\mathfrak{m}M$ is finite dimensional as a k-vector space.*

3. Tensor and Hom

The tensor product is a construction which allows us to study bilinear maps, which are abundant, as if they were linear. Thus enabling the use of all the machinery that we developed so far. It also formalizes the notion of base change and will allow us to study the spectra of more general rings.

Also, we'll see that the functor defined by the tensor product and the Hom functors are closely intertwined by an adjunction, which lies at the heart of much of the theory that follows.

3.1. Bilinear maps and the tensor product

Let A be a commutative ring and M, N be A-modules. As we know, the direct sum $M \oplus N$ is both the product and the coproduct of M and N. As a set $M \oplus N$ is just $M \times N$, where the A-module structure is defined componentwise. Since it is a coproduct, a morphism

$$M \oplus N \to P,$$

where P is another A-module, is determined by morphisms $M \to P$ and $N \to P$. But there is another way to map $M \times N$ to P which is compatible with the A-module structures.

Definition 3.1.1 — Bilinear maps. Let M, N and P be A-modules. A function $\varphi : M \times N \to P$ is said to be *bilinear* if for all $m \in M$ the function

$$\begin{aligned} N &\to P \\ n &\mapsto \varphi(m, n) \end{aligned}$$

is a A-module morphism and for all $n \in N$ the function

$$\begin{aligned} M &\to P \\ m &\mapsto \varphi(m, n) \end{aligned}$$

is also a A-module morphism. We denote by $\mathrm{Hom}_A(M, N; P)$ the set of all bilinear maps $M \times N \to P$.

Although we know lots of things about morphisms of A-modules, we know nothing about bilinear maps. So there ought to be some way to deal with them as if they were in fact linear. Formally, we want a A-module $M \otimes_A N$ and a bilinear map

$$t : M \times N \to M \otimes_A N$$

such that every bilinear map $\varphi : M \times N \to P$ factors uniquely through t

$$\begin{tikzcd}
M \times N \arrow[rr, "\varphi"] \arrow[dr, "t"'] & & P \\
& M \otimes_A N \arrow[ur, "\tilde{\varphi}"'] &
\end{tikzcd}$$

in such a way that the map $\tilde{\varphi}$ is a A-module morphism. In other words, we want the functor $\mathrm{Hom}_A(M, N; -)$ to be representable by a A-module $M \otimes_A N$. Fortunately, it is.

Proposition 3.1.1 Let M and N be A-modules. Then the functor $\mathrm{Hom}_A(M, N; -)$ is representable.

Proof. Consider the module $A^{\oplus(M \times N)}$. This is the free module generated by $M \times N$ and so, by the universal property of free modules, has a natural function

$$j : M \times N \to A^{\oplus(M \times N)}$$

which is universal with respect to all functions from $M \times N$ to any A-module P. Our first task is to transform j into a bilinear map. The natural approach is clearly to quotient everything: let K be the submodule of $A^{\oplus(M \times N)}$ generated by the elements

$$j(m, a_1 n_1 + a_2 n_2) - a_1 j(m, n_1) - a_2 j(m, n_2)$$

and

$$j(a_1 m_1 + a_2 m_2, n) - a_1 j(m_1, n) - a_2 j(m_2, n),$$

where $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $a_1, a_2 \in A$. In this way the composition of j with the natural projection

$$M \times N \to A^{\oplus(M \times N)} \to \frac{A^{\oplus(M \times N)}}{K}$$

becomes a bilinear map. We denote this map by t and its domain by $M \otimes_A N$. The image of (m, n) by t is usually denoted by $m \otimes n$.

We now verify that this satisfies the desired universal property. Let $\varphi : M \times N \to P$ be a bilinear map. By the universal property of the free module, there exists a unique R-linear map $A^{\oplus(M \times N)} \to P$ such that the diagram

$$\begin{tikzcd}
M \times N \arrow[rr, "\varphi"] \arrow[dr, "j"'] & & P \\
& A^{\oplus(M \times N)} \arrow[ur] &
\end{tikzcd}$$

commutes. We can readily verify that K is contained in the kernel of this map and so, by the universal property of quotients, it factors uniquely through

$$\begin{tikzcd}
M \times N \arrow[r, "\varphi"] \arrow[d, "j"'] \arrow[dr, "t"] & P \\
A^{\oplus(M\times N)} \arrow[r] & M \otimes_A N \arrow[u, "\tilde{\varphi}"']
\end{tikzcd}$$

and we obtain our A-module morphism $\tilde{\varphi} : M \otimes_A N \to P$. ☐

Definition 3.1.2 — Tensor product. Let M and N be A-modules. The A-module $M \otimes_A N$ that represents the functor $\mathrm{Hom}_A(M, N; -)$ is the *tensor product* of M and N.

Concretely, the elements of $M \otimes_A N$ are finite linear combinations of the *pure tensors* $m \otimes n$, which satisfy

$$\begin{aligned}
m \otimes (n_1 + n_2) &= m \otimes n_1 + m \otimes n_2 \\
(m_1 + m_2) \otimes n &= m_1 \otimes n + m_2 \otimes n \\
m \otimes (an) &= a(m \otimes n) = (am) \otimes n
\end{aligned}$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $a \in A$. This is precisely the fact that the map $(m, n) \mapsto m \otimes n$ is bilinear. Pure tensors are very useful in concrete calculations. For example, if we want to show that two morphisms $M \otimes_A N \to P$ are equal, it suffices to show that they coincide on pure tensors. We now describe some natural isomorphisms, which we'll use all the time.

Proposition 3.1.2 — Identity element. Let A be a ring and N be a A-module. Then $A \otimes_A N \cong N$.

Proof. Let $t : A \times N \to N$ be the map that sends (a, n) to an. Then, if $\varphi : A \times N \to P$ is a bilinear map, it is clear that

$$\begin{aligned}
\tilde{\varphi} : N &\to P \\
n &\mapsto \varphi(1, n)
\end{aligned}$$

is the only A-linear map that makes the diagram

$$\begin{tikzcd}
A \times N \arrow[rr, "\varphi"] \arrow[dr, "t"'] & & P \\
& N \arrow[ur, "\tilde{\varphi}"'] &
\end{tikzcd}$$

commute. The result now follows from the unicity of universal objects. ☐

Proposition 3.1.3 — Comutativity. Let A be a ring and M, N be A-modules. Then $M \otimes_A N \cong N \otimes_A M$.

Proof. Let $\varphi : M \times N \to P$ be a bilinear map. Then φ factors as

$$\begin{tikzcd}
M \times N \arrow[r] \arrow[rr, bend left, "\varphi"] & N \times M \arrow[r, "\psi"'] & P,
\end{tikzcd}$$

where $\psi(n, m) := \varphi(m, n)$. Since ψ is also bilinear, it factors uniquely through $N \otimes_A M$.

$$\begin{tikzcd}
M \times N \arrow[r] \arrow[rrr, bend left, "\varphi"] & N \times M \arrow[rr, "\psi"'] \arrow[dr] & & P \\
& & N \otimes_A M \arrow[ur] &
\end{tikzcd}$$

But then φ also factors uniquely through $N \otimes_A M$. Once again, the result follows by the unicity of universal objects. ☐

Let N be a fixed A-module. If $\varphi : M_1 \to M_2$ is a morphism of A-modules, then we obtain a map

$$\begin{tikzcd}
M_1 \times N \arrow[r] & M_2 \times N \arrow[r] & M_2 \otimes_A N \\
(m, n) \arrow[r, mapsto] & (\varphi(m), n) \arrow[r, mapsto] & \varphi(m) \otimes n
\end{tikzcd}$$

which is bilinear and so induces a linear map

$$\varphi \otimes N : M_1 \otimes_A N \to M_2 \otimes N$$

which is defined on pure tensors by $m \otimes n \mapsto \varphi(m) \otimes n$ and extended by linearity. Now, if $\psi : M_2 \to M_3$ is another morphism, then $(\psi \circ \varphi) \otimes N$ and $(\psi \otimes N) \circ (\varphi \otimes N)$ both map pure tensors $m \otimes n$ to $\psi(\varphi(m)) \otimes n$ and so they coincide everywhere. In other words, $- \otimes_A N$ is a covariant functor from the category of A-modules to itself.

Now, there's another natural covariant functor from the category of A-modules to itself. We associate to each A-module P the A-module $\mathrm{Hom}_A(N, P)$ and to each morphism $\varphi : P_1 \to P_2$ the morphism $\mathrm{Hom}_A(N, P_1) \to \mathrm{Hom}_A(N, P_2)$ given by $\psi \mapsto \varphi \circ \psi$.

$$\begin{tikzcd}
P_1 \arrow[rr, "\varphi"] & & P_2 \\
& N \arrow[ul, "\psi"] \arrow[ur, "\varphi \circ \psi"'] &
\end{tikzcd}$$

This assignment is functorial since putting two commutative diagrams side-by-side yields a new commutative diagram.

$$\begin{tikzcd} P_1 \arrow[r] & P_2 \arrow[r] & P_3 \\ & N \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

Somewhat surprisingly, the functors $- \otimes_A N$ and $\mathrm{Hom}_A(N, -)$ form a beautiful adjunction!

Theorem 3.1.4 — Tensor-Hom adjunction. Let N be a A-module. Then we have an adjunction

$$\begin{tikzcd} A\text{-Mod} \arrow[r, bend left, "- \otimes_A N"] \arrow[r, phantom, "\perp"] & A\text{-Mod} \arrow[l, bend left, "\mathrm{Hom}_A(N,-)"] \end{tikzcd}$$

In other words, there is an isomorphism of A-modules

$$\mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)) \cong \mathrm{Hom}_A(M \otimes_A N, P)$$

which is natural in M and P.

Proof. This result follows from the simple observation that $\mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P))$ is nothing but $\mathrm{Hom}_A(M, N; P)$. In fact, lets see what it means for a function $\varphi : M \times N \to P$ be bilinear. First of all, for all $m \in M$, the function $n \mapsto \varphi(m, n)$ ought to be linear. In other words, we have a function

$$M \to \mathrm{Hom}_A(N, P).$$

Also, for all $n \in N$, the function $m \mapsto \varphi(m, n)$ is linear. This precisely means that our morphism $M \to \mathrm{Hom}_A(N, P)$ is also linear. That is, φ is naturally an element of $\mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P))$. Since $\mathrm{Hom}_A(M, N; P) \cong \mathrm{Hom}_A(M \otimes_A N, P)$, it follows that

$$\mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)) \cong \mathrm{Hom}_A(M \otimes_A N, P).$$

We leave to the reader the (boring) task of verifying that this isomorphism is natural in M and P. ☐

There's a useful mnemonic for remembering this result. The word *tensor* comes to the left of the *tensor-hom adjunction* and so $- \otimes_A N$ is the left-adjoint. Similarly, the word *hom* comes to the right and so $\mathrm{Hom}_A(N, -)$ is the right-adjoint.

We then have a natural Pavlovian reaction: since right adjoints preserve limits[1], so does $\mathrm{Hom}_A(N, -)$. Similarly, $- \otimes_A N$ preserves colimits. In particular, $\mathrm{Hom}_A(N, -)$ is

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[1] Another mnemonic: **r**ight **a**djoints **p**reserve **l**imits. RAPL!

left-exact and $- \otimes_A N$ is right-exact. By commutativity of the tensor product, $M \otimes_A -$ is also right-exact. Numerous results follow from these considerations. For now we'll focus on those related to the tensor product.

Corollary 3.1.5 — Associativity. Let A be a ring and M, N, P be A-modules. Then $(M \otimes_A N) \otimes_A P \cong M \otimes_A (N \otimes_A P)$.

Proof. Let Q be a A-module. Then multiple applications of the theorem 3.1.4 yield

$$\begin{aligned} \mathrm{Hom}_A((M \otimes_A N) \otimes_A P, Q) &= \mathrm{Hom}_A(M \otimes_A N, \mathrm{Hom}_A(P, Q)) \\ &= \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, \mathrm{Hom}_A(P, Q))) \\ &= \mathrm{Hom}_A(M, \mathrm{Hom}_A(N \otimes_A P, Q)) \\ &= \mathrm{Hom}_A(M \otimes_A (N \otimes_A P), Q). \end{aligned}$$

Since this holds for every Q, the Yoneda lemma implies our desired isomorphism. □

Corollary 3.1.6 — Distributivity. Let $\{M_i\}$ be a collection of A-modules and N be a A-module. Then

$$\left(\bigoplus\nolimits_i M_i\right) \otimes_A N \cong \bigoplus\nolimits_i M_i \otimes_A N.$$

In particular, for finitely generated free modules this says that $A^{\oplus n} \otimes_A A^{\oplus m} \cong A^{\oplus mn}$.

Proof. This follows from the fact that $- \otimes_A N$ preserves colimits. □

Corollary 3.1.7 Let I be an ideal of A and N be a A-module. Then

$$\frac{A}{I} \otimes_A N \cong \frac{N}{IN}.$$

Proof. Since $- \otimes_A N$ is right-exact, the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] & A \arrow[r, two heads] & \frac{A}{I} \arrow[r] & 0 \end{tikzcd}$$

induces an exact sequence

$$\begin{tikzcd} I \otimes_A N \arrow[r] & A \otimes_A N \arrow[r, two heads] & \frac{A}{I} \otimes_A N \arrow[r] & 0. \end{tikzcd}$$

The image of $I \otimes_A N$ in $A \otimes_A N \cong N$ is generated by the image of the pure tensors $a \otimes n$ with $a \in I$ and $n \in N$. This is precisely IN. It follows that this sequence identifies N/IN with $(A/I) \otimes_A N$. □

As an interesting particular case of this last result we consider $N = A/J$, where J is an ideal of A. It then follows that

$$\frac{A}{I} \otimes_A \frac{A}{J} = \frac{A/J}{I(A/J)} = \frac{A/J}{(I+J)/J} = \frac{A}{I+J}.$$

by the third isomorphism theorem.

3.2. Base change

In order to understand how tensor products formalizes the idea of changing the base ring, we have to upgrade our universal property, yielding a stronger adjunction that the one we've just studied. In fact we'll see that the tensor product is capable of factoring a larger class of maps: *balanced maps*.

Definition 3.2.1 — Balanced maps. Let M, N be modules over a commutative ring A and G be an abelian group. A function $\varphi : M \times N \to G$ is said to be *balanced* if it is $\mathbb{Z}$-bilinear and satisfies

$$\varphi(am, n) = \varphi(m, an)$$

for all $m \in M$, $n \in M$ and $a \in A$.

In fact, if A is not commutative, M is a *right*-module and N is a *left*-module, then the same construction works. Everything is this sections also works in the non-commutative case. Nevertheless, it is simpler to just work in the commutative case so that's what we'll do.

Observe that if G is a A-module, then bilinear maps $M \times N \to G$ are automatically balanced. But balanced maps are surely more general so it would seem that we need another object to factor such maps. Luckily, the tensor product does this job as well.

Proposition 3.2.1 Let M and N be A-modules and G be an abelian group. Then, if $\varphi : M \times N \to G$ is a balanced map, there exists a unique group morphism $\tilde{\varphi} : M \otimes_A N \to G$ such that the diagram

$$\begin{tikzcd}
M \times N \arrow[rr, "\varphi"] \arrow[dr, "t"'] & & G \\
& M \otimes_A N \arrow[ur, "\tilde{\varphi}"'] &
\end{tikzcd}$$

commutes.

Proof. Recall that every element of $M \otimes_A N$ can be written as a finite sum of pure tensors. In other words, the morphism of groups

$$\mathbb{Z}^{\oplus(M \times N)} \to M \otimes_A N$$

given on generators by $(m, n) \mapsto m \otimes n$ is surjective. Its kernel K_B is generated by elements of the form

$$\begin{gathered}(m, n_1 + n_2) - (m, n_1) - (m, n_2)\\(m_1 + m_2, n) - (m_1, n) - (m_2, n)\\(am, n) - (m, an)\end{gathered}$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $a \in A$. It follows that we have an isomorphism of abelian groups

$$\frac{\mathbb{Z}^{\oplus(M \times N)}}{K_B} \cong M \otimes_A N.$$

But the abelian group on the left-hand side of this equation is manifestly a solution of our universal problem. The result follows. ☐

As before, this can be reinterpreted by saying that the functor that associates to each abelian group G the set of all balanced maps $M \times N \to G$ is represented by the tensor product $M \otimes_A N$. Its clear that this set of balanced maps is actually an abelian group and that the bijection

$$\{\text{balanced maps } M \times N \to G\} \cong \mathrm{Hom}_{\mathsf{Ab}}(M \otimes_A N, G)$$

is an isomorphism of abelian groups.

Our previous universal property is then recovered as the statement that if G is a A-module and φ is linear, then the induced group morphism $\tilde{\varphi} : M \otimes_A N \to G$ is in fact a morphism of A-modules.

Since we want to change the base ring of modules, we need a notion of a module which has two compatible structures.

Definition 3.2.2 — Bimodules. Let A and B be two commutative rings. A (A, B)-*bimodule* is an abelian group N endowed with compatible A-module and B-module structures in the sense that

$$a(bn) = b(an)$$

for all $a \in A$, $b \in B$ and $n \in N$.

If M is a A-module and N is a (A, B)-bimodule, then the tensor product $M \otimes_A N$ acquires a B-module structure. In fact, $b \in B$ acts on pure tensors by

$$b(m \otimes n) := m \otimes (bn).$$

This action is then extended by linearity. This endows $M \otimes_A N$ with a (A, B)-bimodule structure.

Similarly, if N is a (A, B)-bimodule and P is a B-module, then $\mathrm{Hom}_B(N, P)$ inherits a A-module structure given by

$$(a\varphi)(n) := \varphi(an)$$

for all $a \in A$, $\varphi \in \mathrm{Hom}_B(N, P)$ and $n \in N$. This makes $\mathrm{Hom}_B(N, P)$ a (A, B)-bimodule. As expected, this generalizes our adjunction.

Theorem 3.2.2 — Tensor-Hom adjunction. Let N be a (A, B)-bimodule. Then we have an adjunction

$$\begin{tikzcd} A\text{-Mod} \arrow[r, bend left, "- \otimes_A N"] \arrow[r, phantom, "\perp"] & B\text{-Mod} \arrow[l, bend left, "\mathrm{Hom}_A(N, -)"] \end{tikzcd}$$

In other words, there is an isomorphism of abelian groups

$$\mathrm{Hom}_A(M, \mathrm{Hom}_B(N, P)) \cong \mathrm{Hom}_B(M \otimes_A N, P)$$

which is natural in $M \in A\text{-Mod}$ and $P \in B\text{-Mod}$.

Proof. As we saw in the proof of the original adjunction, the elements of the abelian group $\mathrm{Hom}_A(M, \mathrm{Hom}_B(N, P))$ are in bijection with balanced maps

$$M \times N \to P$$

with respect to the A-module structures of M and N. For any such map, we have an unique induced morphism of abelian groups

$$M \otimes_A N \to P.$$

The B-linearity on $\mathrm{Hom}_B(N, P)$ implies that this map is actually B-linear. In other words, we have a bijection

$$\mathrm{Hom}_A(M, \mathrm{Hom}_B(N, P)) \cong \mathrm{Hom}_B(M \otimes_A N, P).$$

It is clear that this is actually an isomorphism of abelian groups. Moreover, as usual the reader can verify that it is natural. ☐

For this next part, we fix a morphism $f : A \to B$ of commutative rings. We'll study how this ring allows us to change the base ring of a module. In other words, we'll study functors between $A\text{-Mod}$ and $B\text{-Mod}$ which are induced by f.

Seeing B as an A-algebra allows us to form the tensor product $M \otimes_A B$. Since B is not only a A-module but a ring, it is naturally a (A, B)-bimodule and so, as we saw, the tensor product inherits the structure of a B-module. This defines our first functor.

Definition 3.2.3 — Extension of scalars. Let $f^* : A\text{-Mod} \to B\text{-Mod}$ be the functor defined by $f^*(M) := M \otimes_A B$. If $\varphi : M_1 \to M_2$ is a morphism of A-modules, the induced map $f^*(\varphi) : f^*(M_1) \to f^*(M_2)$ is simply $\varphi \otimes B$. We say that f^* is the *extension of scalars* functor.

When dealing with the viewpoint that an A-algebra is a ring B with a compatible A-module structure, we'll usually denote the extension of scalars $f^*(M)$ by M_B and the associated morphism $f^*(\varphi)$ as φ_B.

We can understand this nomenclature in the following way. Let

$$A^{\oplus P} \to A^{\oplus Q} \to M \to 0$$

be a presentation of M. Since the tensor functor is right-exact, tensoring by B gives

$$B^{\oplus P} \to B^{\oplus Q} \to f^*(M) \to 0.$$

In other words, $f^*(M)$ is the module defined by the same generators and relations as M, but with coefficients in B.

▪ **Example 3.2.1** Let K/k be a field extension and V be a finite-dimensional k-vector space. The precedent discussion implies that V_K is a K-vector space with

$$\dim_K V_K = \dim_k V.$$

By far, the most important case is the *complexification* of a real vector space. ▪

The case where f is the usual quotient map sheds some light into Nakayama's lemma.

> **Proposition 3.2.3 — Nakayama's lemma.** Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Also, let M be a finitely generated A-module. Then, $M \otimes_A k = 0$ implies $M = 0$.

Proof. This follows from the isomorphism

$$M \otimes_A k = M \otimes_A \frac{A}{\mathfrak{m}} = \frac{M}{\mathfrak{m}M}$$

and corollary 2.5.5. □

We can also do the opposite operation of extension of scalars: restriction. Recall that a B-module structure on an abelian group N is nothing but a morphism of rings

$$B \to \mathrm{End}_{\mathsf{Ab}}(N).$$

Precomposing with our morphism of rings $f : A \to B$ we obtain a A-module structure on N

$$A \to B \to \mathrm{End}_{\mathsf{Ab}}(N).$$

Explicitly, if $a \in A$ and $n \in N$, the action of a in n is defined by

$$a \cdot n = f(a)n.$$

Since a B-linear morphism is automatically A-linear, we obtain a functor $B\text{-}\mathsf{Mod} \to A\text{-}\mathsf{Mod}$. The next definition summarizes this discussion.

Definition 3.2.4 — Restriction of scalars. Let $f_* : B\text{-Mod} \to A\text{-Mod}$ be the functor which associates to each B-module N the A-module with the same underlying abelian group and A-module structure given by $a \cdot n = f(a)n$ for all $a \in A$ and $n \in N$. The functor f_* acts as the identity on morphisms. We say that f_* is the *restriction of scalars* functor.

If $f : A \to B$ is the inclusion of A as a subring of B, then all we're doing is viewing N as a module on a restricted range of scalars, hence the terminology.

Actually, there's yet another natural functor $A\text{-Mod} \to B\text{-Mod}$.

Definition 3.2.5 — Coextension of scalars. Let $f^! : A\text{-Mod} \to B\text{-Mod}$ be the functor which associates to each A-module M the B-module $\mathrm{Hom}_A(B, M)$. As before, $f^!(M) = \mathrm{Hom}_A(B, M)$ has a natural B-module structure given by $(b_1\varphi)(b_2) := \varphi(b_1b_2)$ for all $\varphi \in f^!(M)$ and $b_1, b_2 \in B$. We say that $f^!$ is the *coextension of scalars* functor.

All these functors now coalesce to form an even prettier adjunction.

Theorem 3.2.4 Let $f : A \to B$ be a morphism of commutative rings. Then f_* is right-adjoint to f^* and left-adjoint to $f^!$.

$$\begin{tikzcd}
A\text{-Mod} \arrow[d, bend right=60, "f^*"'] \arrow[d, bend left=60, "f^!"] \\
B\text{-Mod} \arrow[u, "\dashv\ f_*\ \dashv" description]
\end{tikzcd}$$

In particular, f_* is exact, f^* is right-exact, and $f^!$ is left-exact.

Proof. Let M be a A-module and N be a B-module. Note that $\mathrm{Hom}_B(B, N)$ is canonically isomorphic to N as a B-module and to $f_*(N)$ as a A-module. Thus,

$$\mathrm{Hom}_A(M, f_*(N)) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_B(B, N)) \cong \mathrm{Hom}_B(f^*(M), N)$$

by the tensor-hom adjunction. Since these isomorphisms (of abelian groups) are natural, this proves that f^* is left-adjoint to f_*.

Similarly, $f_*(N) \cong N \otimes_B B$ as A-modules and so

$$\mathrm{Hom}_A(f_*(N), M) \cong \mathrm{Hom}_A(N \otimes_B B, M) \cong \mathrm{Hom}_B(N, f^!(M))$$

by the same adjunction as before. In other words, $f^!$ is right-adjoint to f_*. □

A very important case of extension by scalars is that of a localization, where $f : A \to S^{-1}A$ is the usual localization map.

Corollary 3.2.5 Let S be a multiplicative subset of a commutative ring A and let $f : A \to S^{-1}A$ be the localization map. Then f^* is naturally isomorphic to the localization functor S^{-1}. In particular, $M \otimes_A S^{-1}A \cong S^{-1}M$ for every A-module M.

This is follows directly from a general lemma.

Lemma 3.2.6 Let $F, G : A\text{-Mod} \to \mathsf{Ab}$ be right-exact functors. If $F \to G$ is a natural transformation which is an isomorphism on free modules, then it is an isomorphism of functors.

Proof. We apply the functors F and G to a presentation $A^{\oplus P} \to A^{\oplus Q} \to M \to 0$ of M:

$$\begin{tikzcd}
F(A^{\oplus P}) \arrow[r] \arrow[d] & F(A^{\oplus Q}) \arrow[r] \arrow[d] & F(M) \arrow[r] \arrow[d] & 0 \\
G(A^{\oplus P}) \arrow[r] & G(A^{\oplus Q}) \arrow[r] & G(M) \arrow[r] & 0.
\end{tikzcd}$$

The first two vertical arrows are isomorphisms by supposition. The result now follows by a simple diagram chase. ☐

Proof of the corollary 3.2.5. Both functors are left-adjoints and so preserve colimits. The result then follows by the previous lemma and the fact that they agree on A. ☐

As we saw, if M is a A-module and B is an A-algebra, the tensor product $M \otimes_A B$ is not only a A-module but also a B-module. Carrying this one step further, if C is also a A-algebra, the tensor product $B \otimes_A C$ inherits a natural multiplication given by

$$(b_1 \otimes c_1) \cdot (b_2 \otimes c_2) := (b_1 b_2) \otimes (c_1 c_2),$$

granting $B \otimes_A C$ a A-algebra structure.

The tensor product of algebras is a richer version of its module counterpart. That being so, we hope that it satisfies a powered up universal property. In the case of modules, the maps $B \otimes_A C \to D$ are in bijection with the bilinear maps $B \times C \to D$. Now, the morphisms of A-algebras $B \otimes_A C \to D$ are in bijection with pairs of A-algebra morphisms $B \to D$ and $C \to D$:

$$\mathrm{Hom}_A(B \otimes_A C, D) \cong \mathrm{Hom}_A(B, D) \times \mathrm{Hom}_A(C, D).$$

In other words, the tensor product $B \otimes_A C$ is the coproduct of B and C in the category of commutative A-algebras.

Since we'll later see $\operatorname{Spec}(B \otimes_A C)$ as the fibered product of $\operatorname{Spec} B \to \operatorname{Spec} A$ and $\operatorname{Spec} C \to \operatorname{Spec} A$ (in the category of affine schemes), we prefer to see an algebra as a morphism to rings and say that the tensor product $B \otimes_A C$ is the fibered coproduct in the category of commutative rings.

Its also useful to observe that we finally solved a problem that we posed long ago! Since $\mathbb{Z}$ is initial in Ring, every ring is naturally a $\mathbb{Z}$-algebra and so the coproduct of A and B in the category of commutative rings is the tensor product $A \otimes_{\mathbb{Z}} B$, which we'll often denote simply by $A \otimes B$.

Theorem 3.2.7 — Fibered coproduct. Let $f : A \to B$ and $g : A \to C$ be two morphisms of commutative rings. Then the tensor product $B \otimes_A C$, along with the natural morphisms

$$\begin{aligned} B &\to B \otimes_A C \\ b &\mapsto b \otimes 1 \end{aligned} \qquad\qquad \begin{aligned} C &\to B \otimes_A C \\ c &\mapsto 1 \otimes c \end{aligned}$$

form the fibered coproduct of f and g

$$\begin{tikzcd}
B \otimes_A C & C \arrow[l] \\
B \arrow[u] & A \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

in the category of commutative rings. In other words, $B \otimes_A C$ is the coproduct of B and C in the category of commutative A-algebras.

Proof. The fact that our diagram commutes follows from the equation $a \otimes 1 = 1 \otimes a$ for every $a \in A$. Now, let D be a commutative ring, along with morphisms $\alpha : B \to D$ and $\beta : C \to D$, such that the diagram

$$\begin{tikzcd}
D & C \arrow[l, "\beta"'] \\
B \arrow[u, "\alpha"] & A \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

commutes. We have to show that there exists a unique morphism $B \otimes_A C \to D$ such that the diagram

$$\begin{tikzcd}
D & & \\
& B \otimes_A C \arrow[ul, dashed] & C \arrow[l] \arrow[ull, bend right, "\beta"'] \\
& B \arrow[u] \arrow[uul, bend left, "\alpha"] & A \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

commutes. The map $B \times C \to D$ given by $(b, c) \mapsto \alpha(b)\beta(c)$ is manifestly bilinear and thus factors uniquely into a map $B \otimes_A C \to D$. It is clear that our diagram commutes and so we're done. □

3.3. Fibers of a map between spectra

3.4. Flat modules

3.5. Exterior and symmetric powers

3.6. Hom and duality

3.7. Projective and injective modules

4. Chain conditions

5. Integral extensions

In this chapter we will study the ring-theoretic concepts which generalize the notions of finite and algebraic field extensions. As we'll see, if $A \subset B$ is an integral extension, the associated spectrum map $\operatorname{Spec} B \to \operatorname{Spec} A$ is as simple as we can hope it to be: its a closed map and it is surjective with finite fibers. In this way, we can interpret $\operatorname{Spec} B$ as a finite covering of $\operatorname{Spec} A$.

5.1. Definitions and basic properties

We begin our journey with *finite* morphisms, which are a even simpler class of morphisms.

> **Definition 5.1.1 — Finite morphism.** Let $\varphi : A \to B$ be a morphism of rings. We say that φ is *finite* if B is a finitely generated A-module. If a finite morphism $\varphi : A \hookrightarrow B$ is an inclusion of rings, then we say that $A \subset B$ is a *finite extension*.

As a source of examples, we have that surjective morphisms are automatically finite. In fact, if $\varphi : A \to B$ is surjective, then $1 \in B$ is a generator of B as a A-module. In particular, quotient maps are always finite.

We now prove two simple results about finite morphisms: namely that the class of finite morphisms is closed under composition and base change.

> **Proposition 5.1.1** If $\varphi : A \to B$ and $\psi : B \to C$ are finite morphisms, then so is $\psi \circ \varphi : A \to C$.

Proof. If $b_1, \dots, b_m$ generate B as a A-module and $c_1, \dots, c_n$ generate C as a B-module, then

$$C = \sum_{i=1}^{n} c_i B = \sum_{i=1}^{n} c_i \left(\sum_{j=1}^{m} b_j A \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} c_i b_j A$$

and so C is also finitely generated as a A-module. □

> **Proposition 5.1.2** If $\varphi : A \to B$ is a finite morphism and $f : A \to C$ is any ring morphism, then the base change $f^*(\varphi) : C \to B \otimes_A C$ is also finite.

Proof. If $b_1, \dots, b_m$ generate B as a A-module, then

$$B \otimes_A C = \left(\sum_{j=1}^{m} b_j A \right) \otimes_A C \cong \sum_{j=1}^{m} b_j C$$

and so $B \otimes_A C$ is a finitely generated C-module. □

Below we have the main definition of this chapter.

Definition 5.1.2 — Integral extension. Let $A \subset B$ be an extension of rings. An element $b \in B$ is said to be *integral* over A if it is a root of a monic polynomial in $A[x]$. We say that the extension $A \subset B$ is *integral* if every element of B is integral over A. More generally, we say that a morphism of rings $\varphi : A \to B$ is *integral* is the extension $\varphi(A) \subset B$ is.

In practice, the definition of an integral element is quite difficult to work with. For example, it is not at all obvious that if b_1 and b_2 are integral over A then so is $b_1 + b_2$ and $b_1 b_2$. The following result solves beautifully this problem.

Theorem 5.1.3 Let $A \subset B$ be an extension of rings. Then $b_1, \dots, b_n \in B$ are all integral over A if and only if $A[b_1, \dots, b_n]$ is a finite A-algebra.

Proof. Let $b \in B$ be integral over A. If

$$f = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in A[x]$$

is a monic polynomial such that $f(b) = 0$, then we can use the fact that

$$b^n = -a_{n-1}b^{n-1} - \cdots - a_1 b - a_0$$

to recursively write b^{n+i}, for all $i \geqslant 0$, as a linear combination of $1, b, \dots, b^{n-1}$. That being so, the A-algebra $A[b]$ is generated, as a A-module, by these elements. More generally, if $b_1, \dots, b_n \in B$ are all integral over A, then we prove that $A[b_1, \dots, b_n]$ is a finite A-algebra by induction using proposition 5.1.1.

Conversely, assume that $m_1, \dots, m_r$ are generators of $A[b_1, \dots, b_n]$ as an A-algebra. Then, for any $b \in A[b_1, \dots, b_n]$ we have that

$$\begin{cases} bm_1 = a_{11}m_1 + a_{12}m_2 + \cdots + a_{1r}m_r \\ bm_2 = a_{21}m_1 + a_{22}m_2 + \cdots + a_{2r}m_r \\ \qquad \vdots \\ bm_r = a_{r1}m_1 + a_{r2}m_2 + \cdots + a_{rr}m_r \end{cases}$$

for some $a_{ij} \in A$. Then b is a root of the characteristic polynomial of the matrix (a_{ij}), which is monic and in $A[x]$. □

Observe that, according to this theorem if $b_1, \ldots, b_n \in B$ are integral over A, then so is *any* $b \in A[b_1, \ldots, b_n]$. Indeed, this can be concluded from its proof but it also follows from the fact that $A[b_1, \ldots, b_n, b] = A[b_1, \ldots, b_n]$.

Let's see how this solves our problem. If b_1 and b_2 are integral over A, then $A[b_1, b_2]$ is finite as an A-algebra. But both $b_1 + b_2$ and $b_1 b_2$ lie in $A[b_1, b_2]$ and so our last result shows that they are integral over A. Quite magical, isn't it?

Corollary 5.1.4 — Finite = integral + finite-type. Let $A \subset B$ be an extension of rings. If $A \subset B$ is finite, then it is integral. Conversely, if $A \subset B$ is integral and B is a finite-type A-algebra, then $A \subset B$ is finite.

Proof. If $A \subset B$ is finite and $b \in B$, then we can proceed in exactly the same way as in the proof of the theorem to find that b is a root of the characteristic polynomial of a matrix with coefficients in A. The converse is a particular case of the theorem. □

Corollary 5.1.5 If $A \subset B$ and $B \subset C$ are integral extensions, then so if $A \subset C$.

Proof. Let $c \in C$. Since c is integral over B, it satisfies a relation of the form

$$c^n + b_{n-1} c^{n-1} + \cdots + b_0 = 0,$$

where $b_i \in B$ for all i. This shows that c is integral over $R := A[b_0, \ldots, b_{n-1}]$. In other words, $R[c]$ is a finite R-algebra. But all the b_i are integral over A, so R is a finite A-algebra. The result now follows from proposition 5.1.1 and the previous theorem. □

5.2. Fibers of integral extensions

5.3. Normal rings

5.4. Noether normalization lemma

5.5. Nullstellensätze

Part II.

Homological Algebra

6. Abelian categories

Homological algebra deals extensively with the notions of kernel, image, exact sequences, chain complexes, and the like. This chapter will explain the most general setting, that of abelian categories, in which these concepts make sense. Certainly, the category of A-modules has all the needed characteristics. Going even further, it is true that every abelian category has a fully faithful embedding on A-Mod for some (not necessarily commutative) ring A. However, when it is not too troublesome, we'll study abelian categories "on their own" for we believe that understanding arrow-theoretic arguments and not becoming dependent on a difficult theorem can only be beneficial.

6.1. Additive categories

We begin our quest of understanding which properties a category should have in order for exact sequences to make sense. A first problem is that our category should have a distinguished object corresponding to the trivial module in A-Mod. In order to allow for exact sequences, this object should be initial and final at the same time. We arrive at our first definition.

Definition 6.1.1 Let A be a category. A *zero-object* is an object of A which is both initial and final. We'll always denote zero-objects as 0.

The reader should notice that even reasonable categories may fail to have initial or final objects (the category of fields, for example, has neither). And even if they exist, they may not coincide (as in Set or Ring). Nevertheless, Grp, Ab, and A-Mod are examples of categories possessing zero-objects.

The existence of zero-objects in a category allows us to talk about zero-morphisms.

Definition 6.1.2 Let A be a category with a zero-object 0. A morphism $\varphi : M \to N$ is called a *zero-morphism* if it factors through the zero-object 0. We'll also denote zero-morphisms by 0.

We observe that in a category with a zero-object, there is exactly one zero-morphism from each object M to each object N: it's just the composite of the unique morphism $M \to 0$ with the unique morphism $0 \to N$. In any of the aforementioned categories which possess zero-objects, the zero morphism $M \to N$ is the one sending every element of M to $0 \in N$. Moreover, the composition of a zero-morphism with an arbitrary morphism is again a zero-morphism. Indeed, the composition factors through 0.

In an abstract category, we have no means of defining kernels set-theoretically as the subobjects composed of the elements which are sent to zero. Instead, we define a kernel as a *morphism* by a suitable universal property.

Definition 6.1.3 — Kernel. Let $\varphi : M \to N$ be a morphism in a category A with a zero-object 0. The *kernel* of φ is the equalizer of φ and the zero-morphism. In other words, it is a morphism $\iota : K \to M$ such that, whenever $\zeta : Z \to M$ satisfies $\varphi \circ \zeta = 0$, there exists a unique morphism $Z \to K$ making the diagram

$$\begin{tikzcd}
K \arrow[r, "\iota"] & M \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "0"'] & N \\
Z \arrow[u, dashed] \arrow[ur, "\zeta"'] & &
\end{tikzcd}$$

commute. We denote both K and $\iota : K \to M$ by $\ker \varphi$.

Once again, we observe that kernels are not guaranteed to exist even in reasonable categories. For example, kernels may fail to exist in the category of finitely generated A-modules whenever A is not noetherian.

In any of the previously mentioned categories with zero-objects, the universal property of the kernel is satisfied by the inclusion map from the set-theoretic kernel. This generalizes nicely to the categorical kernel. For that, we need another piece of nomenclature.

Definition 6.1.4 — Subobject. Let M be an object in a category A. We say that two monomorphisms $s : S \to M$ and $t : T \to M$ are equivalent if there exists an isomorphism $S \to T$ making the diagram

$$\begin{tikzcd}
S \arrow[rr, "\sim"] \arrow[dr, hook, "s"'] & & T \arrow[dl, hook', "t"] \\
& M &
\end{tikzcd}$$

commute. In other words, s and t are equivalent if they are isomorphic in the slice category $\mathsf{A} \downarrow M$. A *subobject* of M is an equivalence class for this equivalence relation.

The universal property of kernels implies that all kernels of a morphism $M \to N$ belong to the same isomorphism class in $\mathsf{A} \downarrow M$. Thus, in order to prove that the kernel of $M \to N$ is a subobject of M it suffices to show that kernels are monic.

Proposition 6.1.1 Let $\varphi : M \to N$ be a morphism in a category A with a zero-object 0 and suppose that $\ker \varphi : K \to M$ is its kernel. Then $\ker \varphi$ is a monomorphism and so defines a subobject of M.

The reader should notice that the same proof shows that every equalizer is monic.

Proof. Let $\alpha, \beta : Z \to K$ be two morphisms such that $(\ker \varphi) \circ \alpha = (\ker \varphi) \circ \beta$ and let ζ be their common compositions. By the universal property of kernels, there is a unique morphism $Z \to K$ making the diagram

$$\begin{tikzcd}
K \arrow[r, "\ker \varphi"] & M \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "0"'] & N \\
Z \arrow[u] \arrow[ur, "\zeta"'] & &
\end{tikzcd}$$

commute. But α and β are two such morphisms. It follows that $\alpha = \beta$. □

In most categories in algebra, kernels measure how far a morphism is from being injective. The following propositions shows that the categorical kernel still, in some sense, encodes this information.

Proposition 6.1.2 Let $\varphi : M \to N$ be a monomorphism in a category A with a zero-object 0. Then $\ker \varphi$ is the zero-morphism $0 \to M$.

Proof. Suppose $\zeta : Z \to M$ is a morphism such that $\varphi \circ \zeta = 0$. Since φ is a monomorphism, $\varphi \circ \zeta = 0 = \varphi \circ 0$ means that $\zeta = 0$ and so ζ factors uniquely through the zero-object, making the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "0"'] & N \\
Z \arrow[u, dashed] \arrow[ur, "\zeta"'] & &
\end{tikzcd}$$

commute. This means that $0 \to M$ is the, necessarily unique, kernel of φ. □

Proposition 6.1.3 Let $\varphi : M \to N$ be a morphism in a category A with a zero-object 0. Then φ is a zero-morphism if and only if $\ker \varphi$ is, up to isomorphism, the identity on M.

Proof. Suppose φ is the zero-morphism. Then $\varphi \circ \mathrm{id}_M = 0 \circ \mathrm{id}_M$ and so any morphism $\zeta : Z \to M$ factors uniquely through id_M. Conversely, if id_M is a kernel of φ, then $\varphi = \varphi \circ \mathrm{id}_M = 0$. □

The main problems of the categorical kernel are the fact that they may not exist and, even when they exist, it is not necessarily true that every monomorphism is a kernel, as in A-Mod. For example, in the category of groups, kernels are normal subgroups but monomorphisms correspond to all subgroups. All these problems will be solved in the next section. For now, we observe that the dual notion (which inverses all the arrows) of kernel is just as useful.

Definition 6.1.5 — Cokernel. Let $\varphi : M \to N$ be a morphism in a category A with a zero-object 0. The *cokernel* of φ is the coequalizer of φ and the zero-morphism. In other words, it is a morphism $\pi : N \to C$ such that, whenever $\beta : N \to Z$ satisfies $\beta \circ \varphi = 0$, there exists a unique morphism $C \to Z$ making the diagram

$$\begin{tikzcd}
 & & Z \\
M \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "0"'] & N \arrow[ur, "\beta"] \arrow[r, "\pi"] & C \arrow[u, dashed]
\end{tikzcd}$$

commute. We denote both C and $\pi : N \to C$ by $\operatorname{coker} \varphi$.

Before we prove any properties of the cokernel, we present how it works in some categories, since the reader may be unfamiliar with it.

■ **Example 6.1.1 — Cokernels in A-Mod.** Let $\varphi : M \to N$ be a morphism of A-modules. Here, the cokernel of φ is the quotient map $\pi : N \to N/\operatorname{im} \varphi$, where $\operatorname{im} \varphi$ is the usual set-theoretic image. Indeed, if $\beta : N \to P$ satisfies $\beta \circ \varphi = 0$, then $\operatorname{im} \varphi \subset \ker \beta$ and the universal property of the quotient induces a unique morphism $\widetilde{\beta} : N/\operatorname{im} \varphi \to P$ which makes the diagram

$$\begin{tikzcd}
 & & P \\
M \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "0"'] & N \arrow[ur, "\beta"] \arrow[r, "\pi"] & N/\operatorname{im} \varphi \arrow[u, dashed, "\widetilde{\beta}"']
\end{tikzcd}$$

commute. In other words, $\pi : N \to N/\operatorname{im} \varphi$ satisfies the universal property of the cokernel. ■

■ **Example 6.1.2 — Cokernels in Grp.** Let $\varphi : G \to H$ be a morphism of groups. The same argument as in A-Mod doesn't work as the set-theoretical image may not be a normal subgroup of H. Nevertheless, we may consider the smallest normal subgroup of H containing $\operatorname{im} \varphi$, which we denote by N. Then the cokernel of φ becomes the quotient map $\pi : H \to H/N$. Indeed, if $\beta : H \to H'$ satisfies $\beta \circ \varphi = 0$, then $\operatorname{im} \varphi \subset \ker \beta$ and, since $\ker \beta$ is a normal subgroup of H containing $\operatorname{im} \varphi$, $N \subset \ker \beta$. Now the same argument as before works, showing that $\pi : H \to H/N$ satisfies the universal property of the cokernel. ■

■ **Example 6.1.3 — Cokernels in the category of Banach spaces.** The same problem as before happens frequently in topological settings. In the category of Banach spaces with bounded (continuous) linear maps as morphisms, not every subspace defines a quotient, only the closed ones. A similar reasoning as before shows that the cokernel of a morphism $T : X \to Y$ is the quotient map $Y \to Y/N$, where N is the closure of the set-theoretical image $\operatorname{im} T$ in Y. ■

It is actually the case that, whenever it exists, the cokernel of a morphism $\varphi : M \to N$ is a quotient of N just as the kernel is a subobject of M. In order to make sense of that in an arbitrary category, we invert the arrows in the definition 6.1.4.

Definition 6.1.6 — Quotient object. Let N be an object in a category A. We say that two epimorphisms $p : N \to S$ and $q : N \to T$ are equivalent if there exists an isomorphism $S \to T$ making the diagram

$$\begin{tikzcd}
& N \arrow[dl, "p"'] \arrow[dr, "q"] & \\
S \arrow[rr, "\sim"] & & T
\end{tikzcd}$$

commute. In other words, p and q are equivalent if they are isomorphic in the coslice category $N \downarrow A$. A *quotient object* of N is an equivalence class for this equivalence relation.

As before, it is clear by the universal property that all cokernels of a morphism $M \to N$ belong to the same isomorphism class in $N \downarrow A$. So, by proving that cokernels are epic, we prove that every cokernel is a quotient object.

Proposition 6.1.4 Let $\varphi : M \to N$ a morphism in a category A with a zero-object 0 and suppose that $\pi : N \to C$ is its cokernel. Then π is an epimorphism and so $\operatorname{coker} \varphi$ is a quotient object of N.

Proof. We could do basically the same argument as in the proof of proposition 6.1.1, but we'll use this as an opportunity to understand a powerful idea: the duality principle. Let $\alpha, \beta : C \to D$ be morphisms such that $\alpha \circ \pi = \beta \circ \pi$. Inverting all the arrows, we see that $\pi^{op} : C \to N$ is the kernel of $\varphi^{op} : N \to M$ and $\pi^{op} \circ \alpha^{op} = \pi^{op} \circ \beta^{op}$. Since π^{op} is a monomorphism by proposition 6.1.1, $\alpha^{op} = \beta^{op}$ and so $\alpha = \beta$, proving that π is an epimorphism. $\square$

By inverting all the arrows as above, we can easily prove dual versions of the propositions 6.1.2 and 6.1.3, which we state below.

Proposition 6.1.5 Let $\varphi : M \to N$ be an epimorphism in a category A with a zero-object 0. Then $\operatorname{coker} \varphi$ is the zero morphism $N \to 0$.

Proposition 6.1.6 Let $\varphi : M \to N$ be a morphism in a category A with a zero-object. Then φ is a zero-morphism if and only if $\operatorname{coker} \varphi$ is, up to isomorphism, the identity on N.

Everything we did so far only makes sense given the existence of zero-morphisms in the category under consideration. There's a natural way in which a category may be endowed with such morphisms.

Definition 6.1.7 — Preadditive category. A category A is said to be *preadditive* if each set of morphisms $\mathrm{Hom}_{\mathsf{A}}(M, N)$ is endowed with an abelian group structure, in such a way that the composition maps are bilinear.

The exquisite reader may recognize that this is nothing but a category enriched over Ab. Explicitly, in a preadditive category it makes sense to add or subtract morphisms and this operation satisfies

$$\varphi \circ (\psi_1 + \psi_2) = \varphi \circ \psi_1 + \varphi \circ \psi_2 \qquad \text{and} \qquad (\varphi_1 + \varphi_2) \circ \psi = \varphi_1 \circ \psi + \varphi_2 \circ \psi,$$

whenever those compositions exist.

A preadditive category A may still lack zero-objects. But, given a zero-object, we have two natural notions of zero-morphism $M \to N$: the unique morphism $M \to N$ which factors through the zero object and the identity of $\mathrm{Hom}_{\mathsf{A}}(M, N)$. It is reassuring to know that they coincide.

Proposition 6.1.7 In a preadditive category A, the following conditions are equivalent:

(a) A has an initial object;

(b) A has a final object;

(c) A has a zero-object.

In that case, the zero-morphisms are exactly the identities for the group structure of the hom-sets.

Proof. Clearly, (c) implies both (a) and (b). Since the dual of a preadditive category is also preadditive, it suffices to prove that (a) implies (c). Let I be an initial object. The group $\mathrm{Hom}_{\mathsf{A}}(I, I)$ has only one element and so id_I coincides with the group identity of $\mathrm{Hom}_{\mathsf{A}}(I, I)$. Now, if $\varphi : M \to I$ is any morphism, then

$$\varphi = \mathrm{id}_I \circ \varphi = (\mathrm{id}_I + \mathrm{id}_I) \circ \varphi = \mathrm{id}_I \circ \varphi + \mathrm{id}_I \circ \varphi = \varphi + \varphi$$

and so $\mathrm{Hom}_{\mathsf{A}}(M, I)$ is the trivial group. This proves that I is also a final object. Finally, if A has a zero-object 0, then the groups $\mathrm{Hom}_{\mathsf{A}}(M, 0)$ and $\mathrm{Hom}_{\mathsf{A}}(0, N)$ are reduced to their identities and so, by the fact that composition is bilinear, the zero-morphism $M \to 0 \to N$ is the identity of $\mathrm{Hom}_{\mathsf{A}}(M, N)$. ☐

Observe that, in a preadditive category, two morphisms are equal if and only if their difference in the corresponding hom-set is 0. This implies that a morphism $\varphi : M \to N$ in a preadditive category is a monomorphism if and only if for all $\alpha : Z \to M$,

$$\varphi \circ \alpha = 0 \implies \alpha = 0.$$

Similarly, it is an epimorphism if and only if for all $\beta : N \to Z$,

$$\beta \circ \varphi = 0 \implies \beta = 0.$$

We are now in a position to prove a converse to the propositions 6.1.2 and 6.1.5.

Proposition 6.1.8 Let $\varphi : M \to N$ be a morphism in a preadditive category A. Then φ is a monomorphism if and only if $\ker \varphi$ is the zero-morphism $0 \to M$. Dually, φ is an epimorphism if and only if $\operatorname{coker} \varphi$ is the zero morphism $N \to 0$.

Proof. The fact that a monomorphism has the zero-morphism as its kernel was proved in proposition 6.1.2. Conversely, suppose that $0 \to M$ is a kernel for $\varphi : M \to N$, and let $\zeta : Z \to M$ be a morphism such that $\varphi \circ \zeta = 0$. The universal property implies that ζ factors through $0 \to M$ and so $\zeta = 0$, proving that φ is a monomorphism. The statement about epimorphisms follows by duality. ☐

In some sense, life is simpler in the world of modules, since finite products and coproducts coincide. Fortunately, this is already the case in preadditive categories.

Theorem 6.1.9 Let M and N be two objects in a preadditive category. Given a third object P, the following are equivalent:

(a) there exist natural projections $\pi_M : P \to M$ and $\pi_N : P \to N$ such that P satisfies the universal property of $M \times N$;

(b) there exist natural injections $\iota_M : M \to P$ and $\iota_N : N \to P$ such that P satisfies the universal property of $M \coprod N$;

(c) there exist morphisms $\pi_M : P \to M$, $\pi_N : P \to N$, $\iota_M : M \to P$ and $\iota_N : N \to P$ such that

$$\pi_M \circ \iota_M = \mathrm{id}_M, \quad \pi_N \circ \iota_N = \mathrm{id}_N, \quad \pi_M \circ \iota_N = 0, \quad \pi_N \circ \iota_M = 0,$$
$$\iota_M \circ \pi_M + \iota_N \circ \pi_N = \mathrm{id}_P\,.$$

Moreover, under these conditions we have that

$$\iota_M = \ker \pi_N, \quad \iota_N = \ker \pi_M, \quad \pi_M = \operatorname{coker} \iota_N, \quad \pi_N = \operatorname{coker} \iota_M.$$

If P satisfies any of the conditions above, we say that P is the direct sum $M \oplus N$.

Proof. By duality, it suffices to prove the equivalence of (a) and (c). Given (a), we use the universal property of products to obtain our desired morphisms ι_M and ι_N as the unique morphisms that satisfy $\pi_M \circ \iota_M = \mathrm{id}_M$, $\pi_N \circ \iota_N = \mathrm{id}_N$, $\pi_M \circ \iota_N = 0$ and

$\pi_N \circ \iota_M = 0$:

$$\begin{tikzcd}
& & M \\
M \arrow[r, dashed, "\iota_M"] \arrow[urr, bend left, "\mathrm{id}_M"] \arrow[drr, bend right, "0"'] & P \arrow[ur, "\pi_M"'] \arrow[dr, "\pi_N"] & \\
& & N
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& & M \\
N \arrow[r, dashed, "\iota_N"] \arrow[urr, bend left, "0"] \arrow[drr, bend right, "\mathrm{id}_N"'] & P \arrow[ur, "\pi_M"'] \arrow[dr, "\pi_N"] & \\
& & N.
\end{tikzcd}$$

We then affirm that $\iota_M \circ \pi_M + \iota_N \circ \pi_N = \mathrm{id}_P$. Indeed, observe that the left-hand side satisfies

$$\pi_M \circ (\iota_M \circ \pi_M + \iota_N \circ \pi_N) = \pi_M \circ \iota_M \circ \pi_M + \pi_M \circ \iota_N \circ \pi_N = \pi_M + 0 = \pi_M$$
$$\pi_N \circ (\iota_M \circ \pi_M + \iota_N \circ \pi_N) = \pi_N \circ \iota_M \circ \pi_M + \pi_N \circ \iota_N \circ \pi_N = 0 + \pi_N = \pi_N.$$

But then both $\iota_M \circ \pi_M + \iota_N \circ \pi_N$ and id_P fit in the place of the dotted morphism which makes the diagram

$$\begin{tikzcd}
& & M \\
P \arrow[r, dotted] \arrow[urr, bend left, "\pi_M"] \arrow[drr, bend right, "\pi_N"'] & P \arrow[ur, "\pi_M"'] \arrow[dr, "\pi_N"] & \\
& & N
\end{tikzcd}$$

commute. The uniqueness part of the universal property of products then implies that they are equal, proving (c).

Now, given (c) and an object Q with morphisms $\gamma_M : Q \to M$ and $\gamma_N : Q \to N$, we need to show that there is a unique morphism $\gamma : Q \to P$ making the diagram

$$\begin{tikzcd}
& & M \\
Q \arrow[r, dashed, "\gamma"] \arrow[urr, bend left, "\gamma_M"] \arrow[drr, bend right, "\gamma_N"'] & P \arrow[ur, "\pi_M"'] \arrow[dr, "\pi_N"] & \\
& & N
\end{tikzcd}$$

commute. For the existence, we define $\gamma := \iota_M \circ \gamma_M + \iota_N \circ \gamma_N$. The diagram above then commutes since

$$\pi_M \circ \gamma = \pi_M \circ \iota_M \circ \gamma_M + \pi_M \circ \iota_N \circ \gamma_N = \gamma_M + 0 = \gamma_M,$$
$$\pi_N \circ \gamma = \pi_N \circ \iota_M \circ \gamma_M + \pi_N \circ \iota_N \circ \gamma_N = 0 + \gamma_N = \gamma_N.$$

Moreover, if $\gamma' : Q \to P$ is another morphism making the diagram commute, then,

$$\begin{aligned}\gamma' &= \mathrm{id}_P \circ \gamma' = (\iota_M \circ \pi_M + \iota_N \circ \pi_N) \circ \gamma' \\ &= \iota_M \circ \pi_M \circ \gamma' + \iota_N \circ \pi_N \circ \gamma' \\ &= \iota_M \circ \gamma_M + \iota_N \circ \gamma_N = \gamma.\end{aligned}$$

This proves (a). Assuming all the equivalent conditions for P to be the direct sum $M \oplus N$, we now show that $\iota_M = \ker \pi_N$. Since $\pi_N \circ \iota_M = 0$, it suffices to prove that if $\zeta : Z \to P$ satisfies $\pi_N \circ \zeta = 0$, then there exists a unique morphism $Z \to M$ making the diagram

$$\begin{tikzcd} M \arrow[r, "\iota_M"] & P \arrow[r, shift left, "\pi_N"] \arrow[r, shift right, "0"'] & N \\ Z \arrow[u, dashed] \arrow[ur, "\zeta"'] & & \end{tikzcd}$$

commute. We affirm that $\pi_M \circ \zeta$ is the desired morphism $Z \to M$. Indeed, we observe that

$$\begin{aligned}\pi_M \circ (\iota_M \circ \pi_M \circ \zeta) &= \pi_M \circ \zeta \\ \pi_N \circ (\iota_M \circ \pi_M \circ \zeta) &= 0 = \pi_N \circ \zeta\end{aligned}$$

since $\pi_M \circ \iota_M = \mathrm{id}_M$ and $\pi_N \circ \iota_M = 0$. As before, using the uniqueness part of the universal property of products, we have that $\iota_M \circ \pi_M \circ \zeta = \zeta$, proving that the diagram above commutes. This is the unique morphism making it commute because, as $\pi_M \circ \iota_M = \mathrm{id}_M$, ι_M is a monomorphism.

We can prove that $\iota_N = \ker \pi_M$ in the same way and then $\pi_M = \operatorname{coker} \iota_N$ and $\pi_N = \operatorname{coker} \iota_M$ follow by duality. □

A perk from the fact that direct sums in preadditive categories have both canonical projections and canonical injections is that it allows us to write morphisms using a matrix notation. If M_1, M_2, N_1, N_2 are four objects in a preadditive category, a morphism

$$\varphi : M_1 \oplus M_2 \to N_1 \oplus N_2$$

is completely determined by the four morphisms

$$\begin{aligned}\varphi_{11} &= \pi_1 \circ \varphi \circ \iota_1 : M_1 \to N_1 \\ \varphi_{12} &= \pi_1 \circ \varphi \circ \iota_2 : M_2 \to N_1 \\ \varphi_{21} &= \pi_2 \circ \varphi \circ \iota_1 : M_1 \to N_2 \\ \varphi_{22} &= \pi_2 \circ \varphi \circ \iota_2 : M_2 \to N_2.\end{aligned}$$

Henceforth we will represent such a morphism φ by the matrix

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}.$$

Using the correspondence between A-module morphisms $A \to A$ and elements of A, this is nothing but the matrix notation used in linear algebra to describe A-module morphisms $A^{\oplus n} \to A^{\oplus m}$. Given another morphism

$$\psi : N_1 \oplus N_2 \to P_1 \oplus P_2,$$

the matrix representation of the composition $\psi \circ \varphi$ is simply the matrix product of the individual matrices. Similarly, the sum of two morphisms $M_1 \oplus M_2 \to N_1 \oplus N_2$ is represented by the sum of the individual matrices. It is clear that this notation allows us to describe morphisms of the form

$$\bigoplus_{i=1}^{n} M_i \to \bigoplus_{j=1}^{m} N_j$$

for any positive integers n, m.

Finally, we impose the existence of zero-objects and binary products. This suffices to guarantee the existence of finite products and coproducts, which coincide by the theorem 6.1.9.

> **Definition 6.1.8 — Additive category.** A preadditive category A is *additive* if it has a zero-object and binary products.

The prototypical example of an additive category surely is A-Mod but Ab and the category of Banach spaces with continuous linear maps are also examples of additive categories. Nevertheless, Grp is not additive since finite products and coproducts do not coincide, and neither is the category of Banach spaces with linear contractions as finite products and coproducts are not isometric.

Even though additive categories do not suffer from some of the problems we met before, they may still fail to have kernels or cokernels. For example, the category of finitely generated A-modules, when A is not noetherian, is additive but has morphisms without kernels. Furthermore, even when the additive category in consideration has kernels and cokernels, the usual first isomorphism theorem may not hold. We discuss those questions in the next section.

We finish this section with another interesting consequence of the theorem 6.1.9: the preadditive structure in an additive category is unique.

> **Proposition 6.1.10** Let A be a category with a zero-object and binary products. Then A has at most one abelian group structure on its hom-sets.

Proof. We endow A with any preadditive structure, and then we'll show that the addition of morphisms is actually determined by the limit-colimit structure of A.

Let $\varphi_1, \varphi_2 : M \to N$ be two morphisms in A. We define a map $\alpha : M \to M \oplus M$ by the universal property of products and a map $\beta : N \oplus N \to N$ by the universal property of coproducts:

$$\begin{tikzcd}
 & & M \\
M \arrow[r, dashed, "\alpha"] \arrow[urr, bend left, "\mathrm{id}_M"] \arrow[drr, bend right, "\mathrm{id}_M"'] & M \oplus M \arrow[ur, "\pi_M"'] \arrow[dr, "\pi'_M"] & \\
 & & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
N \arrow[dr, "\iota_N"'] \arrow[drr, bend left, "\mathrm{id}_N"] & & \\
 & N \oplus N \arrow[r, dashed, "\beta"] & N. \\
N \arrow[ur, "\iota'_N"] \arrow[urr, bend right, "\mathrm{id}_N"'] & &
\end{tikzcd}$$

We observe that, by the theorem 6.1.9 and the uniqueness of the universal property of products, the equations

$$\pi_M \circ (\iota_M + \iota'_M) = \pi_M \circ \iota_M + \pi_M \circ \iota'_M = \mathrm{id}_M + 0 = \mathrm{id}_M = \pi_M \circ \alpha,$$
$$\pi'_M \circ (\iota_M + \iota'_M) = \pi'_M \circ \iota_M + \pi'_M \circ \iota'_M = 0 + \mathrm{id}_M = \mathrm{id}_M = \pi'_M \circ \alpha$$

imply that $\alpha = \iota_M + \iota'_M$. The same exact reasoning shows that $\beta = \pi_N + \pi'_N$.

Now, we affirm that the composition $M \to M \oplus M \to N \oplus N \to N$, where the map $\psi : M \oplus M \to N \oplus N$ in the middle is given by

$$\begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix},$$

is the sum $\varphi_1 + \varphi_2$. Indeed, the composition is

$$\begin{aligned}
\beta \circ \psi \circ \alpha &= (\pi_N + \pi'_N) \circ \psi \circ (\iota_M + \iota'_M) \\
&= \pi_N \circ \psi \circ \iota_M + \pi'_N \circ \psi \circ \iota_M + \pi_N \circ \psi \circ \iota'_M + \pi'_N \circ \psi \circ \iota'_M \\
&= \varphi_1 + 0 + 0 + \varphi_2 = \varphi_1 + \varphi_2
\end{aligned}$$

by the very definition of ψ. □

6.2. Abelian categories

As we saw, whenever kernels and cokernels exist, they behave reasonably well. However, their possible lack of existence prevents us from going further. Moreover, despite the fact that kernels are always monomorphisms and cokernels are always epimorphisms, there's no guarantee that every monomorphism is a kernel and that every epimorphism is a cokernel. It just so happens that demanding these properties is enough for us to have the first isomorphism theorem.

Definition 6.2.1 — Abelian category. An additive category A is *abelian* if it possesses kernels and cokernels, if every monomorphism is the kernel of some morphism and if every epimorphism is the cokernel of some morphism.

For now, our only real example of an abelian category is A-Mod and its variants, such as Ab, the category of finitely generated modules over a noetherian ring, the category of finite abelian groups, their opposites, and so forth. But the reader shouldn't worry about having few examples; a plethora of abelian categories lie ahead.

In an abelian category, every monomorphism is the kernel of some morphism. We can actually be more precise.

Proposition 6.2.1 In an abelian category A, every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.

Proof. Let $\varphi : M \to N$ be a monomorphism which is the kernel of some morphism $\beta : N \to Z$. Since A is abelian, φ has a cokernel $\pi : N \to C$. The universal property of the cokernel shows that β factors through π.

$$\begin{tikzcd}
 & & Z \\
M \arrow[r, "\varphi"] & N \arrow[r, "\pi"] \arrow[ur, "\beta"] & C \arrow[u, dashed]
\end{tikzcd}$$

We show that φ satisfies the universal property defining the kernel of π. Let $K \to N$ be a morphism whose composition with π is the zero-morphism.

$$\begin{tikzcd}
 & & Z \\
M \arrow[r, "\varphi"] & N \arrow[r, "\pi"] \arrow[ur, "\beta"] & C \arrow[u] \\
K \arrow[ur] \arrow[urr, bend right, "0"] & &
\end{tikzcd}$$

By the commutativity of the diagram, $K \to N \to Z$ is also the zero-morphism. But φ is the kernel of β and so there exists a unique induced morphism $K \to M$, proving our claim. The statement about epimorphisms follows by duality. □

This proposition implies a quick criterion for deciding when a full subcategory of an abelian category is abelian.

Corollary 6.2.2 Let A be an abelian category and let C be a full subcategory. Suppose that the zero-object of A is in C and that C is closed under binary sums, kernels, and cokernels. Then C is also abelian.

Proof. The only thing we have to verify is that every monomorphism is the kernel of some morphism and that every epimorphism is the kernel of some morphism. Now, let φ be a monomorphism in C. This implies that its kernel in C is the zero-morphism but, since kernels in C and A coincide, φ is also a monomorphism in A. We observe that, as C is closed under cokernels, $\psi := \operatorname{coker}\varphi$ is a morphism in C. Since A is abelian, the preceding proposition implies that φ satisfies the universal property of $\ker\psi$ in A and, a fortiori, in C. This proves that every monomorphism in C is the kernel of some morphism in C. The result about epimorphisms follows by duality. □

Recall that in any category, isomorphisms are both monic and epic. The converse may fail to hold even in usual categories, such as Ring, where the inclusion $\mathbb{Z} \to \mathbb{Q}$ is a monomorphism and an epimorphism but is clearly not an isomorphism. Luckily, the proposition 6.2.1 also implies that the converse holds in abelian categories.

Corollary 6.2.3 Let $\varphi : M \to N$ be a morphism in an abelian category A. Then φ is an isomorphism if and only if it is both a monomorphism and an epimorphism.

Proof. If φ is both monic and epic, its kernel is $0 \to M$ and its cokernel is $N \to 0$. Furthermore, by proposition 6.2.1, φ is the kernel of $N \to 0$ and the cokernel of $0 \to M$. Now consider the diagram below.

$$\begin{tikzcd}
 & & N \arrow[d, "\mathrm{id}_N"] & \\
0 \arrow[r] & M \arrow[r, "\varphi"] & N \arrow[r] & 0
\end{tikzcd}$$

Since $N \to N \to 0$ is the zero morphism and φ is the kernel of $N \to 0$, we obtain a unique morphism $\psi : N \to M$ making the diagram

$$\begin{tikzcd}
 & & N \arrow[d, "\mathrm{id}_N"] \arrow[dl, dashed, "\psi"'] & \\
0 \arrow[r] & M \arrow[r, "\varphi"] & N \arrow[r] & 0
\end{tikzcd}$$

commute. As $\varphi \circ \psi = \mathrm{id}_N$, this shows that φ has a right-inverse. Similarly, the fact that φ is the cokernel of $0 \to M$ implies the existence of a unique morphism $\eta : N \to M$ such that the diagram

$$\begin{tikzcd}
 & M & & \\
0 \arrow[r] & M \arrow[u, "\mathrm{id}_M"] \arrow[r, "\varphi"] & N \arrow[r] \arrow[ul, dashed, "\eta"'] & 0
\end{tikzcd}$$

commutes. It follows that φ has both a left-inverse η and a right-inverse ψ. Thus, $\eta = \psi$ is a two-sided inverse of φ and so φ is an isomorphism. The converse holds in every category. □

We observe that this corollary implies that the category of Banach spaces with bounded linear maps is not abelian. A bounded linear map $T : X \to Y$ is a monomorphism if it's injective and an epimorphism if $\operatorname{im} T$ is dense in Y. But there exists monomorphisms with dense image which are not isomorphisms; the inclusion $\ell_1 \to \ell_2$, for example.

Earlier, we said that demanding every monomorphism to be a kernel and every epimorphism to be a cokernel is enough to guarantee the first isomorphism theorem. In order to understand how we should even enunciate such a result, we have to make sense of images in abelian categories. As with kernels and cokernels, this is best done via a suitable universal property.

Let's translate our intuitive notion of the image of a morphism $\varphi : M \to N$ in Set to a purely arrow-theoretic statement. The main point in Set is that $\operatorname{im} \varphi$ is the smallest subset of N to which we can restrict the codomain of φ to. In other words, we can factor $\varphi : M \to N$ as

$$\begin{tikzcd} M \arrow[r] & \operatorname{im} \varphi \arrow[r, hook] & N, \end{tikzcd}$$

where $\operatorname{im} \varphi \to N$ is injective and $\operatorname{im} \varphi$ is the smallest subset of N which allows this decomposition. Switching to categorical terms, we arrive at the following universal property: the image of $\varphi : M \to N$ is a monomorphism $\iota : K \to N$ such that φ factors through ι and that is initial with these properties. That is, if $L \to N$ is another monomorphism through which φ also factors, then it exists a unique morphism $K \to L$ such that the diagram

$$\begin{tikzcd} & L \arrow[dr, hook] & \\ M \arrow[ur] \arrow[r] \arrow[rr, bend right, "\varphi"'] & K \arrow[u, dashed] \arrow[r, hook, "\iota"] & N \end{tikzcd}$$

commutes. In an arbitrary category, it could very well happen that no morphism $\iota : K \to N$ satisfies this universal property. Luckily, this is never the case in the realm of abelian categories.

> **Proposition 6.2.4** Let $\varphi : M \to N$ be a morphism in an abelian category, and let $\iota : K \to N$ be the kernel of $\operatorname{coker} \varphi$. Then ι is a monomorphism through which φ factors, and it is initial with these properties.

Proof. It is clear that ι is a monomorphism by the fact that it is a kernel. Since $\iota : K \to N$ is the kernel of $\operatorname{coker} \varphi : N \to C_\varphi$, the diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] & N \arrow[r, shift left, "\operatorname{coker} \varphi"] \arrow[r, shift right, "0"'] & C_\varphi \\ K \arrow[ur, "\iota"'] & & \end{tikzcd}$$

commutes. The universal property of the kernel then implies the existence of a morphism $M \to K$ factoring φ through ι. We now show that ι satisfies the desired universal property. Let $\lambda : L \to N$ be another monomorphism through which φ factors, and consider its cokernel $N \to C_\lambda$.

$$\begin{tikzcd}
& L \arrow[dr, hook, "\lambda"] & & \\
M \arrow[ur] \arrow[r] \arrow[rr, bend right, "\varphi"'] & K \arrow[r, hook, "\iota"] & N \arrow[dr, "\operatorname{coker}\lambda"'] & \\
& & & C_\lambda
\end{tikzcd}$$

Since φ factors through λ, the composition $M \to N \to C_\lambda$ is 0. The universal property of $\operatorname{coker}\varphi$ induces a morphism $C_\varphi \to C_\lambda$:

$$\begin{tikzcd}
& L \arrow[dr, hook, "\lambda"] & & \\
M \arrow[ur] \arrow[r] \arrow[rr, bend right, "\varphi"'] & K \arrow[r, hook, "\iota"] & N \arrow[r, "\operatorname{coker}\varphi"] \arrow[dr, "\operatorname{coker}\lambda"'] & C_\varphi \arrow[d, dashed] \\
& & & C_\lambda.
\end{tikzcd}$$

Observe that since $K \to N \to C_\varphi$ is the zero-morphism, so is $K \to N \to C_\lambda$. But λ is a monomorphism, which implies that it is the kernel of $\operatorname{coker}\lambda$. Its universal property then implies the existence of a unique morphism $K \to L$ making the diagram commute. □

Since all there is to know about the image of a morphism φ is encoded in the $\operatorname{im}\varphi = \ker(\operatorname{coker}\varphi)$ mantra, we use it to *define* images from now on.

Definition 6.2.2 — Image. Let $\varphi : M \to N$ be a morphism in an abelian category. Its *image*, denoted $\operatorname{im}\varphi$, is the kernel of $\operatorname{coker}\varphi$.

As it is probably clear by now, the image of a morphism $\varphi : M \to N$ of A-modules is simply the inclusion $I \to N$, where I is the set-theoretical image of φ. Indeed, $\operatorname{coker}\varphi$ is simply $N \to N/I$ and its kernel is nothing but $I \to N$.

Inverting all the arrows, we arrive at the dual notion of the image of a morphism.

Definition 6.2.3 — Coimage. Let $\varphi : M \to N$ be a morphism in an abelian category. Its *coimage*, denoted $\operatorname{coim}\varphi$, is the cokernel of $\ker\varphi$.

By duality, the proposition 6.2.4 gives a universal property for the coimage of a morphism $\varphi : M \to N$ in an abelian category: it is an epimorphism $\pi : M \to C$ such that φ factors through π and such that if $M \to D$ is another epimorphism through which φ also factors, then it exists a unique morphism $D \to C$ making the diagram

$$\begin{tikzcd}
M \arrow[r, two heads, "\pi"'] \arrow[rr, bend left, "\varphi"] \arrow[dr, two heads] & C \arrow[r] & N \\
& D \arrow[u, dashed] \arrow[ur] &
\end{tikzcd}$$

commute.

Now, our sought-for first isomorphism theorem is simply a particular relation between the image and the coimage of a given morphism. In A-Mod, the coimage of a morphism $\varphi : M \to N$ is the quotient map $M \to M/K$, where K is the set-theoretical kernel of φ. The first isomorphism theorem in this context amounts to the fact that we can factor $\varphi : M \to N$ as

$$\begin{tikzcd}
M \arrow[r, "\operatorname{coim}\varphi"'] \arrow[rrr, bend left, "\varphi"] & M/K \arrow[r] & I \arrow[r, "\operatorname{im}\varphi"'] & N,
\end{tikzcd}$$

where the morphism in the middle, induced by φ, is an isomorphism. In this form, the result holds in arbitrary abelian categories.

Theorem 6.2.5 — First isomorphism theorem. Let $\varphi : M \to N$ be a morphism in an abelian category. Then φ can be decomposed as

$$\begin{tikzcd}
M \arrow[r] \arrow[rrr, bend left, "\varphi"] & C \arrow[r, "\sim"] & K \arrow[r] & N,
\end{tikzcd}$$

where $M \to C$ is the coimage of φ, $K \to N$ is its image and $C \to K$ is an isomorphism.

Proof. The universal properties of the image and of the coimage give two decompositions of φ as follows:

$$\begin{tikzcd}
& & K \arrow[dr, hook, "\operatorname{im}\varphi"] & \\
M \arrow[urr, "\alpha"] \arrow[rrr, "\varphi"] \arrow[dr, two heads, "\operatorname{coim}\varphi"'] & & & N. \\
& C \arrow[urr, "\beta"'] & &
\end{tikzcd}$$

In order to use the universal property of $\operatorname{im}\varphi$ to obtain an induced morphism $K \to C$, we must prove that β is a monomorphism. (Similarly, we could prove that α is an epimorphism and use the universal property of $\operatorname{coim}\varphi$.) Since every monomorphism is the kernel of its cokernel, $\ker(\operatorname{coim}\beta) = \ker(\operatorname{coker}(\ker\beta)) = \ker\beta$. It suffices then to show that $\ker(\operatorname{coim}\beta) = 0$. We observe that the composition of $\operatorname{coim}\beta$ and $\operatorname{coim}\varphi$

$$\begin{tikzcd}
M \arrow[r, two heads, "\operatorname{coim}\varphi"] \arrow[rrr, bend right, "\varphi"'] & C \arrow[r, two heads, "\operatorname{coim}\beta"'] \arrow[rr, bend left, "\beta"] & N' \arrow[r] & N
\end{tikzcd}$$

is an epimorphism through which φ factors. The universal property of $\operatorname{coim}\varphi$ then implies that $\operatorname{coim}\beta$ is an isomorphism, concluding that $\ker\beta = 0$ and so β is a monomorphism.

As we said above, the universal property of $\operatorname{im}\varphi$ induces a morphism $\psi : K \to C$ making the diagram

$$\begin{tikzcd}
 & & K \arrow[dr, hook, "\operatorname{im}\varphi"] \arrow[ddl, dashed, "\psi"'] & \\
M \arrow[urr, two heads, "\alpha"] \arrow[dr, two heads, "\operatorname{coim}\varphi"'] & & & N \\
 & C \arrow[urr, hook, "\beta"] & &
\end{tikzcd}$$

commute. Since $\beta \circ \psi = \operatorname{im}\varphi$ is a monomorphism, so is ψ. Similarly, the fact that $\psi \circ \alpha = \operatorname{coim}\varphi$ is an epimorphism implies that ψ has the same property. It follows that ψ is an isomorphism, and so it suffices to consider its inverse to be our desired morphism $C \to K$. □

As we'll see, this theorem even gives an alternative definition of abelian category. For now, suppose that $\varphi : M \to N$ is a morphism in an additive category that possesses kernels and cokernels. In this context, it is *not* true that $\ker(\operatorname{coker}\varphi) : K \to N$ satisfies the universal property of the image of φ[1] but, since $(\operatorname{coker}\varphi) \circ \varphi = 0$, the universal property of kernels implies that φ factors through $\ker(\operatorname{coker}\varphi)$.

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[dr, dashed] & N \\
 & K \arrow[u, "\ker(\operatorname{coker}\varphi)"']
\end{tikzcd}$$

Similarly, the universal property of cokernels implies that $M \to K$ factors through

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[1] For a counterexample, consider the morphism $\varphi : \mathbb{Z} \to \mathbb{Z}$ given by multiplication by 2 in the category of torsion-free abelian groups. The reader may verify that this is an additive category, with kernels and cokernels, and that $\ker(\operatorname{coker}\varphi) = \mathrm{id} : \mathbb{Z} \to \mathbb{Z}$. Then φ is another monomorphism through which φ factors, but there's no morphism induced by the universal property of images.

$\operatorname{coker}(\ker\varphi) : M \to C$ via a morphism $\overline{\varphi} : C \to K$.

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\operatorname{coker}(\ker\varphi)"'] & N \\ C \arrow[r, dashed, "\overline{\varphi}"] & K \arrow[u, "\ker(\operatorname{coker}\varphi)"'] \end{tikzcd}$$

Our previous theorem shows that $\overline{\varphi}$ is an isomorphism whenever we're dealing with an abelian category. Conversely, this property suffices to define an abelian category.

> **Proposition 6.2.6** Let A be an additive category that possesses kernels and cokernels. Then A is abelian if and only if for every morphism $\varphi : M \to N$, the induced morphism $\overline{\varphi} : C \to K$ is an isomorphism.

Proof. One direction was shown in the previous theorem. Conversely, suppose that $\varphi : M \to N$ is a monomorphism. Then $\ker\varphi = 0$ and so φ factors as

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}_M"'] & N \\ M \arrow[r, "\overline{\varphi}"] & K. \arrow[u, "\ker(\operatorname{coker}\varphi)"'] \end{tikzcd}$$

This implies that φ satisfies the universal property of $\ker(\operatorname{coker}\varphi)$. (Since φ and $\ker(\operatorname{coker}\varphi)$ define the same subobjects of N.) By duality, it follows that every epimorphism is a cokernel. □

6.3. Unions and intersections

Let M be an object in a (not necessarily abelian) category A. As we saw in the beginning of this chapter, a subobject of M is an equivalence class of monomorphisms $s : S \to M$. Given another subobject defined by $t : T \to M$, we say that s is *smaller than* t if there exists a morphism $S \to T$, automatically monic, making the diagram

$$\begin{tikzcd} S \arrow[rr, hook] \arrow[dr, hook, "s"'] & & T \arrow[dl, hook', "t"] \\ & M & \end{tikzcd}$$

commute. This is independent of the representatives chosen for each equivalence class. Also, the morphism $S \to T$ is unique whenever it exists. This endows the collection of all subobjects of M with the structure of a partially ordered class.[2] In particular, we are able to define the union and the intersection of a family of subobjects.

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[2] It need not be a set, even when the category in question is abelian. We say that a category is *well-powered* if the subobjects of every object constitute a set.

Definition 6.3.1 Let M be an object of a category A. The *union*, if it exists, of a family of subobjects of M is their supremum in the partially ordered class of subobjects. Similarly, the *intersection* of a family of subobjects is their infimum.

We'll often use the customary symbols $\cup$ and $\cap$ to denote the union and the intersection of subobjects, leaving their target implicit.

In A-Mod, the union of two submodules S and T of a given module M is simply their sum $S+T$. In other words, it's the image of the canonical morphism $S \oplus T \to M$, which sends (s,t) to $s+t$. This description generalizes to arbitrary abelian categories.

Proposition 6.3.1 Let A be an abelian category and $S_i \to M$ be a finite collection of subobjects. The union of those subobjects exists and is given by the image of the natural map $\bigoplus_i S_i \to M$.

Proof. Factoring each $S_i \to M$ through the coproduct and then factoring the resulting morphism through its image we obtain the diagram below.

$$\begin{tikzcd}
 & S_i \arrow[dl] \arrow[dr, hook] & \\
\bigoplus_i S_i \arrow[r] & K \arrow[r, hook] & M
\end{tikzcd}$$

In particular, $K \to M$ is a subobject which is greater than all of the $S_i \to M$. Now, suppose that $T \to M$ is another subobject through which all the $S_i \to M$ factor. The universal property of coproducts induces a dashed morphism making the diagram

$$\begin{tikzcd}
 & S_i \arrow[d] \arrow[ddl] \arrow[ddr, hook] & \\
 & T \arrow[dr, hook] & \\
\bigoplus_i S_i \arrow[r] \arrow[ur, dashed] & K \arrow[r, hook] & M
\end{tikzcd}$$

commute. (The lower triangle commutes by the unicity of the induced morphism $\bigoplus_i S_i \to M$.) Finally, the universal property of images induces a morphism $K \to T$, proving that $K \to M$ is indeed the supremum of the $S_i \to M$. ☐

The same proof shows that the preceding description also works for infinite unions, replacing the direct sums by coproducts, whenever those coproducts exist.

In a wide range of cases (even for some categories very dissimilar to A-Mod, such as Top), the proposition below describes binary intersections.

Proposition 6.3.2 Let A be a category with pullbacks. The intersection of two subobjects $S \to M$ and $T \to M$ exists and is given by their pullback.

Proof. We recall that, in absolute generality, pullbacks preserve monomorphisms. [3, Proposition 2.5.3] That is, if

$$\begin{tikzcd} P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \\ S \arrow[r, "s"] & M \end{tikzcd}$$

is a cartesian diagram and s is a monomorphism, then so is s'. Similarly for t and t', of course. In particular, $P \to M$ is a subobject which is less than $S \to M$ and $T \to M$. Moreover, $P \to M$ is their infimum, due to the universal property of pullbacks. □

Once again, the same proof shows that the intersection of a family of subobjects $S_i \to M$ exists and is given by the limit of the diagram constituted of those morphisms, as long as such limit exists.

Fortunately, abelian categories possess pullbacks and they have simple descriptions. In A-Mod, the pullback of two morphisms $\varphi : M \to P$ and $\psi : N \to P$ is given by submodule of $M \oplus N$ determined by the elements (m, n) satisfying $\varphi(m) = \psi(n)$. Basically the same description works more generally. In particular, the collection of subobjects of every object in an abelian category form a lattice.

Proposition 6.3.3 Let $s : S \to M$ and $t : T \to M$ be two morphisms in an abelian category A. The kernel of the morphism

$$(s, -t) : S \oplus T \to M$$

satisfies the universal property of the pullback $S \times_M T$. Dually, if $s' : N \to S$ and $t' : N \to T$ are two morphisms in A, the cokernel of

$$\begin{pmatrix} s' \\ -t' \end{pmatrix} : N \to S \oplus T$$

satisfies the universal property of the pushout $S \coprod_N T$.

Proof. Let $\pi_S : S \oplus T \to S$ and $\pi_T : S \oplus T \to T$ be the canonical projections. Moreover, denote the kernel of $(s, -t)$ by $\kappa : P \to S \oplus T$, and pose $s' := \pi_T \circ \kappa$, $t' := \pi_S \circ \kappa$. Being more precise, the first statement is that the square

$$\begin{tikzcd} P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \\ S \arrow[r, "s"'] & M \end{tikzcd}$$

is cartesian. We observe that

$$(s, -t) = (s, -t) \circ \mathrm{id}_{S \oplus T} = (s, -t) \circ \begin{pmatrix} \pi_S \\ \pi_T \end{pmatrix} = s \circ \pi_S - t \circ \pi_T.$$

This implies the commutativity of the square above, given that

$$s \circ t' - t \circ s' = s \circ \pi_S \circ \kappa - t \circ \pi_T \circ \kappa = (s, -t) \circ \kappa = 0.$$

We now prove that the square satisfies the universal property of pullbacks. Let $\varphi : Q \to S$ and $\psi : Q \to T$ be such that $s \circ \varphi = t \circ \psi$. Since

$$(s, -t) \circ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = s \circ \varphi - t \circ \psi = 0,$$

the universal property of kernels gives a unique morphism $\mu : Q \to P$ making the diagram

$$\begin{tikzcd} Q \arrow[r, "\mu"] \arrow[dr, "\begin{pmatrix} \varphi \\ \psi \end{pmatrix}"'] & P \arrow[d, "\kappa"] \\ & S \oplus T \end{tikzcd}$$

commute. Moreover, we have that

$$s' \circ \mu = \pi_T \circ \kappa \circ \mu = \pi_T \circ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \psi$$

and, similarly, that $t' \circ \mu = \varphi$. The unicity of these factorizations follows from the unicity in the universal properties of kernels and of products. As usual, the other statement follows by duality. □

As we recalled in the proof of proposition 6.3.2, pullbacks preserve monomorphisms. Dually, pushouts preserve epimorphisms. In abelian categories we have even more.

Corollary 6.3.4 Let A be an abelian category. Suppose that

$$\begin{tikzcd} P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \\ S \arrow[r, "s"] & M \end{tikzcd}$$

is a cartesian diagram in A, and that s is an epimorphism. Then s' is also an epimorphism and the square is also a pushout. Dually, the pushout of a monomorphism is a monomorphism, and the corresponding square is also a pullback.

Proof. We keep the same notations as in the proof of the previous proposition, and begin by proving that $(s, -t)$ is an epimorphism. Let $\rho : M \to N$ be a morphism such that $\rho \circ (s, -t) = 0$. Then, denoting by $\iota_S : S \to S \oplus T$ the natural injection, we have

$$0 = \rho \circ (s, -t) \circ \iota_S = \rho \circ (s \circ \pi_S - t \circ \pi_T) \circ \iota_S = \rho \circ s.$$

This implies that $\rho = 0$, for s is an epimorphism. In particular, $(s, -t) = \operatorname{coker} \kappa$ due to proposition 6.2.1.

Now, let $\sigma : T \to Z$ be a morphism such that $\sigma \circ s' = 0$. Since $s' = \pi_T \circ \kappa$, the universal property of cokernels gives a morphism $\zeta : M \to Z$ making the diagram

$$\begin{tikzcd}
P \arrow[r, "\kappa"] \arrow[dr, "s'"'] & S \oplus T \arrow[r, "{(s,-t)}"] \arrow[d, "\pi_T"] & M \arrow[d, dashed, "\zeta"] \\
& T \arrow[r, "\sigma"] & Z
\end{tikzcd}$$

commute. But the equation

$$\zeta \circ s = \zeta \circ (s, -t) \circ \iota_S = \sigma \circ \pi_T \circ \iota_S = 0$$

implies that $\zeta = 0$, since s is an epimorphism. Finally, the fact that π_T is epic and satisfies $\sigma \circ \pi_T = 0$ implies that $\sigma = 0$, proving that s' is an epimorphism as well.

We now show that our cartesian square is also cocartesian. Let $\eta : S \to Q$ and $\lambda : T \to Q$ be two morphisms making the diagram

$$\begin{tikzcd}
P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \arrow[ddr, bend left, "\lambda"] & \\
S \arrow[r, "s"'] \arrow[drr, bend right, "\eta"'] & M & \\
& & Q
\end{tikzcd}$$

commute. Observe that, by the universal property of pullbacks, there exists a dashed morphism making the diagram

$$\begin{tikzcd}
K \arrow[drr, bend left, "0"] \arrow[dr, dashed] \arrow[ddr, bend right, "\ker s"'] & & & \\
& P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \arrow[ddr, bend left, "\lambda"] & \\
& S \arrow[r, "s"'] \arrow[drr, bend right, "\eta"'] & M & \\
& & & Q
\end{tikzcd}$$

commute. This implies that $\eta \circ \ker s = 0$, and so the universal properties of cokernels (since s is the cokernel of $\ker s$) gives a morphism $\xi : M \to Q$ satisfying $\eta = \xi \circ s$. Moreover, we have that

$$\lambda \circ s' = \eta \circ t' = \xi \circ s \circ t' = \xi \circ t \circ s'.$$

It follows that $\lambda = \xi \circ t$, since s' is an epimorphism. In other words, ξ makes the diagram

$$\begin{tikzcd}
P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \arrow[ddr, bend left, "\lambda"] & \\
S \arrow[r, "s"'] \arrow[drr, bend right, "\eta"'] & M \arrow[dr, "\xi"] & \\
 & & Q
\end{tikzcd}$$

commute. Such a morphism is unique, due to s being an epimorphism. We conclude the result. The other statement follows by duality. □

Given two subobjects $S \to M$ and $T \to M$, we can naturally form the commutative diagram below.

$$\begin{tikzcd}
S \cap T \arrow[r, hook] \arrow[d, hook] & T \arrow[d, hook] \\
S \arrow[r, hook] & S \cup T
\end{tikzcd}$$

Since we can always complete this square into a diagram of the form

$$\begin{tikzcd}
S \cap T \arrow[r, hook] \arrow[d, hook] & T \arrow[d, hook] \arrow[ddr, hook, bend left] & \\
S \arrow[r, hook] \arrow[drr, hook, bend right] & S \cup T \arrow[dr, hook] & \\
 & & M,
\end{tikzcd}$$

the proposition 6.3.2 implies that our original square is always cartesian. The explicit description of pullbacks, along with the explicit description of unions, allows us to go further.

> **Corollary 6.3.5** Let $s : S \to M$ and $t : T \to M$ be two subobjects in an abelian category. The commutative diagram
>
> $$\begin{tikzcd}
> S \cap T \arrow[r, hook, "t'"] \arrow[d, hook, "s'"'] & T \arrow[d, hook] \\
> S \arrow[r, hook] & S \cup T
> \end{tikzcd}$$
>
> is cartesian and cocartesian.

Proof. The discussion above proves that our commutative diagram is cartesian. In order to show that it's also cocartesian, let $\sigma : S \to Q$ and $\tau : T \to Q$ be two morphisms making the diagram

$$\begin{tikzcd}
S \cap T \arrow[r, hook, "t'"] \arrow[d, hook, "s'"'] & T \arrow[d, hook] \arrow[ddr, bend left, "\tau"] & \\
S \arrow[r, hook] \arrow[drr, bend right, "\sigma"'] & S \cup T & \\
 & & Q
\end{tikzcd}$$

commute. The key to proving the existence and uniqueness of an induced morphism $S \cup T \to Q$ is the calculation of the kernel of (s, t). For now, assume it to be

$$\begin{pmatrix} s' \\ -t' \end{pmatrix} : S \cap T \to S \oplus T,$$

and let's see how this solves the problem. The proposition 6.3.1 implies that $S \cup T$ is the target of the coimage of (s, t) (i.e., the cokernel of the morphism above). Since $\sigma \circ s' = \tau \circ t'$, the universal property of cokernels gives our desired morphism.

$$\begin{tikzcd}
 & & Q \\
S \cap T \arrow[r, "\begin{pmatrix} s' \\ -t' \end{pmatrix}"] & S \oplus T \arrow[ur, "(\sigma,\tau)"] \arrow[r, "\operatorname{coim}(s,t)"'] & S \cup T \arrow[u, dashed]
\end{tikzcd}$$

Now, let's prove that the kernel of (s, t) is the one we described by showing that it satisfies the universal property. First of all, the commutativity of the diagram giving $S \cap T$ as the pullback of s and t implies that

$$(s, t) \circ \begin{pmatrix} s' \\ -t' \end{pmatrix} = s \circ s' - t \circ t' = 0.$$

If $\zeta : Z \to S \oplus T$ is any other morphism satisfying $0 = (s, t) \circ \zeta = (s \circ \pi_S + t \circ \pi_T) \circ \zeta$, the diagram

$$\begin{tikzcd}
Z \arrow[drr, bend left, "-\pi_T \circ \zeta"] \arrow[ddr, bend right, "\pi_S \circ \zeta"'] & & \\
 & S \cap T \arrow[r, "t'"] \arrow[d, "s'"'] & T \arrow[d, "t"] \\
 & S \arrow[r, "s"] & M
\end{tikzcd}$$

commutes and the universal property of pullbacks gives a morphism $\zeta' : Z \to S \cap T$ satisfying $\pi_S \circ \zeta = s' \circ \zeta'$ and $\pi_T \circ \zeta = -t' \circ \zeta'$. This implies that

$$\zeta = \begin{pmatrix} s' \\ -t' \end{pmatrix} \circ \zeta',$$

and finishes the proof. □

The preceding result is usually phrased as the motto *binary unions in abelian categories are effective*. This means that, in order to define a morphism $S \cup T \to P$, it suffices to find morphisms $S \to P$ and $T \to P$ which agree on the intersection $S \cap T$.

A final interesting result, which will be the soul of the next few propositions, can also be proved using the same circle of ideas.

Proposition 6.3.6 Let A be an abelian category. Given a commutative square

$$\begin{tikzcd}
P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \\
S \arrow[r, "s"] & M
\end{tikzcd}$$

in A, consider the induced morphisms $k : K' \to K$ and $c : C' \to C$ between the kernels and cokernels of s' and s:

$$\begin{tikzcd}
K' \arrow[r, "\ker s'"] \arrow[d, "k"'] & P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[r, "\operatorname{coker} s'"] \arrow[d, "t"] & C' \arrow[d, "c"] \\
K \arrow[r, "\ker s"'] & S \arrow[r, "s"'] & M \arrow[r, "\operatorname{coker} s"'] & C.
\end{tikzcd}$$

If the original square is cartesian, then k is an isomorphism and c is a monomorphism. Dually, if the original square is cocartesian, then k is an epimorphism and c is an isomorphism.

Proof. Suppose that our square is cartesian. We prove that k is an isomorphism. The universal property of pullbacks gives a dashed morphism, making the diagram

$$\begin{tikzcd}
K \arrow[drr, bend left, "0"] \arrow[dr, dashed, "p"] \arrow[ddr, bend right, "\ker s"'] & & \\
& P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \\
& S \arrow[r, "s"'] & M
\end{tikzcd}$$

commute. Then, since $K \to P \to T$ is zero, the universal property of kernels gives a dashed morphism $k' : K \to K'$ making the diagram

$$\begin{tikzcd}
K' \arrow[r, "\ker s'"] & P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \\
K \arrow[u, dashed, "k'"] \arrow[ur, "p"] \arrow[r, "\ker s"'] & S \arrow[r, "s"'] & M
\end{tikzcd}$$

commute. Checking the commutativity of the previous diagrams, we remark that

$$\begin{aligned} t' \circ (\ker s') \circ k' \circ k = t' \circ p \circ k = (\ker s) \circ k = t' \circ (\ker s') \\ s' \circ (\ker s') \circ k' \circ k = s' \circ p \circ k = 0 \circ k = s' \circ (\ker s'). \end{aligned}$$

In other words, $\ker s'$ and $(\ker s') \circ k' \circ k$ are two morphisms making the diagram

$$\begin{tikzcd} K' \arrow[dr, shift left, "\ker s'"] \arrow[dr, shift right, "(\ker s') \circ k' \circ k"'] & & \\ & P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \\ & S \arrow[r, "s"'] & M \end{tikzcd}$$

commute. The uniqueness in the universal property of pullbacks implies that they're equal. Since $\ker s'$ is a monomorphism, $k' \circ k = \mathrm{id}_{K'}$. Furthermore,

$$(\ker s) \circ k \circ k' = t' \circ (\ker s') \circ k' = \ker s.$$

As $\ker s$ is monic, we have $k \circ k' = \mathrm{id}_K$; proving that k is an isomorphism.

We now suppose our original square to be cocartesian and prove that k is epic. A first observation is that the left square in

$$\begin{tikzcd} P \arrow[r, "\mathrm{coim}\, s'"] \arrow[d, "t'"] & I' \arrow[r, "\mathrm{im}\, s'"] \arrow[d] & T \arrow[d, "t"] \\ S \arrow[r, "\mathrm{coim}\, s"] & I \arrow[r, "\mathrm{im}\, s"] & M, \end{tikzcd}$$

is also cocartesian. (The vertical arrow on the middle exists by the universal property of kernels, using that $\mathrm{im}\, s = \ker(\mathrm{coker}\, s)$, and the left square commutes by the fact that $\mathrm{im}\, s$ is monic.) Indeed, let $I' \to Q$ and $S \to Q$ be two maps making the natural square commute and consider the pushout of $Q \leftarrow I' \to T$:

$$\begin{tikzcd} P \arrow[r, "\mathrm{coim}\, s'"] \arrow[d, "t'"] & I' \arrow[rr, "\mathrm{im}\, s'"] \arrow[d] \arrow[ddr, bend left] & & T \arrow[d, "t"] \arrow[ddr, bend left] & \\ S \arrow[r, "\mathrm{coim}\, s"] \arrow[drr, bend right] & I \arrow[rr, "\mathrm{im}\, s"] & & M & \\ & & Q \arrow[rr] & & Q'. \end{tikzcd}$$

The universal property of pushouts then induces a map $M \to Q'$. The commutativity of the diagram below implies that $K \to S \to Q \to Q'$ is the zero-morphism. Since

$Q \to Q'$ is monic, by the corollary 6.3.4, $K \to S \to Q$ is already zero.

$$\begin{tikzcd}
K' \arrow[r, "\ker s'"] \arrow[d] & P \arrow[r, "\operatorname{coim} s'"] \arrow[d, "t'"] & I' \arrow[rr, "\operatorname{im} s'"] \arrow[d] \arrow[ddr, bend left] & & T \arrow[d, "t"] \arrow[ddr, bend left] & \\
K \arrow[r, "\ker s"] & S \arrow[r, "\operatorname{coim} s"] \arrow[drr, bend right] & I \arrow[rr, "\operatorname{im} s"] & & M \arrow[dr, dashed] & \\
& & & Q \arrow[rr] & & Q'
\end{tikzcd}$$

As $\operatorname{coim} s = \operatorname{coker}(\ker s)$, the universal property of cokernels induces a unique morphism $I \to Q$ making the triangle (1), below, commute.

$$\begin{tikzcd}
P \arrow[r, "\operatorname{coim} s'"] \arrow[d, "t'"] & I' \arrow[rr, "\operatorname{im} s'"] \arrow[d] \arrow[ddr, bend left] & & T \arrow[d, "t"] \arrow[ddr, bend left] & \\
S \arrow[r, "\operatorname{coim} s"] \arrow[drr, bend right] \arrow[r, phantom, shift right=6ex, "(1)"] & I \arrow[rr, "\operatorname{im} s"] \arrow[rr, phantom, "(2)" near start] \arrow[dr] & & M \arrow[dr] \arrow[l, phantom, shift right=6ex, "(3)"] & \\
& & Q \arrow[rr] & & Q'
\end{tikzcd}$$

Since $Q \to Q'$ is monic, to see if the triangle (2) commutes, it suffices to post-compose with $Q \to Q'$. But $I' \to Q \to Q'$ coincides with $I' \to I \to M \to Q'$, so it suffices to see that the square (3) commutes. As $\operatorname{coim} s$ is epic, it suffices to pre-compose with it. Finally the commutativity of the triangle (1), already known, implies the commutativity of (2). We conclude the proof that our square is a pushout.

Now, let $h : K \to L$ be a morphism satisfying $h \circ k = 0$ and consider the diagram

$$\begin{tikzcd}
K' \arrow[r, hook, "\ker s'"] \arrow[d, "k"] & P \arrow[r, "\operatorname{coim} s'"] \arrow[d, "t'"] & I' \arrow[d] \\
K \arrow[r, hook, "\ker s"] \arrow[d, "h"] & S \arrow[r, "\operatorname{coim} s"] \arrow[d] & I \\
L \arrow[r, hook] & N, &
\end{tikzcd}$$

whose lower square is a pushout. We recall that, by the corollary 6.3.4, the morphism $L \to N$ is monic. The commutativity of the diagram implies that $K' \to P \to S \to N$ is the zero-morphism and so, since $\operatorname{coim} s' = \operatorname{coker}(\ker s')$, the universal property of cokernels yields a morphism $I' \to N$ making

$$\begin{tikzcd}
K' \arrow[r, "\ker s'"] & P \arrow[r, "\operatorname{coim} s'"] \arrow[d, "t'"] & I' \arrow[ddl, dashed] \\
& S \arrow[d] & \\
& N &
\end{tikzcd}$$

commute. But then, the universal property of pushouts gives a dashed morphism making the diagram

$$\begin{tikzcd}
K' \arrow[r, "\ker s'"] \arrow[d, "k"] & P \arrow[r, "\operatorname{coim} s'"] \arrow[d, "t'"] & I' \arrow[d] \\
K \arrow[r, "\ker s"] \arrow[d, "h"] & S \arrow[r, "\operatorname{coim} s"] \arrow[d] & I \arrow[dl, dashed] \\
L \arrow[r] & N &
\end{tikzcd}$$

commute. Since $K \to S \to I$ is the zero-morphism, the commutativity of the diagram implies that so is $K \to L \to N$. Finally, the fact that $L \to N$ is monic gives that $h = 0$. The other statements follow by duality. □

We defined an abelian category by imposing the first isomorphism theorem. Somewhat surprising, all the other isomorphism theorems are also true in this generality.

If $t : T \to M$ is a subobject, we'll denote the target of $\operatorname{coker} t$ by M/T, as it would be in $A\text{-}\mathsf{Mod}$. We remark that if $S \to M$ is a subobject containing t, then S/T is naturally a subobject of M/T. That is, there exists a dashed monomorphism making the diagram

$$\begin{tikzcd}
 & & T \arrow[dl, hook'] \arrow[dr, hook] & & \\
 & S \arrow[rr, hook] \arrow[dl] & & M \arrow[dr] & \\
S/T \arrow[rrrr, hook, dashed] & & & & M/T
\end{tikzcd}$$

commute. Indeed, the universal property of the cokernel on the left gives the existence, and the universal property of the cokernel on the right implies that the trapezoid above is a pushout; proving that the dashed morphism is monic.

Proposition 6.3.7 Let $t : T \to M$ be a subobject in an abelian category. Then,

$$\begin{aligned} u : \{\text{subobjects of } M \text{ containing } t\} &\to \{\text{subobjects of } M/T\} \\ (S \to M) &\mapsto (S/T \to M/T) \end{aligned}$$

is a lattice isomorphism. Moreover, if $S \to M$ is a subobject containing t, the objects

$$(M/T)/(S/T) \qquad \text{and} \qquad M/S$$

are isomorphic.

Before we begin the proof, recall that a partially ordered class may be seen naturally as a category. In this context, a lattice is a partially ordered class with binary products and coproducts. Similarly, a morphism of lattices can be seen as a functor preserving such (co)products.

Proof. We define an explicit inverse to u. Consider the function v, which sends a subobject $Q \to M/T$ to the top arrow in the pullback

$$\begin{tikzcd}
P \arrow[r, hook] \arrow[d] & M \arrow[d] \\
Q \arrow[r, hook] & M/T.
\end{tikzcd}$$

Since $T \to M \to M/T$ is zero, the universal property of pullbacks gives a dashed morphism making the diagram

$$\begin{tikzcd}
T \arrow[rrd, hook, bend left, "t"] \arrow[rd, dashed] \arrow[rdd, bend right, "0"'] & & \\
& P \arrow[r, hook] \arrow[d] & M \arrow[d, "\mathrm{coker}\ t"] \\
& Q \arrow[r, hook] & M/T
\end{tikzcd}$$

commute, proving that $P \to M$ contains $T \to M$. It's clear that both u and v are order-preserving. In other words, they are functors. Applying u to the subobject $P \to M$, we obtain the commutative diagram below.

$$\begin{tikzcd}
& T \arrow[ld, hook'] \arrow[rd, hook] & \\
P \arrow[rr, hook] \arrow[rd, two heads] \arrow[ddd, two heads] & & M \arrow[ddd, two heads] \\
& P/T \arrow[rdd, hook] & \\
& & \\
Q \arrow[rr, hook] & & M/T
\end{tikzcd}$$

Observe that the composition $T \to P \to Q \to M/T$ is zero, due to the commutativity of the diagram. Actually, $T \to P \to Q$ is already zero, as $Q \to M/T$ is monic. Then, the universal property of cokernels gives a morphism $P/T \to Q$ making the diagram above commute. This morphism is both monic and epic, by the commutativity of the triangles on its sides. In other words, $u \circ v$ is the identity functor.

Now, let $S \to M$ be a subobject containing $t : T \to M$. We recall that the square

$$\begin{tikzcd}
S \arrow[r, hook] \arrow[d, two heads] & M \arrow[d, two heads] \\
S/T \arrow[r, hook] & M/T
\end{tikzcd}$$

is cocartesian. The corollary 6.3.4 implies that it's also cartesian, proving that $v \circ u$ is also the identity functor. Since u is an equivalence of categories, it preserves products and coproducts. In particular, it's an isomorphism of lattices.

The isomorphism between $(M/T)/(S/T)$ and M/S follows from the proposition 6.3.6, applied to the cocartesian square above. ☐

The last isomorphism theorem also follows from the machinery developed in this section.

> **Proposition 6.3.8** Let $S \to M$ and $T \to M$ be subobjects in an abelian category. Then the objects
> $$(S \cup T)/T \qquad \text{and} \qquad S/(S \cap T)$$
> are isomorphic.

Proof. Since binary unions in abelian categories are effective, the commutative diagram

$$\begin{tikzcd} S \cap T \arrow[r, hook] \arrow[d, hook] & T \arrow[d, hook] \\ S \arrow[r, hook] & S \cup T \end{tikzcd}$$

is a pushout. The result then follows from the same proposition 6.3.6. ☐

6.4. Exactness in abelian categories

After all this foundational work, we can at long last understand how exact sequences work in an abelian category.

> **Definition 6.4.1 — Exact sequence.** Consider a sequence of objects and morphisms in an abelian category:
> $$\cdots \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P \longrightarrow \cdots.$$
> We say that this sequence is *exact* at N if $\ker\psi$ and $\operatorname{im}\varphi$ define the same subobject of N. It is *exact* if it's exact at every object.

As it is the case in $A\text{-}\mathsf{Mod}$, most properties about morphisms can be stated in terms of exact sequences. For example,

$$0 \longrightarrow M \xrightarrow{\varphi} N$$

is an exact sequence if and only if φ is a monomorphism. Likewise,

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P$$

is an exact sequence if and only if φ is a kernel of ψ. Also,

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P \longrightarrow 0.$$

is exact if and only if φ is a kernel of ψ and ψ is cokernel of φ. These last exact sequences are so important that they deserve a name.

Definition 6.4.2 — Short exact sequence. An exact sequence of the form

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P \longrightarrow 0.$$

is said to be a *short exact sequence*.

Another reason for the importance of short exact sequences is that we can check the exactness of an arbitrary sequence by intertwining it with short exact sequences. Let's illustrate this procedure with a sequence of the form

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \xrightarrow{\varphi_3} M_4.$$

Using the theorem 6.2.5, we can enlarge our diagram to be

$$\begin{tikzcd}
 & & C_2 \arrow[dr, "\mathrm{im}\,\varphi_1"] & & & & C_4 \arrow[dr, "\mathrm{im}\,\varphi_3"] & & \\
 & M_1 \arrow[ur, "\mathrm{coim}\,\varphi_1"] \arrow[rr, "\varphi_1"] & & M_2 \arrow[rr, "\varphi_2"] \arrow[dr, "\mathrm{coim}\,\varphi_2"'] & & M_3 \arrow[ur, "\mathrm{coim}\,\varphi_3"] \arrow[rr, "\varphi_3"] & & M_4 \arrow[dr, "\mathrm{coker}\,\varphi_3"] & \\
C_1 \arrow[ur, "\ker\varphi_1"] & & & & C_3 \arrow[ur, "\mathrm{im}\,\varphi_2"'] & & & & C_5.
\end{tikzcd}$$

Using that kernels and images are monic and that cokernels and coimages are epic, we obtain a yet larger diagram which is exact at all the C_i, at M_1, and at M_4.

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & \\
 & & & C_2 \arrow[ur] \arrow[dr, "\mathrm{im}\,\varphi_1"] & & & & C_4 \arrow[ur] \arrow[dr, "\mathrm{im}\,\varphi_3"] & & & \\
 & & M_1 \arrow[ur, "\mathrm{coim}\,\varphi_1"] \arrow[rr, "\varphi_1"] & & M_2 \arrow[rr, "\varphi_2"] \arrow[dr, "\mathrm{coim}\,\varphi_2"'] & & M_3 \arrow[ur, "\mathrm{coim}\,\varphi_3"] \arrow[rr, "\varphi_3"] & & M_4 \arrow[dr, "\mathrm{coker}\,\varphi_3"] & & \\
 & C_1 \arrow[ur, "\ker\varphi_1"] & & & & C_3 \arrow[ur, "\mathrm{im}\,\varphi_2"'] \arrow[dr] & & & & C_5 \arrow[dr] & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & & & & 0
\end{tikzcd}$$

Now, we affirm that our original sequence is exact if and only if those four diagonal sequences are exact. Indeed, the only place where the diagonal sequences could lack exactness is at M_2 and M_3. Being exact at M_2 means that $\ker(\mathrm{coim}\,\varphi_2) = \mathrm{im}(\mathrm{im}\,\varphi_1)$ which is equivalent to $\ker\varphi_2 = \mathrm{im}\,\varphi_1$. The same holds for exactness at M_3, and it's clear that this procedure generalizes to sequences of arbitrary length.

A particularly frequent kind of short exact sequence appears when we consider the direct sum of two objects M and N. Since $M \oplus N$ fulfills both the role of the product and the coproduct of M and N, we have a natural injection $\iota : M \to M \oplus N$ and a natural projection $\pi : M \oplus N \to N$. These objects fit nicely into a sequence

$$0 \longrightarrow M \xrightarrow{\iota} M \oplus N \xrightarrow{\pi} N \longrightarrow 0,$$

which is exact since ι is the kernel of π and π is the cokernel of ι. (Theorem 6.1.9.) This is the prototypical example of a split exact sequence.

Definition 6.4.3 — Split exact sequence. A short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

is *split* if there's a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\sim"] & N \arrow[r] \arrow[d, "\sim"] & P \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & M' \arrow[r, "\iota"] & M' \oplus P' \arrow[r, "\pi"] & P' \arrow[r] & 0
\end{tikzcd}$$

in which all the vertical maps are isomorphisms, ι is the natural injection and π is the natural projection.

Understanding which exact sequences are split will allow us to characterize injective and projective objects, to prove a criterion for when a morphism has a right- or left-inverse, and to gain a refined version of the first isomorphism theorem. The following theorem takes care of these last two tasks.

Theorem 6.4.1 — Splitting lemma. A short exact sequence of the form

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P \longrightarrow 0$$

is split if and only if one of the conditions below is satisfied:

(a) there exists a morphism $\sigma : P \to N$ such that $\psi \circ \sigma = \mathrm{id}_P$;

(b) there exists a morphism $\rho : N \to M$ such that $\rho \circ \varphi = \mathrm{id}_M$.

Proof. If the sequence is split, then by composing the natural injections/projections with the vertical maps in the definition 6.4.3 we obtain the desired morphisms $\sigma : P \to N$ and $\rho : N \to M$.

Conversely, we suppose that (a) holds and prove that the sequence is split. Our approach will be based on the construction of a morphism $\rho : N \to M$ as in (b) such that

$$\rho \circ \varphi = \mathrm{id}_M, \quad \psi \circ \sigma = \mathrm{id}_P, \quad \rho \circ \sigma = 0, \quad \psi \circ \varphi = 0,$$
$$\varphi \circ \rho + \sigma \circ \psi = \mathrm{id}_N \, .$$

This is enough for the theorem 6.1.9 to imply that N is isomorphic to the direct sum of M and P. We already have two of the equations: $\psi \circ \sigma = \mathrm{id}_P$ and $\psi \circ \varphi = 0$.

In order to find a morphism ρ such that $\varphi \circ \rho + \sigma \circ \psi = \mathrm{id}_N$, we consider the morphism $\mathrm{id}_N - \sigma \circ \psi$. Observe that

$$\psi \circ (\mathrm{id}_N - \sigma \circ \psi) = \psi - \underbrace{\psi \circ \sigma}_{\mathrm{id}_P} \circ \psi = 0.$$

The universal property of kernels, by the fact that $\varphi = \ker \psi$, implies the existence of a unique morphism $\rho : N \to M$ such that $\varphi \circ \rho = \mathrm{id}_N - \sigma \circ \psi$, proving another equation.

Finally, we observe that, since φ is a monomorphism,

$$\varphi \circ \rho \circ \varphi = (\mathrm{id}_N - \sigma \circ \psi) \circ \varphi = \varphi - \sigma \circ \underbrace{\psi \circ \varphi}_{0} = \varphi$$

implies that $\rho \circ \varphi = \mathrm{id}_M$. Similarly,

$$\varphi \circ \rho \circ \sigma = (\mathrm{id}_N - \sigma \circ \psi) \circ \sigma = \sigma - \sigma \circ \underbrace{\psi \circ \sigma}_{\mathrm{id}_P} = 0$$

and so $\rho \circ \sigma = 0$, proving the last equation.

The proof that (b) implies that the sequence is split is basically the same. □

As promised, the splitting lemma gives a necessary and sufficient condition for a morphism to have a right- or left-inverse. We recall that a morphism that has a right-inverse is necessarily an epimorphism and that a morphism that has a left-inverse is necessarily a monomorphism.

Corollary 6.4.2 Let $\varphi : M \to N$ be a morphism in an abelian category. Then φ has a left-inverse if and only if the sequence

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\operatorname{coker} \varphi} C \longrightarrow 0$$

is split, and it has a right-inverse if and only if the sequence

$$0 \longrightarrow K \xrightarrow{\ker \varphi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is split.

The splitting lemma also provides a refinement of the first isomorphism theorem. For that, we observe that a morphism $\varphi : M \to N$ determines a sequence

$$0 \longrightarrow K \xrightarrow{\ker \varphi} M \xrightarrow{\operatorname{coim} \varphi} I \longrightarrow 0,$$

which is exact since $\ker \varphi$ is the kernel of $\operatorname{coim} \varphi = \operatorname{coker}(\ker \varphi)$ (every monomorphism is the kernel of its cokernel) and $\operatorname{coim} \varphi$ is the cokernel of $\ker \varphi$. We also recall that, due to the first isomorphism theorem, I is isomorphic to the source of $\operatorname{im} \varphi$.

Corollary 6.4.3 Let $\varphi : M \to N$ be a morphism in an abelian category, let $\ker \varphi : K \to M$ be its kernel and $\operatorname{coim} \varphi : M \to I$ be its coimage. If there exists a morphism $\sigma : I \to M$ such that $(\operatorname{coim} \varphi) \circ \sigma = \mathrm{id}_I$ or a morphism $\rho : M \to K$ such that $\rho \circ \ker \varphi = \mathrm{id}_K$, then $M \cong K \oplus I$.

In the category of finite-dimensional vector spaces over a field, this result holds unconditionally, since two such vector spaces are isomorphic if and only if they have the same dimension. Thus, this corollary follows from the rank-nullity theorem. But, in general abelian categories, the decomposition $M \cong K \oplus I$ need not hold.[3]

6.5. Functors on abelian categories

Just as all the useful morphisms on a group must preserve its structure, so must the useful functors on a preadditive category.

Definition 6.5.1 — Additive functor. Let A and B be two preadditive categories. A functor $F : \mathsf{A} \to \mathsf{B}$ is said to be *additive* if, for all objects M, N in A, the induced map

$$\begin{aligned} \operatorname{Hom}_{\mathsf{A}}(M, N) &\to \operatorname{Hom}_{\mathsf{B}}(F(M), F(N)) \\ \varphi &\mapsto F(\varphi) \end{aligned}$$

is a morphism of groups.

Basically all the functors defined between preadditive categories that we'll encounter are additive. Some examples are $\operatorname{Hom}_{\mathsf{A}}(M, -)$ and, in A-Mod, the tensor product functor $M \otimes_A -$.

There's an interesting criterion for a functor to be additive. For that, we observe that if $F : \mathsf{A} \to \mathsf{B}$ is a functor between additive categories and M, N are two objects of A, then

---

[3] Just take the projection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ in the category of abelian groups, for example.

the universal property of products induces a morphism $F(M \oplus N) \to F(M) \oplus F(N)$:

$$\begin{tikzcd}
 & & F(M) \\
F(M \oplus N) \arrow[r, dashed] \arrow[urr, bend left, "F(\pi_M)"] \arrow[drr, bend right, "F(\pi_N)"'] & F(M) \oplus F(N) \arrow[ur] \arrow[dr] & \\
 & & F(N),
\end{tikzcd}$$

where $F(M) \oplus F(N) \to F(M)$ and $F(M) \oplus F(N) \to F(N)$ are the natural projections. Similarly, the universal property of coproducts induces a morphism $F(M) \oplus F(N) \to F(M \oplus N)$.

Proposition 6.5.1 Let $F : A \to B$ be a functor between additive categories. Then the following are equivalent:

(a) F is additive;

(b) the natural map $F(M) \oplus F(N) \to F(M \oplus N)$ is an isomorphism for every M, N in A;

(c) the natural map $F(M \oplus N) \to F(M) \oplus F(N)$ is an isomorphism for every M, N in A.

Proof. Due to the fact that an additive functor preserves composition and addition of morphisms, the theorem 6.1.9 gives automatically that (a) implies (b) and (c). Also, (b) and (c) are equivalent since the uniqueness part of the universal property of the coproduct

$$\begin{tikzcd}
F(M) \arrow[dr] \arrow[drr, bend left, "F(\iota_M)"] \arrow[drrr, bend left] & & & \\
 & F(M) \oplus F(N) \arrow[r] & F(M \oplus N) \arrow[r] & F(M) \oplus F(N) \\
F(N) \arrow[ur] \arrow[urr, bend right, "F(\iota_N)"'] \arrow[urrr, bend right] & & &
\end{tikzcd}$$

implies that $F(M) \oplus F(N) \to F(M \oplus N) \to F(M) \oplus F(N)$ is the identity map. (Then one of the morphisms is an isomorphism if and only if the other is, in which case they're each other's inverses.)

Now, we assume (b) and (c) and prove (a). Recall from the proof of the proposition 6.1.10 that the sum of two morphisms $\varphi_1, \varphi_2 : M \to N$ can be written as the composition

$$\begin{tikzcd}
M \arrow[r] \arrow[rrr, bend right, "\varphi_1+\varphi_2"'] & M \oplus M \arrow[r, "\begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}"] & N \oplus N \arrow[r] & N.
\end{tikzcd}$$

We apply the functor F and consider the following diagram

$$\begin{tikzcd}
F(M) \arrow[r] \arrow[dr] & F(M \oplus M) \arrow[r, "F\left(\begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}\right)"] & F(N \oplus N) \arrow[r] \arrow[d] & F(N), \\
& F(M) \oplus F(M) \arrow[u] \arrow[r, "\begin{pmatrix} F(\varphi_1) & 0 \\ 0 & F(\varphi_2) \end{pmatrix}"] & F(N) \oplus F(N), \arrow[ur] &
\end{tikzcd}$$

which we claim to be commutative. Observe that the composition of the morphisms on the top is $F(\varphi_1 + \varphi_2)$ and the composition of the morphisms on the bottom is $F(\varphi_1) + F(\varphi_2)$. The commutativity of the diagram then implies (a).

Both triangles commute by the very definition of the morphisms $F(M) \oplus F(M) \to F(M \oplus M)$ and $F(N \oplus N) \to F(N) \oplus F(N)$. The commutativity of the inner square is just as natural, but a little notationally awkward. Let's denote the morphisms involved as follows:

$$\begin{tikzcd}
F(M \oplus M) \arrow[r, "F(\psi)"] & F(N \oplus N) \arrow[d, "\beta"] \\
F(M) \oplus F(M) \arrow[u, "\alpha"] \arrow[r, "\widetilde{\psi}"] & F(N) \oplus F(N).
\end{tikzcd}$$

Recall that $\psi : M \oplus M \to N \oplus N$ is the unique morphism such that

$$\begin{aligned}
\pi_1 \circ \psi \circ \iota_1 &= \varphi_1 & \qquad \pi_1 \circ \psi \circ \iota_2 &= 0 \\
\pi_2 \circ \psi \circ \iota_1 &= 0 & \qquad \pi_2 \circ \psi \circ \iota_2 &= \varphi_2.
\end{aligned}$$

By applying the functor F to these relations and recalling that $F(\iota_i) = \alpha \circ \widetilde{\iota_i}$ and $F(\pi_i) = \widetilde{\pi_i} \circ \beta$, where $\widetilde{\iota_i} : F(M) \to F(M) \oplus F(M)$ and $\widetilde{\pi_i} : F(N) \oplus F(N) \to F(N)$ are the natural inclusions and projections, we get that $\beta \circ F(\psi) \circ \alpha$ satisfies the defining equations for

$$\begin{pmatrix} F(\varphi_1) & F(0) \\ F(0) & F(\varphi_2) \end{pmatrix}.$$

The commutativity of the square (and the end of this proof) then reduces to the fact that $F(0) = 0$ on morphisms. We only prove that $F(0)$ is final, for the proof that it's initial is similar.

A first observation is that, since $0 \in A$ is terminal, the two projections $\pi_1, \pi_2 : 0 \oplus 0 \to 0$ are equal. Our assumption then implies that $F(0 \oplus 0)$ satisfies the universal property of $F(0) \oplus F(0)$ with $F(\pi_1)$ and $F(\pi_2)$ as projections. In particular, $F(\pi_1) = F(\pi_2)$.

Now, for every object P of B, there's at least the zero morphism $P \to F(0)$. So, suppose that there are two such morphisms γ and δ. We have that

$$\gamma - \delta = (F(\pi_1) - F(\pi_2)) \circ \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = 0 \circ \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = 0,$$

finishing the proof. ☐

Our next goal is to prove that if C is a small category and A is an abelian category, then the category of all functors and natural transformations $\mathrm{Fun}(C, A)$ is also abelian. This is a generalization of a fact that will become very important to us in the future: the category of presheaves with values in an abelian category is abelian.

The reader might wonder the raison d'être *of the set-theoretic condition above. If* C *is not small, then the objects of* $\mathrm{Fun}(C, A)$ *doesn't even form a class. If it were a class, then a functor* $C \to A$ *would be a set, since a set is defined to be a collection that is a member of some class. But then we could use the axiom of replacement to deduce that the class of objects of* C *is a set.*

For that, we have to understand how some limits and colimits work in a functor category. The general statement is that "limits and colimits in a functor category are computed pointwise". We prefer to understand concretely the particular cases we're interested in, but the reader can find the general theorem in [3] (proposition 2.15.1) or in [23] (theorem 6.2.5).

We begin by a simple observation: the functor $C \to A$ which sends every object of C to the zero-object of A is a zero-object of $\mathrm{Fun}(C, A)$. Moreover, if F, G are objects of $\mathrm{Fun}(C, A)$, a natural transformation $F \to G$ is a zero-morphism if and only if all its components $F(C) \to G(C)$ are zero-morphisms in A.

Now, let's deal with kernels. Suppose that $\varphi : F \to G$ is a natural transformation in $\mathrm{Fun}(C, A)$. For each $C \in C$, the morphism $\varphi_C : F(C) \to G(C)$ has a kernel $\ker \varphi_C : K(C) \to F(C)$. We observe that this assignment is functorial. If $f : C \to D$ is a morphism in C, then the diagram

$$\begin{tikzcd}
K(C) \arrow[r, "\ker \varphi_C"] & F(C) \arrow[r, "\varphi_C"] \arrow[d, "F(f)"] & G(C) \arrow[d, "G(f)"] \\
K(D) \arrow[r, "\ker \varphi_D"] & F(D) \arrow[r, "\varphi_D"] & G(D)
\end{tikzcd}$$

is commutative and so the universal property of kernels will induce a morphism $K(C) \to K(D)$ making the diagram commute as long as the morphism

$$\begin{tikzcd}
K(C) \arrow[r, "\ker \varphi_C"] & F(C) \arrow[r, "\varphi_C"] \arrow[d, "F(f)"] & G(C) \arrow[d, "G(f)"] \\
K(D) \arrow[r, "\ker \varphi_D"] & F(D) \arrow[r, "\varphi_D"] & G(D)
\end{tikzcd}$$

is zero. But this is evident since the commutativity of the diagram implies that this morphism is equal to

$$\begin{tikzcd}
K(C) \arrow[r, "\ker \varphi_C"] \arrow[rr, bend right, "0"] & F(C) \arrow[r, "\varphi_C"] \arrow[d] & G(C) \arrow[d, "G(f)"] \\
K(D) \arrow[r, "\ker \varphi_D"] & F(D) \arrow[r, "\varphi_D"] & G(D)
\end{tikzcd}$$

Moreover, the uniqueness part of the universal property of kernels shows that if $G : D \to E$ is another morphism in C, then the bigger diagram

$$\begin{tikzcd}
K(C) \arrow[r, "\ker \varphi_C"] \arrow[d, dashed] \arrow[dd, dashed, bend right] & F(C) \arrow[r, "\varphi_C"] \arrow[d, "F(f)"] & G(C) \arrow[d, "G(f)"] \\
K(D) \arrow[r, "\ker \varphi_D"] \arrow[d, dashed] & F(D) \arrow[r, "\varphi_D"] \arrow[d, "F(g)"] & G(D) \arrow[d, "G(g)"] \\
K(E) \arrow[r, "\ker \varphi_E"] & F(E) \arrow[r, "\varphi_E"] & G(E)
\end{tikzcd}$$

commutes. We conclude that $C \mapsto K(C)$ defines a functor $\mathsf{C} \to \mathsf{A}$ and that $K \to F$ is a morphism in $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$ whose composition with $\varphi : F \to G$ is zero. Does it satisfy the universal property of $\ker \varphi$? Let $\zeta : Z \to F$ be another natural transformation which satisfies $\varphi \circ \zeta = 0$. By the universal property of kernels, there exist unique morphisms $Z(C) \to K(C)$ for every object C of C making the diagram

$$\begin{tikzcd}
K(C) \arrow[r, "\ker \varphi_C"] & F(C) \arrow[r, shift left, "\varphi_C"] \arrow[r, shift right, "0"'] & G(C) \\
Z(C) \arrow[u, dashed] \arrow[ru, "\zeta_C"'] & &
\end{tikzcd}$$

commute. These morphisms form a natural transformation since, if $f : C \to D$ is a morphism in C, then the diagram

$$\begin{tikzcd}
K(C) \arrow[rr, "\ker \varphi_C"] \arrow[dr, "K(f)"] & & F(C) \arrow[dr, "F(f)"] & \\
& K(D) \arrow[rr, "\ker \varphi_D"] & & F(D) \\
Z(C) \arrow[uu, dashed] \arrow[uurr, "\zeta_C"] \arrow[dr, "Z(f)"'] & & & \\
& Z(D) \arrow[uu, dashed] \arrow[uurr, "\zeta_D"'] & &
\end{tikzcd}$$

commutes since $\ker \varphi_D$ is a monomorphism. This proves that $K \to F$ satisfies the universal property of $\ker \varphi$. Since $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$ is a preadditive category (with the addition of morphisms given pointwise), this implies that a morphism $\varphi : F \to G$ in $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$ is a monomorphism if and only if $\ker \varphi$ is the zero-morphism $0 \to F$ and if and only if it is a monomorphism pointwise.

It should be clear that the same argument shows that cokernels[4] in $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$ exist and are computed pointwise. Moreover, a morphism $\varphi : F \to G$ in $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$ is an epimorphism if and only if $\operatorname{coker} \varphi$ is the zero-morphism $G \to 0$ and if and only if it is an epimorphism pointwise.

Finally, basically the same arguments show that, if F and G are two objects of $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$, the functor $F \oplus G$ defined by

$$(F \oplus G)(C) := F(C) \oplus G(C) \qquad \text{and} \qquad (F \oplus G)(f) := \begin{pmatrix} F(f) & 0 \\ 0 & G(f) \end{pmatrix}$$

satisfies the universal property of products and coproducts in $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$, with the natural injections and projections being given by the respective pointwise injections and projections.

We're now ready to prove our desired result.

Proposition 6.5.2 Let C be a small category and A be an abelian category. Then the category of all functors and natural transformations $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$ is abelian.

Proof. After all our preliminary work, all there's left to prove is that every monomorphism is the kernel of its cokernel and that every epimorphism is the cokernel of its kernel. This also follows quickly from our previous discussion: if $\varphi : F \to G$ is a monomorphism then its components $\varphi_C : F(C) \to G(C)$, for every object C of C, are monic. Since A is abelian, each φ_C is the kernel of its cokernel. But kernels and cokernels are computed pointwise and so φ is also the kernel of its cokernel. The same argument shows that every epimorphism is the cokernel of its kernel. $\square$

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[4] Or even more general limits and colimits.

The preceding proposition is one of the main results in Grothendieck's seminal paper *Sur Quelques Points d'Algèbre Homologique*[5]. Taking C to be the discrete category with two objects, we obtain that the product category $\mathsf{A} \times \mathsf{A}$ is abelian. Many other families of abelian categories that we'll encounter on the rest of these notes arise in similar fashion.

We illustrate how to apply the corollary 6.2.2 by proving that the category of additive functors is also abelian.

> **Corollary 6.5.3** Let C be a small additive category and A be an abelian category. Then the full subcategory $\mathrm{Add}(\mathsf{C}, \mathsf{A})$ of $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$, composed of additive functors and natural transformations, is abelian.

Proof. It is clear that the zero-object of $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$ is additive, and so it is also the zero-object of $\mathrm{Add}(\mathsf{C}, \mathsf{A})$. If F and G are two additive functors, their direct sum acts on morphisms by

$$(F \oplus G)(f) = \begin{pmatrix} F(f) & 0 \\ 0 & G(f) \end{pmatrix}.$$

Since the sum of morphisms is represented by the sum of matrices, the additivity of both F and G implies that of $F \oplus G$. Finally, we show that, if $\varphi : F \to G$ is a morphism in $\mathrm{Add}(\mathsf{C}, \mathsf{A})$ and $\ker \varphi : K \to F$ is its kernel in $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$, K is an additive functor. Indeed, if $f, g : C \to D$ are two morphisms in C,

$$\begin{aligned} \ker \varphi_D \circ K(f+g) &= F(f+g) \circ \ker \varphi_C = (F(f) + F(g)) \circ \ker \varphi_C \\ &= F(f) \circ \ker \varphi_C + F(g) \circ \ker \varphi_C \\ &= \ker \varphi_D \circ K(f) + \ker \varphi_D \circ K(g) = \ker \varphi_D \circ (K(f) + K(g)), \end{aligned}$$

and so $K(f+g) = K(f) + K(g)$ by the fact that $\ker \varphi_D$ is a monomorphism. The same argument shows that the target of $\operatorname{coker} \varphi$ is also additive. ☐

We'll now delve into the relationship between functors and exact sequences. Unfortunately, being additive does *not* guarantee that a functor preserves exact sequences.[6] For example, consider the exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \frac{\mathbb{Z}}{2\mathbb{Z}} \longrightarrow 0,$$

where the map $\mathbb{Z} \to \mathbb{Z}$ is multiplication by two. Upon tensorization by $\mathbb{Z}/2\mathbb{Z}$ we get the sequence

$$0 \longrightarrow \frac{\mathbb{Z}}{2\mathbb{Z}} \xrightarrow{0} \frac{\mathbb{Z}}{2\mathbb{Z}} \longrightarrow \frac{\mathbb{Z}}{2\mathbb{Z}} \longrightarrow 0,$$

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[5] This paper was published in the Tohoku Mathematical Journal and, even though this journal exists for more than a century, many people refer to this precise paper as being "the Tohoku paper".

[6] Or perhaps that's a blessing, for this issue is at the heart of homological algebra.

which is not exact since the zero-morphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is not a monomorphism. The additive functors that indeed preserve some kind of exact sequences are so special that they deserve a name.

Definition 6.5.2 — Exact functor. Let $F : \mathsf{A} \to \mathsf{B}$ be an additive functor between abelian categories. Then F is said to be *left exact* when it preserves exact sequences of the form

$$0 \longrightarrow M \longrightarrow N \longrightarrow P,$$

right exact when it preserves exact sequences of the form

$$M \longrightarrow N \longrightarrow P \longrightarrow 0,$$

and *exact* when it preserves short exact sequences.

We observe that our discussion right after the definition 6.4.2 implies that an exact functor preserves exact sequences of any length, not only short exact sequences.

Proposition 6.5.4 Let $F : \mathsf{A} \to \mathsf{B}$ be an additive functor between abelian categories. The following equivalences hold:

(a) F is left exact if and only if it preserves finite limits;

(b) F is right exact if and only if it preserves finite colimits;

(c) F is exact if and only if it preserves finite limits and finite colimits.

Proof. By duality, it suffices to prove (a). We observe that a sequence of the form

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P$$

is exact if and only if $\varphi = \ker \psi$. This implies right away that if F preserves finite limits, then it preserves kernels and so it is left exact. For the converse, recall that finite limits can be built up from binary products, terminal objects and equalizers. (Proposition 2.8.2 in [3].) Since F is additive, it preserves binary products and zero-objects. Moreover, if F is left exact, then it preserves kernels. It suffices then to show that F preserves equalizers. But the equalizer of a pair $\varphi, \psi : M \to N$ is simply the kernel of $\varphi - \psi$. The result follows. □

More often than not, what we'll use to prove that a functor is left or right exact is the corollary below, which follows from the good old mottos "right adjoints preserve limits" and its dual "left adjoints preserve colimits".[7]

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[7] We remember that right adjoints preserve limits by the mnemonic *RAPL*.

Corollary 6.5.5 Let $F : A \to B$ be an additive functor between abelian categories. If F is a right adjoint then it is left exact and if F is a left adjoint then it is right exact.

6.6. Diagram chasing

In the abelian category $A\text{-Mod}$ of modules over a ring A, exact sequences have simple characterizations in terms of elements. Indeed, the sequence of A-modules

$$M \xrightarrow{\varphi} N \xrightarrow{\psi} P$$

is exact if and only if $\psi(\varphi(m)) = 0$ for all $m \in M$ and if $\psi(n) = 0$, for some $n \in N$, implies the existence of $m \in M$ such that $n = \varphi(m)$. Using this, proofs involving exact sequences can usually be done by pointing fingers to a diagram and observing the fate of some elements. This technique is called *diagram chasing*.

To illustrate this technique, we prove the following result in two ways; first using universal properties and then, in $A\text{-Mod}$, using diagram chasing.

Proposition 6.6.1 — Four lemma. Consider the following diagram with exact rows in an abelian category A:

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4 .
\end{tikzcd}$$

If β and δ are monomorphisms and α is an epimorphism, then γ is a monomorphism. Dually, if α and γ are epimorphisms and δ is a monomorphism, then β is an epimorphism.

As usual, we prove only the first part of the result, since the second part follows by duality.

Proof using universal properties. Let $\rho : P \to M_3$ be a morphism such that $\gamma \circ \rho = 0$. Our goal is to prove that $\rho = 0$. Since the diagram commutes, the morphism

$$\begin{tikzcd}
 & & P \arrow[d, "\rho"] & \\
M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4
\end{tikzcd}$$

is zero and, as δ is monic, so is $P \to M_3 \to M_4$. The universal property of kernels then implies that ρ factors through the kernel $K \to M_3$ of $M_3 \to M_4$, which coincides with the image of $M_2 \to M_3$ by exactness.

$$\begin{tikzcd}
& & & P \arrow[dl, dashed] \arrow[d, "\rho"] & \\
M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & K \arrow[r] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & N_2 \arrow[rr] & & N_3 \arrow[r] & N_4
\end{tikzcd}$$

We consider the pullback $M_2 \times_K P$ and observe that the commutativity of the diagram implies that the morphism below is zero.

$$\begin{tikzcd}
& M_2 \times_K P \arrow[rr] \arrow[d] & & P \arrow[dl] \arrow[d, "\rho"] & \\
M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & K \arrow[r] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & N_2 \arrow[rr] & & N_3 \arrow[r] & N_4
\end{tikzcd}$$

By the universal property of kernels, $M_2 \times_K P \to M_2 \to N_2$ factors through the kernel $K' \to N_2$ of $N_2 \to N_3$, which coincides with the image of $N_1 \to N_2$ by exactness.

$$\begin{tikzcd}
& & M_2 \times_K P \arrow[rr] \arrow[d] \arrow[ddl, dashed] & & P \arrow[dl] \arrow[d, "\rho"] & \\
M_1 \arrow[rr] \arrow[d, "\alpha"] & & M_2 \arrow[r] \arrow[d, "\beta"] & K \arrow[r] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & K' \arrow[r] & N_2 \arrow[rr] & & N_3 \arrow[r] & N_4
\end{tikzcd}$$

We consider the pullback $M_1 \times_{K'} (M_2 \times_K P)$ and observe that, since β is a monomorphism, the upper square (in black below) commutes.

$$\begin{tikzcd}
M_1 \times_{K'} (M_2 \times_K P) \arrow[rr] \arrow[d] & & M_2 \times_K P \arrow[rr] \arrow[d] \arrow[ddl] & & P \arrow[dl] \arrow[d, "\rho"] & \\
M_1 \arrow[rr] \arrow[d, "\alpha"] & & M_2 \arrow[r] \arrow[d, "\beta"] & K \arrow[r] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & K' \arrow[r] & N_2 \arrow[rr] & & N_3 \arrow[r] & N_4
\end{tikzcd}$$

Remark that both $M_2 \to K$ and $M_1 \to N_1 \to K'$ are epimorphisms. The corollary 6.3.4 then implies that so are the arrows in black below.

$$\begin{tikzcd}
M_1 \times_{K'} (M_2 \times_K P) \arrow[rr] \arrow[d] & & M_2 \times_K P \arrow[rr] \arrow[d] \arrow[ddl] & & P \arrow[d, "\rho"] \arrow[dl] & \\
M_1 \arrow[rr] \arrow[d, "\alpha"] & & M_2 \arrow[r] \arrow[d, "\beta"] & K \arrow[r] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & K' \arrow[r] & N_2 \arrow[rr] & & N_3 \arrow[r] & N_4
\end{tikzcd}$$

The composition of the arrows above with ρ is zero, since $M_1 \to M_2 \to M_3$ is. But the fact that they are epic implies $\rho = 0$, finishing the proof. ☐

We now redo this proof, when $\mathsf{A} = A\text{-}\mathsf{Mod}$, using diagram chase. Observe that, since we're now proving this result for only one category, we can't use a duality argument. (The opposite category of $A\text{-}\mathsf{Mod}$ is rarely a category of modules.) Nevertheless, we'll still only prove the first part below, for our last proof took care of both parts.

Proof by diagram chasing. Let m be an element of M_3 such that $\gamma(m) = 0$. Our goal is to prove that $m = 0$. Observe that m is sent to 0 in N_4 by the composition

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & m \arrow[r] \arrow[d, mapsto, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & N_2 \arrow[r] & 0 \arrow[r, mapsto] & 0.
\end{tikzcd}$$

Since the diagram commutes, m is also sent to 0 by going through the other side of the square

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & m \arrow[r, mapsto] \arrow[d, "\gamma"] & ? \arrow[d, mapsto, "\delta"] \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & 0.
\end{tikzcd}$$

But δ is injective, so m is in the kernel of the morphism $M_3 \to M_4$. (That is, our "?" above is actually zero.) By exactness of the top row, there exists $m' \in M_2$, which is sent to m by $M_2 \to M_3$.

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "\alpha"] & m' \arrow[r, mapsto] \arrow[d, "\beta"] & m \arrow[r] \arrow[d, "\gamma"] & 0 \arrow[d, "\delta"] \\
N_1 \arrow[r] & N_2 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Since the middle square commutes, $n := \beta(m')$ is in the kernel of $N_2 \to N_3$. So, by exactness of the lower row, there exists $n' \in N_1$ whose image through $N_1 \to N_2$ is n.

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "\alpha"] & m' \arrow[r] \arrow[d, mapsto, "\beta"] & m \arrow[r] \arrow[d, "\gamma"] & 0 \arrow[d, "\delta"] \\
n' \arrow[r, mapsto] & n \arrow[r, mapsto] & 0 \arrow[r] & 0
\end{tikzcd}$$

The morphism α is epic, so there exists $m'' \in M_1$ which is sent to n'. This element is actually sent to m' via $M_1 \to M_2$ due to the fact that β is monic. We conclude that m is the image of m'' under the composition $M_1 \to M_2 \to M_3$.

$$\begin{tikzcd}
m'' \arrow[r, mapsto] \arrow[d, "\alpha"] & m' \arrow[r, mapsto] \arrow[d, "\beta"] & m \arrow[r] \arrow[d, "\gamma"] & 0 \arrow[d, "\delta"] \\
n' \arrow[r] & n \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

But this composition is zero, proving the result. ☐

By gluing both versions of the four lemma, we obtain the corollary below.

Corollary 6.6.2 — Five lemma. Consider the following diagram with exact rows in an abelian category A:

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[r] \arrow[d, "\delta"] & M_5 \arrow[d, "\varepsilon"] \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4 \arrow[r] & N_5.
\end{tikzcd}$$

If β and δ are isomorphisms, α is an epimorphism, and ε is a monomorphism, then γ is an isomorphism.

Proof. The first part of the four lemma, applied to the diagram

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4,
\end{tikzcd}$$

yields that γ is a monomorphism. Similarly, the second part of the four lemma, applied to the diagram

$$\begin{tikzcd}
M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[r] \arrow[d, "\delta"] & M_5 \arrow[d, "\varepsilon"] \\
N_2 \arrow[r] & N_3 \arrow[r] & N_4 \arrow[r] & N_5,
\end{tikzcd}$$

yields that γ is an epimorphism. This concludes the proof. $\square$

The preceding discussion hopefully conveyed that proofs by diagram chasing are often simpler than their arrow-theoretic counterparts. It would be great if we could use the same technique even when dealing with abelian categories other than $A\text{-}\mathsf{Mod}$. The theorem below establishes precisely that.

Theorem 6.6.3 — Freyd-Mitchell. Let A be a small abelian category. Then there exists a fully faithful exact embedding of A into $A\text{-}\mathsf{Mod}$ for some (not necessarily commutative) ring A.

While all the necessary prerequisites for the (unfortunately long) proof of this result were already discussed, we prefer to direct the interested reader to the wonderful proof in [4] and confine ourselves to an explanation of how this result is used in practice.

Let $V : \mathsf{A} \to A\text{-}\mathsf{Mod}$ be the functor given by the Freyd-Mitchell theorem. For now, we define a *pseudo-element* m of an object $M \in \mathsf{A}$ to be an element of $V(M)$. We shall abuse notation and write $m \in M$ for this relation. The action of a morphism $\varphi : M \to N$, denoted as $\varphi(m)$, on a pseudo-element m is given simply by $V(\varphi)(m)$. We gather a few properties of those notions.

Proposition 6.6.4 Let A be a small abelian category. If $\varphi : M \to N$ is a morphism in A, we have that:

(a) φ is monic if and only if for all $m \in M$, $\varphi(m) = 0$ implies $m = 0$;

(b) φ is epic if and only if for all $n \in N$, there exists $m \in M$ such that $\varphi(m) = n$;

(c) we may construct a morphism φ by describing its action of pseudo-elements.

Moreover,

(d) two morphisms $\varphi_1, \varphi_2 : M \to N$ are equal if and only if $\varphi_1(m) = \varphi_2(m)$ for all $m \in M$;

(e) a sequence $M \xrightarrow{\varphi} N \xrightarrow{\psi} P$ is exact if and only if $\psi(\varphi(m)) = 0$ for all $m \in M$ and if $\psi(n) = 0$, for some $n \in N$, implies the existence of $m \in M$ such that $n = \varphi(m)$.

Proof. The item (c) translates the fullness of the functor V in theorem 6.6.3, and the item (d) translates its faithfulness. Since V is exact, it preserves finite limits and colimits; this gives one direction on the items (a), (b) and (e). The other direction follows from the fact that a fully faithful functor reflects limits and colimits, which is clear from their universal properties. $\square$

Finally, we address the elephant in the room: most abelian categories are not small.

This is not as bad as it seems, and we explain why. Let A be an abelian category and D be a diagram in A. Consider the sequence

$$\mathsf{B}_0 \subset \mathsf{B}_1 \subset \cdots \subset \mathsf{B}_n \subset \cdots ,$$

where B_0 is the full subcategory of A generated by D and B_{n+1} is the full subcategory of A generated by the limits and colimits of all finite diagrams in B_n. Then

$$\mathsf{B} := \bigcup_{n=0}^{\infty} \mathsf{B}_n$$

is a full subcategory of A stable under finite limits and colimits. In particular, B is an abelian category due to the corollary 6.2.2. If the diagram D is small (which is the case in basically all applications), so is the abelian category B, and then we can apply the theorem 6.6.3 in B.

In a nutshell, the Freyd-Mitchell theorem allows us prove basically every result about exact sequences in abelian categories as if we were in a category of modules. And we may even use duality arguments!

Henceforth, we'll prefer arrow-theoretic constructions whenever they aren't too troublesome, but we will freely use elements when they simplify or shed light on some arguments.

We end this section with arguably the most important diagram chase: the *snake lemma*. Its statement involves a diagram of the form

$$\begin{tikzcd}
 & M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r] & N_3,
\end{tikzcd}$$

whose rows are exact, where we expand the kernels and cokernels of the vertical morphisms and insert the natural morphisms induced from the universal properties:

$$\begin{tikzcd}
 & K_\alpha \arrow[r, dashed] \arrow[d, "\ker \alpha"] & K_\beta \arrow[r, dashed] \arrow[d, "\ker \beta"] & K_\gamma \arrow[d, "\ker \gamma"] & \\
 & M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] \arrow[d, "\operatorname{coker} \alpha"] & N_2 \arrow[r] \arrow[d, "\operatorname{coker} \beta"] & N_3 \arrow[d, "\operatorname{coker} \gamma"] & \\
 & C_\alpha \arrow[r, dashed] & C_\beta \arrow[r, dashed] & C_\gamma. &
\end{tikzcd}$$

We're now in a position to state this important result.

Theorem 6.6.5 — Snake lemma. Consider the following commutative diagram with exact rows in an abelian category:

$$\begin{tikzcd}
 & M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r] & N_3 . &
\end{tikzcd}$$

We denote by K_α, K_β and K_γ the sources of $\ker\alpha$, $\ker\beta$ and $\ker\gamma$. Similarly, C_α, C_β and C_γ denote the targets of the cokernels thereof. Then, there exists a morphism $\delta : K_\gamma \to C_\alpha$ making the sequence

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & K_\alpha \arrow[r] & K_\beta \arrow[r] & K_\gamma \arrow[dll, rounded corners, "\delta"] & & \\
 & & C_\alpha \arrow[r] & C_\beta \arrow[r] & C_\gamma \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

exact, where K is the source of the kernel of $M_1 \to M_2$ and C is the target of the cokernel of $N_2 \to N_3$.

Before we delve into the proof, we observe that, even though there may be many morphisms $\delta : K_\gamma \to C_\alpha$ which satisfy the conclusion above[8], there's a canonical one that will be the one in consideration whenever we talk about the snake lemma.

We construct the morphism δ using elements as follows: let m be an element of K_γ. Since $\ker\gamma$ is a monomorphism, we can view m naturally as an element of M_3. Due to the fact that $M_2 \to M_3$ is an epimorphism, there exists a lift m' of m to M_2, which we then map to N_2 as $\beta(m')$. By the commutativity of the diagram, the image of $\beta(m')$ to N_3 is zero, proving that $\beta(m')$ is in the image of $N_1 \to N_2$. Since the latter is monic, we denote the element of N_1 whose image by $N_1 \to N_2$ is $\beta(m')$ by the same symbol. Finally, $\delta(m)$ is the image of $\beta(m')$ in the cokernel of α.

$$\begin{tikzcd}
 & K_\alpha \arrow[r] \arrow[d, "\ker\alpha"] & K_\beta \arrow[r] \arrow[d, "\ker\beta"] & m \arrow[d, maps to, "\ker\gamma"] & \\
 & M_1 \arrow[r] \arrow[d, "\alpha"] & m' \arrow[r, maps to] \arrow[d, maps to, "\beta"] & m \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \beta(m') \arrow[r, maps to] \arrow[d, maps to, "\operatorname{coker}\alpha"] & \beta(m') \arrow[r] \arrow[d, "\operatorname{coker}\beta"] & N_3 \arrow[d, "\operatorname{coker}\gamma"] & \\
 & \delta(m) \arrow[r] & C_\beta \arrow[r] & C_\gamma &
\end{tikzcd}$$

---

[8] If δ satisfies the conclusion of the snake lemma, then so does $-\delta$.

In order for this morphism to be well-defined, we need to check whether a different choice for the lift m' would change the image $\delta(m)$. If m'' is another choice, then $m' - m''$ is in the kernel of $M_2 \to M_3$ and so in the image of $M_1 \to M_2$. Let $\tilde{m} \in M_1$ be one element mapping to $m' - m''$. Its image in C_α is zero, since $M_1 \to N_1 \to C_\alpha$ is the zero-morphism.

$$\begin{tikzcd}
& \tilde{m} \arrow[r, mapsto] \arrow[d, mapsto, "\alpha"] & m' - m'' \arrow[r, mapsto] \arrow[d, "\beta"] & 0 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \alpha(\tilde{m}) \arrow[r] \arrow[d, mapsto, "\operatorname{coker} \alpha"] & N_2 \arrow[r] \arrow[d, "\operatorname{coker} \beta"] & N_3 \arrow[d, "\operatorname{coker} \gamma"] & \\
& 0 \arrow[r] & C_\beta \arrow[r] & C_\gamma &
\end{tikzcd}$$

The commutativity of the diagram then implies that $\delta(m)$ is independent of the choice of the lift.

Proof of theorem 6.6.5. □

7. Complexes and cohomology

In the previous chapter, we saw that only the most distinguished additive functors turns out to be exact. Nevertheless, the image of an exact sequence

$$M \xrightarrow{\varphi} N \xrightarrow{\psi} P$$

by an additive functor F is still special, for it satisfies $F(\psi) \circ F(\varphi) = F(\psi \circ \varphi) = F(0) = 0$. The sequences of objects and morphisms in which the composition of two consecutive morphisms is zero are called *complexes* and compose the main topic of the present chapter. We'll see that there are many contexts in which associating a particular complex to a mathematical object provides useful information about the aforesaid object.

7.1. Basic definitions

We begin with the precise definition of a complex.

> **Definition 7.1.1 — Complex.** Let A be a (not necessarily abelian) category. A *cochain complex* $(M^\bullet, d^\bullet)$ in A is a sequence of objects and morphisms
>
> $$\cdots \xrightarrow{d^{i-2}} M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} \cdots$$
>
> such that $d^i \circ d^{i-1} = 0$ for all i. The morphisms $d^i : M^i \to M^{i+1}$ are said to be the *differentials* of the complex.

In some applications, it is useful for the indices to be descending. In this case, the indices are usually written as subscripts

$$\cdots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots$$

and the corresponding object is said to be a *chain complex*. Since most of the complexes that we'll encounter are *co*chain complexes, we'll just call them *complexes* and denote them by $M^\bullet$. Of course, we can always set $M^i := M_{-i}$ and see a chain complex $M_\bullet$ as the cochain complex $M^{-\bullet}$.

Also important are the ways complexes can interact with each other. For that, we gather all the complexes in A in a new category $\mathsf{C}(\mathsf{A})$.

Definition 7.1.2 — Category of complexes. Let A be a category. An object in the *category of complexes* $\mathsf{C}(\mathsf{A})$ is a complex in A and a morphism $\psi^\bullet : M^\bullet \to N^\bullet$ is a collection of morphisms $\psi^i : M^i \to N^i$ making the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & M^{i-1} \arrow[r, "d^{i-1}_{M^\bullet}"] \arrow[d, "\psi^{i-1}"] & M^i \arrow[r, "d^i_{M^\bullet}"] \arrow[d, "\psi^i"] & M^{i+1} \arrow[r] \arrow[d, "\psi^{i+1}"] & \cdots \\
\cdots \arrow[r] & N^{i-1} \arrow[r, "d^{i-1}_{N^\bullet}"] & N^i \arrow[r, "d^i_{N^\bullet}"] & N^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

commute in A.

Other usual variants of the category $\mathsf{C}(\mathsf{A})$ may be concocted by considering complexes which are bounded in some sense. For example, we let $\mathsf{C}^+(\mathsf{A})$ denote the full subcategory of $\mathsf{C}(\mathsf{A})$ composed of the complexes $M^\bullet$ which are bounded below, i.e., for which $M^i = 0$ for all $i \ll 0$. Similarly, we consider the categories $\mathsf{C}^-(\mathsf{A})$ of bounded-above complexes and $\mathsf{C}^b(\mathsf{A})$ of complexes which are bounded above and below. A shorthand notation for all these categories is $\mathsf{C}^*(\mathsf{A})$.

Proposition 7.1.1 Let A be an abelian category. Then the categories of complexes $\mathsf{C}^*(\mathsf{A})$ are abelian.

Proof. Due to the corollary 6.2.2, it suffices to prove that $\mathsf{C}(\mathsf{A})$ is abelian. Consider the category Z, which has an object for each integer and a single non-trivial morphism between each consecutive integers (from the smallest to the biggest). Then $\mathsf{C}(\mathsf{A})$ is a full subcategory of $\mathrm{Fun}(\mathsf{Z}, \mathsf{A})$, which is abelian by the proposition 6.5.2. Appealing once again to the corollary 6.2.2, it suffices to see that the category of complexes is closed under direct sums, kernels and cokernels.

Binary direct sums of complexes form another complex since, for all i, the composition

$$M^{i-1} \oplus N^{i-1} \longrightarrow M^i \oplus N^i \longrightarrow M^{i+1} \oplus N^{i+1}$$

is simply given by

$$\begin{pmatrix} d^i_{M^\bullet} \circ d^{i-1}_{M^\bullet} & 0 \\ 0 & d^i_{N^\bullet} \circ d^{i-1}_{N^\bullet} \end{pmatrix} = 0.$$

Moreover, if $\psi^\bullet : M^\bullet \to N^\bullet$ is a morphism of complexes, we have a commutative diagram

$$\begin{tikzcd}
K^{i-1} \arrow[r] \arrow[d, "\ker \psi^{i-1}"] & K^i \arrow[r] \arrow[d, "\ker \psi^i"] & K^{i+1} \arrow[d, "\ker \psi^{i+1}"] \\
M^{i-1} \arrow[r, "d^{i-1}_{M^\bullet}"] & M^i \arrow[r, "d^i_{M^\bullet}"] & M^{i+1}
\end{tikzcd}$$

By the complex condition, $K^{i-1} \to K^i \to K^{i+1} \to M^{i+1}$ is the zero-morphism and, since $\ker \psi^{i+1}$ is a monomorphism, $K^{i-1} \to K^i \to K^{i+1}$ is also already zero, proving that the category of complexes is closed under kernels. A dual argument shows that it is closed under cokernels. ☐

We now observe some natural functors which involve the category of complexes. First of all, our category A can be embedded in $\mathsf{C}^*(\mathsf{A})$. Indeed, the functor $\iota : \mathsf{A} \to \mathsf{C}^*(\mathsf{A})$ which sends an object A of A to the complex

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \underbrace{A}_{\text{degree } 0} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

is fully faithful (it is also exact when A is abelian). Another natural functor on $\mathsf{C}^*(\mathsf{A})$ is the shift functor:

$$\begin{aligned} \mathsf{C}^*(\mathsf{A}) &\to \mathsf{C}^*(\mathsf{A}) \\ M^\bullet &\mapsto M[n]^\bullet, \end{aligned}$$

defined by $M[n]^i := M^{n+i}$ and $d^i_{M[n]^\bullet} := (-1)^n d^{n+i}_{M^\bullet}$. The sign on the differential doesn't change the isomorphism class of the complex but simplifies some other equations.

It's important to observe that the functor $[1]$ *shifts a complex* to the left, *contrary to what may seem natural. This is a very common source of confusion for most people.*

Also, an additive functor between additive categories $\mathsf{F} : \mathsf{A} \to \mathsf{B}$ determines a functor between the categories of complexes

$$\mathsf{C}^*(\mathsf{F}) : \mathsf{C}^*(\mathsf{A}) \to \mathsf{C}^*(\mathsf{B})$$

given by setting the image of $M^\bullet$ to be the complex defined by $\mathsf{F}(M^i)$ and $\mathsf{F}(d^i_{M^\bullet})$. Whenever there's no risk of confusion, we'll denote this functor simply by F.

There's another, even more interesting, functor defined on the category of complexes $\mathsf{C}^*(\mathsf{A})$ when A is abelian. Consider a complex $M^\bullet$. The complex condition $d^i \circ d^{i-1} = 0$ and the universal property of kernels imply that d^{i-1} factors through $\ker d^i$:

$$\begin{tikzcd} \cdots \arrow[r, "d^{i-2}"] & M^{i-1} \arrow[rr, "d^{i-1}"] \arrow[dr, dashed] & & M^i \arrow[r, "d^i"] & M^{i+1} \arrow[r, "d^{i+1}"] & \cdots . \\ & & K^i \arrow[ur, "\ker d^i"'] & & & \end{tikzcd}$$

But $\ker d^i$ is a monomorphism and so the universal property of images yields a unique factorization of $\operatorname{im} d^{i-1}$ through $\ker d^i$:

$$\begin{tikzcd}
 & & I^{i-1} \arrow[dr, "\operatorname{im} d^{i-1}"] \arrow[dd, dashed] & & & \\
\cdots \arrow[r, "d^{i-2}"] & M^{i-1} \arrow[ur] \arrow[rr] \arrow[dr] & & M^i \arrow[r, "d^i"] & M^{i+1} \arrow[r, "d^{i+1}"] & \cdots . \\
 & & K^i \arrow[ur, "\ker d^i"'] & & &
\end{tikzcd}$$

The induced morphism $I^{i-1} \to K^i$ is always a monomorphism (for $\operatorname{im} d^{i-1}$ is) and is epic if and only if the complex is exact at M^i. Thus, its cokernel measures the lack of exactness of the complex at M^i.

Definition 7.1.3 — Cohomology. Let $M^\bullet$ be a complex in an abelian category A. Its i-*th cohomology*, denoted $H^i(M^\bullet)$, is the target of the cokernel of the induced morphism $I^{i-1} \to K^i$ as above.

We affirm that the assignment $M^\bullet \mapsto H^i(M^\bullet)$ defines an additive functor $C^*(A) \to A$. Indeed, let $\psi^\bullet : M^\bullet \to N^\bullet$ be a morphism of complexes. By the universal property of kernels and cokernels, we have induced morphisms

$$\begin{tikzcd}
K^i_{M^\bullet} \arrow[r, "\ker d^i_{M^\bullet}"] \arrow[d, dashed] & M^i \arrow[r, "d^i_{M^\bullet}"] \arrow[d, "\psi^i"] & M^{i+1} \arrow[r, "\operatorname{coker} d^i_{M^\bullet}"] \arrow[d, "\psi^{i+1}"] & C^i_{M^\bullet} \arrow[d, dashed] \\
K^i_{N^\bullet} \arrow[r, "\ker d^i_{N^\bullet}"] & N^i \arrow[r, "d^i_{N^\bullet}"] & N^{i+1} \arrow[r, "\operatorname{coker} d^i_{N^\bullet}"] & C^i_{N^\bullet}.
\end{tikzcd}$$

In order for the universal property of cokernels to induce a morphism $H^i(\psi^\bullet) : H^i(M^\bullet) \to H^i(N^\bullet)$ making the diagram

$$\begin{tikzcd}
 & & I^{i-1}_{M^\bullet} \arrow[dl] \arrow[dr, "\operatorname{im} d^{i-1}_{M^\bullet}"] & & \\
H^i(M^\bullet) \arrow[d, dashed, "H^i(\psi^\bullet)"] & K^i_{M^\bullet} \arrow[l] \arrow[rr, "\ker d^i_{M^\bullet}"] \arrow[d] & & M^i \arrow[r, "d^i_{M^\bullet}"] \arrow[d, "\psi^i"] & M^{i+1} \arrow[d, "\psi^{i+1}"] \\
H^i(N^\bullet) & K^i_{N^\bullet} \arrow[l] \arrow[rr, "\ker d^i_{N^\bullet}"] & & N^i \arrow[r, "d^i_{N^\bullet}"] & N^{i+1} \\
 & & I^{i-1}_{N^\bullet} \arrow[ul] \arrow[ur, "\operatorname{im} d^{i-1}_{N^\bullet}"'] & &
\end{tikzcd}$$

commute, we have to show that the morphism $I^{i-1}_{M^\bullet} \to K^i_{M^\bullet} \to K^i_{N^\bullet} \to H^i(N^\bullet)$ is zero. Since $I^{i-1}_{N^\bullet} \to K^i_{N^\bullet} \to H^i(N^\bullet)$ is the zero-morphism, it suffices to construct a morphism

$I^{i-1}_{M^\bullet} \to I^{i-1}_{N^\bullet}$ which factors $I^{i-1}_{M^\bullet} \to K^i_{M^\bullet} \to K^i_{N^\bullet}$. This morphism is induced by the universal property of kernels using the fact that $\operatorname{im} = \ker(\operatorname{coker})$:

$$\begin{tikzcd}
K^{i-1}_{M^\bullet} \arrow[r, "\ker d^{i-1}_{M^\bullet}"] \arrow[d] & M^{i-1} \arrow[r, "\operatorname{coim} d^{i-1}_{M^\bullet}"] \arrow[d, "\psi^{i-1}"] & I^{i-1}_{M^\bullet} \arrow[r, "\operatorname{im} d^{i-1}_{M^\bullet}"] \arrow[d, dashed] & M^i \arrow[r, "\operatorname{coker} d^{i-1}_{M^\bullet}"] \arrow[d, "\psi^i"] & C^{i-1}_{M^\bullet} \arrow[d] \\
K^{i-1}_{N^\bullet} \arrow[r, "\ker d^{i-1}_{N^\bullet}"] & N^{i-1} \arrow[r, "\operatorname{coim} d^{i-1}_{N^\bullet}"] & I^{i-1}_{N^\bullet} \arrow[r, "\operatorname{im} d^{i-1}_{N^\bullet}"] & N^i \arrow[r, "\operatorname{coker} d^{i-1}_{M^\bullet}"] & C^{i-1}_{N^\bullet}.
\end{tikzcd}$$

The left-hand side of the diagram commutes due to the universal property of cokernels and the fact that $\operatorname{coim} = \operatorname{coker}(\ker)$. The uniqueness of the induced morphism on cohomology implies right-away that H^i preserves the composition of morphisms and that it is additive.

In A-Mod, the i-th cohomology of a complex is simply given by $\ker d^i / \operatorname{im} d^{i-1}$ and, for a morphism $\psi^\bullet : M^\bullet \to N^\bullet$ of complexes, the induced morphism on cohomology is nothing but

$$H^i(\psi^\bullet) : [m] \mapsto [\psi^i(m)].$$

As it will become clear in the next sections, the morphisms of complexes $\psi^\bullet : M^\bullet \to N^\bullet$ which induce an isomorphism in cohomology are important and deserve a name.

Definition 7.1.4 — Quasi-isomorphism. A morphism of complexes $\psi^\bullet : M^\bullet \to N^\bullet$ is said to be a *quasi-isomorphism* if, for all i, the induced morphism $H^i(\psi^\bullet) : H^i(M^\bullet) \to H^i(N^\bullet)$ is an isomorphism.

We observe that we can also see cohomology as a functor $C^*(A) \to C^*(A)$, where the image of a complex $M^\bullet$ is a complex $H^\bullet(M^\bullet)$ which has $H^i(M^\bullet)$ as objects and zero-morphisms as differentials.

An important property of cohomology is that it commutes with exact functors.

Proposition 7.1.2 Let $F : A \to B$ be an exact functor between abelian categories and $M^\bullet$ a complex in A. Then $H^\bullet(F(M^\bullet)) = F(H^\bullet(M^\bullet))$.

Proof. We construct the i-th cohomology of $F(M^\bullet)$. Since F is additive,

$$\begin{tikzcd}
\cdots \arrow[r] & F(M^{i-1}) \arrow[r, "F(d^{i-1})"] & F(M^i) \arrow[r, "F(d^i)"] & F(M^{i+1}) \arrow[r] & \cdots
\end{tikzcd}$$

is indeed a complex. Due to the fact that F preserves finite limits and finite colimits, $F(\operatorname{im} d^{i-1})$ is the image of $F(d^{i-1})$ and $F(\ker d^i)$ is the kernel of $F(d^i)$.

$$\begin{tikzcd}
 & & F(I^{i-1}) \arrow[dr, "F(\operatorname{im} d^{i-1})"] \arrow[dd, dashed] & & & \\
\cdots \arrow[r] & F(M^{i-1}) \arrow[ur] \arrow[rr] \arrow[dr] & & F(M^i) \arrow[r, "F(d^i)"] & F(M^{i+1}) \arrow[r] & \cdots \\
 & & F(K^i) \arrow[ur, "F(\ker d^i)"'] & & &
\end{tikzcd}$$

Moreover, by the uniqueness in the universal property of images, the induced morphism $F(I^{i-1}) \to F(K^i)$ coincides with the image of the induced morphism $I^{i-1} \to K^i$ by F. Then, since F preserves finite colimits, the cokernel of $F(I^{i-1}) \to F(K^i)$ is simply the image of the cokernel of $I^{i-1} \to K^i$ by F, proving that $H^i(F(M^\bullet)) = F(H^i(M^\bullet))$. □

Before we move on, we observe that our definition of the cohomology of a complex is somewhat asymmetrical. Instead of factoring d^{i-1} through $\ker d^i$, we could have factorized d^i through $\operatorname{coker} d^{i-1}$. Then the universal property of coimages induces a morphism $C^{i-1} \to I^i$ making the diagram

$$\begin{tikzcd}
 & & & I^i \arrow[dr] & & \\
\cdots \arrow[r, "d^{i-2}"] & M^{i-1} \arrow[r, "d^{i-1}"] & M^i \arrow[ur, "\operatorname{coim} d^i"] \arrow[rr] \arrow[dr, "\operatorname{coker} d^{i-1}"'] & & M^{i+1} \arrow[r, "d^{i+1}"] & \cdots \\
 & & & C^{i-1} \arrow[uu, dashed] \arrow[ur] & &
\end{tikzcd}$$

commute. Dually to our previous situation, this morphism is always an epimorphism (for $\operatorname{coim} d^i$ is) and is monic if and only if the complex is exact at M^i. Not surprisingly, the source of its kernel is nothing but $H^i(M^\bullet)$.

Proposition 7.1.3 Let $M^\bullet$ be a complex in an abelian category. Then, for every i, there exists a natural morphism $H^i(M^\bullet) \to C^{i-1}$ making the diagram

$$\begin{tikzcd}
 & I^{i-1} \arrow[dd] \arrow[dr, "\operatorname{im} d^{i-1}"] & & I^i \arrow[dr] & \\
M^{i-1} \arrow[ur] \arrow[rr] \arrow[dr] & & M^i \arrow[ur, "\operatorname{coim} d^i"] \arrow[rr] \arrow[dr, "\operatorname{coker} d^{i-1}"] & & M^{i+1} \\
 & K^i \arrow[ur, "\ker d^i"] \arrow[dr] & & C^{i-1} \arrow[uu] \arrow[ur] & \\
 & & H^i(M^\bullet) \arrow[ur, dashed] & &
\end{tikzcd}$$

commute and satisfying the universal property of the kernel of $C^{i-1} \to I^i$.

Proof. Let $\mu : K^i \to C^{i-1}$ be the composition $\operatorname{coker} d^{i-1} \circ \ker d^i$. By the first isomorphism theorem (theorem 6.2.5), the source of $\operatorname{im} \mu$ and the target of $\operatorname{coim} \mu$ are isomorphic. Thus, it suffices to show that $I^{i-1} \to K^i$ is its kernel and $C^{i-1} \to I^i$ is its cokernel.

The composition $I^{i-1} \to K^i \to C^{i-1}$ is zero, for it coincides with $\operatorname{coker} d^{i-1} \circ \operatorname{im} d^{i-1}$. Moreover, if $\zeta : Z \to K^i$ is another morphism whose composition with μ is zero, then $(\operatorname{coker} d^{i-1}) \circ (\ker d^i) \circ \zeta = 0$ and so the universal property of kernels (using that $\operatorname{im} d^{i-1} = \ker(\operatorname{coker} d^{i-1})$) induces a morphism $Z \to I^{i-1}$ making the diagram commute. This shows that $I^{i-1} \to K^i$ is the kernel of μ. That $C^{i-1} \to I^i$ is its cokernel follows by duality. ☐

Beyond satisfying our desire for symmetry, the preceding proposition also gives a very useful exact sequence linking cohomologies of different degrees for free.

Corollary 7.1.4 Let $M^\bullet$ be a complex in an abelian category. Then, for every i, the sequence

$$0 \longrightarrow H^i(M^\bullet) \longrightarrow C^{i-1} \longrightarrow K^{i+1} \longrightarrow H^{i+1}(M^\bullet) \longrightarrow 0,$$

where the morphism in the middle is the composition $C^{i-1} \to I^i \to K^{i+1}$, is exact.

Proof. We already know that the sequence is exact at $H^i(M^\bullet)$ and at $H^{i+1}(M^\bullet)$. Exactness at the other objects means that the kernel of $C^{i-1} \to K^{i+1}$ is $H^i(M^\bullet) \to C^{i-1}$ and that its cokernel is $K^{i+1} \to H^{i+1}(M^\bullet)$.

For the first statement, let $Z \to C^{i-1}$ be a morphism whose composition with $C^{i-1} \to K^{i+1}$ is zero. Since the $C^{i-1} \to K^{i+1}$ is the composition of $C^{i-1} \to I^i$ and $I^i \to K^{i+1}$, and the latter is a monomorphism, it follows that $Z \to C^{i-1} \to I^i$ is zero. But then, since $H^i(M^\bullet) \to C^{i-1}$ is the kernel of $C^{i-1} \to I^i$, there's a unique morphism $Z \to H^i(M^\bullet)$ making the diagram commute. The other statement follows in the same way. ☐

7.2. Exact triangles

One of the main ideas that will motivate our study of homological algebra is the fact that the cohomology functor is not exact, but that somehow we can correct this defect. Let's understand in detail what this means. Consider a short exact sequence of complexes in an abelian category:

$$0 \longrightarrow L^\bullet \xrightarrow{\varphi^\bullet} M^\bullet \xrightarrow{\psi^\bullet} N^\bullet \longrightarrow 0.$$

We recall that, since kernels and cokernels on the category of complexes are computed pointwise, this means that all the components

$$0 \longrightarrow L^i \xrightarrow{\varphi^i} M^i \xrightarrow{\psi^i} N^i \longrightarrow 0.$$

are exact. By the functoriality of H^i, we get a complex

$$0 \longrightarrow H^i(L^\bullet) \longrightarrow H^i(M^\bullet) \longrightarrow H^i(N^\bullet) \longrightarrow 0,$$

which is exact at $H^i(M^\bullet)$ but need not be at the extremities. The first statement will emerge as a particular case of our next theorem, but we can see right away that the cohomology functor need not be exact. Indeed, let $L^\bullet = \iota(\mathbb{Z})[-1]$, $M^\bullet$ be the complex whose only non-zero objects are $M^1 = \mathbb{Z}$ and $M^0 = \mathbb{Z}$, and $N^\bullet$ be the complex whose only non-zero objects are $N^1 = \mathbb{Z}/2\mathbb{Z}$ and $N^0 = \mathbb{Z}$. These complexes fit into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\mathrm{id}_{\mathbb{Z}}"] \arrow[d, "\mathrm{id}_{\mathbb{Z}}"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underbrace{\mathbb{Z}}_{L^1} \arrow[r, "\cdot 2"] & \underbrace{\mathbb{Z}}_{M^1} \arrow[r] & \underbrace{\mathbb{Z}/2\mathbb{Z}}_{N^1} \arrow[r] & 0,
\end{tikzcd}$$

whose rows are exact. Then the complex induced by the functoriality of H^0 is

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 2\mathbb{Z} \longrightarrow 0,$$

which is not exact on the right, and the complex induced by H^1 is

$$0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow 0 \longrightarrow 0,$$

which is not exact on the left.

Considering the lack of exactness of H^i, the next best thing we can hope for is to be able to measure how far it is from being exact at each side. Surprisingly, the objects that measure this lack of exactness are the cohomology objects itself shifted in degree. What follows is arguably the most useful result in homological algebra.

Theorem 7.2.1 — Long exact sequence in cohomology. Consider the following exact sequence of complexes in an abelian category:

$$0 \longrightarrow L^\bullet \xrightarrow{\varphi^\bullet} M^\bullet \xrightarrow{\psi^\bullet} N^\bullet \longrightarrow 0.$$

There exist morphisms $\delta^i : H^i(N^\bullet) \to H^{i+1}(L^\bullet)$ making the diagram

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & H^i(L^\bullet) \arrow[r] & H^i(M^\bullet) \arrow[r] & H^i(N^\bullet) \arrow[dll, dashed, rounded corners, "\delta^i"] & \\
 & H^{i+1}(L^\bullet) \arrow[r] & H^{i+1}(M^\bullet) \arrow[r] & H^{i+1}(N^\bullet) \arrow[r, dashed] & \cdots
\end{tikzcd}$$

a long exact sequence. The δ^i are said to be *connecting morphisms*.

Proof. First of all, we observe that the snake lemma (theorem 6.6.5) implies that the top row in the diagram

$$\begin{tikzcd}
0 \arrow[r] & K^{i+1}_{L^\bullet} \arrow[r] \arrow[d, "\ker d^{i+1}_{L^\bullet}"] & K^{i+1}_{M^\bullet} \arrow[r] \arrow[d, "\ker d^{i+1}_{M^\bullet}"] & K^{i+1}_{N^\bullet} \arrow[d, "\ker d^{i+1}_{N^\bullet}"] & \\
0 \arrow[r] & L^{i+1} \arrow[r, "\varphi^{i+1}"] \arrow[d, "d^{i+1}_{L^\bullet}"] & M^{i+1} \arrow[r, "\psi^{i+1}"] \arrow[d, "d^{i+1}_{M^\bullet}"] & N^{i+1} \arrow[r] \arrow[d, "d^{i+1}_{N^\bullet}"] & 0 \\
0 \arrow[r] & L^{i+2} \arrow[r, "\varphi^{i+2}"] & M^{i+2} \arrow[r, "\psi^{i+2}"] & N^{i+2} \arrow[r] & 0
\end{tikzcd}$$

is exact. Similarly, it implies that the bottom row in the diagram

$$\begin{tikzcd}
0 \arrow[r] & L^{i-1} \arrow[r, "\varphi^{i-1}"] \arrow[d, "d^{i-1}_{L^\bullet}"] & M^{i-1} \arrow[r, "\psi^{i-1}"] \arrow[d, "d^{i-1}_{M^\bullet}"] & N^{i-1} \arrow[r] \arrow[d, "d^{i-1}_{N^\bullet}"] & 0 \\
0 \arrow[r] & L^{i} \arrow[r, "\varphi^{i}"] \arrow[d, "\operatorname{coker} d^{i-1}_{L^\bullet}"] & M^{i} \arrow[r, "\psi^{i}"] \arrow[d, "\operatorname{coker} d^{i-1}_{M^\bullet}"] & N^{i} \arrow[r] \arrow[d, "\operatorname{coker} d^{i-1}_{N^\bullet}"] & 0 \\
 & C^{i-1}_{L^\bullet} \arrow[r] & C^{i-1}_{M^\bullet} \arrow[r] & C^{i-1}_{N^\bullet} \arrow[r] & 0
\end{tikzcd}$$

is exact.

Now, we fit the morphisms $C^{i-1} \to K^{i+1}$ described in the corollary 7.1.4 into a commutative diagram

$$\begin{tikzcd}
 & C^{i-1}_{L^\bullet} \arrow[r] \arrow[d] & C^{i-1}_{M^\bullet} \arrow[r] \arrow[d] & C^{i-1}_{N^\bullet} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K^{i+1}_{L^\bullet} \arrow[r] & K^{i+1}_{M^\bullet} \arrow[r] & K^{i+1}_{N^\bullet}, &
\end{tikzcd}$$

whose rows are exact. One more application of the snake lemma (theorem 6.6.5) provides the desired connecting morphisms. □

As we argued in the proof of the snake lemma (theorem 6.6.5), even though there may be many morphisms $H^i(N^\bullet) \to H^{i+1}(L^\bullet)$ inducing a long exact sequence, there are distinguished ones which are defined as follows: for a class $[n] \in H^i(N^\bullet)$, let m be an element of M^i such that $\psi^i(m) = n$. Then $d^i_{M^\bullet}(m)$ is in the image of φ^{i+1} and we denote its preimage by the same symbol.

$$\begin{tikzcd}
L^i \arrow[r, "\varphi^i"] \arrow[d, "d^i_{L^\bullet}"] & m \arrow[r, mapsto, "\psi^i"] \arrow[d, mapsto, "d^i_{M^\bullet}"] & n \arrow[d, "d^i_{N^\bullet}"] \\
d^i_{M^\bullet}(m) \arrow[r, mapsto, "\varphi^{i+1}"] & d^i_{M^\bullet}(m) \arrow[r, "\psi^{i+1}"] & N^{i+1}
\end{tikzcd}$$

Finally, δ^i is the map which sends $[n]$ to $[d^i_{M^\bullet}(m)]$. Whenever we talk about connecting morphisms, it should be understood that these are the morphisms under consideration.

One important property of the connecting morphisms is that they satisfy a certain naturality condition, which we describe below.

Corollary 7.2.2 Consider the following commutative diagram of complexes in an abelian category

$$\begin{tikzcd}
0 \arrow[r] & L^\bullet \arrow[r] \arrow[d] & M^\bullet \arrow[r] \arrow[d] & N^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L'^\bullet \arrow[r] & M'^\bullet \arrow[r] & N'^\bullet \arrow[r] & 0,
\end{tikzcd}$$

whose rows are exact. Then, for every i, the diagram induced by functoriality and the connecting morphisms

$$\begin{tikzcd}
H^i(N^\bullet) \arrow[r, "\delta^i"] \arrow[d] & H^{i+1}(L^\bullet) \arrow[d] \\
H^i(N'^\bullet) \arrow[r, "\delta'^i"] & H^{i+1}(L'^\bullet)
\end{tikzcd}$$

commutes.

Proof. Let A be the abelian category in question and consider a category $\tilde{A}$, the *arrow category*, whose objects are morphisms in A and whose morphisms between $A \to B$ and $A' \to B'$ are commutative diagrams

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
A' \arrow[r] & B'.
\end{tikzcd}$$

Since $\tilde{A}$ is nothing but $\mathrm{Fun}(T, A)$, where T is a category with two objects and only one non-trivial morphism between them, the proposition 6.5.2 implies that $\tilde{A}$ is abelian.

A complex in $\tilde{A}$ is nothing but a morphism of complexes in A. Denoting the morphism $L^\bullet \to L'^\bullet$ in $C(A)$ by $\tilde{L}^\bullet$, and similarly for the other morphisms, we obtain a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \tilde{L}^\bullet \arrow[r] & \tilde{M}^\bullet \arrow[r] & \tilde{N}^\bullet \arrow[r] & 0
\end{tikzcd}$$

in $C(\tilde{A})$. Then the previous theorem yields morphisms $\tilde{\delta}^i : H^i(\tilde{N}^\bullet) \to H^{i+1}(\tilde{L}^\bullet)$. Since kernels and cokernels are computed pointwise in a functor category (due to the proof of the aforementioned proposition), a morphism $\tilde{\delta}^i : H^i(\tilde{N}^\bullet) \to H^{i+1}(\tilde{L}^\bullet)$ is nothing but a commuting square as desired. □

Due to its somewhat contrived construction, the connecting morphisms doesn't seem to arise in the same fashion as the other morphisms, which are induced from the functoriality of the cohomology functor. This couldn't be further from the truth. We would argue that the long exact sequence in cohomology is simply a shadow of a, perhaps more fundamental, long sequence of complexes. In an ideal world, we would have a morphism of complexes $N^\bullet \to L[1]^\bullet$ and the long exact sequence in cohomology would be nothing but the image of the sequence

$$\cdots \longrightarrow L^\bullet \xrightarrow{\varphi^\bullet} M^\bullet \xrightarrow{\psi^\bullet} N^\bullet \longrightarrow L[1]^\bullet \xrightarrow{\varphi[1]^\bullet} M[1]^\bullet \longrightarrow \cdots$$

under the cohomology functor. This doesn't work.[1] The next best thing would be to find a complex $P^\bullet$, along with a quasi-isomorphism $\rho^\bullet : P^\bullet \to N^\bullet$ making the diagram

$$\begin{tikzcd}
H^i(L^\bullet) \arrow[r, "H^i(\varphi^\bullet)"] \arrow[d, equal] & H^i(M^\bullet) \arrow[r, "H^i(\iota^\bullet)"] \arrow[d, equal] & H^i(P^\bullet) \arrow[r, "H^i(\pi^\bullet)"] \arrow[d, "H^i(\rho^\bullet)"] & H^i(L[1]^\bullet) \arrow[d, equal] \\
H^i(L^\bullet) \arrow[r, "H^i(\varphi^\bullet)"] & H^i(M^\bullet) \arrow[r, "H^i(\psi^\bullet)"] & H^i(N^\bullet) \arrow[r, "\delta^i"] & H^{i+1}(L^\bullet)
\end{tikzcd}$$

commute. In this way, the long exact sequence in cohomology arises, up to isomorphism, as the image of

$$\cdots \longrightarrow L^\bullet \xrightarrow{\varphi^\bullet} M^\bullet \xrightarrow{\iota^\bullet} P^\bullet \xrightarrow{\pi^\bullet} L[1]^\bullet \xrightarrow{\varphi[1]^\bullet} M[1]^\bullet \longrightarrow \cdots$$

under the cohomology functor and connecting morphism δ^i can be described as $H^i(\pi^\bullet) \circ H^i(\rho^\bullet)^{-1}$.

All our hopes and dreams will come true. The reader may recall that there is indeed a natural complex $P^\bullet$ which fits in a short exact sequence

$$0 \longrightarrow M^\bullet \xrightarrow{\iota^\bullet} P^\bullet \xrightarrow{\pi^\bullet} L[1]^\bullet \longrightarrow 0.$$

It is the direct sum $P^\bullet = M^\bullet \oplus L[1]^\bullet$, with its natural injections and projections. We also have a natural morphism $\rho^\bullet : P^\bullet \to N^\bullet$ defined as the composition of the projection $M^\bullet \oplus L[1]^\bullet \to M^\bullet$ with the given morphism $\psi^\bullet : M^\bullet \to N^\bullet$.

Ay, there's the rub! The natural morphism $\rho^\bullet : P^\bullet \to N^\bullet$ need *not* induce an isomorphism on cohomology. For example, consider the following short exact sequence of complexes of abelian groups

$$0 \longrightarrow \iota(\mathbb{Z}) \xrightarrow{\varphi^\bullet} \iota(\mathbb{Z}) \longrightarrow \iota(\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0,$$

---

[1] For example, consider the exact sequence of complexes we used to prove that the cohomology functor is not exact. A morphism of complexes $N^\bullet \to L[1]^\bullet$ inducing the connecting morphism $H^0(N^\bullet) \to H^1(L^\bullet)$ would correspond to a morphism of abelian groups $\mathbb{Z} \to \mathbb{Z}$ whose restriction to $2\mathbb{Z}$ is an isomorphism $2\mathbb{Z} \to \mathbb{Z}$. Such a morphism does not exist.

where ι is the embedding of Ab into $\mathsf{C}(\mathsf{Ab})$ and $\varphi^\bullet$ is the multiplication by 2 map. In this case, the naive direct sum is simply the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \underbrace{\mathbb{Z}}_{\text{degree } 0} \longrightarrow 0 \longrightarrow \cdots$$

whose 0-th cohomology is $\mathbb{Z}$, instead of $\mathbb{Z}/2\mathbb{Z}$. The problem, of course, is that our definition of $P^\bullet$ carries no information about morphisms involved in the original exact sequence. If, in the place of the zero-morphism $\mathbb{Z} \to \mathbb{Z}$ above, it was φ^0, no such problem would arise: the 0-th cohomology would be $\mathbb{Z}/2\mathbb{Z}$ and all the other degrees would be zero.

The preceding discussion suggests that it may be useful to consider a complex with the same objects as $M^\bullet \oplus L[1]^\bullet$ but whose i-th differential is given by

$$\begin{pmatrix} d^i_{M^\bullet} & -\varphi[1]^i \\ 0 & d^i_{L[1]^\bullet} \end{pmatrix} = \begin{pmatrix} d^i_{M^\bullet} & -\varphi^{i+1} \\ 0 & -d^{i+1}_{L^\bullet} \end{pmatrix}.$$

As we shall see, it is this object that will solve all our problems.

Definition 7.2.1 — Mapping cone. Let $\varphi^\bullet : L^\bullet \to M^\bullet$ be a morphism of complexes in an additive category. The *mapping cone* of $\varphi^\bullet$ is the complex $\mathrm{MC}(\varphi)^\bullet$ whose objects are $\mathrm{MC}(\varphi)^i := M^i \oplus L^{i+1}$ and whose i-th differential is[a]

$$\begin{pmatrix} d^i_{M^\bullet} & -\varphi^{i+1} \\ 0 & -d^{i+1}_{L^\bullet} \end{pmatrix}.$$

---

[a] There are different sign conventions in the literature.

Since the composition of morphisms represented by matrices is given by the multiplication of the respective matrices, we have that $d^i_{\mathrm{MC}(\varphi)^\bullet} \circ d^{i-1}_{\mathrm{MC}(\varphi)^\bullet}$ is represented by

$$\begin{pmatrix} d^i_{M^\bullet} & -\varphi^{i+1} \\ 0 & -d^{i+1}_{L^\bullet} \end{pmatrix} \begin{pmatrix} d^{i-1}_{M^\bullet} & -\varphi^{i} \\ 0 & -d^{i}_{L^\bullet} \end{pmatrix} = \begin{pmatrix} d^i_{M^\bullet} \circ d^{i-1}_{M^\bullet} & \varphi^{i+1} \circ d^i_{L^\bullet} - d^i_{M^\bullet} \circ \varphi^i \\ 0 & d^{i+1}_{L^\bullet} \circ d^i_{L^\bullet} \end{pmatrix} = 0,$$

proving that $\mathrm{MC}(\varphi)^\bullet$ is indeed a complex. In an abelian category, the mapping cone inherits a short exact sequence

$$0 \longrightarrow M^\bullet \xrightarrow{\iota^\bullet} \mathrm{MC}(\varphi)^\bullet \xrightarrow{\pi^\bullet} L[1]^\bullet \longrightarrow 0$$

where the natural injections and projections are still morphisms of complexes, even with the new differential. Moreover, there's an induced long sequence of complexes

$$\cdots \longrightarrow L^\bullet \xrightarrow{\varphi^\bullet} M^\bullet \xrightarrow{\iota^\bullet} \mathrm{MC}(\varphi)^\bullet \xrightarrow{\pi^\bullet} L[1]^\bullet \xrightarrow{\varphi[1]^\bullet} M[1]^\bullet \longrightarrow \cdots.$$

In order to properly deal with such long sequences of complexes, we introduce some notation. We denote a long sequence of complexes of the form

$$\cdots \longrightarrow L^\bullet \xrightarrow{\varphi^\bullet} M^\bullet \xrightarrow{\psi^\bullet} N^\bullet \longrightarrow L[1]^\bullet \xrightarrow{\varphi[1]^\bullet} M[1]^\bullet \longrightarrow \cdots$$

as a *triangle*

$$\begin{tikzcd}
 & L^\bullet \arrow[dr, "\varphi^\bullet"] & \\
N^\bullet \arrow[ur, "+1"] & & M^\bullet, \arrow[ll, "\psi^\bullet"]
\end{tikzcd}$$

where the arrow marked by $+1$ indicates that the morphism shifts the degree by one, representing the imposing diagram

$$\begin{tikzcd}
 & & \\
 & L^{i+1} \arrow[dr] & \\
N^{i+1} \arrow[uu, no head] \arrow[uur, no head] & & M^{i+1} \arrow[uu, no head] \arrow[ll] \\
 & L^{i} \arrow[uu, "d^i_{N^\bullet}"] \arrow[dr] & \\
N^{i} \arrow[uu, "d^i_{N^\bullet}"] \arrow[uuur] & & M^{i} \arrow[uu, "d^i_{M^\bullet}"'] \arrow[ll] \\
 & L^{i-1} \arrow[uu, "d^{i-1}_{N^\bullet}"] \arrow[dr] & \\
N^{i-1} \arrow[uu, "d^{i-1}_{N^\bullet}"] \arrow[uuur] & & M^{i-1}. \arrow[uu, "d^{i-1}_{M^\bullet}"'] \arrow[ll] \\
 \arrow[uur] & \arrow[uu] &
\end{tikzcd}$$

A morphism of triangles consists of morphisms $\lambda^\bullet$, $\mu^\bullet$, and $\nu^\bullet$, making the diagram

$$\begin{tikzcd}
L^\bullet \arrow[r, "\varphi^\bullet"] \arrow[d, "\lambda^\bullet"] & M^\bullet \arrow[r, "\psi^\bullet"] \arrow[d, "\mu^\bullet"] & N^\bullet \arrow[r] \arrow[d, "\nu^\bullet"] & L[1]^\bullet \arrow[d, "\lambda[1]^\bullet"] \\
L'^\bullet \arrow[r, "\varphi'^\bullet"] & M'^\bullet \arrow[r, "\psi'^\bullet"] & N'^\bullet \arrow[r] & L'[1]^\bullet
\end{tikzcd}$$

commute. Moreover, a triangle is said to be exact if it arises from a long exact sequence.[2]

In this notation, the plan we outlined before can be encapsulated as the fact that, given a short exact sequence of complexes

$$0 \longrightarrow L^\bullet \xrightarrow{\varphi^\bullet} M^\bullet \xrightarrow{\psi^\bullet} N^\bullet \longrightarrow 0,$$

---

[2] Some references define an *exact triangle* to be what we'll soon call a *distinguished triangle*.

the cohomology functor $H^\bullet$ takes the triangle induced by $\mathrm{MC}(\varphi)^\bullet$ and outputs an exact triangle

$$\begin{tikzcd}
& H^\bullet(L^\bullet) \arrow[dr, "H^\bullet(\varphi^\bullet)"] & \\
H^\bullet(\mathrm{MC}(\varphi)^\bullet) \arrow[ur, "+1"] & & H^\bullet(M^\bullet), \arrow[ll]
\end{tikzcd}$$

which is isomorphic to the triangle arising from the long exact sequence in cohomology

$$\begin{tikzcd}
& H^\bullet(L^\bullet) \arrow[dr, "H^\bullet(\varphi^\bullet)"] & \\
H^\bullet(N^\bullet) \arrow[ur, "+1"] & & H^\bullet(M^\bullet). \arrow[ll, "H^\bullet(\psi^\bullet)"']
\end{tikzcd}$$

We now prove this fact.

Proposition 7.2.3 Consider the following exact sequence of complexes in an abelian category:

$$0 \longrightarrow L^\bullet \xrightarrow{\varphi^\bullet} M^\bullet \xrightarrow{\psi^\bullet} N^\bullet \longrightarrow 0.$$

There exists a quasi-isomorphism $\rho^\bullet : \mathrm{MC}(\varphi)^\bullet \to N^\bullet$ making the diagram

$$\begin{tikzcd}
H^{i-1}(L[1]^\bullet) \arrow[r] \arrow[d, equal] & H^i(M^\bullet) \arrow[r, "H^i(\iota^\bullet)"] \arrow[d, equal] & H^i(\mathrm{MC}(\varphi)^\bullet) \arrow[r, "H^i(\pi^\bullet)"] \arrow[d, "H^i(\rho^\bullet)"] & H^i(L[1]^\bullet) \arrow[d, equal] \\
H^i(L^\bullet) \arrow[r, "H^i(\varphi^\bullet)"] & H^i(M^\bullet) \arrow[r, "H^i(\psi^\bullet)"] & H^i(N^\bullet) \arrow[r, "\delta^i"] & H^{i+1}(L^\bullet)
\end{tikzcd}$$

commute. In particular, the top row is exact.

Proof. As before, $\rho^\bullet$ is simply the composition of the projection $\mathrm{MC}(\varphi)^\bullet \to M^\bullet$ with the given morphism $\psi^\bullet : M^\bullet \to N^\bullet$. This morphism makes the middle square commute due to the fact that the composition $M^\bullet \to \mathrm{MC}(\varphi)^\bullet \to M^\bullet$ is the identity (theorem 6.1.9) and so the diagram below

$$\begin{tikzcd}
M^\bullet \arrow[r, "\iota^\bullet"] \arrow[dd, equal] & \mathrm{MC}(\varphi)^\bullet \arrow[d] \arrow[dd, bend left, "\rho^\bullet"] \\
& M^\bullet \arrow[d, "\psi^\bullet"'] \\
M^\bullet \arrow[r, "\psi^\bullet"] & N^\bullet
\end{tikzcd}$$

commutes. The commutativity of the square on the right means that the composition $\delta^i \circ H^i(\rho^\bullet)$ sends $[(m, l)] \in H^i(\mathrm{MC}(\varphi)^\bullet)$ to $[l] \in H^{i+1}(L^\bullet)$. Now, $H^i(\rho^\bullet)$ sends $[(m, l)]$ to

$[\psi^i(m)]$ and δ^i sends this element to $[l']$, where l' is some element satisfying $\varphi^{i+1}(l') = d^i_{M^\bullet}(m)$. But l itself is one such element, due to the fact that $(m, l) \in \ker d^i_{MC(\varphi)^\bullet}$.

All that remains is to prove that $H^i(\rho^\bullet)$ is an isomorphism for all i. We begin by showing that it's surjective. If $[n] \in H^i(N^\bullet)$, let $m \in M^i$ by any element satisfying $\psi^i(m) = n$. (Such an element indeed exists, for ψ^i is surjective.) The commutativity of the right square

$$\begin{tikzcd}
L^i \arrow[r, "\varphi^i"] \arrow[d, "d^i_{L^\bullet}"] & M^i \arrow[r, "\psi^i"] \arrow[d, "d^i_{M^\bullet}"] & N^i \arrow[d, "d^i_{N^\bullet}"] \\
L^{i+1} \arrow[r, "\varphi^{i+1}"] & M^{i+1} \arrow[r, "\psi^{i+1}"] & N^{i+1}
\end{tikzcd}$$

implies that $d^i_{M^\bullet}(m)$ is in the kernel of ψ^{i+1}, and so $d^i_{M^\bullet}(m) = \varphi^{i+1}(l)$ for some $l \in L^{i+1}$. Since the diagram

$$\begin{tikzcd}
L^i \arrow[r, "\varphi^i"] \arrow[d, "d^i_{L^\bullet}"] & M^i \arrow[d, "d^i_{M^\bullet}"] \\
L^{i+1} \arrow[r, "\varphi^{i+1}"] \arrow[d, "d^{i+1}_{L^\bullet}"] & M^{i+1} \arrow[d, "d^{i+1}_{M^\bullet}"] \\
L^{i+2} \arrow[r, "\varphi^{i+2}"] & M^{i+2}
\end{tikzcd}$$

commutes, $d^{i+1}_{L^\bullet}(l)$ is in the kernel of φ^{i+2}. But φ^{i+2} is injective, proving that $d^{i+1}_{L^\bullet}(l) = 0$. We conclude that

$$d^i_{MC(\varphi)^\bullet}(m, l) = (d^i_{M^\bullet}(m) - \varphi^{i+1}(l), -d^{i+1}_{L^\bullet}(l)) = 0,$$

which implies that $[(m, l)]$ is an element of $H^i(MC(\varphi)^\bullet)$ that is sent to $[n]$ by $H^i(\rho^\bullet)$.

In order to see that $H^i(\rho^\bullet)$ is also injective, suppose that $[(m, l)] \in H^i(MC(\varphi)^\bullet)$ is such that $\psi^i(m) \in \operatorname{im} d^{i-1}_{N^\bullet}$. Then, choose $n \in N^{i-1}$ satisfying $d^{i-1}_{N^\bullet}(n) = \psi^i(m)$ and $m' \in M^{i-1}$ satisfying $\psi^{i-1}(m') = n$. (The latter exists by surjectivity of ψ^{i-1}.) The commutativity of the diagram

$$\begin{tikzcd}
L^{i-1} \arrow[r, "\varphi^{i-1}"] \arrow[d, "d^{i-1}_{L^\bullet}"] & M^{i-1} \arrow[r, "\psi^{i-1}"] \arrow[d, "d^{i-1}_{M^\bullet}"] & N^{i-1} \arrow[d, "d^{i-1}_{N^\bullet}"] \\
L^i \arrow[r, "\varphi^i"] & M^i \arrow[r, "\psi^i"] & N^i
\end{tikzcd}$$

shows that m and $d^{i-1}_{M^\bullet}(m')$ are sent to the same element in N^i by ψ^i. It follows that there exists $l' \in L^i$ satisfying

$$\varphi^i(l') = d^{i-1}_{M^\bullet}(m') - m.$$

Now, both $-d^i_{L^\bullet}(l')$ and l are sent to $d^i_{M^\bullet}(m)$ by φ^{i+1}. The injectivity of φ^{i+1} yields that they're equal. We conclude that $(m, l) = d^{i-1}_{MC(\varphi)^\bullet}(m', l')$; showing that $[(m, l)] = 0$ in $H^i(MC(\varphi)^\bullet)$ and finishing the proof. □

Beyond being conceptually enlightning, the last proposition also provides us with a criterion for a morphism of complexes to be a quasi-isomorphism.

> **Corollary 7.2.4** Let $\varphi^\bullet : L^\bullet \to M^\bullet$ be a morphism of complexes. Then $\varphi^\bullet$ is a quasi-isomorphism if and only if its mapping cone $\mathrm{MC}(\varphi)^\bullet$ is an exact complex.

Proof. If $\mathrm{MC}(\varphi)^\bullet$ is an exact complex, then its cohomology is zero and so the long exact sequence in cohomology

$$\underbrace{H^{i-1}(\mathrm{MC}(\varphi)^\bullet)}_{=0} \longrightarrow H^i(L^\bullet) \xrightarrow{H^i(\varphi^\bullet)} H^i(M^\bullet) \longrightarrow \underbrace{H^i(\mathrm{MC}(\varphi)^\bullet)}_{=0}$$

implies that $\varphi^\bullet$ is a quasi-isomorphism. Conversely, if $\varphi^\bullet$ is a quasi-isomorphism, the long exact sequence in cohomology

$$H^i(L^\bullet) \xrightarrow{H^i(\varphi^\bullet)} H^i(M^\bullet) \xrightarrow{\alpha} H^i(\mathrm{MC}(\varphi)^\bullet) \xrightarrow{\beta} H^{i+1}(L^\bullet) \xrightarrow{H^{i+1}(\varphi^\bullet)} H^{i+1}(M^\bullet)$$

implies that $\ker\alpha = \mathrm{id}_{H^i(M^\bullet)}$, that $\operatorname{im}\alpha = \ker\beta$ and that $\operatorname{im}\beta$ is the zero-morphism. The first and the last pieces of information mean that both α and β are zero-morphisms, and $\operatorname{im}\alpha = \ker\beta$ implies that β is a monomorphism. But then $\ker\beta$ is both the identity on $H^i(\mathrm{MC}(\varphi)^\bullet)$ and the zero-morphism $0 \to H^i(\mathrm{MC}(\varphi)^\bullet)$. It follows that $H^i(\mathrm{MC}(\varphi)^\bullet) = 0$. □

This corollary allows us to prove that quasi-isomorphisms are preserved by exact functors.

> **Corollary 7.2.5** Let $F : \mathsf{A} \to \mathsf{B}$ be an exact functor between abelian categories and $\varphi^\bullet : L^\bullet \to M^\bullet$ be a morphism of complexes in A. If $\varphi^\bullet$ is a quasi-isomorphism, then so is $F(\varphi^\bullet)$.

Proof. Due to the last corollary, it suffices to prove that $\mathrm{MC}(F(\varphi^\bullet)) = F(\mathrm{MC}(\varphi^\bullet))$ is an exact complex. But this follows from the fact that F is exact. □

One aspect of mapping cone of a morphism $\varphi^\bullet : L^\bullet \to M^\bullet$ that we have not yet addressed is the fact that, even though $\mathrm{MC}(\varphi)^\bullet$ is always a complex, the long sequence induced

$$\cdots \longrightarrow L^\bullet \xrightarrow{\varphi^\bullet} M^\bullet \xrightarrow{\iota^\bullet} \mathrm{MC}(\varphi)^\bullet \xrightarrow{\pi^\bullet} L[1]^\bullet \xrightarrow{\varphi[1]^\bullet} M[1]^\bullet \longrightarrow \cdots$$

need not be. This means that our triangles aren't elements of $\mathsf{C}(\mathsf{C}(\mathsf{A}))$. Indeed, the composition

$$L^\bullet \to M^\bullet \to \mathrm{MC}(\varphi)^\bullet$$

sends $l \in L^i$ to $(\varphi^i(l), 0) \in M^i \oplus L^{i+1}$, which isn't always zero unless φ^i is the zero morphism. Moreover, the composition

$$\mathrm{MC}(\varphi)^\bullet \to L[1]^\bullet \to M[1]^\bullet$$

sends $(m, l) \in M^i \oplus L^{i+1}$ to $\varphi^{i+1}(l) \in M^{i+1}$, which also isn't always zero unless φ^{i+1} is the zero morphism.

Notwithstanding the fact that these compositions are usually not zero, they do indeed map to the zero-morphism in cohomology. And they do so for a good reason, which will be the main focus of the next section.

7.3. The homotopic category

The main line of attack in homological algebra to understanding some mathematical object consists of associating some interesting complex to this object and then taking its cohomology. For example, given a smooth manifold M, we associate to it a complex

$$0 \longrightarrow \Omega^0_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \Omega^2_M \xrightarrow{d} \cdots,$$

where Ω^i_M is the $\mathbb{R}$-vector space of differential i-forms on M and d is the exterior derivative. The i-th cohomology of this complex $H^i_{dR}(M)$ is said to be the *de Rham cohomology* of M and is an important invariant of a manifold. Of a more algebraic nature are the modules $\mathrm{Tor}^A_i(M, N)$ which are computed in the following way: we find an exact sequence of A-modules

$$\cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where each P_i is a *projective* module, we tensor by N and take the $-i$-th cohomology of the complex

$$\cdots \longrightarrow P_3 \otimes_A N \longrightarrow P_2 \otimes_A N \longrightarrow P_1 \otimes_A N \longrightarrow P_0 \otimes_A N.$$

(All the omitted objects are supposed to be zero.) Surprisingly, the final result is independent of the choice of the projective modules P_i. We could even take the P_i to be flat modules and the result wouldn't change.

But we can do better! Instead of taking the cohomology of the associated complexes, we can consider them "up to quasi-isomorphism". In this way we retain all the cohomological information while being able to use the tools available for dealing with complexes. Somewhat more formally, we would like to find a category $\mathsf{D}(\mathsf{A})$, with the same objects as $\mathsf{C}(\mathsf{A})$ but where all the quasi-isomorphisms become genuine isomorphisms.

This category, along with its bounded variants $\mathsf{D}^*(\mathsf{A})$ for $* = +, -, b$, indeed exists[3] and it's called the *derived category* of A. This category satisfies a universal property alike that of the localization of modules: it is endowed with an additive functor $\mathsf{C}(\mathsf{A}) \to \mathsf{D}(\mathsf{A})$ such that quasi-isomorphisms in $\mathsf{C}(\mathsf{A})$ are mapped to isomorphisms in $\mathsf{D}(\mathsf{A})$ and which is initial with respect to this property.

It is the derived category that is the natural place to study homological algebra. Nevertheless, there is an intermediate category, the *homotopic category*, that will not only simplify the description of the morphisms in the derived category but also furnish a substitute thereof in important cases. We begin its study now.

Definition 7.3.1 — Homotopy. Let $\varphi^\bullet, \psi^\bullet : L^\bullet \to M^\bullet$ be two morphisms of complexes in an additive category. A *homotopy* between $\varphi^\bullet$ and $\psi^\bullet$ is a collection of morphisms $h^i : L^i \to M^{i-1}$ such that

$$\psi^i - \varphi^i = d^{i-1}_{M^\bullet} \circ h^i + h^{i+1} \circ d^i_{L^\bullet}$$

for all i. If there exists a homotopy between $\varphi^\bullet$ and $\psi^\bullet$, we say that they are *homotopic*, and we denote it by $\varphi^\bullet \sim \psi^\bullet$.

We observe that this is indeed an equivalence relation: reflexivity and symmetry are immediate, and it suffices to sum the homotopies to prove that it is transitive. We also emphasize that the h^i need not form a morphism of complexes $L^\bullet \to M[-1]^\bullet$. In particular, the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & L^{i-1} \arrow[r, "d^{i-1}_{L^\bullet}"] \arrow[dl] \arrow[d, shift right, "\varphi^{i-1}"'] \arrow[d, shift left, "\psi^{i-1}"] & L^i \arrow[r, "d^i_{L^\bullet}"] \arrow[dl, "h^i"'] \arrow[d, shift right, "\varphi^{i}"'] \arrow[d, shift left, "\psi^{i}"] & L^{i+1} \arrow[r] \arrow[dl, "h^{i+1}"'] \arrow[d, shift right, "\varphi^{i+1}"'] \arrow[d, shift left, "\psi^{i+1}"] & \cdots \arrow[dl] \\
\cdots \arrow[r] & M^{i-1} \arrow[r, "d^{i-1}_{M^\bullet}"'] & M^i \arrow[r, "d^i_{M^\bullet}"'] & M^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

need not commute. The next proposition explains how homotopy interacts with the additive structure of $\mathsf{C}(\mathsf{A})$. The reader may remember its first part as saying that "morphisms homotopic to zero form an ideal".

Proposition 7.3.1 Let $\varphi_1^\bullet, \varphi_2^\bullet : L^\bullet \to M^\bullet$ and $\psi_1^\bullet, \psi_2^\bullet : M^\bullet \to N^\bullet$ be morphisms of complexes in an additive category. The following holds.

(a) If $\varphi_1^\bullet \sim 0$ and $\varphi_2^\bullet \sim 0$, then $\varphi_1^\bullet + \varphi_2^\bullet \sim 0$, $\varphi_1^\bullet \circ \alpha^\bullet \sim 0$ and $\beta^\bullet \circ \varphi_1^\bullet \sim 0$ whenever those compositions exist;

(b) if $\varphi_1^\bullet \sim \varphi_2^\bullet$ and $\psi_1^\bullet \sim \psi_2^\bullet$, then $\psi_1^\bullet \circ \varphi_1^\bullet \sim \psi_2^\bullet \circ \varphi_2^\bullet$.

Proof. If $\varphi_1^\bullet \sim 0$ and $\varphi_2^\bullet \sim 0$, then there exists collections of morphisms $h^i, k^i : L^i \to$

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[3] Up to some set-theoretic subtleties, which will be discussed in the next chapter.

M^{i-1} such that

$$\varphi_1^i = d_{M^\bullet}^{i-1} \circ h^i + h^{i+1} \circ d_{L^\bullet}^i \quad \text{and} \quad \varphi_2^i = d_{M^\bullet}^{i-1} \circ k^i + k^{i+1} \circ d_{L^\bullet}^i.$$

Summing these equations, we see that the morphisms $h^i + k^i$ form a homotopy between $\varphi_1^\bullet + \varphi_2^\bullet$ and zero. By composing on the left with $\alpha^\bullet : P^\bullet \to L^\bullet$ we get that

$$\begin{aligned} \varphi_1^i \circ \alpha^i &= d_{M^\bullet}^{i-1} \circ h^i \circ \alpha^i + h^{i+1} \circ d_{L^\bullet}^i \circ \alpha^i \\ &= d_{M^\bullet}^{i-1} \circ (h^i \circ \alpha^i) + (h^{i+1} \circ \alpha^{i+1}) \circ d_{P^\bullet}^i, \end{aligned}$$

proving that $h^i \circ \alpha^i$ is a homotopy between $\varphi_1^\bullet \circ \alpha^\bullet$ and 0. The same argument proves that $\beta^\bullet \circ \varphi_1^\bullet \sim 0$. This establishes (a).

Now, (b) follows from (a) by noticing that

$$\begin{aligned} 0 &\sim \psi_1^\bullet \circ (\varphi_1^\bullet - \varphi_2^\bullet) = \psi_1^\bullet \circ \varphi_1^\bullet - \psi_1^\bullet \circ \varphi_2^\bullet \\ 0 &\sim (\psi_1^\bullet - \psi_2^\bullet) \circ \varphi_2^\bullet = \psi_1^\bullet \circ \varphi_2^\bullet - \psi_2^\bullet \circ \varphi_2^\bullet \end{aligned}$$

and adding the two equations. ☐

There's an important definition which encodes the notion of "isomorphism up to homotopy".

Definition 7.3.2 — Homotopy equivalence. A morphism of complexes $\varphi^\bullet : L^\bullet \to M^\bullet$ is said to be a *homotopy equivalence* if there exists a morphism $\psi^\bullet : M^\bullet \to L^\bullet$ such that $\varphi^\bullet \circ \psi^\bullet \sim \mathrm{id}_{M^\bullet}$ and $\psi^\bullet \circ \varphi^\bullet \sim \mathrm{id}_{L^\bullet}$. If there exists a homotopy equivalence between two complexes, they are said to be *homotopy equivalent*.

Once again, this defines an equivalence relation: reflexivity and symmetry are clear and transitivity follows from the preceding proposition. We observe that, from this point of view, the notion of homotopy equivalence is better behaved then that of quasi-isomorphism as the latter doesn't define an equivalence relation between complexes. Indeed, in $\mathsf{C}(\mathsf{Ab})$ the morphism of complexes

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is a quasi-isomorphism which does not possess an inverse (as there are no non-trivial morphisms $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$), proving that quasi-isomorphism is not a symmetric relation.

Even though the above is only one of the multiple reasons why homotopy equivalence is a more tractable notion than that of quasi-isomorphisms, it would all be for nothing if homotopy weren't a stepping stone to the derived category. The next proposition begins to describe how our plan works.

Proposition 7.3.2 Let $\varphi^\bullet, \psi^\bullet : L^\bullet \to M^\bullet$ be homotopic morphisms of complexes in an abelian category. Then $\varphi^\bullet$ and $\psi^\bullet$ induce the same morphism on cohomology. In particular, every homotopy equivalence is a quasi-isomorphism.

Proof. We prove that $\psi^\bullet - \varphi^\bullet$ induces the zero-morphism in cohomology, i.e., that it sends elements of $\ker d^i_{L^\bullet}$ to elements of $\operatorname{im} d^{i-1}_{M^\bullet}$. But this is clear since

$$\psi^i(l) - \varphi^i(l) = d^{i-1}_{M^\bullet}(h^i(l)) + h^{i+1}(d^i_{L^\bullet}(l)),$$

and the last term vanishes whenever $l \in \ker d^i_{L^\bullet}$. □

This is why the long sequence induced by the mapping cone of a morphism "has a good reason" to become a complex in cohomology: the composition of two consecutive morphisms is not necessarily zero, but they are *homotopic* to zero. This proposition, in the form of the corollary below, also describes why the line of attack described in the beginning of this section works: often we'll associate non-isomorphic complexes to a mathematical object, but they'll turn out to be homotopy equivalent.

Corollary 7.3.3 Let $L^\bullet$ and $M^\bullet$ be homotopy equivalent complexes in an abelian category. Then $H^\bullet(L^\bullet) \cong H^\bullet(M^\bullet)$.

Proof. Due to the preceding proposition, the morphisms which define a homotopy equivalence between $L^\bullet$ and $M^\bullet$ induce inverse morphisms in cohomology. □

There is another aspect where homotopy equivalences are simpler than quasi-isomorphisms: while the latter is only[4] preserved by an exact functor (corollary 7.2.5), the former is preserved by arbitrary additive functors.

Proposition 7.3.4 Let $F : \mathsf{A} \to \mathsf{B}$ be an additive functor between additive categories, and let $\varphi^\bullet, \psi^\bullet : L^\bullet \to M^\bullet$ be homotopic morphisms in $\mathsf{C}(\mathsf{A})$. Then $F(\varphi^\bullet)$ and $F(\psi^\bullet)$ are homotopic in $\mathsf{C}(\mathsf{B})$. Moreover, if $L^\bullet$ and $M^\bullet$ are homotopy equivalent in $\mathsf{C}(\mathsf{A})$, then $F(L^\bullet)$ and $F(M^\bullet)$ are homotopy equivalent in $\mathsf{C}(\mathsf{B})$.

Proof. The second assertion follows immediately from the first. As for the first, observe that if h is a homotopy between $\varphi^\bullet$ and $\psi^\bullet$, then

$$\begin{aligned} F(\psi^i) - F(\varphi^i) &= F(d^{i-1}_{M^\bullet}) \circ F(h^i) + F(h^{i+1}) \circ F(d^i_{L^\bullet}) \\ &= d^{i-1}_{F(M^\bullet)} \circ F(h^i) + F(h^{i+1}) \circ d^i_{F(L^\bullet)}. \end{aligned}$$

This proves that the morphisms $F(h^i)$ define a homotopy between $F(\varphi^\bullet)$ and $F(\psi^\bullet)$. □

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[4] Indeed, if F is not exact, there exists a three-term exact sequence whose image is not exact. But a three-term complex is exact if and only if it is quasi-isomorphic to zero.

This validates our strategy: beginning with a mathematical object to which we associate some interesting complex, we apply some additive functor and then see the result in the derived category. The proposition 7.3.4 and the corollary 7.3.3 shows that any other homotopy equivalent complex would yield the same result at the end.

By identifying homotopic morphisms, we obtain the homotopic category.

Definition 7.3.3 — Homotopic category. Let A be an additive category. The *homotopic category* $\mathsf{K}(\mathsf{A})$ is the category whose objects are complexes in A and whose morphisms are homotopy classes of morphisms of complexes. We define likewise bounded variants $\mathsf{K}^*(\mathsf{A})$, for $* = +, -, b$, thereof.

The part (b) of proposition 7.3.1 implies that indeed $\mathsf{K}^*(\mathsf{A})$ satisfy the axioms of a category, and the part (a) shows that they are moreover preadditive. Since they have a zero-object and binary products, they are even additive. They aren't, through, almost never abelian even if A is. Indeed, we'll soon see that if $\mathsf{K}^*(\mathsf{A})$ is abelian then every short exact sequence in A splits.

We observe that if $F : \mathsf{A} \to \mathsf{B}$ is an additive functor between additive categories, then we have a natural functor $F : \mathsf{K}(\mathsf{A}) \to \mathsf{K}(\mathsf{B})$ by the proposition 7.3.4 and the universal property of quotients.

The reader may recall that our long-term goal is to understand the derived category, which is constructed from $\mathsf{C}(\mathsf{A})$ by inverting all the quasi-isomorphisms. In defining the homotopic category, we have determined a functor

$$\mathsf{C}(\mathsf{A}) \to \mathsf{K}(\mathsf{A})$$

which sends every object to itself and every morphism to its homotopy class. This functor sends every homotopy equivalence to an isomorphism and, as the proposition below shows, is a stepping stone to the derived category.

Proposition 7.3.5 Let $F : \mathsf{C}^*(\mathsf{A}) \to \mathsf{D}$ be an additive functor such that $F(\varphi^\bullet)$ is an isomorphism whenever $\varphi^\bullet$ is a quasi-isomorphism. Then there exists a unique additive functor $\mathsf{K}^*(\mathsf{A}) \to \mathsf{D}$ making the diagram

$$\begin{tikzcd}
\mathsf{C}^*(\mathsf{A}) \arrow[r, "F"] \arrow[d] & \mathsf{D} \\
\mathsf{K}^*(\mathsf{A}) \arrow[ur, dashed] &
\end{tikzcd}$$

commute.

We remark that the natural functor $\mathsf{C}^*(\mathsf{A}) \to \mathsf{K}^*(\mathsf{A})$ *doesn't always sends quasi-isomorphisms to isomorphisms. In other words,* $\mathsf{K}^*(\mathsf{A})$ *need* not *satisfy the universal property of the derived category* $\mathsf{D}^*(\mathsf{A})$.

Proof. We begin by remarking that F sends an exact complex $\mathsf{P}^\bullet \in \mathsf{C}^*(\mathsf{A})$ to the zero-object of D. In this case, $0 : \mathsf{P}^\bullet \to \mathsf{P}^\bullet$ is a quasi-isomorphism and so $\mathsf{F}(0) : \mathsf{F}(\mathsf{P}^\bullet) \to \mathsf{F}(\mathsf{P}^\bullet)$ is an isomorphism. But we may write the identity of $\mathsf{F}(\mathsf{P}^\bullet)$ as $\mathsf{F}(0) \circ \mathsf{F}(0)^{-1} = 0$. Since every morphism going to or out of $\mathsf{F}(\mathsf{P}^\bullet)$ factors through the identity, this implies that $\mathsf{F}(\mathsf{P}^\bullet)$ is both initial and final. That is, $\mathsf{F}(\mathsf{P}^\bullet) = 0$.

Now, we need to show that if $\varphi^\bullet, \psi^\bullet : \mathsf{L}^\bullet \to \mathsf{M}^\bullet$ are homotopic morphisms in $\mathsf{C}^*(\mathsf{A})$, then $\mathsf{F}(\varphi^\bullet) = \mathsf{F}(\psi^\bullet)$. Since F is additive, we may assume $\psi^\bullet = 0$. Moreover, since $-\operatorname{id}_{\mathsf{L}^\bullet}$ is a quasi-isomorphism, the corollary 7.2.4 implies that $\mathrm{MC}(-\operatorname{id}_{\mathsf{L}^\bullet})^\bullet$ is exact. In particular, it suffices to prove that $\varphi^\bullet$ factors through $\mathrm{MC}(-\operatorname{id}_{\mathsf{L}^\bullet})^\bullet$.

We already possess the natural injection $\mathsf{L}^\bullet \to \mathrm{MC}(-\operatorname{id}_{\mathsf{L}^\bullet})^\bullet$ so we only have to define a morphism of complexes $\pi^\bullet : \mathrm{MC}(-\operatorname{id}_{\mathsf{L}^\bullet})^\bullet \to \mathsf{M}^\bullet$ making the diagram

$$\begin{tikzcd}
\mathsf{L}^\bullet \arrow[r, "\varphi^\bullet"] \arrow[d] & \mathsf{M}^\bullet \\
\mathrm{MC}(-\operatorname{id}_{\mathsf{L}^\bullet})^\bullet \arrow[ur, "\pi^\bullet"'] &
\end{tikzcd}$$

commute. With that in mind, consider a homotopy $\mathsf{h}^i : \mathsf{L}^i \to \mathsf{M}^{i-1}$ between $\varphi^\bullet$ and the zero-morphism. We then define our desired morphism $\pi^i : \mathsf{L}^i \oplus \mathsf{L}^{i+1} \to \mathsf{M}^i$ as $(\varphi^i, \mathsf{h}^{i+1})$. It is clear that this makes the diagram above commute. It being a morphism of complexes means that

$$\begin{aligned}
\begin{pmatrix} d_{\mathsf{M}^\bullet}^{i-1} \circ \varphi^{i-1} & d_{\mathsf{M}^\bullet}^{i-1} \circ \mathsf{h}^i \end{pmatrix} &= \begin{pmatrix} \varphi^i & \mathsf{h}^{i+1} \end{pmatrix} \begin{pmatrix} d_{\mathsf{L}^\bullet}^{i-1} & \operatorname{id}_{\mathsf{L}^i} \\ 0 & -d_{\mathsf{L}^\bullet}^i \end{pmatrix} \\
&= \begin{pmatrix} \varphi^i \circ d_{\mathsf{L}^\bullet}^{i-1} & \varphi^i - \mathsf{h}^{i+1} \circ d_{\mathsf{L}^\bullet}^i \end{pmatrix}
\end{aligned}$$

is verified for all i. The first coordinates are equal due to the fact that $\varphi^\bullet : \mathsf{L}^\bullet \to \mathsf{M}^\bullet$ is a morphism of complexes, and the second are equal for the h^i define a homotopy between $\varphi^\bullet$ and zero. This completes the proof. □

Since the cohomology functors $\mathsf{H}^i : \mathsf{C}^*(\mathsf{A}) \to \mathsf{A}$ send quasi-isomorphisms to bona fide isomorphisms, they descend to well-defined functors $\mathsf{K}^*(\mathsf{A}) \to \mathsf{A}$ which will still be denoted by H^i. In particular, it makes sense to ask whether a morphism in $\mathsf{K}^*(\mathsf{A})$ is a quasi-isomorphism or not, and we can construct the derived category by inverting the quasi-isomorphisms *in the homotopic category*. This will turn out to be simpler than going straight from $\mathsf{C}^*(\mathsf{A})$.

7.4. The triangulated structure

As hinted before, even if A is an abelian category, the homotopic category $\mathsf{K}^*(\mathsf{A})$ need not be. So, in order to be able to do homological algebra, we need some sort of substitute in $\mathsf{K}^*(\mathsf{A})$ for short exact sequences. It turns out that triangles behave even better in $\mathsf{K}^*(\mathsf{A})$ than they do in $\mathsf{C}^*(\mathsf{A})$.

The shift functor $[n] : \mathsf{C}^*(\mathsf{A}) \to \mathsf{C}^*(\mathsf{A})$ preserves homotopies and so descends to a functor $\mathsf{K}^*(\mathsf{A}) \to \mathsf{K}^*(\mathsf{A})$ denoted by the same symbol. As before, a *triangle* in $\mathsf{K}^*(\mathsf{A})$

$$\begin{tikzcd}
& L^\bullet \arrow[dr, "\overline{\varphi}"] & \\
N^\bullet \arrow[ur, "+1"] & & M^\bullet, \arrow[ll, "\overline{\psi}"]
\end{tikzcd}$$

or even

$$L^\bullet \xrightarrow{\overline{\varphi}} M^\bullet \xrightarrow{\overline{\psi}} N^\bullet \longrightarrow L[1]^\bullet,$$

is a shorthand for a long sequence of the form

$$\cdots \longrightarrow L^\bullet \xrightarrow{\overline{\varphi}} M^\bullet \xrightarrow{\overline{\psi}} N^\bullet \longrightarrow L[1]^\bullet \xrightarrow{\overline{\varphi}[1]} M[1]^\bullet \longrightarrow \cdots,$$

where the morphisms involved are now those of $\mathsf{K}^*(\mathsf{A})$. Similarly, a morphism of triangles consists of morphisms $\overline{\lambda}$, $\overline{\mu}$, and $\overline{\nu}$ in $\mathsf{K}^*(\mathsf{A})$, making the diagram

$$\begin{tikzcd}
L^\bullet \arrow[r, "\overline{\varphi}"] \arrow[d, "\overline{\lambda}"] & M^\bullet \arrow[r, "\overline{\psi}"] \arrow[d, "\overline{\mu}"] & N^\bullet \arrow[r] \arrow[d, "\overline{\nu}"] & L[1]^\bullet \arrow[d, "\overline{\lambda}[1]"] \\
L'^\bullet \arrow[r, "\overline{\varphi}'"] & M'^\bullet \arrow[r, "\overline{\psi}'"] & N'^\bullet \arrow[r] & L'[1]^\bullet
\end{tikzcd}$$

commute. As in $\mathsf{C}^*(\mathsf{A})$, the triangles arising from mapping cones have a prominent role.

> **Definition 7.4.1 — Distinguished triangles.** A triangle in $\mathsf{K}^*(\mathsf{A})$ is said to be *distinguished* if it is isomorphic to some triangle of the form
>
> $$\begin{tikzcd}
& L^\bullet \arrow[dr, "\varphi"] & \\
\mathrm{MC}(\varphi)^\bullet \arrow[ur, "+1"] & & M^\bullet \arrow[ll]
\end{tikzcd}$$
>
> for a morphism $\varphi^\bullet : L^\bullet \to M^\bullet$ in $\mathsf{C}^*(\mathsf{A})$.

As a first sign that triangles work better in $\mathsf{K}^*(\mathsf{A})$ than they do in $\mathsf{C}^*(\mathsf{A})$, we observe that the identity morphism and the zero-object always defines a distinguished triangle in $\mathsf{K}^*(\mathsf{A})$. This means that even though the mapping cone of the identity morphism is not zero, it is homotopy equivalent to zero.

Lemma 7.4.1 Let $M^\bullet$ be a complex in an additive category A. Then the triangle

$$\begin{tikzcd}
& M^\bullet \arrow[dr, "\overline{\mathrm{id}_{M^\bullet}}"] & \\
0 \arrow[ur, "+1"] & & M^\bullet \arrow[ll]
\end{tikzcd}$$

is distinguished.

Proof. Consider the collection of morphisms $h^i : M^i \oplus M^{i+1} \to M^{i-1} \oplus M^i$ given by the matrices

$$\begin{pmatrix} 0 & 0 \\ -\mathrm{id}_{M^i} & 0 \end{pmatrix}.$$

The composition $d^{i-1}_{\mathrm{MC}(\mathrm{id}_{M^\bullet})^\bullet} \circ h^i + h^{i+1} \circ d^i_{\mathrm{MC}(\mathrm{id}_{M^\bullet})^\bullet}$ is represented by the matrix

$$\begin{pmatrix} d^{i-1}_{M^\bullet} & -\mathrm{id}_{M^i} \\ 0 & -d^i_{M^\bullet} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\mathrm{id}_{M^i} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\mathrm{id}_{M^{i+1}} & 0 \end{pmatrix} \begin{pmatrix} d^i_{M^\bullet} & -\mathrm{id}_{M^{i+1}} \\ 0 & -d^{i+1}_{M^\bullet} \end{pmatrix},$$

which is nothing but the identity of $\mathrm{MC}(\mathrm{id}_{M^\bullet})^i$. It follows that the identity morphism on $\mathrm{MC}(\mathrm{id}_{M^\bullet})^\bullet$ is homotopic to zero, and so the natural morphism $0 \to \mathrm{MC}(\mathrm{id}_{M^\bullet})^\bullet$ is an isomorphism in $\mathsf{K}^*(\mathsf{A})$. ☐

Another useful property of distinguished triangles in $\mathsf{K}^*(\mathsf{A})$ is that they remain distinguished upon rotation. We observe that, while the proof is basically only the definition of a homotopy equivalence, there are a lot of things that need to be verified, and we won't shy away.

Lemma 7.4.2 Let A be an additive category. Consider the following triangles in $\mathsf{K}^*(\mathsf{A})$:

$$\begin{tikzcd}
& L^\bullet \arrow[dr, "\overline{\varphi}"] & \\
N^\bullet \arrow[ur, "\overline{\rho}"] \arrow[ur, phantom, "+1"'] & & M^\bullet \arrow[ll, "\overline{\psi}"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& M^\bullet \arrow[dr, "\overline{\psi}"] & \\
L[1]^\bullet \arrow[ur, "-\overline{\varphi[1]^\bullet}"] \arrow[ur, phantom, "+1"'] & & N^\bullet. \arrow[ll, "\overline{\rho}"]
\end{tikzcd}$$

Then one of the triangles is distinguished if and only if the other is.

Proof. We first suppose that the triangle on the left is of the form

$$\begin{tikzcd}
& L^\bullet \arrow[dr, "\overline{\varphi}"] & \\
\mathrm{MC}(\varphi)^\bullet \arrow[ur, "+1"] & & M^\bullet. \arrow[ll, "\overline{\iota}"']
\end{tikzcd}$$

Since the mapping cone of $\iota^\bullet : M^\bullet \to \mathrm{MC}(\varphi)^\bullet$ is naturally endowed with morphisms $\mathrm{MC}(\varphi)^\bullet \to \mathrm{MC}(\iota)^\bullet$ and $\mathrm{MC}(\iota)^\bullet \to M[1]^\bullet$, it suffices to prove the existence of a homotopy equivalence $L[1]^\bullet \to \mathrm{MC}(\iota)^\bullet$ making the diagram

$$\begin{tikzcd}
M^\bullet \arrow[r, "\overline{\iota}"] \arrow[d, equal] & \mathrm{MC}(\varphi)^\bullet \arrow[r] \arrow[d, equal] & L[1]^\bullet \arrow[r, "-\overline{\varphi}[1]^\bullet"] \arrow[d, dashed] & M[1]^\bullet \arrow[d, equal] \\
M^\bullet \arrow[r, "\overline{\iota}"] & \mathrm{MC}(\varphi)^\bullet \arrow[r] & \mathrm{MC}(\iota)^\bullet \arrow[r] & M[1]^\bullet
\end{tikzcd}$$

commute for the triangle on the right to be distinguished. We define a morphism $L^{i+1} \to M^i \oplus L^{i+1} \oplus M^{i+1}$ by

$$\begin{pmatrix} 0 \\ \mathrm{id}_{L^{i+1}} \\ -\varphi^{i+1} \end{pmatrix}.$$

This indeed describes a morphism of complexes $L[1]^\bullet \to \mathrm{MC}(\iota)^\bullet$ since

$$\underbrace{\begin{pmatrix} d^{i-1}_{M^\bullet} & -\varphi^i & -\mathrm{id}_{M^i} \\ 0 & -d^i_{L^\bullet} & 0 \\ 0 & 0 & -d^i_{M^\bullet} \end{pmatrix}}_{d^{i-1}_{\mathrm{MC}(\iota)^\bullet}} \begin{pmatrix} 0 \\ \mathrm{id}_{L^i} \\ -\varphi^i \end{pmatrix} - \begin{pmatrix} 0 \\ \mathrm{id}_{L^{i+1}} \\ -\varphi^{i+1} \end{pmatrix} d^i_{L[1]^\bullet} = 0.$$

In order to prove that this is a homotopy equivalence, we define a morphism $M^i \oplus L^{i+1} \oplus M^{i+1} \to L^{i+1}$, given by projecting onto the middle coordinate. This is also a morphism of complexes $\mathrm{MC}(\iota)^\bullet \to L[1]^\bullet$ for

$$\begin{pmatrix} 0 & \mathrm{id}_{L^{i+1}} & 0 \end{pmatrix} \begin{pmatrix} d^{i-1}_{M^\bullet} & -\varphi^i & -\mathrm{id}_{M^i} \\ 0 & -d^i_{L^\bullet} & 0 \\ 0 & 0 & -d^i_{M^\bullet} \end{pmatrix} - d^{i-1}_{L[1]^\bullet} \begin{pmatrix} 0 & \mathrm{id}_{L^i} & 0 \end{pmatrix} = 0.$$

The composition $L^{i+1} \to M^i \oplus L^{i+1} \oplus M^{i+1} \to L^{i+1}$ is

$$\begin{pmatrix} 0 & \mathrm{id}_{L^{i+1}} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \mathrm{id}_{L^{i+1}} \\ -\varphi^{i+1} \end{pmatrix} = \mathrm{id}_{L^{i+1}},$$

and the composition $M^i \oplus L^{i+1} \oplus M^{i+1} \to L^{i+1} \to M^i \oplus L^{i+1} \oplus M^{i+1}$,

$$\begin{pmatrix} 0 \\ \mathrm{id}_{L^{i+1}} \\ -\varphi^{i+1} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id}_{L^{i+1}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathrm{id}_{L^{i+1}} & 0 \\ 0 & -\varphi^{i+1} & 0 \end{pmatrix},$$

is homotopic to the identity via the homotopy $h^i : M^i \oplus L^{i+1} \oplus M^{i+1} \to M^{i-1} \oplus L^i \oplus M^i$ given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{id}_{M^i} & 0 & 0 \end{pmatrix}.$$

Indeed, the morphism $d^{i-1}_{\mathrm{MC}(\iota)^\bullet} \circ h^i + h^{i+1} \circ d^i_{\mathrm{MC}(\iota)^\bullet}$ is represented by

$$\begin{pmatrix} d^{i-1}_{M^\bullet} & -\varphi^i & -\mathrm{id}_{M^i} \\ 0 & -d^i_{L^\bullet} & 0 \\ 0 & 0 & -d^i_{M^\bullet} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{id}_{M^i} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{id}_{M^{i+1}} & 0 & 0 \end{pmatrix} \begin{pmatrix} d^i_{M^\bullet} & -\varphi^{i+1} & -\mathrm{id}_{M^{i+1}} \\ 0 & -d^{i+1}_{L^\bullet} & 0 \\ 0 & 0 & -d^{i+1}_{M^\bullet} \end{pmatrix}$$
$$= \begin{pmatrix} -\mathrm{id}_{M^i} & 0 & 0 \\ 0 & 0 & 0 \\ -d^i_{M^\bullet} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d^i_{M^\bullet} & -\varphi^{i+1} & -\mathrm{id}_{M^{i+1}} \end{pmatrix} = \begin{pmatrix} -\mathrm{id}_{M^i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\varphi^{i+1} & -\mathrm{id}_{M^{i+1}} \end{pmatrix}.$$

This proves that we have our desired homotopy equivalence. The square on the right commutes (even in $\mathsf{C}^*(\mathsf{A})$) by the very definition of the morphism $L[1]^\bullet \to \mathrm{MC}(\iota)^\bullet$. As for the one in the middle, we observe that the difference between the two morphisms $\mathrm{MC}(\varphi)^i \to \mathrm{MC}(\iota)^i$ is

$$\begin{pmatrix} \mathrm{id}_{M^i} & 0 \\ 0 & 0 \\ 0 & \varphi^{i+1} \end{pmatrix},$$

which is homotopic to zero via the homotopy $h^i : M^i \oplus L^{i+1} \to M^{i-1} \oplus L^i \oplus M^i$ given by

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -\mathrm{id}_{M^i} & 0 \end{pmatrix}.$$

Indeed, the morphism $d^{i-1}_{\mathrm{MC}(\iota)^\bullet} \circ h^i + h^{i+1} \circ d^i_{\mathrm{MC}(\varphi)^\bullet}$ is represented by

$$\begin{pmatrix} d^{i-1}_{M^\bullet} & -\varphi^i & -\mathrm{id}_{M^i} \\ 0 & -d^i_{L^\bullet} & 0 \\ 0 & 0 & -d^i_{M^\bullet} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -\mathrm{id}_{M^i} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -\mathrm{id}_{M^{i+1}} & 0 \end{pmatrix} \begin{pmatrix} d^i_{M^\bullet} & -\varphi^{i+1} \\ 0 & -d^{i+1}_{L^\bullet} \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{id}_{M^i} & 0 \\ 0 & 0 \\ d^i_{M^\bullet} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -d^i_{M^\bullet} & \varphi^{i+1} \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{M^i} & 0 \\ 0 & 0 \\ 0 & \varphi^{i+1} \end{pmatrix},$$

which is equal to the difference calculated above. This completes the proof that the triangle on the right is distinguished if the left one is. For the converse, we observe that by applying what we just proved to the triangle on the right, supposed distinguished, five times, we arrive at the triangle

$$\begin{tikzcd} & L[2]^\bullet \arrow[dr, "\overline{\varphi}[2]"] & \\ N[2]^\bullet \arrow[ur, "\overline{\rho}[2]", "+1"'] & & M[2]^\bullet, \arrow[ll, "\overline{\psi}[2]"] \end{tikzcd}$$

which is distinguished if and only if the triangle on the left is. □

The next result shows that the mapping cone "almost" defines a functor from the category of morphisms (as seen in the proof of corollary 7.2.2) in $K^*(A)$ to $K^*(A)$ itself.

Lemma 7.4.3 Consider the following commutative diagram in $K^*(A)$ whose rows are distinguished triangles:

$$\begin{tikzcd}
L^\bullet \arrow[r, "\overline{\varphi}"] \arrow[d, "\overline{\lambda}"] & M^\bullet \arrow[r, "\overline{\psi}"] \arrow[d, "\overline{\mu}"] & N^\bullet \arrow[r] & L[1]^\bullet \\
L'^\bullet \arrow[r, "\overline{\varphi}'"] & M'^\bullet \arrow[r, "\overline{\psi}'"] & N'^\bullet \arrow[r] & L'[1]^\bullet.
\end{tikzcd}$$

It exists a (not necessarily unique) morphism $\overline{\nu} : N^\bullet \to N'^\bullet$ making the diagram

$$\begin{tikzcd}
L^\bullet \arrow[r, "\overline{\varphi}"] \arrow[d, "\overline{\lambda}"] & M^\bullet \arrow[r, "\overline{\psi}"] \arrow[d, "\overline{\mu}"] & N^\bullet \arrow[r] \arrow[d, dashed, "\overline{\nu}"] & L[1]^\bullet \arrow[d, "\overline{\lambda}{[1]}"] \\
L'^\bullet \arrow[r, "\overline{\varphi}'"] & M'^\bullet \arrow[r, "\overline{\psi}'"] & N'^\bullet \arrow[r] & L'[1]^\bullet
\end{tikzcd}$$

commute. That is, defining a morphism of triangles.

Proof. By composing with some isomorphisms, if necessary, we may assume that our original diagram is of the form

$$\begin{tikzcd}
L^\bullet \arrow[r, "\overline{\varphi}"] \arrow[d, "\overline{\lambda}"] & M^\bullet \arrow[r] \arrow[d, "\overline{\mu}"] & \mathrm{MC}(\varphi)^\bullet \arrow[r] & L[1]^\bullet \\
L'^\bullet \arrow[r, "\overline{\varphi}'"] & M'^\bullet \arrow[r] & \mathrm{MC}(\varphi')^\bullet \arrow[r] & L'[1]^\bullet.
\end{tikzcd}$$

Since the square on the left commutes in $K^*(A)$, let $h^i : L^i \to M'^{i-1}$ be a collection of morphisms satisfying

$$\mu^i \circ \varphi^i - \varphi'^i \circ \lambda^i = d^{i-1}_{M'^\bullet} \circ h^i + h^{i+1} \circ d^i_{L^\bullet}$$

for all i. We then define our desired morphism $\nu^i : M^i \oplus L^{i+1} \to M'^i \oplus L'^{i+1}$ as

$$\begin{pmatrix} \mu^i & -h^{i+1} \\ 0 & \lambda^{i+1} \end{pmatrix}.$$

This is indeed a morphism of complexes since $\nu^i \circ d^{i-1}_{\mathrm{MC}(\varphi)^\bullet} - d^{i-1}_{\mathrm{MC}(\varphi')^\bullet} \circ \nu^{i-1}$ is represented by

$$\begin{pmatrix} \mu^i & -h^{i+1} \\ 0 & \lambda^{i+1} \end{pmatrix} \begin{pmatrix} d^{i-1}_{M^\bullet} & -\varphi^i \\ 0 & -d^i_{L^\bullet} \end{pmatrix} - \begin{pmatrix} d^{i-1}_{M'^\bullet} & -\varphi'^i \\ 0 & -d^i_{L'^\bullet} \end{pmatrix} \begin{pmatrix} \mu^{i-1} & -h^i \\ 0 & \lambda^i \end{pmatrix}$$
$$= \begin{pmatrix} \mu^i \circ d^{i-1}_{M^\bullet} & -\mu^i \circ \varphi^i + h^{i+1} \circ d^i_{L^\bullet} \\ 0 & -\lambda^{i+1} \circ d^i_{L^\bullet} \end{pmatrix} - \begin{pmatrix} d^{i-1}_{M'^\bullet} \circ \mu^{i-1} & -d^{i-1}_{M'^\bullet} \circ h^i - \varphi'^i \circ \lambda^i \\ 0 & -d^i_{L'^\bullet} \circ \lambda^i \end{pmatrix},$$

which is nothing but the zero matrix. This morphism makes the square on the middle commute due to the fact that

$$\begin{pmatrix} \mu^i & -h^{i+1} \\ 0 & \lambda^{i+1} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{M^i} \\ 0 \end{pmatrix} = \begin{pmatrix} \mu^i \\ 0 \end{pmatrix}$$

is equal to the composition of $\mu^i : M^i \to M'^i$ with the natural injection $M'^i \to M'^i \oplus L'^{i+1}$. Similarly, the square on the right commutes as

$$\begin{pmatrix} 0 & \mathrm{id}_{L'^{i+1}} \end{pmatrix} \begin{pmatrix} \mu^i & -h^{i+1} \\ 0 & \lambda^{i+1} \end{pmatrix} = \begin{pmatrix} 0 & \lambda^{i+1} \end{pmatrix}$$

coincides with the composition of the natural projection $M^i \oplus L^{i+1} \to L^{i+1}$ with $\lambda^{i+1} : L^{i+1} \to L'^{i+1}$. □

For an example of the lack of uniqueness, let $\overline{\nu} : M^\bullet \oplus L[1]^\bullet \to M^\bullet \oplus L[1]^\bullet$ be the morphism defined by

$$\begin{pmatrix} \overline{\mathrm{id}_{M^\bullet}} & \overline{\varphi} \\ 0 & \overline{\mathrm{id}_{L[1]^\bullet}} \end{pmatrix},$$

where $\varphi^\bullet$ is any morphism $L[1]^\bullet \to M^\bullet$. This morphism makes the diagram, whose rows are distinguished triangles,

$$\begin{tikzcd}
L^\bullet \arrow[r, "\overline{0}"] \arrow[d, equal] & M^\bullet \arrow[r] \arrow[d, equal] & M^\bullet \oplus L[1]^\bullet \arrow[r] \arrow[d, "\overline{\nu}"] & L[1]^\bullet \arrow[d, equal] \\
L^\bullet \arrow[r, "\overline{0}"] & M^\bullet \arrow[r] & M^\bullet \oplus L[1]^\bullet \arrow[r] & L[1]^\bullet
\end{tikzcd}$$

commute. This lack of uniqueness was the main motivation behind Grothendieck's unpublished 1991 manuscript *Les Dérivateurs*, which has almost 2000 pages.

Somewhat surprisingly, the lemmata that precedes amounts to essentially all the information needed to do homological algebra in $K^*(A)$. In our context, this was first formalized in *Jean-Louis Verdier*'s 1967 thesis as the notion of *triangulated category*, which we now present.

We begin with an additive category K endowed with an additive isomorphism of categories[5] $T : K \to K$ modeling the shift functor in $K^*(A)$. As before, a *triangle* in K is a diagram of the form

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L),$$

---

[5] In some texts the functor T is only required to be an equivalence of categories, instead of a genuine isomorphism. The resulting theory is more complicated as it is 2-categorical.

and a morphism of triangles is simply a commutative diagram

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "\lambda"] & M \arrow[r] \arrow[d, "\mu"] & N \arrow[r] \arrow[d, "\nu"] & T(L) \arrow[d, "T(\lambda)"] \\
L' \arrow[r] & M' \arrow[r] & N' \arrow[r] & T(L').
\end{tikzcd}$$

We also specify a set of *distinguished triangles* that should satisfy the axioms below.[6]

(TR1) (a) Every triangle that is isomorphic to a distinguished triangle is also distinguished.

(b) For every morphism $\varphi : L \to M$ in K there is a distinguished triangle

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L).$$

(c) For every object M the triangle

$$M \xrightarrow{\mathrm{id}_M} M \longrightarrow 0 \longrightarrow T(M)$$

is distinguished.

(TR2) A triangle

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L)$$

is distinguished if and only if the triangle

$$M \xrightarrow{\psi} N \xrightarrow{\rho} T(L) \xrightarrow{-T(\varphi)} T(M)$$

is distinguished.

(TR3) Given a commutative diagram in K

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "\lambda"] & M \arrow[r] \arrow[d, "\mu"] & N \arrow[r] & T(L) \\
L' \arrow[r] & M' \arrow[r] & N' \arrow[r] & T(L'),
\end{tikzcd}$$

whose rows are distinguished triangles, there's a morphism $\nu : N \to N'$ making the diagram

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "\lambda"] & M \arrow[r] \arrow[d, "\mu"] & N \arrow[r] \arrow[d, "\nu"] & T(L) \arrow[d, "T(\lambda)"] \\
L' \arrow[r] & M' \arrow[r] & N' \arrow[r] & T(L')
\end{tikzcd}$$

commute.

---

[6] Actually TR3 and half of TR2 follow from the rest of the axioms. The interested reader can check [26].

(TR4) Suppose we are given these three distinguished triangles:

$$L \xrightarrow{\varphi} M \longrightarrow P \longrightarrow T(L),$$

$$M \xrightarrow{\psi} N \longrightarrow R \longrightarrow T(M),$$

$$L \xrightarrow{\psi\circ\varphi} N \longrightarrow Q \longrightarrow T(L).$$

Then there exists a distinguished triangle

$$P \longrightarrow Q \longrightarrow R \longrightarrow T(P)$$

making the diagram

$$\begin{tikzcd}
L \arrow[rr, bend left, "\psi\circ\varphi"] \arrow[dr, "\varphi"'] & & N \arrow[rr, bend left] \arrow[dr] & & R \arrow[rr, dashed, bend left] \arrow[dr] & & T(P) \\
& M \arrow[ur, "\psi"'] \arrow[dr] & & Q \arrow[ur, dashed] \arrow[dr] & & T(M) \arrow[ur] & \\
& & P \arrow[ur, dashed] \arrow[rr, bend right] & & T(L) \arrow[ur] & &
\end{tikzcd}$$

commute.

The object we're left with is a *triangulated category*.

Definition 7.4.2 — Triangulated category. A *triangulated category* is an additive category K, endowed with an additive automorphism $T : K \to K$ and a set of distinguished triangles satisfying the axioms TR1 to TR4 above.

By now, the reader probably wonders what is the axiom TR4 for. We affirm that it is a sort of palliative solution to the lack of uniqueness in the induced morphism of axiom TR3. Indeed, for every morphism $\varphi : L \to M$, the axiom TR1(b) gives an abstract mapping cone P defining a distinguished triangle

$$L \xrightarrow{\varphi} M \longrightarrow P \longrightarrow T(L).$$

Similarly, this axiom gives an abstract mapping cone R to a morphism $\psi : M \to N$, and an abstract mapping cone Q to the composition $\psi \circ \varphi : L \to N$. Naturally, we wonder how Q relates to P and R. The axiom TR4 affirms simply that they fit into a distinguished triangle

$$P \longrightarrow Q \longrightarrow R \longrightarrow T(P).$$

We leave a study of triangulated categories for the next section and end this one by proving that indeed $K^*(A)$ are triangulated categories. Once again, this isn't difficult at all, but there are a myriad of things that need to be verified.

Theorem 7.4.4 Let A be an additive category. Then the homotopic categories $K^*(A)$ are triangulated.

Proof. After our preliminary work, the only axiom that remains to be proven is the last one. For that we may suppose $P^\bullet = \mathrm{MC}(\varphi)^\bullet$, $R^\bullet = \mathrm{MC}(\psi)^\bullet$ and $Q^\bullet = \mathrm{MC}(\psi \circ \varphi)^\bullet$. We define morphisms $\alpha^i : P^i \to Q^i$ and $\beta^i : Q^i \to R^i$ as

$$\begin{pmatrix} \psi^i & 0 \\ 0 & \mathrm{id}_{L^{i+1}} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & \varphi^{i+1} \end{pmatrix},$$

respectively. The α^i define a morphism of complexes since $\alpha^i \circ d^{i-1}_{P^\bullet} - d^{i-1}_{Q^\bullet} \circ \alpha^{i-1}$ is represented by

$$\begin{aligned} &\begin{pmatrix} \psi^i & 0 \\ 0 & \mathrm{id}_{L^{i+1}} \end{pmatrix} \begin{pmatrix} d^{i-1}_{M^\bullet} & -\varphi^i \\ 0 & -d^i_{L^\bullet} \end{pmatrix} - \begin{pmatrix} d^{i-1}_{N^\bullet} & -\psi^i \circ \varphi^i \\ 0 & -d^i_{L^\bullet} \end{pmatrix} \begin{pmatrix} \psi^{i-1} & 0 \\ 0 & \mathrm{id}_{L^i} \end{pmatrix} \\ &= \begin{pmatrix} \psi^i \circ d^{i-1}_{M^\bullet} & -\psi^i \circ \varphi^i \\ 0 & -d^i_{L^\bullet} \end{pmatrix} - \begin{pmatrix} d^{i-1}_{N^\bullet} \circ \psi^{i-1} & -\psi^i \circ \varphi^i \\ 0 & -d^i_{L^\bullet} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Similarly, the β^i define a morphism of complexes since $\beta^i \circ d^{i-1}_{Q^\bullet} - d^{i-1}_{R^\bullet} \circ \beta^{i-1}$ is represented by

$$\begin{aligned} &\begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & \varphi^{i+1} \end{pmatrix} \begin{pmatrix} d^{i-1}_{N^\bullet} & -\psi^i \circ \varphi^i \\ 0 & -d^i_{L^\bullet} \end{pmatrix} - \begin{pmatrix} d^{i-1}_{N^\bullet} & -\psi^i \\ 0 & -d^i_{M^\bullet} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^{i-1}} & 0 \\ 0 & \varphi^i \end{pmatrix} \\ &= \begin{pmatrix} d^{i-1}_{N^\bullet} & -\psi^i \circ \varphi^i \\ 0 & -\varphi^{i+1} \circ d^i_{L^\bullet} \end{pmatrix} - \begin{pmatrix} d^{i-1}_{N^\bullet} & -\psi^i \circ \varphi^i \\ 0 & -d^i_{M^\bullet} \circ \varphi^i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

We also define a morphism $\gamma^\bullet : R^\bullet \to P[1]^\bullet$ as the composition $R^\bullet \to M[1]^\bullet \to P[1]^\bullet$. We must now verify that

$$P^\bullet \xrightarrow{\alpha^\bullet} Q^\bullet \xrightarrow{\beta^\bullet} R^\bullet \xrightarrow{\gamma^\bullet} P[1]^\bullet.$$

is a distinguished triangle and that those morphisms fit into the commutative diagram of the axiom TR4. For clarity, we number the relevant parts of this diagram and rewrite it here.

$$\begin{tikzcd}
L^\bullet \arrow[rr, bend left, "\psi^\bullet \circ \varphi^\bullet"] \arrow[dr, "\varphi^\bullet"'] & (1) & N^\bullet \arrow[rr, bend left] \arrow[dr] & (3) & R^\bullet \arrow[rr, bend left, "\gamma^\bullet"] \arrow[dr] & (6) & P[1]^\bullet \\
& M^\bullet \arrow[ur, "\psi^\bullet"'] \arrow[dr] & (2) & Q^\bullet \arrow[ur, "\beta^\bullet"'] \arrow[dr] & (5) & M[1]^\bullet \arrow[ur] & \\
& & P^\bullet \arrow[ur, "\alpha^\bullet"'] \arrow[rr, bend right] & (4) & L[1]^\bullet \arrow[ur] & &
\end{tikzcd}$$

The triangles (1) and (6) commute by the very definition of the morphisms involved. The square (2) commutes since

$$\begin{pmatrix} \mathrm{id}_{N^i} \\ 0 \end{pmatrix} \psi^i = \begin{pmatrix} \psi^i & 0 \\ 0 & \mathrm{id}_{L^{i+1}} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{M^i} \\ 0 \end{pmatrix}.$$

The triangle (3) commutes since

$$\begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & \varphi^{i+1} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^i} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{N^i} \\ 0 \end{pmatrix}.$$

The triangle (4) commutes since

$$\begin{pmatrix} 0 & \mathrm{id}_{L^{i+1}} \end{pmatrix} \begin{pmatrix} \psi^i & 0 \\ 0 & \mathrm{id}_{L^{i+1}} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id}_{L^{i+1}} \end{pmatrix}.$$

Finally, the square (5) commutes since

$$\begin{pmatrix} 0 & \mathrm{id}_{M^{i+1}} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & \varphi^{i+1} \end{pmatrix} = \varphi^{i+1} \begin{pmatrix} 0 & \mathrm{id}_{L^{i+1}} \end{pmatrix}.$$

In order to show that the triangle we defined is distinguished we'll define morphisms $\rho^\bullet : \mathrm{MC}(\alpha)^\bullet \to R^\bullet$ and $\sigma^\bullet : R^\bullet \to \mathrm{MC}(\alpha)^\bullet$ determining an isomorphism of triangles

$$\begin{tikzcd}
P^\bullet \arrow[r, "\alpha^\bullet"] \arrow[d, equal] & Q^\bullet \arrow[r, "\beta^\bullet"] \arrow[d, equal] & R^\bullet \arrow[r, "\gamma^\bullet"] \arrow[d, shift left, "\sigma^\bullet"] & P[1]^\bullet \arrow[d, equal] \\
P^\bullet \arrow[r, "\alpha^\bullet"] & Q^\bullet \arrow[r] & \mathrm{MC}(\alpha)^\bullet \arrow[r] \arrow[u, shift left, "\rho^\bullet"] & P[1]^\bullet.
\end{tikzcd}$$

The morphisms $\rho^i : N^i \oplus L^{i+1} \oplus M^{i+1} \oplus L^{i+2} \to N^i \oplus M^{i+1}$ and $\sigma^i : N^i \oplus M^{i+1} \to N^i \oplus L^{i+1} \oplus M^{i+1} \oplus L^{i+2}$ are defined as

$$\begin{pmatrix} \mathrm{id}_{N^i} & 0 & 0 & 0 \\ 0 & \varphi^{i+1} & \mathrm{id}_{M^{i+1}} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{M^{i+1}} \\ 0 & 0 \end{pmatrix},$$

respectively. They define morphisms of complexes since $\rho^{i+1} \circ d^i_{\mathrm{MC}(\alpha)^\bullet} - d^i_{R^\bullet} \circ \rho^i$ is represented by

$$\begin{pmatrix} \mathrm{id}_{N^{i+1}} & 0 & 0 & 0 \\ 0 & \varphi^{i+2} & \mathrm{id}_{M^{i+2}} & 0 \end{pmatrix} \begin{pmatrix} d^i_{N^\bullet} & -\psi^{i+1} \circ \varphi^{i+1} & -\psi^{i+1} & 0 \\ 0 & -d^{i+1}_{L^\bullet} & 0 & -\mathrm{id}_{L^{i+2}} \\ 0 & 0 & -d^{i+1}_{M^\bullet} & \varphi^{i+2} \\ 0 & 0 & 0 & d^{i+2}_{L^\bullet} \end{pmatrix}$$

$$- \begin{pmatrix} d^i_{N^\bullet} & -\psi^{i+1} \\ 0 & -d^{i+1}_{M^\bullet} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^i} & 0 & 0 & 0 \\ 0 & \varphi^{i+1} & \mathrm{id}_{M^{i+1}} & 0 \end{pmatrix} =$$

$$\begin{pmatrix} d^i_{N^\bullet} & -\psi^{i+1} \circ \varphi^{i+1} & -\psi^{i+1} & 0 \\ 0 & -\varphi^{i+2} \circ d^{i+1}_{L^\bullet} & -d^{i+1}_{M^\bullet} & 0 \end{pmatrix} - \begin{pmatrix} d^i_{N^\bullet} & -\psi^{i+1} \circ \varphi^{i+1} & -\psi^{i+1} & 0 \\ 0 & -d^{i+1}_{M^\bullet} \circ \varphi^{i+1} & -d^{i+1}_{M^\bullet} & 0 \end{pmatrix}$$

and $\sigma^{i+1} \circ d^i_{R^\bullet} - d^i_{MC(\alpha)^\bullet} \circ \sigma^i$ is represented by

$$\begin{pmatrix} \mathrm{id}_{N^{i+1}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{M^{i+2}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d^i_{N^\bullet} & -\psi^{i+1} \\ 0 & -d^{i+1}_{M^\bullet} \end{pmatrix} -$$

$$\begin{pmatrix} d^i_{N^\bullet} & -\psi^{i+1} \circ \varphi^{i+1} & -\psi^{i+1} & 0 \\ 0 & -d^{i+1}_{L^\bullet} & 0 & -\mathrm{id}_{L^{i+2}} \\ 0 & 0 & -d^{i+1}_{M^\bullet} & \varphi^{i+2} \\ 0 & 0 & 0 & d^{i+2}_{L^\bullet} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{M^{i+1}} \\ 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} d^i_{N^\bullet} & -\psi^{i+1} \\ 0 & 0 \\ 0 & -d^{i+1}_{M^\bullet} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} d^i_{N^\bullet} & -\psi^{i+1} \\ 0 & 0 \\ 0 & -d^{i+1}_{M^\bullet} \\ 0 & 0 \end{pmatrix}.$$

In both cases the result is the zero matrix. We now affirm that the morphisms $\rho^\bullet$ and $\sigma^\bullet$ define a homotopy equivalence. The composition $\rho^\bullet \circ \sigma^\bullet$ is equal to the identity morphism on $R^\bullet$ as

$$\begin{pmatrix} \mathrm{id}_{N^i} & 0 & 0 & 0 \\ 0 & \varphi^{i+1} & \mathrm{id}_{M^{i+1}} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{M^{i+1}} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & \mathrm{id}_{M^{i+1}} \end{pmatrix},$$

and the morphism $\sigma^\bullet \circ \rho^\bullet - \mathrm{id}_{MC(\alpha)^\bullet}$, represented by

$$\begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{M^{i+1}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^i} & 0 & 0 & 0 \\ 0 & \varphi^{i+1} & \mathrm{id}_{M^{i+1}} & 0 \end{pmatrix} - \begin{pmatrix} \mathrm{id}_{N^i} & 0 & 0 & 0 \\ 0 & \mathrm{id}_{L^{i+1}} & 0 & 0 \\ 0 & 0 & \mathrm{id}_{M^{i+1}} & 0 \\ 0 & 0 & 0 & \mathrm{id}_{L^{i+2}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\mathrm{id}_{L^{i+1}} & 0 & 0 \\ 0 & \varphi^{i+1} & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{id}_{L^{i+2}} \end{pmatrix},$$

is homotopic to zero via the homotopy $h^i : N^i \oplus L^{i+1} \oplus M^{i+1} \oplus L^{i+2} \to N^{i-1} \oplus L^i \oplus M^i \oplus L^{i+1}$ given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathrm{id}_{L^{i+1}} & 0 & 0 \end{pmatrix}.$$

Indeed, the composition $d^{i-1}_{\mathrm{MC}(\alpha)^\bullet} \circ h^i + h^{i+1} \circ d^i_{\mathrm{MC}(\alpha)^\bullet}$ is represented by the matrix

$$\begin{pmatrix} d^{i-1}_{N^\bullet} & -\psi^i \circ \varphi^i & -\psi^i & 0 \\ 0 & -d^i_{L^\bullet} & 0 & -\mathrm{id}_{L^{i+1}} \\ 0 & 0 & -d^i_{M^\bullet} & \varphi^{i+1} \\ 0 & 0 & 0 & d^{i+1}_{L^\bullet} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathrm{id}_{L^{i+1}} & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathrm{id}_{L^{i+2}} & 0 & 0 \end{pmatrix} \begin{pmatrix} d^i_{N^\bullet} & -\psi^{i+1} \circ \varphi^{i+1} & -\psi^{i+1} & 0 \\ 0 & -d^{i+1}_{L^\bullet} & 0 & -\mathrm{id}_{L^{i+2}} \\ 0 & 0 & -d^{i+1}_{M^\bullet} & \varphi^{i+2} \\ 0 & 0 & 0 & d^{i+2}_{L^\bullet} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\mathrm{id}_{L^{i+1}} & 0 & 0 \\ 0 & \varphi^{i+1} & 0 & 0 \\ 0 & d^{i+1}_{L^\bullet} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -d^{i+1}_{L^\bullet} & 0 & -\mathrm{id}_{L^{i+2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\mathrm{id}_{L^{i+1}} & 0 & 0 \\ 0 & \varphi^{i+1} & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{id}_{L^{i+2}} \end{pmatrix},$$

which coincides with the one representing $\sigma^\bullet \circ \rho^\bullet - \mathrm{id}_{\mathrm{MC}(\alpha)^\bullet}$. It remains only to show that $\rho^\bullet$ defines a morphism of triangles. That is, that the associated diagram commutes. The composition of the natural injection $Q^\bullet \to \mathrm{MC}(\alpha)^\bullet$ with $\rho^\bullet$ is given by

$$\begin{pmatrix} \mathrm{id}_{N^i} & 0 & 0 & 0 \\ 0 & \varphi^{i+1} & \mathrm{id}_{M^{i+1}} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & \mathrm{id}_{L^{i+1}} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & \varphi^{i+1} \end{pmatrix},$$

which is nothing but β^i. Since $\sigma^\bullet$ is the inverse of $\rho^\bullet$ in $\mathsf{K}^*(\mathsf{A})$, it suffices to show that the composition of $\sigma^\bullet$ with the natural projection $\mathrm{MC}(\alpha)^\bullet \to \mathsf{P}[1]^\bullet$ is $\gamma^\bullet$. This holds since

$$\begin{pmatrix} 0 & 0 & \mathrm{id}_{M^{i+1}} & 0 \\ 0 & 0 & 0 & \mathrm{id}_{L^{i+2}} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{M^{i+1}} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id}_{M^{i+1}} \\ 0 & 0 \end{pmatrix},$$

which is equal to γ^i. The proof is at long last over. □

7.5. Triangulated categories

After proving that the homotopic categories are triangulated in the last section, we now delve into the world of triangulated categories. The formal results we'll obtain will not only be valid and useful for the homotopic categories, but also for the derived category in the next chapter.

We begin by understanding what are the natural functors between triangulated categories, preserving their extra structure.

Definition 7.5.1 — Triangulated functor. Let (K, T) and (K', T') be triangulated categories. A *triangulated functor* from K to K' is an additive functor $F : K \to K'$, together with a natural isomorphism $\tau : F \circ T \to T' \circ F$, such that for every distinguished triangle

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L),$$

in K, the triangle

$$F(L) \xrightarrow{F(\varphi)} F(M) \xrightarrow{F(\psi)} F(N) \xrightarrow{\tau_L \circ F(\rho)} T'(F(L))$$

is distinguished in K'.

Whenever we say that two triangulated categories are equivalent, it is to be understood that the functor defining the equivalence of categories is triangulated. Also, if $F : A \to B$ is an additive functor between additive categories, then the induced functor $F : K(A) \to K(B)$ is triangulated. Indeed, an additive functor commutes both with mapping cones and with the shift functor.

Recall that, given a morphism $\varphi : L \to M$ in a triangulated category K, the axiom TR1 gives a distinguished triangle

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L).$$

As in the homotopy category, we say that N is the cone of φ. We'll soon see that it is unique up to isomorphism. For now, this will allow us to define the natural notion of a triangulated (full) subcategory.

Definition 7.5.2 — Triangulated subcategory. Let (K, T) be a triangulated category. A *triangulated subcategory* of K is a full additive subcategory $C \subset K$, which is closed under cones and under the action of T. That is, the cone of a morphism in C is in C and $T(L) \in C$ whenever $L \in C$.

Surely, if C is a triangulated subcategory of (K, T), the restriction of T to C and the collection of distinguished triangles in K whose objects are in C gives a structure of triangulated category to C. Moreover, the inclusion functor $C \to K$ is triangulated.

As we observed before, the long sequence induced by the mapping cone of a morphism is a complex in the homotopic category (but not in the category of complexes). This generalizes to triangulated categories. In particular, it follows that the category of complexes cannot be triangulated (with respect to the usual shift functor and mapping cones).

Proposition 7.5.1 Let K be a triangulated category and

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L)$$

be a distinguished triangle. Then the compositions $\psi \circ \varphi$, $\rho \circ \psi$ and $T(\varphi) \circ \rho$ are zero.

Proof. The axiom TR1 says that the cone of the identity morphism id_L is zero. So, by the axiom TR3, we have a dashed arrow making the diagram

$$\begin{tikzcd}
L \arrow[r, "\mathrm{id}_L"] \arrow[d, "\mathrm{id}_L"] & L \arrow[r] \arrow[d, "\varphi"] & 0 \arrow[r] \arrow[d, dashed] & T(L) \arrow[d, "T(\mathrm{id}_L)"] \\
L \arrow[r, "\varphi"'] & M \arrow[r, "\psi"'] & N \arrow[r, "\rho"'] & T(L)
\end{tikzcd}$$

commute. This proves that $\psi \circ \varphi = 0$. Now, the axiom TR2 says that the triangles

$$M \xrightarrow{\psi} N \xrightarrow{\rho} T(L) \xrightarrow{-T(\varphi)} T(M)$$

and

$$N \xrightarrow{\rho} T(L) \xrightarrow{-T(\varphi)} T(M) \xrightarrow{-T(\psi)} T(N)$$

are distinguished. So, by applying what we just proved to these triangles, we obtain $\rho \circ \psi = 0$ and $T(\varphi) \circ \rho = 0$. ☐

Duality arguments abound in category theory, as we clearly saw in the chapter about abelian categories. In order to use such arguments in our present context, we need to know that the opposite of a triangulated category is also triangulated.

Proposition 7.5.2 Let K be a triangulated category and let $D : K \to K^{op}$ be the contravariant functor sending each object to itself and inverting all the arrows. We define an additive isomorphism of categories $T^{op} : K^{op} \to K^{op}$ as $D \circ T^{-1} \circ D^{-1}$ and we say that a triangle of K^{op} is distinguished if it is of the form

$$N \xrightarrow{D(\psi)} M \xrightarrow{D(\varphi)} L \xrightarrow{D(-T^{-1}(\rho))} T^{op}(N),$$

where

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L)$$

is a distinguished triangle in K. Then K^{op} is a triangulated category.

Considering that the proof of this result amounts only to a formal verification of the axioms, and that it won't add new useful ideas or techniques to the arsenal of the reader, we won't write it here. In case the reader wants to see it anyway, a full proof is available online on [27].

We also remark that the collection of distinguished triangles in K^{op} is motivated by the fact that the axiom TR2 says that a triangle

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L)$$

is distinguished if and only if its "reverse rotation"

$$T^{-1}(N) \xrightarrow{-T^{-1}(\rho)} L \xrightarrow{\varphi} M \xrightarrow{\psi} N$$

is. By inverting the triangle above, we obtain a distinguished triangle in the opposite category.

In the lingo of triangulated categories, the content of the proposition 7.2.3 is that the functor $H^\bullet : K^*(A) \to C^*(A)$ sends distinguished triangles to exact triangles. We axiomatize this behavior.

Definition 7.5.3 Let K be a triangulated category and A be an abelian category. We say that an additive functor $H : K \to A$ is *cohomological* is, for every distinguished triangle

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L),$$

the sequence

$$H(L) \xrightarrow{H(\varphi)} H(M) \xrightarrow{H(\psi)} H(N)$$

is exact in A.

Since we can use the axiom TR2 to rotate our distinguished triangles, we obtain a (infinite) sequence of distinguished triangles

$$\begin{tikzcd}
L \arrow[r, "\varphi"] & M \arrow[r, "\psi"] & N \arrow[r, "\rho"] & T(L) & & & \\
& M \arrow[r, "\psi"] & N \arrow[r, "\rho"] & T(L) \arrow[r, "-T(\varphi)"] & T(M) & & \\
& & N \arrow[r, "\rho"] & T(L) \arrow[r, "-T(\varphi)"] & T(M) \arrow[r, "-T(\psi)"] & T(N) & \\
& & & T(L) \arrow[r, "-T(\varphi)"] & T(M) \arrow[r, "-T(\psi)"] & T(N) \arrow[r, "-T(\rho)"] & T^2(L).
\end{tikzcd}$$

Moreover, we can make sure that in each triangle the first two morphisms don't have a minus sign. For example, the commutative diagram

$$\begin{tikzcd}
N \arrow[r, "\rho"] \arrow[d, "\mathrm{id}_N"] & T(L) \arrow[r, "-T(\varphi)"] \arrow[d, "\mathrm{id}_{T(L)}"] & T(M) \arrow[r, "-T(\psi)"] \arrow[d, "-\mathrm{id}_{T(M)}"] & T(N) \arrow[d, "T(\mathrm{id}_N)"] \\
N \arrow[r, "\rho"'] & T(L) \arrow[r, "T(\varphi)"'] & T(M) \arrow[r, "T(\psi)"'] & T(N)
\end{tikzcd}$$

shows that the third triangle is isomorphic to a triangle with the same objects but whose first two morphisms "don't have a minus sign". By applying a cohomological functor H, we obtain a long exact sequence associated with our original distinguished triangle

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & H(L) \arrow[r, "H(\varphi)"] & H(M) \arrow[r, "H(\psi)"] & H(N) \arrow[dll, rounded corners, "H(\rho)"] & \\
& H(T(L)) \arrow[r, "H(T(\varphi))"] & H(T(M)) \arrow[r, "H(T(\psi))"] & H(T(N)) \arrow[r, dashed] & \cdots .
\end{tikzcd}$$

As we just hinted, the functor $H^i : K^*(A) \to A$, for all i, is cohomological. But it isn't by all means the only one. The proposition below gives two other cohomological functors which will allow the use of the Yoneda lemma to study triangulated categories.

Proposition 7.5.3 Let K be a triangulated category. Then, the functors

$$\mathrm{Hom}_K(P, -) : K \to \mathsf{Ab} \qquad \text{and} \qquad \mathrm{Hom}_K(-, P) : K^{op} \to \mathsf{Ab},$$

for every object P of K, are cohomological.

Proof. We'll only prove the covariant statement, for $\mathrm{Hom}_K(-, P) = \mathrm{Hom}_{K^{op}}(P, -)$ implies the other. Consider the following distinguished triangle in K:

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L).$$

In order to show that $\mathrm{Hom}_K(P, -)$ is cohomological, we need to prove that the induced sequence

$$\mathrm{Hom}_K(P, L) \longrightarrow \mathrm{Hom}_K(P, M) \longrightarrow \mathrm{Hom}_K(P, N)$$

is exact. Since $\psi \circ \varphi = 0$, due to the proposition 7.5.1, it suffices to show that for every $\alpha : P \to M$ such that $\psi \circ \alpha = 0$, there exists a morphism $\beta : P \to L$ such that $\alpha = \varphi \circ \beta$.

Now, consider the diagram below:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d] & T(P) \arrow[r, "-\mathrm{id}_{T(P)}"] & T(P) \arrow[d, "T(\alpha)"] \\
M \arrow[r, "\psi"'] & N \arrow[r, "\rho"'] & T(L) \arrow[r, "-T(\varphi)"'] & T(M).
\end{tikzcd}$$

Its lower row is a distinguished triangle, since it is nothing but our original triangle rotated with help of the axiom TR2. The upper row is also a distinguished triangle by the axioms TR1 and TR2. The axiom TR3 gives a morphism $T(P) \to T(L)$ making it commute which, since T is fully-faithful, is of the form $T(\beta)$ for exactly one $\beta : P \to L$. Since the square on the right commutes, $T(\alpha) = T(\varphi) \circ T(\beta) = T(\varphi \circ \beta)$. This implies that $\alpha = \varphi \circ \beta$ and finishes the proof. □

We now prove a couple of interesting corollaries. The one below is a form of the five lemma for triangulated categories.

Corollary 7.5.4 Consider the following morphism of distinguished triangles in a triangulated category K:

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "\lambda"] & M \arrow[r] \arrow[d, "\mu"] & N \arrow[r] \arrow[d, "\nu"] & T(L) \arrow[d, "T(\lambda)"] \\
L' \arrow[r] & M' \arrow[r] & N' \arrow[r] & T(L').
\end{tikzcd}$$

If two of the vertical morphisms λ, μ and ν are isomorphism, then so is the third.

Proof. Without loss of generality, we may suppose that λ and μ are isomorphisms. Let P be an object of K and $H := \mathrm{Hom}_K(P, -)$. By applying H, we get a commutative diagram of abelian groups

$$\begin{tikzcd}
H(L) \arrow[r] \arrow[d, "H(\lambda)"] & H(M) \arrow[r] \arrow[d, "H(\mu)"] & H(N) \arrow[r] \arrow[d, "H(\nu)"] & H(T(L)) \arrow[r] \arrow[d, "H(T(\lambda))"] & H(T(M)) \arrow[d, "H(T(\mu))"] \\
H(L') \arrow[r] & H(M') \arrow[r] & H(N') \arrow[r] & H(T(L')) \arrow[r] & H(T(M'))
\end{tikzcd}$$

which, due to the proposition above and its preceding discussion, has exact rows. The five lemma (proposition 6.6.2) then implies that $H(\nu)$ is an isomorphism of abelian groups and, in particular, of sets. Since this holds for every P, the Yoneda lemma implies that ν is an isomorphism. □

If the reader prefers to avoid the Yoneda lemma, we can arrive at the same conclusion in a direct way. Since

$$\begin{aligned} H(\nu) : \mathrm{Hom}_K(P, N) &\to \mathrm{Hom}_K(P, N') \\ \alpha &\mapsto \nu \circ \alpha \end{aligned}$$

is an isomorphism for all P, we can take $P = N'$ and conclude that there is some $\alpha : N' \to N$ such that $\nu \circ \alpha = \mathrm{id}_{N'}$. That is, ν has a right inverse. The same argument with the contravariant hom functor gives a left inverse to ν, proving that it is an isomorphism.

One corollary of the result above is that the cone of a morphism is unique up to isomorphism.

Corollary 7.5.5 Let K be a triangulated category and $\varphi : L \to M$ be a morphism in K. Then the cone N of φ is unique up to isomorphism.

Proof. Suppose that N' is another cone of φ. The axiom TR3 gives a morphism $\nu : N \to N'$ making the diagram

$$\begin{tikzcd}
L \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}_L"] & M \arrow[r, "\psi"] \arrow[d, "\mathrm{id}_M"] & N \arrow[r, "\rho"] \arrow[d, dashed, "\nu"] & T(L) \arrow[d, "T(\mathrm{id}_L)"] \\
L \arrow[r, "\varphi"'] & M \arrow[r] & N' \arrow[r] & T(L)
\end{tikzcd}$$

commute. The preceding corollary then implies that ν is an isomorphism. □

We observe that the non-uniqueness in the induced morphism of the axiom TR3 implies that the isomorphism ν above is not necessarily unique. In particular, the cone of φ is not functorial in φ. As discussed right after the definition 7.4.2, this is the *raison d'être* of the axiom TR4.

Corollary 7.5.6 Let K be a triangulated category and

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L)$$

be a distinguished triangle. Then φ is an isomorphism if and only if N is isomorphic to the zero object.

Proof. Suppose that φ is an isomorphism, and let $\varphi^{-1} : M \to L$ be its inverse. Since two of the vertical morphisms in the diagram (induced by the axiom TR3)

$$\begin{tikzcd}
L \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}_L"] & M \arrow[r, "\psi"] \arrow[d, "\varphi^{-1}"] & N \arrow[r, "\rho"] \arrow[d] & T(L) \arrow[d, "T(\mathrm{id}_L)"] \\
L \arrow[r, "\mathrm{id}_L"'] & L \arrow[r] & 0 \arrow[r] & T(L)
\end{tikzcd}$$

are isomorphisms, so is $N \to 0$. Conversely, suppose that N is isomorphic to zero. By rotating backwards our distinguished triangle, we obtain the diagram below

$$\begin{tikzcd}
T^{-1}(N) \arrow[r, "-T^{-1}(\rho)"] \arrow[d] & L \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}_L"] & M \arrow[r, "\psi"] \arrow[d, dashed] & N \arrow[d] \\
0 \arrow[r] & L \arrow[r, "\mathrm{id}_L"'] & L \arrow[r] & 0,
\end{tikzcd}$$

whose rows are distinguished triangles. The dashed morphism, induced by the axiom TR3, is an isomorphism by the proposition above. The commutativity of the diagram then implies that so is φ. □

We're now in position to explain why the homotopic category (and the derived category) are usually not abelian. The reader may remember the next result as saying that "in a triangulated category, monomorphisms and epimorphisms split".

Proposition 7.5.7 Let K be a triangulated category. If $\varphi : L \to M$ is a monomorphism, then there exists $\rho : M \to L$ such that $\rho \circ \varphi = \mathrm{id}_L$. Dually, if $\psi : M \to N$ is an epimorphism, then there exists $\sigma : N \to M$ such that $\psi \circ \sigma = \mathrm{id}_N$.

Proof. Suppose that $\varphi : L \to M$ is a monomorphism. By the axioms TR1(b) and TR2, there exists a distinguished triangle of the form

$$T^{-1}(N) \longrightarrow L \xrightarrow{\varphi} M \longrightarrow N.$$

Due to the proposition 7.5.1, the composition $T^{-1}(N) \to L \to M$ is zero. But, since φ is a monomorphism, it follows that $T^{-1}(N) \to L$ is also zero. As $\mathrm{Hom}_{\mathsf{K}}(-, L)$ is cohomological, we get an exact sequence

$$\mathrm{Hom}_{\mathsf{K}}(M, L) \longrightarrow \mathrm{Hom}_{\mathsf{K}}(L, L) \xrightarrow{0} \mathrm{Hom}_{\mathsf{K}}(T^{-1}(N), L),$$

which implies that $\mathrm{Hom}_{\mathsf{K}}(M, L) \to \mathrm{Hom}_{\mathsf{K}}(L, L)$ is surjective. In particular, there exists $\rho : M \to L$ such that $\rho \circ \varphi = \mathrm{id}_L$. The other statement follows by duality. ☐

The proposition above says, in particular, that if $\mathsf{K}(\mathsf{A})$ is abelian, then every exact sequence splits, due to the splitting lemma (theorem 6.4.1). In fact, this also implies that every exact sequence in A splits.

Corollary 7.5.8 Let A be an abelian category and suppose that $\mathsf{K}(\mathsf{A})$ is abelian. Then every exact sequence in A splits.

Proof. Let $\varphi : A \to B$ be a monomorphism in A and see this morphism in $\mathsf{K}(\mathsf{A})$. Since we suppose that the homotopy category is abelian, we can factor φ as $\operatorname{im}\varphi \circ \operatorname{coim}\varphi$ in $\mathsf{K}(\mathsf{A})$. As $\operatorname{im}\varphi$ is a monomorphism and $\operatorname{coim}\varphi$ is an epimorphism, the preceding proposition gives morphisms ρ and σ such that $\rho \circ \operatorname{im}\varphi = \mathrm{id}$ and $(\operatorname{coim}\varphi) \circ \sigma = \mathrm{id}$.

Let $\alpha = \sigma \circ \rho : B \to A$. Observe that α is in A, since A embeds fully faithfully in $\mathsf{K}(\mathsf{A})$, and that

$$\begin{aligned}
\varphi \circ \alpha \circ \varphi &= (\operatorname{im}\varphi \circ \operatorname{coim}\varphi) \circ (\sigma \circ \rho) \circ (\operatorname{im}\varphi \circ \operatorname{coim}\varphi) \\
&= \operatorname{im}\varphi \circ \underbrace{(\operatorname{coim}\varphi \circ \sigma)}_{\mathrm{id}} \circ \underbrace{(\rho \circ \operatorname{im}\varphi)}_{\mathrm{id}} \circ \operatorname{coim}\varphi \\
&= \operatorname{im}\varphi \circ \operatorname{coim}\varphi = \varphi.
\end{aligned}$$

But φ is a monomorphism in A and so $\alpha \circ \varphi = \mathrm{id}_A$. The result then follows by the splitting lemma. ☐

This result was proved only for the homotopy category, since we are yet to see the formal definition of the derived category. But the reader will realize in due time that the same argument also proves that if $\mathsf{D}(\mathsf{A})$ is abelian, then every exact sequence in A splits.[7]

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[7] An abelian category where every exact sequence splits is said to be *semisimple*.

8. The derived category

As hinted in the previous chapter, our goal is to eventually study the derived category $\mathsf{D}(\mathsf{A})$, which will be constructed from the homotopy category $\mathsf{K}(\mathsf{A})$ by inverting all the quasi-isomorphisms. Unlike homotopy equivalences, quasi-isomorphisms does not define an equivalence relation, precluding us from defining $\mathsf{D}(\mathsf{A})$ by quotienting the hom-sets as we did in the homotopy category. We shall need more powerful machinery; the localization of categories.

8.1. Localization of categories

The main idea of this section is very simple: given a category C and a collection of morphisms S in C, we will define a category $S^{-1}\mathsf{C}$, along with a functor $Q : \mathsf{C} \to S^{-1}\mathsf{C}$ sending all elements of S to isomorphisms in $S^{-1}\mathsf{C}$, and such that Q is universal with this property. In other words, we'll establish the following theorem.

Theorem 8.1.1 Let C be a category and S a collection of morphisms in C. Then there exists a category $S^{-1}\mathsf{C}$ and a functor $Q : \mathsf{C} \to S^{-1}\mathsf{C}$ satisfying the following properties:

(a) for every $s \in S$, $Q(s)$ is an isomorphism in $S^{-1}\mathsf{C}$;

(b) if $F : \mathsf{C} \to \mathsf{D}$ is a functor such that $F(s)$ is an isomorphism for every $s \in S$, there exists a unique functor $S^{-1}\mathsf{C} \to \mathsf{D}$ making the diagram

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "F"] \arrow[d, "Q"'] & \mathsf{D} \\
S^{-1}\mathsf{C} \arrow[ur, dashed] &
\end{tikzcd}$$

commute.

Moreover, $S^{-1}\mathsf{C}$ is unique up to a unique isomorphism.

We say that $S^{-1}\mathsf{C}$ is the *localization* of C with respect to S. Before going on to the proof of this result, it is useful to understand how the explicit construction of $S^{-1}\mathsf{C}$ works. Let's begin by posing that $S^{-1}\mathsf{C}$ should have the same objects as C. As for the morphisms, if M and N are objects of C, we define a *path* from M to N to be a diagram of the form

$$M \xrightarrow{f_0} L_1 \xleftarrow{s_1} L_2 \xleftarrow{s_2} \cdots \xrightarrow{f_{n-1}} L_n \xleftarrow{s_n} N,$$

where $L_1, \dots, L_n$ are objects of C, the arrows f_i to the right are morphisms of C, and the arrows s_i to the left are elements of S. We denote such a path symbolically as $s_n^{-1} \circ f_{n-1} \circ \cdots \circ s_2^{-1} \circ s_1^{-1} \circ f_0$. Now, in order for this representation to function as it should, we define an equivalence relation on paths by imposing that compositions behave well

$$L_{i-1} \xrightarrow{f_{i-1}} L_i \xrightarrow{f_i} L_{i+1} \quad \text{is equivalent to} \quad L_{i-1} \xrightarrow{f_i \circ f_{i-1}} L_{i+1},$$

that we may ignore identities

$$L_{i-1} \xrightarrow{f_{i-1}} L_i \xrightarrow{\mathrm{id}_{L_i}} L_i \xrightarrow{f_i} L_{i+1} \quad \text{is equivalent to} \quad L_{i-1} \xrightarrow{f_{i-1}} L_i \xrightarrow{f_i} L_{i+1},$$

and that arrows to the left correspond to inverses

$$\begin{matrix} M \xrightarrow{s} N \xleftarrow{s} M \\ N \xleftarrow{s} M \xrightarrow{s} N \end{matrix} \quad \text{are equivalent to} \quad \begin{matrix} M \xrightarrow{\mathrm{id}_M} M \\ N \xrightarrow{\mathrm{id}_N} N. \end{matrix}$$

We then define a morphism $M \to N$ in the localization $S^{-1}\mathsf{C}$ to be an equivalence class of paths from M to N. Composition of morphisms is given simply by concatenation. Moreover, the identity morphism id_M in $S^{-1}\mathsf{C}$ of an object M is the equivalence class of the path

$$M \xrightarrow{\mathrm{id}_M} M.$$

Finally, the functor $Q : \mathsf{C} \to S^{-1}\mathsf{C}$ is given by the identity on objects and sends a morphism $f : M \to N$ to the equivalence class of the path

$$M \xrightarrow{f} N.$$

We now verify all the formal details for the proof of our theorem.

Proof of theorem 8.1.1. First and foremost, we remark that we have indeed defined an equivalence relation on paths and that $S^{-1}\mathsf{C}$ is indeed a category. Also, the image $Q(s)$ of any morphism $s : M \to N$ in S is indeed an isomorphism in $S^{-1}\mathsf{C}$, whose inverse is represented by

$$N \xleftarrow{s} M.$$

As for the universal property, let $F : \mathsf{C} \to \mathsf{D}$ be a functor such that $F(s)$ is an isomorphism for every $s \in S$. We define a functor $G : S^{-1}\mathsf{C} \to \mathsf{D}$ which is equal to F on objects and sends the equivalence class of a path

$$M \xrightarrow{f_0} L_1 \xleftarrow{s_1} L_2 \xleftarrow{s_2} \cdots \xrightarrow{f_{n-1}} L_n \xleftarrow{s_n} N$$

to the composition

$$\mathsf{F}(M) \xrightarrow{F(f_0)} \mathsf{F}(L_1) \xrightarrow{F(s_1)^{-1}} \mathsf{F}(L_2) \xrightarrow{F(s_2)^{-1}} \cdots \xrightarrow{F(f_{n-1})} \mathsf{F}(L_n) \xrightarrow{F(s_n)^{-1}} \mathsf{F}(N).$$

Since functors preserve composition and identities, this is independent of the choice of representative. This functor indeed satisfies $F = G \circ Q$ and, by construction, is uniquely determined by F. The uniqueness of the localization follows as usual from universal properties. □

As a quick corollary, we observe that localization behaves well with relation to the opposite category.

> **Corollary 8.1.2** Let C be a category and S a collection of morphisms in C. The category $(S^{-1}\mathsf{C})^{\mathrm{op}}$ is isomorphic to the localization of C^{op} with respect to S^{op}.

Proof. Consider the functor $Q^{\mathrm{op}} : \mathsf{C}^{\mathrm{op}} \to (S^{-1}\mathsf{C})^{\mathrm{op}}$. It is clear that Q^{op} sends elements of S^{op} to isomorphisms. Now, if $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ is a functor sending elements of S^{op} to isomorphisms, its opposite $F^{\mathrm{op}} : \mathsf{C} \to \mathsf{D}^{\mathrm{op}}$ sends elements of S to isomorphisms and so factors through the localization $S^{-1}\mathsf{C}$:

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "F^{\mathrm{op}}"] \arrow[d, "Q"'] & \mathsf{D}^{\mathrm{op}} \\
S^{-1}\mathsf{C}. \arrow[ur, dashed] &
\end{tikzcd}$$

The image of the diagram above by the opposite category functor gives the existence of a unique functor $(S^{-1}\mathsf{C})^{\mathrm{op}} \to \mathsf{D}$ making the diagram

$$\begin{tikzcd}
\mathsf{C}^{\mathrm{op}} \arrow[r, "F"] \arrow[d, "Q^{\mathrm{op}}"'] & \mathsf{D} \\
(S^{-1}\mathsf{C})^{\mathrm{op}} \arrow[ur, dashed] &
\end{tikzcd}$$

commute. The uniqueness of the localization then yields the desired result. □

The homotopy category $\mathsf{K}(\mathsf{A})$ is already the localization of $\mathsf{C}(\mathsf{A})$ with respect to the collection of homotopy equivalences. In addition, we'll define the derived category $\mathsf{D}(\mathsf{A})$ as the localization of $\mathsf{C}(\mathsf{A})$ (or, as we've seen, $\mathsf{K}(\mathsf{A})$) with respect to the collection of quasi-isomorphisms. Before going any further, let's check another interesting example.

■ **Example 8.1.1 — Lie's third theorem.** Let LieGrp be the category of *connected* Lie groups and LieAlg be the category of finite-dimensional Lie algebras. The *tangent space at the identity* functor

$$\mathsf{LieGrp} \to \mathsf{LieAlg}$$

is faithful and essentially surjective, but it isn't an equivalence of categories. Indeed, if $\varphi : G \to G'$ is a covering map, its differential at the identity $d_e\varphi : \mathfrak{g}' \to \mathfrak{g}$ is an isomorphism.

There are two ways of turning this functor into an equivalence of categories. Perhaps the simplest way is to restrict its domain to the full subcategory of *simply connected* Lie groups. But another way is to simply localize LieGrp with respect to all covering maps.[1] Then the universal property of localization gives an equivalence of categories between this localization and LieAlg. ■

A huge collection of examples are of the following form.

■ **Example 8.1.2 — Reflective localization.** Let D be a full subcategory of C. We say that D is a *reflective subcategory* if the inclusion functor $i : \mathsf{D} \to \mathsf{C}$ admits a left adjoint $r : \mathsf{C} \to \mathsf{D}$. Let S be the collection of morphisms in C which are sent to an isomorphism by r. Then D is equivalent to the localization $S^{-1}\mathsf{C}$. (Proposition 5.3.1 in [3].)

A plethora of examples of localization are of this form. The functor $\mathsf{Grp} \to \mathsf{Ab}$ sending a group to its abelianization identifies Ab as a localization of Grp. Similarly, the fraction field functor $\mathsf{IntDom} \to \mathsf{Fld}$, from the category of integral domains and injective morphisms to the category of fields, identifies Fld as a localization of IntDom. The reader which already has some knowledge of algebraic geometry may appreciate that both the sheafification functor and the functor

$$\begin{aligned} \mathsf{Sch} &\to \mathsf{Aff} \\ X &\mapsto \operatorname{Spec}\Gamma(X, \mathscr{O}_X), \end{aligned}$$

from the category of schemes to the category of affine schemes, are examples of reflective localization. ■

There are two issues with our notion of localization that ought to be addressed. Firstly, the localization of a locally small category need not be locally small.[2] This may be a problem for applying the Yoneda lemma, for example. Fortunately, almost all the localizations we are interested in will be locally small. (We'll soon see that the derived category of a *Grothendieck* abelian category is locally small.)

Another problem with our notion of localization is that, if C is additive, it isn't clear if $S^{-1}\mathsf{C}$ is also additive or not. Indeed, how can we sum paths? We can solve this problem by forcing every path from M to N to be equivalent to a path of the form

$$\begin{tikzcd} & L & \\ M \arrow[ur, "f"] & & N, \arrow[ul, "s"'] \end{tikzcd}$$

---

[1] The reader may wonder if the composition of covering maps is still a covering map. Somewhat surprisingly, this is false in general, but it holds for manifolds due to the theorem 2.11 in [19].

[2] Or, using the formalism of Grothendieck universes, the localization of a category need not exist in our fixed universe.

which we call a *roof*. We'll then conclude that any two roofs can be written with the same morphism s on the right, allowing their sum.

Definition 8.1.1 — Multiplicative system. Let C be a category and S be a collection of morphisms in C. We say that S is a *left multiplicative system* if it satisfies:

(LMS1) S is stable under composition and contains all the identities of C.

(LMS2) For any pair of morphisms $f : L \to N$ in C and $s : L \to M$ in S, there exists $g : M \to L'$ in C and $t : N \to L'$ in S making the diagram

$$\begin{tikzcd}
& L' & \\
M \arrow[ur, dashed, "g"] & L \arrow[l, "s"'] \arrow[r, "f"] & N \arrow[ul, dashed, "t"']
\end{tikzcd}$$

commute.

(LMS3) For every pair of morphisms $f, g : L \to L'$ in C and $s : M \to L$ in S such that $f \circ s = g \circ s$, there exists $t : L' \to N$ in S such that $t \circ f = t \circ g$.

The conditions for a *right multiplicative* system, denoted RMS, are the same with all the arrows reversed. We say that S is a *multiplicative system* if it's both a right and a left multiplicative system.

While the axiom LMS3 may seem somewhat technical, the other two axioms are precisely what we need in order for every morphism in $S^{-1}\mathsf{C}$ to be represented by a roof. Indeed, if S is a left multiplicative system, the axiom LMS2 allows us to gather all the inverse arrows on the right side of the path and the axiom LMS1 says that all these inverse arrows become one single element of S.

Even better, we can detect equivalence of paths without ever leaving the realm of roofs. Formally, there exists an equivalence relation $\sim_L$ on roofs which induces a dashed isomorphism making the diagram

$$\begin{tikzcd}
\{\text{roofs from } M \text{ to } N\} \arrow[r, hook] \arrow[d, two heads] & \{\text{paths from } M \text{ to } N\} \arrow[d, two heads] \\
\{\text{roofs from } M \text{ to } N\}/\sim_L \arrow[r, dashed, "\sim"] & \mathrm{Hom}_{S^{-1}\mathsf{C}}(M, N)
\end{tikzcd}$$

commute. We say that two roofs

$$\begin{tikzcd}
& L_1 & & & & L_2 & \\
M \arrow[ur, "f_1"] & & N \arrow[ul, "s_1"'] & \text{and} & M \arrow[ur, "f_2"] & & N \arrow[ul, "s_2"']
\end{tikzcd}$$

are $\sim_L$ equivalent if there exists an object L in C and morphisms $p_1 : L_1 \to L$, $p_2 : L_2 \to L$ making the diagram

$$\begin{tikzcd}
 & & L & & \\
 & L_1 \arrow[ur, dashed, "p_1"] & & L_2 \arrow[ul, dashed, "p_2"'] & \\
M \arrow[ur, "f_1"] \arrow[urrr, "f_2"'] & & & & N \arrow[ulll, "s_1"] \arrow[ul, "s_2"']
\end{tikzcd}$$

commute and such that $p_2 \circ s_2 = p_1 \circ s_1$ is in S. We observe that, if S is a right multiplicative system, every morphism in $S^{-1}C$ can be represented by a *trough*

$$\begin{tikzcd}
M & & N \\
 & L \arrow[ul, "s"] \arrow[ur, "f"'] &
\end{tikzcd}$$

and we have a similar relation $\sim_R$ for such diagrams. The next proposition proves all these claims. Since a right multiplicative system on C is nothing but a left multiplicative system on C^{op}, we'll henceforth only cite and prove results for left multiplicative systems, for analogous results hold by duality.

Proposition 8.1.3 Let S be a left multiplicative system in a category C, and let M, N be two objects of C. Then $\sim_L$ is an equivalence relation on the collection of roofs from M to N. Moreover, the canonical morphism sending a roof to a morphism in $S^{-1}C$ descends to the quotient defining an isomorphism

$$\mathrm{Hom}_{S^{-1}C}(M, N) \cong \{\text{roofs from } M \text{ to } N\}/\sim_L .$$

Proof. The relation $\sim_L$ is clearly reflexive and symmetric. As for transitivity, suppose that

$$\begin{tikzcd}
 & L_1 & & & & L_2 & & & & L_2 & & & & L_3 & \\
M \arrow[ur, "f_1"] & & N \arrow[ul, "s_1"'] & \sim_L & M \arrow[ur, "f_2"] & & N \arrow[ul, "s_2"'] & \text{and} & M \arrow[ur, "f_2"] & & N \arrow[ul, "s_2"'] & \sim_L & M \arrow[ur, "f_3"] & & N \arrow[ul, "s_3"'] .
\end{tikzcd}$$

This means that there exists objects L_{12} and L_{23}, along with morphisms p_1, p_2, p_2', and p_3 making the diagrams

$$\begin{tikzcd}
 & & L_{12} & & & & & & L_{23} & & \\
 & L_1 \arrow[ur, dashed, "p_1"] & & L_2 \arrow[ul, dashed, "p_2"'] & & \text{and} & & L_2 \arrow[ur, dashed, "p_2'"] & & L_3 \arrow[ul, dashed, "p_3"'] & \\
M \arrow[ur, "f_1"] \arrow[urrr, "f_2"'] & & & & N \arrow[ulll, "s_1"] \arrow[ul, "s_2"'] & & M \arrow[ur, "f_2"] \arrow[urrr, "f_3"'] & & & & N \arrow[ulll, "s_2"] \arrow[ul, "s_3"']
\end{tikzcd}$$

commute. Moreover, $p_2 \circ s_2 = p_1 \circ s_1$ and $p_3 \circ s_3 = p_2' \circ s_2$ are in S. In particular, the axiom LMS2 gives the existence of the dashed morphisms making the diagram

$$\begin{tikzcd}
& L & \\
L_{12} \arrow[ur, dashed, "p"] & & L_{23} \arrow[ul, dashed, "s"'] \\
& N \arrow[ul, "p_2 \circ s_2"] \arrow[ur, "p_2' \circ s_2"'] &
\end{tikzcd}$$

commute. Now, the axiom LMS3 gives an object L' and a morphism $t : L \to L'$ in S such that $t \circ p \circ p_2 = t \circ s \circ p_2'$. This morphism makes the diagram

$$\begin{tikzcd}
& & L' & & \\
& L_1 \arrow[ur, "t \circ p \circ p_1"] & & L_3 \arrow[ul, "t \circ s \circ p_3"'] & \\
M \arrow[ur, "f_1"] \arrow[urrr, "f_3"] & & & & N \arrow[ul, "s_3"'] \arrow[ulll, "s_1"]
\end{tikzcd}$$

commute. Indeed,

$$t \circ p \circ p_1 \circ f_1 = t \circ p \circ p_2 \circ f_2 = t \circ s \circ p_2' \circ f_2 = t \circ s \circ p_3 \circ f_3$$

and

$$t \circ p \circ p_1 \circ s_1 = t \circ p \circ p_2 \circ s_2 = t \circ s \circ p_2' \circ s_2 = t \circ s \circ p_3 \circ s_3.$$

Moreover, $t \circ s \circ p_3 \circ s_3$ is in S since $p_3 \circ s_3$, s, and t are. This finishes the proof that $\sim_L$ is transitive.

We now prove that the natural morphism sending a roof from M to N to the associated morphism $M \to N$ in $S^{-1}C$ descents to the quotient by $\sim_L$. Let

$$\begin{tikzcd}
& L_1 & & & & L_2 & \\
M \arrow[ur, "f_1"] & & N \arrow[ul, "s_1"'] & \text{and} & M \arrow[ur, "f_2"] & & N \arrow[ul, "s_2"']
\end{tikzcd}$$

be $\sim_L$ equivalent roofs. This means that there exists p_1 and p_2 such that

$$p_1 \circ f_1 = p_2 \circ f_2$$
$$p_1 \circ s_1 = p_2 \circ s_2$$

and such that the latter is in S. Then, denoting by $\sim$ the equivalence of paths, we have that

$$\begin{aligned} s_1^{-1} \circ f_1 &\sim (p_2 \circ s_2)^{-1} \circ (p_1 \circ s_1) \circ s_1^{-1} \circ f_1 \\ &\sim (p_2 \circ s_2)^{-1} \circ p_1 \circ f_1 \\ &\sim (p_2 \circ s_2)^{-1} \circ p_2 \circ f_2 \\ &\sim (p_2 \circ s_2)^{-1} \circ p_2 \circ s_2 \circ s_2^{-1} \circ f_2 = s_2^{-1} \circ f_2, \end{aligned}$$

proving that $s_1^{-1} \circ f_1$ and $s_2^{-1} \circ f_2$ define the same morphism in $S^{-1}\mathsf{C}$. In other words, we have a dashed map making the diagram

$$\begin{tikzcd}
\{\text{roofs from } M \text{ to } N\} \arrow[r, hook] \arrow[d, two heads] & \{\text{paths from } M \text{ to } N\} \arrow[d, two heads] \\
\{\text{roofs from } M \text{ to } N\}/\sim_L \arrow[r, dashed] & \mathrm{Hom}_{S^{-1}\mathsf{C}}(M, N)
\end{tikzcd}$$

commute. As we already saw, every morphism in $S^{-1}\mathsf{C}$ can be represented by a roof, meaning that this map is surjective. In order to prove that it's injective as well, it suffices to find a left inverse.

Consider a path from M to N like this:

$$\begin{tikzcd}
M & \bullet \arrow[l] & \bullet \arrow[l] \arrow[r] & \bullet & \bullet \arrow[l] \arrow[r] & \bullet \arrow[r] & \bullet & N. \arrow[l]
\end{tikzcd}$$

As a first step, we compose all the composable morphisms; yielding something like the path below.

$$\begin{tikzcd}
M & \bullet \arrow[l] \arrow[r] & \bullet & \bullet \arrow[l] \arrow[r] & \bullet & N \arrow[l]
\end{tikzcd}$$

Then, we apply successively the axiom LMS2 to all pairs of arrows of the form $\bullet \leftarrow \bullet \rightarrow \bullet$.

$$\begin{tikzcd}
 & & \bullet & & & \\
 & \bullet \arrow[ur] & & \bullet \arrow[ul] & & \\
M \arrow[ur] & \bullet \arrow[l] \arrow[r] & \bullet \arrow[ul] \arrow[ur] & \bullet \arrow[l] \arrow[r] & \bullet \arrow[ul] & N \arrow[l]
\end{tikzcd}$$

Finally, we compose the outermost arrows to yield a roof. We want to declare this roof as the image of a morphism

$$\{\text{paths from } M \text{ to } N\} \to \{\text{roofs from } M \text{ to } N\}/\sim_L .$$

Since the axiom LMS2 only gives the existence, but not the unicity, of the morphisms that we used, we need to verify that our definition is independent of any choices. In other words, if $M \xleftarrow{s} L \xrightarrow{f} N$ is a path and we have two commutative diagrams

$$\begin{tikzcd} & L_1 & \\ M \arrow[ur, "g_1"] & L \arrow[l, "s"'] \arrow[r, "f"] & N \arrow[ul, "t_1"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & L_2 & \\ M \arrow[ur, "g_2"] & L \arrow[l, "s"'] \arrow[r, "f"] & N \arrow[ul, "t_2"'] \end{tikzcd}$$

coming from the axiom LMS2, we need to show that the roofs $t_1^{-1} \circ g_1$ and $t_2^{-1} \circ g_2$ are $\sim_L$-equivalent.

By the same axiom LMS2, there exists $p_1 : L_1 \to P$ in C and $p_2 : L_2 \to P$ in S making the diagram

$$\begin{tikzcd} & P & \\ L_1 \arrow[ur, "p_1"] & N \arrow[l, "t_1"'] \arrow[r, "t_2"] & L_2 \arrow[ul, "p_2"'] \end{tikzcd}$$

commute. Since $p_1 \circ t_1 = p_2 \circ t_2$, we have that

$$p_1 \circ g_1 \circ s = p_1 \circ t_1 \circ f = p_2 \circ t_2 \circ f = p_2 \circ g_2 \circ s,$$

and so the axiom LMS3 gives a morphism $t : P \to Q$ in S satisfying $t \circ p_1 \circ g_1 = t \circ p_2 \circ g_2$. In other words, making the diagram

$$\begin{tikzcd} & & Q & & \\ & L_1 \arrow[ur, "t \circ p_1"] & & L_2 \arrow[ul, "t \circ p_2"'] & \\ M \arrow[ur, "g_1"] \arrow[urrr, "g_2"'] & & & & N \arrow[ul, "t_2"'] \arrow[ulll, "t_1"] \end{tikzcd}$$

commute. This proves that the roofs $t_1^{-1} \circ g_1$ and $t_2^{-1} \circ g_2$ are $\sim_L$-equivalent.

We now prove that this morphism descends to the quotient, defining a morphism

$$\mathrm{Hom}_{S^{-1}C}(M, N) \to \{\text{roofs from } M \text{ to } N\}/ \sim_L,$$

which is clearly the desired left inverse. Since the equivalence relation $\sim$ on paths is generated by four simple equivalences, it suffices to verify that the roofs assigned to those equivalent paths coincide. Now, it's manifest that the roofs assigned to

$$L_{i-1} \xrightarrow{f_{i-1}} L_i \xrightarrow{f_i} L_{i+1} \quad \text{and} \quad L_{i-1} \xrightarrow{f_i \circ f_{i-1}} L_{i+1}$$

coincide, and that the roofs assigned to

$$L_{i-1} \xrightarrow{f_{i-1}} L_i \xrightarrow{\mathrm{id}_{L_i}} L_i \xrightarrow{f_i} L_{i+1} \quad \text{and} \quad L_{i-1} \xrightarrow{f_{i-1}} L_i \xrightarrow{f_i} L_{i+1}$$

coincide. Indeed, the first step is to compose all composable morphisms. The roofs assigned to

$$M \xrightarrow{s} N \xleftarrow{s} M \quad \text{and} \quad M \xrightarrow{\mathrm{id}_M} M$$

are $\sim_L$ equivalent due to the commutativity of the diagram

$$\begin{tikzcd}
 & & N & & \\
 & N \arrow[ur, "\mathrm{id}_N"] & & M \arrow[ul, "s"'] & \\
M \arrow[ur, "s"] \arrow[urrr, "\mathrm{id}_M"] & & & & M. \arrow[ulll, "s"'] \arrow[ul, "\mathrm{id}_M"']
\end{tikzcd}$$

Finally, we finish the proof by proving that the roofs assigned to

$$N \xleftarrow{s} M \xrightarrow{s} N \quad \text{and} \quad N \xrightarrow{\mathrm{id}_N} N$$

are $\sim_L$ equivalent. The axiom LMS2 gives morphisms $g : N \to L$ and $t : N \to L$, with the latter in S, satisfying $g \circ s = t \circ s$. Then, the axiom LMS3 gives a morphism $p : L \to P$ in S satisfying $p \circ g = p \circ t$. In particular, the diagram

$$\begin{tikzcd}
 & & P & & \\
 & L \arrow[ur, "p"] & & N \arrow[ul, "p \circ t"'] & \\
N \arrow[ur, "g"] \arrow[urrr, "\mathrm{id}_N"] & & & & N \arrow[ulll, "t"'] \arrow[ul, "\mathrm{id}_N"']
\end{tikzcd}$$

commutes and $p \circ t \circ \mathrm{id}_N$ is in S; yielding the result. □

In a similar vein, the composition of morphisms can also be seen without leaving the realm of roofs. Indeed, given two roofs

$$\begin{tikzcd}
 & L_1 & \\
M \arrow[ur, "f_1"] & & N \arrow[ul, "s_1"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & L_2 & \\
N \arrow[ur, "f_2"] & & P, \arrow[ul, "s_2"']
\end{tikzcd}$$

we may use the axiom LMS2 to find morphisms $g : L_1 \to L$ in C and $t : L_2 \to L$ in S

making the diagram

$$\begin{tikzcd}
& & L & & \\
& L_1 \arrow[ur, "g"] & & L_2 \arrow[ul, "t"'] & \\
M \arrow[ur, "f_1"] & & N \arrow[ul, "s_1"'] \arrow[ur, "f_2"] & & P \arrow[ul, "s_2"']
\end{tikzcd}$$

commute. The composition of our given roofs is none other than the roof $(t \circ s_2)^{-1} \circ (g \circ f_1)$ concocted from the outermost arrows.

8.2. Localization of additive and abelian categories

The categories which interested us in the previous two chapters were often endowed with some extra structure or satisfied some good properties. This section is then dedicated to the study the localizations of (pre)additive and abelian categories.

Our first task is to understand how can we sum two roofs in a preadditive category. In precisely the same way that we sum fractions by writing them with a common denominator, we can write any two roofs with a single morphism s on the right.

Proposition 8.2.1 Let S be a left multiplicative system in a category C. Every two morphisms $M \to N$ in $S^{-1}C$ may be written as the equivalence classes of $s^{-1} \circ f$ and $s^{-1} \circ g$ for suitable morphisms f, g in C and $s \in S$.

Proof. Consider the following two roofs from M to N:

$$\begin{tikzcd}
& L_1 & & & & L_2 & \\
M \arrow[ur, "p_1"] & & N \arrow[ul, "s_1"'] & \text{and} & M \arrow[ur, "p_2"] & & N. \arrow[ul, "s_2"']
\end{tikzcd}$$

The axiom LMS2 gives morphisms $q : L_1 \to Q$ in C and $t : L_2 \to Q$ in S making the diagram

$$\begin{tikzcd}
& Q & \\
L_1 \arrow[ur, "q"] & N \arrow[l, "s_1"'] \arrow[r, "s_2"] & L_2 \arrow[ul, "t"']
\end{tikzcd}$$

commute. We affirm that the choice $s = t \circ s_2$, $f = q \circ p_1$, and $g = t \circ p_2$ works. Indeed the diagrams

$$\begin{tikzcd}
 & & Q & & \\
 & L_1 \arrow[ur, "q"] & & Q \arrow[ul, "\mathrm{id}_Q"'] & \\
M \arrow[ur, "p_1"] \arrow[urrr, "f"] & & & & N \arrow[ul, "s"'] \arrow[ulll, "s_1"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & Q & & \\
 & L_2 \arrow[ur, "t"] & & Q \arrow[ul, "\mathrm{id}_Q"'] & \\
M \arrow[ur, "p_2"] \arrow[urrr, "g"] & & & & N \arrow[ul, "s"'] \arrow[ulll, "s_2"']
\end{tikzcd}$$

commute. □

Besides allowing the sum of two morphisms in a localization of a preadditive category, the above writing also allows us to easily decide whether two morphisms in the localization are equal. We claim that two morphisms in $S^{-1}C$ represented by the roofs

$$\begin{tikzcd}
 & L & \\
M \arrow[ur, "f_1"] & & N \arrow[ul, "s"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & L & \\
M \arrow[ur, "f_2"] & & N \arrow[ul, "s"']
\end{tikzcd}$$

are equal if and only if there exists a morphism $q : L \to L$ in C such that $q \circ s \in S$ and $q \circ f_1 = q \circ f_2$. Indeed, both roofs are equivalent if and only if there exist morphisms $p_1 : L \to L'$ and $p_2 : L \to L'$ making the diagram

$$\begin{tikzcd}
 & & L' & & \\
 & L \arrow[ur, dashed, "p_1"] & & L \arrow[ul, dashed, "p_2"'] & \\
M \arrow[ur, "f_1"] \arrow[urrr, "f_2"] & & & & N \arrow[ul, "s"'] \arrow[ulll, "s"']
\end{tikzcd}$$

commute and such that $p_2 \circ s = p_1 \circ s$ is in S. If there exists such a morphism q, we may take $p_1 = p_2 = q$. Conversely, the axiom LMS3 gives a morphism $t \in S$ such that $t \circ p_2 = t \circ p_1$ and we may take q to be this common morphism.

> **Corollary 8.2.2** Let S be a left multiplicative system in a preadditive category A. Then $S^{-1}A$ is also preadditive, and the localization functor $Q : A \to S^{-1}A$ is additive. Moreover, if B is another preadditive category and $F : A \to B$ is an additive functor such that $F(s)$ is an isomorphism for every $s \in S$, the induced functor $S^{-1}A \to B$ is also additive. If A is additive, then so is $S^{-1}A$.

Proof. Given two roofs $s^{-1} \circ f : M \to N$ and $s^{-1} \circ g : M \to N$, we define their sum to be $s^{-1} \circ (f+g)$. In order for this operation to descend to $\mathrm{Hom}_{S^{-1}A}(M, N)$, we need to show that $\sim_L$-equivalent roofs give rise to the same sum. Now, by the preceding discussion, if we change $s^{-1} \circ g$ to $s^{-1} \circ h$, there exists a morphism q in A such that $q \circ g = q \circ h$ and $q \circ s \in S$. But then the same discussion implies that $s^{-1} \circ (f + g) \sim_L s^{-1} \circ (f + h)$, for

$$q \circ (f + g) = q \circ (f + h) \qquad \text{and} \qquad q \circ s \in S.$$

This same criterion shows readily that composition is bilinear. Since the localization functor Q sends a morphism $f : M \to N$ in A to $\mathrm{id}_N^{-1} \circ f$, our definition of sum makes Q additive.

If B is another preadditive category and $F : \mathsf{A} \to \mathsf{B}$ is an additive functor such that $F(s)$ is an isomorphism for every $s \in S$, the induced functor $S^{-1}\mathsf{A} \to \mathsf{B}$ sends a roof $s^{-1} \circ f$ to $F(s)^{-1} \circ F(f)$. Its additivity then follows from the additivity of F.

Finally, if A is additive, the localization functor Q sends the zero-object of A to a zero-object of $S^{-1}\mathsf{A}$. Also, the theorem 6.1.9 gives that the image by Q of a direct sum in A defines a direct sum in $S^{-1}\mathsf{A}$. ☐

We're finally able to explain the rationale behind the nomenclature and notation used in this section.

■ Example 8.2.1 — Localization of noncommutative rings. Let A be a (not necessarily commutative) ring. We define a category A which only has one object $*$ and such that $\mathrm{Hom}_{\mathsf{A}}(*, *) = A$. This is a preadditive category and, in this context, a left multiplicative system S on A is a subset of A such that

(a) S is multiplicatively closed and contains 1;

(b) for every $a \in A$ and $s \in S$, the set $As \cap Sa$ is nonempty;

(c) for every $a \in A$ and $s \in S$, if $as = 0$, then $ta = 0$ for some $t \in S$.

A particular case of the preceding corollary proves the existence of localizations for left multiplicative systems on any ring. This is a very important result on noncommutative ring theory. ■

For our next result, let C be a category and S be a left multiplicative system on C. Given an object N in C, we define a category N/S whose objects are morphisms $N \to L$ in S and whose morphisms are commutative diagrams

$$\begin{tikzcd} & N \arrow[dl] \arrow[dr] & \\ L_1 \arrow[rr] & & L_2, \end{tikzcd}$$

where the arrow $L_1 \to L_2$ is in C. A reasonable explanation for the axiom LMS3, which seemed technical at first glance, is that it makes the category N/S filtered. While finite limits don't usually commute with colimits; in Set[3] they do whenever the colimit in question is filtered.

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[3] Or in any *algebraic category*.

Proposition 8.2.3 Let S be a left multiplicative system in a category C. Then we may write $\mathrm{Hom}_{S^{-1}\mathsf{C}}(M, N)$ as the filtered colimit

$$\operatorname*{colim}_{(N \to L) \in N/S} \mathrm{Hom}_{\mathsf{C}}(M, L).$$

In particular, the localization functor $Q : \mathsf{C} \to S^{-1}\mathsf{C}$ commutes with finite colimits. Similarly, if S is a right multiplicative system, Q commutes with finite limits.

Proof. The category N/S being filtered means that: it has at least one object, that for any two objects $N \to L_1$ and $N \to L_2$ in N/S we have a third object $N \to L$ with morphisms

$$\begin{tikzcd} & N \arrow[dl] \arrow[dr] & \\ L_1 \arrow[rr] & & L \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & N \arrow[dl] \arrow[dr] & \\ L_2 \arrow[rr] & & L, \end{tikzcd}$$

and that for every pair of morphisms $L_1 \rightrightarrows L_2$ in N/S, there exists a morphism $L_2 \to L$ in N/S making the diagram

$$\begin{tikzcd} & N \arrow[dl] \arrow[dr] \arrow[drr, bend left] & & \\ L_1 \arrow[rr, shift left] \arrow[rr, shift right] & & L_2 \arrow[r, dashed] & L \end{tikzcd}$$

commute. All those properties follow readily from the axioms of a left multiplicative system. Now, recall that filtered colimits in the category of sets have simple descriptions. Namely,

$$\operatorname*{colim}_{(N \to L) \in N/S} \mathrm{Hom}_{\mathsf{C}}(M, L) = \left(\coprod_{(N \to L) \in N/S} \mathrm{Hom}_{\mathsf{C}}(M, L) \right) \Big/ \sim,$$

where $M \to L_1$ is equivalent to $M \to L_2$ if there's morphisms $L_1 \to L \leftarrow L_2$ in N/S making the diagram

$$\begin{tikzcd} & L_1 \arrow[dr] & \\ M \arrow[ur] \arrow[dr] & & L \\ & L_2 \arrow[ur] & \end{tikzcd}$$

commute. This is precisely the equivalence relation on roofs; proving that the map sending a roof to the associated equivalence class in the colimit is injective. It's clear that this map is also surjective.

Now, let $\mathsf{D} : \mathsf{I} \to \mathsf{C}$ be a finite diagram whose colimit exists. Then,

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{S}^{-1}\mathsf{C}}\left(Q\left(\operatorname*{colim}_{I\in\mathsf{I}} D(I)\right), N\right) &= \operatorname*{colim}_{(N\to L)\in N/S} \mathrm{Hom}_{\mathsf{C}}\left(\operatorname*{colim}_{I\in\mathsf{I}} D(I), L\right) \\
&= \operatorname*{colim}_{(N\to L)\in N/S} \lim_{I\in\mathsf{I}} \mathrm{Hom}_{\mathsf{C}}(D(I), L) \\
&= \lim_{I\in\mathsf{I}} \operatorname*{colim}_{(N\to L)\in N/S} \mathrm{Hom}_{\mathsf{C}}(D(I), L) \\
&= \lim_{I\in\mathsf{I}} \mathrm{Hom}_{\mathsf{S}^{-1}\mathsf{C}}\left(Q(D(I)), N\right),
\end{aligned}$$

and the Yoneda lemma gives that $Q\left(\operatorname{colim}_{I\in\mathsf{I}} D(I)\right) = \operatorname{colim}_{I\in\mathsf{I}} Q(D(I))$. The other statement follows by duality. $\square$

The fact that we may write the hom-sets as a filtered colimit may lead us to think that the localization functor should commute with arbitrary colimits and finite limits, *where it's actually the opposite.*

Since Ab has all filtered colimits, this gives another proof that the localization $S^{-1}\mathsf{A}$ is additive, whenever A is additive and S is a left multiplicative system. If A is actually abelian and S is a "two-sided" multiplicative system, we have even more.

Corollary 8.2.4 Let S be a multiplicative system in an abelian category A. Then $S^{-1}\mathsf{A}$ is also abelian, and the localization functor $Q : \mathsf{A} \to S^{-1}\mathsf{A}$ is exact. Moreover, if B is another abelian category and $F : \mathsf{A} \to \mathsf{B}$ is an exact functor such that $F(s)$ is an isomorphism for every $s \in S$, the induced functor $S^{-1}\mathsf{A} \to \mathsf{B}$ is also exact.

Proof. The corollary 8.2.2 implies that $S^{-1}\mathsf{A}$ is additive. We affirm that it has kernels (that it has cokernels then follows by duality). Let

$$\begin{tikzcd}
 & L & \\
M \arrow[ur, "f"] & & N \arrow[ul, "s"']
\end{tikzcd}$$

be a morphism in $S^{-1}\mathsf{A}$. Since $Q(s)$ is an isomorphism, the kernel of this roof exists if and only if the kernel of $Q(f)$ exists. (In this case, the kernel of the roof coincides with $\ker Q(f)$.) But the preceding proposition gives that $Q(\ker f) = \ker Q(f)$; proving that $S^{-1}\mathsf{A}$ has kernels.

For completeness sake, let us be precise about what happens with cokernels. Let $\varphi : M \to N$ be the roof above, and let $L \to C$ be the cokernel of $Q(f)$. (Which is also $Q(\operatorname{coker} f)$, due to the preceding proposition.) Since $Q(s)$ is an isomorphism, we obtain that

$$N \xrightarrow{Q(s)} L \xrightarrow{\operatorname{coker} Q(f)} C$$

satisfies the universal property of $\operatorname{coker}\varphi$. Alternatively, we could work with troughs and write the cokernel of $\varphi = g \circ t^{-1}$ simply as $\operatorname{coker} Q(g)$.

In order to finish the proof that $S^{-1}A$ is abelian, we'll use the proposition 6.2.6. We remark that the previous discussion implies that

$$\begin{aligned}\ker(\operatorname{coker} Q(f)) &= Q(s) \circ \ker(\operatorname{coker}\varphi)\\ \operatorname{coker}(\ker Q(f)) &= \operatorname{coker}(\ker\varphi).\end{aligned}$$

In particular, we apply the localization functor Q to the square in proposition 6.2.6 to obtain the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "Q(f)"] \arrow[dd, "\operatorname{coker}(\ker\varphi)"'] & L \\
 & N \arrow[u, "Q(s)"'] \\
C' \arrow[r, "Q(\bar{f})"'] & K'. \arrow[u, "\ker(\operatorname{coker}\varphi)"'] \arrow[uu, bend right, "\ker(\operatorname{coker} Q(f))"']
\end{tikzcd}$$

By reordering, we conclude that the diagram

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "\operatorname{coker}(\ker\varphi)"'] & N \\
C' \arrow[r, "Q(\bar{f})"] & K' \arrow[u, "\ker(\operatorname{coker}\varphi)"']
\end{tikzcd}$$

commutes and that $Q(\bar{f})$ is an isomorphism, for $\bar{f}$ is. The same proposition 6.2.6 then implies the result.

The localization functor $Q : A \to S^{-1}A$ is exact since it commutes with finite (co)limits (by the preceding proposition). If $F : A \to B$ is an exact functor such that $F(s)$ is an isomorphism for every $s \in S$, we have an induced functor $H : S^{-1}A \to B$ by the universal property of localization. We affirm that it's, moreover, exact. Let

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P \longrightarrow 0$$

be an exact sequence in $S^{-1}A$, where $\varphi = s^{-1} \circ f$ and $\psi = t^{-1} \circ g$. Since ψ is an epimorphism, $\operatorname{coker} Q(g)$ is the zero-morphism and so $Q(\operatorname{im} g) = \ker(\operatorname{coker} Q(g))$ is an isomorphism. The universal property of kernels induces a morphism $\alpha : M \to K$ in $S^{-1}A$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[dd, "\alpha"] & N \arrow[r, "\psi"] \arrow[dd, equal] & P \arrow[r] \arrow[d, "Q(t)"] & 0 \\
 & & & L' \arrow[d, "Q(\operatorname{im} g)^{-1}"] & \\
0 \arrow[r] & K \arrow[r, "Q(\ker g)"] & N \arrow[r, "Q(\operatorname{coim} g)"] & I \arrow[r] & 0
\end{tikzcd}$$

commute, and the snake lemma implies that it's an isomorphism. In other words, our exact sequence is isomorphic to the image under Q of a short exact sequence in A. Finally, the fact that $F = H \circ Q$ is exact implies that so is H. □

There's another point of view which is often used when dealing with localizations of abelian categories. For that we need the definition below.

Definition 8.2.1 — Thick subcategory. Let A be an abelian category. We say that a non-empty full subcategory C of A is *thick* if for any short exact sequence in A

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0,$$

N is in C if and only if M and P are.

Due to the corollary 6.2.2, a thick subcategory is always abelian. The discussion after the definition 6.4.2 implies that if C is a thick subcategory of A and

$$M \longrightarrow N \longrightarrow P$$

is an exact sequence, then N is in C if M and P are. Conversely, let C be a non-empty full subcategory of A where, for every exact sequence as above, N is in C whenever M and P are. If Q is any object of C, the sequence

$$Q \longrightarrow 0 \longrightarrow Q$$

is exact, and so $0 \in \mathsf{C}$. Then, we may split a short exact sequence into three even shorter exact sequences and obtain that C is thick. We'll use both characterizations interchangeably.

The *raison d'être* of such subcategories is the result below.

Proposition 8.2.5 Let A be an abelian category. Given a multiplicative system S in A, the full subcategory C_S, composed of the objects which are isomorphic to 0 in $S^{-1}\mathsf{A}$, is thick. Conversely, given a thick subcategory C, the collection S_{C} of all morphisms φ in A such that $\ker \varphi$ and $\operatorname{coker} \varphi$ are in C is a multiplicative system.

Proof. Let S be a multiplicative system in A and consider a short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

in A. Since the localization functor Q is exact, the sequence

$$0 \longrightarrow Q(M) \longrightarrow Q(N) \longrightarrow Q(P) \longrightarrow 0$$

in $S^{-1}\mathsf{A}$ is also exact. Now, by exactness, $Q(N)$ is zero precisely when both $Q(M)$ and $Q(P)$ are. In other words, N is in C_S if and only if M and P are; proving that C_S is thick.

Conversely, let C be a thick subcategory of A. By duality, it suffices to show that S_{C} satisfies the axioms of a *left* multiplicative system. Since $0 \in \mathsf{C}$, S_{C} contains all identities. Also, if $f : M \to N$ and $g : N \to P$ are morphisms in S_{C}, the snake lemma applied to the diagram

$$\begin{tikzcd}
 & M \arrow[r, "f"] \arrow[d, "g \circ f"] & N \arrow[r] \arrow[d, "g"] & \operatorname{coker} f \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P \arrow[r, "\mathrm{id}_P"] & P \arrow[r] & 0 &
\end{tikzcd}$$

yields a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \ker f \arrow[r] & \ker(g \circ f) \arrow[r] & \ker g \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- (\tikztotarget.north) \tikztonodes}] & \\
 & \operatorname{coker} f \arrow[r] & \operatorname{coker}(g \circ f) \arrow[r] & \operatorname{coker} g \arrow[r] & 0.
\end{tikzcd}$$

Then, the fact that C is thick implies that $\ker(g \circ f)$ and $\operatorname{coker}(g \circ f)$ are in C, proving that $g \circ f \in S_{\mathsf{C}}$, and so S_{C} satisfies LMS1.

Given a pair of morphisms $f : L \to N$ in A and $s : L \to M$ in S_{C}, we consider their pushout

$$\begin{tikzcd}
L \arrow[r, "s"] \arrow[d, "f"] & M \arrow[d, "g"] \\
N \arrow[r, "t"] & L'.
\end{tikzcd}$$

The axiom LMS2 will follow as soon as we prove that t is in S_{C}. The proposition 6.3.6 gives an isomorphism $\operatorname{coker} s \to \operatorname{coker} t$ and an epimorphism $\ker s \to \ker t$, implying that $\ker t, \operatorname{coker} t \in \mathsf{C}$.

In order to obtain the axiom LMS3, consider morphisms $f : L \to L'$ in A and $s : M \to L$ in S_{C} satisfying $f \circ s = 0$. As in proposition 7.1.3, there's a natural epimorphism

$$\operatorname{coker} s \to \operatorname{coim} f,$$

proving that $\operatorname{coim} f \in \mathsf{C}$. We recall that, as objects of A, $\operatorname{coim} f = \operatorname{im} f$ by the first isomorphism theorem. Then, we may define t as the natural quotient $L' \to \operatorname{coker} f$, which is in S_{C} for $\ker t = \operatorname{im} f$ and $\operatorname{coker} t = 0$. ☐

Given a thick subcategory C, the collection of morphisms S_{C} is no ordinary multiplicative system. It satisfies an even stronger property; namely, the morphisms in A that are sent to isomorphisms in $S_{\mathsf{C}}^{-1}\mathsf{A}$ are *precisely* the elements of S_{C}. Indeed, suppose that $Q(f)$ is an isomorphism, and let $t^{-1} \circ g$ be an inverse to it. Since $t^{-1} \circ g \circ f$ is the identity morphism, there exists morphisms h in A and s in S_{C} making the diagram

$$\begin{tikzcd}
 & & L' & & \\
 & L \arrow[ur, "h"] & & M \arrow[ul, "s"'] & \\
M \arrow[ur, "g\circ f"] \arrow[urrr, "\mathrm{id}_M"] & & & & M \arrow[ul, "\mathrm{id}_M"'] \arrow[ulll, "t"']
\end{tikzcd}$$

commute. That is, satisfying $s = h \circ t$ and $s = h \circ g \circ f$. We apply the first equation to the long exact sequence used in the proof of the preceding proposition to conclude that $h \in S_{\mathsf{C}}$. Then, the same exact sequence applied to the other equation implies that $\ker f \in \mathsf{C}$. Finally, we may write the inverse of $Q(f)$ as a trough and do exactly the same reasoning to conclude that $\operatorname{coker} f \in \mathsf{C}$. façam essa conta pls

The multiplicative systems S which satisfy the property that $Q(f)$ is an isomorphism in $S^{-1}\mathsf{A}$ if and only if $f \in S$ are said to be *saturated*. Such multiplicative systems are in one-to-one correspondence with thick subcategories.

Corollary 8.2.6 The operations $\mathsf{C} \mapsto S_{\mathsf{C}}$ and $S \mapsto \mathsf{C}_S$ define a bijection between thick subcategories and saturated multiplicative systems.

Proof. ☐

Motivated by the results above, we define the *quotient* A/C of an abelian category A by a thick subcategory C as the localization $S_{\mathsf{C}}^{-1}\mathsf{A}$. These quotients are often called *Serre quotients* in the literature.

Given an exact functor $F : \mathsf{A} \to \mathsf{B}$ between abelian categories, its *kernel* is the full subcategory of A composed of the objects whose image by F is zero. It's clear that the kernel of an exact functor is thick. As in basically every algebraic category, the existence of quotients gives the converse. In this case, our last results imply that every thick subcategory C of A is the kernel of some exact functor. Namely, the quotient / localization functor $Q : \mathsf{A} \to \mathsf{A}/\mathsf{C}$.

We may also rephrase the universal property of localization obtained in the corollary 8.2.4 using the point of view of quotients. If $F : \mathsf{A} \to \mathsf{B}$ is an exact functor whose kernel contains C, then F factors uniquely through the quotient A/C.

In due time, we'll see that many interesting abelian categories are Serre quotients of A-Mod, for some (not necessarily commutative) ring A. (Theorem 10.3.1.) We present two other examples of Serre quotients.

■ **Example 8.2.2** Let S be a multiplicative subset of a ring A and C be the category of A-modules whose elements are annihilated by some element of S. (These A-modules are often said to have *S-torsion*.) It's clear that C is a thick subcategory of A-Mod. We affirm that $S^{-1}A$-Mod is canonically equivalent to A-Mod/C.

Let $f : A \to S^{-1}A$ (resp. $Q : A\text{-Mod} \to A\text{-Mod}/C$) be the localization map (resp. functor). The functor $f^* : A\text{-Mod} \to S^{-1}A\text{-Mod}$, which sends M to $M \otimes_A S^{-1}A \cong S^{-1}M$, is exact and maps elements of C to zero. The universal property then implies that it descends to an exact functor $\tilde{f^*} : A\text{-Mod}/C \to S^{-1}A\text{-Mod}$.

Denoting by $f_* : S^{-1}A\text{-Mod} \to A\text{-Mod}$ the restriction of scalars functor, the adjunction $f^* \dashv f_*$ gives rise to another adjunction $\tilde{f^*} \dashv Q \circ f_*$. The unit of the latter is a natural isomorphism, which proves that $\tilde{f^*}$ is fully faithful. Finally, it's also essentially surjective since the restriction of scalars of a $S^{-1}A$-module N to A is sent to an isomorphic copy of N. This finishes the proof.

In particular, the quotient of Ab by the thick subcategory of torsion groups is equivalent to $\mathbb{Q}$-Vect. ■

The reader that already knows some algebraic geometry may be pleased to know that the basic theory of quasicoherent sheaves on projective schemes may be phrased using Serre quotients.

■ **Example 8.2.3** Let A be a $\mathbb{N}$-graded ring, which is finitely generated by A_1 as an A_0-algebra, and $X = \operatorname{Proj} A$. We denote by A-GrMod the category of graded A-modules M *such that* $\bigoplus_{d>n} M_d$ *is finite for some* n. The usual tilde functor

$$\begin{aligned} r : A\text{-GrMod} &\to \mathsf{QCoh}(X) \\ M &\mapsto \widetilde{M} \end{aligned}$$

is exact and its kernel, denoted by $A\text{-GrMod}_0$, is composed by the modules M satisfying $M_d = 0$ for all d large enough. [16, Proposition 2.7.3] The tilde functor r admits a right adjoint $\Gamma_\bullet$, defined by

$$\Gamma_\bullet(\mathscr{F}) := \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathscr{F}(n)),$$

which is fully faithful due to the fact that the counit

$$\widetilde{\Gamma_\bullet(\mathscr{F})} \to \mathscr{F}$$

is a natural isomorphism. Then, the formalism of example 8.1.2 implies that r factors through the quotient and that

$$A\text{-GrMod}/A\text{-GrMod}_0 \to \mathsf{QCoh}(X)$$

is an equivalence of categories. ■

8.3. Localization of triangulated categories

As we just saw, in good cases, the localization of an abelian category is still abelian. A similar theory exists for triangulated categories, and it forms the basis for the study of derived categories.

If (K, T) is a triangulated category and S is a multiplicative system, we already know that $S^{-1}K$ is additive. In order for $T : K \to K$ to descend to the localization making the diagram

$$\begin{tikzcd} K \arrow[r, "T"] \arrow[d, "Q"'] & K \arrow[d, "Q"] \\ S^{-1}K \arrow[r, dashed] & S^{-1}K \end{tikzcd}$$

commute, we need to impose that $T(s) \in S$ for all $s \in S$. We denote the induced functor $S^{-1}K \to S^{-1}K$ by T_S. If, moreover, $T^{-1}(s)$ lies in S whenever $s \in S$, then T^{-1} also descends to $S^{-1}K$ and defines an inverse to $T_S : S^{-1}K \to S^{-1}K$; proving that it's an additive isomorphism of categories. This motivates the first part of the definition below.

Definition 8.3.1 Let (K, T) be a triangulated category and consider the following axioms on a collection of morphisms S in K:

(TMS1) We have that $T^i(s) \in S$ for all $i \in \mathbb{Z}$, whenever $s \in S$.

(TMS2) Given a commutative diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d, "\lambda"] & M \arrow[r] \arrow[d, "\mu"] & N \arrow[r] & T(L) \\ L' \arrow[r] & M' \arrow[r] & N' \arrow[r] & T(L'), \end{tikzcd}$$

whose rows are distinguished triangles and whose columns are in S, there's a morphism $\nu : N \to N'$ in S making the diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d, "\lambda"] & M \arrow[r] \arrow[d, "\mu"] & N \arrow[r] \arrow[d, "\nu"] & T(L) \arrow[d, "T(\lambda)"] \\ L' \arrow[r] & M' \arrow[r] & N' \arrow[r] & T(L') \end{tikzcd}$$

commute.

A multiplicative system S satisfying the axioms above is said to be *compatible with the triangulated structure.*

The second part of this definition, a twisted form of the axiom TR3, will be necessary to prove that the localization $S^{-1}K$ still satisfies the axioms of a triangulated category. We remark in the next proposition that the axioms for a multiplicative system are not all independent.

Proposition 8.3.1 Let (K, T) be a triangulated category and S be a collection of morphisms in K. If S satisfies the axioms LMS1, TMS1 and TMS2, then it also satisfies LMS2. Similarly, if it satisfies RMS1, TMS1 and TMS2, it also satisfies RMS2.

Proof. ☐

Since we want the localization functor $Q : \mathsf{K} \to \mathsf{S}^{-1}\mathsf{K}$ to be triangulated, we have no choice but to declare a triangle in $\mathsf{S}^{-1}\mathsf{K}$ to be distinguished if it's isomorphic to the image by Q of a distinguished triangle in K.

Proposition 8.3.2 Let (K, T) be a triangulated category and S be a multiplicative system compatible with the triangulated structure in K. Then the aforementioned structure makes $\mathsf{S}^{-1}\mathsf{K}$ a triangulated category and $Q : \mathsf{K} \to \mathsf{S}^{-1}\mathsf{K}$ a triangulated functor.

Proof. ☐

As with (pre)additive and abelian categories, the localization of a triangulated category also inherits a better universal property.

Corollary 8.3.3 Let $F : \mathsf{K} \to \mathsf{K}'$ be a triangulated functor and S be a multiplicative system in K compatible with the triangulated structure. If $F(s)$ is an isomorphism whenever $s \in \mathsf{S}$, then the induced functor $\mathsf{S}^{-1}\mathsf{K} \to \mathsf{K}'$ is also triangulated.

Proof. ☐

We remark that the proof of the corollary 8.1.2 now implies that $(\mathsf{S}^{-1}\mathsf{K})^{\mathrm{op}}$ is isomorphic to the localization of K^{op} with respect to S^{op}.

The localization of a triangulated category also inherits a universal property with respect to cohomological functors.

Corollary 8.3.4 Let K be a triangulated category, A be an abelian category, and $H : \mathsf{K} \to \mathsf{A}$ be a cohomological functor. Suppose that S be a multiplicative system in K compatible with the triangulated structure. If $H(s)$ is an isomorphism whenever $s \in \mathsf{S}$, then the induced functor $\mathsf{S}^{-1}\mathsf{K} \to \mathsf{A}$ is also cohomological.

Proof. ☐

Recall that a multiplicative system S in a category C is said to be saturated if the morphisms in C that are sent to isomorphisms in $\mathsf{S}^{-1}\mathsf{C}$ are precisely the elements of S. The next proposition gives a source of (saturated) multiplicative systems in triangulated categories.

Proposition 8.3.5 Let (K, T) be a triangulated category, A be an abelian category, and $\mathsf{H} : \mathsf{K} \to \mathsf{A}$ be a cohomological functor. The collection

$$S := \left\{ s \text{ morphism in } \mathsf{K} \mid \mathsf{H}(\mathsf{T}^i(s)) \text{ is an isomorphism for all } i \in \mathbb{Z} \right\}$$

is a saturated multiplicative system compatible with the triangulated structure.

Proof. ☐

As with abelian categories, there's another point of view to the localization of triangulated categories that's occasionally useful.

Definition 8.3.2 — Thick subcategory. Let K be a triangulated category. We say that a triangulated subcategory C of K is *thick* if whenever $M \oplus N$ is isomorphic to an object of C, so are M and N.

Notice that the kernel (the full subcategory composed of the objects whose image is isomorphic to zero) of a triangulated functor F is always thick. Indeed, if $F(M \oplus N) = F(M) \oplus F(N)$ is zero, so are $F(M)$ and $F(N)$.

Somewhat similarly to the proposition 8.2.5, we have a dictionary between triangulated subcategories and multiplicative systems compatible with the triangulated structure. This next proposition is the only place, so far, where we need the axiom TR4 and it's Verdier's original motivation for it. [37, §2.2.12]

Proposition 8.3.6 Let K be a triangulated category. Given a multiplicative system compatible with the triangulation S in K, the kernel C_S of the localization functor $Q : \mathsf{K} \to S^{-1}\mathsf{K}$ is a triangulated subcategory of K. Conversely, given a triangulated subcategory C of K, the collection S_C of all morphisms in K whose cone is in C is a multiplicative system compatible with the triangulation.

Proof. ☐

Since the kernel of a triangulated functor is always a thick subcategory, there's no chance for the preceding proposition to yield a one-to-one correspondence between triangulated subcategories and multiplicative systems compatible with the triangulation. It does, however, yield a bijection when restricted to thick subcategories.

Corollary 8.3.7 We use the notations of the proposition above. If S is saturated, then C_S is thick. Moreover, the operations $\mathsf{C} \mapsto S_\mathsf{C}$ and $S \mapsto \mathsf{C}_S$ define a bijection between thick subcategories and saturated multiplicative systems compatible with the triangulation.

Proof. ☐

As with abelian categories, we define the *quotient* K/C of a triangulated category K by a (not necessarily thick) triangulated subcategory C as the localization $S_C^{-1}K$. These quotients are often called *Verdier quotients* in the literature.

We remark that the kernel of the localization / quotient functor $Q : K \to K/C$ *need* not *be* C. *It's actually the smallest thick subcategory of* K *containing* C. *In particular, the kernel is* C *if the latter is thick.*

The universal properties of corollaries 8.3.3 and 8.3.4 may be translated naturally to this point of view. A triangulated functor $F : K \to K'$ whose kernel contains C factors uniquely through the quotient K/C. A similar universal property holds for cohomological functors.

8.4. The derived category

Definition 8.4.1 cat derivada

■ **Example 8.4.1 — Semisimple categories.** ■

Proposition 8.4.1 provar que a inclusão natural $A \to D(A)$ identifica A com a subcategoria plena de $D(A)$ composta pelos complexos com cohomologia só em grau 0.

Proof. □

Proposition 8.4.2 seq exatas curtas em C(A) (em particular em A) viram triângulos em D(A)

Proof. □

Definition 8.4.2 $D_B(A)$

Proposition 8.4.3 $D_B(A)$ é uma subcategoria triangulada plena de $D(A)$.

Proof. □

8.5. Resolutions

Our main motivation for the definition of the homotopy category is that it is a stepping stone to the derived category, where all quasi-isomorphisms become invertible. Since homotopy equivalences are already invertible in the homotopy category, we may wonder if there are complexes for which quasi-isomorphisms and homotopy equivalences coincide.

We begin by recalling in the context of abelian categories the notions of projective and injective objects.

Definition 8.5.1 Let A be an abelian category. We say that an object P of A is *projective* if the functor $\mathrm{Hom}_{\mathsf{A}}(P,-)$ is exact. Similarly, an object Q is said to be *injective* if the functor $\mathrm{Hom}_{\mathsf{A}}(-,Q)$ is exact. We denote by $\mathsf{Proj}(\mathsf{A})$, resp. $\mathsf{Inj}(\mathsf{A})$, the full subcategory of A composed by the projective, resp. injective, objects.

As projective objects become injective in the opposite category, we'll mainly consider injective objects in this section; the statements about projective objects will follow by duality. We remark, however, that projective and injective objects in a given abelian category may behave very differently.

There are two usual ways to rephrase the definition above. For the first, recall that the Hom functor preserves limits in both variables. (Proposition 6.2.2 in [23].) The proposition 6.5.4 then implies that both its covariant and contravariant forms are left exact. In particular, an object Q is injective if and only if $\mathrm{Hom}_{\mathsf{A}}(-,Q)$ is right exact. This yields the proposition below.

Proposition 8.5.1 An object Q of an abelian category A is injective if and only if, for every monomorphism $\varphi : A \to B$, any morphism $A \to Q$ factors through φ:

$$\begin{tikzcd}
 & Q & \\
0 \arrow[r] & A \arrow[u] \arrow[r, "\varphi"'] & B. \arrow[ul, dashed]
\end{tikzcd}$$

Proof. By our discussion above, Q is injective if and only if whenever a sequence

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\varphi"] & B
\end{tikzcd}$$

is exact, then so is its image through $\mathrm{Hom}_{\mathsf{A}}(-,Q)$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{A}}(B,Q) \arrow[r] & \mathrm{Hom}_{\mathsf{A}}(A,Q) \arrow[r] & 0.
\end{tikzcd}$$

But epimorphisms in Ab are precisely the surjective maps. This is exactly the condition of the proposition. ☐

The splitting lemma (theorem 6.4.1) gives yet another characterization of injective objects.

Corollary 8.5.2 An object Q of an abelian category A is injective if and only if every short exact sequence of the form

$$0 \longrightarrow Q \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

splits.

Proof. The splitting lemma says that our short exact sequence splits if and only if there exists a morphism $\rho : B \to Q$ such that $\rho \circ \varphi = \mathrm{id}_Q$. If Q is injective, the preceding proposition gives the desired $\rho : B \to Q$:

$$\begin{tikzcd}
 & Q & \\
0 \arrow[r] & Q \arrow[u, "\mathrm{id}_Q"] \arrow[r, "\varphi"'] & B. \arrow[ul, dashed, "\rho"']
\end{tikzcd}$$

Conversely, let $A \to B$ be a monomorphism and $A \to Q$ be any morphism. By taking their pushout P (and using the corollary 6.3.4), we obtain the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r] & P \\
0 \arrow[r] & A \arrow[u] \arrow[r] & B. \arrow[u]
\end{tikzcd}$$

Since $Q \to P$ fits into a short exact sequence (with its cokernel), we obtain a morphism $P \to Q$ giving us the condition of the proposition above. ☐

Definition 8.5.2 K-injective / projective complexes

Proposition 8.5.3

Theorem 8.5.4 os Hom da cat derivada são iguais aos da cat de homotopia.

Proposition 8.5.5 injetivo implica k-injetivo

Definition 8.5.3 Resolution

Theorem 8.5.6 existencia de resoluções para D(A) onde A é Grothendieck; logo D(A) é equivalente a K(K-injetivos)

Corollary 8.5.7 nesse contexto D(A) é localmente pequena (talvez eu tenha que provar isso antes)

9. Derived functors

9.1. The 2-categorical point of view

9.2. Derived functors

9.3. Deformations

9.4. Tor and Ext

9.5. Spectral sequences

10. Existence of resolutions

10.1. Grothendieck categories

10.2. Brown representability

10.3. Gabriel-Popescu

Theorem 10.3.1 — Gabriel-Popescu.

10.4. Existence of resolutions for modules

10.5. Existence of resolutions in general

Part III.

Sheaf Theory

11. Sheaves and presheaves

Sheaf theory was conceived by the French mathematician Jean Leray while he was held captive in the German camp Oflag XVII as a prisoner of war. There he gave a course in algebraic topology introducing the ideas of sheaves and spectral sequences. In loose terms, a sheaf is a tool for systematically tracking locally defined data attached to the open sets of a topological space.

Sheaf theory forms an abstract machinery which describes much of modern mathematics. We'll be able to put topological and smooth manifolds, complex manifolds, schemes, topological coverings, vector bundles and a myriad of other objects in the same standpoint.

11.1. Presheaves

Before any abstract definitions, let us consider the prototypical example of a sheaf: the sheaf of continuous maps on a topological space X. For every open set $U \subset X$, our space has a naturally associated ring $C(U)$ consisting of the continuous functions $U \to \mathbb{R}$. Note that if V is an open set contained in U, then the restriction $f|_V$ of a continuous function $f : U \to \mathbb{R}$ is again continuous. In other words, we have a morphism of rings given by

$$\begin{aligned} \mathrm{res}_{U,V} : C(U) &\to C(V) \\ f &\mapsto f|_V. \end{aligned}$$

Moreover, if $W \subset V \subset U$ are three open sets, it is clear that restricting a function $f \in C(U)$ to V and then restricting to W is the same thing as restricting directly to W. In other words, the restriction maps satisfy $\mathrm{res}_{U,W} = \mathrm{res}_{V,W} \circ \mathrm{res}_{U,V}$. The data of all the rings $C(U)$ along with their restriction maps $\mathrm{res}_{U,V}$ constitutes the *sheaf of continuous functions on* X.

Generalizing these notions we obtain the concept of a *presheaf*.

Definition 11.1.1 — Presheaf. Let X be a topological space. We denote by Open_X the category whose objects are open sets $U \subset X$, and whose morphisms are the inclusion maps. Then, a *presheaf* $\mathscr{F}$ over X with values in a concrete category C is a contravariant functor $\mathscr{F} : \mathrm{Open}_X \to \mathsf{C}$.

Let's unpack this definition. If U is an open set in X, then we are given an object $\mathscr{F}(U)$ of C whose elements will usually be called *sections* of $\mathscr{F}$ over U. Moreover, if $V \subset U$ is a pair of nested open sets, we have a morphism $\mathrm{res}_{U,V} : \mathscr{F}(U) \to \mathscr{F}(V)$ in C which is called a *restriction map*. The functor axioms amount to the conditions that $\mathrm{res}_{U,U}$ ought to be the identity map of $\mathscr{F}(U)$ and that, if $W \subset V \subset U$ are three nested open sets, then $\mathrm{res}_{U,W} = \mathrm{res}_{V,W} \circ \mathrm{res}_{U,V}$.

Naturally, if $s \in \mathscr{F}(U)$ we'll usually write $s|_V$ for $\mathrm{res}_{U,V}(s)$. We'll also write $\Gamma(U, \mathscr{F})$ for the object $\mathscr{F}(U)$ of C and we'll say that an element of $\Gamma(X, \mathscr{F})$ is a *global section*. All this terminology can be explained by the following example.

■ **Example 11.1.1 — Vector bundle.** Let $\pi : E \to X$ be a vector bundle of topological spaces and U an open set of X. A *section* s of π over U is a continuous function $s : U \to E$ such that $\pi \circ s = \mathrm{id}_U$. The data of all the sections s over all open sets U forms a presheaf of rings on X. ■

There's a key aspect of our prototypical example which is not present in the definition of a presheaf: the fact that being continuous is a local property. More precisely, if $\{U_i\}$ is an open cover of an open set $U \subset X$, then functions $f_i \in C(U_i)$ which coincide on the intersections $U_i \cap U_j$ can be *glued* to a unique function $f \in C(U)$ such that $f|_{U_i} = f_i$ for all i. Indeed, it suffices to define $f(x)$ as being $f_i(x)$ whenever $x \in U_i$. This is the content of the definition below.

Definition 11.1.2 — Sheaf. A presheaf $\mathscr{F}$ over a topological space X with values in a concrete category with products C is a *sheaf* if, whenever $U \subset X$ is an open set and $\{U_i\}$ is an open cover of U, the product of the restrictions $\alpha : \mathscr{F}(U) \to \prod_i \mathscr{F}(U_i)$ is the equalizer of

$$\prod_i \mathscr{F}(U_i) \overset{\beta_1}{\underset{\beta_2}{\rightrightarrows}} \prod_{ij} \mathscr{F}(U_i \cap U_j),$$

where β_1, β_2 are defined by $\beta_1((s_i)_i) = (s_i|_{U_i \cap U_j})_{i,j}$ and $\beta_2((s_i)_i) = (s_j|_{U_i \cap U_j})_{i,j}$.

Before we go any further, our abstract formalism deserves a categorical digression. In basically all interesting cases, we'll deal with categories of models of *algebraic theories*. An algebraic theory is characterized by the existence of one or several operations which are defined everywhere and satisfy axioms expressed by equalities. In particular, Set, Grp, Ab, Ring, CRing, A-Mod and A-Alg are categories of models of algebraic theories but the categories of fields or of integral domains aren't. These categories satisfy a myriad of important properties which we know axiomatize.

Definition 11.1.3 — Algebraic category. An *algebraic category* is a concrete category C whose underlying set functor u creates limits, filtered colimits and reflects isomorphisms.

In other words, limits and filtered colimits always exist in C and are given by *any* construction that maps to the usual construction in Set. Particularly, C has a final object $\{*\}$ whose underlying set is a singleton. Also, a presheaf $\mathscr{F}$ is a sheaf if and only if the underlying presheaf of sets (that is, $u \circ \mathscr{F}$) is a sheaf. Indeed, suppose that $u \circ \mathscr{F}$ is a sheaf of sets. I.e., that

$$\begin{tikzcd} u(\mathscr{F}(U)) \arrow[r, "u(\alpha)"] & \prod_i u(\mathscr{F}(U_i)) \arrow[r, shift left, "u(\beta_1)"] \arrow[r, shift right, "u(\beta_2)"'] & \prod_{ij} u(\mathscr{F}(U_i \cap U_j)) \end{tikzcd}$$

is an equalizer diagram. If E is the equalizer of β_1 and β_2 in C, there's a canonical morphism $\mathscr{F}(U) \to E$ by the universal property of the equalizer. But $u(\mathscr{F}(U)) \to u(E)$ is an isomorphism and u reflects isomorphisms. We conclude that $\mathscr{F}$ is a sheaf. The converse follows simply by functoriality. One can read about all of this in [4]. Hereafter, we'll deal exclusively with algebraic categories.

Now, without further ado, let's understand how this definition encodes our intuition that a sheaf should be defined by local data. If $\{*\}$ is the final object of C, then the diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\alpha"] & \prod_i \mathscr{F}(U_i) \arrow[r, shift left, "\beta_1"] \arrow[r, shift right, "\beta_2"'] & \prod_{ij} \mathscr{F}(U_i \cap U_j) \\ \{*\} \arrow[ur, "\gamma"'] & & \end{tikzcd}$$

commutes precisely when the sections in the image of $*$ by γ coincide on the intersections. Accordingly, the existence of a map $\{*\} \to \mathscr{F}(U)$ which makes the diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\alpha"] & \prod_i \mathscr{F}(U_i) \arrow[r, shift left, "\beta_1"] \arrow[r, shift right, "\beta_2"'] & \prod_{ij} \mathscr{F}(U_i \cap U_j) \\ \{*\} \arrow[u] \arrow[ur, "\gamma"'] & & \end{tikzcd}$$

commute means that there exists a section $s \in \mathscr{F}(U)$ (the image of $*$ in $\mathscr{F}(U)$) such that $s|_{U_i} = s_i$ for all i. The section is unique exactly when there is a unique such morphism. In other words, a presheaf $\mathscr{F}$ is a sheaf when given sections $s_i \in \mathscr{F}(U_i)$ which coincide on the intersections, there is *a unique* $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

More often than not, the verification that a presheaf is indeed a sheaf comes in two steps: we first verify that sections which coincide on the intersections glue and we verify that there's at most one way of doing so. This translates in the following axioms which a sheaf must satisfy.

i. (Identity axiom) if $s, t \in \mathscr{F}(U)$ coincide over U_i for all i, then $s = t$;

ii. (Gluability axiom) if $s_i \in \mathscr{F}(U_i)$ coincide on the intersections $U_i \cap U_j$, then there is some $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

We make some final observations. Firstly, since the empty product is a final object of C, the equalizer diagram associated with the empty covering shows that a sheaf necessarily satisfies that $\mathscr{F}(\varnothing)$ is the final object of C. In particular, $\mathscr{F}(\varnothing)$ is a singleton. Also, when C is an abelian category, we can rephrase the definition of a sheaf slightly by saying that a presheaf $\mathscr{F}$ over X is a sheaf if and only if the sequence

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\alpha} \prod_i \mathscr{F}(U_i) \xrightarrow{\beta} \prod_{ij} \mathscr{F}(U_i \cap U_j),$$

where $\beta = \beta_1 - \beta_2$, is exact whenever $U \subset X$ is an open set and $\{U_i\}$ is an open cover of U. The identity axiom is encoded by imposing α to be a monomorphism and the gluability axiom is encoded by $\ker \beta = \operatorname{im} \alpha$.

It is clear that the presheaves of continuous maps on a topological space and of sections of a vector bundle are indeed sheaves. We now present some other examples and non-examples.

■ **Example 11.1.2 — Constant presheaf.** Let X be a topological space, and S a set. We define a presheaf of sets $\mathscr{F}$ over X by declaring $\mathscr{F}(U)$ to be S for every open set $U \subset X$. It is a presheaf where the restriction maps are simply the identity function.

If S is not a singleton, this is not a sheaf since $\mathscr{F}(\varnothing)$ is not a final object of Set. This fails to be a sheaf even if we declare $\mathscr{F}(\varnothing) = \{*\}$, where $\{*\}$ is a one-element set. In fact, consider $X = \{a, b\}$ with the discrete topology and suppose that S has more than one element. Since $\{a\}$ and $\{b\}$ form an open cover of X,

$$\begin{tikzcd} \mathscr{F}(X) \arrow[r] & \mathscr{F}(\{a\}) \times \mathscr{F}(\{b\}) \arrow[r, shift left] \arrow[r, shift right] & \varnothing \end{tikzcd}$$

should be an equalizer diagram. In other words, we should have that $\mathscr{F}(\{a\} \cup \{b\}) = \mathscr{F}(\{a\}) \times \mathscr{F}(\{b\})$, which does not happen. ■

Let's see how we could define a sheaf over $X = \{a, b\}$. Giving a presheaf of sets on X amounts to choosing sets $\mathscr{F}(\varnothing)$, $\mathscr{F}(\{a\})$, $\mathscr{F}(\{b\})$ and $\mathscr{F}(X)$ and restriction maps

$$\begin{tikzcd} \mathscr{F}(X) \arrow[r] \arrow[d] & \mathscr{F}(\{b\}) \arrow[d] \\ \mathscr{F}(\{a\}) \arrow[r] & \mathscr{F}(\varnothing) \end{tikzcd}$$

(All the other restriction maps are determined by those four.) If $\mathscr{F}$ is to be a sheaf, then $\mathscr{F}(\varnothing)$ has to be a singleton $\{*\}$. This determines the restriction maps $\mathscr{F}(\{a\}), \mathscr{F}(\{b\}) \to \{*\}$. Now, let $\mathscr{F}(\{a\}) = S_1$ and $\mathscr{F}(\{b\}) = S_2$. As before, this forces $\mathscr{F}(X)$ to be $S_1 \times S_2$. The construction below is what we get setting $S = S_1 = S_2$.

■ **Example 11.1.3 — Constant sheaf.** We endow S with the discrete topology and define a sheaf $\underline{S}$ whose sections over an open set $U \subset X$ are the continuous maps $U \to S$. This is the *constant sheaf* associated with S. More generally, if S is an object in some concrete category C, then $\underline{S}$ is a sheaf with values in C. ■

We now see how nonlocal notions lead to presheaves which are not sheaves.

■ **Example 11.1.4 — Presheaf of bounded functions.** Let X be a topological space. For every open set $U \subset X$, we define $B(U)$ to be the set of bounded functions $U \to \mathbb{R}$. This determines a presheaf of sets on X. Nevertheless, it usually isn't a sheaf. For example, when $X = \mathbb{R}$ the inclusions $(-n, n) \hookrightarrow \mathbb{R}$, for $n \geqslant 1$, are bounded but don't glue in order to form a bounded function on X. ■

The same exact phenomenon arises in the presheaf of L^1 functions on the real line, with its usual Lebesgue measure. This presheaf satisfies the identity axiom but fails to satisfy the gluability axiom.

For a more exotic example, where the restriction maps are not exactly the restriction of functions, we turn to the theory of distributions. The interested reader may check the beautiful book [7].

■ **Example 11.1.5 — Sheaf of distributions.** Let $X = \mathbb{R}^n$. Recall that, for every open set $U \subset \mathbb{R}^n$, $\mathcal{D}(U)$ is the set $C_c^\infty(U)$, of all compactly supported smooth functions on U, with its natural locally convex topology. Assigning $\mathcal{D}'(U)$, the continuous dual of $\mathcal{D}(U)$, to each open set $U \subset \mathbb{R}^n$ defines a presheaf where the restriction is given by

$$\begin{aligned} \mathrm{res}_{U,V} : \mathcal{D}'(U) &\to \mathcal{D}'(V) \\ u &\mapsto u \circ E_{U,V}, \end{aligned}$$

where $E_{U,V} : \mathcal{D}(V) \to \mathcal{D}(U)$ is the operator which extends by zero a given smooth function compactly supported in V to a smooth function compactly supported in U.

One can prove that $\mathcal{D}'$ is indeed a sheaf of locally convex topological vector spaces. Similarly, the tempered distributions $\mathcal{S}'$ form a presheaf. Observe that $\mathcal{S}'$ is a subpresheaf of $\mathcal{D}'$ when both are regarded as presheaves of vector spaces but it is *not* a subpresheaf when they are regarded as presheaves of locally convex spaces since the topology of $\mathcal{S}'(U)$ is not the topology inherited by $\mathcal{D}'(U)$. The tempered distributions also don't form a sheaf, since being in the Schwartz space is not a local property. ■

Those interested in knowing more about the interpretation of distributions (in $\mathbb{R}^n$ or even in manifolds) as presheaves can check [6], which has everything that one might ever want.

11.2. Morphisms and stalks

In this section we'll define morphisms of presheaves and sheaves, establishing their categories. Our definition of a presheaf as a functor makes it clear what a morphism of presheaves should be.

Definition 11.2.1 — Morphism of (pre)sheaves. Let $\mathscr{F}$ and $\mathscr{G}$ be presheaves over a topological space X with values in a category C. A *morphism* $\varphi : \mathscr{F} \to \mathscr{G}$ is nothing but a natural transformation of functors. Explicitly, it is the data of maps $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$, for all open sets $U \subset X$, such that the diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\mathrm{res}_{U,V}"'] & \mathscr{G}(U) \arrow[d, "\mathrm{res}_{U,V}"] \\
\mathscr{F}(V) \arrow[r, "\varphi_V"] & \mathscr{G}(V)
\end{tikzcd}$$

commutes whenever $V \subset U$ is a pair of nested open sets. We denote the category of presheaves over X with values in C by C_X^{pre}. A morphism of sheaves is just a morphism of the underlying presheaves. In other words, sheaves over X with values in C form a full subcategory C_X of C_X^{pre}.

As usual, a map $\varphi : \mathscr{F} \to \mathscr{G}$ between presheaves is an isomorphism if it has a two-sided inverse. That is, if there exists a map $\psi : \mathscr{G} \to \mathscr{F}$ such that $\varphi \circ \psi = \mathrm{id}_{\mathscr{G}}$ and $\psi \circ \varphi = \mathrm{id}_{\mathscr{F}}$. In this case we can evaluate these identities at an open set $U \subset X$ to obtain that if φ is an isomorphism then so is φ_U for each U. Conversely, if each φ_U is an isomorphism, let ψ_U be their corresponding inverses. Then the diagram

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[r, "\psi_U"] \arrow[d, "\mathrm{res}_{U,V}"'] & \mathscr{F}(U) \arrow[d, "\mathrm{res}_{U,V}"] \\
\mathscr{G}(V) \arrow[r, "\psi_V"] & \mathscr{F}(V)
\end{tikzcd}$$

commutes since $\varphi_V \circ \mathrm{res}_{U,V} = \mathrm{res}_{U,V} \circ \varphi_U$ implies that $\mathrm{res}_{U,V} \circ \psi_U = \psi_V \circ \mathrm{res}_{U,V}$ by composing on the left by ψ_V and on the right by ψ_U. In other words, we conclude that $\varphi : \mathscr{F} \to \mathscr{G}$ is an isomorphism if and only if $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is an isomorphism for every open set $U \subset X$. We shall investigate later the extent to which this result is true with regard to monomorphisms and epimorphisms.

For the next definition, we observe that we can naturally *restrict* presheaves in the following way: if $\mathscr{F}$ is a presheaf over a topological space X and $U \subset X$ is an open set, then we can define $\mathscr{F}|_U(V) = \mathscr{F}(V)$ for every open set $V \subset U$.

Definition 11.2.2 — (Pre)sheaf Hom. Let $\mathscr{F}$ and $\mathscr{G}$ be presheaves over a topological space X with values in a category C. We define a presheaf $\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})$ as

$$\Gamma(U, \underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})) = \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U),$$

where the right side is composed by the morphisms of presheaves between $\mathscr{F}|_U$ and $\mathscr{G}|_U$. If $\varphi \in \Gamma(U, \underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G}))$, its restriction to an open set $V \subset U$ is defined to be the data of maps $\varphi_W : \mathscr{F}(W) \to \mathscr{G}(W)$ for every open set $W \subset V$.

The definition of $\Gamma(U, \underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G}))$ *is* not $\mathrm{Hom}_{C}(\mathscr{F}(U), \mathscr{G}(U))$. *The latter does not admit natural restriction maps so this is not even a presheaf.*

We could prove right away that $\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})$ is a sheaf if $\mathscr{G}$ is, and we encourage the reader to do it. But we'll leave it to the future, where this result fits nicely within the theory.

There's also the natural notion of subpresheaf.

Definition 11.2.3 — Subpresheaf. Let $\mathscr{F}$ be a presheaf over a topological space X. A *subpresheaf* of $\mathscr{F}$ is a presheaf $\mathscr{G}$ such that $\mathscr{G}(U)$ is a subobject of $\mathscr{F}(U)$ for every open set $U \subset X$ and such that the diagram

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[r, hook] \arrow[d, "\mathrm{res}_{U,V}"'] & \mathscr{F}(U) \arrow[d, "\mathrm{res}_{U,V}"] \\
\mathscr{G}(V) \arrow[r, hook] & \mathscr{F}(V)
\end{tikzcd}$$

commutes for every pair of nested open sets $V \subset U$. In other words, the restriction maps of $\mathscr{G}$ are the restriction of the restriction maps of $\mathscr{F}$. A *subsheaf* is a subpresheaf which satisfies the same conditions and is moreover a sheaf.

We present a simple example.

■ **Example 11.2.1** Let $X = \mathbb{R}$, and let C^r be the subsheaf of the sheaf of continuous functions on X consisting of the functions which are r times continuously differentiable. The differential operator $D = d/dx$ defines a map of sheaves of $\mathbb{R}$-vector spaces $D : C^r \to C^{r-1}$. We observe that this is *not* a morphism of sheaves of $\mathbb{R}$-algebras. ■

By its very definition, sections on a sheaf are defined by local data. We will explore this property using the notion of *stalk*, which capture all the important information in a neighborhood of a point.

Definition 11.2.4 — Stalk. Let $\mathscr{F}$ be a presheaf over a topological space X with values in an algebraic category C. We define the *stalk* of $\mathscr{F}$ at $p \in X$ as the (filtered) colimit of all $\mathscr{F}(U)$ over open sets $U \subset X$ containing p:

$$\mathscr{F}_p := \operatorname{colim} \mathscr{F}(U).$$

The elements of $\mathscr{F}_p$ are called *germs*. If $p \in U$ and $s \in \mathscr{F}(U)$, the image s_p of s in $\mathscr{F}_p$ is said to be the *germ* of s at p.

Concretely, under our supposition over C, the filtered colimit which defines the stalk can be constructed as

$$\left(\coprod_{U \ni x} \mathscr{F}(U) \right) \Big/ \sim,$$

where $(s, U) \sim (t, V)$[1] if there exists an open set $W \subset U \cap V$ containing p such that $s|_W = t|_W$. Indeed, this is nothing but the construction of a filtered colimit in Set.

The stalks illustrate clearly the interest of considering sheaves instead of presheaves. For example, sections of a sheaf are determined by its germs. Precisely, this means that if $s, t \in \mathscr{F}(U)$ are such that $s_p = t_p$ for all $p \in U$, then $s = t$. Indeed, if $s_p = t_p$, then s and t coincide over a neighborhood of p. If this happens for every $p \in U$, then there's an open cover $\{U_i\}$ of U such that $s|_{U_i} = t|_{U_i}$ for every i. The identity axiom then implies that $s = t$. In other words, the natural map

$$\mathscr{F}(U) \to \prod_{p \in U} \mathscr{F}_p$$

is injective (in particular, it is monic).

Now, if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, the very definition of morphism implies that the diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[rr, "\varphi_U"] \arrow[dd] \arrow[dr] & & \mathscr{G}(U) \arrow[dd] \arrow[dr] & \\
& \mathscr{F}_p \arrow[rr, dashed] & & \mathscr{G}_p \\
\mathscr{F}(V) \arrow[rr, "\varphi_V"] \arrow[ur] & & \mathscr{G}(V) \arrow[ur] &
\end{tikzcd}$$

commutes and so, by the universal property of colimits, there is a unique induced map $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$, which is given in our construction by

$$\begin{aligned} \varphi_p : \mathscr{F}_p &\to \mathscr{G}_p \\ \overline{(s, U)} &\mapsto \overline{(\varphi_U(s), U)}. \end{aligned}$$

The assignment $\mathscr{F} \mapsto \mathscr{F}_p$ and $\varphi \mapsto \varphi_p$ defines a functor $\mathsf{C}_X^{\text{pre}} \to \mathsf{C}$. Also, by restriction we obtain a functor $\mathsf{C}_X \to \mathsf{C}$.

We now answer a question that we left open not long ago.

Proposition 11.2.1 Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves over X with values in C. Then the following are equivalent:

(a) φ is a monomorphism in C_X;

(b) φ_p is a monomorphism in C for every $p \in X$;

(c) φ_U is a monomorphism in C for every open set $U \subset X$.

---

[1] Formally this should be denoted as $(s, \mathscr{F}(U))$ but (s, U) is the usual notation.

Proof. Recall that $\varphi : \mathscr{F} \to \mathscr{G}$ is a monomorphism if and only if

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\mathrm{id}_{\mathscr{F}}"] \arrow[d, "\mathrm{id}_{\mathscr{F}}"'] & \mathscr{F} \arrow[d, "\varphi"] \\ \mathscr{F} \arrow[r, "\varphi"] & \mathscr{G} \end{tikzcd}$$

is a pullback square. Since filtered colimits commute with finite limits[2], this shows that (a) implies (b). Supposing (b), let $U \subset X$ be an open set and consider the commutative diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[d, hook] & \mathscr{G}(U) \arrow[d, hook] \\ \prod_{p \in U} \mathscr{F}_p \arrow[r] & \prod_{p \in U} \mathscr{G}_p, \end{tikzcd}$$

where the arrow at the bottom is the product of the stalks of φ. By supposition, the vertical arrows and the bottom one are monomorphisms. It follows that φ_U is monic, proving (c). Finally, we suppose that (c) holds and consider the parallel morphisms

$$\begin{tikzcd} \mathscr{H} \arrow[r, shift left, "\psi"] \arrow[r, shift right, "\psi'"'] & \mathscr{F} \arrow[r, "\varphi"] & \mathscr{G}. \end{tikzcd}$$

Since φ_U is a monomorphism in C for every open set $U \subset X$, it follows that

$$\begin{tikzcd} \mathscr{H}(U) \arrow[r, shift left, "\psi_U"] \arrow[r, shift right, "\psi'_U"'] & \mathscr{F}(U) \arrow[r, "\varphi_U"] & \mathscr{G}(U) \end{tikzcd}$$

are parallel morphisms in C and so $\psi'_U = \psi_U$ for every U. Then we have that $\psi' = \psi$ and so φ is a monomorphism in C_X. ☐

In the preceding proof, we used in an essential way the fact that $\mathscr{F}(U) \to \prod_{p \in U} \mathscr{F}_p$ is a monomorphism. This suggests that it may not be true that if φ_p is an epimorphism for every $p \in X$ then φ_U is an epimorphism for every open set $U \subset X$.

Proposition 11.2.2 Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves over X with values in C. Then the following are equivalent:

(a) φ is an epimorphism in C_X;

(b) φ_p is an epimorphism in C for every $p \in X$.

Proof. As in the last proposition, recall that $\varphi : \mathscr{F} \to \mathscr{G}$ is an epimorphism if and only if

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & \mathscr{G} \arrow[d, "\mathrm{id}_{\mathscr{G}}"] \\ \mathscr{G} \arrow[r, "\mathrm{id}_{\mathscr{G}}"] & \mathscr{G} \end{tikzcd}$$

---

[2] This is true in Set and thus in C under our hypothesis.

is a pushout square. Since colimits commute with colimits, this shows that (a) implies (b). Conversely, let

$$\begin{tikzcd}\mathscr{F} \arrow[r, "\varphi"] & \mathscr{G} \arrow[r, shift left, "\psi"] \arrow[r, shift right, "\psi'"'] & \mathscr{H}.\end{tikzcd}$$

be parallel morphisms. Taking stalks we obtain that $\psi'_p = \psi_p$ for every $p \in X$. Since the diagram

$$\begin{tikzcd}\mathscr{F}(U) \arrow[r, hook] \arrow[d, shift right, "\psi'_U"'] \arrow[d, shift left, "\psi_U"] & \prod_{p\in U}\mathscr{F}_p \arrow[d] \\ \mathscr{G}(U) \arrow[r, hook] & \prod_{p\in U}\mathscr{G}_p,\end{tikzcd}$$

where the arrow on the right is the product of the $\psi'_p = \psi_p$ for every $p \in U$, commutes for every $U \subset X$, it follows that $\psi'_U = \psi_U$ and so $\psi' = \psi$. ☐

Monomorphisms and epimorphisms in $\mathsf{C}_X^{\text{pre}}$ *have somewhat different descriptions. As we'll later see, a morphism* φ *of presheaves is monic (resp. epic) if and only if* φ_U *is monic (resp. epic) for every open set* U.

We now observe that the same proof that we used in proposition 11.2.1 shows that if φ_U is an epimorphism in C for every open set $U \subset X$, then φ is an epimorphism in C_X. Nevertheless, there do exist epimorphisms φ such that φ_U is not epic for some $U \subset X$.

■ **Example 11.2.2** Let $\mathscr{F}$ be the sheaf of sets constituted of nonvanishing continuous functions on $\mathbb{C}$ and consider the morphism $\varphi : \mathscr{F} \to \mathscr{F}$ which sends a function f to its square f^2. Since $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is the universal cover of $\mathbb{C} \setminus \{0\}$, $\varphi_U : \mathscr{F}(U) \to \mathscr{F}(U)$ is an epimorphism when U is simply connected. In particular, locally every function has a square root and so φ_p is an epimorphism for every $p \in \mathbb{C}$. It follows that φ is an epimorphism but φ_U is not necessarily epic. For example, when $U = \mathbb{C} \setminus \{0\}$. ■

We'll soon see that this "failure" is precisely where *sheaf cohomology* is born.

11.3. Sheafification

When dealing with sheaves of abelian groups, A-modules or, more generally, with values in any abelian category, we'll wish to extend some natural notions to the category of sheaves. For example, if $\mathscr{F}$ and $\mathscr{G}$ are sheaves of abelian groups, then

$$(\mathscr{F} \oplus \mathscr{G})(U) := \mathscr{F}(U) \oplus \mathscr{G}(U) \qquad \text{for every open set } U$$

defines a sheaf. Most other constructions don't have the same luck. The presheaf defined by $\mathscr{F}(U) \otimes \mathscr{G}(U)$ may not be a sheaf even if $\mathscr{F}$ and $\mathscr{G}$ are. Images, quotients and infinite direct sums also suffer from the same fate. For dealing with all those cases, we need a systematic way of constructing sheaves from presheaves. The following universal property describes this process.

Definition 11.3.1 — Sheafification. Let $\mathscr{F}$ be a presheaf over a topological space X with values in C. The *sheafification* of $\mathscr{F}$ is the initial map among the morphisms $\mathscr{F} \to \mathscr{G}$ where $\mathscr{G}$ is a sheaf. In other words, the sheafification is a morphism $\mathscr{F} \to \widetilde{\mathscr{F}}$, where $\widetilde{\mathscr{F}}$ is a sheaf, such that for every other sheaf $\mathscr{G}$ and every morphism $\mathscr{F} \to \mathscr{G}$ there exists a unique morphism $\widetilde{\mathscr{F}} \to \mathscr{G}$ such that the diagram

$$\begin{tikzcd}
\mathscr{F} \arrow[r] \arrow[dr] & \widetilde{\mathscr{F}} \arrow[d, dashed] \\
 & \mathscr{G}
\end{tikzcd}$$

commutes.

As usual, if it exists, the sheafification is unique up to a unique isomorphism. If $\mathscr{F}$ is already a sheaf, the identity morphism $\mathrm{id} : \mathscr{F} \to \mathscr{F}$ clearly satisfies the universal property of the sheafification. Finally, we observe that the assignment $\mathscr{F} \mapsto \widetilde{\mathscr{F}}$ defines a functor. Indeed, if $\mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, the universal property gives a induced morphism

$$\begin{tikzcd}
\mathscr{F} \arrow[r] \arrow[d] & \widetilde{\mathscr{F}} \arrow[d, dashed] \\
\mathscr{G} \arrow[r] & \widetilde{\mathscr{G}}
\end{tikzcd}$$

Hereafter, we'll refer to both the morphism $\mathscr{F} \to \widetilde{\mathscr{F}}$ and the sheaf $\widetilde{\mathscr{F}}$ as the sheafification of $\mathscr{F}$.

Even before we construct the sheafification, let's understand what may preclude a presheaf from being a sheaf. First of all, a presheaf can possibly not satisfy the gluability axiom. That is, it can have local sections that don't glue to form a global section.

■ **Example 11.3.1** Let S^1 be the unit circle and $p, q \in S^1$ two distinct points. Consider the presheaf of continuous functions on S^1 which have the same image at p and q. If U is a neighborhood of p not containing q, the sections over U are nothing but the continuous functions on U. In this fashion, we can have two sections f and g, defined respectively over neighborhoods of p and q, such that $f(p) \neq g(q)$. These sections cannot possibly glue to a global section. ■

Whenever this happens, what the sheafification does is simply to add the missing sections. For example, the sheafification of the presheaf considered above is the sheaf of continuous functions on S^1. Similarly, the sheafification of the presheaf of bounded functions in example 11.1.4 is the sheaf constituted of locally bounded functions.

The only other possibility for a presheaf to not be a sheaf is if it does not satisfy the identity axiom. That is, if there are distinct sections which are locally equal.

▪ **Example 11.3.2** Let X be a topological space. We define a presheaf $\mathscr{F}$ over X by imposing $\mathscr{F}(X) = \{a, b\}$ and $\mathscr{F}(U) = \{a\}$, whenever $U \neq X$. A restriction map $\mathrm{res}_{U,V} : \mathscr{F}(U) \to \mathscr{F}(V)$ is simply the constant function if $V \neq X$ and, of course, $\mathrm{res}_{X,X} = \mathrm{id}_X$. If $\{U_i\}$ is an open cover of X composed by proper subsets, then a and b are different global sections which coincide over U_i, for all i. In other words, $\mathscr{F}$ fails the identity axiom. ▪

Here the sheafification removes the unnecessary sections. This last case is a little less relevant since it is the gluability axiom that often fails. For example, if $\mathscr{F}$ is a presheaf whose sections over an open set U are functions from U to a fixed set Y and whose restriction maps are the restrictions of those functions, then $\mathscr{F}$ satisfies the identity axiom.

In order to explicitly construct the sheafification, we observe that if $\mathscr{F}$ is a subpresheaf of a sheaf $\mathscr{H}$, then its sheafification is almost tautological.

Lemma 11.3.1 Let $\mathscr{F}$ be a subpresheaf of a sheaf $\mathscr{H}$. Then the sheafification of $\mathscr{F}$ is the subsheaf of $\mathscr{H}$ defined by

$$\widetilde{\mathscr{F}}(U) = \{s \in \mathscr{H}(U) \mid s \text{ locally lies in } \mathscr{F}\},$$

where we say that a section $s \in \mathscr{H}(U)$ locally lies in $\mathscr{F}$ if there exists an open cover $\{U_i\}$ of U such that $s|_{U_i} \in \mathscr{F}(U_i)$ for every i.

Proof. First of all, we observe that $\widetilde{\mathscr{F}}$ is indeed a sheaf. The identity axiom is satisfied since it is satisfied by $\mathscr{H}$. Furthermore, if $\{U_i\}$ is an open cover of U and we have sections $s_i \in \widetilde{\mathscr{F}}(U_i)$ which coincide on the intersections, then there is some $s \in \mathscr{H}(U)$ such that $s|_{U_i} = s_i$ for all i. But s locally lies in $\mathscr{F}$ and so $s \in \widetilde{\mathscr{F}}(U)$. Thus the gluability axiom is also verified.

Now, we show that the inclusion $\mathscr{F} \hookrightarrow \widetilde{\mathscr{F}}$ satisfies the universal property. Indeed, let $\mathscr{G}$ be a sheaf and $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism. If U is an open set and $s \in \widetilde{\mathscr{F}}(U)$, there's an open cover $\{U_i\}$ of U such that $s|_{U_i} \in \mathscr{F}(U_i)$ for every i. But then we have induced sections $\varphi_{U_i}(s|_{U_i}) \in \mathscr{G}(U_i)$ which glue uniquely into a section of $\mathscr{G}(U)$. This defines the unique induced morphism $\widetilde{\mathscr{F}} \to \mathscr{G}$. □

While not every presheaf is a subpresheaf of a sheaf (such a presheaf always satisfies the identity axiom, for example), there's a canonical map from any presheaf to a sheaf. Indeed, we can associate to any presheaf $\mathscr{F}$ a sheaf $\Pi(\mathscr{F})$ whose sections are given by

$$\Gamma(U, \Pi(\mathscr{F})) := \prod_{p \in U} \mathscr{F}_p$$

and whose restriction maps are defined as the natural projections

$$\mathrm{res}_{U,V} : \prod_{p\in U} \mathscr{F}_p \to \prod_{p\in V} \mathscr{F}_p$$

which throws away the components at points of U not lying in V. We show that this is truly a sheaf. If $U = U_1 \cap U_2$ and

$$s_1 = (s_{1,p})_{p\in U_1} \in \Gamma(U_1, \Pi(\mathscr{F})), \qquad s_2 = (s_{2,p})_{p\in U_2} \in \Gamma(U_2, \Pi(\mathscr{F}))$$

coincide over $U_1 \cap U_2$, then $s_{1,p} = s_{2,p}$ for all $p \in U_1 \cap U_2$. It follows that we can concatenate s_1 and s_2 to obtain a unique section over U. It is clear that this procedure works with an arbitrary open cover of U. This is called the *Godement sheaf* associated to $\mathscr{F}$.

The canonical map $\kappa_{\mathscr{F}} : \mathscr{F} \to \Pi(\mathscr{F})$ is the one sending a section $s \in \mathscr{F}(U)$ to the element $(s_p)_{p\in U} \in \prod_{p\in U} \mathscr{F}_p$. This construction is functorial in $\mathscr{F}$, for if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, one has stalkwise maps $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ and, by taking appropriate products of these, we obtain a map $\Pi(\varphi) : \Pi(\mathscr{F}) \to \Pi(\mathscr{G})$ which makes the diagram

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\kappa_{\mathscr{F}}"] \arrow[d, "\varphi"'] & \Pi(\mathscr{F}) \arrow[d, "\Pi(\varphi)"] \\
\mathscr{G} \arrow[r, "\kappa_{\mathscr{G}}"] & \Pi(\mathscr{G})
\end{tikzcd}$$

commute. Moreover, the images of $\kappa_{\mathscr{F}}$ and $\kappa_{\mathscr{G}}$ are naturally subpresheaves of $\Pi(\mathscr{F})$ and $\Pi(\mathscr{G})$, respectively. Restricting $\Pi(\varphi)$ to $\operatorname{im}\kappa_{\mathscr{F}}$ we obtain a morphism between the images with makes the diagram

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\kappa_{\mathscr{F}}"] \arrow[d, "\varphi"'] & \operatorname{im}\kappa_{\mathscr{F}} \arrow[r, hook] \arrow[d, dashed] & \Pi(\mathscr{F}) \arrow[d, "\Pi(\varphi)"] \\
\mathscr{G} \arrow[r, "\kappa_{\mathscr{G}}"] & \operatorname{im}\kappa_{\mathscr{G}} \arrow[r, hook] & \Pi(\mathscr{G})
\end{tikzcd}$$

commute. It is clear that $\Pi(\mathrm{id}_{\mathscr{F}}) = \mathrm{id}_{\Pi(\mathscr{F})}$ and that $\Pi(\psi \circ \varphi) = \Pi(\psi) \circ \Pi(\varphi)$. Thus Π defines a functor from the category of presheaves on X to the category of sheaves on X. This construction allow us to sheafify arbitrary presheaves.

Proposition 11.3.2 Let $\mathscr{F}$ be a presheaf over a topological space X with values in C. Then the image of $\kappa_{\mathscr{F}} : \mathscr{F} \to \Pi(\mathscr{F})$ is a subpresheaf of $\Pi(\mathscr{F})$ whose sheafification coincides with the sheafification of $\mathscr{F}$.

Proof. Let $\mathscr{G}$ be a sheaf and $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism. Since $\mathscr{G}$ is a sheaf, the canonical map $\kappa_{\mathscr{G}} : \mathscr{G} \to \operatorname{im}\kappa_{\mathscr{G}}$ is an isomorphism. Then, by the lemma 11.3.1, we have a unique induced morphism making the diagram

$$\begin{tikzcd}
\mathscr{F} \arrow[r] \arrow[d, "\varphi"] & \operatorname{im} \kappa_{\mathscr{F}} \arrow[r] \arrow[d] & \widetilde{\operatorname{im} \kappa_{\mathscr{F}}} \arrow[dl, dashed] \\
\mathscr{G} \arrow[r, "\sim"] & \operatorname{im} \kappa_{\mathscr{G}} &
\end{tikzcd}$$

commute. In other words, the composition $\mathscr{F} \to \operatorname{im} \kappa_{\mathscr{F}} \to \widetilde{\operatorname{im} \kappa_{\mathscr{F}}}$ satisfies the universal property of sheafification. ☐

This proposition gives an explicit formula for the sheafification of $\mathscr{F}$. For every open set $U \subset X$, the sheafification $\widetilde{\mathscr{F}}$ of $\mathscr{F}$ is given by

$$\begin{aligned}
\widetilde{\mathscr{F}}(U) &= \{ s \in \Gamma(U, \Pi(\mathscr{F})) \mid s \text{ locally lies in } \operatorname{im} \kappa_{\mathscr{F}} \} \\
&= \left\{ (s_p)_{p \in U} \in \prod_{p \in U} \mathscr{F}_p \;\middle|\; \begin{array}{l} \text{for every } p \in U \text{, there exists a neighbor-} \\ \text{hood } V \subset U \text{ of } p \text{ and a section } t \in \mathscr{F}(V) \\ \text{such that } s_q = t_q \text{ for every } q \in V \end{array} \right\}
\end{aligned}$$

and the restriction maps are induced from those of $\Pi(\mathscr{F})$. Nevertheless, it isn't very practical to use this explicit characterization of the sheafification so what we'll often use are the two results below.

Proposition 11.3.3 Let $\mathscr{F}$ be a presheaf over a topological space X with values in C. The sheafification map $\mathscr{F} \to \widetilde{\mathscr{F}}$ induces an isomorphism of stalks $\mathscr{F}_p \xrightarrow{\sim} \widetilde{\mathscr{F}}_p$ for every $p \in X$.

Proof. We construct an inverse to the natural morphism $\mathscr{F}_p \to \widetilde{\mathscr{F}}_p$. Let $\overline{(s, U)} \in \widetilde{\mathscr{F}}_p$. By the construction above, there exists a neighborhood $V \subset U$ of p and a section $t \in \mathscr{F}(V)$ whose stalks coincide with the components of $s|_V$. We let $\overline{(t, V)} \in \mathscr{F}_p$ be the image of $\overline{(s, U)}$. It is clear that this is indeed an inverse of $\mathscr{F}_p \to \widetilde{\mathscr{F}}_p$. ☐

The following result is a simple reformulation of the universal property of sheafification but is marked as a theorem since it will allow us to understand the role of sheafification in the category of sheaves.

Theorem 11.3.4 The sheafification functor is left adjoint to the forgetful functor from sheaves to presheaves.

This clarifies many things. Since the inclusion of the category of sheaves in the category of presheaves is right adjoint, it preserves limits. That is, a limit in the category of sheaves, when it exists, must be the corresponding limit in the category of presheaves. But more is true! This functor is also fully faithful, which implies that it also *creates* limits. In other words, a limit in the category of sheaves exists precisely when it exists in the category of presheaves, in which case they coincide. That's why, when we'll deal with sheaves with values in an abelian category, kernels and finite direct sums won't need to be sheafified.

It gets even better! The sheafification functor is left adjoint and so preserves colimits. This implies that, in order to construct a colimit of sheaves, we can construct the same colimit in the category of presheaves and then sheafify. That's why, in an abelian category, the sheafification of cokernels (in particular, quotients), images, and infinite direct sums will all satisfy their respective universal properties in the category of sheaves.

In fact, the sheafification functor also preserves finite limits. We won't prove this now but it follows from Grothendieck's plus construction, which defines the sheafification as a filtered colimit, which then commutes with finite limits. This is the content of proposition 13.4.1.

11.4. Direct and inverse images

So far we were dealing with presheaves on a fixed topological space X. In this section we'll see how, given a continuous map $f : X \to Y$, we can transfer presheaves from X to Y and conversely. Indeed, we'll define functors

$$\begin{tikzcd} \mathsf{C}_X^{\mathrm{pre}} \arrow[r, bend left, "f_*"] & \mathsf{C}_Y^{\mathrm{pre}} \arrow[l, bend left, "f^{-1}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathsf{C}_X \arrow[r, bend left, "f_*"] & \mathsf{C}_Y. \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

Moreover, we'll see that in both cases they are adjoint. We begin with the direct image.

Definition 11.4.1 — Direct image. Let $f : X \to Y$ be a continuous map between topological spaces and $\mathscr{F}$ be a presheaf on X. We define the *direct image* of $\mathscr{F}$ by f to be the presheaf $f_*\mathscr{F}$ on Y whose sections over an open set $U \subset Y$ are

$$\Gamma(U, f_*\mathscr{F}) := \Gamma(f^{-1}(U), \mathscr{F})$$

and whose restriction maps are those of $\mathscr{F}$.

This presheaf is also called the *pushforward* of $\mathscr{F}$ by f. We observe that, as a functor, $f_*\mathscr{F}$ is nothing but the restriction of $\mathscr{F} : \mathrm{Open}_X \to \mathsf{C}$ to the full subcategory of Open_X whose objects are the preimages of open sets in Y. In particular, if $\mathscr{F}$ is a sheaf, then so is $f_*\mathscr{F}$.

Here are two simple examples which appear frequently.

■ **Example 11.4.1 — Skyscraper sheaf.** Let $i_p : \{p\} \hookrightarrow X$ be the inclusion of a point in X. If S is a set, the direct image of the constant sheaf $\underline{S}$ by i_p is the *skyscraper sheaf* $i_{p,*}\underline{S}$.

If $U \subset X$ is an open set, the sections of $i_{p,*}\underline{S}$ over U are given by

$$i_{p,*}\underline{S}(U) = \begin{cases} S & \text{if } p \in U \\ \{*\} & \text{if } p \notin U \end{cases},$$

where $\{*\}$ is any singleton. (Strictly speaking, $*$ should be the unique map $\varnothing \to S$ but it doesn't really matter.) Its stalks are simply

$$(i_{p,*}\underline{S})_q = \begin{cases} S & \text{if } q \in \overline{\{p\}} \\ \{*\} & \text{if } q \notin \overline{\{p\}} \end{cases}.$$

When S is an object of a category C, the same descriptions apply by changing $\{*\}$ to the final object of C. ■

Dually, we have the following example.

■ **Example 11.4.2** Let $\{*\}$ be a one-point topological space and $f : X \to \{*\}$ the only possible continuous map. A sheaf on $\{*\}$ is nothing but the data of an object of C. This shows that the category of sheaves over $\{*\}$ is equivalent to C. Under this equivalence, f_* is nothing but the global sections functor. Indeed, if $\mathscr{F}$ is a sheaf we have that

$$\Gamma(\{*\}, f_*\mathscr{F}) = \Gamma(X, \mathscr{F}).$$

In other words, $f_*\mathscr{F} = \underline{\Gamma(X, \mathscr{F})}$ as sheaves over $\{*\}$. ■

If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves over X, then we have an induced morphism $f_*\varphi : f_*\mathscr{F} \to f_*\mathscr{G}$ of presheaves over Y. Indeed, its components are simply $\varphi_{f^{-1}(U)}$ for all open sets $U \subset Y$. It is clear that $f_* \mathrm{id}_{\mathscr{F}} = \mathrm{id}_{f_*\mathscr{F}}$ and that $f_*(\varphi \circ \psi) = f_*\varphi \circ f_*\psi$, so f_* indeed defines a functor $\mathsf{C}_X^{\mathrm{pre}} \to \mathsf{C}_Y^{\mathrm{pre}}$ which restricts to a functor $\mathsf{C}_X \to \mathsf{C}_Y$.

The direct image is not only functorial in $\mathscr{F}$ but also in f. Indeed, if $g : Y \to Z$ is another continuous map, we have that $g_*(f_*\mathscr{F}) = (g \circ f)_*\mathscr{F}$ and that $g_*(f_*\varphi) = (g \circ f)_*\varphi$ whenever $\mathscr{F}$ is a presheaf over X and φ is a morphism of presheaves over X. In other words, $(g \circ f)_*$ is the composition of the functors g_* and f_*.

Also, if $p \in X$ and $V \subset U$ is a pair of nested neighborhoods of $f(p) \in Y$ we have a commutative diagram

$$\begin{tikzcd}
\Gamma(U, f_*\mathscr{F}) \arrow[dr] \arrow[drr, bend left] \arrow[dd, "\mathrm{res}_{U,V}"'] & & \\
& (f_*\mathscr{F})_{f(p)} \arrow[r, dashed] & \mathscr{F}_p \\
\Gamma(V, f_*\mathscr{F}) \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

which induces a morphism $(f_*\mathscr{F})_{f(p)} \to \mathscr{F}_p$. In general, it is neither injective nor surjective. But there's one simple case where it is indeed an isomorphism: when f is an embedding of topological spaces. Indeed, in this case there's an inverse $g : f(X) \to X$ of $f : X \to f(X)$ which yields a morphism

$$\mathscr{F}_p = (g_*(f_*\mathscr{F}))_{g(f(p))} \to (f_*\mathscr{F})_{f(p)}$$

that is the inverse of $(f_*\mathscr{F})_{f(p)} \to \mathscr{F}_p$.

We give a last example which illustrates the usefulness of this construction.

■ **Example 11.4.3** Let M and N be smooth manifolds along with their respective sheaves of smooth functions C^∞_M and C^∞_N. If $f : M \to N$ is a smooth map and $s \in C^\infty_N(U)$, then the composition

$$f^{-1}(U) \xrightarrow{f} U \xrightarrow{s} \mathbb{R}$$

is a smooth function on $f^{-1}(U) \subset M$. That is, a section of $f_*C^\infty_M(U)$. In this way, f induces a morphism of sheaves

$$f^\sharp : C^\infty_N \to f_*C^\infty_M.$$

But more is true! If f is hereafter only supposed continuous and the composition $s \circ f$ is in $f_*C^\infty_M(U)$ whenever $U \subset N$ is open and $s \in C^\infty_N(U)$, then f is smooth. Indeed, if $p \in M$, it suffices to choose s_i to be the coordinates of a chart about $f(p)$ in N. In this case, $s_i \circ f \in f_*C^\infty_M(U)$ implies that there exists a chart (V, t) around p such that

$$s_i \circ f \circ t^{-1} : t(f^{-1}(U) \cap V) \to \mathbb{R}$$

is smooth at $t(p)$. This is precisely what it means to say that f is smooth at p. ■

As for the inverse image, we would wish to do something similar to the direct image. Unfortunately for us, if $\mathscr{G}$ is a presheaf on Y, assigning $\mathscr{G}(f(U))$ to every open set $U \subset X$ does not define a presheaf on X since $f(U)$ may not be open. Nevertheless, we can approximate $f(U)$ by open sets.

Definition 11.4.2 — Inverse image presheaf. Let $f : X \to Y$ be a continuous map between topological spaces and $\mathscr{G}$ be a presheaf on Y. We define the *inverse image* of $\mathscr{G}$ by f to be the presheaf $f^{-1}\mathscr{G}$ on X whose sections over an open set $U \subset X$ are

$$\Gamma(U, f^{-1}\mathscr{G}) := \operatorname{colim} \mathscr{G}(V),$$

where the (filtered) colimit is taken over the open sets $V \subset Y$ containing $f(U)$. The restriction maps are those induced from $\mathscr{G}$.

Most books define the inverse image to be the sheafification *of what we've defined. So before the reader begins screaming in anger, rest assured that we'll eventually do the same. But for now this definition will suffice.*

We explain in more detail how the restriction maps from $\mathscr{G}$ induce those of $f^{-1}\mathscr{G}$. If $V \subset U$ is a pair of nested open sets in X, then any open set $W \subset Y$ containing $f(U)$ automatically contains $f(V)$. Thus, if $W' \subset W$ are open sets in Y containing $f(U)$, the universal property of the colimit yields an induced morphism

$$\begin{tikzcd}
\mathscr{G}(W) \arrow[dr] \arrow[drr, bend left] \arrow[dd, "\mathrm{res}_{W,W'}"'] & & \\
& \Gamma(U, f^{-1}\mathscr{G}) \arrow[r, dashed] & \Gamma(V, f^{-1}\mathscr{G}), \\
\mathscr{G}(W') \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

which is our desired restriction map $\mathrm{res}_{U,V} : \Gamma(U, f^{-1}\mathscr{G}) \to \Gamma(V, f^{-1}\mathscr{G})$.

Similarly, the universal property of the colimit shows the functoriality of this construction. If $\varphi : \mathscr{G} \to \mathscr{F}$ is a morphism of presheaves over Y and $U \subset X$ is an open set, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{G}(W) \arrow[rr, "\varphi_W"] \arrow[dr] \arrow[dd] & & \mathscr{F}(W) \arrow[dr] \arrow[dd] & \\
& \Gamma(U, f^{-1}\mathscr{G}) \arrow[rr, dashed] & & \Gamma(U, f^{-1}\mathscr{F}), \\
\mathscr{G}(W') \arrow[ur] \arrow[rr, "\varphi_{W'}"] & & \mathscr{F}(W) \arrow[ur] &
\end{tikzcd}$$

whenever $W' \subset W \subset Y$ is a pair of nested open sets containing $f(U)$, which induces a unique morphism $f^{-1}\varphi : \Gamma(U, f^{-1}\mathscr{G}) \to \Gamma(U, f^{-1}\mathscr{F})$. Once again, the unicity of the induced morphism implies that $f^{-1}\,\mathrm{id}_{\mathscr{G}} = \mathrm{id}_{f^{-1}\mathscr{G}}$ and that $f^{-1}(\varphi \circ \psi) = f^{-1}\varphi \circ f^{-1}\psi$, so f^{-1} indeed defines a functor $\mathsf{C}_Y^{\mathrm{pre}} \to \mathsf{C}_X^{\mathrm{pre}}$.

Let's check some examples.

■ **Example 11.4.4** Let $i_p : \{p\} \hookrightarrow X$ be the inclusion of a point in X. If $\mathscr{G}$ is a presheaf on X, its inverse image by i_p is given by

$$\Gamma(\{p\}, i_p^{-1}\mathscr{G}) = \mathscr{G}_p \qquad \text{and} \qquad \Gamma(\varnothing, i_p^{-1}\mathscr{G}) = \mathscr{G}(\varnothing).$$

This is basically the constant presheaf of example 11.1.2. ■

Somewhat more generally, we can use the inverse image to restrict a presheaf to an arbitrary subset.

■ **Example 11.4.5** Let S be a subset of a topological space X and $i_S : S \hookrightarrow X$ be its inclusion. If $\mathscr{G}$ is a presheaf over X, we say that $i_S^{-1}\mathscr{G}$ is its restriction to S. This names comes from the fact if S is open, $i_S^{-1}\mathscr{G}$ is precisely the restriction $\mathscr{G}|_S$. ■

We now prove the main result of this section. It says that, if $f : X \to Y$ is a continuous map, giving a morphism $f^\sharp : \mathscr{G} \to f_*\mathscr{F}$ is the same thing as giving a morphism $f^\flat : f^{-1}\mathscr{G} \to \mathscr{F}$. These sharps and flats, which will be systematically used once we begin talking about ringed spaces, explain why we call this theorem a *musical adjunction*.

Theorem 11.4.1 — Musical adjunction. Let $f : X \to Y$ be a continuous map between topological spaces. Then we have an adjunction

$$\begin{tikzcd} C_X^{\mathrm{pre}} \arrow[r, bend left, "f_*"] \arrow[r, phantom, "\top"] & C_Y^{\mathrm{pre}} \arrow[l, bend left, "f^{-1}"] \end{tikzcd}.$$

In other words, if $\mathscr{F}$ is a presheaf on X and $\mathscr{G}$ is a presheaf on Y, there is an isomorphism

$$\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$$

which is natural in $\mathscr{F}$ and $\mathscr{G}$.

Proof. If $f^\flat : f^{-1}\mathscr{G} \to \mathscr{F}$ is a morphism of presheaves on X, we define the corresponding morphism $f^\sharp : \mathscr{G} \to f_*\mathscr{F}$ of presheaves on Y by letting $f^\sharp_V$, for an open set $V \subset Y$, be the composition

$$\mathscr{G}(V) \longrightarrow \Gamma(f^{-1}(V), f^{-1}\mathscr{G}) \xrightarrow{f^\flat_{f^{-1}(V)}} \Gamma(f^{-1}(V), \mathscr{F}) = f_*\mathscr{F}(V),$$

where the first arrow is the natural map of the colimit. Conversely, let $f^\sharp : \mathscr{G} \to f_*\mathscr{F}$ be a morphism of presheaves on Y. We fix $U \subset X$ and let $V' \subset V$ be a pair of nested open sets in Y containing $f(U)$. The universal property of colimits then induces our morphism $f^\flat : f^{-1}\mathscr{G} \to \mathscr{F}$.

$$\begin{tikzcd}
\mathscr{G}(V) \arrow[rr, "f^\sharp_V"] \arrow[dr] \arrow[dd] & & f_*\mathscr{F}(V) \arrow[dr, "\mathrm{res}_{f^{-1}(V),U}"] \arrow[dd] & \\
& f^{-1}\mathscr{G}(U) \arrow[rr, dashed] & & \mathscr{F}(U) \\
\mathscr{G}(V') \arrow[ur] \arrow[rr, "f^\sharp_{V'}"] & & f_*\mathscr{F}(V') \arrow[ur, "\mathrm{res}_{f^{-1}(V'),U}"'] &
\end{tikzcd}$$

By taking $V = f(U)$ in the diagram above, we get that

$$\begin{tikzcd}
\mathscr{G}(V) \arrow[rr, "f^\sharp_V"] \arrow[dr] & & f_*\mathscr{F}(V) \\
& \Gamma(f^{-1}(V), f^{-1}\mathscr{G}) \arrow[ur, "f^\flat_{f^{-1}(V)}"'] &
\end{tikzcd}$$

commutes, proving that $f^\sharp \mapsto f^\flat \mapsto f^\sharp$ is indeed the identity. Conversely, let $f^\flat : f^{-1}\mathscr{G} \to \mathscr{F}$ be a morphism of presheaves on X, $U \subset X$ be an open set and $V' \subset V$ be a pair of nested open sets in Y containing $f(U)$. By the very definition of morphism, the diagram

$$\begin{tikzcd}
\Gamma(f^{-1}(V), f^{-1}\mathscr{G}) \arrow[r, "f^\flat_{f^{-1}(V)}"] \arrow[d] & f_*\mathscr{F}(V) \arrow[d] \\
f^{-1}\mathscr{G}(U) \arrow[r, "f^\flat_U"] & \mathscr{F}(U) \\
\Gamma(f^{-1}(V'), f^{-1}\mathscr{G}) \arrow[r, "f^\flat_{f^{-1}(V)}"] \arrow[u] & f_*\mathscr{F}(V') \arrow[u]
\end{tikzcd}$$

commutes and then so does

$$\begin{tikzcd}
\mathscr{G}(V) \arrow[r] \arrow[dd] \arrow[rr, bend left, "f^\sharp_V"] & \Gamma(f^{-1}(V), f^{-1}\mathscr{G}) \arrow[r, "f^\flat_{f^{-1}(V)}"] \arrow[d] & f_*\mathscr{F}(V) \arrow[d] \\
 & f^{-1}\mathscr{G}(U) \arrow[r, "f^\flat_U"] & \mathscr{F}(U) \\
\mathscr{G}(V') \arrow[r] \arrow[rr, bend right, "f^\sharp_{V'}"'] & \Gamma(f^{-1}(V'), f^{-1}\mathscr{G}) \arrow[r, "f^\flat_{f^{-1}(V)}"] \arrow[u] & f_*\mathscr{F}(V'). \arrow[u]
\end{tikzcd}$$

It now follows from the unicity of the induced morphism from the colimit that $f^\flat \mapsto f^\sharp \mapsto f^\flat$ is also the identity. □

After any adjunction we have our usual Pavlovian reaction: since f_* is a right adjoint, it preserves limits and so, in an abelian category, is left-exact. Similarly, f^{-1} preserves colimits and, in an abelian category, is right-exact.

In case the reader has trouble remembering which functor is the left and which functor is the right adjoint, here's a useful mnemonic: the inverse image is defined as a colimit and we have a handle on the morphisms going *out* of a colimit. That is, f^{-1} has to be a left-adjoint.[3]

Here are some other corollaries that follow easily from this adjunction. Firstly we observe that, as before, the inverse image is functorial not only on $\mathscr{G}$ but also on f.

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[3] There is a particular case in which f^{-1} is also a right adjoint. But I hope the reader gets the point...

Corollary 11.4.2 Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps between topological spaces. Then the functors $f^{-1} \circ g^{-1}$ and $(g \circ f)^{-1}$ are isomorphic.

Proof. Recall the equality of functors $g_* \circ f_* = (g \circ f)_*$. Then, if $\mathscr{F}$ is a presheaf on X and $\mathscr{H}$ is a presheaf on Z, we have that

$$\begin{aligned}\mathrm{Hom}_X((g \circ f)^{-1}\mathscr{H}, \mathscr{F}) &\cong \mathrm{Hom}_Z(\mathscr{H}, g_*(f_*\mathscr{F}))\\ &\cong \mathrm{Hom}_Y(g^{-1}\mathscr{H}, f_*\mathscr{F})\\ &\cong \mathrm{Hom}_X(f^{-1}(g^{-1}\mathscr{H}), \mathscr{F}).\end{aligned}$$

Since this holds naturally in $\mathscr{F}$ and $\mathscr{H}$, the Yoneda lemma implies our result. ☐

While the definition of the inverse image is more complicated than that of the direct image, now we have isomorphic stalks without further conditions.

Corollary 11.4.3 Let $f : X \to Y$ be a continuous map between topological spaces on Y and $\mathscr{G}$ be a presheaf on Y. Then the stalks $\mathscr{G}_{f(p)}$ and $(f^{-1}\mathscr{G})_p$ are isomorphic.

Proof. Letting the maps in the previous corollary be $i_p : \{p\} \hookrightarrow X$ and $f : X \to Y$, we have that

$$(f^{-1}\mathscr{G})_p = \Gamma(\{p\}, i_p^{-1}(f^{-1}\mathscr{G})) \cong \Gamma(\{p\}, (f \circ i_p)^{-1}\mathscr{G}) = \mathscr{G}_{f(p)}$$

by the calculation that we did in example 11.4.4. ☐

Another reason why the inverse image is more complicated than the direct image is that the inverse image of a sheaf is not necessarily a sheaf as we can see in the next example.

■ **Example 11.4.6** Let Y be a topological space, X be the disjoint union of two copies of Y and $f : X \to Y$ be the quotient map which identifies the two copies of Y. Let also $\mathscr{G}$ be a sheaf on Y. If $V \subset Y$ is an open set and $U = f^{-1}(V)$, we have that $\Gamma(U, f^{-1}\mathscr{G}) = \mathscr{G}(V)$. Since U is the disjoint union of two copies V_1 and V_2 of V, if $f^{-1}\mathscr{G}$ is a sheaf, we should have that

$$\mathscr{G}(V) = \Gamma(U, f^{-1}\mathscr{G}) = \Gamma(V_1, f^{-1}\mathscr{G}) \times \Gamma(V_2, f^{-1}\mathscr{G}) = \mathscr{G}(V) \times \mathscr{G}(V).$$

As this need not be true, $f^{-1}\mathscr{G}$ is usually not a sheaf. ■

This prompts the definition below.

Definition 11.4.3 — Inverse image sheaf. Let $f : X \to Y$ be a continuous map between topological spaces and $\mathscr{G}$ be a sheaf on Y. We define the *inverse image* $f^{-1}\mathscr{G}$ of $\mathscr{G}$ by f to be the sheafification of the inverse image presheaf of definition 11.4.2. Since both notions are denoted in the same way and have the same name, unless explicitly stated, from now on this will be what we mean by the inverse image of a sheaf.

The inverse image so defined is the composition of the functors

$$\begin{tikzcd} \mathsf{C}_Y \arrow[r, hook] & \mathsf{C}_Y^{\mathrm{pre}} \arrow[r] & \mathsf{C}_X^{\mathrm{pre}} \arrow[r] & \mathsf{C}_X, \end{tikzcd}$$

where the leftmost arrow is the inclusion of the category of sheaves in the category of presheaves, the next one is our old definition of inverse image, and the rightmost arrow is the sheafification functor. As one could expect, the musical adjunction still holds.

Theorem 11.4.4 — Musical adjunction. Let $f : X \to Y$ be a continuous map between topological spaces. Then we have an adjunction

$$\begin{tikzcd} \mathsf{C}_X \arrow[r, bend left, "f_*"] \arrow[r, phantom, "\top"] & \mathsf{C}_Y. \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

In other words, if $\mathscr{F}$ is a sheaf on X and $\mathscr{G}$ is a sheaf on Y, there is an isomorphism

$$\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$$

which is natural in $\mathscr{F}$ and $\mathscr{G}$.

Proof. Since the sheafification functor is a left adjoint of the inclusion functor, this follows from the musical adjunction on presheaves. ☐

We observe that it is still true that the functors $f^{-1} \circ g^{-1}$ and $(g \circ f)^{-1}$ are isomorphic. Indeed, the proof of corollary 11.4.2 used nothing but the adjunction and the fact that the analogous result for direct images holds. Moreover, the stalks $\mathscr{G}_{f(p)}$ and $(f^{-1}\mathscr{G})_p$ are still naturally isomorphic since the sheafification functor preserves stalks. (Proposition 11.3.3.)

As the category of sheaves over a singleton is equivalent to C, the particular case $f = i_p : \{p\} \hookrightarrow X$ of the musical adjunction is already interesting by itself.

Corollary 11.4.5 Let X be a topological space and $p \in X$. Then the stalk functor is left adjoint to the skyscraper sheaf functor.

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "i_{p,*}(-)"] \arrow[r, phantom, "\top"] & \mathsf{C}_X \arrow[l, bend left, "(-)_p"] \end{tikzcd}$$

In particular, the stalk functor preserves colimits.

Proof. This follows from the examples 11.4.1 and 11.4.4. ☐

11.5. Sheaves on a base

Defining a presheaf over a topological space X with values in C involves giving a plethora of information. Namely, an object of C for every open set $U \subset X$. Since sheaves are somewhat more constrained, we may hope that it suffices to define a sheaf by giving the images of the open sets in a base of the topology. This suggests the following notion.

Definition 11.5.1 — Presheaf on a base. Let X be a topological space and $\mathcal{B}$ be a base of its topology. We denote by $\mathrm{Open}_{\mathcal{B}}$ the full subcategory of Open_X whose objects are the elements of $\mathcal{B}$. Then, a *presheaf* F *on* $\mathcal{B}$ with values in C is a contravariant functor $F : \mathrm{Open}_{\mathcal{B}} \to \mathsf{C}$.

As before, the restriction maps satisfy the conditions imposed by the functor axioms and we say that the elements of $F(B)$ are sections. Also, we define the stalk of F at $p \in X$ as the (filtered) colimit of all $F(B)$ over the basic open sets $B \in \mathcal{B}$ containing p:

$$F_p := \operatorname{colim} F(B).$$

Naturally, the elements of F_p are called germs.

The only definition that has to be slightly modified is that of a sheaf, since the intersection of basic open sets need not be in the base.

Definition 11.5.2 — Sheaf on a base. Let X be a topological space and $\mathcal{B}$ be a base of its topology. A presheaf F on $\mathcal{B}$ with values in C is a *sheaf* if, whenever $B \in \mathcal{B}$, $\{B_i\}$ is a cover of B by basic open sets and B_{ij} are basic open sets contained in $B_i \cap B_j$, the product of the restrictions $\alpha : F(B) \to \prod_i F(B_i)$ is the equalizer of

$$\begin{tikzcd} \prod_i F(B_i) \arrow[r, shift left, "\beta_1"] \arrow[r, shift right, "\beta_2"'] & \prod_{ij} F(B_{ij}), \end{tikzcd}$$

where β_1, β_2 are defined by $\beta_1((s_i)_i) = (s_i|_{B_{ij}})_{i,j}$ and $\beta_2((s_i)_i) = (s_j|_{B_{ij}})_{i,j}$.

Just as one could expect, asking a presheaf F on $\mathcal{B}$ to be a sheaf is precisely the same thing as demanding the following axioms whenever $B \in \mathcal{B}$ and $\{B_i\}$ is a cover of B by basic open sets:

i. (Identity axiom) if $s, t \in F(B)$ coincide over B_i for all i, then $s = t$;

ii. (Gluability axiom) if $s_i \in F(B_i)$ coincide on any basic open set $B_{ij} \subset B_i \cap B_j$, then there is some $s \in F(B)$ such that $s|_{B_i} = s_i$ for all i.

It is clear that if $\mathcal{B}$ is closed by intersections, which will be the case quite frequently, these follow from the usual sheaf axioms applied to coverings of basic open sets by other basic opens.

We now prove our desired result. The main idea behind it is the fact that in order to determine the image of a sheaf over an open set, it suffices to know the images of basic open sets. Indeed, if $\mathscr{F}$ is a sheaf, our concrete construction of the sheafification gives an isomorphism between $\mathscr{F}(U)$ and

$$\widetilde{\mathscr{F}}(U) = \left\{ (s_p)_{p\in U} \in \prod_{p\in U} \mathscr{F}_p \;\middle|\; \begin{array}{l} \text{for every } p \in U\text{, there exists a basic neigh-} \\ \text{borhood } B \subset U \text{ of } p \text{ and } t \in \mathscr{F}(B) \text{ such that} \\ s_q = t_q \text{ for every } q \in B \end{array} \right\}.$$

This will be used to *define* the sheaf associated with a given sheaf on a base. We also observe that, as in the section about sheafification, there's a Godement sheaf $\Pi(F)$ associated to a presheaf on a base F by the same rule as before

$$\Gamma(U, \Pi(F)) := \prod_{p\in U} F_p.$$

This is a sheaf defined on every open set; not just the basic ones.

Theorem 11.5.1 Let X be a topological space, $\mathcal{B}$ be a base of its topology and F be a sheaf on $\mathcal{B}$ with values in C. Then there is a unique, up to unique isomorphism, sheaf $\mathscr{F}$ over X which coincides with F over $\mathcal{B}$. Also, both sheaves have the same stalks.

Proof. As we've just said, we define the extended sheaf $\mathscr{F}$ to be the subpresheaf of $\Pi(F)$ given by

$$\mathscr{F}(U) := \left\{ (s_p)_{p\in U} \in \prod_{p\in U} F_p \;\middle|\; \begin{array}{l} \text{for every } p \in U\text{, there exists a basic neigh-} \\ \text{borhood } B \subset U \text{ of } p \text{ and } t \in F(B) \text{ such that} \\ s_q = t_q \text{ for every } q \in B \end{array} \right\}$$

for every open set $U \subset X$. Since $\Pi(F)$ is a sheaf, it suffices to show that gluing sections in $\mathscr{F}$ we obtain a section of $\Pi(F)$ that is actually in $\mathscr{F}$. For that, let $\{U_i\}$ be an open cover of U and let $s_i = (s_{i,p})_{p\in U_i} \in \mathscr{F}(U_i)$ be sections which agree on the intersections. As $\Pi(F)$ is a sheaf, we find an element $s = (s_p)_{p\in U} \in \prod_{p\in U} F_p$ which restricts to s_i on U_i. We show that $s \in \mathscr{F}(U)$. Pick $p \in U$. Then $p \in U_i$ for some i. Since $s_i \in \mathscr{F}(U_i)$, there exists a basic neighborhood $B \subset U_i \subset U$ of p and $t \in F(B)$ such that $s_{i,q} = t_q$ for every $q \in B$. As $s_q = s_{i,q}$, it follows that $s \in \mathscr{F}(U)$. In other words, $\mathscr{F}$ is a sheaf.

Now, if B is a basic open set, the natural morphism $F(B) \to \mathscr{F}(B)$ is injective by the identity axiom and surjective by the gluability axiom. It follows that it is a bijection and then an isomorphism, since C reflects isomorphisms.

Finally, if $\mathscr{F}'$ is another sheaf extending F, then $\mathscr{F}'(B) = F(B) = \mathscr{F}(B)$ for every $B \in \mathcal{B}$ and so $\mathscr{F}' = \mathscr{F}$, since the basic open sets determine the sheaves. The proof that both sheaves have the same stalks is identical to showing that the sheafification has the same stalks. $\square$

As one could hope, we can not only extend sheaves on a base but also morphisms of sheaves on a base. For that, we need the definition of a morphism of presheaves on a base.

Definition 11.5.3 — Morphism of (pre)sheaves on a base. Let X be a topological space and $\mathcal{B}$ be a base of its topology. Also, let F and G be presheaves on $\mathcal{B}$. A *morphism* $\varphi : F \to G$ is nothing but a natural transformation of functors. Explicitly, it is the data of maps $\varphi_B : F(B) \to G(B)$, for all basic open sets $B \subset X$, such that the diagram

$$\begin{tikzcd} F(B) \arrow[r, "\varphi_B"] \arrow[d, "\mathrm{res}_{B,B'}"'] & G(B) \arrow[d, "\mathrm{res}_{B,B'}"] \\ F(B') \arrow[r, "\varphi'_B"] & G(B') \end{tikzcd}$$

commutes whenever $B' \subset B$ is a pair of nested basic open sets. As usual, a morphism of sheaves is nothing but a morphism of the underlying presheaves.

Now we indeed have a relative version of theorem 11.5.1.

Theorem 11.5.2 Let X be a topological space and $\mathcal{B}$ be a base of its topology. If $\varphi : F \to G$ is a morphism of sheaves on $\mathcal{B}$, then there exists a unique morphism $\widetilde{\varphi} : \mathscr{F} \to \mathscr{G}$ between the induced sheaves which coincides with φ over the basic open sets.

Proof. We affirm that the restriction of $\Pi(\varphi) : \Pi(F) \to \Pi(G)$ to $\mathscr{F}$ is the desired $\widetilde{\varphi}$. First of all, we verify that its image is contained in $\mathscr{G}$. If $U \subset X$ is an open set and $s = (s_p)_{p \in U} \in \mathscr{F}(U)$, we need to show that $\Pi(\varphi)_U(s) = (\varphi_p(s_p))_{p \in U}$ is in $\mathscr{G}(U)$. Fix $p \in X$. Since $s \in \mathscr{F}(U)$, there exists a basic neighborhood $B \subset U$ of p and $t \in F(B)$ such that $s_q = t_q$ for every $q \in B$. But then $\varphi_B(t) \in G(B)$ is such that $\varphi_q(s_q) = \varphi_B(t)_q$ for every $q \in B$, establishing our result.

Now, we observe that $\widetilde{\varphi}$ coincides with φ over a basic open set B. More precisely, it fits into the commutative diagram below.

$$\begin{tikzcd} F(B) \arrow[r, "\sim"] \arrow[d, "\varphi_B"'] & \mathscr{F}(B) \arrow[d, "\widetilde{\varphi}_B"] \\ G(B) \arrow[r, "\sim"] & \mathscr{G}(B) \end{tikzcd}$$

Indeed, this is nothing but the fact that $\varphi_B(s)_p = \varphi_p(s_p)$ for every $p \in B$. Finally, the unicity follows by noticing that the stalks determine the morphism and that the induced sheaves have the same stalks as the sheaves on a base. □

In other words, the natural restriction functor from the category of sheaves over X to the category of sheaves on $\mathcal{B}$ is an equivalence of categories. The quasi-inverse functor was given by theorem 11.5.1.

As an application of this equivalence of categories, we'll learn how to glue sheaves and morphisms thereof. We begin with the latter, whose proof doesn't really needs the theory that we've just did but which we'll be useful in gluing sheaves.

> **Proposition 11.5.3** Let $\{U_i\}$ be an open cover of a topological space X. Also, let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ be a sheaf over X. If $\varphi_i : \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$ are morphisms which restrict to the same map on the intersections, then there exists a unique morphism $\varphi : \mathscr{F} \to \mathscr{G}$ whose restriction to U_i agrees with φ_i for every i.

Proof. Let $V \subset X$ be an open set. The sets $V_i := V \cap U_i$ cover V and so we have a diagram

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[r] \arrow[d, dashed] & \prod_i \mathscr{F}(V_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "(\varphi_i)_{V_i}"] & \prod_{ij} \mathscr{F}(V_i \cap V_j) \\
\mathscr{G}(V) \arrow[r] & \prod_i \mathscr{G}(V_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{ij} \mathscr{G}(V_i \cap V_j).
\end{tikzcd}$$

It commutes precisely because the φ_i restrict to the same map on the intersections. The universal property of the equalizer gives a unique induced morphism. This defines our unique $\varphi : \mathscr{F} \to \mathscr{G}$. If V is a subset of U_i, the fact that the square on the left commutes implies that $\varphi_V = (\varphi_i)_V$ and so $\varphi|_{U_i} = \varphi_i$. ☐

Just as a quick digression, this proposition is precisely a result we had promised not long ago.

> **Corollary 11.5.4** Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ be a sheaf over a topological space X. Then $\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})$ is a sheaf.

We now delve into gluing sheaves. For that we define the rigorously what it really means to *glue* sheaves.

> **Definition 11.5.4 — Gluing data.** Let $\{U_i\}$ be an open cover of a topological space X. For each i, let $\mathscr{F}_i$ be a sheaf over U_i. Also, for each i, j, let
>
> $$\varphi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \to \mathscr{F}_j|_{U_i \cap U_j}$$
>
> be an isomorphism of sheaves over $U_i \cap U_j$. Assume in addition that for every i, j, k the following diagram (this is called the *cocycle condition*)
>
> $$\begin{tikzcd}
\mathscr{F}_i|_{U_i \cap U_j \cap U_k} \arrow[r, "\varphi_{ij}"] \arrow[d, "\varphi_{ik}"'] & \mathscr{F}_j|_{U_i \cap U_j \cap U_k} \arrow[dl, "\varphi_{jk}"] \\
\mathscr{F}_k|_{U_i \cap U_j \cap U_k} &
\end{tikzcd}$$
>
> commutes. We say that the collection of $(\mathscr{F}_i, \varphi_{ij})$ is a *gluing data* for the cover $\{U_i\}$.

The plan for the following proposition is simple: the collection $\mathcal{B}$ of the open sets $U \subset X$ which are contained in one of the U_i forms a base for the topology of X. If $U \in \mathcal{B}$, we wish to define a sheaf on $\mathcal{B}$ by setting $F(U) = \mathscr{F}_i(U)$ whenever $U \subset U_i$. The isomorphisms φ_{ij} imply that different choices of i yield isomorphic objects. The problem, of course, is that giving an isomorphism class for each U is not enough to define a presheaf.

Proposition 11.5.5 Let $\{U_i\}$ be an open cover of a topological space X. If $(\mathscr{F}_i, \varphi_{ij})$ is a gluing data for this cover, then there exists a unique sheaf $\mathscr{F}$ over X together with isomorphisms $\varphi_i : \mathscr{F}|_{U_i} \to \mathscr{F}_i$ such that the diagram

$$\begin{tikzcd}
\mathscr{F}|_{U_i \cap U_j} \arrow[r, "\varphi_i"] \arrow[d, "\varphi_j"'] & \mathscr{F}_i|_{U_i \cap U_j} \arrow[dl, "\varphi_{ij}"] \\
\mathscr{F}_j|_{U_i \cap U_j} &
\end{tikzcd}$$

commutes for every i, j.

Proof. As we've just said, let $\mathcal{B}$ the collection of the open sets $U \subset X$ which are contained in one of the U_i. For each $p \in X$, we choose an k such that $p \in U_k$ and then define F_p to be the stalk of $\mathscr{F}_k$ at p. The maps φ_{ki} induce isomorphisms $\varphi_{i,p} : F_p \to (\mathscr{F}_i)_p$ for every i such that $p \in U_i$. We observe that the cocycle condition implies that $\varphi_{j,q} = (\varphi_{ij})_q \circ \varphi_{i,q}$ for all $q \in U_i \cap U_j$.

Now, we define a sheaf F over $\mathcal{B}$ to be

$$F(U) := \left\{ (s_p)_{p\in U} \in \prod_{p\in U} F_p \;\middle|\; \begin{array}{l} \text{for every } p \in U \text{, there exists a neighborhood} \\ V \subset U \text{ of } p \text{ and } t \in \mathscr{F}_i(V) \text{ such that } \varphi_{i,q}(s_q) = \\ t_q \text{ for every } q \in V \end{array} \right\}$$

whenever $U \subset U_i$ for some i. This is independent of the choice of i. Indeed, if $U \subset U_i \cap U_j$ and $t \in \mathscr{F}_i(V)$ is such that $\varphi_{i,q}(s_q) = t_q$ for every $q \in V$, then $t' := (\varphi_{ij})_V(t) \in \mathscr{F}_j(V)$ is such that

$$t'_q = (\varphi_{ij})_q(t_q) = (\varphi_{ij})_q(\varphi_{i,q}(s_q)) = \varphi_{j,q}(s_q)$$

for every $q \in V$.

We let $\mathscr{F}$ be the extension of F given by theorem 11.5.1. If $U \subset U_i$ for some i, the product of $\varphi_{i,p} : F_p \to (\mathscr{F}_i)_p$ for every $p \in U$

$$\begin{aligned} \prod_{p\in U} F_p &\to \prod_{p\in U} (\mathscr{F}_i)_p \\ (s_p)_{p\in U} &\mapsto (\varphi_{i,p}(s_p))_{p\in U} \end{aligned}$$

restricts to give an isomorphism $F(U) \to \widetilde{\mathscr{F}_i}(U) \cong \mathscr{F}_i(U)$. The equivalence of categories studied in this section then gives our isomorphism $\varphi_i : \mathscr{F}|_{U_i} \to \mathscr{F}_i$. The same construction shows that $\varphi_{j,q} = (\varphi_{ij})_q \circ \varphi_{i,q}$ for all $q \in U_i \cap U_j$ implies the desired commutative diagram.

Finally, if there exists another sheaf $\mathscr{G}$ with isomorphisms $\psi_i : \mathscr{G}|_{U_i} \to \mathscr{F}_i$ satisfying the given conditions, the isomorphisms $\psi_i^{-1} \circ \varphi_i : \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$ restrict to the same map on the intersections since the diagram

$$\begin{tikzcd}
\mathscr{F}|_{U_i\cap U_j} \arrow[r, "\varphi_i"] \arrow[d, "\varphi_j"'] & \mathscr{F}_i|_{U_i\cap U_j} \arrow[dl, "\varphi_{ij}"'] \\
\mathscr{F}_j|_{U_i\cap U_j} & \mathscr{G}|_{U_i\cap U_j} \arrow[l, "\psi_j"] \arrow[u, "\psi_i"']
\end{tikzcd}$$

commutes. Proposition 11.5.3 then implies that the $\psi_i^{-1} \circ \varphi_i$ glue to form an isomorphism $\mathscr{F} \to \mathscr{G}$. □

11.6. Sheaves with values in an abelian category

In this section we'll learn how to deal with the categories of presheaves and sheaves when they have values in an abelian category. The reader who is uneasy with the notion of an abelian category shouldn't worry. We'll run over the main points as they are needed. Of course, the part on homological algebra contains more information about these ideas.

We begin with the notion of an *additive category*.

Definition 11.6.1 — Additive category. A locally small category C is said to be *additive* if it has an object 0 which is both initial and final (hereafter called a *zero-object*), if it has finite products and finite coproducts, and if each set of morphisms $\mathrm{Hom}_C(A, B)$ is endowed with an abelian group structure, in such a way that the composition maps are bilinear. A functor between additive categories is *additive* if it preserves the abelian group structures.

Explicitly, in an additive category it makes sense to add or subtract morphisms and this operation satisfies

$$\varphi \circ (\psi_1 + \psi_2) = \varphi \circ \psi_1 + \varphi \circ \psi_2 \qquad \text{and} \qquad (\varphi_1 + \varphi_2) \circ \psi = \varphi_1 \circ \psi + \varphi_2 \circ \psi$$

whenever those compositions exist. Moreover, there is a *zero-morphism*, also denoted 0, between any two objects and so two morphisms φ and ψ are equal if and only if $\varphi - \psi = 0$. This allows us to define kernels by a universal property.

Definition 11.6.2 — Kernel. Let $\varphi : A \to B$ be a morphism in an additive category C. The *kernel* of φ is the equalizer of φ and the zero-morphism. In other words, it is a morphism $i : K \to A$ such that, whenever $\zeta : Z \to A$ satisfies $\varphi \circ \zeta = 0$, there exists a unique morphism $Z \to K$ making the diagram

$$\begin{tikzcd} K \arrow[r, "i"] & A \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "0"'] & B \\ Z \arrow[u, dashed] \arrow[ru, "\zeta"'] & & \end{tikzcd}$$

commute. We denote both K and $i : K \to A$ by $\ker \varphi$.

In a general one cannot talk about inclusions in an arbitrary category. That's why the kernel is indeed a morphism. *We also observe that kernels are not guaranteed to exist in an additive category. For example, kernels not necessarily exist in the category of finitely generated* A*-modules, which is additive.*

While it doesn't make sense to say that $i : K \to A$ is injective, it is indeed a monomorphism.[4] By the bilinearity of composition, it suffices to prove that if $j : Z \to K$ is such that $i \circ j = 0$, then $j = 0$. In this case, $\varphi \circ (i \circ j) = 0$ and so there exists a unique induced morphism $Z \to K$ such that the diagram

$$\begin{tikzcd} K \arrow[r, "i"] & A \arrow[r, "\varphi"] & B \\ Z \arrow[u, dashed] \arrow[ru, "i \circ j"'] & & \end{tikzcd}$$

commutes. But both $j : Z \to \ker \varphi$ and 0 are such morphisms. The unicity then implies that $j = 0$. Thus i is always a monomorphism.

Let's see how this works in a concrete case.

■ **Example 11.6.1 — Kernels in A-Mod.** The prototypical example of an additive category is surely the category of modules over a ring A. In this case, the kernel of a linear map $\varphi : M \to N$ is the inclusion $\ker \varphi \to A$ of the usual kernel in A. Indeed, if $\zeta : P \to M$ is a linear map such that $\varphi \circ \zeta = 0$, then $\zeta(p) \in \ker \varphi$ for all $p \in P$. This implies that $\zeta : P \to M$ factors through $\ker \varphi \to A$, showing that $\ker \varphi \to A$ satisfies the universal property of the kernel. ■

Recall that in A-Mod a morphism $\varphi : M \to N$ is monic if and only if $\ker \varphi = 0$. This generalizes to additive categories, given that the morphism possesses a kernel.

Proposition 11.6.1 Let $\varphi : A \to B$ be a morphism in an additive category. Then φ is a monomorphism if and only if $0 \to A$ is its kernel.

---

[4] As every equalizer is.

Proof. Suppose that φ is a monomorphism with a kernel $i : K \to A$. Since $\varphi \circ i = 0$, it follows that $i = 0$. We affirm that $0 \to A$ satisfies the universal property of the kernel. Indeed, let $\zeta : Z \to A$ be a morphism such that $\varphi \circ \zeta = 0$. As i is a kernel, ζ factors through $i = 0$, proving that $\zeta = 0$ and so, in particular, it factors uniquely through $0 \to A$.

Conversely, suppose that $0 \to A$ is a kernel for $\varphi : A \to B$, and let $\zeta : Z \to A$ be a morphism such that $\varphi \circ \zeta = 0$. The universal property implies that ζ factors through $0 \to A$ and so $\zeta = 0$, proving that φ is a monomorphism. □

Inverting all the arrows, we have the dual definition of a cokernel.

Definition 11.6.3 — Cokernel. Let $\varphi : A \to B$ be a morphism in an additive category C. The *cokernel* of φ is the coequalizer of φ and the zero-morphism. In other words, it is a morphism $\pi : B \to C$ such that, whenever $\beta : B \to Z$ satisfies $\beta \circ \varphi = 0$, there exists a unique morphism $C \to Z$ making the diagram

$$\begin{tikzcd}
 & & Z \\
A \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "0"'] & B \arrow[ur, "\beta"] \arrow[r, "\pi"] & C \arrow[u, dashed]
\end{tikzcd}$$

commute. We denote both C and $\pi : B \to C$ by $\operatorname{coker} \varphi$.

As before, cokernels are automatically epimorphisms (as every coequalizer). Once again, let's see how this works in A-Mod.

■ **Example 11.6.2 — Cokernels in A-Mod.** Let $\varphi : M \to N$ be a morphism of A-modules. Here, the cokernel of φ is the quotient map $\pi : N \to N/\operatorname{im}\varphi$, where $\operatorname{im}\varphi$ is the usual set-theoretic image. Indeed, if $\beta : N \to P$ satisfies $\beta \circ \varphi = 0$, then $\operatorname{im}\varphi \subset \ker\beta$ and the universal property of the quotient induces a unique morphism $\tilde{\beta} : N/\operatorname{im}\varphi \to P$ which makes the diagram

$$\begin{tikzcd}
 & & P \\
M \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "0"'] & N \arrow[ur, "\beta"] \arrow[r, "\pi"] & N/\operatorname{im}\varphi \arrow[u, dashed, "\tilde{\beta}"']
\end{tikzcd}$$

commute. In other words, $\pi : N \to N/\operatorname{im}\varphi$ satisfies the universal property of the cokernel. ■

In A-Mod, a morphism $\varphi : M \to N$ is an epimorphism if and only if $N/\operatorname{im}\varphi$ vanishes. As expected, this generalizes to morphisms which possess cokernels in additive categories.

Proposition 11.6.2 Let $\varphi : A \to B$ be a morphism in an additive category. If φ has a cokernel, then it is an epimorphism if and only if $B \to 0$ is its cokernel.

Proof. We could do a similar proof to the one in proposition 11.6.1 but this result actually follows from it. Indeed, the opposite category of an additive category is still additive. Now φ is an epimorphism if and only if the opposite arrow $\varphi^{op} : B \to A$ is a monomorphism. By proposition 11.6.1 this happens precisely when $0 \to B$ is the kernel of φ^{op} in the opposite category. Reversing all the arrows once again, this happens if and only if $B \to 0$ is the cokernel of φ. ☐

Going one step further, we arrive at the main definition of this section.

Definition 11.6.4 — Abelian category. An additive category C is *abelian* if kernels and cokernels exist in C, if every monomorphism is the kernel of some morphism, and if every epimorphism is the cokernel of some morphism.

As we saw, in an additive category kernels are monomorphisms and cokernels are epimorphisms. But there is no guarantee that monomorphisms should necessarily be kernels and epimorphisms should be cokernels, as it happens with modules. In the end, we simply demand these additional features explicitly.

■ **Example 11.6.3** As usual the category of A-modules is abelian. If A is Noetherian, then so is every A-module, proving that the category of finitely generated A-modules is also abelian in this case. In particular the categories of abelian groups, of vector spaces and of finite dimensional vector spaces are abelian. ■

We now imposed that every monomorphism should be the kernel of some morphism. A priori this could be anything. Luckily we can describe it explicitly.

Proposition 11.6.3 In an abelian category C, every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.

Proof. Let $\varphi : A \to B$ be a monomorphism which is the kernel of some morphism $\beta : B \to Z$. Since C is abelian, φ has a cokernel $\pi : B \to C$. The universal property of the cokernel shows that β factors through π.

$$\begin{tikzcd} & & Z \\ A \arrow[r, "\varphi"] & B \arrow[r, "\pi"] \arrow[ur, "\beta"] & C \arrow[u, dashed] \end{tikzcd}$$

We show that φ satisfies the universal property defining the kernel of π. Let $K \to B$

be a morphism whose composition with π is the zero-morphism.

$$\begin{tikzcd}
 & & Z \\
A \arrow[r, "\varphi"] & B \arrow[r, "\pi"] \arrow[ur, "\beta"] & C \arrow[u] \\
K \arrow[ur] \arrow[urr, bend right, "0"] & &
\end{tikzcd}$$

By the commutativity of the diagram, $K \to B \to Z$ is also the zero-morphism. But φ is the kernel of β and so there exists a unique induced morphism $K \to A$, proving our claim. The statement about epimorphisms follows in the same way. □

After this long digression, we get back to our main goal of understanding how this all works in categories of presheaves. If C is an abelian category and X is a topological space, we'll see that $\mathsf{C}_X^{\mathrm{pre}}$ is also abelian. The main point behind this statement is the fact that "limits and colimits in a functor category are computed pointwise". Let's see what this means.

For now, we fix small categories I and O and a locally small category C. We denote by $\mathrm{Fun}(\mathsf{O}, \mathsf{C})$ the category whose objects are functors $\mathsf{O} \to \mathsf{C}$ and whose morphisms are natural transformations. Given an object U of O, we have a functor

$$\begin{aligned} \mathrm{ev}_U : \mathrm{Fun}(\mathsf{O}, \mathsf{C}) &\to \mathsf{C} \\ F &\mapsto F(U) \end{aligned}$$

called *evaluation* at U. This allows us to precisely state what it means for limits in a functor category to be computed pointwise.

Theorem 11.6.4 — Limits in a functor category. Let $D : \mathsf{I} \to \mathrm{Fun}(\mathsf{O}, \mathsf{C})$ be a diagram, and suppose that for each $U \in \mathsf{O}$, the diagram $\mathrm{ev}_U \circ D : \mathsf{I} \to \mathsf{C}$ has a limit. Then there is a cone on D whose image under ev_U is a limit cone on $\mathrm{ev}_U \circ D$ for each $U \in \mathsf{O}$. Moreover, any such cone on D is a limit cone.

We won't prove this result, but it isn't particularly difficult. As usual in category theory, it's just a matter of unwinding definitions. The reader can check [23] for a proof. We'll use the particular case where O is the category Open_X of open sets of a topological space X. In this context, the theorem implies, for example, that if C has binary products then so does $\mathsf{C}_X^{\mathrm{pre}}$. Moreover, the product of two presheaves $\mathscr{F}$ and $\mathscr{G}$ is the one given by

$$(\mathscr{F} \times \mathscr{G})(U) = \mathscr{F}(U) \times \mathscr{G}(U).$$

This same procedure works with all limits. Of course, theorem 11.6.4 has a dual, stating that colimits in a functor category are also computed pointwise. In particular, these theorems prove that the category of presheaves with values in an additive category is also additive. Indeed, terminal objects, finite products and finite products are instances of limits and colimits. Furthermore, the abelian group structure on $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ is given pointwise by that of $\mathrm{Hom}_{\mathsf{C}}(\mathscr{F}(U), \mathscr{G}(U))$.

Finally, if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves with values in an abelian category C, these same results also imply that the morphisms $\ker \varphi_U \to \mathscr{F}(U)$ and $\mathscr{G}(U) \to \operatorname{coker} \varphi_U$ define morphisms of presheaves which satisfy the respective universal properties in $\mathsf{C}_X^{\mathrm{pre}}$. Let's dig a little deeper in the inner workings of these notions. Let $V \subset U$ be a pair of nested open sets. This gives the commutative diagram below.

$$\begin{tikzcd}
\ker \varphi_U \arrow[r, "i_U"] & \mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\mathrm{res}_{U,V}"'] & \mathscr{G}(U) \arrow[d, "\mathrm{res}_{U,V}"] \\
\ker \varphi_V \arrow[r, "i_V"] & \mathscr{F}(V) \arrow[r, "\varphi_V"] & \mathscr{G}(V)
\end{tikzcd}$$

We'll obtain the restriction map $\ker \varphi_U \to \ker \varphi_V$ using the universal property of the kernel of φ_V. It suffices then to show that this composition

$$\begin{tikzcd}
\ker \varphi_U \arrow[r, "i_U"] & \mathscr{F}(U) \arrow[d, "\mathrm{res}_{U,V}"'] & \\
 & \mathscr{F}(V) \arrow[r, "\varphi_V"] & \mathscr{G}(V)
\end{tikzcd}$$

is the zero-morphism. But, by the commutativity of our diagram, this is the same as

$$\begin{tikzcd}
\ker \varphi_U \arrow[r, "i_U"'] \arrow[rr, bend left, "0"] & \mathscr{F}(U) \arrow[r, "\varphi_U"'] & \mathscr{G}(U) \arrow[d, "\mathrm{res}_{U,V}"] \\
 & & \mathscr{G}(V).
\end{tikzcd}$$

The universal property then yields the restriction map

$$\begin{tikzcd}
\ker \varphi_U \arrow[r, "i_U"] \arrow[d, dashed] & \mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\mathrm{res}_{U,V}"'] & \mathscr{G}(U) \arrow[d, "\mathrm{res}_{U,V}"] \\
\ker \varphi_V \arrow[r, "i_V"] & \mathscr{F}(V) \arrow[r, "\varphi_V"] & \mathscr{G}(V),
\end{tikzcd}$$

which shows that, not only $U \mapsto \ker \varphi_U$ defines a presheaf but that the collection of all the $\ker \varphi_U \to \mathscr{F}(U)$ defines a morphism of presheaves. In the same fashion, the universal property of the cokernel gives the restriction maps below

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\mathrm{res}_{U,V}"'] & \mathscr{G}(U) \arrow[r, "\pi_U"] \arrow[d, "\mathrm{res}_{U,V}"] & \operatorname{coker} \varphi_U \arrow[d, dashed] \\
\mathscr{F}(V) \arrow[r, "\varphi_V"] & \mathscr{G}(V) \arrow[r, "\pi_V"] & \operatorname{coker} \varphi_V,
\end{tikzcd}$$

which shows that $U \mapsto \operatorname{coker} \varphi_U$ defines a presheaf and the collection of all the $\mathscr{G}(U) \to \operatorname{coker} \varphi_U$ defines a morphism of presheaves $\mathscr{G} \to \operatorname{coker} \varphi$ which satisfies the universal property of the cokernel in C_X^{pre}.

We now finish the proof of the theorem below.

Theorem 11.6.5 Let C be an abelian category and X be a topological space. Then the category C_X^{pre} of presheaves is also abelian.

Proof. All that remains is to show that every monomorphism in C_X^{pre} is the kernel of some morphism, and that every epimorphism is the cokernel of some morphism. Once again, this follows from theorem 11.6.4 and its dual counterpart.

We first observe that a morphism $\varphi : \mathscr{F} \to \mathscr{G}$ is monic if and only if all the components φ_U are monomorphisms. Indeed, if all the φ_U are monomorphisms we can apply the definition of a monomorphism pointwise to see that φ is also monic. Conversely, if φ is monic, then

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\mathrm{id}_{\mathscr{F}}"] \arrow[d, "\mathrm{id}_{\mathscr{F}}"'] & \mathscr{F} \arrow[d, "\varphi"] \\
\mathscr{F} \arrow[r, "\varphi"] & \mathscr{G}
\end{tikzcd}$$

is a pullback square and so

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\mathrm{id}_{\mathscr{F}(U)}"] \arrow[d, "\mathrm{id}_{\mathscr{F}(U)}"'] & \mathscr{F}(U) \arrow[d, "\varphi_U"] \\
\mathscr{F}(U) \arrow[r, "\varphi_U"] & \mathscr{G}(U)
\end{tikzcd}$$

is a pullback square for every open set U since limits are computed pointwise.

Now, if $\varphi : \mathscr{F} \to \mathscr{G}$ is monic then all the φ_U are the kernels of their cokernels. If $V \subset U$ is a pair of nested open sets, we have the following commutative diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\mathrm{res}_{U,V}"'] & \mathscr{G}(U) \arrow[r, "\pi_U"] \arrow[d, "\mathrm{res}_{U,V}"] & \operatorname{coker} \varphi_U \arrow[d, "\mathrm{res}_{U,V}"] \\
\mathscr{F}(V) \arrow[r, "\varphi_V"] & \mathscr{G}(V) \arrow[r, "\pi_V"] & \operatorname{coker} \varphi_V
\end{tikzcd}$$

which shows that φ is the kernel of $\pi : \mathscr{G} \to \operatorname{coker} \varphi$. The same arguments show that every epimorphism is the cokernel of its kernel. $\square$

We go back to general abelian categories for a while, since there are two important properties of these categories that we want to talk about. The first one is the fact that being an isomorphism is equivalent to being both monic and epic.

Proposition 11.6.6 Let $\varphi : A \to B$ be a morphism in an abelian category C, and assume that φ is both a monomorphism and an epimorphism. Then φ is an isomorphism.

Proof. Since φ is both monic and epic, its kernel is $0 \to A$ and its cokernel is $B \to 0$. Furthermore, by proposition 11.6.3, φ is the kernel of $B \to 0$ and the cokernel of $0 \to A$. Now consider the diagram below.

$$\begin{tikzcd}
 & & B \arrow[d, "\mathrm{id}_B"] & \\
0 \arrow[r] & A \arrow[r, "\varphi"] & B \arrow[r] & 0
\end{tikzcd}$$

Since $B \to B \to 0$ is the zero morphism and φ is the kernel of $B \to 0$, we obtain a unique morphism $\psi : B \to A$ making the diagram

$$\begin{tikzcd}
 & & B \arrow[d, "\mathrm{id}_B"] \arrow[dl, dashed, "\psi"'] & \\
0 \arrow[r] & A \arrow[r, "\varphi"] & B \arrow[r] & 0
\end{tikzcd}$$

commute. As $\varphi \circ \psi = \mathrm{id}_B$, this shows that φ has a right-inverse. Similarly, the fact that φ is the cokernel of $0 \to A$ implies the existence of a unique morphism $\eta : B \to A$ such that the diagram

$$\begin{tikzcd}
 & A & & \\
0 \arrow[r] & A \arrow[u, "\mathrm{id}_A"] \arrow[r, "\varphi"] & B \arrow[ul, dashed, "\eta"'] \arrow[r] & 0
\end{tikzcd}$$

commutes. It follows that φ has both a left-inverse η and a right-inverse ψ. Thus, $\eta = \psi$ is a two-sided inverse of φ and so φ is an isomorphism. ☐

This evidently holds in all the categories of example 11.6.3 since in every one of them monomorphisms are injective and epimorphisms are surjective. Therefore φ is bijective and, as in any algebraic category, bijective morphisms are isomorphisms. Nevertheless, it is reassuring to know that this is true in any abelian category. We also observe that this statement does *not* hold even in fairly mundane categories, such as Ring, for example.

The second property that we want to talk about is the fact that, in an abelian category, finite products and finite coproducts coincide. We enunciate this as a proposition, where we denote coproducts as direct sums.

Proposition 11.6.7 Let A and B be two objects in an abelian category C. Then $A \times B$ and $A \oplus B$ are naturally isomorphic.

Sketch of proof. Let's understand how this works. By its very definition, a coproduct is endowed with morphisms

$$A \xrightarrow{i_A} A \oplus B \xleftarrow{i_B} B.$$

In general, $A \oplus B$ can't satisfy the universal property of the coproduct since morphisms $\pi_A : A \oplus B \to A$ and $\pi_B : A \oplus B \to B$ need not exist. But in an abelian category we can obtain such morphisms using zero-morphisms and the universal property of coproducts:

$$\begin{tikzcd}
A \arrow[dr, "i_A"'] \arrow[drr, bend left, "\mathrm{id}_A"] & & \\
& A \oplus B \arrow[r, dashed, "\pi_A"] & A \\
B \arrow[ur, "i_B"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[dr, "i_A"'] \arrow[drr, bend left, "0"] & & \\
& A \oplus B \arrow[r, dashed, "\pi_B"] & B. \\
B \arrow[ur, "i_B"] \arrow[urr, bend right, "\mathrm{id}_B"'] & &
\end{tikzcd}$$

Then the universal property of products shows that there exists a unique morphism $A \oplus B \to A \times B$ making the diagram

$$\begin{tikzcd}
& & A \\
A \oplus B \arrow[r, dashed] \arrow[urr, bend left, "\pi_A"] \arrow[drr, bend right, "\pi_B"'] & A \times B \arrow[ur] \arrow[dr] & \\
& & B
\end{tikzcd}$$

commute. This is the desired isomorphism. We won't prove that but, as usual, it's nothing but a whole lot of universal properties. The reader is encouraged to prove that π_B is the cokernel of i_A, i_A is the kernel of π_B, and to use that to show that $0 \to A \oplus B$ is the kernel of $A \oplus B \to A \times B$, concluding that it is a monomorphism. Similarly, one can show that it is an epimorphism, establishing the result. The interested reader can also check theorem 6.1.9 for a (different) complete proof. ☐

We now explore how we can define the arrow-theoretic image of a morphism in an abelian category. For that, let's translate our intuitive notion of $\operatorname{im} \varphi$ in Set to a purely arrow-theoretic statement. The main point in Set is that $\operatorname{im} \varphi$ is the smallest subset of B to which we can restrict the codomain of φ to. In other words, we can factor $\varphi : A \to B$ as

$$\begin{tikzcd} A \arrow[r] & \operatorname{im}\varphi \arrow[r, hook] & B, \end{tikzcd}$$

where $\operatorname{im}\varphi \to B$ is injective and $\operatorname{im}\varphi$ is the smallest subset of B which allows this decomposition. Switching to categorical terms, we arrive at the following universal property: the image of $\varphi : A \to B$ is a monomorphism $i : K \to B$ such that φ factors through i and that is initial with these properties. That is, if $L \to B$ is another monomorphism through which φ also factors, then it exists a unique morphism $K \to L$ such that the diagram

$$\begin{tikzcd} & L \arrow[dr, hook] & \\ A \arrow[ur] \arrow[r] \arrow[rr, bend right, "\varphi"'] & K \arrow[u, dashed] \arrow[r, hook, "i"] & B \end{tikzcd}$$

commutes. In an abelian category, it could well happen that no morphism $i : K \to B$ satisfies this universal property. Luckily, this is never the case.

Proposition 11.6.8 Let $\varphi : A \to B$ be a morphism in an abelian category, and let $i : K \to B$ be the kernel of $\operatorname{coker}\varphi$. Then i is a monomorphism through which φ factors, and it is initial with these properties.

Proof. It is clear that i is a monomorphism by the fact that it is a kernel. Since $i : K \to B$ is the kernel of the cokernel $\pi_\varphi : B \to C_\varphi$ of φ, the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] & B \arrow[r, shift left, "\pi_\varphi"] \arrow[r, shift right, "0"'] & C_\varphi \\ K \arrow[ur, "i"'] & & \end{tikzcd}$$

commutes. The universal property of the kernel then implies the existence of a morphism $A \to K$ factoring φ through i. We now show that i satisfies the desired universal property. Let $\lambda : L \to B$ be another monomorphism through which φ factors, and consider its cokernel $\pi_\lambda : B \to C_\lambda$.

$$\begin{tikzcd} & L \arrow[dr, hook, "\lambda"] & & \\ A \arrow[ur] \arrow[r] \arrow[rr, bend right, "\varphi"'] & K \arrow[r, hook, "i"] & B \arrow[dr, "\pi_\lambda"'] & \\ & & & C_\lambda \end{tikzcd}$$

Since φ factors through λ, the composition $A \to B \to C_\lambda$ is 0. The universal property of $\operatorname{coker}\varphi$ induces a morphism $C_\varphi \to C_\lambda$:

$$\begin{tikzcd}
& L \arrow[dr, hook, "\lambda"] & & \\
A \arrow[ur] \arrow[r] \arrow[rr, bend right, "\varphi"'] & K \arrow[r, hook, "i"] & B \arrow[r, "\pi_\varphi"] \arrow[dr, "\pi_\lambda"'] & C_\varphi \arrow[d, dashed] \\
& & & C_\lambda .
\end{tikzcd}$$

Observe that since $K \to B \to C_\varphi$ is the zero-morphism, so is $K \to B \to C_\lambda$. But λ is a monomorphism, which implies that it is the kernel of π_λ. Its universal property then implies the existence of a unique morphism $K \to L$ making the diagram commute. ☐

This motivates the definition below.

Definition 11.6.5 — Image. Let $\varphi : A \to B$ be a morphism in an abelian category. Its *image*, denoted $\operatorname{im}\varphi$, is the kernel of $\operatorname{coker}\varphi$.

Just as a reality-check, we verify that this works as intended in the concrete case of A-modules.

■ **Example 11.6.4 — Images in A-Mod.** Let $\varphi : M \to N$ be a morphism of A-modules. Recall that the cokernel of φ is the projection map $N \to N/\operatorname{im}\varphi$, where $\operatorname{im}\varphi$ is the usual set-theoretic image. The kernel of this morphism is surely $\operatorname{im}\varphi \to N$, which is the inclusion of the set-theoretic image in N. ■

We now have everything we need to talk about exact sequences.

Definition 11.6.6 — Exact sequence. Consider a sequence of objects and morphisms in an abelian category:

$$\cdots \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow \cdots .$$

We say that this sequence is *exact* at B if $\psi \circ \varphi = 0$ and $\operatorname{coker}\varphi \circ \ker\psi = 0$.

Let's see that this definition indeed encodes what we know about exact sequences. The first condition $\psi \circ \varphi = 0$ implies that φ factors through $\ker\psi$.

$$\begin{tikzcd}
\cdots \arrow[r] & A \arrow[rr, "\varphi"] \arrow[dr, dashed] & & B \arrow[r, "\psi"] & C \arrow[r] & \cdots \\
& & \ker\psi \arrow[ur, hook] & & &
\end{tikzcd}$$

Since $\ker\psi$ is a monomorphism, the universal property of $\operatorname{im}\varphi$ yields a unique factorization of $\operatorname{im}\varphi$ through $\ker\psi$.

$$\begin{tikzcd}
 & & \operatorname{im}\varphi \arrow[dr, hook] \arrow[dd, dashed] & & & \\
\cdots \arrow[r] & A \arrow[ur] \arrow[rr] \arrow[dr] & & B \arrow[r, "\psi"] & C \arrow[r] & \cdots . \\
 & & \ker\psi \arrow[ur, hook] & & &
\end{tikzcd}$$

Similarly, the second condition tells us that $\ker\psi$ factors through $\ker(\operatorname{coker}\varphi) = \operatorname{im}\varphi$, proving that $\operatorname{im}\varphi = \ker\psi$. Conversely, if $\operatorname{im}\varphi = \ker\psi$, then $\psi \circ \varphi : A \to C$ is the zero-morphism since $\ker\psi \to B \to C$ is. Also, in this case we have that $\operatorname{coker}\varphi \circ \ker\psi$ is nothing but $\operatorname{coker}\varphi \circ \operatorname{im}\varphi$, which is 0 since $\operatorname{im}\varphi = \ker(\operatorname{coker}\varphi)$. In other words, we have proved the result below.

Proposition 11.6.9 Consider a sequence of objects and morphisms in an abelian category:

$$\cdots \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow \cdots .$$

Then this sequence is exact at B if and only if ψ has a kernel that is an image of φ.

Once again, we go back to understanding how this all works in the category of presheaves. Let $\mathscr{F}$ and $\mathscr{G}$ be presheaves over a topological space X with values in an abelian category C. One more time, theorem 11.6.4 and its dual counterpart are all that we need to know. The presheaf $\mathscr{F} \oplus \mathscr{G}$ defined pointwise by

$$(\mathscr{F} \oplus \mathscr{G})(U) := \mathscr{F}(U) \oplus \mathscr{G}(U)$$

satisfies the universal property of both the product and the coproduct in $\mathsf{C}_X^{\mathrm{pre}}$. A priori there's no reason for C to have infinite products or coproducts but, if they exist then $\mathsf{C}_X^{\mathrm{pre}}$ also has infinite products or coproducts, and they are still computed pointwise, even though they may not coincide.

If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, then the assignment $U \mapsto \operatorname{im}\varphi_U$ defines a presheaf which satisfies the universal property of $\operatorname{im}\varphi$ in $\mathsf{C}_X^{\mathrm{pre}}$. Since $\operatorname{im}\varphi = \ker(\operatorname{coker}\varphi)$, the restriction maps of precisely those of the kernel presheaf. Namely, if $V \subset U$ is a pair of nested open sets in X, precisely the same argument that we used to describe the restriction maps of the kernel presheaf show the existence of a induced morphism $\operatorname{im} \varphi_U \to \operatorname{im} \varphi_V$ making the diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] \arrow[d, "\mathrm{res}_{U,V}"'] \arrow[rr, bend left, "\varphi_U"] & \operatorname{im} \varphi_U \arrow[r, hook] \arrow[d, dashed] & \mathscr{G}(U) \arrow[r] \arrow[d, "\mathrm{res}_{U,V}"] & \operatorname{coker} \varphi_U \arrow[d, "\mathrm{res}_{U,V}"] \\
\mathscr{F}(V) \arrow[r] \arrow[rr, bend right, "\varphi_V"'] & \operatorname{im} \varphi_V \arrow[r, hook] & \mathscr{G}(V) \arrow[r] & \operatorname{coker} \varphi_V
\end{tikzcd}$$

commute. Surely, if C is also an algebraic category[5] then monomorphisms are injective, and we can identify $\operatorname{im} \varphi_U$ and $\operatorname{im} \varphi_V$ as subsets of $\mathscr{G}(U)$ and $\mathscr{G}(V)$. In this case, the restriction maps become precisely those of $\mathscr{G}$.

Exact sequences of presheaves are even simpler. Consider a sequence of presheaves with values in an abelian category:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathscr{F} \arrow[r, "\varphi"] & \mathscr{G} \arrow[r, "\psi"] & \mathscr{H} \arrow[r] & \cdots .
\end{tikzcd}$$

As we saw in proposition 11.6.9, this sequence is exact at $\mathscr{G}$ if and only if $\operatorname{im} \varphi = \ker \psi$. But images and kernels are computed pointwise, so this holds if and only if $\operatorname{im} \varphi_U = \ker \psi_U$ for every open set $U \subset X$. In other words, our original sequence is exact at $\mathscr{G}$ if and only if

$$\begin{tikzcd}
\cdots \arrow[r] & \mathscr{F}(U) \arrow[r, "\varphi_U"] & \mathscr{G}(U) \arrow[r, "\psi_U"] & \mathscr{H}(U) \arrow[r] & \cdots
\end{tikzcd}$$

is exact at $\mathscr{G}(U)$ whenever U is an open set of X.

We pass our attention to the category of sheaves. In particular, we'll prove the theorem below.

Theorem 11.6.10 Let C be an abelian algebraic category and X be a topological space. Then the category C_X of sheaves is also abelian.

Now the answer to all our prayers will be the theorem 11.3.4, which says that the sheafification functor is left adjoint to the forgetful functor from sheaves to presheaves. As we remarked in the section about the sheafification, not only the inclusion of the category of sheaves in the category of presheaves preserves limits, but it also creates limits, since this functor is fully faithful. In other words, a limit in C_X exists if and only if it exists in C_X^{pre}, in which case they coincide. As an example, if $\mathscr{F}$ and $\mathscr{G}$ are sheaves then presheaf defined as

$$(\mathscr{F} \oplus \mathscr{G})(U) = \mathscr{F}(U) \oplus \mathscr{G}(U)$$

---

[5] It does *not* suffice that C is concrete.

for every open set $U \subset X$, satisfies not only the universal property of products in $\mathsf{C}_X^{\mathrm{pre}}$ but also in C_X. In particular, it is also a sheaf. Similarly, if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, the presheaf kernel $\ker \varphi$ is also a sheaf and satisfies its universal property in C_X.

Also, the sheafification functor preserves colimits. This implies that, in order to construct a colimit of sheaves, we can construct the same colimit in the category of presheaves and then sheafify. In particular, the sheafification of the cokernels, direct sums and images that we constructed in this section all satisfy their respective universal properties in the category of sheaves. Hereafter, when dealing with sheaves, we'll denote by $\operatorname{coker} \varphi$ the sheafification of the presheaf cokernel and similarly to all the other notions that need to be sheafified.

All that remains to prove in order to obtain theorem 11.6.10 is the fact that every monomorphism is the kernel of its cokernel and that every epimorphism is the cokernel of its kernel. For that, we'll need to talk about stalks. The main result is the proposition below, which basically says that filtered colimits are exact.

Proposition 11.6.11 Let C be an abelian algebraic category and X be a topological space. Then, for every $p \in X$, the stalk functor

$$\begin{aligned} (-)_p : \mathsf{C}_X &\to \mathsf{C} \\ \mathscr{F} &\mapsto \mathscr{F}_p \end{aligned}$$

is exact.

We observe that, since the image sheaf is the sheafification of the image presheaf, exact sequences in C_X and in $\mathsf{C}_X^{\mathrm{pre}}$ are different things.

Proof. Consider a short exact sequence of sheaves:

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H} \longrightarrow 0.$$

The fact that this sequence is exact amounts precisely to imposing that φ is a monomorphism, ψ is an epimorphism and $\ker \psi = \operatorname{im} \varphi$. Propositions 11.2.1 and 11.2.2 imply that φ_p is also a monomorphism and ψ_p is also an epimorphism. Also, since filtered colimits commute with finite limits and colimits in an algebraic category, taking kernels and images commute with taking stalks, proving that

$$\ker \psi_p = (\ker \psi)_p = (\operatorname{im} \varphi)_p = \operatorname{im} \varphi_p.$$

In other words, the sequence of stalks

$$0 \longrightarrow \mathscr{F}_p \xrightarrow{\varphi_p} \mathscr{G}_p \xrightarrow{\psi_p} \mathscr{H}_p \longrightarrow 0.$$

is also exact. ☐

Actually, more is true. If the induced sequence on the stalks is exact for every $p \in X$, then the original sequence is also exact. Indeed, the same propositions 11.2.1 and 11.2.2 imply that φ_p is a monomorphism and ψ_p is an epimorphism if and only if φ and ψ are. Similarly, $\ker\psi = \operatorname{im}\varphi$ holds if and only if $\ker\psi_p = \operatorname{im}\varphi_p$ for every p. This allows us to generalize the previous proposition.

> **Corollary 11.6.12** Let $f : X \to Y$ be a continuous map. Then the inverse image functor $f^{-1} : \mathsf{C}_Y \to \mathsf{C}_X$ is exact.

Proof. Consider a short exact sequence of sheaves over Y:

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H} \longrightarrow 0.$$

Taking stalks at $f(p)$ yields another exact sequence:

$$0 \longrightarrow \mathscr{F}_{f(p)} \longrightarrow \mathscr{G}_{f(p)} \longrightarrow \mathscr{H}_{f(p)} \longrightarrow 0.$$

Now, recall that $\mathscr{F}_{f(p)} \cong (f^{-1}\mathscr{F})_p$ and that this isomorphism is natural in $\mathscr{F}$ (by corollary 11.4.2). In other words, the sequence

$$0 \longrightarrow (f^{-1}\mathscr{F})_p \longrightarrow (f^{-1}\mathscr{G})_p \longrightarrow (f^{-1}\mathscr{H})_p \longrightarrow 0.$$

is also exact. Since this holds for every $p \in X$, the result follows by the previous discussion. ☐

We now finish our proof of the theorem 11.6.10 by showing that indeed every monomorphism is the kernel of its cokernel and that every epimorphism is the cokernel of its kernel.

Proof of theorem 11.6.10. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a monomorphism of sheaves. Being a kernel of its cokernel is equivalent to demanding the sequence

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \longrightarrow \operatorname{coker}\varphi$$

to be exact. Since the sequence induced on the stalks is indeed exact, so is the sequence above. The same argument proves that every epimorphism is the cokernel of its kernel. ☐

As some last remarks, we observe that since a sequence of the form

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H}$$

is exact if and only if $\varphi = \ker \psi$. In particular, right adjoints preserve such exact sequences. As usual, we say that a functor that preserves exact sequences of this form is *left exact*. This should become another Pavlovian reaction: right adjoints are left exact. Similarly, a sequence of the form

$$\mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H} \longrightarrow 0$$

is exact if and only if $\psi = \operatorname{coker} \varphi$. A functor that preserves such a sequence is said to be *right exact*. Particularly, left adjoints are right exact.

A notable case is the direct image functor f_* which is left exact. Somewhat more interesting is the case of the functor $\Gamma(U, -)$ which takes a sheaf to its sections over an open set U. Since the inclusion of the category of sheaves in the category of presheaves is right adjoint to the sheafification functor, an exact sequence of sheaves of the form

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H}$$

is still exact in the category of presheaves. But kernels and images are computed pointwise in the category of presheaves, so this means that the sequence

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\varphi_U} \mathscr{G}(U) \xrightarrow{\psi_U} \mathscr{H}(U)$$

is exact whenever U is an open set. In other words, the functor $\Gamma(U, -)$ is left exact.

12. Ringed spaces

Nowadays it has become clear that most "geometric spaces" are best described as objects in a certain category that are locally isomorphic to a class of standard geometric spaces. For instance, a smooth manifold is nothing but a topological space which is locally diffeomorphic to an open set of $\mathbb{R}^n$. In this chapter we'll construct the category of ringed spaces which will contain basically all the geometric objects that we want to study in these notes.

12.1. Basic definitions

Definition 12.1.1 — Ringed space. A *ringed space* is a pair $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a sheaf of rings over X, hereafter called *structure sheaf*. A morphism of ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a pair $(f, f^\sharp)$, where $f : X \to Y$ is a continuous map and $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is a morphism of sheaves. We denote the category of ringed spaces by RS.

We'll often denote a ringed space $(X, \mathscr{O}_X)$ simply by X and a morphism $(f, f^\sharp)$ by f when there's no possibility of confusion. Recall that the musical adjunction (theorem 11.4.4) says that the datum of a morphism of sheaves $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is equivalent to the datum of a morphism $f^\flat : f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$.

We remark that a morphism $f : X \to Y$ of ringed spaces induces morphisms on the stalks of the structure sheaves as follows. If $p \in X$, we use the identification $(f^{-1}\mathscr{O}_Y)_p = \mathscr{O}_{Y,f(p)}$ to obtain a morphism

$$f^\flat_p : \mathscr{O}_{Y,f(p)} = (f^{-1}\mathscr{O}_Y)_p \to \mathscr{O}_{X,p}.$$

Somewhat more explicitly, if $V \subset U$ is a pair of nested open sets containing $f(p)$ in Y, then the universal property of the coproduct induces a morphism $\mathscr{O}_{Y,f(p)} \to \mathscr{O}_{X,p}$

$$\begin{tikzcd}
\mathscr{O}_Y(U) \arrow[rr, "f^\sharp_U"] \arrow[dd] \arrow[dr] & & \Gamma(U, f_*\mathscr{O}_X) \arrow[dd] \arrow[dr] & \\
& \mathscr{O}_{Y,f(p)} \arrow[rr, dashed] & & \mathscr{O}_{X,p} \\
\mathscr{O}_Y(V) \arrow[rr, "f^\sharp_V"] \arrow[ur] & & \Gamma(V, f_*\mathscr{O}_X). \arrow[ur] &
\end{tikzcd}$$

The proof of the musical adjunction shows that this coincides with the morphism $f^{\flat}_p$ described above.

The simplest examples of ringed spaces are a topological space with its sheaf of continuous functions and a singleton endowed with the constant sheaf. We talk about a more interesting example below.

■ **Example 12.1.1 — Manifolds as ringed spaces.** A smooth manifold, endowed with its natural sheaf of smooth functions is a ringed space. Moreover, the example 11.4.3 shows that a morphism of smooth manifolds is always a morphism of ringed spaces. In other words, the category of smooth manifolds is a subcategory of RS.

But this is not a *full* subcategory! In other words, there are morphisms of ringed spaces between manifolds which are not smooth maps. Indeed, in a morphism $(f, f^{\sharp})$ of ringed spaces, the morphism of sheaves $f^{\sharp}$ need not be composition by the map f. In the sequence, we will impose a condition on morphisms of ringed spaces that will solve this problem. ■

Other examples of ringed spaces that will appear eventually in these notes are schemes, formal schemes, analytic spaces and other classes of manifolds, such as topological and complex manifolds. These notions will be dealt with in due time.

In some sense, ringed spaces are not the geometric spaces that we wish to work with. In such space we want to think about the sections of the structure sheaf as functions. A reasonable property to ask of such functions is that those which do not vanish at a point p should be invertible in some neighborhood of p. In other words, all the elements of the stalk not contained in the ideal of functions vanishing at p are units. This implies that the stalk is a local ring. (Proposition 2.5.1.) We arrive at the definition below.

Definition 12.1.2 — Locally ringed space. A *locally ringed space* is a ringed space $(X, \mathscr{O}_X)$ such that for all $p \in X$ the stalk $\mathscr{O}_{X,p}$ is a local ring. We denote by $\mathfrak{m}_p$ the maximal ideal of $\mathscr{O}_{X,p}$ and by $\kappa(p)$ the residue field $\mathscr{O}_{X,p}/\mathfrak{m}_p$. A morphism of locally ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces $(f, f^{\sharp})$ such that for all $p \in X$ the induced morphism on stalks

$$f^{\flat}_p : \mathscr{O}_{Y,f(p)} \to \mathscr{O}_{X,p}$$

is a morphism of local rings. We denote by LRS the category of locally ringed spaces.

We recall that the morphism of rings $f^{\flat}_p : \mathscr{O}_{Y,f(p)} \to \mathscr{O}_{X,p}$ is said to be local if $f^{\flat}_p(\mathfrak{m}_{f(p)}) \subset \mathfrak{m}_p$. In our intuitive image, this means that if a function on Y vanishes on $f(p)$, then its image in X must vanish at p, which indeed holds when $f^{\sharp}$ is defined by composing with f.

All the examples of ringed spaces (with perhaps the exception of a singleton with a constant sheaf of rings) that we described above are indeed locally ringed spaces.

One could hope that the morphisms of locally ringed spaces between smooth manifolds now correspond precisely to smooth maps. This is not the case, the main problem being that the structure sheaves of manifolds are not only rings but $\mathbb{R}$- or $\mathbb{C}$-algebras. This brings us to the definition below.

Definition 12.1.3 Fix a locally ringed space S. A *locally ringed space over* S is a locally ringed space X endowed with a morphism $X \to S$. A morphism $X \to Y$ of locally ringed spaces over S is a morphism of locally ringed spaces $X \to Y$ such that the diagram

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\
& S &
\end{tikzcd}$$

commutes. We denote by LRS/S the category of locally ringed spaces over S. In other words, LRS/S is the slice category $\mathsf{LRS} \downarrow S$.

Let's see how this works. Suppose that $(X, \mathscr{O}_X)$ is a locally ringed space. Saying that $\mathscr{O}_X$ is a sheaf of k-algebras means that, whenever $U \subset X$ is an open set, we have a natural morphism of rings $k \to \mathscr{O}_X(U)$ compatible with the restriction maps. In particular, the diagram

$$\begin{tikzcd}
& \mathscr{O}_X(X) \arrow[dd, "\mathrm{res}_{X,U}"] \\
k \arrow[ur] \arrow[dr] & \\
& \mathscr{O}_X(U)
\end{tikzcd}$$

commutes, proving that all the morphisms $k \to \mathscr{O}_X(U)$ are actually determined by $k \to \mathscr{O}_X(X)$. We can rephrase this condition in yet another way. If we consider k as the locally ringed space whose underlying topological space is a singleton $\{*\}$ and whose structure sheaf is the constant sheaf $\underline{k}$, such a morphism is precisely the data of a morphism of locally ringed spaces $X \to k$. Indeed, there is a single continuous map $f : X \to \{*\}$ and a morphism

$$f^\sharp : \underline{k} \to f_*\mathscr{O}_X$$

is simply a morphism of rings $A \to \mathscr{O}_X(X)$ by example 11.4.2. We finally get our desired result.

Proposition 12.1.1 Let Man be the category of smooth manifolds. Then the inclusion $\mathsf{Man} \to \mathsf{LRS}/\mathbb{R}$ is fully faithful. In other words, Man is a full subcategory of $\mathsf{LRS}/\mathbb{R}$.

Proof. Let $(f, f^\sharp) : (M, C^\infty_M) \to (N, C^\infty_N)$ be a morphism of locally ringed spaces over $\mathbb{R}$. By our discussion in example 11.4.3, it suffices to show that $f^\sharp : C^\infty_N \to f_* C^\infty_M$ is given by composition with f. We observe that the maximal ideal of the stalk $C^\infty_{M,p}$ is composed by the functions which vanish at p. It follows that

$$\begin{aligned} C^\infty_{M,p} &\to \mathbb{R} \\ \overline{s} &\mapsto s(p) \end{aligned}$$

is a surjective morphism of $\mathbb{R}$-algebras which induces an isomorphism $\kappa(p) \to \mathbb{R}$. Similarly, $\kappa(f(p))$ is also isomorphic to $\mathbb{R}$. Now, let s be a section of $C^\infty_N(U)$ and observe the commutative diagram below.

$$\begin{tikzcd}
s \arrow[d, mapsto] & C^\infty_N(U) \arrow[r, "f^\sharp_U"] \arrow[d] & C^\infty_M(f^{-1}(U)) \arrow[d] & f^\sharp_U(s) \arrow[d, mapsto] \\
s_{f(p)} \arrow[d, mapsto] & C^\infty_{N,f(p)} \arrow[r, "f^\flat_p"] \arrow[d] & C^\infty_{M,p} \arrow[d] & (f^\sharp_U(s))_p \arrow[d, mapsto] \\
s(f(p)) & \mathbb{R} \arrow[r] & \mathbb{R} & f^\sharp_U(s)(p)
\end{tikzcd}$$

Since the identity is the only morphism of $\mathbb{R}$-algebras $\mathbb{R} \to \mathbb{R}$, it follows that $f^\sharp_U(s) = s \circ f$, concluding the proof. □

Similarly, the category of topological manifolds is a full subcategory of LRS/$\mathbb{R}$ and the category of complex manifolds is a full subcategory of LRS/$\mathbb{C}$. The proof is precisely the same.

This result allows us to *define* a smooth manifold as a locally ringed space X over $\mathbb{R}$, whose underlying topological space is both Hausdorff and second countable, which satisfies the following condition: every point $p \in X$ has a neighborhood $U \subset X$ that is isomorphic to an open set of $\mathbb{R}^n$, endowed with its natural sheaf of smooth functions, as locally ringed spaces over $\mathbb{R}$. Topological and complex manifolds admit descriptions akin to this one.

In order to check that some locally ringed space is a manifold, we have to construct some isomorphisms. Fortunately, there is an useful criterion for that.

Proposition 12.1.2 Let X and Y be locally ringed spaces and $f : X \to Y$ be an isomorphism of ringed spaces. Then f is also an isomorphism of locally ringed spaces.

Proof. We have to prove that if $f^\flat_p : \mathscr{O}_{Y,f(p)} \to \mathscr{O}_{X,p}$ is an isomorphism of rings, then $f^\flat_p(\mathfrak{m}_{f(p)}) \subset \mathfrak{m}_p$. Or, equivalently, that $\mathfrak{m}_{f(p)} = (f^\flat_p)^{-1}(\mathfrak{m}_p)$. Since $f^\flat_p$ is surjective, the preimage of $\mathfrak{m}_p$ is a maximal ideal of $\mathscr{O}_{Y,f(p)}$. The result follows as $\mathfrak{m}_{f(p)}$ is the only such ideal. □

In other words, in order to verify that a morphism $f : X \to Y$ between locally ringed spaces is an isomorphism, it suffices to show that the underlying continuous map is an homeomorphism and that $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is an isomorphism of sheaves.

Another useful result gives a criterion for a morphism of ringed spaces to be a monomorphism. For that, recall that a monomorphism in the category of topological spaces is simply an injective continuous function.

Proposition 12.1.3 Let $f : X \to Y$ be a morphism of ringed spaces such that the underlying map of topological spaces is a monomorphism and such that either $f^\flat : f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ or $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is an epimorphism. Then f is a monomorphism in the category of ringed spaces.

Proof. Let $(g_1, g_1^\sharp), (g_2, g_2^\sharp) : (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$ be morphisms of ringed spaces such that the diagram

$$\begin{tikzcd}(Z, \mathscr{O}_Z) \arrow[r, shift left, "{(g_1,g_1^\sharp)}"] \arrow[r, shift right, "{(g_2,g_2^\sharp)}"'] & (X, \mathscr{O}_X) \arrow[r, "{(f,f^\sharp)}"] & (Y, \mathscr{O}_Y)\end{tikzcd}$$

commutes. Passing to the diagram of underlying continuous maps, we get that $g_1 = g_2$. Henceforth, we'll denote both g_1 and g_2 by g. By the musical adjunction, the diagram of sheaves over Z

$$\begin{tikzcd}(f \circ g)^{-1}\mathscr{O}_Y \arrow[r] & g^{-1}\mathscr{O}_X \arrow[r, shift left] \arrow[r, shift right] & \mathscr{O}_Z\end{tikzcd}$$

is also commutative. Passing to the stalks we obtain the commutative diagram

$$\begin{tikzcd}\mathscr{O}_{Y,f(g(p))} \arrow[r, "f^\flat_{g(p)}"] & \mathscr{O}_{X,g(p)} \arrow[r, shift left] \arrow[r, shift right] & \mathscr{O}_{Z,p}.\end{tikzcd}$$

If $f^\flat$ is an epimorphism, then so are its stalks. Therefore, the morphisms on the right coincide, proving that so do the morphisms

$$\begin{tikzcd}g^{-1}\mathscr{O}_X \arrow[r, shift left] \arrow[r, shift right] & \mathscr{O}_Z\end{tikzcd}$$

Another application of the musical adjunction implies that $g_1^\sharp = g_2^\sharp$. We'll now show that if $f^\sharp$ is an epimorphism, then so is $f^\flat$, reducing this case to the one already proved. Since f is injective, the direct image functor f_* is fully faithful. By the musical adjunction, we have isomorphisms

$$\mathrm{Hom}_X(\mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_Y(f_*\mathscr{F}, f_*\mathscr{G}) \cong \mathrm{Hom}_X(f^{-1}f_*\mathscr{F}, \mathscr{G})$$

which are natural in $\mathscr{F}, \mathscr{G} \in \mathsf{Ring}_X$. The Yoneda lemma then implies that the counit $\varepsilon : f^{-1}f_* \to \mathrm{id}$ is an isomorphism. Finally, the fact that f^{-1} is left adjoint and so preserves colimits implies that if $f^\sharp$ is an epimorphism, then so is

$$\begin{tikzcd}
f^{-1}\mathscr{O}_Y \arrow[r, "f^{-1}f^\sharp"'] \arrow[rr, bend left, "f^\flat"] & f^{-1}f_*\mathscr{O}_X \arrow[r, "\varepsilon_{\mathscr{O}_X}"'] & \mathscr{O}_X.
\end{tikzcd}$$

The result follows. □

In particular, the same result holds for morphisms of locally ringed spaces, locally ringed spaces over S, manifolds, schemes, etc.

12.2. The structure sheaf of an affine scheme

In this section we'll construct a natural structure sheaf on the spectrum of a ring.

One of Grothendieck's most fruitful ideas was to observe that elements of a ring A are naturally "functions" on $\operatorname{Spec} A$. We consider the image of $\mathfrak{p} \in \operatorname{Spec} A$ by $f \in A$ to be $f \mod \mathfrak{p}$. For example, the image of $(x-a) \in \operatorname{Spec} \mathbb{C}[x]$ by $f \in \mathbb{C}[x]$ is simply $f(a) \mod (x-a)$. More generally, we want the elements of A_f to be functions on $D(f)$.

As the distinguished open sets form a base of the Zariski topology, we can define a presheaf on $X = \operatorname{Spec} A$ by declaring its values only on those sets, where it would be desirable to define $\mathscr{O}_X(D(f))$ to be A_f. The only hindrance is that $\mathscr{O}_X(D(f))$ should *not* depend on f but only on $D(f)$. It is indeed true that if $D(f) = D(g)$ then A_f is isomorphic to A_g. Nevertheless, as in proposition 11.5.5, giving an isomorphism class for each open set is not enough to define a presheaf. This is solved by the lemma below.

Lemma 12.2.1 Let A be a ring and $f \in A$. If $S_{D(f)}$ is the multiplicative set composed by those $s \in A$ such that $D(f) \subset D(s)$, then $S_{D(f)}^{-1}A$ is isomorphic to A_f.

Proof. Since $f \in S_{D(f)}$, the universal property of localization induces a map $\varphi : A_f \to S_{D(f)}^{-1}A$, that we'll show to be an isomorphism. If $\varphi(a/f^n) = 0$, then $sa = 0$ for some $s \in S_{D(f)}$. Corollary 2.4.5 implies that $f^m = cs$, for some $c \in A$ and some positive integer m. But then $f^m a = csa = 0$, which implies that $a/1 = 0$ in A_f. In other words, φ is injective. To see that it is surjective, take any $a/s \in S_{D(f)}^{-1}A$. As before we have that $f^m = cs$, for some $c \in A$ and some positive integer m. This implies that

$$\frac{a}{s} = \frac{ca}{cs} = \frac{ca}{f^m}$$

is the image of $ca/f^m \in A_f$ by φ. □

We then define a presheaf on the base of distinguished open sets by declaring $\mathscr{O}_X(D(f))$ to be $S^{-1}_{D(f)}A$. If $D(g) \subset D(f)$, then $S_{D(f)} \subset S_{D(g)}$ and so $\mathscr{O}_X(D(g))$ is a further localization of $\mathscr{O}_X(D(f))$. We define the localization map

$$\mathscr{O}_X(D(f)) \to \mathscr{O}_X(D(g))$$

to be the restriction map. It is clear that this defines a presheaf on the base. Hereafter we'll systematically use the isomorphisms $\mathscr{O}_X(D(f)) \cong A_f$ and

$$\mathrm{res}_{D(f),D(g)} : A_f \to (A_f)_g = A_g.$$

Finally, we verify that $\mathscr{O}_X$ is indeed a sheaf on the base of distinguished open sets.

Theorem 12.2.2 The presheaf $\mathscr{O}_X$ just defined is a sheaf on the base of distinguished open sets. Thus, it determines a sheaf on $X = \operatorname{Spec} A$.

Proof. Since $D(f)$ is naturally identified to $\operatorname{Spec} A_f$, it suffices to consider the case of an open covering $\{D(f_i)\}_{i \in I}$ of $\operatorname{Spec} A$. We then suppose that $\operatorname{Spec} A = \bigcup_{i \in I} D(f_i)$ or, equivalently, that the ideal generated by the f_i is the entire ring A.

Let's verify the identity axiom. By quasi-compactness, there's a finite subset of I, which we name $\{1, \ldots, n\}$, such that $\operatorname{Spec} A = \bigcup_{i=1}^n D(f_i)$. Let $s \in A$ be a global section such that $s|_{D(f_i)} = 0$ in A_{f_i} for every $i = 1, 2, \ldots, n$. We want to show that $s = 0$. For every such i, there exists an integer m_i such that $f_i^{m_i} s = 0$. By considering the maximum of all these integers, there exists m satisfying $f_i^m s = 0$ for $i = 1, \ldots, n$. Since $\operatorname{Spec} A = \bigcup_{i=1}^n D(f_i^m)$, there are $r_i \in A$ such that $\sum_{i=1}^n r_i f_i^m = 1$. Then,

$$s = \left(\sum_{i=1}^n r_i f_i^m \right) s = \sum_{i=1}^n r_i (f_i^m s) = 0.$$

We now show the gluability axiom. For now, let's suppose $I = \{1, \ldots, n\}$ finite. Let $a_i / f_i^{l_i} \in A_{f_i}$ be a collection of sections that coincide over $A_{f_i f_j}$. We'll do a couple of simplifications. Firstly we define $g_i := f_i^{l_i}$. Using that $D(f_i) = D(g_i)$, we can simplify the notation by considering our sections to be $a_i / g_i \in A_{g_i}$. The fact that those sections coincide over $A_{g_i g_j}$ means that there exist positive integers m_{ij} such that

$$(g_i g_j)^{m_{ij}} (g_j a_i - g_i a_j) = 0$$

for every i, j. Taking the maximum m of all these integers, we simplify again:

$$(g_i g_j)^m (g_j a_i - g_i a_j) = 0.$$

Let $b_i := a_i g_i^m$ and $h_i := g_i^{m+1}$. Since $D(h_i) = D(g_i) = D(f_i)$, we can do one last simplification: our sections are $b_i / h_i \in A_{h_i}$ such that

$$h_j b_i = h_i b_j.$$

Since $\operatorname{Spec} A = \bigcup_{i=1}^{n} D(h_i)$, there are $r_i \in A$ such that $\sum_{i=1}^{n} r_i h_i = 1$. Then,

$$r := \sum_{i=1}^{n} r_i b_i$$

is the element of A that restricts to b_j/h_j on A_{h_j}. Indeed,

$$rh_j = \sum_{i=1}^{n} r_i b_i h_j = b_j \sum_{i=1}^{n} r_i h_i = b_j.$$

Finally, if I is infinite, we can use the quasi-compactness of $\operatorname{Spec} A$ to choose a finite subcover $\operatorname{Spec} A = \bigcup_{i=1}^{n} D(f_i)$. By our preceding construction, there exists a global section $r \in A$ whose restriction to A_{f_i} is $a_i/f_i^{l_i}$ for every $i = 1, \dots, n$. If $k \in I \backslash \{1, \dots, n\}$, we apply the same construction to obtain a global section $r' \in A$ whose restriction to A_{f_i} is $a_i/f_i^{l_i}$ for every $i \in \{1, \dots, n, k\}$. By the identity axiom, $r = r'$. We conclude that r restricts to $a_i/f_i^{l_i}$ for all $i \in I$ as desired. ☐

It may seem that it's difficult to obtain an explicit description of $\mathscr{O}_X(U)$ when U is not a distinguished open set. Fortunately, this is not the case.

■ **Example 12.2.1 — Affine plan minus the origin.** mostrar que a gente consegue calcular isso só usando a base ■

Proposition 12.2.3 Let A be a ring and $X = \operatorname{Spec} A$. Then the stalk of $\mathscr{O}_X$ at a prime ideal $\mathfrak{p}$ is $A_{\mathfrak{p}}$. In particular, it is a local ring.

Proof. ☐

Definition 12.2.1 — Affine scheme. An *affine scheme* is a locally ringed space X which is isomorphic to $\operatorname{Spec} A$ for some ring A. We define the category Aff of affine schemes as the full subcategory of LRS whose objects are affine schemes.

Proposition 12.2.4 Let A be an integral domain. We define $\mathscr{F}$ to be the presheaf of rings given by

$$\mathscr{F}(U) := \bigcap_{\mathfrak{p} \in U} A_{\mathfrak{p}},$$

whenever U is a nonempty open subset of $\operatorname{Spec} A$. If $V \subset U$, we define the restriction $\operatorname{res}_{U,V} : \mathscr{F}(U) \to \mathscr{F}(V)$ to be the inclusion. Then $\mathscr{F}$ is a sheaf isomorphic to the structure sheaf of $\operatorname{Spec} A$.

Proof. ☐

Theorem 12.2.5 — Tate. Let X be a locally ringed space and A a ring. Then the natural map

$$\mathrm{Hom}_{\mathsf{LRS}}(X, \operatorname{Spec} A) \to \mathrm{Hom}_{\mathsf{Ring}}(A, \Gamma(X, \mathscr{O}_X)),$$

which maps $f : X \to \operatorname{Spec} A$ to $f^\sharp_{\operatorname{Spec} A} : A \to \mathscr{O}_X(X)$, is bijective. This bijection is natural in X and A, proving that the global section functor is left adjoint to Spec.

Proof. ☐

Corollary 12.2.6 Let X be a locally ringed space. Then, the canonical map $X \to \operatorname{Spec} \Gamma(X, \mathscr{O}_X)$ is universal among the maps from X to affine schemes. That is, if $X \to \operatorname{Spec} A$ is a morphism of locally ringed spaces, there exists a unique morphism $\operatorname{Spec} \Gamma(X, \mathscr{O}_X) \to \operatorname{Spec} A$ such that the diagram

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & \operatorname{Spec} A \\
& \operatorname{Spec} \Gamma(X, \mathscr{O}_X) \arrow[ur, dashed] &
\end{tikzcd}$$

commutes.

Proof. ☐

Corollary 12.2.7 The category of affine schemes is anti-equivalent to the category of rings.

Proof. ☐

Corollary 12.2.8 Let $A \to B$ and $A \to C$ be morphisms of rings. Then the fibered product

$$\operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} C$$

is given by $\operatorname{Spec}(B \otimes_A C)$ in Aff. In particular, the product of $\operatorname{Spec} B$ and $\operatorname{Spec} C$ in Aff is $\operatorname{Spec}(B \otimes C)$. Moreover, the products and fiber products in LRS are the same as in Aff.

Proof. ☐

Corollary 12.2.9 Let X be a locally ringed space and suppose that X is the coproduct $U \coprod V$, where U and V are open in X. If U and V are affine schemes, then so is X.

Proof. ☐

Definition 12.2.2 — Scheme.

12.3. Limits and colimits of ringed spaces

12.4. Open and closed immersions

Proposition 12.4.1 Let X and Y be locally ringed spaces and $\{U_i\}$ an open cover of X. If $f_i : U_i \to Y$ are morphisms of locally ringed spaces which restrict to the same map on the intersections, then there exists a unique morphism $f : X \to Y$ whose restriction to U_i agrees with f_i for every i.

Proof. ☐

Definition 12.4.1 — Gluing data.

Proposition 12.4.2 gluing (locally) ringed spaces (over S?)

Proof. ☐

Corollary 12.4.3 Gluing schemes

Proof. ☐

Corollary 12.4.4 gluing manifolds

12.5. $\mathscr{O}_X$-modules

12.6. Tangent spaces

13. Sheaf cohomology

reescrever essa merda toda KKK fazer toda a cohomologia como um caso particular dos 4 funtores.

13.1. Derived functor cohomology

Proposition 13.1.1 Let X be a ringed space. The category of sheaves of $\mathscr{O}_X$-modules on X has enough injectives.

Proof. □

mas não tem projetivos o suficiente!

Definition 13.1.1 — Sheaf cohomology. Let X be a ringed space. We denote by $\mathsf{D}^*(X)$, for $* = \varnothing, +, -, b$, the derived category of the category of $\mathscr{O}_X$-modules. If U is an open subset of X, the functor $\Gamma(U, -)$ is left exact and so gives rise to

$$\mathsf{R}\Gamma(U,-) : \mathsf{D}^+(X) \to \mathsf{D}^+(\Gamma(U, \mathscr{O}_X)),$$

whose i-th cohomology is denoted by $H^i(U,-)$.

We observe that, even though we'll often apply $H^i(U,-)$ to a single sheaf of $\mathscr{O}_X$-modules, this functor takes bounded below complexes of sheaves (objects of $\mathsf{D}^+(X)$) as input. In the past, the cohomology of complexes of sheaves was called *hypercohomology* but, since the formalism of derived categories deals just as well with complexes as with single sheaves, we'll not need to differentiate between sheaf cohomology and hypercohomology. The example below provides some motivation for considering the cohomology of complexes of sheaves.

■ **Example 13.1.1 — De Rham cohomology.** Let M be a smooth manifold. For every open set $U \subset M$, we denote by $\Omega^i_M(U)$ the $\mathbb{R}$-vector space of differential i-forms over U. These vector spaces form a sheaf Ω^i_M and, together with the exterior derivative, they coalesce into the *de Rham complex* $\Omega^\bullet_M$. The i-th *de Rham cohomology*, $H^i_{dR}(M)$, is defined as being $H^i(M, \Omega^\bullet_M)$.

We'll soon see that the existence of partitions of unity implies that the sheaves Ω^i_M are acyclic and so the de Rham cohomology may be computed as $H^i(\Gamma(M, \Omega^\bullet_M))$, which is its usual definition in differential geometry. Nevertheless, when dealing with complex manifolds or schemes, partitions of unity may not exist and we'll be obliged to consider the cohomology of the de Rham complex. ■

The reader should recall that the derived functor $\mathsf{R}\Gamma(U,-)$ is a triangulated functor. That is, it sends distinguished triangles in $\mathsf{D}^+(X)$ to distinguished triangles in $\mathsf{D}^+(\Gamma(U,\mathscr{O}_X))$, which in turn are sent to exact triangles in $\Gamma(U,\mathscr{O}_X)\text{-}\mathsf{Mod}$ by H^i. In particular, a short exact sequence of bounded below complexes of $\mathscr{O}_X$-modules

$$0 \longrightarrow \mathscr{F}^\bullet \longrightarrow \mathscr{G}^\bullet \longrightarrow \mathscr{H}^\bullet \longrightarrow 0$$

induces a long exact sequence of $\Gamma(U,\mathscr{O}_X)$-modules

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(U,\mathscr{F}^\bullet) \arrow[r] & H^i(U,\mathscr{G}^\bullet) \arrow[r] & H^i(U,\mathscr{H}^\bullet) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& H^{i+1}(U,\mathscr{F}^\bullet) \arrow[r] & H^{i+1}(U,\mathscr{G}^\bullet) \arrow[r] & H^{i+1}(U,\mathscr{H}^\bullet) \arrow[r] & \cdots .
\end{tikzcd}$$

Before we continue our adventure into the world of sheaf cohomology, we remark that there are two possible ambiguities in the definition 13.1.1. The first is that we have considered $\Gamma(U,-)$ as a functor $\mathscr{O}_X\text{-}\mathsf{Mod} \to \Gamma(U,\mathscr{O}_X)\text{-}\mathsf{Mod}$, while it also makes sense to view it as a functor $\mathsf{Ab}_X \to \mathsf{Ab}$. Another possible ambiguity is that, for a bounded below complex of $\mathscr{O}_X$-modules $\mathscr{F}^\bullet$, it is not a priori obvious that $H^i(U,\mathscr{F}^\bullet) = H^i(U,\mathscr{F}|_U^\bullet)$.[1]

None of these ambiguities pose real threats. The former will be addressed in the next section while we deal with the latter now.

> **Proposition 13.1.2** Let X be a ringed space and let $\mathscr{F}^\bullet$ be a bounded below complex of $\mathscr{O}_X$-modules. If U is an open subset of X, the complexes
>
> $$\mathsf{R}\Gamma(U,\mathscr{F}^\bullet) \qquad \text{and} \qquad \mathsf{R}\Gamma(U,\mathscr{F}|_U^\bullet)$$
>
> coincide. In particular, $H^i(U,\mathscr{F}^\bullet) = H^i(U,\mathscr{F}|_U^\bullet)$ for all i.

Proof. ☐

falar que se $V \subset U$, então existe uma aplicação de restrição natural $H^i(U,\mathscr{F}^\bullet) \to H^i(V,\mathscr{F}^\bullet)$, dando a $U \mapsto H^i(U,\mathscr{F}^\bullet)$ a estrutura de um prefeixe de $\mathscr{O}_X$-módulos.

> **Definition 13.1.2 — Flasque sheaf.** Let X be a topological space. A presheaf $\mathscr{F}$ over X with values in a concrete category is said to be *flasque* if, whenever $V \subset U$ is a pair of nested open sets in X, the restruction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective.

---

[1] For that we have to prove that if $\mathscr{I}^\bullet \to \mathscr{F}^\bullet$ is an injective resolution, then so is $\mathscr{I}|_U^\bullet \to \mathscr{F}|_U^\bullet$.

Proposition 13.1.3 Let X be a ringed space and let $U \subset X$ be an open subset. Any injective $\mathscr{O}_X$-module is flasque and any flasque $\mathscr{O}_X$-module is acyclic for $R\Gamma(U, -)$.

Proof. ☐

Definition 13.1.3 — Higher direct image. Let $f : X \to Y$ be a morphism of ringed spaces. The functor f_* is left exact and so gives rise to the *derived direct image*

$$Rf_* : D^+(X) \to D^+(Y).$$

The i-th cohomology of $Rf_*\mathscr{F}^\bullet$, where $\mathscr{F}^\bullet$ is a bounded below complex of $\mathscr{O}_X$-modules, is denoted by $R^if_*\mathscr{F}^\bullet$ and is called the i-th *higher direct image*.

a gente pode descrever as imagens diretas superiores usando cohomologia

Proposition 13.1.4 Let $f : X \to Y$ be a morphism of ringed spaces and let $\mathscr{F}^\bullet$ be a bounded below complex of $\mathscr{O}_X$-modules. The sheafification of the presheaf

$$U \mapsto H^i(f^{-1}(U), \mathscr{F}^\bullet)$$

coincides with the higher direct image $R^if_*\mathscr{F}^\bullet$.

Proof. ☐

dar exemplos de pq que isso não é um feixe
talvez falar aqui que módulos flasque são acíclicos para a imagem direta derivada.

13.2. Functoriality of cohomology

explicar pq que o resultado abaixo é útil.

Proposition 13.2.1 Let $f : X \to Y$ be a morphism of ringed spaces. Then the diagram

$$\begin{tikzcd} D^+(X) \arrow[r, "R\Gamma(X,-)"] \arrow[d, "Rf_*"'] & D^+(\Gamma(X, \mathscr{O}_X)) \arrow[d] \\ D^+(Y) \arrow[r, "R\Gamma(Y,-)"] & D^+(\Gamma(Y, \mathscr{O}_Y)) \end{tikzcd}$$

commutes.

Proof. ☐

Corollary 13.2.2 tanto faz derivar o funtor seção global $\mathscr{O}_X$-Mod $\to \Gamma(X, \mathscr{O}_X)$-Mod ou o funtor seção global $\mathsf{Ab}_X \to \mathsf{Ab}$. O mesmo pra imagem direta superior. (E isso vale pra complexos)

Proof. ☐

Corollary 13.2.3 Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Then $Rg_* \circ Rf_* = R(g \circ f)_*$ as functors $D^+(X) \to D^+(Z)$.

Proof. ☐

Corollary 13.2.4 — Leray spectral sequence. Let $f : X \to Y$ be a morphism of ringed spaces and $\mathscr{F}^\bullet$ be a bounded below complexes of $\mathscr{O}_X$-modules. There is a spectral sequence whose second page is

$$(R^p\Gamma \circ R^q f_*)(\mathscr{F}^\bullet) = H^p(Y, R^q f_* \mathscr{F}^\bullet)$$

and which converges to $R^{p+q}(\Gamma \circ f_*)(\mathscr{F}^\bullet) = H^{p+q}(X, \mathscr{F}^\bullet)$.

Proof. ☐

Corollary 13.2.5 Let $f : X \to Y$ be a morphism of ringed spaces and let $\mathscr{F}$ be a $\mathscr{O}_X$-module. If $R^i f_* \mathscr{F} = 0$ for all $i > 0$, then the natural map

$$H^i(Y, f_*\mathscr{F}) \to H^i(X, \mathscr{F})$$

is an isomorphism for all i.

Proof. ☐

Corollary 13.2.6 — Mayer-Vietoris. Let X be a ringed space and suppose that $X = U \cup V$ is the union of two open subsets. For every $\mathscr{O}_X$-module $\mathscr{F}$, there exists a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(X, \mathscr{F}) \arrow[r] & H^i(U, \mathscr{F}) \oplus H^i(V, \mathscr{F}) \arrow[r] & H^i(U \cap V, \mathscr{F}) \arrow[dll, rounded corners] \\
& H^{i+1}(X, \mathscr{F}) \arrow[r] & H^{i+1}(U, \mathscr{F}) \oplus H^{i+1}(V, \mathscr{F}) \arrow[r] & H^{i+1}(U \cap V, \mathscr{F}) \arrow[r] & \cdots,
\end{tikzcd}$$

which is functorial in $\mathscr{F}$.

Proof. ☐

dar exemplos (provavelmente ainda não dá pra calcular nada, mas pelo menos explicar pq que isso é útil)

13.3. Torsors, extensions and invertible sheaves

Proposition 13.3.1 Let X be a ringed space and $\mathscr{F}$ be a $\mathscr{O}_X$-module. There is a canonical isomorphism of abelian groups

$$\operatorname{Ext}^1_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{F}) \to H^1(X, \mathscr{F})$$

which associates to the extension

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{G} \arrow[r] & \mathscr{O}_X \arrow[r] & 0 \end{tikzcd}$$

the image of $1 \in \Gamma(X, \mathscr{O}_X)$ in $H^1(X, \mathscr{F})$ given by the long exact sequence in cohomology.

Proof. ☐

explicar a relação entre torsores e fibrados principais
(explicar que simply transitive = transitive + free)

Definition 13.3.1 — Torsor. Let X be a topological space and $\mathscr{G}$ be a sheaf of groups on X. A *$\mathscr{G}$-torsor* is a sheaf of sets $\mathscr{F}$ on X, whose stalks are non-empty, and endowed with simply transitive actions

$$\mathscr{G}(U) \times \mathscr{F}(U) \to \mathscr{F}(U),$$

for every open set $U \subset X$, compatible with the restriction maps. A morphism of $\mathscr{G}$-torsors is a morphism of sheaves of sets compatible with the $\mathscr{G}$-actions. We say that $\mathscr{G}$, endowed with action by left-multiplication, is *the trivial* $\mathscr{G}$-torsor.

explicar que todo morfismo é um isomorfismo e que um $\mathscr{G}$-torsor $\mathscr{F}$ é trivial se e somente se ele possui uma seção global.

explicar a operação de grupo no conjunto de classes de isomorfismo de $\mathscr{G}$ torsores.

Proposition 13.3.2 Let X be a ringed space and $\mathscr{F}$ be a $\mathscr{O}_X$-module. There is a canonical isomorphism of abelian groups between $H^1(X, \mathscr{F})$ and the group of isomorphism classes of $\mathscr{F}$-torsors.

Proof. ☐

Proposition 13.3.3 Let X be a ringed space. There is a canonical isomorphism of abelian groups between $H^1(X, \mathscr{O}_X^*)$ and the Picard group $\operatorname{Pic}(X)$.

Proof. ☐

13.4. Čech cohomology

Essa seção é meio padrão também. A única peculiaridade provavelmente vai ser que eu vou usar cohomologia de Cech para ilustrar a construção ++ de Grothendieck, que é útil pois prova que o funtor de feixeficação não só preserva colimites (pois é adjunto à esquerda) como também preserva limites finitos.

> **Proposition 13.4.1** The sheafification functor preserves colimits and finite limits. In particular, when dealing with sheaves with values in an abelian category, the sheafification functor is exact.

Proof. □

14. Verdier Duality

14.1. Separated locally proper maps

14.2. Proper direct image

14.3. Proper inverse image

14.4. Constructible sheaves

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