L-functions, Tate's thesis

And all that *jazz*

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Summary

L-functions

Riemann zeta function

Consider the *Riemann zeta function* ζ, defined for $\Re(s) > 1$ by the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

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We have that

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

for $\Re(s) > 1$.

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for $\Re(s) > 1$.

In particular ζ has no zeros on the half-plane $\Re(s) > 1$.

Riemann zeta function - Product formula

Observe that

$$\begin{aligned} \zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots \\ 2^{-s}\zeta(s) &= \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \dots \end{aligned}$$

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so that

$$(1 - 2^{-s})\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \dots$$

has no terms of the form n^{-s} with n even.

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$$(1 - 3^{-s})(1 - 2^{-s})\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \dots$$

has no terms with factors of 2 or 3.

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has no terms with factors of 2 or 3. We continue *ad infinitum*.

Riemann zeta function - Functional equation

We set $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

Proposition - Functional equation

The function ζ has an analytic extension to $\mathbb{C}$, holomorphic except for a simple pole at $s = 1$ with residue 1. Moreover, we have that $\xi(s) = \xi(1-s)$ for all $s \in \mathbb{C}$.

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In particular, since Γ has no zeros and simple poles at the negative integers, ζ has simple zeros at the even negative integers. All other zeros should be on the strip $0 \leq \Re(s) \leq 1$.

Riemann zeta function - Functional equation

Let $\theta(u) := \sum_{n\in\mathbb{Z}} \exp(-\pi n^2 u)$. Since the Fourier transform of $x \mapsto \exp(-\pi x^2 u)$ is $y \mapsto \exp(-\pi y^2/u)/\sqrt{u}$, the Poisson formula gives that

$$\theta(1/u) = \sqrt{u}\theta(u).$$

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We'll consider a variant

$$\tilde{\theta}(u) := \sum_{n=1}^{\infty} \exp(-\pi n^2 u) = \frac{\theta(u) - 1}{2}$$

which satisfies

$$\tilde{\theta}(1/u) = \sqrt{u}\tilde{\theta}(u) + \frac{1}{2}(\sqrt{u} - 1).$$

Riemann zeta function - Functional equation

By the definition of the gamma and the zeta functions,

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-t} \underbrace{t^{s/2}\pi^{-s/2}n^{-s}}_{(t/\pi n^2)^{s/2}} \frac{\mathrm{d}t}{t}.$$

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Letting $t = \pi n^2 u$ in the integral,

$$\xi(s) = \int_0^{\infty} \left(\sum_{n=1}^{\infty} \exp(-\pi n^2 u) \right) u^{s/2} \frac{\mathrm{d}u}{u} = \int_0^{\infty} \tilde{\theta}(u) u^{s/2} \frac{\mathrm{d}u}{u}.$$

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Riemann zeta function - Functional equation

We divide the interval $(0,\infty)$ into $(0,1)$ and $(1,\infty)$. Then we make a change of variables ($u \mapsto 1/u$) on the $(0,1)$ part, yielding

$$\xi(s) = \int_1^\infty \tilde{\theta}(1/u)u^{-s/2}\frac{\mathrm{d}u}{u} + \int_1^\infty \tilde{\theta}(u)u^{s/2}\frac{\mathrm{d}u}{u}.$$

Riemann zeta function - Functional equation

We divide the interval $(0, \infty)$ into $(0, 1)$ and $(1, \infty)$. Then we make a change of variables ($u \mapsto 1/u$) on the $(0, 1)$ part, yielding

$$\xi(s) = \int_1^\infty \tilde{\theta}(1/u) u^{-s/2} \frac{\mathrm{d}u}{u} + \int_1^\infty \tilde{\theta}(u) u^{s/2} \frac{\mathrm{d}u}{u}.$$

Then the functional equation of $\tilde{\theta}$ gives

$$\xi(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \tilde{\theta}(u) \left\{ u^{s/2} + u^{(1-s)/2} \right\} \frac{\mathrm{d}u}{u}.$$

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The RHS is holomorphic on $\mathbb{C} \setminus \{0, 1\}$ and clearly symmetric with respect to $s \mapsto 1 - s$.

Natural generalizations

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$$L(\chi, s) := \sum_{n=1}^{\infty} \frac{\chi(s)}{n^s}.$$

Let K be a number field. We define Dedekind's zeta function

$$\zeta_K(s) := \sum_{I \subset \mathcal{O}_K} \frac{1}{N(I)^s},$$

where the sum goes through all non-zero ideals I of $\mathcal{O}_K$, and $N(I) = \#\mathcal{O}_K/I$.

Why should we care?

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As for the Dedekind zeta function, we can write it as

$$\zeta_K(s) := \sum_{I \subset \mathcal{O}_K} \frac{1}{N(I)^s} = \sum_{n=1}^{\infty} \frac{r_K(n)}{n^s},$$

where $r_K(n)$ is the number of ideals of norm n.

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There's also the analytic class number formula, relating $\operatorname{res}_1 \zeta_K$ and h_K.

Product formulae

We have that

$$L(\chi, s) = \prod_{p} \frac{1}{1 - \chi(p)p^{-s}} \qquad \text{and} \qquad \zeta_K(s) = \prod_{\mathfrak{p} \subset \mathcal{O}_K} \frac{1}{1 - N(\mathfrak{p})^{-s}}.$$

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However, establishing a functional equation and a meromorphic continuation is a difficult task!

The local side of the force

Quasi-characters

Definition

Let K be a local field. A (multiplicative) *quasi-character* is a continuous morphism $\chi : K^\times \to \mathbb{C}^\times$. It's a *character* if $\chi(K^\times) \subset S^1$.

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We denote have a natural short exact sequence

$$1 \to U_K \to K^\times \to |K^\times| \to 1,$$

and we say that a character χ is *unramified* if $\chi|_{U_K} = 1$.

Quasi-characters

Proposition

Every quasi-character χ can be written as $\eta|\cdot|^s$, where η is a character and $s \in \mathbb{C}$. We say that $\sigma = \Re(s)$ is the *exponent* of χ.

(While the decomposition is not unique, the real part of s is always the same.)

Non-archimedean local L-factors

Recall that the ring of integers $\mathscr{O}_K$ of a non-archimedean local field K is a discrete valuation ring with uniformizing parameter ϖ. We define $L(\chi)$ to be

$$\frac{1}{1-\chi(\varpi)}$$

is χ is unramified and $L(\chi) = 1$ otherwise.

Archimedean local L-factors

Proposition

The quasi-characters of $\mathbb{R}$ are of the form $\chi_{a,s}(x) = x^{-a}|x|^s$ for some $a \in \{0, 1\}$ and $s \in \mathbb{C}$.

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We define the local L-factor associated with such characters as

$$L(\chi_{a,s}) := \pi^{-s/2}\Gamma(s/2).$$

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Proposition

The quasi-characters of $\mathbb{C}$ are of the form $\chi_{a,b,s}(z) = z^{-a}\overline{z}^{-b}||z||^s$ for some $a, b \in \mathbb{Z}$ with $\min(a, b) = 0$ and $s \in \mathbb{C}$.

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We define the local L-factor associated with such characters as

$$L(\chi_{a,b,s}) := 2(2\pi)^{-s}\Gamma(s).$$

Reality check

Let χ be the character $|\cdot|^s$ on $\mathbb{Q}_p$. It's always unramified (actually every unramified character is like this) and $\chi(\varpi) = p^{-s}$.

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$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\prod_p \frac{1}{1-p^{-s}} = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

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$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\prod_p \frac{1}{1-p^{-s}} = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Also, let K be a number field and $\chi : K_v^\times \to \mathbb{C}^\times$ be given by $|\cdot|^s$. The product of the finite local L-factors is

$$\zeta_K(s) = \prod_{\mathfrak{p}_v \subset \mathcal{O}_K} \frac{1}{1-N(\mathfrak{p}_v)^{-s}}.$$

Let K be a local field. A *Schwartz-Bruhat* function is a Schwartz function if K is archimedean or a locally constant function with compact support if K is nonarchimedean.

Local zeta integrals

Let K be a local field. A *Schwartz-Bruhat* function is a Schwartz function if K is archimedean or a locally constant function with compact support if K is nonarchimedean.

For those functions f, we define the *local zeta integral*

$$Z(f, \chi) := \int_{K^\times} f(x)\chi(x) \, \mathrm{d}^\times x.$$

Local zeta integrals

Theorem

Let f be a SB function on K and $\chi = \eta|\cdot|^s$ with exponent σ. Then,

1. $Z(f, \chi)$ is absolutely convergent if $\sigma > 0$.

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2. $Z(f, \chi)$ has a meromorphic continuation of $\mathbb{C}$.

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Theorem

Let f be a SB function on K and $\chi = \eta|\cdot|^s$ with exponent σ. Then,

1. $Z(f, \chi)$ is absolutely convergent if $\sigma > 0$.
2. $Z(f, \chi)$ has a meromorphic continuation of $\mathbb{C}$.
3. There exists a nonvanishing holomorphic function $\varepsilon(\chi)$ such that

$$L(\chi)Z(\hat{f}, \chi^\vee) = \epsilon(\chi)L(\chi^\vee)Z(f, \chi),$$

where $\chi^\vee := \chi^{-1}|\cdot|$.

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Let K be a global field. A Schwartz-Bruhat function on $\mathbb{A}_K$ is a product of Schwartz-Bruhat functions on K_v for all places v of K.

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Theorem - Poisson formula

Let f be a SB function on $\mathbb{A}_K$. Then,

$$\sum_{\gamma \in K} f(\gamma + x) = \sum_{\gamma \in K} \hat{f}(\gamma + x)$$

for all $x \in \mathbb{A}_K$.

The Riemann-Roch theorem

Corollary - Riemann-Roch

Let f be a SB function on $\mathbb{A}_K$. Then,

$$\sum_{\gamma \in K} f(\gamma x) = \frac{1}{|x|} \sum_{\gamma \in K} \hat{f}(\gamma x^{-1})$$

for all $x \in \mathbb{I}_K$.

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for all $x \in \mathbb{I}_K$.

Indeed, it suffices to apply the Poisson formula to $h(y) := f(xy)$ since $\hat{h}(\gamma) = \hat{f}(\gamma x^{-1})/|x|$.

The Riemann-Roch theorem

Corollary 2 - Riemann-Roch

Let C be a smooth projective curve over a finite field. Then, for all divisors D

$$\ell(D) - \ell(K - D) = \deg(D) - g + 1,$$

where K is the canonical divisor and g is the genus of C.

The global side of the force

Grössencharakters (german for fancy idelic characters)

Let K be a global field. Recall that $K^\times \hookrightarrow \mathbb{I}_K$ and that the quotient $\mathbb{I}_K/K^\times$ is the *idèle class group* C_K.

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If χ is a Dirichlet character, the composition

$$\mathbb{I}_\mathbb{Q} \twoheadrightarrow \widehat{\mathbb{Z}}^\times \twoheadrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times$$

defines a quasi-character.

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If χ is a Dirichlet character, the composition

$$\mathbb{I}_\mathbb{Q} \twoheadrightarrow \widehat{\mathbb{Z}}^\times \twoheadrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times$$

defines a quasi-character. Another example is $|\cdot|^s$ for $s \in \mathbb{C}$. (Recall the product formula!)

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Proposition

Every quasi-character χ can be written as $\eta|\cdot|^s$, where η is a character and $s \in \mathbb{C}$. We say that $\sigma = \Re(s)$ is the *exponent* of χ.

(While the decomposition is not unique, the real part of s is always the same.)

Global zeta integrals

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$$Z(f, \chi) = \int_{\mathbb{I}_K} f(x)\chi(x) \, \mathrm{d}^\times x.$$

As before, if $\chi = \eta |\cdot|^s$ with exponent σ, $Z(f, \chi)$ is absolutely convergent for $\sigma > 1$.

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1. $Z(f, \chi)$ has a meromorphic extension to $\mathbb{C}$.
2. We have the functional equation $Z(f, \chi) = Z(\hat{f}, \chi^\vee)$.
3. Its residue at 0 is $\mathrm{Vol}(C_K^1)\hat{f}(0)$.

We write

$$Z_t(f, \chi) := \int_{\mathbb{I}_K^1} f(tx)\chi(tx)\, \mathrm{d}^\times x$$

so that

$$Z(f, \chi) = \int_0^\infty Z_t(f, \chi)\, \frac{\mathrm{d}t}{t}.$$

Proof of the functional equation

Using the Riemann-Roch theorem, we obtain a functional equation:

$$Z_t(f,\chi) = Z_{t^{-1}}(\hat{f},\chi^\vee) + \hat{f}(0)\int_{C_K^1} \chi^\vee(x/t)\,\mathrm{d}^\times x - f(0)\int_{C_K^1} \chi(tx)\,\mathrm{d}^\times x.$$

The result then follows as before by the formula above and the fact that

$$\begin{aligned} Z(f,\chi) &= \int_0^1 Z_t(f,\chi)\,\frac{\mathrm{d}t}{t} + \int_1^\infty Z_t(f,\chi)\,\frac{\mathrm{d}t}{t} \\ &= \int_1^\infty Z_{t^{-1}}(f,\chi)\,\frac{\mathrm{d}t}{t} + \int_1^\infty Z_t(f,\chi)\,\frac{\mathrm{d}t}{t}. \end{aligned}$$

Let K be a global field and χ be a quasi-character. We pose

$$\Lambda(\chi, s) = \prod_v L(\chi_v|\cdot|^s) \qquad \text{and} \qquad L(\chi, s) = \prod_{v \text{ finite}} L(\chi_v|\cdot|^s),$$

where $\chi_v := \chi|_{K_v}$. Both define holomorphic functions on the half plane $\Re(s) > 1$.

Theorem

Let χ be a character of a global field K. Then $L(\chi, s)$ admits a meromorphic continuation to $\mathbb{C}$ and we have

$$\Lambda(\chi^\vee, 1-s) = \varepsilon(\chi, s)\Lambda(\chi, s),$$

where $\varepsilon(\chi, s) := \prod_v \varepsilon(\chi_v|\cdot|^s)$.

Questions?