# The Hodge Decomposition

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For a lot of algebraic geometers, the Hodge decomposition follows from an analytic result (our theorem 4.7) which seems like it's best treated as a black-box. The goal of these notes is to demystify its proof.

## 1 A digression into the world of analysis

Let $X$ be an open subset of $\mathbb{R}^n$. As is well known to most readers, the natural place to do analysis are the Lebesgue spaces $L^p(X)$, instead of the space of continuous functions $\mathcal{C}(X)$.[1] Indeed, the former is the completion of the latter, with respect to the $\|\cdot\|_p$ norm. Now, we would like to find $L^p$ solutions to differential equations. A first problem is that it isn't even clear what the derivative of a $L^p$ function should mean.

The modern solution to this is the following. We define the *space of distributions* $\mathcal{D}'(X)$ as the topological dual of the space $\mathcal{D}(X) := \mathcal{C}_c^\infty(X)$ of smooth functions with compact support. The Lebesgue space $L^p(X)$ injects naturally into $\mathcal{D}'(X)$ via the map

$$f \mapsto \left(\varphi \mapsto \int_X f(x)\varphi(x)\,dx\right).$$

Motivated by this inclusion, we define the derivative $\partial_i u$ of a distribution $u \in \mathcal{D}'(X)$ by the formula $\partial_i u(\varphi) := -u(\partial_i \varphi)$. (If $u$ is given by a bonafide function on $X$, this formula is just integration by parts.) As usual, we are going to use multi-indices to write higher-order derivatives.

The problem which arises, then, is that the derivative of a $L^p$ function need not be in $L^p$. This leads to the following definition. (As usual in analysis, we write $D_\alpha$ to mean $(-i)^{|\alpha|}\partial_\alpha$. This simplifies quite a lot of formulas.)

**Definition 1.1** We define the *Sobolev space* $W^{k,p}(X)$ as the set of all $f \in L^p(X)$ such that $D_\alpha f \in L^p(X)$ for all $|\alpha| \leqslant k$, endowed with the norm $\|f\|_{k,p}^2 := \sum_{|\alpha| \leqslant k} \|D_\alpha f\|_p^2$.

[1] Every function on these notes is supposed to be complex-valued.

These are always Banach spaces for all $k \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$. When $p = 2$, they are even Hilbert spaces. Since that is our main case of interest, we will use the shorthand $W^k(X) := W^{k,2}(X)$. (It is common in the analysis literature to denote $W^{k,2}(X)$ by $H^k(X)$, but we are going to avoid it for obvious reasons.)

In practice one can often pretend the elements of $W^{k,p}(X)$ are good old functions due to the fact that $\mathcal{C}^\infty(X) \cap W^{k,p}(X)$ is a dense subset of $W^{k,p}(X)$, for all $k \in \mathbb{N}$ and $1 \leqslant p < \infty$. (Up to some technicalities, the basic regularization technique, by convoluting a singular function with appropriate test functions, also works here. [AF03, Theorem 3.17])

When $X = \mathbb{R}^n$, we can write the norm $\|\cdot\|_k := \|\cdot\|_{k,2}$ in a particularly convenient way. The Plancherel formula gives that

$$\|f\|_{k,2}^2 = \sum_{|\alpha| \leqslant k} \int_{\mathbb{R}^n} |D_\alpha f(x)|^2 \, dx = \sum_{|\alpha| \leqslant k} \int_{\mathbb{R}^n} |\xi^\alpha \hat{f}(\xi)|^2 \, d\xi.$$

Since there's a positive constant $c$, depending only on $n$ and $k$, such that

$$c(1 + |\xi|^2)^k \leqslant \sum_{|\alpha| \leqslant k} (\xi^\alpha)^2 \leqslant (1 + |\xi|^2)^k,$$

(one can write $(1 + |\xi|^2)^k = \sum_{|\alpha| \leqslant k} c_\alpha (\xi^\alpha)^2$ for some positive integers $c_\alpha$ and take $c$ to be the inverse of $\max_{|\alpha| \leqslant k} c_\alpha$) it follows that the norm $\|\cdot\|_k$ is equivalent to the norm defined by

$$f \mapsto \left( \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \cdot (1 + |\xi|^2)^k \, d\xi \right)^{1/2}.$$

We'll use both norms interchangeably. Of course, the same description also works when $X$ is a torus.

All that we need to know about these Sobolev spaces is how they relate to other spaces of functions in analysis. This first result morally says that if a $L^2$ function has enough distributional derivatives, then at least some of those derivatives are continuous.

**Proposition 1.1 — Sobolev lemma.** For all $k > n/2 + l$, there's a natural continuous inclusion $W^k(\mathbb{R}^n) \hookrightarrow \mathcal{C}^l(\mathbb{R}^n)$.

**Proof.** Let $f \in W^k(\mathbb{R}^n)$. Since $D_\alpha$ sends $W^k(\mathbb{R}^n)$ to $W^{k-|\alpha|}(\mathbb{R}^n)$, it suffices to consider the case $l = 0$. Recall that if the Fourier transform $\hat{f}$ is in $L^1(\mathbb{R}^n)$, then the Fourier inversion formula applies. Finally, the Riemann-Lebesgue lemma (or simply the dominated convergence theorem) says that $f$ is continuous.

Let us show that this is the case here. By Cauchy-Schwarz,

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)| \, d\xi \leqslant \left[ \underbrace{\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \cdot (1 + |\xi|^2)^k \, d\xi}_{A} \right]^{1/2} \left[ \underbrace{\int_{\mathbb{R}^n} (1 + |\xi|^2)^{-k} \, d\xi}_{B} \right]^{1/2}.$$

Our previous discussion shows that $A$ is finite. (For $f$ is in $W^k(\mathbb{R}^n)$.) We can also calculate $B$ by writing $\xi = r\omega$, where $r = |\xi|$ and $\omega \in S^{n-1}$. Indeed,

$$\int_{\mathbb{R}^n} (1+|\xi|^2)^{-k}\, d\xi = \int_{S^{n-1}} \int_0^\infty (1+r^2)^{-k} r^{n-1}\, dr\, d\omega = \mathrm{Vol}(S^{n-1}) \int_0^\infty (1+r^2)^{-k} r^{n-1}\, dr.$$

Now, the integral $\int_0^\infty (1+r^2)^{-k} r^{n-1}\, dr$ is comparable to $\int_1^\infty r^{-2k+n-1}\, dr$, which converges precisely for $-2k+n-1 < -1$. I.e., for $k > n/2$. □

Let's recall the definition of a *compact* linear map $T : V \to W$ between two normed spaces. We say that $T$ is compact if the image of a bounded subset of $V$ is relatively compact (i.e., has compact closure) in $W$. When $W$ is a Hilbert space, it is true that a linear map is compact if and only if it is a limit, under the operator norm, of finite-rank operators.[2]

As for our next comparison between function spaces, we remark that $W^{k+1,p}(X)$ is clearly a subspace of $W^{k,p}(X)$ for all $k$ and $p$. A result of Rellich and Kondrashov says that, when $X$ is compact with $\mathcal{C}^1$ boundary, this inclusion is compact. Our next result is a particular case of this, which suffices for our needs.

**Proposition 1.2 — Rellich.** Let $T$ be a $n$-dimensional torus. Then the natural inclusion $j : W^{k+1}(T) \hookrightarrow W^k(T)$ is compact.

**Proof.** The Pontryagin dual of $T$ is $\mathbb{Z}^n$ and so we have a Fourier series map $L^2(T) \to L^2(\mathbb{Z}^n)$. Let $s \in \mathbb{R}$ and consider the operator $T_s : L^2(T) \to L^2(T)$ defined by the formula

$$T_s(f) = \left( x \mapsto \sum_{\lambda \in \mathbb{Z}^n} (1+|\lambda|^2)^{-s/2} \hat{f}(\lambda) e^{i\lambda\cdot x} \right).$$

As before, the Plancherel theorem implies that a $L^2$ function $f$ is in $W^k(T)$ if and only if

$$\sum_{\lambda\in\mathbb{Z}^n} |\hat{f}(\lambda)|^2 (1+|\lambda|^2)^k < \infty.$$

In particular, since the Fourier transform of $T_s(f)$ is $\lambda \mapsto (1+|\lambda|^2)^{-s/2}\hat{f}(\lambda)$, the operator $T_k$ defines an isomorphism $L^2(T) \xrightarrow{\sim} W^k(T)$. (A quick calculation shows that the image of $T_k$ is contained in $W^k(T)$. Conversely, if $f \in W^k(T)$, then the bound above implies that $T_{-k}(f)$ is in $L^2(T)$ and so $f = T_k(T_{-k}(f))$ is in the image of $T_k$.)

Finally, we observe that $T_{k+1} = T_1 \circ T_k$ and so the following diagram

$$\begin{tikzcd} L^2(T) \arrow[rr, "T_k"] \arrow[dr, "T_{k+1}"'] & & W^k(T) \arrow[dl, "T_1"] \\ & W^{k+1}(T) & \end{tikzcd}$$

[2]So, in some sense, compact operators generalize those of finite-rank in more or less the same way that quasi-coherent sheaves generalize coherent ones.

commutes. In particular, it shows that $T_1 : W^k(T) \to W^{k+1}(T)$ is an isomorphism. Now, the composition $j \circ T_1 : W^k(T) \to W^k(T)$ is a limit of finite-rank operators; and so is compact. Our result follows. $\square$

## 2 Elliptic operators

For the purposes of this section, let's say that a *differential operator of order* $d$ is an expression of the form $\sum_{|\alpha| \leqslant d} a_\alpha D_\alpha$, where the $a_\alpha$ are functions on $X$, and $a_\alpha \neq 0$ for at least one $\alpha$ with $|\alpha| = d$. A particularly important example for us is the *Laplacian* $\Delta := \sum_{i=1}^n \partial_i^2 = -\sum_{i=1}^n D_i^2$. It satisfies the following, rather miraculous, property.

**Lemma 2.1** Let $f$ be a $L^2$ function on $\mathbb{R}^n$ and suppose that $\Delta f \in L^2(\mathbb{R}^n)$. Then $f \in W^2(\mathbb{R}^n)$.

**Proof.** Let $u = f - \Delta f$ and observe that $\widehat{u} = (1 + |\xi|^2)\widehat{f}$. By our supposition, $\widehat{u} \in L^2$ and so $f \in W^2(\mathbb{R}^n)$. (Recall that $f$ is in $W^2(\mathbb{R}^n)$ if and only if $(1 + |\xi|^2)\widehat{f}$ is in $L^2(\mathbb{R}^n)$.) $\square$

We remark how surprising this is. This is absolutely not true if we replace the Laplacian by most other differential operators of order 2. Take $\partial_1^2$ acting on $\mathbb{R}^2$, for example: we may choose some function $f$ which is the sum of a very regular function on the first variable and an absolutely rough function on the second variable.

However, the Laplacian is far from being the only differential operator with this property. If $P$ is a differential operator of order 2 with constant coefficients, the Fourier transform of $Pf$ is of the form $\sigma_P(\xi)\widehat{f}$ for some polynomial $\sigma_P(\xi)$. As the proof above shows, the important property is that $\sigma_P(\xi)$ is "comparable" to $|\xi|^2$. In particular, it has to be nonzero for all $\xi \neq 0$. This leads to the following definition.

**Definition 2.1** Let $P = \sum_{|\alpha| \leqslant d} a_\alpha D_\alpha$ be a differential operator of order $d$ on $X$. The *principal symbol* $\sigma_P(x, \xi)$ of $P$ is defined as being the function on $X \times \mathbb{R}^n$ given by $(x, \xi) \mapsto \sum_{|\alpha| = d} a_\alpha(x)\xi^\alpha$. We say that $P$ is *elliptic* if $\sigma_P(x, \xi)$ is nonzero for all $x \in X$ and $\xi \in \mathbb{R}^n \setminus \{0\}$.

Observe that, since its symbol is $-|\xi|^2$, the Laplacian is clearly an elliptic operator. The main theorem of this section confirms our claim that an analog of lemma 2.1 holds for elliptic operators in general.

**Theorem 2.2 — Gårding's inequality.** Let $P$ be an elliptic differential operator of order $d$. Then, for every $f \in L^2(\mathbb{R}^n)$ such that $Pf \in W^k(\mathbb{R}^n)$, we have

$$\|f\|_{k+d} \leqslant C_k(\|Pf\|_k + \|f\|_0),$$

where $C_k$ is a positive constant depending only on $k$. In particular, $f \in W^{k+d}(\mathbb{R}^n)$.

**Sketch of proof.** We'll content ourselves with the case in which $P$ has constant coefficients and only terms of order $d$. (I.e., no terms containing $D_\alpha$ for $|\alpha| < d$.) Since the symbol of $P$ is independent of $x \in \mathbb{R}^n$, we will denote it by $\sigma_P(\xi)$. Now, by ellipticity, there is a positive constant $c$ such that

$$|\sigma_P(\xi)|^2 \geqslant c|\xi|^{2d}$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$. (Write $\xi = r\omega$, where $r = |\xi| > 0$ and $\omega \in \mathbb{R}^n$ has unit norm. By our supposition, the function $f(r, \omega) = |\sigma_P(r\omega)|^2/r^{2d}$ is constant on $r$. By compactness of the unit ball, $f$ attains a non-zero minimum $c$.) In particular,

$$\|Pf\|_k^2 = \int_{\mathbb{R}^n} |\sigma_P(\xi)\hat{f}(\xi)|^2(1+|\xi|^2)^k \, d\xi \geqslant c\int_{\mathbb{R}^n} |\xi|^{2d}|\hat{f}(\xi)|^2(1+|\xi|^2)^k \, d\xi.$$

We conclude that there exists a positive constant $c'$ such that

$$\begin{aligned}(\|Pf\|_k + \|f\|_0)^2 &\geqslant \|Pf\|_k^2 + \|f\|_0^2 \\ &\geqslant c\int_{\mathbb{R}^n} |\xi|^{2d}|\hat{f}(\xi)|^2(1+|\xi|^2)^k \, d\xi + \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \, d\xi \\ &\geqslant \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \left[c|\xi|^{2d}(1+|\xi|^2)^k + 1\right] d\xi \\ &\geqslant c'\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \left[(1+|\xi|^2)^{k+d}\right] d\xi = c'\|f\|_{k+d},\end{aligned}$$

where the last inequality was given by the lemma below. This implies the result in our particular case. The general case actually follows from this calculation. See [War83, Prop. 6.29]. □

We needed the following lemma in the previous proof.

**Lemma 2.3** Let $k, d \in \mathbb{N}$ and $c > 0$ be a real number. Then, there exists a constant $c' > 0$, depending on $c$, $k$, and $d$, such that $cx^d(1+x)^k + 1 \geqslant c'(1+x)^{k+d}$ for all $x > 0$.

**Proof.** Let $f(x) = [cx^d(1+x)^k + 1]/(1+x)^{k+d}$. It's clear that $f(x) \to c$ as $x \to \infty$. In particular, there exists a positive constant $M$ such that $|f(x) - c| < c/2$ for all $x > M$. Even more particularly, this gives that $f(x) > c/2$ for all $x > M$. Now, the interval $[0, M]$ is compact and so $f$ attains a minimum $c_M > 0$ there. It follows that $f(x) \geqslant \min(c/2, c_M) > 0$ for all $x > 0$. □

## 3 Vector-valued functions

For the sake of simplicity of exposition, we have only considered functions of the form $X \to \mathbb{C}$ so far. Nevertheless, up to keeping track of a little more data, every single definition and result also works in the same way for vector-valued functions. (And with the same proofs!) In this section, we will explain this straightforward generalization. (A reader with some faith will lose nothing by skipping to the next section.)

If $r \in \mathbb{N}$, the Sobolev space $W^{k,p}(X, \mathbb{C}^r)$ is simply the set of all $f \in L^p(X, \mathbb{C}^r)$ such that $D_\alpha f \in L^p(X, \mathbb{C}^r)$ for all $|\alpha| \leqslant k$, endowed with the norm

$$\|(f_1, \ldots, f_r)\|_{k,p}^2 := \sum_{|\alpha| \leqslant k} \sum_{i=1}^{r} \|D_\alpha f_i\|_p^2.$$

This coincides with the $\ell_2$-direct sum $W^{k,p}(X)^{\oplus r}$. In particular, $W^k(X, \mathbb{C}^r) := W^{k,2}(X, \mathbb{C}^r)$ is still a Hilbert space.

As before, the Plancherel theorem gives an equivalent expression to the norm of $W^k(\mathbb{R}^n, \mathbb{C}^r)$ and of $W^k(T, \mathbb{C}^r)$, for a torus $T$. Moreover, the proofs of propositions 1.1 and 1.2 (Sobolev and Rellich's lemmas) translate verbatim to this context.

A differential operator is still defined as an expression of the form $P = \sum_{|\alpha| \leqslant d} a_\alpha D_\alpha$, but now the $a_\alpha$ are $r \times s$ *matrices* of (one-variable) functions on $X$ (for a differential operator which sends $r$ functions to $s$ functions). Similarly, its principal symbol $\sigma_P(x, \xi)$ is a matrix of functions on $X \times \mathbb{R}^n$ and we say that $P$ is elliptic if this matrix is invertible for all $x \in X$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. (In particular, this forces $r = s$.)

Gårding's inequality also basically works as before by remarking that $P$ being elliptic implies that $|\sigma_P(x, \xi)\nu| > 0$ for all $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n \setminus \{0\}$, and $\nu \in \mathbb{C}^n \setminus \{0\}$. Under the supposition that $P$ has constant coefficients and only terms of order $d$ (as before), we use the compactness to conclude that there is a positive constant $c$ such that $|\sigma_P(\xi)\nu| > c$ for all $\xi$ and $\nu$ in the unit sphere. It follows that $|\sigma_P(\xi)\nu| \geqslant c|\xi|^d|\nu|$ for all $\xi$ and $\nu$. That is what we needed to continue the proof.

## 4 Analysis on manifolds

From now on, let $(X, g)$ be a compact oriented Riemannian manifold of dimension $n$, and let $(E, h)$ be a rank $r$ Hermitian vector bundle endowed with a connection $\nabla$. (We recall that every manifold admits a Riemannian metric, every complex vector bundle admits a Hermitian metric, and every Hermitian vector bundle admits a compatible connection.)

Let's see how all of the structure above allows us to define the same objects as before. Given two measurable sections $f, g$ of $E$, we define their $L^2$ inner product as $\langle f, g\rangle_{L^2}$ as being

$$\int_X h(f, g) \operatorname{vol}_g .$$

In particular, we also have the norm $\|f\|_{L^2} := \sqrt{\langle f, f\rangle_{L^2}}$ and the Hilbert space $L^2(X, E)$ composed of the measurable sections $f$ satisfying $\|f\|_{L^2} < \infty$, modulo the sections with zero norm.

One may endow $\Gamma(X, E)$ with a natural locally convex topology[3] and define $\mathcal{D}'(X, E)$

[3]See [BC09, Section 2.3] for more details.

to be the continuous dual of $\Gamma(X, E)$. There's a natural inclusion of $L^2(X, E)$ in $\mathcal{D}'(X, E)$ given by

$$f \mapsto (\varphi \mapsto \langle \varphi, f \rangle_{L^2}).$$

In order to define Sobolev spaces, we would like to extend the action of $\nabla$ to a map $\mathcal{D}'(X, E) \to \mathcal{D}'(X, E \otimes T^*X)$. On a compact subset $X$ of $\mathbb{R}$, we remarked that integration by parts implies that

$$\int_X f'(x)\varphi(x)\,dx = -\int_X f(x)\varphi'(x)\,dx,$$

and so we could define $u'(\varphi)$ as $-u(\varphi')$. Given our generality, we would like to find a map $\nabla^* : \Gamma(X, E \otimes T^*X) \to \Gamma(X, E)$, playing the role of $-d/dx$ in the example above, satisfying $\langle \varphi, \nabla f \rangle_{L^2} = \langle \nabla^* \varphi, f \rangle_{L^2}$ for every $f \in \Gamma(X, E)$ and $\varphi \in \Gamma(X, E \otimes T^*X)$. (For then we can define $\nabla u(\varphi)$ to be $u(\nabla^* \varphi)$.) This map $\nabla^* : \Gamma(X, E \otimes T^*X) \to \Gamma(X, E)$ is said to be the *formal adjoint* of $\nabla$.

The formal adjoint exists and is a useful construction for more general differential operators, so we make a little digression into their study. For this, it's actually convenient to see complex vector bundles as locally free sheaves of modules over $\mathcal{C}^\infty_X$.[4]

**Definition 4.1** Let $E, F$ be two complex vector bundles. A differential operator $P : E \to F$ is a $\mathbb{C}$-linear map of sheaves.

Hopefully this definition is as extraordinary to the reader as it is to the writer. In other contexts (complex manifolds and smooth algebraic varieties, for example), we define inductively differential operators: a $\mathbb{C}$-linear map of sheaves $P : E \to F$ is a differential operator of order at most $d$ if $[P, f]$ is a differential operator of order at most $d - 1$ for every function $f$, and the zero-map is the only differential operator of order $-1$. As it turns out, over smooth manifolds all of this is unnecessary.

**Proposition 4.1** Let $P : E \to F$ be a differential operator. Then, if $\varphi : U \xrightarrow{\sim} \mathbb{R}^n$ is a chart trivializing $E$ and $F$, the composition

$$\begin{tikzcd}
\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^r) \arrow[r, "\varphi^{-1}"] \arrow[d, dashed] & \mathcal{C}^\infty(U, \mathbb{C}^r) \arrow[r, "\sim"] & \Gamma(U, E) \arrow[d, "P_U"] \\
\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^s) & \mathcal{C}^\infty(U, \mathbb{C}^s) \arrow[l, "\varphi"'] & \Gamma(U, F) \arrow[l, "\sim"']
\end{tikzcd}$$

is a differential operator of degree, in our previous sense.

The result above is Theorem 3.3.8 in [Nar85]. Its proof is not so bad, but we prefer to avoid it in order to arrive at our main result as quickly as possible. In any case, the untrustworthy reader can take the characterization above as a definition of a differential

[4] Recall that for us every function is complex-valued!

operator. One can take either the inductive point of view to define the order of a differential operator, or the local point of view above. (Both coincide, of course.)

We're now in position to prove the existence of our desired formal adjoint.

**Proposition 4.2** Let $P : E \to F$ be a differential operator. Then, there exists a unique differential operator $P^* : F \to E$ such that

$$\langle Pf, g\rangle_{L^2} = \langle f, P^*g\rangle_{L^2}$$

for all local sections $f$ and $g$ (of $E$ and $F$, respectively). Moreover, $P^*$ has the same order as $P$.

**Proof.** If $P$ has two formal adjoints $P^*$ and $P'$, then $\langle f, P^*g\rangle_{L^2} = \langle f, P'g\rangle_{L^2}$ for all local sections $f$ and $g$. In particular, $\langle f, P^*g - P'g\rangle_{L^2} = 0$ and so $P^*g = P'g$ for all $g$. A consequence of this uniqueness is the following. Let $U$ and $V$ be two open subsets of $X$ and suppose that there exist adjoints $(P_U)^*$ and $(P_V)^*$. Then, the restrictions of both adjoints to $U \cap V$ coincide. (Indeed, both are adjoints of $P_{U\cap V}$.) Now, a partition of unity argument implies that it suffices to construct $P^*$ on the open sets of some covering.

By our previous paragraph, we may suppose that $X = \mathbb{R}^n$ and that $E, F$ are trivial. Furthermore, the Gram-Schmidt process yields isometric local trivializations and so we may suppose that $E, F$ have the standard metrics. (But we may not suppose that $X$ has the usual metric.) Also, by compactness, we can assume that $f$ and $g$ have compact support.

The local description of $P$ says that it's a sum of compositions of matrices of $C^\infty$ functions with vector fields. If $A$ is such a matrix, it's clear that the adjoint of the operator $f \mapsto Af$ is its usual hermitian transpose. It suffices then to find the formal adjoint of a vector field $X$.

Recall that $\operatorname{div} X$ is the unique scalar function such that $\mathcal{L}_X \operatorname{vol}_g = \operatorname{div} X \operatorname{vol}_g$. By Cartan's magic formula and Stokes' theorem, we have that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} d\iota_X(f \cdot \overline{g} \operatorname{vol}_g) = \int_{\mathbb{R}^n} \mathcal{L}_X(f \cdot \overline{g} \operatorname{vol}_g) \\ &= \int_{\mathbb{R}^n} (Xf) \cdot \overline{g} \operatorname{vol}_g + \int_{\mathbb{R}^n} f \cdot (X\overline{g}) \operatorname{vol}_g + \int_{\mathbb{R}^n} (f \cdot \overline{g}) \underbrace{\mathcal{L}_X(\operatorname{vol}_g)}_{\operatorname{div} X \operatorname{vol}_g}. \end{aligned}$$

In particular, it follows that $\langle Xf, g\rangle_{L^2} = \langle f, (-X - \operatorname{div} X)g\rangle_{L^2}$; proving that $-X - \operatorname{div} X$ is the formal adjoint of $X$. $\square$

As a consequence of the existence of formal adjoints, we can extend a differential operator $P : E \to F$ to a map $P : \mathcal{D}'(X, E) \to \mathcal{D}'(X, F)$ by defining $Pu(\varphi) := u(P^*\varphi)$. Now, if $\nabla_g : T^*X \to T^*X$ is the (dual of the) Levi-Civita connection and $\nabla : E \to E \otimes T^*X$ is our connection on $E$, we denote by $\nabla^j$ the composition

$$E \xrightarrow{\nabla} E \otimes T^*X \xrightarrow{\nabla\otimes\nabla_g} E \otimes (T^*X)^{\otimes 2} \xrightarrow{\nabla\otimes(\nabla_g)^{\otimes 2}} \cdots \xrightarrow{\nabla\otimes(\nabla_g)^{\otimes(j-1)}} E \otimes (T^*X)^{\otimes j}.$$

At long last, we define the Sobolev space $W^k(X, E)$ as the set of all $f \in L^2(X, E)$ such that $\nabla^j f$ is in $L^2(X, E \otimes (T^*X)^{\otimes j})$ for all $j \leqslant k$. The Sobolev norm is simply

$$\|f\|_k^2 = \sum_{j=0}^{k} \|\nabla^j f\|_{L^2}^2.$$

One also has a local point of view on these Sobolev spaces.

**Proposition 4.3** Pick a finite atlas $(\varphi_i : U_i \xrightarrow{\sim} V_i \subset \mathbb{R}^n)_{i\in I}$ trivializing $E$, and a partition of unity $(\mu_i)_{i\in I}$ subordinate to this cover. Furthermore, consider a norm on $\Gamma(X, E)$ given by

$$f \mapsto \left( \sum_{i\in I} \|(\mu_i f) \circ \varphi_i^{-1}\|_{k,i}^2 \right)^{1/2},$$

where $\|\cdot\|_{k,i}$ is the Sobolev norm on $W^k(V_i, \mathbb{C}^r)$. Then, the completion of $\Gamma(X, E)$ with respect to this norm is $W^k(X, E)$.

The proof of this result is just a (rather long but) straightforward computation and may be found in [BH22, Section 9.3].

The precedent proposition explains many things. Going from the abstract definition to the local one, we see that $W^k(X, E)$ is independent of the metrics in $X$ and $E$, and even from the connection $\nabla$. In the other direction, we obtain that the norm $\|\cdot\|_{\mathrm{loc}}$ is independent, up to equivalence, from the choices of atlas, trivializations and partitions of unity. Finally, this point of view also allows us to generalize local statements to Sobolev spaces on vector bundles.

**Proposition 4.4** The following holds:

(a) The space $\Gamma(X, E)$ is dense in $W^k(X, E)$.

(b) For all $k > n/2 + l$, there's a natural continuous inclusion $W^k(X, E) \hookrightarrow \mathcal{C}^l(X, E)$.

(c) The natural inclusion $W^{k+1}(X, E) \hookrightarrow W^k(X, E)$ is compact.

**Proof.** The first claim is a general property of completions. As for the second, let $f \in W^k(X, E)$. Using the notation of the previous result, the proposition 1.1 implies that for every $x \in X$, there exists a neighborhood $U_i$ of it such that $f \circ \varphi_i^{-1}$ is (almost everywhere equal to) a $\mathcal{C}^l$ function on $\mathbb{R}^n$. The second claim follows.

For the last part, let $(f_n)$ be a bounded sequence in $W^{k+1}(X, E)$. Denote by $\rho : \mathbb{R}^n \to T$ any homeomorphism to the $n$-dimensional torus $T$. Rellich's lemma gives, for each $i$, a subsequence of $((\mu_i f_n) \circ \varphi_i^{-1} \circ \rho^{-1})_n$ which converges in $W^k(T, \mathbb{C}^r)$. Let $(n_k)_k$ be a set of indexes such that, for all $i$, the sequence $((\mu_i f_{n_k}) \circ \varphi_i^{-1} \circ \rho^{-1})_k$ converges in $W^k(T, \mathbb{C}^r)$ and denote the limit by $g_i$. The previous proposition implies that $(f_{n_k})_k$ converges in $W^k(X, E)$ to $\sum_{i\in I} \mu_i (g_i \circ \rho \circ \varphi_i)$. □

Now, one would like to apply functional analysis (along with all our machinery) to the study of differential operators. In particular, we would like to extend differential operators to Sobolev spaces.

**Proposition 4.5** Let $P : E \to F$ be a differential operator of order $d$. Then, for all $k \in \mathbb{N}$, $P : \mathcal{D}'(X, E) \to \mathcal{D}'(X, F)$ restricts to a bounded operator $W^{k+d}(X, E) \to W^k(X, F)$. Moreover, if $P^*$ is the formal adjoint of $P$, the equality $\langle Pf, g\rangle_{L^2} = \langle f, P^*g\rangle_{L^2}$ still holds for every $f \in W^d(X, E)$ and $g \in W^d(X, F)$.

**Proof.** We can work in coordinates and prove the boundedness of $W^{k+d}(X, E) \to W^k(X, F)$ on $\mathbb{R}^n$. Observe that for every $|\alpha| \leqslant d$, the sum defining $\|D_\alpha f\|_k^2$ is contained in the sum defining $\|f\|_{k+d}^2$. Moreover, by compactness, we may suppose the (matrices of) functions $a_\alpha$ appearing in the local presentation of $P$ to be bounded, and so

$$\left\| \sum_{|\alpha| \leqslant d} a_\alpha D_\alpha f \right\|_k^2 \leqslant \sum_{|\alpha| \leqslant d} \|a_\alpha D_\alpha f\|_k^2 \leqslant \sum_{|\alpha| \leqslant d} \|a_\alpha\|_\infty^2 \|D_\alpha f\|_k^2 \leqslant \left[ \sum_{|\alpha| \leqslant d} \|a_\alpha\|_\infty^2 \right] \|f\|_{k+d}^2.$$

This implies that $P : W^{k+d}(X, E) \to W^k(X, F)$ is bounded. The last statement follows by density. $\square$

In order to globalize the notion of an elliptic differential operator to manifolds, we need to define the symbol in an invariant way.

**Definition 4.2** Let $P : E \to F$ be a differential operator of order $d$ and $x \in X$. Its *symbol* $\sigma_P : E(x) \to F(x)$ is the linear map defined in the following way. Let $\xi \in T_x^*X$, and $e \in E(x)$. We pick $f \in \mathcal{C}^\infty(X, \mathbb{R})$ with $f(x) = 0$ and $d_x f = \xi$, and pick $s \in \Gamma(X, E)$ with $s(x) = e$. Then we pose $\sigma_P(\xi)e := (1/d!)P(f^d s)(x)$.

A first remark is that $\sigma_P(\xi)e$ only depends on $\xi$ and $e$. If $s(x) = 0$, then $f^d s$ has a zero of order $> d$ at $x$, hence $P(f^d s)(x) = 0$; proving that $\sigma_P(\xi)e$ is independent of $s$. Moreover, if $g \in \mathcal{C}^\infty(X, \mathbb{R})$ with $g(x) = 0$ and $d_x g = \xi$, then $f^d - g^d$ is annihilated by any differential operator of order at most $d$.[5] Finally, if $E = F$ is the trivial bundle over $\mathbb{R}^n$, this definition recovers our previous one.

As before, we say that $P$ is *elliptic* if $\sigma_P : E(x) \to F(x)$ is injective for every $x \in X$ and $\xi \in T_x^*X \setminus \{0\}$. It's true that $\sigma_{P^*} = \sigma_P^*$, where the second asterisk denotes the hermitian adjoint. In particular, $P$ is elliptic if and only if $P^*$ is.

The last result from the first two sections which remains to be globalized is Gårding's inequality. Since being elliptic is clearly a local notion, the result below is a corollary of propositions 4.3 and 2.2.

[5] Let $\mathfrak{m}_x$ be the unique maximal ideal of $\mathcal{C}_x^\infty(X, \mathbb{R})$. Since $d_x(f - g) = 0$, we have that $f \equiv g \pmod{\mathfrak{m}_x^2}$. It follows that $f^d - g^d \in \mathfrak{m}_x^{d+1}$.

**Proposition 4.6** Let $P : E \to F$ be an elliptic differential operator of order $d$. Then, for every $f \in L^2(X, E)$ such that $Pf \in W^k(X, F)$, we have

$$\|f\|_{k+d} \leqslant C_k(\|Pf\|_k + \|f\|_0),$$

where $C_k$ is a positive constant depending only on $k$. In particular, $f \in W^{k+d}(X, E)$.

We're finally in position to prove the main analytical theorem of these notes.

**Theorem 4.7** Let $P : \Gamma(X, E) \to \Gamma(X, E)$ be an elliptic differential operator of order $d$, and let $P^*$ be its formal adjoint. Then,

(a) $\ker P$ is finite-dimensional;

(b) $\operatorname{im} P$ is closed in $\Gamma(X, E)$, and of finite codimension;

(c) There's an orthogonal decomposition $\Gamma(X, E) = \ker P \oplus \operatorname{im} P^*$ in $L^2(X, E)$.

We begin the proof of this result with a purely functional-analytic lemma.

**Lemma 4.8** Let $H_0, H_1, H_2$ be Hilbert spaces, $T : H_0 \to H_1$ be a bounded linear map, and $K : H_0 \to H_2$ be a compact linear map. Moreover, suppose that there's a positive constant $C$ such that

$$\|x\|_0 \leqslant C(\|Tx\|_1 + \|Kx\|_2)$$

for all $x \in H_0$. Then $\operatorname{im} T$ is closed and $\ker T$ is finite-dimensional.

**Proof of the lemma.** Let's prove that $\ker T$ is finite-dimensional. Let $B$ be the closed unit ball in $\ker T$ and let $(x_n)$ be a sequence in $B$. By compactness of $K$, the sequence $(Kx_n)$ has a converging subsequence $(Kx_{n_i})$. But our inequality gives that $\|x_{n_i} - x_{n_j}\|_0 \leqslant C\|Kx_{n_i} - Kx_{n_j}\|_2$ and so $(x_{n_i})$ is Cauchy, proving that $(x_{n_i})$ converges and that $B$ is compact. The Riesz' lemma now implies that $\ker T$ is finite-dimensional.

To prove that $\operatorname{im} T$ is closed, let $(y_n)$ be a sequence in $\operatorname{im} T$ converging in $H_1$ to $y$. Since $\ker T$ is closed, we may write $H_0 = \ker T \oplus (\ker T)^\perp$. In particular, there's a sequence $(x_n)$ in $(\ker T)^\perp$ such that $y_n = Tx_n$ for every $n$.

We affirm that there's a positive constant $c$ such that $\|Tx_n\|_1 \geqslant c\|x_n\|$ for all $n$. Otherwise, for every $i \in \mathbb{N}$ we would have some $z_i \in \{x_n/\|x_n\|\}$ such that $\|Tz_i\|_1 \to 0$. Since $K$ is compact, we can assume that $(Kz_i)$ converges. Then, the inequality

$$\|z_i - z_j\|_0 \leqslant C(\|T(z_i - z_j)\|_1 + \|K(z_i - z_j)\|_2)$$

implies that $(z_i)$ is a Cauchy sequence and so converges to some $z \in H_0$. This $z$ must have absolute value equal to 1, since all of the $z_i$ have. It should be in $\ker T$, for $T$ is continuous. And it must be in $(\ker T)^\perp$, since the $x_n$ all are. This is an absurd.

Finally, our inequality implies that $\|x_n - x_m\|_0 \leqslant c^{-1}\|T(x_n - x_m)\|_1$. In particular, $(x_n)$ is a Cauchy sequence and so converges to $x \in H_0$. The continuity of T implies that $Tx = y$ and so $y \in \operatorname{im} T$. □

Let $k \in \mathbb{N}$ and let $\widetilde{P} : W^d(X, E) \to L^2(X, E)$ be the extension of P to Sobolev spaces. Gårding's inequality and Rellich's lemma allows us to pick K to be the natural inclusion $W^d(X, E) \to L^2(X, E)$ and T to be $\widetilde{P}$. We can also apply the preceding lemma to an extension $\widetilde{P}^*$ of the formal adjoint $P^*$.

**Proof of the theorem 4.7.** (a) We affirm that $\ker P = \ker \widetilde{P}$. (In particular, $\ker P$ is finite-dimensional.) Clearly $\ker P \subset \ker \widetilde{P}$. Now, suppose that $f \in \ker \widetilde{P}$. Since $\widetilde{P}f$ is in $W^k(X, E)$ for every $k \in \mathbb{N}$, Gårding's inequality implies that $f \in W^{k+d}(X, E)$. The Sobolev lemma then shows that $f \in \Gamma(X, E)$ and so $f \in \ker P$.[6]

(c) We will split the proof of this item into three claims.

- Claim 1: $\ker P = (\operatorname{im} P^*)^\perp$ inside $L^2(X, E)$. Let's show that $\ker P \subset (\operatorname{im} P^*)^\perp$. If $f \in \ker P$ and $g = P^*h$, for some $h \in \Gamma(X, E)$, we have that

  $$\langle f, g\rangle_{L^2} = \langle f, P^*h\rangle_{L^2} = \langle Pf, h\rangle_{L^2} = 0$$

  and so $f \in (\operatorname{im} P^*)^\perp$. Conversely, suppose that $f \in L^2(X, E)$ and that $\langle \widetilde{P}f, h\rangle_{L^2} = \langle f, P^*h\rangle_{L^2} = 0$ for every $h \in \Gamma(X, E)$. By density, it's even true that $\langle \widetilde{P}f, h\rangle_{L^2} = 0$ for $h \in L^2(X, E)$. Taking $h = \widetilde{P}f$ we conclude that $f \in \ker \widetilde{P} = \ker P$.

- Claim 2: $L^2(X, E) = \ker P \oplus \operatorname{im} \widetilde{P}^*$. The same density argument as before gives that $(\operatorname{im} P^*)^\perp = (\operatorname{im} \widetilde{P}^*)^\perp$. But $\operatorname{im} \widetilde{P}^*$ is closed in $L^2(X, E)$ (by lemma 4.8) and the result follows.[7]

- Claim 3: $\Gamma(X, E) = \ker P \oplus \operatorname{im} P^*$. It's clear that the right-hand side is contained in the left-hand side. Now, if $f \in \Gamma(X, E)$, the previous claim allows us to write $f = g + \widetilde{P}^*h$, where $g \in \ker P$ and $h \in W^d(X, E)$. Since $\widetilde{P}^*h = f - g \in \Gamma(X, E)$, Gårding's inequality implies that $h \in W^{k+d}(X, E)$ for all $k \in \mathbb{N}$. The Sobolev lemma then gives $h \in \Gamma(X, E)$, proving the claim.

(b) The result follows from items (a) and (c) applied to $P^*$. (Indeed, if $W$ is a subspace of an inner product space V such that $V = W \oplus W^\perp$, then $W$ is closed in V.) □

Now, its a formal business to deduce the Hodge decomposition for Kahler manifolds. We refer to [Voi02] for more details.

[6]These kinds of arguments are usually called *elliptic bootstrapping*.

[7]It's a standard result that if M is a closed subspace of a Hilbert space H, then $H = M \oplus M^\perp$.

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