LIE GROUPS, LIE ALGEBRAS AND REPRESENTATION THEORY

GABRIEL RIBEIRO

This work is the result of numerous meetings between me and my advisor José J. Ramón-Marí, to whom I'm grateful. Since our focus here is on Lie groups and representation theory, standard definitions and proofs about topology, algebra and analysis will be omitted. For our purposes a (differentiable) manifold is a Hausdorff, second countable, locally Euclidean topological space endowed with a maximal atlas. More information about manifolds can be found in Tu's work: An Introduction To Manifolds.[5]

1. Lie Groups

Definition 1 (Lie Group)**.** A group G is said to be a *Lie Group* if it is also a finite-dimensional manifold and the maps

$$(x, y) \mapsto xy \qquad \text{and} \qquad x \mapsto x^{-1}$$

are both C^∞.

Most of the Lie groups that will be in this work are made by matrices, however $\mathbb{R}^n$, $\mathbb{R}^\times$ and S^1 are examples of other kinds of Lie groups.

Example 1 (General Linear Group). The (real) general linear group

$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}$$

is defined as the group of all real matrices with non-null determinant.

Since it is an open subset of $M_n(\mathbb{R})$, it is a manifold.[1] As the multiplication map is polynomial in the entries of both matrices, it is C^∞. Cramer's rule implies that the inversion map, given by

$$(A^{-1})_{ij} = \frac{1}{\det A}(-1)^{i+j}((j,i) - \text{minor of } A),$$

is also C^∞. Hence $GL_n(\mathbb{R})$ is a Lie Group. The complex general linear group $GL_n(\mathbb{C})$ is defined analogously. The special linear groups $SL_n(\mathbb{R})$ and $SL_n(\mathbb{C})$ are the subgroups of $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$ of matrices with unit determinant.

Example 2 (Orthogonal and Unitary Groups). Let $A \in M_n(\mathbb{R})$ be a real matrix. The requirement that the linear transformation determined by A preserves inner products leads us to consider orthogonal matrices, since for any vectors $x, y \in \mathbb{R}^n$

$$\langle Ax, Ay\rangle = \langle A^T Ax, y\rangle = \langle x, A^T Ay\rangle.$$

[1] Note that $\{0\}$ is a closed subset of $\mathbb{R}$ with the order topology. Hence $GL_n(\mathbb{R}) = M_n(\mathbb{R}) \setminus \det^{-1}(0)$ is an open subset of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ with the product topology.

Since $A^T A = I$ implies $\det A = \pm 1$, we say that the orthogonal matrices satisfying $\det A = 1$ are *rotations* and those satisfying $\det A = -1$ are *improper isometries*. We then define the *orthogonal group* as

$$O(n) = \{A \in M_n(\mathbb{R}) \mid A^T A = I\}.$$

A (very) notable subgroup of $O(n)$ is the *special orthogonal group*, defined as

$$SO(n) = \{A \in M_n(\mathbb{R}) \mid A^T A = I \text{ and } \det A = 1\}.$$

The *unitary group* is the analogue of $O(n)$ on $\mathbb{C}^n$ defined by

$$U(n) = \{A \in M_n(\mathbb{C}) \mid A^* A = I\},$$

where A^* is the complex adjoint of A. The *special unitary group*, $SU(n)$, is the subset of $U(n)$ made of matrices of unit determinant.

We shall now prove that all the orthogonal and unitary groups are manifolds (and hence Lie groups). We give proofs only for the orthogonal groups since the proofs for the unitary groups are analogous.

Theorem 1. *The orthogonal group, $O(n)$, and the special orthogonal group, $SO(n)$, are manifolds.*

Proof. Let $\phi : M_n(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R})$, where $\mathrm{Sym}_n(\mathbb{R})$ is the space of all real $n \times n$ symmetric matrices, such that $\phi(X) = X^T X$. By the regular value theorem if $I \in \mathrm{Sym}_n(\mathbb{R})$ is a regular value of ϕ then $O(n) = \phi^{-1}(I)$ is a submanifold of $M_n(\mathbb{R})$.

To prove that I is a regular value of ϕ it suffices to show that $\mathrm{d}\phi_A$ is surjective for all $A \in O(n)$. The derivative $\mathrm{d}\phi_A$ is given by

$$\mathrm{d}\phi_A(H) = A^T H + H^T A.$$

It is clear that $\mathrm{d}\phi_A(H)$ is symmetric. Hence if $\mathrm{d}\phi_A(H) = B$, then

$$H = \frac{1}{2}(A^T)^{-1} B = \frac{1}{2} AB$$

is a particular solution.

The fact that $SO(n)$ is a manifold then follows easily by noticing that $\det : O(n) \to \{-1, 1\}$ is a continuous function and $\{1\}$ is open in $\{-1, 1\}$. □

Corollary 1.1. *The tangent space of both $O(n)$ and $SO(n)$ at the identity is $\mathfrak{so}(n)$, i.e., the space of all $n \times n$ skew-symmetric matrices. Hence, $\dim O(n) = \dim SO(n) = n(n-1)/2$.*

Proof. Follows readily from the fact that $T_I O(n) = T_I SO(n) = \ker \mathrm{d}\phi_I$. □

The following powerful theorem, which we shall not prove, was first proved in its full generality by Élie Cartan and proved for matrix Lie groups by John Von Neumann.

Theorem 2 (Closed Subgroup Theorem)**.** *Let G be a Lie group. Then the inclusion $H \subset G$ is an embedding if H is closed in G. Moreover, the converse also holds.*

Since $\mathbb{R}$ is a T_1 space, $\{1\}$ is closed and hence so is $SL_n(\mathbb{R}) = \det^{-1}(1)$ (here we consider the determinant as a function from $GL_n(\mathbb{R})$ to $\mathbb{R}$). That is, $SL_n(\mathbb{R})$ is a manifold. The result for $SL_n(\mathbb{C})$ is analogous.

2. Topological Facets of Some Lie Groups

In this section we investigate a few important topological properties which are satisfied by some Lie groups. Since Lie groups are manifolds, we shall make no distinction between path-connectedness and connectedness.

Theorem 3. *All the orthogonal and unitary groups are compact.*

Proof. Note that all these groups are bounded since as $|A_{ij}| \leq 1$ for orthogonal or unitary matrices A. Since Heine-Borel holds in $\mathbb{R}^k$ and there's an obvious homeomorphism between $M_n(\mathbb{R})$ or $M_n(\mathbb{C})$ and $\mathbb{R}^k$ it suffices to show that the groups are closed.

The orthogonal group is closed in $M_n(\mathbb{R})$ as $O(n) = \phi^{-1}(I)$ and ϕ is clearly continuous. (It is polynomial in the entries of the matrix.)

The unitary group is closed in $M_n(\mathbb{C})$ as $U(n) = \widetilde{\phi}^{-1}(I)$, where $\widetilde{\phi}(X) = X^*X$.

For $SO(n)$ and $SU(n)$ it is enough to show that they are closed subspaces of $O(n)$ and $U(n)$, respectively.[2] But that is obvious as they can be characterized as $\det^{-1}(1)$. The result follows. $\square$

Theorem 4. *$GL_n(\mathbb{C})$ is connected.*

Proof. Let A and B be two matrices in $GL_n(\mathbb{C})$. Consider the expression $\det(\lambda A + (1-\lambda)B) = 0$. The Fundamental Theorem of Algebra implies that there are only a finite number of solutions to this expression (none of which are $\lambda = 0$ or $\lambda = 1$). Since $\mathbb{C} \setminus S$ is connected for any finite set $S \subset \mathbb{C}$ there is a function $\lambda : [0,1] \to \mathbb{C}$ which avoids all the solutions of $\det(\lambda A + (1-\lambda)B) = 0$. This induces a continuous path from A to B in $GL_n(\mathbb{C})$. It follows that $GL_n(\mathbb{C})$ is path-connected and thence connected. $\square$

Theorem 5. *$SO(n)$ is connected.*

Proof. It is a well-known fact that a special orthogonal matrix A can be written as

$$A = U \begin{bmatrix} B(\theta_1) & & & \\ & \ddots & & \\ & & B(\theta_k) & \\ & & & I_p \end{bmatrix} U^{-1},$$

where $B(\theta_i)$ is the 2×2 matrix

$$\begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) \end{bmatrix},$$

U is a orthogonal matrix and p is a positive integer. All the other elements in the block-diagonal matrix are zeros. We then see that the function

$$f(t) = U \begin{bmatrix} B(t\theta_1) & & & \\ & \ddots & & \\ & & B(t\theta_k) & \\ & & & I_p \end{bmatrix} U^{-1}$$

[2]Let X be a topological space and Y be a subspace. Recall from basic topology that $H \subset Y$ is closed in Y if and only if $H = Y \cap F$ for some closed set F in X. Hence, if H is closed in Y and Y is closed in X, then H is closed in X.

is a path from $I = f(0)$ to $A = f(1)$ in $SO(n)$. Since path-connectedness is transitive, the result follows. $\square$

All the special unitary groups are simply connected. However the proof does not appear to be trivial so I'll just prove the $n = 2$ case.

Theorem 6. *$SU(2)$ is simply connected.*

Proof. A quick calculation shows that $SU(2) \cong S^3$. Thence, the result follows as a simple corollary. $\square$

Corollary 6.1. *$SL_2(\mathbb{C})$ is simply connected.*

The bulk of this proof lies in the Gram-Schmidt procedure to construct a deformation retraction from $SL_2(\mathbb{C})$ to $SU(2)$. Since our focus is not on algebraic topology, this proof will be omitted.

3. Lie Algebras

Usually it is not trivial to understand the structure endowed by a Lie group. However, analyzing vector spaces is frequently straightforward. It would be truly wonderful if we could understand Lie groups in terms of vector spaces. Fortunately, that is exactly the case.

Definition 2 (Lie Algebra)**.** A Lie algebra $\mathfrak{g}$ is a (real or complex) vector space endowed with a skew-symmetric, bilinear form $[\cdot,\cdot]$ which satisfies the Jacobi identity

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0,$$

for all $x, y, z \in \mathfrak{g}$.

Given a matrix Lie group G, we shall assign a Lie algebra $\mathfrak{g}$ to it as follows:

$$\mathfrak{g} = \{X \in M_n(\mathbb{C}) \mid e^{tX} \in G \text{ for all } t \in \mathbb{R}\}, \qquad [X,Y] = XY - YX.$$

To see that this is, in fact, a Lie algebra we need the following lemma

Lemma 1 (Lie Product Formula)**.** *Let $A, B \in M_n(\mathbb{C})$ be complex matrices. Then, the following formula holds:*

$$e^{A+B} = \lim_{m\to\infty} \left(e^{A/m}e^{B/m}\right)^m.$$

Proof. Since $\sum (A/m)^k/k!$ converges absolutely (as all norms are equivalent, this holds with every norm) to $e^{A/m}$,

$$e^{A/m}e^{B/m} = I + \frac{A}{m} + \frac{B}{m} + O\left(\frac{I}{m^2}\right).$$

As $e^{A/m}e^{B/m} \to I$, $e^{A/m}e^{B/m}$ is in the domain of the logarithm for large enough m. Hence,

$$\begin{aligned}
\log\left(e^{A/m}e^{B/m}\right) &= \log\left(I + \frac{A}{m} + \frac{B}{m} + O\left(\frac{I}{m^2}\right)\right) \\
&= \frac{A}{m} + \frac{B}{m} + O\left(\left|\left|\frac{A}{m} + \frac{B}{m} + O\left(\frac{I}{m^2}\right)\right|\right|^2\right) \\
&= \frac{A}{m} + \frac{B}{m} + O\left(\frac{I}{m^2}\right).
\end{aligned}$$

Exponentiating the logarithm:

$$\left(e^{A/m}e^{B/m}\right)^m = \left(\exp\left(\frac{A}{m}+\frac{B}{m}+O\left(\frac{I}{m^2}\right)\right)\right)^m$$
$$= \exp\left(A+B+O\left(\frac{I}{m}\right)\right).$$

The result follows. $\square$

It is clear that, if $X \in \mathfrak{g}$ then cX is also an element of $\mathfrak{g}$ for all $c \in \mathbb{R}$. The fact that if $X, Y \in \mathfrak{g}$, then so does $X+Y$ follows from our lemma and from the closedness of G. Also, note that

$$[X,Y] = \frac{\mathrm{d}}{\mathrm{d}t}e^{tX}Ye^{-tX}\bigg|_{t=0} = \lim_{h\to 0}\frac{e^{hX}Ye^{-hX}-Y}{h}.$$

Since $(e^{hX}Ye^{-hX}-Y)/h \in \mathfrak{g}$, so does $[X,Y]$. Thence, $\mathfrak{g}$ is a Lie algebra.

We shall now compute the Lie algebras of the classical Lie groups. It should be clear that, since e^X is always invertible for any $X \in M_n(\mathbb{C})$ (since $\det e^X = e^{\operatorname{tr} X}$), the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ of $GL_n(\mathbb{R})$ is $M_n(\mathbb{R})$. Similarly, $\mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})$.

Theorem 7. *The Lie algebra of $SL_n(\mathbb{C})$ consists of all $n\times n$ complex matrices with trace zero. Moreover, the Lie algebra of $SL_n(\mathbb{R})$ consists of all $n\times n$ real matrices with trace zero.*

Proof. If $X \in M_n(\mathbb{C})$ is such that $e^{tX} \in SL_n(\mathbb{C})$ for all real t,

$$\operatorname{tr} X = \frac{\mathrm{d}}{\mathrm{d}t}e^{t\operatorname{tr} X}\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\det e^{tX}\bigg|_{t=0} = 0.$$

Conversely, if $\operatorname{tr} X = 0$ then,

$$\det(e^{tX}) = e^{t\operatorname{tr} X} = 1.$$

Hence, $e^{tX} \in SL_n(\mathbb{C})$ for all $t \in \mathbb{R}$. The result follows. Similarly one can prove that $\mathfrak{sl}_n(\mathbb{R})$ consists of all traceless real matrices. $\square$

As we saw in this proof, the "determinant 1" condition of the Lie group adds the "trace 0" at its correspondent Lie algebra.

Theorem 8. *The Lie algebra of $U(n)$ consists of all skew-hermitian $n\times n$ complex matrices. Moreover, the Lie algebra of $SU(n)$ consists of all skew-hermitian $n\times n$ complex matrices with trace zero.*

Proof. Given a complex matrix X, e^{tX} is unitary if and only if

$$\left(e^{tX}\right)^* = \left(e^{tX}\right)^{-1} = e^{-tX}.$$

Since $(e^{tX})^* = e^{tX^*}$, e^{tX} if unitary if and only if

$$\left(e^{tX}\right)^* = e^{-tX},$$

Which happens for all $t \in \mathbb{R}$ precisely when X is skew-hermitian. The fact that every element of $\mathfrak{su}(n)$ is traceless follows from the same argument used in the proof of Theorem 7. $\square$

A similar argument shows that a real matrix X is an element of $\mathfrak{o}(n)$ if and only if X is skew-symmetric. Since every such matrix is traceless, $\mathfrak{so}(n) = \mathfrak{o}(n)$.

The preceding discussion should clarify our notation in Corollary 1.1. The reader might ask if it is a coincidence that $T_I O(n) = \mathfrak{o}(n)$ and $T_I SO(n) = \mathfrak{so}(n)$. The following theorem provides an answer.

Theorem 9. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Then, a matrix X is an element of $\mathfrak{g}$ if and only if there exists a smooth curve γ in G such that $\gamma(0) = I$ and $\gamma'(0) = X$. That is, $\mathfrak{g} = T_I G$.*

Unfortunately, the proof of this important result is rather technical. Since this work is a quick introduction to Lie groups, Lie algebras and representation theory this proof is beyond our scope. A wonderful discussion of this result can be found in Hall's[3] Corollary 3.46.

Lastly, another technical result that will be useful in our studies of representation theory is the following theorem about characterization of the elements of a connected Lie group.

Theorem 10. *Let G be a connected matrix Lie group with corresponding Lie algebra $\mathfrak{g}$. Then every element A of G can be written as*

$$A = e^{X_1} e^{X_2} \dots e^{X_m},$$

for some $X_1, X_2, \cdots, X_m \in \mathfrak{g}$.

4. Basic Representation Theory

Linear actions of groups arise in various branches of both mathematics and physics. Our goal in this section is to understand all the ways a fixed group can act as a group of endomorphisms. As the notation suggests, given a vector space V we denote by $GL(V)$ the Lie group composed by automorphisms of V and endowed with the inherited topology of its identification with $GL_n(\mathbb{R})$. We also denote by $\mathfrak{gl}(V)$ the Lie algebra composed by endomorphisms of V and endowed with the bracket $[X, Y] = XY - YX$.

Definition 3 (Homomorphisms)**.** We call a group homomorphism between Lie groups a Lie group homomorphism if it is continuous.[3] A linear map ϕ between Lie algebras is said to be a Lie algebra homomorphism if $\phi([x, y]) = [\phi(x), \phi(y)]$ for every x, y in its domain.

Definition 4 (Representations)**.** Let G be a Lie group. A linear representation of G is a Lie group homomorphism

$$\rho : G \to GL(V),$$

where V is a finite-dimensional vector space.

Now let $\mathfrak{g}$ be a Lie algebra. A linear representation of $\mathfrak{g}$ is a Lie algebra homomorphism

$$\pi : \mathfrak{g} \to \mathfrak{gl}(V).$$

If a representation is one-to-one, is said to be *faithful*. If v is a vector of V and $g \in G$, we'll generally denote the vector $\rho(g)v$ by $g \cdot v$. If the base field of V is $\mathbb{C}$, we'll say that it is a complex representation. Analogously, if the base field of V is $\mathbb{R}$, it is said to be a real representation.

[3] As one can show, every such map is smooth.

Similarly to the way we understand integers based on their decomposition as products of prime numbers, we shall study a particular class of representations named irreducible representation.

Definition 5. A representation ρ of a Lie group G is said to be irreducible if $\rho(g)$ has only V and $\{0\}$ as invariant subspaces for all $g \in G$. A irreducible representation of a Lie algebra is defined analogously.

If we know a representation of a Lie group G, one may ask if we can find a representation of its associated Lie algebra $\mathfrak{g}$. The following theorem affirms that the answer is affirmative. The converse of this result also holds if G is simply connected.

Theorem 11. *Let ρ be a representation of G acting on a finite-dimensional vector space V. Then, there exists a unique representation π of $\mathfrak{g}$ acting on the same vector space such that*

$$\rho(e^X) = e^{\pi(X)}$$

for all $X \in \mathfrak{g}$. The representation π is the pushforward of ρ at identity. That is,

$$\pi(X) = \frac{\mathrm{d}}{\mathrm{dt}}\rho(e^{tX})\bigg|_{t=0}$$

for all $X \in \mathfrak{g}$. Moreover, if G is simply connected, then given a representation π of $\mathfrak{g}$ there exists an unique representation ρ of G acting on the same space with satisfies $\rho(e^X) = e^{\pi(X)}$.

The proof of this result follows from a discussion on flows of left-invariant fields related by a morphism of Lie groups and is not simple at all. Hence it will be omitted. In the light of the preceding theorem and the following result we see that if a matrix Lie group is compact and simply connected it is enough to understand the irreducible representations of its associated Lie algebra.

Theorem 12. *Let G be a connected matrix Lie group with Lie algebra $\mathfrak{g}$. Let ρ be a representation of G and π be the associated representation of $\mathfrak{g}$. Then ρ is irreducible if and only if π is.*

Proof. Suppose ρ is irreducible and let W be a subspace of V that is invariant under $\pi(X)$ for all $X \in \mathfrak{g}$. Now let A be an arbitrary element of G. Since A can be written as $e^{X_1} \dots e^{X_m}$ and W is invariant under $\pi(X_j)$, it will also be invariant under $\exp(\pi(X_j))$. Hence W is invariant under

$$\rho(A) = \rho(e^{X_1} \dots e^{X_m}) = \rho(e^{X_1}) \dots \rho(e^{X_m}) = e^{\pi(X_1)} \dots e^{\pi(X_m)}.$$

Since ρ is irreducible, W is either V or $\{0\}$. Hence π is irreducible.

The converse follows by noticing that if W is an invariant subspace under $\rho(X)$, it is invariant under $\rho(\exp tX)$ for all $X \in \mathfrak{g}$ and all $t \in \mathbb{R}$. Hence W is invariant under

$$\pi(X) = \frac{\mathrm{d}}{\mathrm{dt}}\rho(e^{tX})\bigg|_{t=0}.$$

The result follows. $\square$

The simplest example of a matrix Lie group which is compact and simply connected is $SU(2) \cong S^3$. Thence, we'll devote some time understanding the irreducible representations of $\mathfrak{su}(2)$. The complexification of a vector space will be of great aid in this task.[4]

Theorem 13. *Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ be its complexification. Then every complex representation π of $\mathfrak{g}$ has a unique extension to a representation of $\mathfrak{g}_{\mathbb{C}}$, also denoted π. Furthermore, π is irreducible as a representation of $\mathfrak{g}$ if and only if it is irreducible as a representation of $\mathfrak{g}_{\mathbb{C}}$.*

Proof. We define the extension map by $\pi(X + iY) = \pi(X) + i\pi(Y)$. If π' in any such extension then

$$\begin{aligned}\pi'(X + iY) &= \pi'(X + iO) + \pi'(O + iY)\\ &= \pi'(X + iO) + i\pi'(Y + iO)\\ &= \pi(X) + i\pi(Y).\end{aligned}$$

That is, the extension presented is unique.

A subspace W of V is invariant under $\pi(X + iY)$ if and only if it is invariant under $\pi(X)$ and under $\pi(Y)$. Hence the claim about irreducibility follows. □

Definition 6 (Intertwining Maps)**.** Let G be a matrix Lie group, ρ be a representation of G acting on a vector space V and σ be a representation of G acting on W. A linear map $\phi : V \to W$ is said to be an intertwining map of representations if

$$\phi(\rho(g)v) = \sigma(g)\phi(v),$$

for all $g \in G$ and all $v \in V$. The definition of intertwining maps of Lie algebra representations is analogous. If ϕ is an intertwining map and ϕ is invertible, then it is said to be an isomorphism and the related representations are said to be isomorphic.

Using our usual action notation the defining equation becomes $\phi(g \cdot v) = g \cdot \phi(v)$, where the g on the left side is acting on V and on the right side it is acting on W.

Our goal in the next section is to determine, up to isomorphism, all the irreducible representations of $SU(2)$.

5. The Irreducible Representations of $SU(2)$

As was discussed in the previous section, to classify all the irreducible representations of $SU(2)$ it is enough to classify the irreducible representations of $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{su}(2)_{\mathbb{C}}$. We'll use the following basis of $\mathfrak{sl}_2(\mathbb{C})$:

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which satisfy the commutation relations

$$\begin{aligned}[][H, X] &= 2X,\\ [H, Y] &= -2Y,\\ [X, Y] &= H.\end{aligned}$$

[4]The complexification $V_{\mathbb{C}}$ of a vector space V is the vector space $V \otimes_{\mathbb{R}} \mathbb{C}$. We write an arbitrary element of $V_{\mathbb{C}}$ as $v_1 + iv_2 := v_1 \otimes 1 + v_2 \otimes i$.

The next lemma will furnish a better understanding of the structure of the linear representations of $\mathfrak{sl}_2(\mathbb{C})$.

Lemma 2. *Let π be a complex representation of $\mathfrak{sl}_2(\mathbb{C})$. If u is an eigenvector of $\pi(H)$ with correspondent eigenvalue $\alpha \in \mathbb{C}$, then either $\pi(X)u = 0$ or $\pi(X)u$ is an eigenvector of $\pi(H)$ with eigenvalue $\alpha + 2$.*

Similarly, either $\pi(Y)u = 0$ or $\pi(Y)u$ is an eigenvector of $\pi(H)$ with eigenvalue $\alpha - 2$.

Proof. Since $[\pi(H), \pi(X)] = \pi([H, X]) = 2\pi(X)$, it follows that

$$\begin{aligned}\pi(H)\pi(X)u &= \pi(X)\pi(H)u + 2\pi(X)u \\ &= \pi(X)(\alpha u) + 2\pi(X)u \\ &= (\alpha + 2)\pi(X)u.\end{aligned}$$

The result follows since the argument of $\pi(Y)$ is similar. □

We are now able to prove our so desired classification of the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$.

Theorem 14. *For any integer $m \geq 0$ there is an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ with dimension $m + 1$. Moreover, any two representations with the same dimension are isomorphic.*

For this proof we'll pick an arbitrary representation and determine it up to isomorphism. Then it is just a matter of computation to show that for every integer $m \geq 0$ the representations so determined actually are irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$.

Proof. Let π be a irreducible finite-dimensional complex representation of $\mathfrak{sl}_2(\mathbb{C})$. Since $\mathbb{C}$ is algebraically closed, let u be an eigenvector of $\pi(H)$ with eigenvalue α. From the previous lemma, it follows that

$$\pi(H)\pi(X)^k u = (\alpha + 2k)\pi(X)^k u.$$

The $\pi(X)^k u$'s cannot all be nonzero as that would imply $\pi(H)$ has infinitely many eigenvalues. Thus, there is a non-negative integer N such that $\pi(X)^N u \neq 0$ but $\pi(X)^{N+1} u = 0$. Then we set $u_0 = \pi(X)^N u$ and $\lambda = \alpha + 2N$. Then,

$$\begin{aligned}\pi(H)u_0 &= \lambda u_0, \\ \pi(X)u_0 &= 0.\end{aligned}$$

We then define

$$u_k = \pi(Y)^k u_0$$

for $k \geq 0$. Our lemma then implies

$$\pi(H)u_k = (\lambda - 2k)u_k,$$

from where it follows that

$$\pi(X)u_k = k[\lambda - (k - 1)]u_{k-1}.$$

Moreover, for the same argument used earlier, the u_k's cannot all be nonzero. Thence, there exists a non-negative integer m such that $u_k = \pi(Y)^k u_0 \neq 0$ for all $k \leq m$ but $u_{m+1} = \pi(Y)^{m+1} u_0 = 0$. Therefore, from our previous equation it follows that

$$0 = \pi(X)u_{m+1} = (m + 1)(\lambda - m)u_m$$

and so $\lambda = m$. Since the vectors $u_0, \dots, u_m$ are eigenvectors of $\pi(H)$ with distinct eigenvalues, the are linearly independent. Moreover, their span is invariant under $\pi(X)$, $\pi(Y)$ and $\pi(H)$. We conclude that, as π is irreducible, $u_0, \dots, u_m$ is a basis of V. To sum up, every irreducible finite-dimensional complex representation of $\mathfrak{sl}_2(\mathbb{C})$ is of the form

$$\begin{aligned}\pi(H)u_k &= (m-2k)u_k,\\ \pi(X)u_k &= \begin{cases} k[m-(k-1)]u_{k-1} & \text{if } k>0 \\ 0 & \text{if } k=0 \end{cases},\\ \pi(Y)u_k &= \begin{cases} u_{k+1} & \text{if } k<m \\ 0 & \text{if } k=m \end{cases}.\end{aligned}$$

Conversely, let V be any complex $(m+1)$-dimensional vector space with $\{u_0, \dots, u_m\}$ as a basis. Then it is just a matter of computation to show that the preceding equations define a irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. $\square$

Now, we shall explicitly present a list of irreducible representations of $SU(2)$ to finish our classification.

Let V_m be the space of homogeneous polynomials of degree m on two complex variables. For each $U \in SU(2)$ define a endomorphism $\rho_m(U)$ by the formula

$$[\rho_m(U)f](z) = f(U^{-1}z),$$

where $z \in \mathbb{C}^2$. Then ρ_m is a representation of $SU(2)$. Clearly, as no two of the V_m's have the same dimension, no two of the ρ_m are isomorphic.

Now for $X \in \mathfrak{su}(2)$, the associated representation π_m of $\mathfrak{su}(2)$ is given by

$$[\pi_m(X)f](z) = \frac{\mathrm{d}}{\mathrm{d}t} f(e^{-tX}z)\bigg|_{t=0}.$$

The chain rule then implies

$$[\pi_m(X)f](z_1, z_2) = -\frac{\partial f}{\partial z_1}(X_{11}z_1 + X_{12}z_2) - \frac{\partial f}{\partial z_2}(X_{21}z_1 + X_{22}z_2).$$

We can extend this representation to a representation of $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{su}(2)_{\mathbb{C}}$ by the same formula with $X \in \mathfrak{sl}_2(\mathbb{C})$. Take H, X, Y as a basis of $\mathfrak{sl}_2(\mathbb{C})$ and $u_k = z_1^{m-k} z_2^k$ as a basis of V_m. Then,

$$\begin{aligned}\pi_m(H)u_k &= (-m+2k)u_k,\\ \pi_m(X)u_k &= (k-m)u_{k+1},\\ \pi_m(Y)u_k &= -ku_{k-1}.\end{aligned}$$

The *theoremata* of section 4 then implies that every irreducible representation of $SU(2)$ is isomorphic to one of the ρ_m's described above.

6. Coverings

As we saw, if G is a simply connected Lie group, every representation of $\mathfrak{g}$ can be exponentiated to a representation of G. If G is not simply connected it can be useful to find another Lie group $\widetilde{G}$ that has the same associated Lie algebra as G and is simply connected. This motivates our next definition.

Definition 7 (Universal Cover)**.** Let G be a connected Lie group. A simply connected Lie group H together with a Lie group homomorphism $\Phi : H \to G$ such that the associated Lie algebra homomorphism $\phi : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra isomorphism. The homomorphism Φ is then called a covering map.

It is not hard to prove that every two universal covers of a connected Lie group are isomorphic, hence it is reasonable to speak of the universal cover of G.

As we saw earlier, the group $SU(2)$ is isomorphic to the unit sphere S^3. We'll identify S^3 as the group of unit quaternions. I claim that $SU(2)$ is the universal cover of $SO(3)$.

Theorem 15. *There is a* 2*-to-*1 *group homomorphism from* $SU(2)$ *to* $SO(3)$*. Since* $\mathfrak{su}(2) \cong \mathfrak{so}(3)$*,* $SU(2)$ *is the universal cover of* $SO(3)$*. It is said that such map is a double covering.*

In the following proof I assume that the reader is acquainted with the fact that a rotation of $\mathbb{R}^3$ through an angle α about the axis u is given by the conjugation map

$$q \mapsto tqt^{-1}, \qquad \text{where} \qquad t = \cos\frac{\alpha}{2} + u \sin\frac{\alpha}{2}.$$

Obviously we identified $\mathbb{R}^3$ to the imaginary quaternions $i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$.

Proof. The required map is given by

$$\begin{aligned} \Phi : SU(2) &\to SO(3) \\ t &\mapsto (q \mapsto tqt^{-1}). \end{aligned}$$

A simple calculation confirms that Φ is a group homomorphism. The fact that Φ is 2-to-1 is a consequence of the $\alpha/2$ factor in the conjugation map. □

As a corollary of the fundamental isomorphism theorem for groups we get that:

Corollary 15.1. *The groups* $SO(3)$ *and* $SU(2)/\{\pm I\}$ *are isomorphic.*

Since $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ it is natural to ask whether one of the π's described in Theorem 14 can become a representation of $SO(3)$. It can be proved that a irreducible representation of $SO(3)$ comes from one of the π's if and only if m is even. In the physics literature such representations are labelled by the parameter $l = m/2$, where l is called the *spin* of the representation. A electron has spin 1/2, which means that it is described by a representation of $SU(2)$. In practice, this implies that a rotation of 360° applied to the electron wave function gives back the negative of the original function. This is illustrated cleverly by Dirac's belt trick.

A similar argument shows that for $t_1, t_2 \in SU(2)$ the map $(t_1, t_2) \mapsto (q \mapsto t_1 q t_2^{-1})$ is a double covering of $SO(4)$ and hence $SU(2) \times SU(2)$ is its universal cover. Also, $SO(4) \cong SU(2) \times SU(2)/\{(I, I), (-I, -I)\}$.

7. Applications to Physics

It is of particular interest to physicists the study of the group $O(3, 1)$, denominated *Lorentz group* after Dutch physicist Hendrik A. Lorentz, whose action preserves the symmetric bilinear form b given by

$$b(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4.$$

That is, we are interested in the matrices $A \in GL_4(\mathbb{R})$ such that $b(Ax, Ay) = b(x, y)$ for all $x, y \in \mathbb{R}^4$. Consider the matrix $g = \operatorname{diag}(1, 1, 1, -1)$. It should be clear that $b(x, y) = \langle x, gy \rangle$, where $\langle \cdot, \cdot \rangle$ is the canonical inner product on $\mathbb{R}^4$. Hence $A \in O(3, 1)$ if and only if $A^T g A = g$. Taking the determinant of this equation gives $\det A = \pm 1$. The group $SO(3, 1)$ is composed by the matrices of $O(3, 1)$ with unit determinant.

Similarly to the proof of Theorem 8, one can show that the Lie algebras of both $O(3, 1)$ and $SO(3, 1)$ are given by the matrices X which satisfy $gX^T g = -X$. The group $O(3, 1)$ has four connected components. We'll denote by $SO^{\uparrow}(3, 1)$ its connected component which contains the identity. Consider the mapping M given by

$$x \in \mathbb{R}^4 \mapsto M(x) = \begin{bmatrix} x_1 + x_4 & x_2 - ix_3 \\ x_2 + ix_3 & x_1 - x_4 \end{bmatrix}.$$

As one can clearly see, $M(x)$ is hermitian. We then define an action of $SL_2(\mathbb{C})$ on the space of hermitian matrices by

$$(A, X) \mapsto A \cdot X = AXA^*,$$

where $A \in SL_2(\mathbb{C})$ and X is hermitian. Note that $\det(A \cdot M(x)) = \det(M(x)) = b(x, x)$. Hence, the matrix T_A associated with the endomorphism $x \mapsto A \cdot M(x)$ is an element of $O(3, 1)$. It can be easily checked that $A \mapsto T_A$ is a homomorphism of Lie groups with kernel $\{\pm I\}$. Since $SL_2(\mathbb{C})$ is connected, its image is the connected component of the identity in $O(3, 1)$, namely, $SO^{\uparrow}(3, 1)$. We conclude that

Theorem 16. *The mapping $A \mapsto T_A$ is a double covering of $SO^{\uparrow}(3, 1)$ by $SL_2(\mathbb{C})$. Hence, $SO^{\uparrow}(3, 1) \cong SL_2(\mathbb{C})/\{\pm I\}$.*

8. Acknowledgements

I would like to thank my advisor José J. Ramón-Marí for all his effort. The numerous meetings, emails and notes (he may have written over 400 pages of notes in the last 2 years just for me) were incredibly helpful in making me understand difficult concepts that are generally taught at graduate level.

It should be emphasized that only the elementary proofs were created by myself (namely, Theorems 3, 4, 6, 7, 8 and Corollaries 1.1 and 15.1). The remainder of this work is just my view of the relevant parts of [4] and [3].

References

[1] G. B. Folland. *Quantum Field Theory: A Tourist Guide for Mathematicians.* Mathematical Surveys and Monographs. American Mathematical Society, 2008.
[2] W. Fulton and J. Harris. *Representation Theory: A First Course.* Graduate Texts in Mathematics. Springer, 1999.
[3] B. C. Hall. *Lie Groups, Lie Algebras and Representations.* Graduate Texts in Mathematics. Springer, 2003.
[4] J. J. Ramón-Marí. Notes on Lie groups, Lie algebras, representation theory, manifolds, differential topology and finite groups, 2017.
[5] L. W. Tu. *An Introduction To Manifolds.* Universitext. Springer New York, 2011.

Instituto Militar de Engenharia, Rio de Janeiro, Brasil.
E-mail: gabriel_ribeiro1707@hotmail.com