TANNAKIAN FORMALISM FOR D-MODULES

Advisor : Javier FRESÁN

Gabriel RIBEIRO

ÉCOLE POLYTECHNIQUE
UNIVERSITÉ PARIS-SACLAY

INTRODUCTION

Presentation of the topic

Given a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$, perhaps the fundamental question in number theory is to ask if the equation $f = 0$ has solutions in $\mathbb{Z}^n$. A related, but equally important, question concerns the solutions of $f \equiv t \pmod{p}$ for a prime number p and $t \in \mathbb{Z}/p\mathbb{Z}$. We then consider the map $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}$ defined as

$$t \mapsto \mathrm{Sol}(f, p, t) := \#\{\text{solutions of } f \equiv t \pmod{p}\}.$$

We lose no information passing to its Fourier transform

$$\psi \mapsto \sum_{t \in \mathbb{Z}/p\mathbb{Z}} \psi(t)\,\mathrm{Sol}(f, p, t) = \sum_{x \in (\mathbb{Z}/p\mathbb{Z})^n} \psi(f(x))$$

and, under the isomorphism $\widehat{\mathbb{Z}/p\mathbb{Z}} \cong \mathbb{Z}/p\mathbb{Z}$, this results in

$$t \mapsto \sum_{x \in (\mathbb{Z}/p\mathbb{Z})^n} \exp\left(\frac{2\pi i t f(x)}{p}\right).$$

This is what is called an *exponential sum*. Using the machinery created by Deligne and Laumon to understand the Weil conjecture, we can use the Grothendieck trace formula to show that the sum above is nothing but the trace of

$$\widehat{\mathbf{R}f_!\underline{\mathbb{Q}_\ell}},$$

where ℓ is a prime number different than p and the hat signifies the Fourier-Deligne transform of the ℓ-adic sheaf $\mathbf{R}f_!\underline{\mathbb{Q}_\ell}$.

In this case, $\widehat{\mathbf{R}f_!\underline{\mathbb{Q}_\ell}}$ is a lisse ℓ-adic sheaf over a dense open set U of $\mathbb{A}^1$. In other words, it *is* a continuous ℓ-adic representation ρ of the étale fundamental group $\pi_1(U)$. Now, the Zariski closure of $\rho(\pi_1(U \times \mathbb{F}_p))$ is an algebraic group encoding most of the information of our exponential sum. We call this a *monodromy group*.

For other classes of sums, when dealing with algebraic varieties other than $\mathbb{A}^1$, or when our base field is of characteristic 0, the method above breaks down mainly due to the lack of a suitable monodromy group.

Lots of progress have been made in this direction. Arguably the most impressive result is the paper [5], which also explains in detail the other main lines of progress in this problem. In a simpler setting, N. Katz, O. Gabber and F. Loeser studied a monodromy group for perverse sheaves on $\mathbb{G}_m$. (In [14] and in [7].)

Due to the similarities between perverse sheaves and $\mathcal{D}$-modules, it has long been folklore that a similar monodromy group could be defined for $\mathcal{D}$-modules on $\mathbb{G}_m$ in the case of characteristic 0. The present work aims to prove (at least parts of) this result in detail.

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CONTENTS

1 TANNAKIAN FORMALISM

Given a locally compact abelian group G, Pontryagin duality shows that we can recover G from its character group $\hat{G}$. Tadao Tannaka and Mark Grigorievich Krein then extended this result to non-commutative compact groups by replacing $\hat{G}$ with the category of linear representations of G. Finally, inspired by the formalism of Galois categories, Grothendieck showed that a similar result holds for algebraic groups. This later theory was formalized by Saavedra-Rivano in what is now known as the *tannakian formalism*, and which is the subject of this chapter.

Our endeavor will begin by defining some axioms encoding properties of the category of (finite-dimensional) linear representations of an algebraic group. Then the main result of the theory is that every such category is equivalent to the category of finite-dimensional representations of an affine group scheme. Even though this group scheme need not be algebraic, we'll see that it is always a limit of algebraic groups. In other words, it is *pro-algebraic*.

1.1 TENSOR CATEGORIES

We will begin our adventure by modeling some properties of the tensor product in vector spaces. Let $\mathbf{C}$ be a category and consider a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, whose image at two objects M and N will be denoted as $M \otimes N$. An *associativity constraint* is a functorial isomorphism

$$\varphi_{-,-,-} : - \otimes (- \otimes -) \to (- \otimes -) \otimes -$$

such that, for all objects $M, N, P, Q \in \mathbf{C}$, the diagram

$$\begin{array}{ccccc} & & M \otimes (N \otimes (P \otimes Q)) & & \\ & \swarrow & & \searrow & \\ M \otimes ((N \otimes P) \otimes Q) & & & & (M \otimes N) \otimes (P \otimes Q) \\ \downarrow & & & & \downarrow \\ (M \otimes (N \otimes P)) \otimes Q & & \longrightarrow & & ((M \otimes N) \otimes P) \otimes Q \end{array}$$

commutes. This is called the *pentagon axiom*. Similarly, a *commutativity constraint* is a functorial isomorphism

$$\psi_{-,\cdot} : - \otimes \cdot \to \cdot \otimes -$$

such that, for all objects $M, N \in \mathbf{C}$, $\psi_{N,M} \circ \psi_{M,N} : M \otimes N \to M \otimes N$ is the identity morphism. We say than an associativity constraint and a commutativity constraint are compatible if, for all objects $M, N, P \in \mathbf{C}$, the diagram

$$\begin{array}{ccc} & M \otimes (N \otimes P) \longrightarrow (M \otimes N) \otimes P & \\ \swarrow & & \searrow \\ M \otimes (P \otimes N) & & P \otimes (M \otimes N) \\ \searrow & & \swarrow \\ & (M \otimes P) \otimes N \longrightarrow (P \otimes M) \otimes N & \end{array}$$

commutes. Not surprisingly, this is called the *hexagon axiom*. Finally, a pair (U, u) consisting of an object U of $\mathbf{C}$ and an isomorphism $u : U \to U \otimes U$ is said to be an *identity object* if the functor $M \mapsto U \otimes M$ is an equivalence of categories.

Definition 1.1 — Tensor category. A tuple $(\mathbf{C}, \otimes, \varphi, \psi)$, consisting of a category $\mathbf{C}$, a bifunctor $\otimes$ and compatible associativity and commutativity constraints φ and ψ, is said to be a *tensor category* if $\mathbf{C}$ contains an identity object.

In the lingo of category theory, this is nothing but a symmetric monoidal category. If our category $\mathbf{C}$ is k-linear, we require the functor $\otimes$ to be bilinear. We remark that any two identity objects are canonically isomorphic, so we'll always chose one and denote it by $(1, e)$. Moreover, these axioms guarantee that the tensor product of any finite family of objects is well-defined up to a unique isomorphism.

Definition 1.2 — Tensor functor. Let $(\mathbf{C}, \otimes, \varphi, \psi)$ and $(\mathbf{C}', \otimes', \varphi', \psi')$ be tensor categories. A *tensor functor* is a pair (F, c) consisting of a functor $F : \mathbf{C} \to \mathbf{C}'$ and a functorial isomorphism $c_{-,-} : F(-) \otimes F(-) \to F(- \otimes -)$ such that, for all $M, N, P \in \mathbf{C}$ the diagrams

$$\begin{array}{ccccc} FM \otimes (FN \otimes FP) & \xrightarrow{\mathrm{id} \otimes c} & FM \otimes F(N \otimes P) & \longrightarrow & F(M \otimes (N \otimes P)) \\ \downarrow{\scriptstyle \varphi'} & & & & \downarrow{\scriptstyle F(\varphi)} \\ (FM \otimes FN) \otimes FP & \xrightarrow{c \otimes \mathrm{id}} & F(M \otimes N) \otimes FP & \longrightarrow & F((M \otimes N) \otimes P) \end{array}$$

and

$$\begin{array}{ccc} FM \otimes FN & \xrightarrow{c} & F(M \otimes N) \\ \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle F(\psi)} \\ FN \otimes FM & \xrightarrow{c} & F(N \otimes M). \end{array}$$

commute. Moreover, we impose that $(F(U), F(u))$ is an identity object of $\mathbf{C}'$ whenever (U, u) is an identity object of $\mathbf{C}$.

These axioms imply that we have a functorial isomorphism

$$F\left(\bigotimes_{i\in I} M_i\right) \to \bigotimes_{i\in I} F(M_i),$$

for every finite collection of objects $(M_i)_{i\in I}$. We also define what it means for a natural transformation to *preserve the tensor structure*. Such natural transformations will be particularly important in our study.

Definition 1.3 — Morphism of tensor functors. Let (F, c) and (G, d) be tensor functors $\mathbf{C} \to \mathbf{C}'$. A *morphism of tensor functors* is a natural transformation $\lambda : F \to G$ making the diagram

$$\begin{array}{ccc} \bigotimes_{i\in I} F(M_i) & \longrightarrow & F\left(\bigotimes_{i\in I} M_i\right) \\ \downarrow & & \downarrow \\ \bigotimes_{i\in I} G(M_i) & \longrightarrow & G\left(\bigotimes_{i\in I} M_i\right) \end{array}$$

commute for every finite collection $(M_i)_{i\in I}$ of objects in $\mathbf{C}$. We denote by $\mathrm{Hom}^{\otimes}(F, G)$ the collection of morphisms of tensor functors $F \to G$.

There is a particular related construction that we'll need. If k is a field and A is a k-algebra, let $\varphi_A : k\text{-}\mathbf{Vect} \to A\text{-}\mathbf{Mod}$ be the base change functor, which sends V to $V \otimes_k A$. If $\omega : \mathbf{C} \to k\text{-}\mathbf{Vect}$ is a tensor functor on a rigid tensor category (which will be defined below), we'll denote by $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ the functor $k\text{-}\mathbf{Alg} \to \mathbf{Grp}$ defined by

$$\underline{\mathrm{Aut}}^{\otimes}(\omega)(A) := \mathrm{Hom}^{\otimes}(\varphi_A \circ \omega, \varphi_A \circ \omega),$$

where the group operation is given by composition. We require $\mathbf{C}$ to be rigid since in this case the category of tensor functors $\mathbf{C} \to k\text{-}\mathbf{Vect}$ is a grupoid. In other words, every morphism of tensor functors is an isomorphism. That's why we write Aut instead of End. A (very) detailed proof of this result may be found in [23].

The main theorem of this chapter will say that, under some reasonable conditions, the functor above will be represented by an affine group scheme.

1.2 RIGID TENSOR CATEGORIES

In the previous section, we dealt with axioms that were satisfied by the category of all vector spaces. Our goal now is to specialize to properties that are exclusive to *finite-dimensional* vector spaces or representations.

Definition 1.4 — Internal hom. Let **C** be a tensor category and let $M, N \in \mathbf{C}$ be two objects. If the functor $\mathrm{Hom}(- \otimes M, N)$ is representable, we denote by $\underline{\mathrm{Hom}}(M,N)$ the representing object and by

$$\mathrm{ev}_{M,N} : \underline{\mathrm{Hom}}(M,N) \otimes M \to N$$

the morphism corresponding to $\mathrm{id}_{\underline{\mathrm{Hom}}(M,N)}$.

Let's be explicit. The representability condition means that we have an isomorphism

$$\mathrm{Hom}(T, \underline{\mathrm{Hom}}(M,N)) \to \mathrm{Hom}(T \otimes M, N)$$

which is functorial in T. In particular, taking $T = \underline{\mathrm{Hom}}(M,N)$, we obtain a morphism

$$\mathrm{Hom}(\underline{\mathrm{Hom}}(M,N), \underline{\mathrm{Hom}}(M,N)) \to \mathrm{Hom}(\underline{\mathrm{Hom}}(M,N) \otimes M, N).$$

The *evaluation map* $\mathrm{ev}_{M,N}$ is the image of $\mathrm{id}_{\underline{\mathrm{Hom}}(M,N)}$ under this morphism.

■ **Example 1.1** In A-**Mod**, the usual tensor-hom adjunction implies that the internal hom $\underline{\mathrm{Hom}}(M,N)$ always exists and is given by $\mathrm{Hom}_A(M,N)$, with its natural A-module structure. In this context, the evaluation map is given simply by

$$\begin{aligned} \mathrm{Hom}_A(M,N) \otimes M &\to N \\ f \otimes m &\mapsto f(m). \end{aligned}$$

This is the motivation for its name. ■

If it exists, the *dual* $M^\vee$ of an object M is defined as $\underline{\mathrm{Hom}}(M, 1)$. As before, we have an evaluation map $\mathrm{ev}_M : M^\vee \otimes M \to 1$. The representability condition in this case is

$$\mathrm{Hom}(T, M^\vee) \to \mathrm{Hom}(T \otimes M, 1).$$

If, moreover, the dual of $M^\vee$ exists, the representability condition above associates a natural morphism

$$i_M : M \to M^{\vee\vee}$$

to the (twisted) evaluation map $\mathrm{ev}_M \circ \psi : M \otimes M^\vee \to 1$. Our object M is said to be *reflexive* if i_M is an isomorphism.

Finally, we observe that, given two collections of objects $(M_i)_{i\in I}$ and $(N_i)_{i\in I}$, the morphism

$$\left(\bigotimes_{i\in I} \underline{\mathrm{Hom}}(M_i, N_i)\right) \otimes \left(\bigotimes_{i\in I} M_i\right) \xrightarrow{\sim} \bigotimes_{i\in I} (\underline{\mathrm{Hom}}(M_i, N_i) \otimes M_i) \xrightarrow{\otimes \mathrm{ev}} \bigotimes_{i\in I} N_i$$

determines, via the representability condition, a morphism

$$\bigotimes_{i\in I} \underline{\mathrm{Hom}}(M_i, N_i) \to \underline{\mathrm{Hom}}\left(\bigotimes_{i\in I} M_i, \bigotimes_{i\in I} N_i\right),$$

whenever all the internal homs in sight exist.

After this somewhat long digression, we are able to specify the axioms which the categories of finite-dimensional vector spaces or representations satisfy.

Definition 1.5 — Rigid tensor category. A tensor category $(\mathbf{C}, \otimes)$ is said to be *rigid* if all the internal homs exist, if the morphism

$$\underline{\mathrm{Hom}}(M_1, N_1) \otimes \underline{\mathrm{Hom}}(M_2, N_2) \to \underline{\mathrm{Hom}}(M_1 \otimes M_2, N_1 \otimes N_2)$$

is an isomorphism for all objects $M_1, M_2, N_1, N_2 \in \mathbf{C}$, and if all objects of $\mathbf{C}$ are reflexive.

A particular, but important, case of the isomorphism above is given by

$$M^\vee \otimes N \xrightarrow{\sim} \underline{\mathrm{Hom}}(M, N),$$

which follows from taking $M_1 = M$, $N_1 = M_2 = 1$, and $N_2 = N$. This isomorphism allows us to define the *trace* of an element of $\mathrm{End}(M)$. We take the composition

$$\underline{\mathrm{End}}(M) \cong M^\vee \otimes M \xrightarrow{\mathrm{ev}_M} 1$$

and apply the functor $\mathrm{Hom}(1, -)$.[1] This defines our trace morphism

$$\mathrm{tr} : \mathrm{End}(M) \to \mathrm{End}(1).$$

When $\mathbf{C}$ is the category of finite-dimensional k-vector spaces with its usual tensor product, the ring $\mathrm{End}(1) = \mathrm{End}(k)$ is nothing but k and this becomes the usual trace morphism.

We end this section citing a simple criterion for recognizing rigid tensor categories.

Proposition 1.1 Let $\mathbf{C}$ be a k-linear abelian category and let $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ be a k-bilinear functor. Given a faithful exact k-linear functor $\omega : \mathbf{C} \to k$-**Vect**, a functorial isomorphism $\varphi_{-,-,-} : - \otimes (- \otimes -) \to (- \otimes -) \otimes -$, and a functorial isomorphism $\psi_{-,\cdot} : - \otimes \cdot \to \cdot \otimes -$, suppose that they satisfy the following properties:

a) $\omega \circ \otimes = \otimes \circ (\omega \times \omega)$;

b) $\omega(\varphi_{-,-,-})$ is the usual associativity constraint in k-**Vect**;

c) $\omega(\psi_{-,\cdot})$ is the usual commutativity constraint in k-**Vect**;

d) there exists an identity object U in $\mathbf{C}$ such that $k \xrightarrow{\sim} \mathrm{End}(U)$ and $\dim_k \omega(U) = 1$;

e) if $\dim_k \omega(L) = 1$, then there exists an object $L^{-1} \in \mathbf{C}$ such that $L \otimes L^{-1} = U$.

Then $(\mathbf{C}, \otimes, \varphi, \psi)$ is a rigid abelian tensor category.

The reader may find a proof (actually two proofs) of this result on [4].

[1] We remark that $\mathrm{Hom}(1, \underline{\mathrm{Hom}}(M, N)) \cong \mathrm{Hom}(1 \otimes M, N) = \mathrm{Hom}(M, N)$.

1.3 NEUTRAL TANNAKIAN CATEGORIES

We finally arrive at the main definition of this chapter.

Definition 1.6 — Tannakian category. A rigid k-linear abelian tensor category C such that $\mathrm{End}(1) = k$ is said to be a *neutral tannakian category* if there exists an exact faithful tensor functor $\omega : \mathsf{C} \to k\text{-}\mathsf{Vect}$. We say that such a functor is a *fibre functor*.

Actually there is a more general notion of tannakian category (as in [4]) but, as we won't need it, we will refer to neutral tannakian categories simply as *tannakian categories*.

▪ **Example 1.2 — Trivial examples.** The category of finite-dimensional k-vector spaces, with the identity functor, and the category of finite-dimensional linear representations of a group, with the forgetful functor, are tannakian. ▪

▪ **Example 1.3 — Local systems.** Let X be a path-connected, locally path-connected and locally simply connected topological space. The category $\mathsf{Loc}_k(X)$, composed of local systems (locally constant sheaves) of finite-dimensional k-vector spaces, endowed with the functor

$$\mathscr{F} \mapsto \mathscr{F}_x$$

which takes the stalk of a local system at $x \in X$, is tannakian. ▪

The result below is the most important result on this chapter.

Theorem 1.2 — Main theorem on tannakian categories. Let $(\mathsf{C}, \otimes)$ be a tannakian category with a fiber functor ω. Then,

a) the functor $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is represented by an affine group scheme G;

b) the functor $\mathsf{C} \to G\text{-}\mathsf{Rep}$ determined by ω is an equivalence of tensor categories.

We call the group scheme G associated with a tannakian category a *tannakian group*. Since, at least for the writer, the proof of this result is not very illuminating, we'll prefer to focus on some interesting consequences of this theorem. Still, the reader may find this result, with its proof, as theorem 2.11 in [4].

▪ **Example 1.4 — Algebraic hull.** Let G be an abstract group and k be a field. As we've seen, the category G-**Rep** of finite-dimensional k-representations of G is tannakian. The theorem above then shows that G-**Rep** is equivalent to the category of finite-dimensional representations of a *group scheme* G^{alg}. We say that this is the *algebraic hull* of G. ▪

In the example above, and in general, the group schemes that arise via the main theorem of tannakian categories are usually huge. However, they have quotients which are more manageable. This suggests the definition below.

Definition 1.7 Let $(\mathbf{C}, \otimes)$ be a tannakian category and $M \in \mathbf{C}$. We denote by $\langle M \rangle$ the smallest full subcategory of $\mathbf{C}$ which is closed under direct sums, tensor products, duals, and subquotients.

In the context above, the subcategory $\langle M \rangle$, along with the restriction of a fiber functor ω of $\mathbf{C}$ to it, is still tannakian. The group G representing $\underline{\mathrm{Aut}}^{\otimes}_{\mathbf{C}}(\omega)$ acts naturally on $\omega(M)$, defining a morphism[2]

$$G \to \mathrm{GL}(\omega(M)).$$

Its *raison d'être* is the proposition below.

Proposition 1.3 Let $\mathbf{C}$ be a tannakian category, with fibre functor ω, tannakian group G, and let $M \in \mathbf{C}$. The image of the morphism $G \to \mathrm{GL}(\omega(M))$ is a closed subgroup of $\mathrm{GL}(\omega(M))$ which coincides with the tannakian group associated with the category $\langle M \rangle$.

This result is contained in the proof of proposition 2.8 in [4]. For now, let's denote the image of the morphism $G \to \mathrm{GL}(\omega(M))$ as G_M. These groups allow us to understand G in the following way.

We order the subcategories of the form $\langle M \rangle$ by inclusion. If $\langle N \rangle$ is contained in $\langle M \rangle$, then $N \in \langle M \rangle$ and so $\langle M \rangle = \langle M \oplus N \rangle$. In particular, we obtain a map

$$G_M \to G_N.$$

Then G is nothing but the limit of these groups.

Proposition 1.4 Let $\mathbf{C}$ be a tannakian category with tannakian group G. Then $G = \lim G_M$ as the limit runs over the subcategories of the form $\langle M \rangle$. In particular, G is a *pro-algebraic* group.

Proof. First of all, we have surjections $G \to G_M$ for every object M of $\mathbf{C}$. Also, if $\langle N \rangle$ is contained in $\langle M \rangle$, then those surjections make the diagram

$$\begin{array}{ccc} & G & \\ \swarrow & & \searrow \\ G_M & \longrightarrow & G_N \end{array}$$

commute. The universal property of the limit then gives a surjection

$$G \to \lim G_M.$$

This morphism is also injective since if $g \in G$ is sent to the identity of $\lim G_M$, then g acts trivially on $\omega(M)$ for every M and so is the identity of G. □

[2]Here we see $\mathrm{GL}(\omega(M))$ as a group scheme. That is, it the the functor which sends a k-algebra A to the group $\mathrm{Aut}(\omega(M) \otimes_k A)$, the group of invertible endomorphisms of A-modules on $\omega(M) \otimes_k A$.

2
D-MODULES

In this chapter, we'll see how we can use the machinery of sheaf theory and homological algebra to understand differential equations. The main idea is the following. Let X be the affine space $\mathbb{A}^n(\mathbb{C}) = \mathbb{C}^n$ and let $A = \mathbb{C}[x_1, \dots, x_n]$ be its ring of global functions. We denote by D the (non-commutative) $\mathbb{C}$-algebra generated by $x_1, \dots, x_n$ and $\partial_1, \dots, \partial_n$, where

$$[x_i, x_j] = 0, \qquad [\partial_i, \partial_j] = 0, \quad \text{and} \quad [\partial_i, x_j] = \delta_{ij}.$$

This ring acts on A, making it a left D-module. Now, we would like to understand the solutions f of the differential equation

$$Lf = 0,$$

where $L \in D$ and $f \in A$. Following Sato, we consider the left D-module $M = D/DL$ and observe that

$$\begin{aligned} \mathrm{Hom}_D(M, A) &= \mathrm{Hom}_D(D/DL, A) \\ &\cong \{\varphi \in \mathrm{Hom}_D(D, A) \mid \varphi(L) = 0\}. \end{aligned}$$

The latter is isomorphic to the solution space $\{f \in A \mid Lf = 0\}$, since $\mathrm{Hom}_D(D, A) \cong A$ via $\varphi \mapsto \varphi(1)$ and

$$Lf = L\varphi(1) = \varphi(L) = 0.$$

The study of such differential equations is then subsumed under the study of the functor $\mathrm{Hom}_D(\_, A)$ from cyclic D-modules to $\mathbb{C}$-vector spaces. More generally, we can study systems of linear partial differential equations by expanding the domain of the aforementioned functor to finitely presented modules.

2.1 BASIC DEFINITIONS

Now that we understand the main motivation, we'll sheafify the preceding discussion and consider more general spaces. In everything that follows, X is a smooth scheme over an algebraically closed field k of characteristic zero and $\mathscr{O}_X$ is its structure sheaf. We recall that $\mathscr{O}_X$ can be seen naturally as a subsheaf of $\underline{\mathrm{End}}(\mathscr{O}_X)$ under the identification

$$\begin{aligned} \mathscr{O}_X &\to \underline{\mathrm{End}}(\mathscr{O}_X) \\ f &\mapsto (g \mapsto fg). \end{aligned}$$

Finally, we also denote by Θ_X the tangent sheaf, defined as the subsheaf of $\underline{\mathrm{End}}(\mathscr{O}_X)$ determined by those $\theta \in \underline{\mathrm{End}}(\mathscr{O}_X)$ such that

$$\theta(fg) = \theta(f)g + f\theta(g), \quad \text{for all } f, g \in \mathscr{O}_X.$$

We're now in position to define the main object of this chapter.

Definition 2.1 — Sheaf of differential operators. The *sheaf of differential operators* $\mathcal{D}_X$ is the subsheaf of k-algebras of $\underline{\mathrm{End}}(\mathscr{O}_X)$ generated by $\mathscr{O}_X$ and Θ_X. A $\mathcal{D}_X$-module M is said to be *quasi-coherent* if its quasi-coherent as an $\mathscr{O}_X$-module. We denote the full subcategory of $\mathbf{D}^*(\mathcal{D}_X)$ consisting of complexes whose cohomology sheaves are quasi-coherent as $\mathbf{D}^*_{\mathrm{qc}}(\mathcal{D}_X)$. A result of Bernstein shows that this is equivalent to the derived category of $\mathcal{D}_X$-**QCoh**.

While we'll try to be general when it isn't too troublesome, in this text we care mainly about quasi-coherent $\mathcal{D}_X$-modules over affine schemes. In particular, a quasi-coherent $\mathcal{D}_X$-module over $\mathbb{A}^n$ (resp. $\mathbb{G}_m$) is simply $\widetilde{M}$, where M is a $k[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$-module[3] (resp. $k[x, x^{-1}, \partial]$-module).

A natural way to think about $\mathcal{D}_X$-modules is as $\mathscr{O}_X$-modules endowed with some extra structure. Usually, this structure is an action of $\mathcal{D}_X$. The following propositions gives another useful possibility.

Proposition 2.1 Let M be an $\mathscr{O}_X$-module. Giving a left $\mathcal{D}_X$-module structure on M extending the $\mathscr{O}_X$-module structure is equivalent to giving a k-linear morphism

$$\begin{aligned} \nabla : \Theta_X &\to \underline{\mathrm{End}}(M) \\ \theta &\mapsto \nabla_\theta, \end{aligned}$$

satisfying the relations $\nabla_{f\theta}(m) = f\nabla_\theta(m)$, $\nabla_\theta(fm) = \theta(f)m + f\nabla_\theta(m)$ and $\nabla_{[\theta_1,\theta_2]}(m) = [\nabla_{\theta_1}, \nabla_{\theta_2}](m)$ for all $f \in \mathscr{O}_X$, $m \in M$ and $\theta, \theta_1, \theta_2 \in \Theta_X$. The left $\mathcal{D}_X$-module structure and the morphism ∇ are related by $\theta \cdot m = \nabla_\theta(m)$.

Proof. Writing $\nabla_\theta(m)$ as $\theta \cdot m$, the relations above become the Leibniz rule and the linearity of the $\mathcal{D}_X$-action. □

Of course, an analogous statement for right $\mathcal{D}_X$-module structures also holds, provided that the relation between the right $\mathcal{D}_X$-module structure and the morphism ∇' becomes $m \cdot \theta = -\nabla'_\theta(m)$. This will be useful in the example below.

■ **Example 2.1 — Canonical sheaf.** The canonical sheaf $\omega_X := \Omega^n_X$, where $n = \dim X$, has a natural right $\mathcal{D}_X$-module structure via the Lie derivative. An element $\theta \in \Theta_X$ acts on

[3] The generators of this algebra are understood to satisfy $[x_i, x_j] = 0$, $[\partial_i, \partial_j] = 0$, and $[\partial_i, x_j] = \delta_{ij}$.

$\omega \in \omega_X$ as $-(\operatorname{Lie}\theta)\omega$, where

$$((\operatorname{Lie}\theta)\omega)(\theta_1,\ldots,\theta_n) := \theta(\omega(\theta_1,\ldots,\theta_n)) - \sum_{i=1}^{n} \omega(\theta_1,\ldots,[\theta,\theta_i],\ldots,\theta_n).$$

A quick calculation shows that the Lie derivative satisfies the relations of the preceding proposition, proving that this indeed gives a right $\mathcal{D}_X$-module structure on ω_X. ■

Using this example, we obtain a way to turn left $\mathcal{D}_X$-modules into right $\mathcal{D}_X$-modules, and vice versa.

Corollary 2.2 Let X be a smooth scheme over an algebraically closed field k of characteristic zero. Denote by $\mathcal{D}_X^{\mathrm{op}}$-**Mod** the category of *right* $\mathcal{D}_X$-modules. Then the functors

$$\begin{aligned} \mathcal{D}_X\text{-}\mathbf{Mod} &\to \mathcal{D}_X^{\mathrm{op}}\text{-}\mathbf{Mod} \\ M &\mapsto \omega_X \otimes_{\mathcal{O}_X} M \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{D}_X^{\mathrm{op}}\text{-}\mathbf{Mod} &\to \mathcal{D}_X\text{-}\mathbf{Mod} \\ N &\mapsto \omega_X^\vee \otimes_{\mathcal{O}_X} N \end{aligned}$$

are quasi-inverse and define an equivalence of categories.

Proof. Let M be a left $\mathcal{D}_X$-module and N be a right $\mathcal{D}_X$ module. We begin by making explicit the actions of $\theta \in \Theta_X$ on $\omega_X \otimes_{\mathcal{O}_X} M$ and $\omega_X^\vee \otimes_{\mathcal{O}_X} N \cong \underline{\operatorname{Hom}}_{\mathcal{O}_X}(\omega_X, N)$:

$$\begin{aligned} (\omega \otimes m)\cdot\theta &= (\operatorname{Lie}\theta)\omega \otimes m - \omega \otimes (\theta\cdot m) \\ (\theta\cdot\varphi)(\omega) &= \varphi(\omega)\cdot\theta + \varphi((\operatorname{Lie}\theta)\omega), \end{aligned}$$

for $\omega \in \omega_X$, $m \in M$, and $\varphi \in \underline{\operatorname{Hom}}_{\mathcal{O}_X}(\omega_X, N)$. The equivalence of categories then follows from the associativity of the tensor product and the fact that $\omega_X^\vee \otimes_{\mathcal{O}_X} \omega_X \cong \mathcal{O}_X$. □

2.2 HOLONOMIC $\mathcal{D}$-MODULES

Given a left $\mathcal{D}_X$-module M, the natural candidate for its dual is $\underline{\operatorname{Hom}}_{\mathcal{D}_X}(M, \mathcal{D}_X)$. But this is a *right* $\mathcal{D}_X$-module, which forces us to consider $\underline{\operatorname{Hom}}_{\mathcal{D}_X}(M, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega_X^\vee$. This motivates the definition below.

Definition 2.2 — Duality functor. Let X be a smooth scheme over k. The *duality functor* $\mathbb{D}_X : \mathbf{D}^b(\mathcal{D}_X) \to \mathbf{D}^b(\mathcal{D}_X)^{\mathrm{op}}$ is defined as $\mathbb{D}_X M^\bullet := \left(\mathbf{R}\underline{\operatorname{Hom}}_{\mathcal{D}_X}(M^\bullet, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega_X^\vee\right)[\dim X]$.

As it is to be expected with any duality, it doesn't behave well without some finiteness condition on the allowed objects. While coherence suffices for most purposes, we need something stronger in order to have the full six functor formalism. The right notion is that of *holonomicity*.

Definition 2.3 — Holonomic modules. Let X be a smooth scheme over k and M be a coherent $\mathcal{D}_X$-module. We say that M is *holonomic* if its dual is concentrated in degree zero. We denote by $\mathcal{D}_X$-**hMod** the category of holonomic $\mathcal{D}_X$-modules and by $\mathbf{D}^b_h(\mathcal{D}_X)$ the full subcategory of $\mathbf{D}^b(\mathcal{D}_X)$ consisting of the complexes whose cohomology is holonomic. It is a result of Beilinson that this coincides with the bounded derived category of holonomic $\mathcal{D}_X$-modules.

This is *not* the usual definition of holonomicity but it will allow us to get away with avoiding much of the foundational theory of $\mathcal{D}$-modules. It is this definition which explains the shift on the duality functor. The reader may find the usual definition, with a proof that it is equivalent to this one, in the section 2.6 of [11].

It is clear from the definition that the duality functor $\mathbb{D}_X$ sends elements of $\mathbf{D}^b_h(\mathcal{D}_X)$ to elements of $\mathbf{D}^b_h(\mathcal{D}_X)^{\mathrm{op}}$. It defines moreover an equivalence of categories.

Proposition 2.3 Let X be a smooth scheme over k. Then $\mathbb{D}_X^2 \cong \mathrm{Id}$ on $\mathbf{D}^b_h(\mathcal{D}_X)$.[a]

[a] This result actually holds, with the same proof, for complexes with coherent cohomology, but this is the setting in which we'll work.

Proof. Given $M^\bullet \in \mathbf{D}^b_h(\mathcal{D}_X)$, we'll construct a morphism $M^\bullet \to \mathbb{D}_X^2 M^\bullet$. First of all, remark that

$$\begin{aligned}
\mathbb{D}_X^2 M^\bullet &= \left(\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathbb{D}_X M^\bullet, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega_X^\vee\right)[\dim X] \\
&= \left(\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}\left(\left(\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(M^\bullet, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega_X^\vee\right)[\dim X], \mathcal{D}_X\right) \otimes_{\mathcal{O}_X} \omega_X^\vee\right)[\dim X] \\
&\cong \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(M^\bullet, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega_X^\vee, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega_X^\vee.
\end{aligned}$$

We may suppose that we're dealing with locally projective finitely generated modules and so this is also isomorphic to

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X^{\mathrm{op}}}(\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(M^\bullet, \mathcal{D}_X), \mathcal{D}_X).$$

Under the tensor-hom adjunction, the evaluation morphism

$$M^\bullet \otimes_k \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(M^\bullet, \mathcal{D}_X) \to \mathcal{D}_X$$

gives rise to our desired

$$M^\bullet \to \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X^{\mathrm{op}}}(\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(M^\bullet, \mathcal{D}_X), \mathcal{D}_X) \cong \mathbb{D}_X^2 M^\bullet.$$

This is an isomorphism since we can argue affine-locally, where $M^\bullet$ may be replaced by a locally free finitely generated $\mathcal{D}_X$-module. In this setting the conclusion is clear. □

2.3 SIX FUNCTOR FORMALISM

• PROPER INVERSE IMAGE

Let $f : X \to Y$ be a morphism between smooth schemes over k. Our first mission in this chapter is to define the inverse image of a left $\mathcal{D}_Y$-module N. As an $\mathscr{O}_X$-module, it'll be defined as

$$f^\dagger N := \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}N.$$

We then endow $f^\dagger N$ of a left $\mathcal{D}_X$-module structure in the following way. Writing the dual of $\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\Omega^1_Y \to \Omega^1_X$ as

$$\begin{aligned} \Theta_X &\to \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\Theta_Y \\ \theta &\mapsto \tilde{\theta}, \end{aligned}$$

the left $\mathcal{D}_X$-module structure on $f^\dagger N$ is given by

$$\theta \cdot (f \otimes m) := \theta(f) \otimes m + f\tilde{\theta}(m)$$

for $\theta \in \Theta_X$, $f \in \mathscr{O}_X$ and $m \in f^{-1}N$. In particular, the $\mathscr{O}_X$-module

$$f^\dagger \mathcal{D}_Y = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathcal{D}_Y$$

is naturally a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule. This bimodule, hereafter denoted $\mathcal{D}_{X\to Y}$, allows for a cleaner definition of the inverse image.

Definition 2.4 — Inverse image. Let $f : X \to Y$ be a morphism between smooth schemes over k and N be a left $\mathcal{D}_Y$-module. Its *inverse image* $f^\dagger N$ is defined to be $\mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}N$.

We remark that this coincides with our previous definition since

$$\begin{aligned} \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}N &= (\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathcal{D}_Y) \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}N \\ &= \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}N. \end{aligned}$$

Nevertheless, it now becomes clear that $f^\dagger$ is a right exact functor from $\mathcal{D}_Y$-**Mod** to $\mathcal{D}_X$-**Mod**. As usual, we denote its left derived functor

$$\begin{aligned} \mathbf{D}^b(\mathcal{D}_Y) &\to \mathbf{D}^b(\mathcal{D}_X) \\ N^\bullet &\mapsto \mathcal{D}_{X\to Y} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{D}_Y} f^{-1}N^\bullet \end{aligned}$$

as $\mathbf{L}f^\dagger$. (Recall that f^{-1} is an exact functor over the categories of sheaves.) While $\mathbf{L}f^\dagger$ *isn't* one of the six functors, one simple variation is.

Definition 2.5 — Proper inverse image. Let $f : X \to Y$ be a morphism between smooth schemes over k. The *proper inverse image functor* $f^! : \mathbf{D}^b(\mathcal{D}_Y) \to \mathbf{D}^b(\mathcal{D}_X)$ is defined as $\mathbf{L}f^\dagger[\dim X - \dim Y]$.

An important fact, which we won't prove, is that the proper direct image functor preserves the holonomic categories. That is, it sends $\mathbf{D}^b_h(\mathcal{D}_Y)$ into $\mathbf{D}^b_h(\mathcal{D}_X)$. The reader may find a proof of this on the section 3.2 of [11].

• DIRECT IMAGE

Now we will reap the rewards for having proven an equivalence of category between left and right $\mathcal{D}_X$-modules. Since we have our $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule $\mathcal{D}_{X\to Y}$, given a *right* $\mathcal{D}_X$-module M, we obtain a right $f^{-1}\mathcal{D}_Y$-module $M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$ and then a right $\mathcal{D}_Y$-module

$$f_*(M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}).$$

But we can use our equivalence of categories to make this procedure work for *left* $\mathcal{D}_X$-modules! Indeed, we define the direct image $f_\dagger$ for left $\mathcal{D}_X$-modules to be the composition

$$\mathcal{D}_X\text{-}\mathbf{Mod} \longrightarrow \mathcal{D}_X^{\mathrm{op}}\text{-}\mathbf{Mod} \longrightarrow \mathcal{D}_Y^{\mathrm{op}}\text{-}\mathbf{Mod} \longrightarrow \mathcal{D}_Y\text{-}\mathbf{Mod},$$

where the middle arrow is the procedure described above. Precisely, $f_\dagger M$, for a left $\mathcal{D}_X$-module M, is given by

$$\omega_Y^\vee \otimes_{\mathcal{O}_Y} f_*((\omega_X \otimes_{\mathcal{O}_X} M) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}).$$

As with the inverse image, we simplify this expression by observing that

$$(\omega_X \otimes_{\mathcal{O}_X} M) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \cong (\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y}) \otimes_{\mathcal{D}_X} M$$

as right $f^{-1}\mathcal{D}_Y$-modules, where the action is now on the middle factor. This implies

$$\begin{aligned}\omega_Y^\vee \otimes_{\mathcal{O}_Y} f_*((\omega_X \otimes_{\mathcal{O}_X} M) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) &= \omega_Y^\vee \otimes_{\mathcal{O}_Y} f_*((\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y}) \otimes_{\mathcal{D}_X} M) \\ &= f_*((\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^\vee) \otimes_{\mathcal{D}_X} M).\end{aligned}$$

We denote the $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bimodule $\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^\vee$ as $\mathcal{D}_{Y\leftarrow X}$, thus obtaining a cleaner definition of the direct image.

Definition 2.6 — Direct image. Let $f : X \to Y$ be a morphism between smooth schemes over k and M be a left $\mathcal{D}_X$-module. Its *direct image* $f_\dagger M$ is defined to be $f_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} M)$.

We observe that $f_\dagger$ is a composition of a right exact functor $\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} -$ with a left exact functor f_*, and so is neither left nor right exact. As usual, we solve this problem by working in the bounded derived category.

Definition 2.7 — Direct image. Let $f : X \to Y$ be a morphism between smooth schemes over k. The *(derived) direct image functor* $f_* : \mathbf{D}^b(\mathcal{D}_X) \to \mathbf{D}^b(\mathcal{D}_Y)$ is defined as $f_* M^\bullet := \mathbf{R} f_*(D_{Y\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} M^\bullet)$.

We denote the (derived) direct image functor with the same symbol as the direct image of sheaves, even though we used a different symbol for the direct image of $\mathcal{D}$-modules. Hopefully the context will leave no possible doubt.

Once again, the derived direct image preserves the holonomic categories and a proof of this result may be found on the section 3.2 of [11]. When f is an affine morphism (in particular, when X is affine), this functor is particularly well behaved.

Proposition 2.4 Let $f : X \to Y$ be an affine morphism between smooth schemes over k. Then f_* is right t-exact with respect to the usual t-structure on the derived category. That is, it sends $\mathbf{D}^b_{qc}(\mathcal{D}_X)^{\leq 0}$ to $\mathbf{D}^b_{qc}(\mathcal{D}_X)^{\leq 0}$.

Proof. Since the statement is local on Y, we may suppose that $Y = \operatorname{Spec} B$ and so $X = \operatorname{Spec} A$. Moreover, the modules involved are quasi-coherent and so we may work with their global sections. In this context, f_* becomes the functor which sends a complex $M^\bullet$ in $\mathbf{D}^b(\mathcal{D}_A)$ to

$$\Gamma(X, D_{Y\leftarrow X}) \otimes^{\mathbf{L}}_{\mathcal{D}_A} M^\bullet \in \mathbf{D}^b(\mathcal{D}_B),$$

where $\mathcal{D}_A := \Gamma(X, \mathcal{D}_X)$ and $\mathcal{D}_B := \Gamma(Y, \mathcal{D}_Y)$. This is the left derived functor of a right exact functor and so is naturally right t-exact. □

Finally, we can relate our two operations via the following theorem which will be stated without proof.

Theorem 2.5 — Base change theorem. Let $f : X \to S$ and $g : Y \to S$ be two morphisms of smooth schemes over k and consider the fiber product

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\tilde{f}} & Y \\ \downarrow{\scriptstyle \tilde{g}} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S. \end{array}$$

If $X \times_S Y$ is also smooth, then there exists an isomorphism

$$g^! \circ f_* \cong \tilde{f}_* \circ \tilde{g}^! : \mathbf{D}^b_{\mathrm{qc}}(\mathcal{D}_X) \to \mathbf{D}^b_{\mathrm{qc}}(\mathcal{D}_Y)$$

of functors.

The interested reader may find this result, with a proof, in the section 1.7 of [11].

• TENSOR AND HOM

Given two $\mathcal{D}_X$-modules M and N, their tensor product $M \otimes_{\mathcal{O}_X} N$ has a natural $\mathcal{D}_X$-module structure (via the Leibniz rule) and so this defines a bifunctor

$$\mathcal{D}_X\text{-}\mathbf{Mod} \times \mathcal{D}_X\text{-}\mathbf{Mod} \to \mathcal{D}_X\text{-}\mathbf{Mod}.$$

Moreover, this bifunctor is right exact with respect to each of its factors and so gives rise to a derived counterpart

$$- \otimes^{\mathbf{L}}_{\mathcal{O}_X} - : \mathbf{D}^b(\mathcal{D}_X) \times \mathbf{D}^b(\mathcal{D}_X) \to \mathbf{D}^b(\mathcal{D}_X).$$

Similarly, if X, Y are two smooth schemes over k and $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$ are the natural projections, we may define the external tensor product $M \boxtimes N$ of a $\mathcal{D}_X$-module M and a $\mathcal{D}_Y$-module N as

$$M \boxtimes N := p_1^\dagger M \otimes_{\mathcal{O}_{X \times X}} p_2^\dagger N.$$

As it is already the case in the underlying $\mathcal{O}$-modules, the bifunctor $- \boxtimes -$ is exact with respect to both factors and so descends to the derived categories

$$- \boxtimes - : \mathbf{D}^b(\mathcal{D}_X) \times \mathbf{D}^b(\mathcal{D}_Y) \to \mathbf{D}^b(\mathcal{D}_{X \times Y}).$$

We remark that both the usual tensor product and the exterior tensor product functors preserve the categories of holonomic complexes.

The proposition below gives a useful relation between those two functors.

Proposition 2.6 Let X be a smooth scheme over k and let $\Delta_X : X \to X \times X$ be the diagonal morphism. Then there is an isomorphism

$$M^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} N^\bullet \cong \mathbf{L}\Delta_X^\dagger (M^\bullet \boxtimes N^\bullet),$$

which is natural in $M^\bullet, N^\bullet \in \mathbf{D}^b(\mathcal{D}_X)$.

Proof. Since the exterior tensor product preserves flatness, it suffices to prove this statement on the level of modules. Then,

$$\begin{aligned}
\Delta_X^\dagger(M \boxtimes N) &= \Delta_X^\dagger(p_1^\dagger M \otimes_{\mathcal{O}_{X \times X}} p_2^\dagger N) \\
&\cong \Delta_X^\dagger(p_1^\dagger M) \otimes_{\mathcal{O}_X} \Delta_X^\dagger(p_2^\dagger N) \\
&= (p_1 \circ \Delta_X)^\dagger M \otimes_{\mathcal{O}_X} (p_2 \circ \Delta_X)^\dagger N \\
&= M \otimes_{\mathcal{O}_X} N,
\end{aligned}$$

which concludes the proof. □

The preceding proposition suggests a possible variant of the tensor product functor. Indeed, given two complexes of $\mathcal{D}_X$-modules in $\mathbf{D}^b(\mathcal{D}_X)$, we may consider the *proper tensor product* $M^\bullet \otimes^!_{\mathcal{O}_X} N^\bullet$, defined as $\Delta_X^!(M^\bullet \boxtimes N^\bullet)$. This will be useful in the definition below.

Definition 2.8 — Internal Hom. Let $M^\bullet$ and $N^\bullet$ be complexes in $\mathbf{D}^b_h(\mathcal{D}_X)$. We define their *internal hom* $\mathrm{Hom}(M^\bullet, N^\bullet)$ as $\mathbb{D}_X M^\bullet \otimes^!_{\mathscr{O}_X} N^\bullet$.

Observe that this has a natural structure of left $\mathcal{D}_X$-module. I *think* that this should be the right adjoint of the derived tensor product $- \otimes^{\mathbf{L}}_{\mathscr{O}_X} - : \mathbf{D}^b_h(\mathcal{D}_X) \times \mathbf{D}^b_h(\mathcal{D}_X) \to \mathbf{D}^b_h(\mathcal{D}_X)$ but I couldn't prove it nor I found anything on the literature.

• Inverse image and proper direct image

In this section we define two new functors and we study some of their properties.

Definition 2.9 — Inverse image and proper direct image. Let $f : X \to Y$ be a morphism between smooth schemes over k. We define the *(derived) inverse image functor* $f^* : \mathbf{D}^b_h(\mathcal{D}_Y) \to \mathbf{D}^b_h(\mathcal{D}_X)$ to be $\mathbb{D}_X \circ f^! \circ \mathbb{D}_Y$ and the *proper direct image functor* $f_! : \mathbf{D}^b_h(\mathcal{D}_X) \to \mathbf{D}^b_h(\mathcal{D}_Y)$ to be $\mathbb{D}_Y \circ f_* \circ \mathbb{D}_X$.

We remark that the definitions above does indeed make sense since, even though $f^!$ and f_* need not send coherent modules to coherent modules, they do preserve holonomicity. Another simple observation is the fact that

$$\mathbb{D}_X \circ f^* = f^! \circ \mathbb{D}_Y \qquad \text{and} \qquad \mathbb{D}_Y \circ f_! = f_* \circ \mathbb{D}_X,$$

which follow from the proposition 2.3. The *raison d'être* of these functors is the result below.

Theorem 2.7 Let $f : X \to Y$ be a morphism between smooth schemes over k. Then we have isomorphisms

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_Y}(M^\bullet, f_* N^\bullet) \cong \mathbf{R}f_*\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(f^* M^\bullet, N^\bullet)$$

and

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_Y}(f_! M^\bullet, N^\bullet) \cong \mathbf{R}f_*\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(M^\bullet, f^! N^\bullet),$$

which are natural on $M^\bullet \in \mathbf{D}^b_h(\mathcal{D}_Y)$ and $N^\bullet \in \mathbf{D}^b_h(\mathcal{D}_X)$.

Proof. The proposition 2.3 implies that

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(M^\bullet, f^! N^\bullet) \cong (\omega_X \otimes^{\mathbf{L}}_{\mathscr{O}_X} \mathbb{D}_X M^\bullet) \otimes^{\mathbf{L}}_{\mathcal{D}_X} f^! N^\bullet[-\dim X].$$

Writing out the expression defining the proper inverse image, we have the isomorphisms

$$\begin{aligned}
\mathbf{R}f_*\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(M^\bullet, f^! N^\bullet) &\cong \mathbf{R}f_*\left((\omega_X \otimes^{\mathbf{L}}_{\mathscr{O}_X} \mathbb{D}_X M^\bullet) \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{D}_Y} f^{-1}N^\bullet\right)[-\dim Y] \\
&\cong \mathbf{R}f_*\left((\omega_X \otimes^{\mathbf{L}}_{\mathscr{O}_X} \mathbb{D}_X M^\bullet) \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{D}_{X\to Y}\right) \otimes^{\mathbf{L}}_{\mathcal{D}_Y} N^\bullet[-\dim Y] \\
&\cong (\omega_Y \otimes^{\mathbf{L}}_{\mathscr{O}_Y} f_*\mathbb{D}_X M^\bullet) \otimes^{\mathbf{L}}_{\mathcal{D}_Y} N^\bullet[-\dim Y] \\
&\cong (\omega_Y \otimes^{\mathbf{L}}_{\mathscr{O}_Y} \mathbb{D}_Y f_! M^\bullet) \otimes^{\mathbf{L}}_{\mathcal{D}_Y} N^\bullet[-\dim Y].
\end{aligned}$$

Finally, the same proposition 2.3 shows that this is nothing but $\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_Y}(f_! M^\bullet, N^\bullet)$. The other isomorphism follows by duality. $\square$

The *local* counterpart of this result is the classical adjunction of the six functor formalism.

Corollary 2.8 Let $f : X \to Y$ be a morphism between smooth schemes over k. Then f^* is the left adjoint of f_* and $f_!$ is the left adjoint of $f^!$.

Proof. By applying the functor $H^0(\mathbf{R}\Gamma(Y, -))$ to the isomorphisms of the preceding theorem, we obtain isomorphisms

$$\mathrm{Hom}_{\mathbf{D}^b_h(\mathcal{D}_Y)}(M^\bullet, f_* N^\bullet) \cong \mathrm{Hom}_{\mathbf{D}^b_h(\mathcal{D}_X)}(f^* M^\bullet, N^\bullet)$$

and

$$\mathrm{Hom}_{\mathbf{D}^b_h(\mathcal{D}_Y)}(f_! M^\bullet, N^\bullet) \cong \mathrm{Hom}_{\mathbf{D}^b_h(\mathcal{D}_X)}(M^\bullet, f^! N^\bullet),$$

which are natural on the complexes $M^\bullet \in \mathbf{D}^b_h(\mathcal{D}_Y)$ and $N^\bullet \in \mathbf{D}^b_h(\mathcal{D}_X)$. This gives the desired adjunctions. $\square$

The result below yields of counterpart of the proposition 2.4 for the proper direct image.

Proposition 2.9 Let $f : X \to Y$ be an affine morphism between smooth schemes over k. Then $f_!$ is left t-exact with respect to the usual t-structure on the derived category. That is, it sends $\mathbf{D}^b_h(\mathcal{D}_X)^{\geq 0}$ to $\mathbf{D}^b_h(\mathcal{D}_X)^{\geq 0}$.

Proof. A complex in $\mathbf{D}^b_h(\mathcal{D}_X)^{\geq 0}$ is sent to $\mathbf{D}^b_h(\mathcal{D}_X)^{\leq 0}$ via $\mathbb{D}_X$, then to $\mathbf{D}^b_h(\mathcal{D}_Y)^{\leq 0}$ via f_* and, finally, to $\mathbf{D}^b_h(\mathcal{D}_Y)^{\geq 0}$ via $\mathbb{D}_Y$. The final result is nothing but the image of $f_!$. $\square$

Finally, we relate the two notions of direct image via a natural morphism.

Theorem 2.10 Let $f : X \to Y$ be a (separated and finite-type) morphism between (quasi-compact and quasi-separated) smooth schemes over k. Then there is a morphism of functors

$$\mathbf{D}^b_h(\mathcal{D}_X) \underset{f_*}{\overset{f_!}{\Downarrow}} \mathbf{D}^b_h(\mathcal{D}_Y),$$

which is an isomorphism whenever f is proper.

Following Katz, we call $f_! \to f_*$ the *forget supports* map. For the proof of this theorem, we'll assume the following result: if $f : X \to Y$ is a proper morphism, then we have a natural isomorphism of functors $f_* \circ \mathbb{D}_X \to \mathbb{D}_Y \circ f_*$. This is the theorem 2.7.2 on [11], and its proof needs a lot of stuff that we didn't had the time (nor space) to cover here.

Proof. Due to Nagata's compactification theorem (as in [1]), we can factor f into an open immersion followed by a proper map:

$$X \overset{f}{\underset{i}{\hookrightarrow}} \bar{X} \xrightarrow[p]{} Y, \qquad f = p \circ i.$$

The result cited above then gives an isomorphism

$$p_! = \mathbb{D}_Y \circ p_* \circ \mathbb{D}_{\bar{X}} \xrightarrow{\sim} \mathbb{D}_Y^2 \circ p_* \cong p_*.$$

Now, if $M^\bullet \in \mathbf{D}_h^b(\mathcal{D}_X)$, the adjunction in corollary 2.8 implies that

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}_h^b(\mathcal{D}_{\bar{X}})}(i_! M^\bullet, i_* M^\bullet) &\cong \mathrm{Hom}_{\mathbf{D}_h^b(\mathcal{D}_X)}(M^\bullet, i^! i_* M^\bullet) \\ &\cong \mathrm{Hom}_{\mathbf{D}_h^b(\mathcal{D}_X)}(M^\bullet, M^\bullet), \end{aligned}$$

from which we define a morphism $i_! M^\bullet \to i_* M^\bullet$ corresponding to the identity map in $\mathrm{Hom}_{\mathbf{D}_h^b(\mathcal{D}_X)}(M^\bullet, M^\bullet)$. □

3
THE TENSOR STRUCTURE

In this chapter, we'll construct a natural convolution on a category of holonomic $\mathcal{D}$-modules over $\mathbb{G}_m$. In everything that follows, $\mathbb{G}_m$ is $\operatorname{Spec} k[x, x^{-1}]$ for an algebraically closed field of characteristic zero k, $\mathcal{D}$ denotes the sheaf of differential operators of $\mathbb{G}_m$ (or its global sections), $m : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ is the multiplication map, and $\operatorname{inv} : \mathbb{G}_m \to \mathbb{G}_m$ is the inverse map.

3.1 THE MELLIN TRANSFORM

As we commented in the previous chapter, a quasi-coherent $\mathcal{D}$-module over the multiplicative group $\mathbb{G}_m$ is nothing but $\widetilde{M}$, where M is a $k[x, x^{-1}, \partial]$-module.[4] A surprisingly useful point of view consists of viewing this as the k-algebra $k[x, x^{-1}, \theta]$, where $\theta := x\partial$.

Inspired by the Fourier transform on ℓ-adic sheaves over the affine line, which allowed G. Laumon to simplify Deligne's proof of the Weil conjectures, we define a multiplicative analog thereof. As a first step, we consider the (iso)morphism of k-algebras

$$\begin{aligned} k[x, x^{-1}, \theta] &\to k[s, \tau, \tau^{-1}] \\ x &\mapsto \tau \\ \theta &\mapsto -s, \end{aligned}$$

where $k[s, \tau, \tau^{-1}]$ is the quotient of $k\langle s, \tau, \eta \rangle$ by the relations $\tau\eta = \eta\tau = 1$ and $\tau s = (s+1)\tau$. This gives $k[s, \tau, \tau^{-1}]$ the structure of a $k[x, x^{-1}, \theta]$-algebra and allows the definition below.

Definition 3.1 — Mellin transform. Let M be a $k[x, x^{-1}, \theta]$-module. We define its *Mellin transform* to be

$$\mathcal{M} := k[s, \tau, \tau^{-1}] \otimes_{k[x, x^{-1}, \theta]} M.$$

We also consider the variant $\mathcal{M}(s) := k(s) \otimes_{k[s]} \mathcal{M}$.

Of course, $\mathcal{M}$ is nothing but M but seen as a $k[s, \tau, \tau^{-1}]$-module. The importance of the variant $\mathcal{M}(s)$ lies within its dimension. A first result about this dimension is the fact that it is finite whenever M is holonomic. Indeed, we can show that (as in the lemme 1.2.2 of [19])

$$\mathcal{M}(s) = p_*(\mathcal{M} \otimes_k k(s)x^s),$$

where $p : \operatorname{Spec} k(s)[x, x^{-1}] \to \operatorname{Spec} k(s)$ is the projection, and so the result follows from the fact that p_* preserves holonomicity.

[4] We recall that this is the quotient of $k\langle x, y, \partial \rangle$ by the relations $xy = yx = 1$ and $\partial x - x\partial = 1$.

For a more interesting relation, we have to define some concepts. If M is a holonomic $\mathcal{D}_X$-module over any smooth scheme X, its *Euler characteristic* $\chi(X, M)$ is defined as the Euler characteristic of the *de Rham complex*

$$\mathrm{DR}(M) := \Omega_X^\bullet \otimes_{\mathcal{O}_X} M[\dim X].$$

In our context, where $X = \mathbb{G}_m$ and M is a $k[x, x^{-1}, \theta]$-module, this complex has a particularly simple expression:

$$0 \longrightarrow M \xrightarrow{\theta} \underbrace{M}_{\text{degree } 0} \longrightarrow 0$$

and so $\chi(\mathbb{G}_m, M)$ is simply $\dim \operatorname{coker} \theta - \dim \ker \theta$. Surprisingly, this number coincides with the dimension of $\mathcal{M}(s)$.

Proposition 3.1 Let M be a holonomic $\mathcal{D}$-module. Then $\chi(\mathbb{G}_m, M) = \dim_{k(s)} \mathcal{M}(s)$. In particular, $\chi(\mathbb{G}_m, M) \geq 0$. Moreover, $\chi(\mathbb{G}_m, M) = 0$ if and only if M is a successive extension of the *Kummer sheaves* $\mathcal{D}/\mathcal{D}(\theta - a)$ for $a \in k$.

Proof. Passing to the Mellin transform, we observe that $\chi(\mathbb{G}_m, M)$ is also the Euler characteristic of the complex

$$\mathbf{L}i^\dagger \mathcal{M} : 0 \longrightarrow \mathcal{M} \xrightarrow{s} \underbrace{\mathcal{M}}_{\text{degree } 0} \longrightarrow 0,$$

where $i : \{0\} \to \operatorname{Spec} k[s]$ is the natural inclusion. Now, if $\mathcal{M}$ *were* a finitely generated $k[s]$-module, we would have $\chi(\mathbf{L}i^\dagger \mathcal{M}) = \dim_{k(s)} \mathcal{M}(s)$ due to the following lemma, which will be proved after we finish this proof.

Lemma 3.2 Let $\mathcal{M}$ be a finitely generated $k[s]$-module. Then,

$$\dim_k \operatorname{coker} s - \dim_k \ker s = \dim_{k(s)} k(s) \otimes_{k[s]} \mathcal{M},$$

where s is the multiplication by s map.

While $\mathcal{M}$ is *not* a finitely generated $k[s]$-module in general, we may hope to find a finitely generated $k[s]$-submodule $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N}(s) = \mathcal{M}(s)$ and $\chi(\mathbf{L}i^\dagger \mathcal{N}) = \chi(\mathbf{L}i^\dagger \mathcal{M})$. This is possible and will finish our proof.

Let $\mathcal{N}_0$ be any finitely generated $k[s]$-submodule of $\mathcal{M}$ such that $\mathcal{N}_0(s) = \mathcal{M}(s)$[5] and define inductively

$$\mathcal{N}_{k+1} := \tau^{-(k+1)} \mathcal{N}_0 + \mathcal{N}_k + \tau^{k+1} \mathcal{N}_0.$$

We remark a couple of simple facts about those finitely generated $k[s]$-modules.

[5] We can take $\mathcal{N}_0$ to be the free $k[s]$-module generated by any $k(s)$-base of $\mathcal{M}(s)$.

a) We have that $\mathcal{N}_k(s) = \mathcal{M}(s)$ for all k. Indeed, this follows inductively using that

$$\begin{aligned}\mathcal{N}_{k+1}(s) &= \tau^{-(k+1)}\mathcal{N}_0(s) + \mathcal{N}_k(s) + \tau^{k+1}\mathcal{N}_0(s)\\ &= \tau^{-(k+1)}\mathcal{M}(s) + \mathcal{M}(s) + \tau^{k+1}\mathcal{M}(s) = \mathcal{M}(s).\end{aligned}$$

b) There exists a polynomial $b(s) \in k[s]$ such that $b(s)\mathcal{N}_1/\mathcal{N}_0 = \{0\}$. Indeed, if $\mathcal{N}_1/\mathcal{N}_0$ is generated by $m_1, \dots, m_r$, we have that $m_i/1 \in \mathcal{M}(s) = \mathcal{N}_0(s)$ and so $m_i/1$ is a $k(s)$-linear combination of elements of $\mathcal{N}_0$. Clearing denominators, we obtain a polynomial $b_i(s)$ such that $b_i(s)m_i \in \mathcal{N}_0$. The product of all the $b_i(s)$ is our desired $b(s)$.

c) For all $k > 0$, $b(s-k)b(s+k)\mathcal{N}_{k+1}/\mathcal{N}_k = \{0\}$. Indeed, by the previous item $b(s)\tau n_1 + b(s)\tau^{-1}n \in \mathcal{N}_0$ for all $n, n' \in \mathcal{N}_0$. Then,

$$b(s-k)b(s+k)\left(\tau^{k+1}n + \tau^{-(k+1)}n'\right) = \tau^k \underbrace{b(s-2k)b(s)\tau n}_{\in \mathcal{N}_0} + \tau^{-k}\underbrace{b(s+2k)b(s)\tau^{-1}n'}_{\in \mathcal{N}_0}$$

is in $\mathcal{N}_k$, proving the result.

d) For large enough k, the induced morphism on the quotients

$$s : \frac{\mathcal{N}_{k+1}}{\mathcal{N}_k} \to \frac{\mathcal{N}_{k+1}}{\mathcal{N}_k}$$

is an isomorphism. Indeed, for large enough k, s and $b(s-k)b(s+k)$ are coprime. Then, by Bézout's theorem, there exist $p(s)$ and $q(s)$ such that

$$sp(s) + b(s-k)b(s+k)q(s) = 1.$$

Then, if $n \in \mathcal{N}_{k+1}$, we can multiply the equation above by n on the right to obtain that

$$s\,(p(s)n) = n$$

on the quotient $\mathcal{N}_{k+1}/\mathcal{N}_k$, proving that the morphism is surjective. The same equation shows that it is injective.

Finally, for some k as in the item (d), we prove that the morphism of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{N}_k & \xrightarrow{s} & \mathcal{N}_k & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \mathcal{M} & \xrightarrow{s} & \mathcal{M} & \longrightarrow & 0 \end{array}$$

is a quasi-isomorphism. We denote by $\mathcal{N}_k^\bullet$ and $\mathcal{M}^\bullet$ the respective complexes and consider the short exact sequence given by

$$0 \longrightarrow \mathcal{N}_k^\bullet \hookrightarrow \mathcal{M}^\bullet \twoheadrightarrow \mathcal{M}^\bullet/\mathcal{N}_k^\bullet \longrightarrow 0.$$

By the induced long exact sequence in cohomology, it suffices to prove that $\mathcal{M}^\bullet/\mathcal{N}_k^\bullet$ has trivial cohomology. That is, that the morphism

$$s : \mathcal{M}/\mathcal{N}_k \to \mathcal{M}/\mathcal{N}_k$$

is an isomorphism. But this follows from the item (d), proving that $\mathcal{N}_k$ is our desired finitely generated $k[s]$-submodule of $\mathcal{M}$ satisfying $\mathcal{N}_k(s) = \mathcal{M}$ (by the item (a)) and $\chi(\mathbf{L}i^\dagger\mathcal{N}_k) = \chi(\mathbf{L}i^\dagger\mathcal{M})$. This concludes the proof of the first part. The reader can find the second part in the lemma 3.7.5 of [15]. □

The proof above is essentially a highly commented version of the one by F. Loeser and C. Sabbah in [18]. We now prove the lemma that we needed. Observe that the finiteness hypothesis is essential for the use of the structure theorem for finite modules over PIDs.

Proof of the lemma 3.2. Since $k[s]$ is a PID, the result is additive and trivially true for $\mathcal{M} = k[s]$, we may assume that $\mathcal{M} = k[s]/(p)$, where $p \in k[s]$ is a non-zero polynomial. In this case,

$$\dim_{k(s)} k(s) \otimes_{k[s]} \mathcal{M} = 0$$

and so it suffices to prove that $\dim_k \operatorname{coker} s = \dim_k \ker s$. Now, this follows from the fact that the sequence

$$0 \longrightarrow \ker s \hookrightarrow \mathcal{M} \xrightarrow{s} \mathcal{M} \twoheadrightarrow \operatorname{coker} s \longrightarrow 0$$

is exact and from the (alternate) additivity of dimensions in exact sequences of finite dimensional k-vector spaces. □

Before ending this chapter, we remark a possible different proof of the proposition 3.1. Let $\iota_k : \mathcal{N}_k \to \mathcal{M}$ be the inclusion and consider the following morphism of complexes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{N}_k & \xrightarrow{s} & \mathcal{N}_k & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota_k} & & \downarrow{\scriptstyle \iota_k} & & \\ 0 & \longrightarrow & \mathcal{M} & \xrightarrow{s} & \mathcal{M} & \longrightarrow & 0. \end{array}$$

We think that ι_k determines a quasi-isomorphism if k is large enough. Its mapping cone is given by

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{N}_k & \longrightarrow & \mathcal{M} \oplus \mathcal{N}_k & \longrightarrow & \mathcal{M} & \longrightarrow 0 \\ & n & \longmapsto & (n, -sn) & & & \\ & & & (m, n) & \longmapsto & sm + n. & \end{array}$$

The arrow on the left is clearly injective. Exactness on the middle means that

$$\text{if } sm + n = 0, \text{ then there is } \bar{n} \in \mathcal{N}_k \text{ such that } \begin{cases} m = \bar{n} \\ n = -s\bar{n} \end{cases} .$$

But $sm = -n \in \mathcal{N}_k$ implies that m itself is in $\mathcal{N}_k$ and so we can take $\bar{n} = m$, proving that the cone is also exact in the middle. If we could prove that the arrow on the right is surjective, which doesn't seem difficult, that would furnish another proof of the proposition.

3.2 CONVOLUTION OF HOLONOMIC MODULES

Definition 3.2 — Convolution. If $M^\bullet$ and $N^\bullet$ are complexes in $\mathbf{D}_h^b(\mathcal{D})$, we define their *convolution* to be

$$M^\bullet * N^\bullet := m_*(M^\bullet \boxtimes N^\bullet) \in \mathbf{D}_h^b(\mathcal{D}).$$

Similarly, their *!-convolution* $M^\bullet *_! N^\bullet$ is defined as $m_!(M^\bullet \boxtimes N^\bullet) \in \mathbf{D}_h^b(\mathcal{D})$.

We remark that both convolutions are exchanged by duality. That is, $\mathbb{D}(M^\bullet * N^\bullet) = \mathbb{D}M^\bullet *_! \mathbb{D}N^\bullet$ and $\mathbb{D}(M^\bullet *_! N^\bullet) = \mathbb{D}M^\bullet * \mathbb{D}N^\bullet$. We also comment that both notions of convolution are commutative, associative, and have an identity object given by the Dirac module $\delta_1 := \mathcal{D}/\mathcal{D}(x-1)$.

Now, we have two problems. First and foremost, it won't always be the case that the convolution of two holonomic $\mathcal{D}$-modules will result in another $\mathcal{D}$-module (that is, a complex concentrated in degree 0). Also, it is not clear which of the two notions of convolution should be used. As it turns out, both problems are related.

Proposition 3.3 Let M, N be holonomic $\mathcal{D}$-modules. If the forget supports map

$$m_*(M \boxtimes N) \to m_!(M \boxtimes N)$$

is an isomorphism, then $M * N \cong M *_! N$ is a holonomic $\mathcal{D}$-module.

Proof. Since $\mathbb{G}_m$ is affine, the multiplication map $m : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ is also affine. In this case, the proposition 2.4 implies that m_* is right t-exact and the proposition 2.9 implies that $m_!$ is left t-exact, with respect to the usual t-structures. Then, as the forget supports map is an isomorphism, it follows that $M * N \cong M *_! N$ is a complex concentrated in degree 0. □

Somewhat surprisingly, the problems we are discussing are related to the Euler characteristic $\chi(\mathbb{G}_m, M)$ of a holonomic $\mathcal{D}$-module M. We begin the exploration of this relation.

Definition 3.3 — Negligible modules. We say that a holonomic $\mathcal{D}$-module M is *negligible* if $\chi(\mathbb{G}_m, M) = 0$ and we denote the full subcategory of $\mathcal{D}$-**hMod** composed of negligible modules by **Neg**.

We recall that a non-empty full subcategory **C** of an abelian category **A** is said to be *thick* if, whenever

$$0 \to M \to N \to P \to 0$$

is an exact sequence in $\mathbf{A}$, the object N is in $\mathbf{C}$ if and only if M and P are. The motivation for defining such categories is that, in this case, we can define a *quotient category* $\mathbf{A}/\mathbf{C}$ which has the same objects as $\mathbf{A}$ and whose hom-sets are given by

$$\mathrm{Hom}_{\mathbf{A}/\mathbf{C}}(M,N) := \operatorname{colim} \mathrm{Hom}_{\mathbf{A}}(M', N/N'),$$

where the colimit runs over all subobjects M' of M and N' of N such that both M/M' and N' are in $\mathbf{C}$. It follows that $\mathbf{A}/\mathbf{C}$ is abelian, that the natural functor $Q : \mathbf{A} \to \mathbf{A}/\mathbf{C}$ is exact and that $Q(M)$ is a zero object in $\mathbf{A}/\mathbf{C}$ if and only if $M \in \mathbf{C}$.[8] As we shall see, the category of negligible objects is thick.

Proposition 3.4 The category of negligible modules **Neg** is thick.

Proof. We observe that, since the Euler characteristic is additive, it suffices to prove that $\chi(\mathbb{G}_m, M) \geq 0$ for every holonomic $\mathcal{D}$-module M. But this follows from the proposition 3.1 on the previous section. □

We then consider the quotient category $\mathcal{D}$-**hMod**/**Neg**. The main idea is that, in this category, both notions of convolution will make sense and coincide. For that, we'll need to once again make a detour to the derived category.

We denote by $\overline{\mathbf{Neg}}$ the full subcategory of $\mathbf{D}^b_h(\mathcal{D})$ composed of the complexes whose cohomology is negligible. The next two propositions were proven in the context of perverse sheaves by O. Gabber and F. Loeser in [7]. There's also a proof of the second proposition in the Séminaire Bourbaki 1141 by Javier Fresán [6].

Proposition 3.5 Let $M^\bullet$ and $N^\bullet$ be two complexes in $\mathbf{D}^b_h(\mathcal{D})$. Then $M^\bullet * N^\bullet$ is in $\overline{\mathbf{Neg}}$ if either $M^\bullet$ of $N^\bullet$ is in $\overline{\mathbf{Neg}}$. The same holds for the !-convolution.

In particular, the universal property of the localization implies that both notions of convolution descend to the quotient category $\mathcal{D}$-**hMod**/**Neg**.

Proposition 3.6 Let $M^\bullet$ and $N^\bullet$ be two complexes in $\mathbf{D}^b_h(\mathcal{D})$. Then the cone of the forget supports map

$$M^\bullet * N^\bullet \to M^\bullet *_! N^\bullet$$

is in $\overline{\mathbf{Neg}}$.

Combining the two previous propositions, we get our desired result.

Corollary 3.7 Both notions of convolution descend to the quotient category $\mathcal{D}$-**hMod**/**Neg**, where they coincide.

3.3 AN EQUIVALENCE OF CATEGORIES

Due to the fact that it is often easier to deal with subcategories than with quotients, in his book Rigid Local Systems [16], N. Katz considers the following subcategory of $\mathcal{D}$-**hMod**. We also follow his exposition in [14].

Definition 3.4 We denote by **P** the full subcategory of $\mathcal{D}$-**hMod** composed of the holonomic $\mathcal{D}$-modules M such that for every $\mathcal{D}$-module N, both convolutions $M * N$ and $M *_! N$ are in $\mathcal{D}$-**hMod**.

We remark that in the context of perverse sheaves, Katz proved in [16] that a perverse sheaf is an element of **P** if and only if it has no subobjects nor quotients which are of the form $\mathcal{L}_\chi[1]$, where $\mathcal{L}_\chi$ is the *Kummer sheaf* associated to a continuous character χ of the tame fundamental group of $\mathbb{G}_m$. We believe the analogous result to be true in the $\mathcal{D}$-module world. Namely, that a $\mathcal{D}$-module is in **P** if and only if it has no subobjects nor quotients which are of the form $\mathcal{D}/\mathcal{D}(\theta - a)$ for $a \in k$.

This category **P** is tailor-made so that we can have a working notion of convolution without needing to pass to the quotient, as in the previous section.

Definition 3.5 — Middle convolution. Let M and N be two $\mathcal{D}$-modules in **P**. We define their *middle convolution* $M *_{\text{mid}} N$ to be the image

$$M *_{\text{mid}} N := \operatorname{im}(M *_! N \to M * N)$$

in $\mathcal{D}$-**hMod** of the natural forget supports map.

A priori it's not clear whether the middle convolution of two elements of **P** is still in **P** or not. N. Katz gave an affirmative response in [16] for perverse sheaves. We believe the same result to be true for holonomic $\mathcal{D}$-modules and, for lack of time, we assume it from now on.

We observe that, due to the exactness of the exterior tensor product, the functors $- * N$ and $- *_! N$ are exact in $\mathcal{D}$-**hMod** whenever N is in **P**. This will allow us to prove the associativity of the middle convolution (the commutativity being clear). But first we need a lemma.

Lemma 3.8 Let **A** and **B** be two abelian categories and let $F, G : \mathbf{A} \to \mathbf{B}$ be functors between them, related by a natural transformation $\varphi : F \to G$. Then the image functor $\operatorname{im}\varphi : \mathbf{A} \to \mathbf{B}$, given by

$$(\operatorname{im}\varphi)_A = \operatorname{im}(\varphi_A : F(A) \to G(A)),$$

is *end-exact*. That is, it sends monomorphisms to monomorphisms and epimorphisms to epimorphisms.

Proof. The proof consists of two simple applications of the snake lemma. We begin with a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in **A**. Its image by the functors F and G define a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(A) & \longrightarrow & F(B) & \longrightarrow & F(C) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \varphi_A} & & \big\downarrow{\scriptstyle \varphi_B} & & \big\downarrow{\scriptstyle \varphi_C} & & \\
0 & \longrightarrow & G(A) & \longrightarrow & G(B) & \longrightarrow & G(C) & \longrightarrow & 0
\end{array}$$

which, by the snake lemma, gives rise to the following exact sequence:

$$\begin{array}{l}
0 \longrightarrow \ker \varphi_A \longrightarrow \ker \varphi_B \longrightarrow \ker \varphi_C \xrightarrow{\ \delta\ } \\
\quad \longrightarrow \operatorname{coker} \varphi_A \longrightarrow \operatorname{coker} \varphi_B \longrightarrow \operatorname{coker} \varphi_C \longrightarrow 0.
\end{array}$$

Then, the morphism $\ker \varphi_B \to \ker \varphi_C$ factorizes through the kernel of the connecting morphism δ, yielding the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker \varphi_A & \longrightarrow & \ker \varphi_B & \longrightarrow & \ker \delta & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & F(A) & \longrightarrow & F(B) & \longrightarrow & F(C) & \longrightarrow & 0,
\end{array}$$

whose rows are exact and whose columns are monic. Another application of the snake lemma gives the short exact sequence

$$0 \longrightarrow F(A)/\ker \varphi_A \longrightarrow F(B)/\ker \varphi_B \longrightarrow F(C)/\ker \delta \longrightarrow 0$$

and the first isomorphism theorem identifies the morphism on the left with

$$\operatorname{im} \varphi_A \to \operatorname{im} \varphi_B,$$

which is then monic. Moreover, since $F(C)/\ker \delta$ surjects into

$$F(C)/\ker \varphi_C \cong \operatorname{im} \varphi_C,$$

we can factor $\operatorname{im} \varphi_B \to \operatorname{im} \varphi_C$ into a composition

$$\operatorname{im} \varphi_B \twoheadrightarrow F(C)/\ker \delta \twoheadrightarrow F(C)/\ker \varphi_C \xrightarrow{\sim} \operatorname{im} \varphi_C$$

of three epimorphisms, which is then also an epimorphism. □

In particular, for a holonomic $\mathcal{D}$-module N in the category **P**, the functor $- *_{\mathrm{mid}} N$ is end-exact. We'll capitalize on this in the following proposition. For that, we observe that the forget supports map $M *_! N \to M * N$ factors

$$M *_! N \twoheadrightarrow M *_{\mathrm{mid}} N \hookrightarrow M * N$$

as a composition of a surjective map with an injective map.

Proposition 3.9 Let M, N, P be three holonomic $\mathcal{D}$-modules in **P**. Then the natural isomorphism

$$(M *_{\mathrm{mid}} N) *_{\mathrm{mid}} P = M *_{\mathrm{mid}} (N *_{\mathrm{mid}} P)$$

holds. That is, middle convolution on **P** is associative.

Proof. It suffices to prove that the left-hand side $(M *_{\mathrm{mid}} N) *_{\mathrm{mid}} P$ is the image of the natural map

$$M *_! N *_! P \to M * N * P.$$

We factor the forget supports map above as a composition of two surjective maps

$$(M *_! N) *_! P \twoheadrightarrow (M *_{\mathrm{mid}} N) *_! P \twoheadrightarrow (M *_{\mathrm{mid}} N) *_{\mathrm{mid}} P$$

and two injective maps

$$(M *_{\mathrm{mid}} N) *_{\mathrm{mid}} P \hookrightarrow (M *_{\mathrm{mid}} N) * P \hookrightarrow (M * N) * P.$$

This proves that $(M *_{\mathrm{mid}} N) *_{\mathrm{mid}} P$ is the image of $M *_! N *_! P$ in $M * N * P$. The same reasoning shows that $M *_{\mathrm{mid}} (N *_{\mathrm{mid}} P)$ coincides with the same image, proving that they are equal. □

We arrive at the main result of this section.

Theorem 3.10 The composition of the inclusion $\mathbf{P} \to \mathcal{D}$-**hMod** with the quotient map $\mathcal{D}$-**hMod** $\to \mathcal{D}$-**hMod**/**Neg** is an equivalence of categories. Moreover, under this equivalence, the image of the middle convolution on **P** is sent to the convolution on the quotient $\mathcal{D}$-**hMod**/**Neg**.

Before going to its proof, we observe that **P** inherits from this equivalence a structure of abelian category that is somewhat twisted. Namely, a sequence of morphisms in **P**

$$0 \longrightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} P \longrightarrow 0$$

is exact if and only if α is injective, β is surjective, $\beta \circ \alpha = 0$ and $\ker \beta / \operatorname{im} \alpha$ is negligible. With this structure of abelian category, the middle convolution becomes a biexact functor.

We now divide the proof of the theorem 3.10 in two propositions. The first of them is purely formal.

Let $\mathbf{A}$ be an abelian category and $\mathbf{C}$ a thick subcategory. Given an object A of $\mathbf{A}$, we denote by $A_{\mathbf{C}}$ the largest subobject of A in $\mathbf{C}$ and by $A^{\mathbf{C}}$ the smallest subobject B of A such that A/B is in $\mathbf{C}$. We write $[\mathbf{A} : \mathbf{C}]$ for the full subcategory of $\mathbf{A}$ whose objects A satisfy $A_{\mathbf{C}} = 0$ and $A^{\mathbf{C}} = A$.

Proposition 3.11 Let $\mathbf{A}$ be an abelian category and $\mathbf{C}$ a thick subcategory. Using the notations above, consider the functor

$$\begin{aligned} T : \mathbf{A} &\to [\mathbf{A} : \mathbf{C}] \\ A &\mapsto (A^{\mathbf{C}} \oplus A_{\mathbf{C}})/A_{\mathbf{C}}. \end{aligned}$$

This functor factors through $\mathbf{A}/\mathbf{C}$, yielding a quasi-inverse

$$S : \mathbf{A}/\mathbf{C} \to [\mathbf{A} : \mathbf{C}]$$

of the restriction of the quotient functor $\mathbf{A} \to \mathbf{A}/\mathbf{C}$ to $[\mathbf{A} : \mathbf{C}]$. In particular, these functors define an equivalence of categories between $\mathbf{A}/\mathbf{C}$ and $[\mathbf{A} : \mathbf{C}]$.

Proof. Given a morphism $\alpha : A \to B$ in $\mathbf{A}$, we remark that $\alpha(A_{\mathbf{C}})$ is a subobject of $B_{\mathbf{C}}$. Moreover, $\alpha(A^{\mathbf{C}})$ is also a subobject of $B^{\mathbf{C}}$. Indeed, the morphism

$$A/\alpha^{-1}(B^{\mathbf{C}}) \to B/B^{\mathbf{C}}$$

is monic, which implies that $A/\alpha^{-1}(B^{\mathbf{C}}) \in \mathbf{C}$ and so $A^{\mathbf{C}} \subset \alpha^{-1}(B^{\mathbf{C}})$. We conclude that T is indeed a functor.

In order to prove that T factors through the quotient functor $Q : \mathbf{A} \to \mathbf{A}/\mathbf{C}$, we have to show that if $\ker \alpha$ and $\operatorname{coker} \alpha$ are in $\mathbf{C}$, then $T(\alpha)$ is an isomorphism. Now, let $\alpha : A \to B$ be a morphism in $\mathbf{A}$ such that $\ker \alpha$ is in $\mathbf{C}$. Since $\alpha^{-1}(B_{\mathbf{C}})$ is an extension of $B_{\mathbf{C}}$ by $\ker \alpha$, it is an object of $\mathbf{C}$ containing $A_{\mathbf{C}}$. It follows that $\alpha^{-1}(B_{\mathbf{C}}) = A_{\mathbf{C}}$ and so $T(\alpha)$ is a monomorphism. Dually, if $\operatorname{coker} \alpha \in \mathbf{C}$ then $T(\alpha)$ is a monomorphism. We conclude that T factors through a functor $S : \mathbf{A}/\mathbf{C} \to [\mathbf{A} : \mathbf{C}]$:

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{T} & [\mathbf{A} : \mathbf{C}] \\ {\scriptstyle Q}\downarrow & \nearrow{\scriptstyle S} & \\ \mathbf{A}/\mathbf{C}. & & \end{array}$$

Let's denote by $I : [\mathbf{A} : \mathbf{C}] \to \mathbf{A}$ the inclusion functor. If $A \in \mathbf{A}$, the canonical morphisms $A^{\mathbf{C}} \to A$ and $A^{\mathbf{C}} \to I(T(A))$ descend to a functorial isomorphism $A \cong I(T(A))$ in $\mathbf{A}/\mathbf{C}$. This proves that $Q \circ I \circ S \cong \mathrm{id}$. Since it is clear that $S \circ Q \circ I = \mathrm{id}$, it follows that the functor $Q \circ I : [\mathbf{A} : \mathbf{C}] \to \mathbf{A}/\mathbf{C}$ defines an equivalence of categories. □

In order to conclude the proof of the theorem 3.10, it suffices then to attain the proposition below, which follows from the aforementioned characterization of **P** by Katz and the proposition 3.1.

Proposition 3.12 Let M be a holonomic $\mathcal{D}$-module. Then $M_{\mathbf{Neg}} = 0$ and $M^{\mathbf{Neg}} = M$ if and only if, for every holonomic $\mathcal{D}$-module N, both convolutions $M * N$ and $M *_! N$ are in $\mathcal{D}$-**hMod**. In other words, $\mathbf{P} = [\mathcal{D}\text{-}\mathbf{hMod} : \mathbf{Neg}]$.

Given all the work that we've done so far and having the proposition 1.1 in mind, we have to define a fibre functor $\omega : \mathbf{P} \to k\text{-}\mathbf{Vect}$ which satisfies

a) there exists an identity object U in **P** such that $k \cong \mathrm{End}(U)$ and $\dim_k \omega(U) = 1$;

b) whenever $\dim_k \omega(L) = 1$, there exists an object $L^\vee \in \mathbf{P}$ such that $L \otimes L^\vee = U$;

c) $\omega(M *_{\mathrm{mid}} N) = \omega(M) \otimes_k \omega(N)$ for every $M, N \in \mathbf{P}$,

in order to prove that **P** is a tannakian category. We affirm that the fibre functor ω defined by

$$M \mapsto H^0(\mathbb{A}^1, j_! M),$$

where $j : \mathbb{G}_m \to \mathbb{A}^1$ is the natural inclusion, along with $U = \delta_1$ and $L^\vee = \mathrm{inv}^* \mathbb{D}(L)$, satisfy our requirements, yielding the theorem below.

Theorem 3.13 The structure described above makes **P** a tannakian category.

The analog proof for perverse sheaves over $\mathbb{G}_m$ is described in detail on the appendix to Katz' book Convolution and Equidistribution [14] and so will me omitted here. The author regrets this choice and plans to convert this proof to the world of $\mathcal{D}$-modules soon.

While we're still here, we remark that the proposition 1.3 gives rise to a algebraic group G_M (a closed subgroup of $\mathrm{GL}(\omega(M))$) for every holonomic $\mathcal{D}$-module M in the category **P**. The existence of such a group allows us to understand a plethora of equidistribution results, which are well explained in the Séminaire Bourbaki 1141 [5], which was given by my advisor Javier Fresán.

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