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REAL AND COMPLEX ANALYSIS

A SOLUTION MANUAL

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Mathematics is the art of giving the same name to different things.

— Henri Poincaré

In memory of Walter Rudin.

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CONTENTS

1

ABSTRACT INTEGRATION

Exercise 1.1 Does there exist an infinite σ-algebra which has only countably many members?

■ Solution Let X be a measurable space with an infinite σ-algebra $\mathfrak{M}$. Our approach will be very similar to the proof that $[0, 1]$ is compact. (If you do not remember this proof, continue reading. We're not going to use anything other than its main idea.) Since $\mathfrak{M}$ is infinite, there exists at least one set $E \in \mathfrak{M}$ other than X and $\varnothing$. Surely, E^c is also in the sigma-algebra $\mathfrak{M}$.

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ E & & E^c \end{tikzcd}$$

Now, E and E^c are measurable spaces by themselves. (Their sigma-algebras are the intersection of the sets in $\mathfrak{M}$ with E or E^c.) As $\mathfrak{M}$ is infinite, at least one of the sigma-algebras of E or E^c is infinite. With this sigma-algebra we can repeat the procedure, obtaining a sequence (E_n) of sets in $\mathfrak{M}$. (We write E_n for the set that has an infinite sigma-algebra.)

$$\begin{tikzcd} & & X \arrow[dl] \arrow[dr, dotted] & \\ & E_1 \arrow[dl, dotted] \arrow[dr] & & E_1^c \\ E_2^c & & E_2 \arrow[dl] \arrow[dr, dotted] & \\ & E_3 \arrow[dl] \arrow[dr, dotted] & & E_3^c \\ {} & & {} & \end{tikzcd}$$

We observe that $\{E_n \setminus E_{n+1} : n \in \mathbb{N}\}$ is a infinite collection of disjoint non-empty measurable sets. Since $\mathfrak{M}$ should contain every possible union of those sets, and choosing one such union amounts to picking a subset of $\mathbb{N}$, the cardinality of $\mathfrak{M}$ is at least

$$|\mathcal{P}(\mathbb{N})| = \mathfrak{c}.$$

It follows that there isn't an infinite σ-algebra with countably many members. ■

Exercise 1.2 Prove an analogue of Theorem 1.8 for n functions.

Theorem 1.8 Let u and v be real measurable functions on a measurable space X, let Φ be a continuous mapping of the plane into a topological space Y, and define

$$h(x) = \Phi(u(x), v(x))$$

for $x \in X$. Then $h : X \to Y$ is measurable.

■ Solution The desired analogue is the following.

Let $u_1, u_2, \ldots, u_n$ be real measurable functions on a measurable space X, let Φ be a continuous mapping of $\mathbb{R}^n$ into a topological space Y, and define

$$h(x) = \Phi(u_1(x), u_2(x), \ldots, u_n(x))$$

for $x \in X$. Then $h : X \to Y$ is measurable.

As with Theorem 1.8, it is enough to prove

$$f(x) = (u_1(x), u_2(x), \ldots, u_n(x))$$

is measurable. Let $B = I_1 \times I_2 \times \cdots \times I_n$ be an open box, and each I_k an open interval. Then

$$f^{-1}(B) = u_1^{-1}(I_1) \cap u_2^{-1}(I_2) \cap \cdots \cap u_n^{-1}(I_n)$$

is a measurable set, and since every open set V is a countable union of open boxes B_i, then

$$f^{-1}(V) = f^{-1}\left(\bigcap_{i=1}^{\infty} B_i\right) = \bigcap_{i=1}^{\infty} f^{-1}(B_i)$$

is measurable. ■

Exercise 1.3 Prove that if f is a real function on a measurable space X such that $\{x : f(x) \geq r\}$ is measurable for every rational r, then f is measurable.

■ Solution Given $\alpha \in \mathbb{R}$, let $\{r_n\}$ be a decreasing sequence of rational numbers such that $\lim r_n = \alpha$. Then

$$f^{-1}((\alpha, \infty)) = \bigcap f^{-1}((r_n, \infty))$$

is measurable. Analogously, if $\{s_n\}$ is an increasing sequence of rational numbers such that $\lim s_n = \alpha$, then

$$f^{-1}([\alpha, \infty)) = \bigcap f^{-1}((s_n, \infty))$$

is measurable. Hence, given $a < b \in \mathbb{R}$, the set

$$f^{-1}((a, b)) = f^{-1}((a, \infty)) \backslash f^{-1}([b, \infty))$$

is also measurable.

Since every open set in $\mathbb{R}$ is a countable union of open intervals, for every open set $V \subset \mathbb{R}$, $f^{-1}(V)$ is measurable, and we conclude f is measurable. ■

EXERCISE 1.4 Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[-\infty, \infty]$, and prove the following assertions :

(a) $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$.

(b) $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$, provided none of the sums is of the form $\infty - \infty$.

(c) If $a_n \leq b_n$ for all n, then $\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$.

Show by an example that strict inequality can hold in (b).

■ SOLUTION

(a) For all $n \in \mathbb{N}$ we have that

$$\sup_{i\geq n}\{-a_i\} = -\inf_{i\geq n}\{a_i\}.$$

The result then follows by taking the limit $n \to \infty$.

(b) Clearly, for all $n \in \mathbb{N}$,

$$\sup_{i\geq n}\{a_i + b_i\} \leq \sup_{i\geq n}\{a_i\} + \sup_{i\geq n}\{b_i\}.$$

The result then follows by taking the limit $n \to \infty$.

(c) As $a_n \leq b_n$ for all $n \in \mathbb{N}$, we have that

$$\inf_{i\geq n}\{a_i\} \leq \inf_{i\geq n}\{b_i\}.$$

The result then follows by taking the limit $n \to \infty$.

For the strict inequality in (b) we can take $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. ■

EXERCISE 1.5

(a) Suppose $f : X \to [-\infty, \infty]$ and $g : X \to [-\infty, \infty]$ are measurable. Prove that the sets

$$\{x : f(x) < g(x)\}, \quad \{x : f(x) = g(x)\}$$

are measurable.

(b) Prove that the set of limit points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

■ SOLUTION

(a) Since f and g are measurable, $h = g - f$ is also measurable, and

$$h^{-1}((0, \infty]) = \{x \colon f(x) < g(x)\},$$
$$h^{-1}(\{0\}) = \{x \colon f(x) = g(x)\}$$

are measurable.

(b) Let f_n be the sequence of measurable functions. Then the functions $f = \liminf_{n\to\infty} f_n$ and $g = \limsup_{n\to\infty} f_n$ are also measurable, and the set of points at which the sequence converges is

$$\{x\colon f(x) = g(x)\}$$

which is measurable by (a). ■

Exercise 1.6 Let X be an uncountable set, let $\mathfrak{M}$ be the collection of all sets $E \subset X$ such that either E or E^c is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second. Prove that $\mathfrak{M}$ is a σ-algebra in X and that μ is a measure in $\mathfrak{M}$.

■ Solution Let us first prove $\mathfrak{M}$ is a σ-algebra.

In fact, $X^c = \varnothing$, and $X \in \mathfrak{M}$. If $A \in \mathfrak{M}$, then A or A^c is countable, i.e., A^c or $(A^c)^c$ is countable, therefore $A^c \in \mathfrak{M}$. Finally, if $A_n \in \mathfrak{M}$, we divide in two cases. If every A_n is countable, then $\cup A_n$ is also countable. Otherwise, some A_k is uncountable. In this case, A_k^c is countable, and $\left(\bigcup A_n\right)^c \subset A_k^c$ is countable.

To prove μ is a measure, we again divide in two cases. If every A_n is countable, $\bigcup A_n$ is also countable, therefore $\mu(\bigcup A_n) = 0 = \sum \mu(A_n)$. On the other hand, if some A_k is uncountable (and only A_k can be uncountable, since the sets are disjoint and A_k^c is countable), then $(\bigcup A_n)^c$ is countable and $\mu(\bigcup A_n) = 1 = \sum \mu(A_n)$. ■

Exercise 1.7 Suppose $f_n\colon X \to [0,\infty]$ is measurable for $n = 1, 2, \ldots$, $f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0$, $f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$, and $f_1 \in L_1(\mu)$. Prove that then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f d\mu$$

and show that this condition does *not* follow if the condition "$f_1 \in L^1(\mu)$" is omitted.

■ Solution Lebesgue's Dominated Convergence Theorem implies that (since the f_n are bounded by $f_1 \in L^1(\mu)$)

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

This condition is necessary since if $X = \mathbb{R}$ and $f_n = \chi_{[n,\infty)}$, then f is the zero function. Hence, $\int_X f \, d\mu = 0$ even though $\inf_X f_n \, d\mu = \infty$ for all $n \in \mathbb{N}$. ■

Exercise 1.8 Put $f_n = \chi_E$ if n is odd, $f_n = 1 - \chi_E$ is n is even. What is the relevance of this example to Fatou's lemma?

■ Solution Note that for every $x \in X$, $f_n(x)$ is either the sequence $\{0, 1, 0, 1, \ldots\}$ or $\{1, 0, 1, 0, \ldots\}$, hence $\liminf_{n\to\infty} f_n = 0$. On the other hand, $\liminf_{n\to\infty} \int_X f_n \, d\mu = \min\{\mu(E), 1 - \mu(E)\}$. Therefore, if we let

$E \in \mathfrak{M}$ be a set with $\mu(E) \in (0,1)$ we see the inequality in Fatou's lemma can be strict. ■

EXERCISE 1.9 Suppose μ is a positive measure on X, $f : X \to [0,\infty]$ is measurable, $\int_X f\, d\mu = c$, where $0 < c < \infty$, and α is a constant. Prove that

$$\lim_{n\to\infty} \int_X n \cdot \log\left(1 + \left(\frac{f}{n}\right)^{\alpha}\right) d\mu = \begin{cases} \infty, & \text{if } \ 0 < \alpha < 1; \\ c, & \text{if } \ \alpha = 1; \\ 0, & \text{if } \ 1 < \alpha < \infty. \end{cases}$$

Hint: If $\alpha \geq 1$, the integrands are dominated by αf. If $\alpha < 1$, Fatou's lemma can be applied.

■ SOLUTION We will follow the hint. Let $\alpha \geq 1$, $x \in X$ and $y = f(x)/n$. In this case we will estimate limits to the integrands and we will use the Lebesgue's dominated convergence theorem. First we will prove that $1 + y \leq e^y$ using the definition given in the prologue of the book.

$$1 + y \leq 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{y^n}{n!} = \exp(y) = e^y.$$

Now we will show that $1 + y^{\alpha} \leq (1+y)^{\alpha}$. For this purpose define $\psi : (0,\infty) \to R^1$ such that $\psi(t) = (1+t)^{\alpha} - t^{\alpha} - 1$, $\forall t \in (0,\infty)$. It is straightforward that ψ is differentiable and its derivative is $\psi' : (0,\infty) \to R^1$ such that $\psi'(t) = \alpha[(1+t)^{\alpha-1} - t^{\alpha-1}]$, $\forall t \in (0,\infty)$. Since $\alpha - 1 \geq 0$ and exponentials are monotonically strictly increasing positive functions, we have that

$$(1+t)^{\alpha-1} \geq t^{\alpha-1} \ \Rightarrow \ \psi'(t) \geq 0,$$

for all $t \in (0,\infty)$. Thus ψ is a non-decreasing function and it follows that $\psi(t) \geq \lim_{s\to 0} \psi(s) = 0$, $\forall t \in (0,\infty)$. In particular,

$$(1+y)^{\alpha} - y^{\alpha} - 1 = \psi(y) \geq 0 \ \Rightarrow \ (1+y)^{\alpha} \geq 1 + y^{\alpha}.$$

Therefore, $1 + y^{\alpha} \leq (1+y)^{\alpha} \leq (e^y)^{\alpha} = e^{\alpha y}$. Applying the (strictly increasing) natural logarithm function gives us

$$\log\left(1 + \left[\frac{f(x)}{n}\right]^{\alpha}\right) = \log(1 + y^{\alpha}) \leq \alpha y = \alpha \cdot \frac{f(x)}{n}$$

and it follows that $n \cdot \log\left(1 + \left[\frac{f(x)}{n}\right]^{\alpha}\right) \leq \alpha f(x)$.

For each $n = 1,2,3...$, let $\varphi_n : X \to [0,\infty]$ be such that, $\forall x \in X$, $\varphi_n(x) = n \cdot \log\left(1 + \left[\frac{f(x)}{n}\right]^{\alpha}\right)$. By theorem 1.7 (b), $\{\varphi_n\}$ is a sequence of measurable functions on X. If $\alpha = 1$, then, for all $x \in X$,

$$f(x) = 0 \ \Rightarrow \ \lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} n \cdot \log\left(1 + \frac{f(x)}{n}\right) = 0 = f(x)$$

and, if $f(x) > 0$, we change variables $\xi = n/f(x)$ to compute

$$\begin{aligned}
\lim_{n\to\infty} \varphi_n(x) &= \lim_{n\to\infty} n \cdot \log\left(1 + \frac{f(x)}{n}\right) \\
&= \lim_{n\to\infty} \log\left(\left[1 + \frac{f(x)}{n}\right]^n\right) \\
&= \log\left(\lim_{n\to\infty}\left[1 + \frac{f(x)}{n}\right]^n\right) \\
&= \log\left(\left[\lim_{\xi\to\infty}\left(1 + \frac{1}{\xi}\right)^{\xi}\right]^{f(x)}\right) \\
&= \log\left(e^{f(x)}\right) \\
&= f(x).
\end{aligned}$$

Where $\lim\limits_{\substack{\xi\to\infty \\ \xi\in R^1}}\left(1+\dfrac{1}{\xi}\right)^{\xi} = \lim\limits_{\substack{k\to\infty \\ k=1,2,3,\ldots}}\left(1+\dfrac{1}{k}\right)^{k} = e = \sum\limits_{k=0}^{\infty}\dfrac{1}{k!}$ can easilly be shown using the squeeze theorem and theorem 3.31 of the book *Principles of Mathematical Analysis* 3rd ed. (pg. 64) by the same author. Thus $\alpha = 1 \;\Rightarrow\; \lim\limits_{n\to\infty} \varphi_n = f$.

If $1 < \alpha < \infty$, then, for all $x \in X$,

$$f(x) = 0 \;\Rightarrow\; \lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} n \cdot \log\left(1 + \left[\frac{f(x)}{n}\right]^{\alpha}\right) = 0$$

and, if $f(x) > 0$, again change variables $s = [f(x)]^{\alpha}/n^{\alpha}$ to compute

$$\lim_{n\to\infty} \frac{\log\left(1 + \frac{[f(x)]^{\alpha}}{n^{\alpha}}\right)}{[f(x)]^{\alpha}/n^{\alpha}} = \lim_{s\to 0} \frac{\log(1+s) - \log(1)}{s} = (\log)'(1) = 1,$$

then notice that $\lim\limits_{n\to\infty} \dfrac{[f(x)]^{\alpha}}{n^{\alpha-1}} = 0$ to conclude

$$\begin{aligned}
\lim_{n\to\infty} \varphi_n(x) &= \lim_{n\to\infty} n \cdot \log\left(1 + \left[\frac{f(x)}{n}\right]^{\alpha}\right) \\
&= \lim_{n\to\infty} \frac{[f(x)]^{\alpha}}{n^{\alpha-1}} \cdot \frac{\log\left(1 + \frac{[f(x)]^{\alpha}}{n^{\alpha}}\right)}{[f(x)]^{\alpha}/n^{\alpha}} \\
&= 0.
\end{aligned}$$

Thus $1 < \alpha < \infty \;\Rightarrow\; \lim\limits_{n\to\infty} \varphi_n = 0$.

Since $|\varphi_n(x)| \leq \alpha f(x)$, $\forall x \in X$, for all $n = 1, 2, 3, \ldots$, and $\alpha f \in L^1(\mu)$, by the Lebesgue's dominated convergence theorem it follows that

$$\lim_{n\to\infty} \int_X \varphi_n \, d\mu = \int_X \left(\lim_{n\to\infty} \varphi_n\right) d\mu = \begin{cases} \int_X f \, d\mu = c, & \text{if } \alpha = 1; \\ \int_X 0 \, d\mu = 0, & \text{if } 1 < \alpha < \infty. \end{cases}$$

Let $0 < \alpha < 1$ and $x \in X$. Now we will compute the limit inferior of the sequence $\{\varphi_n\}$ to use Fatou's lemma. If $f(x) = 0$, then

$$\begin{aligned}
\liminf_{n\to\infty} \varphi_n(x) &= \liminf_{n\to\infty} \; n \cdot \log\left(1 + \left[\frac{f(x)}{n}\right]^\alpha\right) \\
&= \liminf_{n\to\infty} \; n \cdot \log(1+0) \\
&= \liminf_{n\to\infty} 0 \\
&= 0.
\end{aligned}$$

If $f(x) > 0$, again we have

$$\lim_{n\to\infty} \frac{\log\left(1 + \frac{[f(x)]^\alpha}{n^\alpha}\right)}{[f(x)]^\alpha / n^\alpha} = (\log)'(1) = 1,$$

but now $\alpha - 1 < 0$, which implies

$$\lim_{n\to\infty} \frac{[f(x)]^\alpha}{n^{\alpha-1}} = \lim_{n\to\infty} n^{1-\alpha}[f(x)]^\alpha = \infty.$$

It follows that

$$\liminf_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} \frac{[f(x)]^\alpha}{n^{\alpha-1}} \cdot \frac{\log\left(1 + \frac{[f(x)]^\alpha}{n^\alpha}\right)}{[f(x)]^\alpha / n^\alpha} = \infty.$$

Let $E = f^{-1}(\{0\})$. We got the function $\liminf\limits_{n\to\infty} \varphi_n : X \to [0, \infty]$ such that, $\forall x \in X$,

$$\liminf_{n\to\infty} \varphi_n(x) = \begin{cases} 0, & \text{if } \; x \in E; \\ \infty, & x \notin E. \end{cases}$$

That is, $\liminf\limits_{n\to\infty} \varphi_n = \infty \cdot \chi_{E^c}$. Since $\int_X f \, d\mu = c > 0$ we have that $\mu(E^c) > 0$. Hence

$$\int_X \left(\liminf_{n\to\infty} \varphi_n\right) d\mu = \int_X \infty \cdot \chi_{E^c} \, d\mu = \int_{E^c} \infty \, d\mu = \infty \cdot \mu(E^c) = \infty.$$

By Fatou's Lemma,

$$\infty = \int_X \left(\liminf_{n\to\infty} \varphi_n\right) d\mu \leq \liminf_{n\to\infty} \int_X \varphi_n \, d\mu.$$

Therefore $\lim\limits_{n\to\infty} \int_X \varphi_n \, d\mu = \liminf\limits_{n\to\infty} \int_X \varphi_n \, d\mu = \infty$ and we are done. ∎

EXERCISE 1.10 Suppose $\mu(X) < \infty$, $\{f_n\}$ is a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X. Prove that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu,$$

and show that the hypothesis "$\mu(X) < \infty$" cannot be omitted.

■ SOLUTION Let $\epsilon > 0$. By the uniform convergence of $\{f_n\}$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon, \qquad \text{for all } n \geq N \text{ and all } x \in X.$$

By the reverse triangle inequality, $|f(x)| < |f_N(x)| + \epsilon$ and $|f_n(x)| < |f(x)| + \epsilon$. Thus

$$|f_n(x)| < |f_N(x)| + 2\epsilon, \qquad \text{for all } n \geq N \text{ and all } x \in X.$$

We conclude that

$$|f_n(x)| \leq \max\{|f_1(x)|, \ldots, |f_{N-1}(x)|, |f_N(x)| + 2\epsilon\}.$$

Denoting the function on the right side by g, we have that g is bounded and hence in $L^1(\mu)$ since $\mu(X)$ is finite. Lebesgue's Dominated Convergence Theorem now applies.

We show that the hypothesis "$\mu(X) < \infty$" is necessary by considering $X = \mathbb{N}$ endowed with the counting measure. If $f_n(x) = 1/n$, then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{1}{n} = \infty$$

while $\int_X f \, d\mu = 0$ since $f = 0$. ■

The set A is usually denoted by $\limsup_{k\to\infty} E_k$.

EXERCISE 1.11 Show that

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

in Theorem 1.41 and hence prove the theorem without any reference to integration.

When μ is a probability, viz. $\mu(X) = 1$, this result is called Borel-Cantelli lemma.

THEOREM 1.41 Let $\{E_k\}$ be a sequence of measurable sets in X, such that

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty.$$

Then almost all $x \in X$ lie in at most finitely many of the sets E_k.

■ SOLUTION Let A be as in the proof of Theorem 1.41, then

$$\begin{aligned} x \in A &\iff x \text{ lie in infinitely many } E_k \\ &\iff \forall n, \exists k \geq n \text{ such that } x \in E_k \\ &\iff \forall n \; x \in \bigcup_{k=n}^{\infty} E_k \\ &\iff x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k. \end{aligned}$$

Let $A_n = \bigcup_{k=n}^{\infty} E_k$. Then $A = \bigcap_{n=1}^{\infty} A_n$, $A_n \in \mathfrak{M}$,

$$A_1 \supset A_2 \supset A_3 \supset \cdots,$$

and $\mu(A_1) \leq \sum_{n=1}^{\infty} \mu(E_k) < \infty$. Therefore,

$$\begin{aligned}\mu(A) &= \lim_{n\to\infty} \mu(A_n) \\ &\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(E_k) \\ &= 0\end{aligned}$$

and the result follows. ∎

Exercise 1.12 Suppose $f \in L^1(\mu)$. Prove that to each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f|\,d\mu < \varepsilon$ whenever $\mu(E) < \delta$.

∎ Solution Consider the family of functions $g_n : X \to (-\infty, \infty)$, defined by

$$g_n(x) = \begin{cases} |f(x)|, & \text{if } |f(x)| \leq n \\ n, & \text{otherwise.} \end{cases}$$

All functions g_n are measurable, and as a sequence it converges to $|f|$, pointwise. Therefore, using Lebesgue's monotone convergence theorem, given any $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\left| \int_E |f|\,d\mu - \int_E g_{n_0}\,d\mu \right| < \frac{\varepsilon}{2}.$$

On the other hand, for each $n = 1, 2, 3, \ldots$, g_n is bounded by n. Therefore, if s denotes a simple function such that $0 \leq s \leq g_{n_0}$ with constant values α_i on sets A_i, the sums used to define the integral of g_{n_0} satisfy

$$\sum_i \alpha_i \mu(A_i \cap E) \leq n_0 \mu(E).$$

If $\delta = \frac{\varepsilon}{2n_0}$ and $\mu(E) < \delta$, then $n_0\mu(E) < \frac{\varepsilon}{2}$, and since the definition of the integral of g_0 over E is the supremum of the sums of the form above, one has that

$$\int_E g_{n_0}\,d\mu < \frac{\varepsilon}{2}$$

whenever $\mu(E) < \delta$.

Now, using the triangle inequality,

$$\begin{aligned}\int_E |f|\,d\mu &= \left| \int_E |f|\,d\mu - \int_E g_{n_0}\,d\mu + \int_E g_{n_0}\,d\mu \right| \\ &\leq \left| \int_E |f|\,d\mu - \int_E g_{n_0}\,d\mu \right| + \left| \int_E g_{n_0}\,d\mu \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

as it was wanted to prove. ∎

EXERCISE 1.13 Show that Proposition 1.24(c) is also true when $c = \infty$.

■ SOLUTION We recall Proposition 1.24(c): if $f \geq 0$ and c is a constant, $0 \leq c < \infty$, then

$$\int_E cf \, d\mu = c \int_E f \, d\mu$$

for every measurable set E.

There are two cases when $c = \infty$. Firstly, if $\int_E f \, d\mu = 0$, then $f = 0$ a.e. and so $cf = 0$ almost everywhere. We conclude that

$$\int_E cf \, d\mu = 0 = c \int_E f \, d\mu.$$

Otherwise, there exist a number $\epsilon > 0$ and a measurable set E such that $\mu(E) > 0$ and $f(x) > \epsilon$ for all $x \in E$. (If this weren't true, we would have $f(x) < \epsilon$ for all $\epsilon > 0$ and thus $f = 0$ a.e..) Then,

$$\int_E cf \, d\mu > \epsilon \int_E c \, d\mu = \infty.$$

The result follows since $c \int_E f \, d\mu = \infty$. ■

2

POSITIVE BOREL MEASURES

EXERCISE 2.1 Let $\{f_n\}$ be a sequence of real nonnegative functions on $\mathbb{R}^1$, and consider the following four statements:

(a) If f_1 and f_2 are upper semicontinuous, then $f_1 + f_2$ is upper semicontinuous.

(b) If f_1 and f_2 are lower semicontinuous, then $f_1 + f_2$ is lower semicontinuous.

(c) If each f_n is upper semicontinuous, then $\sum_1^\infty f_n$ is upper semicontinuous.

(d) If each f_n is lower semicontinuous, then $\sum_1^\infty f_n$ is lower semicontinuous.

Show that three of these are true and that one is false. What happens if the word "nonnegative" is omitted? Is the truth of the statements affected is $\mathbb{R}^1$ is replaced by a general topological space?

■ SOLUTION

(a) We wish to prove that the set $\{x \in \mathbb{R}^1 : f_1(x) + f_2(x) < \alpha\}$ is open for every real α. This set can be written as the union of

$$\{x \in \mathbb{R}^1 : f_1(x) < \beta\} \cap \{x \in \mathbb{R}^1 : f_2(x) < \alpha - \beta\}$$

for all real β. Since each one of these sets is open, so is their union. It follows that $f_1 + f_2$ is upper semicontinuous.

(b) The same argument utilized in (a) works here.

(c) This is not true in general. As W. Rudin remarked in page 38, characteristic functions of closed sets are upper semicontinuous. We then let $f_n = \chi_{[1/(n+1),1/n]}$ for $n \geq 1$. Since $\sum_1^\infty f_n = \chi_{(0,1]} + \chi_{\{1/k : k \geq 2\}}$, which is *not* upper semicontinuous, this is a counterexample. (Observe that $\{x \in \mathbb{R}^1 : \sum_1^\infty f_n < 1/2\} = \mathbb{R}^1 \setminus (0,1]$, which is not open.)

(d) Let $s_k = \sum_{n=1}^k f_n$. By (b), s_k is lower semi-continuous. Since s_k is increasing, $\sup_k s_k = \lim_k s_k = \sum_{n=1}^\infty f_n$. Now, as the supremum of any collection of lower semicontinuous functions is lower semicontinuous, the result follows.

Roughly speaking, a function f is upper semicontinuous if for all x, the function values for arguments near x are either close to $f(x)$ or are less than $f(x)$. An analogous characterization holds for lower semicontinuous functions.

In each of the cases where the result is true, we have not used the fact that the functions are defined on $\mathbb{R}^1$. As for the nonnegativeness, the first two items remain unchanged. However, (d) is no longer true!

We present a counter-example. Since multiplying a upper semicontinuous function by a negative number turns it into a lower semicontinuous function, the functions $f_1 := \chi_{(-1,1)}$ and $f_n := -\chi_{[1/(n+1),1/n]}$, for $n > 1$, are all lower semicontinuous. Then, since $\sum_1^\infty f_n = \chi_{(-1,0]} + \chi_{(1/2,1)} - \chi_{\{1/k\,:\,k \geq 3\}}$, their sum is not lower semi-continuous. (Observe that $\{x \in \mathbb{R}^1 : \sum_1^\infty f_n > 1/2\} = (-1,0] \cup (1/2,1)$ is not open.) ■

Remark. If X is a metric space, a function f from X to the extended real line is lower semicontinuous if and only if

$$\liminf_{x \to p} f(x) \geq f(p)$$

for all $p \in X$. Similarly, it is upper semicontinuous if and only if

$$\limsup_{x \to p} f(x) \leq f(p)$$

for all $p \in X$.

Exercise 2.2 Let f be an arbitrary complex function on $\mathbb{R}^1$, and define

$$\begin{aligned} \varphi(x,\delta) &= \sup\{|f(s) - f(t)| : s,t \in (x-\delta, x+\delta)\}, \\ \varphi(x) &= \inf\{\varphi(x,\delta) : \delta > 0\}. \end{aligned}$$

Prove that φ is upper semicontinuous, that f is continuous at a point x if and only if $\varphi(x) = 0$, and hence that the set of points of continuity of an arbitrary complex function is a G_δ.

Formulate and prove and analogous statement for general topological spaces in place of $\mathbb{R}^1$.

■ Solution Let x be in the set $V = \{x : \varphi(x) < \alpha\}$. Then there exists δ such that $\varphi(x,\delta) < \alpha$. In this case, for every $y \in (x-\delta, x+\delta)$, take $\delta' := \delta - |y-x| > 0$. Since $(y-\delta', y+\delta') \subset (x-\delta, x+\delta)$, it follows that $\varphi(y) \leq \varphi(y,\delta') \leq \varphi(x,\delta) < \alpha$ and $y \in V$, which implies V is open and φ is upper semicontinuous.

Let us prove the characterization of continuity points.

($\implies$) If f is continuous in x then, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|x-t| < \delta \implies |f(x) - f(t)| < \epsilon$. Hence, if $s,t \in (x-\delta, x+\delta)$, then $|f(s) - f(t)| \leq |f(s) - f(x)| + |f(x) - f(t)| < 2\epsilon$. Since ϵ is arbitrary, $\varphi(x) = 0$.

($\impliedby$) Since $\varphi(x) = 0$, for every $\epsilon > 0$, we can find $\delta > 0$ such that $\varphi(x,\delta) < \epsilon$. Taking $s = x$ we get $|x-t| < \delta \implies |f(x) - f(t)| < \epsilon$ proving the continuity of f at x.

Let $V_n := \{x : \varphi(x) < \frac{1}{n}\}$. Then f is continuous in x if and only if $x \in \bigcap_{n=1}^\infty V_n$, which is a G_δ set.

The desired analogue is the following.

Let f be an arbitrary complex function on a topological space X, and define

$$\varphi(V) = \sup\{|f(s) - f(t)| : s, t \in V\}$$
$$\varphi(x) = \inf\{\varphi(V) : V \text{ neighborhood of } x\}.$$

Then φ is upper semicontinuous, f is continuous at a point x if and only if $\varphi(x) = 0$, and hence the set of points of continuity of an arbitrary complex function is a G_δ.

Let x be in the set $W = \{x : \varphi(x) < \alpha\}$. Then there exists a neighborhood of V of x such that $\varphi(V) < \alpha$. In this case, for every $y \in V$, V is also a neighborhood of y therefore

$$\varphi(y) \leq \varphi(V) < \alpha$$

and $y \in W$, which implies W is open and φ is upper semicontinuous.

Let us prove the characterization of continuity points.

($\Longrightarrow$) If f is continuous in x then, for every $\epsilon > 0$, there exists a neighborhood V of x such that $f(V) \subset B(f(x), \epsilon)$. Hence, if $s, t \in V$, then $|f(s) - f(t)| < \operatorname{diam} B(f(x), \epsilon) = 2\epsilon$. Since ϵ is arbitrary, $\varphi(x) = 0$.

($\Longleftarrow$) Since $\varphi(x) = 0$, for every $\epsilon > 0$, we can find V such that $\varphi(V) < \epsilon$. Taking $s = x$ we get $t \in V \implies f(t) \in B(f(x), \epsilon)$ proving the continuity of f at x.

Let $W_n := \{x : \varphi(x) < \frac{1}{n}\}$. Then f is continuous in x if and only if $x \in \bigcap_{n=1}^{\infty} W_n$, which is a G_δ set. ∎

EXERCISE 2.3 Let X be a metric space, with metric ρ. For any nonempty $E \subset X$, define

$$\rho_E(x) = \inf\{\rho(x, y) : y \in E\}.$$

Show that ρ_E is a uniformly continuous function on X. If A and B are disjoint nonempty closed subsets of X, examine the relevance of the function

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

∎ SOLUTION First of all, note that

$$\rho(x, z) \leq \rho(x, y) + \rho(y, z)$$

for all $x, y \in X$ and $z \in E$. Thus

$$\rho_E(x) - \rho_E(y) \leq \rho(x, y)$$

and with a symmetric argument, we concluded

$$|\rho_E(x) - \rho_E(y)| \leq \rho(x, y).$$

In other words, ρ_E is a Lipschitz function and therefore, an uniformly function.

■

EXERCISE 2.4 Examine the proof of the Riesz theorem and prove the following two statements:

(a) If $E_1 \subset V_1$ and $E_2 \subset V_2$, where V_1 and V_2 are disjoint open sets, then $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$, even if E_1 and E_2 are not in $\mathfrak{M}$.

(b) If $E \in \mathfrak{M}_F$, then $E = N \cup K_1 \cup K_2 \cup \cdots$, where $\{K_i\}$ is a disjoint countable collection of compact sets and $\mu(N) = 0$.

■ SOLUTION

Recall that for any subset E, $\mu(E)$ is defined as the infimum of $\mu(V)$ for open sets V containing E.

(a) Firstly, by the first step of the proof we have that $\mu(E_1 \cup E_2) \leq \mu(E_1) + \mu(E_2)$ for all subsets of X. Now, let V be any open set containing $E_1 \cup E_2$. We may assume that V is a subset of $V_1 \cup V_2$, as $V \cap (V_1 \cup V_2)$ is still an open set containing $E_1 \cup E_2$. Since V_1 and V_2 are disjoint,

$$\mu(V) = \mu(V \cap V_1) + \mu(V \cap V_2) \geq \mu(E_1) + \mu(E_2).$$

Taking the infimum of all open V containing $E_1 \cup E_2$, we have that

$$\mu(E_1 \cup E_2) \geq \mu(E_1) + \mu(E_2).$$

The result follows.

Recall that $\mathfrak{M}_F$ is the class of all E with finite measure such that $\mu(E)$ is equal to the supremum of $\mu(K)$ for compact sets K contained in E.

(b) Let $E_1 = E$. By step V in the proof, there is a compact K_1 and an open set V_1 such that

$$K_1 \subset E_1 \subset V_1 \qquad \text{and} \qquad \mu(V_1 - K_1) < 1.$$

Step VI now implies that $E_2 := E_1 - K_1$ is in $\mathfrak{M}_F$. Inductively we find compacts K_n and open sets V_n such that

$$K_n \subset E_n \subset V_n \qquad \text{and} \qquad \mu(V_n - K_n) < 1/n$$

for $E_n := E_{n-1} - K_{n-1}$. We now set $N = E - \bigcup_n K_n$. By construction, $\{K_n\}$ is a disjoint countable collection of compact sets, all contained in E. Lastly, since $N \subset V_n - K_n$, $\mu(N) < 1/n$ for all n. The result follows. ■

EXERCISE 2.5 Let E be a Cantor's familiar "middle thirds" set. Show that $m(E) = 0$, even though E and $\mathbb{R}^1$ have the same cardinality.

We use the notation of Rudin's *Principles of Mathematical Analysis*, excluding only the fact that it denotes Cantor's set by P.

■ SOLUTION Let E_n be the n-th step in the construction of the Cantor set. Since $E = \bigcap_n E_n$ and $E_{n+1} \subset E_n$, it follows that $m(E) = \lim_n m(E_n)$. Furthermore, E_n is the union of 2^n disjoint intervals, each one of length $1/3^n$. This implies that $m(E_n) = (2/3)^n$ and so $m(E) = 0$. Now, Rudin's first book, *Principles of Mathematical Analysis*, shows that E is a perfect subset of $\mathbb{R}$. We conclude that E and $\mathbb{R}$ have the same cardinality. ∎

EXERCISE 2.6 Construct a totally disconnected compact set $K \subset \mathbb{R}^1$ such that $m(K) > 0$. (K is to have no connected subset consisting of more than one point.)

If v is lower semicontinuous and $v \leq \chi_K$, show that actually $v \leq 0$. Hence χ_K cannot be approximated from below by lower semicontinuous functions, in the sense of the Vitali-Carathéodory theorem.

∎ SOLUTION We'll construct a variation of the familiar Cantor set by removing middle fourths instead of middle thirds. This variation, and modifications thereof, is named *fat Cantor set* or *Smith-Volterra-Cantor set*.

As described, we begin with $K_0 = [0, 1]$. Inductively we define K_{n+1} to by removing the middle fourth of every connected component of K_n. That is, we remove the middle open interval of length $1/4^{n+1}$ of each one of the 2^n connected components of K_n. Finally, we let

$$K = \bigcap_{n=0}^{\infty} K_n.$$

Being the intersection of closed sets and bounded, K is compact. Also, since in every step we divide each connected subset into two pieces, K has no connected subset consisting of more than one point. Lastly, since we started with a set of measure 1 and in each step removed 2^n intervals of length $1/4^{n+1}$, we have that

$$m(K) = 1 - \sum_{n=0}^{\infty} \frac{2^n}{4^{n+1}} = \frac{1}{2}.$$

If v is lower semicontinuous and $v \leq \chi_K$, then the set

$$\{x \in \mathbb{R}^1 : v(x) > 0\}$$

is open and a subset of K. The fact that K contains no intervals implies that this set is empty. It follows that $v \leq 0$. ∎

EXERCISE 2.7 If $0 < \epsilon < 1$, construct an open set $E \subset [0, 1]$ which is dense in $[0, 1]$, such that $m(E) = \epsilon$. (To say that A is dense in B means that the closure of A contains B.)

∎ SOLUTION Using variations of the fat Cantor set constructed in the preceding exercise we can have totally disconnected compact subsets of $[0, 1]$ with any measure in $(0, 1)$.

Let K be such a set with measure $1 - \epsilon$. Since K is closed and totally disconnected, its complement $[0, 1] - K$ is the desired open subset of

Removing intervals of length α^{n+1} in each step we get a set with measure $(1 - 3\alpha)/(1 - 2\alpha)$, which can be any number from 0 to 1 by choosing some $\alpha \in (0, 1/3)$.

$[0,1]$ which is dense and has measure ϵ. In fact, it is dense since K has empty interior and the closure of the complement is the complement of the interior. ∎

EXERCISE 2.8 Construct a Borel set $E \subset \mathbb{R}^1$ such that

$$0 < m(E \cap I) < m(I)$$

for every nonempty segment I. Is it possible to have $m(E) < \infty$ for such a set?

This result is basically an article by W. Rudin himself. (*Amer. Math. Monthly, vol. 90, no. 1, 1983, pp. 41–42.*) We adapted his proof (done for the interval $[0,1]$ in the place of the real line).

∎ SOLUTION Let $V_1, V_2, \ldots$ be a countable base for the topology of $\mathbb{R}$. Since every interval contains a fat Cantor set, we define A_1 and B_1 to be disjoint fat Cantor sets contained in V_1. As $V_2 - (A_1 \cup B_1)$ is open, it contains another two disjoint fat Cantor sets A_2 and B_2. Inductively we construct disjoint fat Cantor sets A_n and B_n, subsets of

$$V_n - \left(\bigcup_{k=1}^{n-1} (A_k \cup B_k) \right).$$

We set

$$E = \bigcup_{n=1}^{\infty} A_n.$$

If I is a nonempty segment, then it contains at least one V_n. So, it contains A_n and B_n. Thus,

$$0 < m(A_n) \leq m(E \cap I) < m(E \cap I) + m(B_n) \leq m(I),$$

where the last inequality follows from the fact that B_n and A_m are disjoint for all m.

Finally, E can be constructed in such a way to have a finite measure. Following the discussion about fat Cantor sets in the preceding exercises, we can make $\{m(A_n)\}$ be a summable sequence. Then $m(E) \leq \sum_n m(A_n) < \infty$. ∎

EXERCISE 2.9 Construct a sequence of continuous functions f_n on $[0,1]$ such that $0 \leq f_n \leq 1$, such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0,$$

but such that the sequence $\{f_n(x)\}$ converges for no $x \in [0,1]$.

∎ SOLUTION For each $n = 1, 2, 3, \ldots$ and each $m = 0, 1, \ldots, n-1$ let

$$V_m^n = \left(-1 + m \cdot \frac{3}{n}, -1 + (m+1) \cdot \frac{3}{n} \right) = \left(\frac{3m-n}{n}, \frac{3m-n+3}{n} \right)$$

and

$$\begin{aligned} K_m^n &= \left[-1 + m\cdot\frac{3}{n} + \frac{1}{2n}, -1 + (m+1)\cdot\frac{3}{n} - \frac{1}{2n}\right] \\ &= \left[\frac{6m-2n+1}{2n}, \frac{6m-2n+5}{2n}\right]. \end{aligned}$$

It is easy to see that

$$K_m^n \subset V_m^n \subset (-1,2) \subset [-1,2] \quad (m = 0,1,\ldots,n-1 \text{ and } n = 1,2,3,\ldots).$$

Fixing a positive integer n a an integer m with $0 \le m \le n-1$, the construction made explicit in the proof of the Urysohn's Lemma (theorem 2.12) provides a function $g_m^n \in C_c[-1,2]$ such that $K_m^n \prec g_m^n \prec V_m^n$. Let f_m^n be the restriction of g_m^n to $[0,1]$, for all $n = 1,2,3,\ldots$ and all $m = 0,1,\ldots,n-1$. The sequence $\{f_m^n\}$ is exactly what we are looking for. Indeed, for each positive integer n and all $m = 0,1,\ldots,n-1$, we have $\chi_{[0,1]\cap K_m^n} \le f_m^n \le \chi_{[0,1]\cap V_m^n}$, which implies

$$0 \le \int_0^1 f_m^n \, dx \le \int_0^1 \chi_{V_m^n} \, dx \le \int_{-1}^2 \chi_{V_m^n} \, dx = \frac{3}{n}.$$

Then clearly $\lim_{n\to\infty} \int_0^1 f_m^n(x)\, dx = 0$.

It is straightforward that $K_m^n \subset \left(-1, \frac{2m+1}{m+1}\right]$, for all $m = 0,1,2,\ldots$ and all $n = m+1, m+2, \ldots$ So $\bigcup_{n=m+1}^{\infty} K_m^n \subset \left(-1, \frac{2m+1}{m+1}\right]$, for all $m = 0,1,2,\ldots$ We claim that

$$\bigcup_{n=m+1}^{\infty} K_m^n = \left(-1, \frac{2m+1}{m+1}\right], \quad m = 0,1,2,\ldots$$

First notice that $K_m^{m+2} \cap K_m^{m+1} = \left[\frac{2m-1}{m+2}, \frac{2m+1}{m+1}\right] \ne \varnothing$, since $2m^2 + m - 1 = (2m-1)(m+1) < (2m+1)(m+2) = 2m^2 + 5m + 2$. Suppose that $K_m^{m+(k+1)} \cap K_m^{m+k} = [] \ne \varnothing$

Indeed, let $x \in \left(-1, \frac{2m+1}{m+1}\right]$.

■

Exercise 2.10 If $\{f_n\}$ is a sequence of continuous functions on $[0,1]$ such that $0 \le f_n \le 1$ and such that $f_n(x) \to 0$ as $n \to \infty$, for every $x \in [0,1]$, then

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0.$$

Try to prove this without using any measure theory of any theorems about Lebesgue integration. (This is to impress you with the power of the Lebesgue integral. A nice proof was given by W. F. Eberlein in *Communications on Pure and Applied Mathematics*, vol. X, pp. 357-360, 1957.)

■ SOLUTION Let us first state some basic results about integration. Remember the space of continuous functions on $[0,1]$ is an inner product space with the inner product $(f,g) := \int_0^1 f(x)g(x)\,dx$, and denote $\|f\|_\infty := \sup_{[0,1]} f(x)$, $\|f\|_1 := \int_0^1 |f|\,dx$ and $\|f\|_2 := (f,f)^{\frac{1}{2}} = \left(\int_0^1 f(x)^2\,dx\right)^{\frac{1}{2}}$. Then, by the linear algebra classes,

(a) *(Cauchy-Schwarz Inequality)* $|(f,g)| \le \|f\|_2 \cdot \|g\|_2$,

(b) *(Parallelogram Law)* $\|f+g\|_2^2 + \|f-g\|_2^2 = 2\left(\|f\|_2^2 + \|g\|_2^2\right)$,

(c) $\|f\|_1 = (|f|,1) \le \|f\|_2 \cdot \|1\|_2 = \|f\|_2$,

and by the basics of the integration theory

$$\|f\|_1 \le \|f\|_\infty,$$
$$\|f\|_2 \le \|f\|_\infty^2.$$

We will also need the following lemma.

LEMMA. If $|f| \le \sum_{n=1}^{\infty} |f_n|$, then

$$\int_0^1 |f(x)|\,dx \le \sum_{n=1}^{\infty} |f_n(x)|\,dx.$$

Its proof is simple: given $\epsilon > 0$, for each $x \in [0,1]$ there exists an $N(x)$ such that $|f(x)| < \sum_{n=1}^{N(x)} |f_n(x)| + \epsilon$. Since all of these functions are continuous, there exists an neighborhood $U(x)$ of x such that there inequality is still valid. Since the set is compact, there is a finite cover $\{U(x_i)\}$ and if we let $N = \max\{N(x_i)\}$, then $|f| < \sum_{n=1}^{N} |f_n| + \epsilon$, which implies the desired result.

Let's return to the problem. Since $\|f_n\|_\infty \le 1$, then $\|f_n\|_1 \le 1$ and its $\limsup$ is well-defined. Moreover, it is sufficient to prove the $\limsup$ is equal to 0 to conclude the result. Suppose $\{\|f_n\|_1\}$ converges to the $\limsup$ (otherwise pass to a subsequence). Define K_n as the *convex hull* of the set $\{f_m : m \ge n\}$ (i.e., the set of all functions of the form $\sum a_j f_{m_j}$ for $a_j \ge 0$, $\sum a_j = 1$ and $m_j \ge n$. Clearly, any sequence $\{g_n\}$ such that $g_n \in K_n$ satisfies the same hypothesis of f_n. Let $d_n := \inf\{\|g\|_2 : g \in K_n\}$. Since $K_{n+1} \subset K_n$, then $d_n \le d_{n+1} \le 1$, and $d = \lim_n d_n$ is well-defined. Now choose $g_n \in K_n$ such that $\|g_n\|_2 \le d_n + \frac{1}{n}$.

LEMMA. $\lim_{n,m} \|g_n - g_m\|_2 = 0$.

Its proof is a famous trick in the Hilbert Space Theory. By the parallelogram law,

$$\|g_n - g_m\|_2^2 = 2\left(\|g_n\|_2^2 + \|g_m\|_2^2\right) - 4\left\|\frac{g_n + g_m}{2}\right\|_2^2.$$

If $n \geq m$, then $\frac{g_n+g_m}{2} \in K_m$ and

$$\left\| \frac{g_n + g_m}{2} \right\| \geq d_m.$$

Thus

$$\|g_n - g_m\|_2^2 \leq 2\left(\left(d_n + \frac{1}{n}\right)^2 + \left(d_m + \frac{1}{m}\right)^2\right) - 4d_m^2 \to 4d^2 - 4d^2 = 0,$$

as $m, n \to \infty$ (and $n \geq m$).

Now, let $\{h_n\}$ be a subsequence of $\{g_n\}$ such that

$$\sum_{n=1}^{\infty} \|h_n - h_{n+1}\|_2 < \infty.$$

(Take, for example, k_n such that $\|g_i - g_j\|_2 < \frac{1}{2^n}$ if $i, j \geq k_n$). Since $\lim h_n = 0$, then $h_n = \sum_{m=n}^{\infty}(h_m - h_{m+1})$ and $|h_n| = \sum_{m=n}^{\infty}|h_m - h_{m+1}|$, and we can use the first lemma to conclude

$$\|h_n\|_1 \leq \sum_{m=n}^{\infty} \|h_m - h_{m+1}\| \to 0$$

as $n \to \infty$. Since, by hypothesis, $\|h_n\| \to \limsup_n\{\|f_n\|_1\}$, we conclude the solution. ∎

EXERCISE 2.11 Let μ be a regular Borel measure on a compact Hausdorff space X; assume $\mu(X) = 1$. Prove that there is a compact $K \subset X$ (the *carrier* or *support* of μ) such that $\mu(K) = 1$ but $\mu(H) < 1$ for every proper compact subset H of K. *Hint:* Let K be the intersection of all compact K_α with $\mu(K_\alpha) = 1$; show that every open set V which contains K also contains some K_α. Regularity of μ is needed; compare Exercise 18. Show that K^c is the largest open set in X whose measure is 0.

∎ SOLUTION Let $K = \bigcap K_\alpha$ be the intersection of all compact K_α such that $\mu(K_\alpha) = 1$. By definition, if H is a proper compact subset of K, then $\mu(H) < 1$. In this case, it follows also that K^c is the largest open set in X whose measure is 0, otherwise if V is open and $V \not\subset K^c$, then $V^c \cap K$ is a compact set strictly contained in K whose measure is 1. It is only left to prove $\mu(K) = 1$. We'll need the following theorem, present in Rudin's Principles of Mathematical Analysis (although only for metric spaces).

THEOREM 2.1 Finite Intersection Characterization. If $\{K_\alpha\}$ is a collection of compact subsets of a compact topological space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.

Let V be an open set containing K. Then $\{K_\alpha\}$ and V^c form a collection of compact sets whose intersection is empty. Therefore there exists a finite subcollection whose intersection is finite.

Since $\mu(K_{i_1} \cap \cdots \cap K_{i_n}) = 1$ (and, therefore, this set is not empty), this subcollection must contain V^c, and $\tilde{K} = K_{i_1} \cap \cdots \cap K_{i_n}$ is a compact set such that $\mu(\tilde{K}) = 1$ and $\tilde{K} \subset V^c = \varnothing$, i.e., $\tilde{K} \subset V$. Therefore $\mu(V) = 1$. Since $K \subset V$ is an arbitrary open set, then

$$\mu(K) = \inf\{\mu(V) : K \subset V, V \text{ is open}\}$$

and the proof is complete. ∎

Remark. The support of a measure is something more general. For *any* measure μ in a topological space (X, τ), we define

$$\operatorname{supp}(\mu) = \{x \in X : x \in V \in \tau \implies \mu(V) > 0\}.$$

This idea is important to carry measure-theoretic properties of f to topological properties of $f\big|_{\operatorname{supp}(\mu)}$.

For example, in Dynamical Systems, it is the bridge between Ergodic Theory (which studies measure-preserving transformations) and Topological Dynamics (which studies continuous transformations).

EXERCISE 2.12 Show that every compact subset of R^1 is the support of a Borel measure.

∎ SOLUTION Let $K \subset R^1$ be compact. Since R^1 is a metric space, K is separable. Let $S \subset K$ be a countable dense subset:

$$S = \bigcup_{i=1}^{\infty} p_i.$$

Now let μ_i be the unit mass (Borel) measure concentrated at p_i. Define $\mu : \mathscr{B} \to [0, \infty]$ as the sum of these measures:

$$\mu := \sum_{i=1}^{\infty} \mu_i.$$

An enumerable sum of measures is again a measure, therefore μ is a Borel measure.

Since S was dense in K and is the set where $\mu > 0$, $\operatorname{supp}\mu = K$, which is what was to be proved. ∎

EXERCISE 2.13 Is it true that every compact subset of $\mathbb{R}^1$ is the support of a continuous function? If not, can you describe the class of all compact sets in $\mathbb{R}^1$ which are supports of continuous functions? Is your description valid in other topological spaces?

∎ SOLUTION Since the support of a continuous function f is the closure of

$$\{x : f(x) \neq 0\} = f^{-1}(\mathbb{R}^1 \setminus \{0\}),$$

every support is the closure of an open set. This obviously is not the case for a singleton, for example. However, every compact which is the closure of an open set is the support of a continuous function. In fact, if V is open and $K = \overline{V}$ is compact, then $x \mapsto d(x, \mathbb{R}^1 \setminus V)$ is continuous with support K.

The same argument holds for functions $f : X \to \mathbb{R}^1$, where X is a metric space. Also, this characterization holds for every subset of X, not just the compact ones.

To study this property for more general spaces, let us first introduce some definitions.

Definition Let X be a topological space.

(a) The space X is called *normal* if every two disjoint closed sets of X have disjoint open neighborhoods.

(b) The space X is called T_4 if it is normal and Hausdorff.

(c) The space X is called *completely normal* or *hereditarily normal* if every subspace of X with the subspace topology is normal.

(d) The space X is called *completely* T_4 or T_5 if it is *completely normal* and Hausdorff.

(e) The space X is called *perfectly normal* if it is normal and every open set is F_σ (equivalently, if every closed set is G_δ).

(f) The space X is called *perfectly* T_4 or T_6 if it is perfectly normal and Hausdorff.

We can now enunciate Urysohn's lemma in its full generality.

LEMMA. *(Urysohn's Lemma).* If A and B are closed sets in a normal space X, then there exists a continuous function $f\colon X \to [0,1]$ such that $f(a) = 0$ for every $a \in A$ and and $f(b) = 1$ for every $b \in B$.

In particular, Theorem 2.12 is valid for every normal space (even if the compact subset K is changed to closed subset).

With this result in mind, we can prove this characterization is always valid for every perfectly normal space X. In fact, let V be an open set of X and $\{C_n\}$ be the sequence of closed sets such that $\bigcup_n C_n = V$. By Urysohn's lemma, for each C_n, there exists a function $f_n\colon X \to [0,1]$ such that $f_n(x) = 1$ for every $x \in C_n$ and the support of f_n lies is V. Therefore, the functions $f\colon X \to [0,1]$ defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{f_n(x)}{2^n}$$

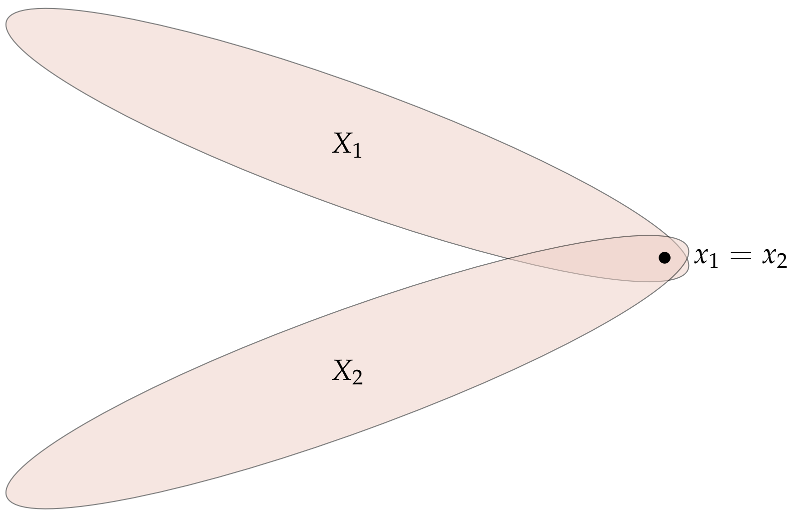
is a continuous function, by Weierstrass M-test, and its support is $\overline{V}$. Moreover, since every metric space is always perfectly normal, this case includes the previous one.

As the reader may have noticed, there is an even more general characterization, which we have essentially proved.

> LEMMA. A subset A of a normal space X is the support of a continuous function $f\colon X \to [0,1]$ if and only if it is the closure of an open F_σ-set.

To find a strange pathology for our initial conjecture, C is the support of a continuous function if and only if it is the closure of an open set, we will find an open set which is not F_σ in a normal space.

Let ω_1 be the first uncountable ordinal. Then $\Omega = S(\omega_1) = \omega_1 \cup \{\omega_1\}$ is the second uncountable ordinal (its successor), and we can endow with the order topology with $\max \Omega = \omega_1$. This space is Hausdorff, as any other space with the order topology, and compact. In fact, let $\mathcal{C}$ be a open cover of Ω. Since $\omega_1 \in \Omega$, there exists $a_0 \in \Omega$ such that $(a_0, \omega_1]$ is in the open cover, further there exists $a_1 < a_0$ such that $a_0 \in (a_1, b_1)$ (for some b_1) or $a_0 \in (a_1, \omega_1]$, and so on. Since Ω is well-ordered, this process must stop. Finally, there exists an open set of the form $[0, b)$ containing 0, and it must also contain a_n because it is minimal in our procedure; hence we have found our finite subcover. Moreover - and this is the most important step - any continuous function $f\colon \Omega \to \mathbb{R}$ is constant on a neighborhood of ω_1. To prove this, let $f(\omega_1) = \alpha$ and consider the sets $f^{-1}\left(\left(\alpha - \frac{1}{n}, \alpha + \frac{1}{n}\right)\right) = A_n \supset (a_n, \omega_1]$ with $a_n \leq a_{n+1}$. If $a_n \to \omega_1$, then $\omega_1 = \bigcup_n a_n$, which is impossible, since the sets a_n are countable, but ω_1 is uncountable. Therefore, $\sup a_n = a < \omega_1$, and $(a, \omega_1] = \bigcap_n (a_n, \omega_1] \subset f^{-1}(\alpha)$. Furthermore, Ω is a T_5-space, as is every space endowed with the order topology (in particular, it is normal). Now let X_1 and X_2 be two disjoint copies of Ω and x_1 and x_2 be its maxima, respectively, and let X be the quotient of $X_1 \cup X_2$ with respect to the equivalence relation $x_1 \sim x_2$ ("glue" these points).



Then X is normal, since union of normal spaces is normal, and quotient by closed set/equivalence relation is also normal. Moreover, X contains a copy of X_1 in it, viz. $X_1 = \{x \in X_1 : x < x_1\} \cup \{x_1\} = \overline{\{x \in X_1 : x < x_1\}}$, which is compact and the closure of an open set. Nonetheless, this copy is not the support of any continuous function. Indeed, suppose X_1 is the closure of a continuous function $f\colon X \to \mathbb{R}$. Then $f(x) = 0$ in the interior of X_2. As we have seen, this implies $f(x_1) = 0$. Yet f is constant in a neighborhood of x_1, therefore the support of f is strictly smaller then X_1. This happens because the interior of X_1 is not F_σ, otherwise, as we have argued above, $\omega_1 = \bigcup_n a_n$ for some a_n countable.

Moreover, there is some related result for different spaces.

LEMMA. Let X be a locally compact Hausdorff space. A compact space K is the support of a continuous function if and only if it is the closure of an open F_σ-set.

Observe the condition is necessary for the set to be the support of a function in any topological space. In fact, it is the closure of an open set, by the first observation in this solution, and this open set is F_σ, since

$$f^{-1}(\mathbb{R}^1 \setminus \{0\}) = \bigcup_{n=1}^{\infty} f^{-1}(\mathbb{R}^1 \setminus (-1/n, 1/n)).$$

Reciprocally, if K is compact, this property is sufficient. In effect, for each $x \in K$, let $V(x)$ be and open neighborhood with compact closure. Since K is compact and these neighborhoods cover K, there exists a finite cover whose union we call V, which satisfies that $\overline{V}$ is compact. Moreover, K is the closure of an open F_σ-set in $\overline{V}$ and, since $\overline{V}$ is Hausdorff and compact, it is also normal. Hence by what we have already proved, K is the support of a continuous function defined in $\overline{V}$, but we can extend this function to the whole space defining it to be 0 outside $\overline{V}$, and we have proved the lemma.

Nonetheless, this characterization may not be useful or even true in some more general spaces. For example, first note that there are topological spaces such that there does not exists nonempty compact spaces K such that K is the support of a continuous function. For example, endow $\mathbb{R}$ with the *co-countable topology* defined by $E \subset \mathbb{R}$ is open if and only if $E = \mathbb{R}$ or E^c is countable (we leave to the reader the task of showing this is in fact a topology). In this topology the only compact subsets are the finite ones. On the other hand, if V is a non-empty open set, then $\overline{V}$ must be the whole space X. Thus the support of a continuous function is either $\varnothing$ or X, and no non-empty compact set can be the support of a continuous function (more generally, every real-valued continuous function in this space is constant).

Actually, this is a particular case of a more general result.

LEMMA. Let X is a topological space and $x \in X$ such that for every open set V and every neighborhood $N(x)$ of x, $V \cap N(x) \neq$. Then every continuous function $f \colon X \to \mathbb{R}$ is constant. (Actually, this holds for every continuous function $f \colon X \to Y$ for any Hausdorff space Y.)

In fact, suppose $y \in X$ is such that $f(y) \neq f(x)$. Then there exists disjoint neighborhoods U, V of $f(y)$ and of $f(x)$, but then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint. A contradiction, since $f^{-1}(V)$ is a neighborhood of x.

Thus we may try to find some pathological spaces with this property. In fact, given a topological space X, define its *one-point compactification* (also called *Alexandroff compactification*) X^* as the set $X^* = X \cup \{\infty\}$ for some $\infty \notin X$ endowed with the topology generated by the open sets of X and the sets $X^* - K$ for some compact set $K \subset X$. Out final example will be $\mathbb{Q}^*$, the one-point compactification of the rational numbers. Note that every compact set K in $\mathbb{Q}$ has empty interior. In fact, if $(a, b) \subset K$, let α be some irrational number in (a, b). Then the function $f \colon K \to \mathbb{R}$ defined by $f(x) = \frac{1}{x-\alpha}$ is continuous and unbounded. Now, observe that $\mathbb{Q}^*$ satisfies the hypothesis of our lemma with $x = \infty$. In fact, if $V \cap N(\infty) = \varnothing$ for some open set V and some neighborhood $N(\infty) = \mathbb{Q}^* - K$, then $V \subset K$, which cannot happen. Therefore every real-valued continuous function on $\mathbb{Q}^*$ is constant. On the other hand, given any open set $V \subset \mathbb{Q}$ which is not dense, for example $V = (0, 1) \cap \mathbb{Q}$, then $\mathrm{cl}_{\mathbb{Q}^*}(V) = \mathrm{cl}(V) \cup \{\infty\} \neq \mathbb{Q}^*$ is not the support of a continuous function. Thus we have found a compact T_1 topological space $\mathbb{Q}^*$ and a compact subset $([0, 1] \cap \mathbb{Q}) \cup \{\infty\}$ whose interior is an open F_σ-set (since $\mathbb{Q}$ is countable). ∎

EXERCISE 2.14 Let f be a real-valued Lebesgue measurable function on $\mathbb{R}^k$. Prove that there exist Borel functions g and h such that $g(x) = h(x)$ a.e. $[m]$, and $g(x) \leq f(x) \leq h(x)$ for every $x \in \mathbb{R}^k$.

∎ SOLUTION Assume first that $f \geq 0$ and let $\{s_n\}, E_{n,i}$ and F_n be as in the proof of Theorem 1.17. Since $\mathbb{R}^k$ is σ-compact, we can use Theorem 2.17 to conclude there are $A_{n,i}$ and $B_{n,i}$ Borel measurable sets such that $A_{n,i} \subset E_{n,i} \subset B_{n,i}$ and $m(B_{n,i} - A_{n,i}) = 0$, and similarly for $C_n \subset F_n \subset D_n$. Therefore, if we define

$$r_n = \sum_{i=0}^{n2^n} \frac{i-1}{2^n} \chi_{A_{n,i}} + n\chi_{C_n}$$

$$t_n = \sum_{i=0}^{n2^n} \frac{i-1}{2^n} \chi_{B_{n,i}} + n\chi_{D_n},$$

then r_n and t_n are Borel measurable functions such that $r_n(x) = t_n(x)$ a.e. $[m]$ and $r_n \leq s_n \leq t_n$. By the Lebesgue's Monotone Convergence Theorem, r_n and t_n converge to Borel measurable functions g and h, respectively, such that $g(x) = h(x)$ a.e $[m]$ and $g(x) \leq f(x) \leq h(x)$ for every $x \in \mathbb{R}^k$. In the general case, write $f = f^+ - f^-$, construct g^+ and h^+ for f^+ and g^- and h^- for f^-. Then $g = g^+ - h^-$ and $h = h^+ - g^-$ are the desired functions. ∎

EXERCISE 2.15 It is easy to guess the limits of

$$\int_0^n \left(1 - \tfrac{x}{n}\right)^n e^{\frac{x}{2}}\, dx \text{ and } \int_0^n \left(1 + \tfrac{x}{n}\right)^n e^{-2x}\, dx$$

as $n \to \infty$. Prove that your guesses are correct.

∎ SOLUTION Firstly, consider the functions

$$F_n(x) = \chi_{[0,n]} \left(1 - \tfrac{x}{n}\right)^n e^{\frac{x}{2}}$$

and

$$G_n(x) = \chi_{[0,n]} \left(1 + \tfrac{x}{n}\right)^n e^{-2x},$$

for all $n \in \mathbb{N}$. Then for all $x \in \mathbb{R}$,

Remember that $1 + x \leq e^x$ and $\lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^n = e^{-x}$ for all $x \in \mathbb{R}$

$$|F_n(x)| = \left|\chi_{[0,n]} \left(1 - \frac{x}{n}\right)^n e^{\frac{x}{2}}\right| \leq e^{-\frac{x}{2}}$$
$$|G_n(x)| = \left|\chi_{[0,n]} \left(1 + \frac{x}{n}\right)^n e^{-2x}\right| \leq e^{-x}$$

and

$$\lim_{n\to\infty} |F_n(x)| = e^{-\frac{x}{2}}$$
$$\lim_{n\to\infty} |G_n(x)| = e^{-x}$$

Applying Lebesgue's Dominated Convergence Theorem, it follows that

$$\lim_{n\to\infty} \int_0^\infty F_n(x)\, dx = \int_0^\infty \lim_{n\to\infty} F_n(x)\, dx = \int_0^\infty e^{-\frac{x}{2}}\, dx = 2$$

and

$$\lim_{n\to\infty} \int_0^\infty G_n(x)\, dx = \int_0^\infty \lim_{n\to\infty} G_n(x)\, dx = \int_0^\infty e^{-x}\, dx = 1.$$

The result follows. ∎

EXERCISE 2.16 Why is $m(Y) = 0$ in the proof of Theorem 2.20(e)?

∎ SOLUTION Surely we could partition Y into k-cells, thicken those cells a small bit and then show that $m(Y) < \epsilon$ for all $\epsilon > 0$. However, this seems to be horrible to write down and not very enlightening. So we'll take a different approach, which consists of a simple application of a lemma that is interesting by itself.

LEMMA. Let $(X, \mathfrak{M}, \mu)$ be a σ-finite measure space and $\{E_i : i \in I\}$ be a collection of disjoint measurable subsets with positive measure. Then I is at most countable.

If you never saw sums over a arbitrary collection of positive numbers before, take a look at section 4.15. The main fact we use here is that these sums converge only if the number of positive numbers is at most countable.

Proof. We write X as a union of a countable collection of sets with finite measure

$$X = \bigcup_{k \in K} X_k$$

and, for a fixed $k \in K$, let $a_i := \mu(E_i \cap X_k)$. Observe that, since

$$\sum_{i \in I} a_i = \sup_{\substack{J \subset I \\ J \text{ finite}}} \sum_{j \in J} a_j = \sup_{\substack{J \subset I \\ J \text{ finite}}} \mu\left(\bigcup_{j \in J} E_j \cap X_k\right) \leq \mu(X_k) < \infty,$$

there is at most a countable number of the a_i that are positive. In other words, inasmuch as a countable union of (at most) countable sets is (at most) countable, the set

$$\{(i,k) \in I \times K : \mu(E_i \cap X_k) > 0\}$$

is at most countable. As the E_i have positive measure, we know that for every $i \in I$ there exists some $k \in K$ such that $\mu(E_i \cap X_k) > 0$. The result follows. □

Now, lets suppose that $m(Y) > 0$ and let $x \in \mathbb{R}^k$ be a vector not in Y. The collection $\{\alpha x + V : \alpha \in \mathbb{R}\}$ satisfies all the criteria of our lemma (by translation invariance). Thus, having $m(Y) > 0$ would imply that $\mathbb{R}$ is at most countable, which is absurd! ■

Remark. Using Theorem 2.20(e), there's a much simpler proof of the fact that a proper subspace Y of $\mathbb{R}^k$ has measure zero. Let T be a linear transformation with Y as image. Since

$$m(Y) = m(T(\mathbb{R}^k)) = |\det T|\, m(\mathbb{R}^k)$$

and T does not have full rank (so $\det T = 0$), the result follows.

EXERCISE 2.17 Define the distance between points (x_1, y_1) and (x_2, y_2) in the plane to be

$$|y_1 - y_2| \text{ if } x_1 = x_2, \quad 1 + |y_1 - y_2| \text{ if } x_1 \neq x_2.$$

Show that this is indeed a metric, and that the resulting metric space X is locally compact.

If $f \in C_c(X)$, let $x_1, x_2, \ldots, x_n$ be those values of x for which $f(x,y) \neq 0$ for at least one y (there are only finitely many such x!), and define

$$\Lambda f = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f(x_j, y)\, dy.$$

Let μ be the measure associated with this Λ by Theorem 2.14. If E is the x-axis, show that $\mu(E) = \infty$ although $\mu(K) = 0$ for every compact $K \subset E$.

■ SOLUTION Let $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ be the distance defined above, we'll prove it is a metric.

(a) $d((x_1, y_1), (x_2, y_2)) = 0 \implies (x_1, y_1) = (x_2, y_2)$. In fact, a distance less than 1 implies $x_1 = x_2$, and

$$|y_1 - y_2| = d((x_1, y_1), (x_2, y_2)) = 0.$$

Therefore $(x_1, y_1) = (x_2, y_2)$.

(b) $d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1))$. In fact, this follows from the symmetry of the equality and from the fact that

$$|y_1 - y_2| = |y_2 - y_1|.$$

(c) To prove the triangular inequality

$$d((x_1, y_1), (x_3, y_3)) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)),$$

we just have to separate in cases. For example, if $x_1 = x_2 = x_3$, then the inequality becomes $|y_1 - y_3| \leq |y_1 - y_2| + |y_2 - y_3|$, which is true. The other cases are analogous.

Observe that if $x_1 \neq x_2$, then $d((x_1, y_1), (x_2, y_2)) \geq 1$, and the balls of radius less than 1 are vertical open intervals. Therefore we have the following characterization:

A set $E \subset \mathbb{R}^2$ is open in the topology induced by d if and only if its intersection with every vertical line is an open subset of the line.

($\implies$) Let V be an open set in the topology induced by d. For any $x = (\xi_1, \xi_2) \in V$, there exists r' such that $B_d(x, r) \in V$ for any $r < r'$ (where $B_d(x, r)$ is the ball of center x and radius r with the metric d). Let ℓ be a vertical line such that its intersection with E is nonempty (otherwise the assertion in vacuously true). Taking $x \in \ell \cap E$ and $r < \min\{1, r'\}$, then $\{\xi_1\} \times (\xi_2 - r, \xi_2 + r) \subset \ell \cap E$, and this intersection is open.

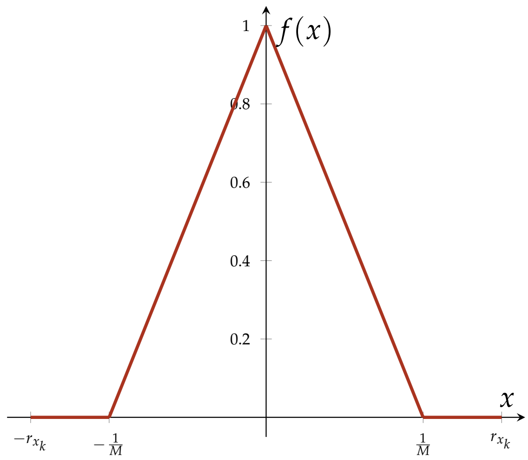
($\impliedby$) Let V be a set such that its intersection with every vertical line is an open subset of the line and $x = (\xi_1, \xi_2) \in V$ be an arbitrary element. If $\ell = \{y = (\eta_1, \eta_2) \in \mathbb{R}^2 : \eta_1 = \xi_1\}$, then there exists $r > 0$ such that $\{\xi_1\} \times (\xi_2 - r, \xi_2 + r) \subset \ell \cap E$. Therefore, if $r' < \min\{1, r\}$, then $B_d(x, r') = \{\xi_1\} \times (\xi_2 - r', \xi_2 + r') \subset E$ and E is open in the topology induced by d.

To see this metric space is locally compact, we just note that for any $x = (\xi_1, \xi_2) \in \mathbb{R}^2$, the set $\{\xi_1\} \times (\xi_2 - 1, \xi_2 + 1)$ is an open set, by our characterization, and its closure is the closed segment $\{\xi_1\} \times [\xi_2 - 1, \xi_2 + 1]$, which is compact, since it is compact with respect to the vertical line.

Notice that any compact set K in this topology is the finite union of vertical compact sets, otherwise the cover consisting of the vertical lines containing an element of K would not have a finite subcover.

Hence, by the definition of support, if $f \in C_c(X)$, there exist only finitely many points $x_1, x_2, \ldots, x_n$ such that $f(x, y) \neq 0$ for at least one y.

Now let Λf be the functional defined above, μ be the measure associated with Λ by Theorem 2.14 and E be the x-axis. By our characterization, f is continuous with respect to d if and only if f restricted to the vertical line is a continuous function as we usually know. An open set $E \subset V$ if and only if it contains a set of the form $U = \bigcup_{x \in \mathbb{R}} \{x\} \times (-r_x, r_x)$. Since $\mathbb{R}$ is noncountable, there exist a natural number $M \in \mathbb{N}$ and a sequence $\{x_k\}$ such that $r_{x_k} > \frac{1}{M}$ for every $k \in \mathbb{N}$. Let f_n be identically zero except in the lines $\ell_k = \{x_k\} \times \mathbb{R}$ for $k \leq n$, where it is defined in the following way.



Then $f_n \in C_c(X)$, $f_n \prec V$ and $\Lambda f_n = \frac{n}{M} \to \infty$ as $n \to \infty$. Therefore $\mu(V) = \infty$. Since V is arbitrary, using the outer regularity, $\mu(E) = \infty$. On the other hand, every $K \subset E$ consists of finite points $K = \{(x_1, 0), (x_2, 0), \ldots, (x_n, 0)\}$. Given $\epsilon > 0$, let $V = \bigcup \{x_k\} \times (-\epsilon, \epsilon)$ be an open set containing K. Then, for any $f \prec V$,

$$\Lambda f = \sum_{k=1}^{n} \int_{-\infty}^{\infty} f(x_k, y)\, dy \leq n \cdot 2\epsilon,$$

hence $\mu(V) \leq 2n\epsilon$. Since ϵ is arbitrary, $\mu(K) = 0$. ■

EXERCISE 2.18

■ SOLUTION Gu ■

Theorem 2.14 is usually called *Riesz-Markov-Kakutani Representation Theorem*. As Rudin points out, the first form of the theorem was proved by Riesz, in 1909, with $X = [0, 1]$, Markov proved it for compact spaces, in 1938, as we are about to do it, and, finally, Kakutani proved it for locally compact, in 1941.

EXERCISE 2.19 Go through the proof of Theorem 2.14, assuming X to be compact (of even compact metric) rather than just locally compact, and see what simplifications you can find.

■ SOLUTION First, let us restate Theorem 2.14 with the desired simplifications.

Let X be a compact Hausdorff space, and let Λ be a positive linear functional on $C(X)$. Then there exists a σ-algebra in X which contains all Borel sets in X, and there exists a unique positive finite measure μ on $\mathfrak{M}$ which represents Λ in the sense that:

(a) $\Lambda f = \int_X f\, d\mu$ for every $f \in C(X)$,

and which has the following additional properties:

(b) For every $E \in \mathfrak{M}$, we have

$$\mu(E) = \inf\{\mu(V) : E \subset V,\, V \text{ open}\}.$$

(c) For every $E \in \mathfrak{M}$, we have

$$\mu(E) = \sup\{\mu(K) : K \subset E,\, K \text{ compact}\}.$$

(d) If $E \in \mathfrak{M}$, $A \subset E$, and $\mu(E) = 0$, then $A \in \mathfrak{M}$.

The proof of the uniqueness is equal to the one given, and the definition of μ must also be equal, which is clearly monotone. Nonetheless, since for every $E \subset X$, $\mu(E) \leq \mu(X) = \Lambda(1) < \infty$. Therefore we can define $\mathfrak{M}$ as the class of all $E \subset X$ such that

$$\mu(E) = \sup\{\mu(K) \colon K \subset E,\, K \text{ compact}\}.$$

By definition, (b) and (d) hold. We'll analyze each step.

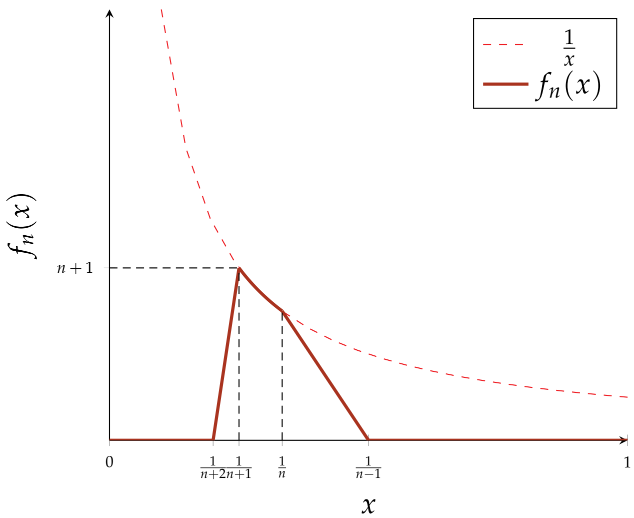
STEP I. ■

EXERCISE 2.20 Find continuous functions $f_n \colon [0, 1] \to [0, \infty)$ such that $f_n(x) \to 0$ for all $x \in [0, 1]$ as $n \to \infty$, $\int_0^1 f_n(x)\, dx \to 0$, but $\sup_n f_n$ is not in L^1. (This shows that the conclusion of the dominated convergence theorem may hold even when part of its hypothesis is violated.)

■ SOLUTION Let $f_n \colon [0, 1] \to [0, \infty)$ be defined in the following way:

$$f_n(x) = \begin{cases} 0, \text{ if } 0 \leq x \leq \frac{1}{n+2}, \\ (n+1)^2((n+2)x - 1), \text{ if } \frac{1}{n+2} \leq x \leq \frac{1}{n+1}, \\ \frac{1}{x}, \text{ if } \frac{1}{n+1} \leq x \leq \frac{1}{n}, \\ n^2(1 - (n-1)x), \text{ if } \frac{1}{n} \leq x \leq \frac{1}{n-1}, \\ 0, \text{ if } \frac{1}{n-1} \leq x \leq 1. \end{cases}$$

In following picture, we have draw the function f_n.



These functions are defined so that the following holds:

(a) each f_n is continuous. In fact, this should be "obvious" from the picture.

(b) $\lim_n f_n(x) = 0$. In fact, for each fixed x and for $n > \frac{1}{x} + 1$, $f_n(x) = 0$.

(c) $\sup_n f_n$ is not in L^1. In fact,

$$\sup_n f_n(x) = \begin{cases} \frac{1}{x}, \text{ if } 0 < x \leq 1, \\ 0, \text{ if } x = 0. \end{cases}$$

(d) $\int_0^1 f_n(x)\,dx \to 0$ as $n \to \infty$. In fact, $f_n(x) \leq (n+1)\chi_{[\frac{1}{n+2},\frac{1}{n-1}]}(x)$, and

$$(n+1)\int_0^1 \chi_{[\frac{1}{n+2},\frac{1}{n-1}]}(x)\,dx = \frac{3(n+1)}{(n-1)(n+2)} \to 0.$$

■

EXERCISE 2.21 If X is compact and $f : X \to (-\infty, \infty)$ is upper semicontinuous, prove that f attains its maximum at some point of X.

■ SOLUTION If f is upper semicontinuous, then for $n = 1, 2, \ldots$ the set

$$A_n = \{x \in X : f(x) < n\}$$

is open, and their union for all n forms an open cover of X. Since X is compact, it is possible to take a finite subcover

$$X = \bigcup_{j=1}^{k} A_{i_j}$$

for some positive integer j. Because there is a maximal index i_j, one sees that f is bounded above. Therefore $\sup f < \infty$.

Assume that f did not reach its supremum on X, then consider

$$B_n = \left\{x \in X : f(x) < \sup f - \frac{1}{n}\right\};$$

these sets are open, and the union

$$\bigcup_{i=1}^{\infty} B_i$$

forms an open cover of X which does not have finite subcover. This is a contradiction with the fact that X is compact. Therefore f must assume its maximum in X. ∎

EXERCISE 2.22 Suppose that X is a metric space, with metric d, and that $f : X \to [0, \infty]$ is lower semicontinuous, $f(p) < \infty$ for at least one $p \in X$. For $n = 1, 2, 3, \ldots$, $x \in X$, define

$$g_n(x) = \inf\{f(p) + nd(x, p) : p \in X\}$$

and prove that

(i) $|g_n(x) - g_n(y)| \leq nd(x, y)$,

(ii) $0 \leq g_1 \leq g_2 \leq \ldots \leq f$,

(iii) $g_n(x) \to f(x)$ as $n \to \infty$, for all $x \in X$.

Thus f is the pointwise limit of an increasing sequence of continuous functions. (Note that the converse is almost trivial.)

■ SOLUTION

(i) Let $x, y, p \in X$. By the triangular inequality,

$$\begin{aligned} nd(x,y) &\geq nd(x,p) - nd(y,p) \\ &= (f(p) + nd(x,p)) - (f(p) + nd(y,p)). \end{aligned}$$

Taking the infimum from both sides,

$$nd(x,y) \geq g_n(x) - g_n(y).$$

Similarly, $nd(x,y) \geq g_n(y) - g_n(x)$. The result follows.

(ii) Clearly $g_1 \geq 0$. Also, since $d(x,p)$ is always a positive number,

$$\begin{aligned} g_{n+1}(x) &= \inf\{f(p) + (n+1)d(x,p) : p \in X\} \\ &\geq \inf\{f(p) + nd(x,p) : p \in X\} = g_n(x) \end{aligned}$$

for all $n = 1, 2, 3, \ldots$. Lastly,

$$\begin{aligned} g_n(x) &= \inf\{f(p) + nd(x,p) : p \in X\} \\ &\leq f(x) + nd(x,x) = f(x) \end{aligned}$$

for all n.

(iii) Let $\epsilon > 0$ and $x \in X$. By the monotone convergence theorem, $\{g_n\}$ converges pointwise to a function $h \leq f$. Now, for each n we pick a point $x_n \in X$ such that

$$f(x_n) \leq f(x_n) + nd(x, x_n) \leq g_n(x) + \epsilon.$$

Since f is positive, it follows that (we used that $f(x_n) \geq 0$ and that $g_n(x) \leq f(x)$)

$$d(x, x_n) \leq \frac{g_n(x) + \epsilon - f(x_n)}{n} \leq \frac{f(x) + \epsilon}{n}.$$

That is, x_n converges to x. Using the lower semicontinuity of f (see the remark in Exercise 2.1), we see that

$$f(x) \leq \liminf_{n\to\infty} f(x_n) \leq \lim_{n\to\infty} [g_n(x) + \epsilon] = h(x) + \epsilon.$$

We conclude that $f = h$ and the result follows. ∎

EXERCISE 2.23 Suppose V is open in $\mathbb{R}^k$ and μ is a finite Borel measure on $\mathbb{R}^k$. Is the function that sends x to $\mu(V + x)$ necessarily continuous? lower semicontinuous? upper semicontinuous?

■ SOLUTION First observe that the function is not necessarily upper semicontinuous. In fact, let δ be the *Dirac measure* defined by

$$\delta(E) = \begin{cases} 1, & \text{if } 0 \in E \\ 0, & \text{if } 0 \notin E. \end{cases}$$

Then $\delta(V + x) < \frac{1}{2}$ if and only if $x \notin -V$, therefore it is an closed set. Nonetheless, the function is always lower semicontinuous. Let $x \in \mathbb{R}^k$ and $\epsilon > 0$. Since every open set is σ-compact, there exists a compact $K \subset V + x$ such that $\mu(K) > \mu(V + x) - \epsilon$. Since $(V + x)^c$ is closed, the distance between K and $(V + x)^c$ is $d > 0$. Therefore, if $|y - x| < d$, then $K \subset V + y$ and $\mu(V + y) > \mu(V + x) - \epsilon$ and the function is lower semicontinuous, as we desired to prove. ∎

Remark. Let μ be any measure on the Borel sets of a topological space X. For any $x \in X$, we can define a new measure by $\mu_x(E) = \mu(E + x)$. Moreover, we can *always* define a very natural topology in the set of measures: the coarsest topology such that for every continuous function $\varphi\colon X \to \mathbb{R}$, the functional

$$\mu \mapsto \int_X \varphi \, d\mu$$

is continuous (this topology is called the *weak* topology*).

In this topology, if $x_n \to x$, then $\mu_{x_n} \to \mu_x$. Moreover, for every open set V, and for every sequence of measures μ_n converging to μ,

$$\liminf_n \mu_n(V) \geq \mu(V),$$

from which would follow the exercise.

EXERCISE 2.24 A *step function* is , by definition, a finite linear combination of characteristic functions of bounded interval of $\mathbb{R}^1$. Assume $f \in L^1(\mathbb{R}^1)$, and prove that there is a sequence g_n of step functions so that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x) - g_n(x)|\, dx = 0$$

■ SOLUTION Let U be a open set in $\mathbb{R}$, so we have that so we can split U as a countable disjoint union of open intervals. Then

$$\chi_U = \sum_{n=1}^{\infty} \chi_{I_n},$$

where χ_U is the characteristic function, $U = \bigcup_{n=1}^{n=\infty} I_n$, and I_n is a open interval, for all $n \in \mathbb{N}$. Suppose that I_{n_0} is unbounded for some $n_0 \in \mathbb{N}$ (the other case are similar), then

$$\lim_{k\to\infty} \int_{\mathbb{R}} \left| \chi_{I_{n_0}}(x) - \chi_{I_{n_0} \cap (-k,k)(x)} \right| \, dx = 0$$

and note that

$$(F_k)_U = \left(\sum_{n=1, n\neq n_0}^{k} \chi_{I_n}(x) \right) + \chi_{I_{n_0} \cap (-k,k)(x)}$$

is a increasing sequence of step functions, and

$$\lim_{k\to\infty} |\chi_U(x) - (F_k)_U(x)| = \lim_{k\to\infty} \sum_{k\leq n, n\neq n_0} \chi_{I_n}(x) = 0,$$

thus by Lebesgue's Monotone Convergence Theorem we have that

$$\lim_{k\to\infty} \int_{\mathbb{R}} |\chi_U(x) - (F_k)_U(x)| \, dx = 0$$

By regularity of Lebesgue's measure we have that for all measurable set E there is a sequence of open sets U_n such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |\chi_E(x) - \chi_{U_n}(x)| \, dx = 0.$$

And now applying Cantor's Diagonal Argument, it follows that exist a sequence of step functions $G_k = (F_{n_k})_{U_{m_k}}$, such that

$$\lim_{k\to\infty} \int_{\mathbb{R}} |\chi_E - G_k| \, dx = 0$$

then, we show that the set of step functions is dense in the set of simple functions. Lastly, let be $f \in L^1(\mathbb{R})$, by Theorem 1.17 we have that there are two increasing sequences of simple functions, c_n and s_n, such that for all $x \in \mathbb{R}$

$$\lim_{n\to\infty} c_n(x) = f^+(x)$$

and

$$\lim_{n\to\infty} s_n(x) = f^-(x).$$

So

$$\lim_{k\to\infty} \int_{\mathbb{R}} |f - c_n + s_n| \, dx = 0,$$

in other words, the set of simple functions is dense in $L^1(\mathbb{R})$. We conclude that the set of step functions is dense in $L^1(\mathbb{R})$ and the result follows.

■

Exercise 2.25

(i) Find the smallest constant c such that

$$\log(1+e^t) < c + t \qquad (0 < t < \infty).$$

(ii) Does

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \log\{1 + e^{nf(x)}\} \, dx$$

exist for every real $f \in L^1$? If it exists, what is it?

■ Solution

(i) Let $g(t) = \log(1+e^t) - t$. Since $g'(t) = -1/(1+e^t)$ is always negative, we have that

$$g(t) < \lim_{x\to 0^+} g(x) = \log(2)$$

for all $0 < t < \infty$. It follows that $c = \log(2)$.

(ii) Let X be the set of all $x \in [0,1]$ such that $f(x) > 0$. By the inequality in (i),

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \int_X \log\{1 + e^{nf(x)}\} \, dx &\le \lim_{n\to\infty} \frac{1}{n} \int_X \{\log(2) + nf(x)\} \, dx \\ &= \int_X f(x) \, dx. \end{aligned}$$

Also, since $\log\{1 + e^{nf(x)}\} > \log\{e^{nf(x)}\} = nf(x)$ for all x,

$$\lim_{n\to\infty} \frac{1}{n} \int_X \log\{1 + e^{nf(x)}\} \, dx \ge \int_X f(x) \, dx.$$

Thus,

$$\lim_{n\to\infty} \frac{1}{n} \int_X \log\{1 + e^{nf(x)}\} \, dx = \int_X f(x) \, dx.$$

Lastly, since $\log(1+e^t) \le \log(2)$ for $t \le 0$,

$$\frac{1}{n} \int_{[0,1]\setminus X} \log\{1 + e^{nf(x)}\} \, dx \to 0.$$

This implies that the limit in the exercise's statement always exists and is equal to $\int_X f(x) \, dx$. ■

3 L^p-SPACES

EXERCISE 3.1 Prove that the supremum of any collection of convex functions on (a, b) is convex on (a, b) (if it is finite) and that pointwise limits of sequences of convex functions are convex. What can you say about upper and lower limits of sequences of convex functions?

■ SOLUTION Let $\{\varphi_\alpha\}$ be a collection of convex functions on (a, b) and $\varphi = \sup_\alpha \varphi_\alpha$. Then, for all $x, y \in (a, b)$, $\lambda \in [0, 1]$ and all α,

$$(1-\lambda)\varphi(x) + \lambda\varphi(y) \geq (1-\lambda)\varphi_\alpha(x) + \lambda\varphi_\alpha(y) \geq \varphi_\alpha((1-\lambda)x + \lambda y).$$

Since the supremum is the least upper bound,

$$(1-\lambda)\varphi(x) + \lambda\varphi(y) \geq \varphi((1-\lambda)x + \lambda y).$$

In other words, φ is convex. To prove that the limit φ of a sequence $\{\varphi_n\}$ of convex functions is convex we need just to take the limit $n \to \infty$ in the inequality

$$\varphi_n((1-\lambda)x + \lambda y) \leq (1-\lambda)\varphi_n(x) + \lambda\varphi_n(y).$$

The upper limit φ of a sequence $\{\varphi_n\}$ of convex functions is also convex since, as we just proved, the supremum and the pointwise limit of a sequence of convex functions is convex.

Nevertheless, the lower limit of a sequence of convex functions need not be convex! For example, if $\varphi_n(x) = (-1)^n x$, then its lower limit is $\liminf_n \varphi_n(x) = -|x|$, which is not convex. ■

EXERCISE 3.2 If φ is convex on (a, b) and if ψ is convex and nondecreasing on the range of φ, prove that $\psi \circ \varphi$ is convex on (a, b). For $\varphi > 0$, show that the convexity of $\log \varphi$ implies the convexity of φ, but not vice versa.

■ SOLUTION Let $\lambda \in [0, 1]$, $x, y \in (a, b)$. Since φ is convex, $\varphi((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$. Since ψ is nondecreasing and convex

$$\begin{aligned} \psi \circ \varphi((1-\lambda)x + \lambda y) &\leq \psi((1-\lambda)\varphi(x) + \lambda\varphi(y)) \\ &\leq (1-\lambda)\psi \circ \varphi(x) + \lambda\psi \circ \varphi(y), \end{aligned}$$

and the composition is convex. Since $\log \varphi$ is convex and $\psi = \exp$ is convex and nondecreasing on $\mathbb{R}$, $\exp \circ \log \varphi = \varphi$ is convex. Nonetheless, let $\varphi(x) = x$. Then φ is convex, but $\log \varphi = \log x$ is not. ■

Remark. A function φ such that $\log\varphi$ is convex is called a logarithmically convex function. This fact can also by written as

$$\varphi((1-\lambda)x+\lambda y)\le\varphi(x)^{1-\lambda}\varphi(y)^{\lambda}.$$

Examples of such functions are the L^p-norms with respect to $\frac{1}{p}$ (this is the *Riesz-Thorin Interpolation Theorem*) and the Gamma function for positive real numbers. Furthermore, if $p(x)=a_nx^n+\cdots+a_1x+a_0$ is a polynomial with positive coefficients whose roots are real and distinct, then

$$\frac{a_i^2}{\binom{n}{i}^2}>\frac{a_{i-1}}{\binom{n}{i-1}}\frac{a_{i+1}}{\binom{n}{i+1}}$$

which implies the function

$$f(i)=\frac{a_i}{\binom{n}{i}}.$$

is strictly logarithmically convex.

EXERCISE 3.3

■ SOLUTION Gu ■

EXERCISE 3.4 Suppose f is a complex measurable function on X, μ is a positive measure on X, and

$$\varphi(p)=\int_X|f|^p\,d\mu=\|f\|_p^p\qquad(0<p<\infty).$$

Let $E=\{p:\varphi(p)<\infty\}$. Assume $\|f\|_\infty>0$.

(a) If $r<p<s, r\in E$, and $s\in E$, prove that $p\in E$.

(b) Prove that $\log\varphi$ is convex in the interior of E and that φ is continuous on E.

(c) By (a), E is connected. Is E necessarily open? Closed? Can E consist of a single point? Can E be any connected subset of $(0,\infty)$?

(d) If $r<p<s$, prove that $\|f\|_p\le\max(\|f\|_r,\|f\|_s)$. Show that this implies the inclusion $L^r(\mu)\cap L^s(\mu)\subset L^p(\mu)$.

(e) Assume that $\|f\|_r<\infty$ for some $r<\infty$ and prove that

$$\|f\|_p\to\|f\|_\infty\quad\text{as p }\to\infty.$$

■ SOLUTION L ■

EXERCISE 3.5 Assume, in addition to the hypotheses of Exercise 4, that
$$\mu(X) = 1.$$

(a) Prove that $\|f\|_r \leq \|f\|_s$ if $0 \leq r < s \leq \infty$.

(b) Under what conditions does it happen that $0 < r < s \leq \infty$ and $\|f\|_r = \|f\|_s$?

(c) Prove that $L^r(\mu) \supset L^s(\mu)$ if $0 < r < s$. Under what conditions do these two spaces contain the same functions?

(d) Assume that $\|f\|_r < \infty$ for some $r > 0$, and prove that
$$\lim_{p \to 0} \|f\|_p = \exp\left\{ \int_X \log|f| \, d\mu \right\}$$
if $\exp\{-\infty\}$ is defined to be 0.

■ SOLUTION

(a) Let $p = \frac{s}{r} > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then
$$\int_X f^r \cdot 1 \, d\mu \leq \left(\int_X f^{rp} \, d\mu \right)^{\frac{1}{p}} \left(\int_X 1^q \, d\mu \right)^{\frac{1}{q}} = \left(\int_X f^s \, d\mu \right)^{\frac{r}{s}}.$$
Therefore
$$\|f\|_r \leq \|f\|_s.$$

(b) Since we used Hölder inequality, equality holds if and only if $\exists \alpha \in \mathbb{C}$ such that $f^{rp} = \alpha$ a.e., i.e., f is constant a.e.

(c) By item (a), if $\|f\|_s < \infty$, then $\|f\|_r < \infty$ for every $r < s$. Therefore $L^r(\mu) \supset L^s(\mu)$. On the other hand the following holds.

> If $0 < r < s$, then $L^r(\mu) \subset L^s(\mu)$ if and only if there exists $c > 0$ such that $\mu(E) > 0$ implies $\mu(E) \geq c$.

In fact,

$(\implies)$ If for every $\epsilon > 0$, there exists E such that $\mu(E) < \epsilon$, consider the following construction: Let $E_1 = X$ be the whole space; given E_n, let E_{n+1} be a space such that $\mu(E_{n+1}) < \mu(E_n)/3$. Since $\sum_{i=1}^{\infty} \frac{1}{3^i} = \frac{1}{2}$, if we denote $A_n = E_n - \bigcup_{k>n} E_k$, then
$$0 < \mu(E_n)/2 < \mu(A_n) \leq \mu(E_n) < 1/3^{n-1}$$
and we have a partition of X with sets of positive measure. If
$$f \colon X \to \mathbb{R}$$

Therefore, for $0 < r < s$, $L^r(\mu) \supset L^s(\mu)$ if and only if X does not contains sets of arbitrarily large measures, and $L^r(\mu) \subset L^s(\mu)$ if and only if X does not contains sets of arbitrarily small measure be defined by

$$f(x) = \frac{1}{\mu(A_n)} \frac{1}{n^{\frac{2}{r+s}}}, \text{ if } x \in A_n,$$

then

$$\int_X |f|^s \, d\mu = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2s}{r+s}}} < \infty,$$

but

$$\int_X |f|^r \, d\mu = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2r}{r+s}}} = \infty.$$

Therefore $L^s(\mu)$ is not a subset of $L^r(\mu)$.

($\Longleftarrow$) Let $f \in L^s(\mu)$ and $E_n = \{x \in X : f(x) \geq n\}$. It follows easily that $\mu(E_n) \to 0$ as $n \to \infty$. Therefore it follows that there exists n_0 such that $\mu(E_{n_0}) = 0$ and $f(x) \leq n_0$ a.e. and since $\mu(X) = 1$, this means f is in L^r for every $0 < r \leq \infty$.

(d) Since we are dealing only with absolute values, all functions in this item are positive. Let $s = \sum_i c_i \chi_{E_i}$ be a complex, measurable, simple function such that $\mu(\{x \colon s(x) \neq 0\}) < \infty$. Then

$$\exp\left\{\int_X \log|s| \, d\mu\right\} = \exp\left\{\sum_i \log(c_i)\mu(E_i)\right\}$$

and

$$\|s\|_p = \left(\int_X s \, d\mu\right)^{\frac{1}{p}} = \exp\left\{\frac{1}{p} \log\left(\sum_i c_i^p \mu(E_i)\right)\right\}.$$

Since the expression inside the exp is of the form $\frac{0}{0}$ and exp is a continuous function, using l'Hôpital rule,

$$\begin{aligned}
\lim_{p \to 0} \|s\|_p &= \exp\left\{\lim_{p \to 0} \frac{\sum_i \log(c_i) c_i^p \mu(E_i)}{\sum_i c_i^p \mu(E_i)}\right\} \\
&= \exp\left\{\sum_i \log(c_i)\mu(E_i)\right\} \\
&= \exp\left\{\int_X \log|s| \, d\mu\right\}.
\end{aligned}$$

Since S is dense in L^1 (and $\|\cdot\|$ is a continuous function), the result follows for every $f \in L^1(\mu)$. In fact, let $f \in L^1(\mu)$ and $\{s_n\}$ be as in Theorem 1.17. Then $\{s_n\}$ is a monotone sequence of complex, measurable, simple functions such that $s_n \to f$. Then $\lim_{n\to\infty} \|s_n\|_p = \|f\|_p$ and $\lim_{n\to\infty} \int_X \log|s_n| \, d\mu = \int_X \log|f| \, d\mu$.

Taking the limit when $p \to 0$ (since all terms are positive, we can exchange the limits),

$$\begin{aligned}
\lim_{p\to 0}\|f\|_p &= \lim_{p\to 0}\lim_{n\to\infty}\|s_n\|_p \\
&= \lim_{n\to\infty}\lim_{p\to 0}\|s_n\|_p \\
&= \lim_{n\to\infty}\exp\left\{\int_X \log|s_n|\,d\mu\right\} \\
&= \exp\left\{\int_X \log|f|\,d\mu\right\}.
\end{aligned}$$

Finally, let f be such that $\|f\|_r < \infty$, and let $\tilde{f} = f^r$. Then $\tilde{f} \in L^1(\mu)$,

$$\|f\|_p = \left(\int_X f^p\,d\mu\right)^{\frac{1}{p}} = \left(\left(\int_X \tilde{f}^{\frac{p}{r}}\,d\mu\right)^{\frac{1}{p/r}}\right)^{\frac{1}{r}} = \left(\|\tilde{f}\|_{\frac{p}{r}}\right)^{\frac{1}{r}}$$

and

$$\begin{aligned}
\lim_{p\to 0}\|f\|_p &= \lim_{\frac{p}{r}\to 0}\left(\|\tilde{f}\|_{\frac{p}{r}}\right)^{\frac{1}{r}} \\
&= \exp\left\{\frac{1}{r}\int_X \log|\tilde{f}|\,d\mu\right\} \\
&= \exp\left\{\int_X \log|f|\,d\mu\right\},
\end{aligned}$$

therefore the claim is proved. ∎

Remark. If μ is the uniform distribution in $\{1,2,\ldots,n\}$ and $f(i) = a_i$, then the last inequality implies the following discrete analogue: For any non-negative real numbers $\{a_1, a_2, \ldots, a_n\}$,

$$\lim_{p\to 0}\left(\frac{a_1^p + a_2^p \cdots + a_n^p}{n}\right)^{\frac{1}{p}} = \sqrt[n]{a_1 a_2 \cdots a_n}.$$

EXERCISE 3.6 Let m be Lebesgue measure on $[0,1]$ and define $\|f\|_p$, with respect to m. Find all functions Φ on $[0,\infty]$ such that the relation

$$\Phi\left(\lim_{p\to\infty}\|f\|_p\right) = \int_0^1 (\Phi\circ f)\,dm$$

holds for every bounded, measurable, positive f. Show first that

$$c\Phi(x) + (1-c)\Phi(1) = \Phi(x^c).$$

Compare with Exercise 5(d) .

■ SOLUTION

$$c\Phi(x) + (1-c)\Phi(1) =$$

■

EXERCISE 3.7 For some measures, the relation $r < s$ implies $L^r(\mu) \subset L^s(\mu)$; for others, the inclusion is reversed; and there are some for which $L^r(\mu)$ does not contain $L^s(\mu)$ if $r \neq s$. Give examples of these situations, and find conditions on μ under which these situations will occur.

It is *very* important to be careful with the quantifiers here. The correct statement is: $(\mu(X) < \infty)$ is equivalent to (for all $r < s$, $L^s(\mu) \subset L^r(\mu)$.)

■ SOLUTION If, in Exercise 3.5, we had $\mu(X) < \infty$ in the place of $\mu(X) = 1$, it would follow from the same argument that

$$\|f\|_r \leq \|f\|_s \, \mu(X)^{(s-r)/sr}.$$

This implies that $L^s(\mu) \subset L^r(\mu)$ as long as X does not contain sets of arbitrarily large measure. Conversely, if $L^s(\mu) \subset L^r(\mu)$ for all $r < s$, we take $s = \infty$ and use the inclusion to conclude that $1 \in L^r(\mu)$ (since it is bounded). It follows that

$$\mu(X) = \|1\|_r < \infty.$$

In the same Exercise 3.5, we proved that the reverse inclusion happens if and only if there exists $c > 0$ such that $\mu(E) > 0$ implies $\mu(E) \geq c$.

For some examples, if $X = [0,1]$, then $L^s([0,1]) \subset L^r([0,1])$ for $r < s$. The counting measure on $\mathbb{N}$ provides an example for the reverse inclusion. Finally, since $\mathbb{R}$ has sets with arbitrarily big and small measures, $L^s(\mathbb{R})$ does not contain $L^s(\mathbb{R})$ if $r \neq s$. ■

EXERCISE 3.8 If g is a positive function on $(0,1)$ such that $g(x) \to \infty$ as $x \to 0$, then there is a convex function h on $(0,1)$ such that $h \leq g$ and $h(x) \to \infty$ as $x \to 0$. True or false? Is the problem changed if $(0,1)$ is replaced by $(0,\infty)$ and $x \to 0$ is replaced by $x \to \infty$

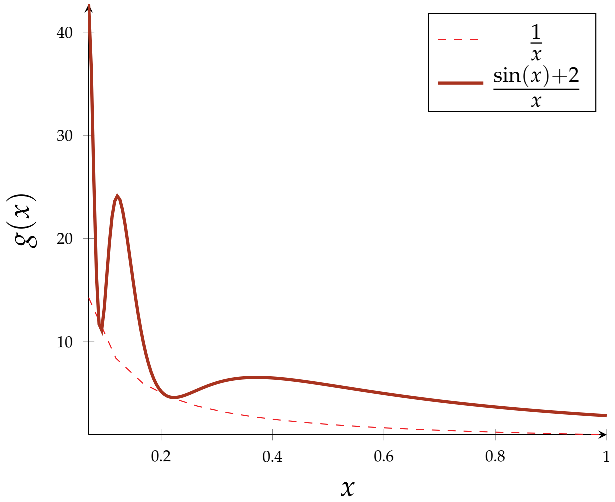
■ SOLUTION Let $(x_1, y_1) = (1, 0)$ and a_1 be defined by

$$a_1 := \sup\{a \in [0,\infty) \mid \forall x \in (0,1), a(1-x) \leq g(x)\}$$

and define $(x_2, y_2) = \left(\frac{1}{2}, \frac{a_1}{2}\right)$. Inductively, given (x_n, y_n), define a_n by

$$a_n := \sup\{a \in [0,\infty) \mid \forall x \in (0, x_n), y_n + a(x_n - x) \leq g(x)\}$$

and define $(x_{n+1}, y_{n+1}) = \left(\frac{1}{n+1}, y_n + \frac{a_n}{n(n+1)}\right)$. Finally, let $h\colon (0,1) \to (0,\infty)$ be defined as the piecewise linear function connecting the points (x_n, y_n). By construction, $h(x) \leq g(x)$ for every $x \in (0,1)$, and since the slope in (x_{n+1}, x_n) is $-a_n$, which is non-decreasing, then h is convex. Therefore, it is only left to prove $\lim_{x\to 0} h(x) = \infty$. It suffices to prove $y_n \to \infty$. In fact, given any y_k, there exists an $\epsilon > 0$ such that if $x < \epsilon$, then $g(x) > y_k + 2$. Let n be such that $\frac{1}{n} < \epsilon$. If $y_n > y_k + 1$ we are done, otherwise the line between $(0, y_k + 2)$ and (x_n, y_n) has slope $-a$ for some $a \in [0, \infty)$ and therefore, since $y_n > y_k$, then $y_2 n > y_n + 1$, proving the sequence goes to infinity.



Nonetheless, the problem is false if $(0,1)$ is replaced by $(0,\infty)$ and $x \to 0$ is replaced by $x \to \infty$. In fact, let $g(x) = \log x$, h be any convex function and $a = h(1)$, Since $h(x) \to \infty$ as $x \to \infty$, there exists r such that $h(r+1) \geq a + 1$. Then for any $n \in \mathbb{N}$,

$$\frac{h(nr+1) - h(r+1)}{nr} \geq \frac{h(r+1) - h(1)}{r} = \frac{1}{r}.$$

Therefore $h(nr+1) \geq n + a + 1$. In particular

$$\lim_{n\to\infty} \frac{g(nr+1)}{h(nr+1)} \leq \frac{\log n}{n + a + 1} = 0$$

and $h(x) \leq g(x)$ cannot happen for every x. ∎

EXERCISE 3.9

∎ SOLUTION Gu ∎

EXERCISE 3.10 Suppose $f_n \in L^p(\mu)$, for $n = 1, 2, 3, \ldots$, and that $\|f_n - f\| \to 0$ and $f_n \to g$ almost everywhere, as $n \to \infty$. What relation exists between f and g?

∎ SOLUTION First of all, notice that L_p convergence does not imply pointwise convergence. Take the sequence of functions

$$f_n(x) = \exp(-n^2 x^2),$$

and the usual Lebesgue measure on $\mathbb{R}$. This family of functions is a peak of fixed height at the origin, but "sharper" as $n \to \infty$, as the figure shows. Analytically,

$$\begin{aligned} \|f_n - 0\| &= \left(\int_{-\infty}^{\infty} |f_n|^p \, d\mu \right)^{1/p} \\ &= \left(\int_{-\infty}^{\infty} e^{-n^2 x^2 p} dx \right)^{1/p} \end{aligned}$$

$$()$$

■

EXERCISE 3.11 Suppose $\mu(\Omega) = 1$, and suppose f and g are positive measurable functions on Ω such that $fg \geq 1$. Prove that

$$\int_\Omega f \, d\mu \cdot \int_\Omega g \, d\mu \geq 1.$$

■ SOLUTION Since f and g are positives, we can take the square root, and get the following inequality $f^{1/2} g^{1/2} \geq 1$. Using Cauchy-Schwarz inequality,

$$\int_\Omega f \, d\mu \cdot \int_\Omega g \, d\mu \geq \left(\int_\Omega f^{2/1} g^{1/2} \right)^2 \geq 1$$

and we're done. ■

EXERCISE 3.12 Suppose that $\mu(\Omega) = 1$, and $h : \Omega \to [0, \infty]$ is measurable. If

$$A = \int_\Omega h \, d\mu,$$

prove that

$$\sqrt{1 + A^2} \leq \int_\Omega \sqrt{1 + h^2} \, d\mu \leq 1 + A.$$

If μ is Lebesgue measure on $[0, 1]$ and if h is continuous, $h = f'$, the above inequalities have a simple geometry interpretation. From this, conjecture (for general) under what conditions on h equality can hold in either of the above inequalities, and prove your conjecture.

■ SOLUTION M ■

EXERCISE 3.13 Under what conditions on f and g does equality hold in the conclusions of Theorems 3.8 and 3.9? You may have to treat the cases $p = 1$ and $p = \infty$ separately.

■ SOLUTION If $1 < p < \infty$, Theorem 3.8 is nothing but Hölder's inequality. As we saw just after its proof, equality holds if and only if

$$\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q} \qquad \text{almost everywhere.}$$

Also, if $p = \infty$ (if $p = 1$, then $q = \infty$ and the argument is the same), equality in the conclusion of Theorem 3.8 means that

$$\int_X |fg|\, d\mu = \|f\|_\infty \int_X |g|\, d\mu.$$

Since $|f| \leq \|f\|_\infty$ almost everywhere, it follows that

$$|fg| = \|f\|_\infty |g| \qquad \text{almost everywhere.}$$

This happens if and only if $|f| = \|f\|_\infty$ for almost all x such that $g(x) \neq 0$.

For Theorem 3.9, if $1 < p < \infty$, examining the proof of Minkowski's inequality we see that equality holds if and only if we have equality in both uses of Hölder's inequality *and*

$$|f + g| = |f| + |g|.$$

The first condition holds if and only if both $|f|^p$ and $|g|^p$ are multiples of $|f + g|^{(p-1)q}$ almost everywhere. In other words, if there exists constants α and β, not both zero, such that $\alpha|f|^p = \beta|g|^p$ a.e. Using this and the fact that $|f + g| = |f| + |g|$, which holds if and only if f and g have the same argument in the complex plane, we conclude that either $g = 0$ or there exists a positive constant λ such that $f = \lambda g$ almost everywhere.

If $p = 1$, we clearly have $|f + g| = |f| + |g|$ almost everywhere and then we have the same result as before. However, there isn't a simple characterization of the functions which satisfy the equality in Theorem 3.9 when $p = \infty$. For example, $f = \chi_{(0,3)}$ and $g = \chi_{(1,2)}$ satisfy the equality even without being zero nor multiples of each other. ■

EXERCISE 3.14 Suppose $1 < p < \infty$, $f \in L^p = L^p((0, \infty))$, relative to Lebesgue measure, and

$$F(x) = \frac{1}{x} \int_0^x f(t)\, dt \qquad (0 < x < \infty).$$

(a) Prove Hardy's inequality

$$\|F\|_p \leq \frac{p}{p-1} \|f\|_p$$

which shows that the mapping $f \to F$ carries L^p into L^p.

(b) Prove that equality holds only if $f = 0$ a.e.

(c) Prove that the constant $p/(p-1)$ cannot be replaced by a smaller one.

(d) If $f > 0$ and $f \in L^1$, prove that $F \notin L^1$.

Suggestions: (a) Assume first that $f \geq 0$ and $f \in C_c((0,\infty))$. Integration by parts gives

$$\int_0^\infty F^p(x)\,dx = -p\int_0^\infty F^{p-1}(x)xF'(x)\,dx.$$

Note that $xF' = f - F$, and apply Hölder's inequality to $\int F^{p-1}f$. Then derive the general case. (c) Take $f(x) = x^{-1/p}$ on $[1, A]$, $f(x) = 0$ elsewhere, for large A. See also Exercise 14, Chap. 8.

■ SOLUTION

(a) Let $f \geq 0$ and $f \in C_c((0,\infty))$. Then $F \geq 0$ and $F \in L^p((0,\infty))$. Since $xF(x) = \int_0^x f(t)\,dt$ and f is continuous,

$$xF'(x) + F(x) = f(x).$$

On the other hand, using integration by parts

$$\int_0^\infty F^p(x)\,dx = -p\int_0^\infty F^{p-1}(x)xF'(x)\,dx.$$

Hence

$$\begin{aligned}\int_0^\infty F^p(x)\,dx &= -p\int_0^\infty F^{p-1}(x)(f(x) - F(x))\,dx \\ &= -p\int_0^\infty F^{p-1}(x)f(x)\,dx + p\int_0^\infty F^p(x)\,dx,\end{aligned}$$

in other words,

$$\int_0^\infty F^p(x)\,dx = \frac{p}{p-1}\int_0^\infty F^{p-1}f\,dx.$$

Using Hölder inequality, if $\frac{1}{p} + \frac{1}{q} = 1$ then $p = (p-1)q$, $F^{p-1} \in L^q((0,\infty))$ and

$$\int_0^\infty F^{p-1}f\,dx \leq \left(\int F^p\,dx\right)^{\frac{1}{q}}\left(\int_0^\infty f^p\,dx\right)^{\frac{1}{p}}.$$

Therefore

$$\left(\int_0^\infty F^p(x)\,dx\right)^{\frac{1}{p}} \leq \frac{p}{p-1}\left(\int_0^\infty f^p\,dx\right)^{\frac{1}{p}},$$

i.e.

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p.$$

Since $C_c((0,\infty))$ is dense in $L^p((0,\infty))$ and $\|\cdot\|$ is a continuous function, then the result follows for all positive functions $f \in L^p((0,\infty))$. If $f \in L^p((0,\infty))$, then $|f| \in L^p((0,\infty))$ is a positive function, and if

$$\tilde{F}(x) = \frac{1}{x}\int_0^x |f(t)|\,dt,$$

then $|F| \leq \tilde{F}$ and

$$\|F\|_p \leq \|\tilde{F}\|_p \leq \frac{p}{p-1}\||f|\|_p = \frac{p}{p-1}\|f\|_p,$$

as we desired to prove.

(b) By the last argument in (a), if equality holds for f, then it also holds for $|f|$, hence we can suppose $f \geq 0$.

(c) Let $A > 1$ and $f(x) = x^{-1/p}\chi_{[1,A]}$. Then $\|f\|_p^p = \int_1^A x^{-1}\,dx = \log A$. On the other hand,

$$F(x) = \begin{cases} 0, \text{ if } 0 < x \leq 1 \\ \frac{p}{p-1}\left(x^{-\frac{1}{p}} - x^{-1}\right), \text{ if } 1 \leq x \leq A, \\ \frac{p\left(A^{1-\frac{1}{p}}-1\right)}{p}\frac{1}{x}, \text{ if } A \leq x. \end{cases}$$

Hence,

$$\begin{aligned} \|F\|_p^p &\geq \frac{p^p}{(p-1)^p}\int_1^A \left(x^{\frac{1}{p}-x^{-1}}\right)^p\,dx \\ &= \frac{p^p}{(p-1)^p}\int_1^A x^{-1}\left(1-x^{\frac{1}{p}-1}\right)^p\,dx \\ &\geq \frac{p^p}{(p-1)^p}\int_1^A x^{-1}\left(1-px^{\frac{1}{p}-1}\right)\,dx \\ &= \frac{p^p}{(p-1)^p}\log A + \frac{p^{p+2}}{(p-1)^{p+1}}\left(1+A^{\frac{1}{p}-1}\right), \end{aligned}$$

from where we conclude, if $\|F\|_p \leq C_p\|f\|_p$, then

$$\frac{p^p}{(p-1)^p} \leq \liminf_{A\to\infty}\frac{\|F\|_p^p}{\|f\|_p^p} \leq C_p^p,$$

which means $\frac{p}{p-1} \leq C_p$.

(d) Since $f \in L^1$, $\exists M > 0$ such that if $x \geq M$, then

$$\int_0^x f(t)\,dt \geq \frac{\|f\|_1}{2}.$$

Therefore, for $x \geq M$, $F(x) \geq \frac{\|f\|_1}{2x}$. Since $\frac{1}{x}$ is not integrable in (M,∞) for any $M > 0$, $F \notin L^1$.

■

Remark. Hardy's inequality has the following discrete analogue:

If $\{a_n\}_n$ is a sequence of non-negative real numbers, and $p > 1$, then

$$\sum_{n=1}^{\infty}\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^p \leq \left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}a_n^p.$$

Taking $a_i = a_i^{\frac{1}{p}}$ and letting $p \to \infty$, we conclude *Carleman's Inequality*

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 a_2 \cdots a_n} \leq e \sum_{n=1}^{\infty} a_n.$$

EXERCISE 3.15

■ SOLUTION Gu ■

EXERCISE 3.16

■ SOLUTION L ■

EXERCISE 3.17

(a) If $0 < p < \infty$, put $\gamma_p = \max(1, 2^{p-1})$, and show that

$$|\alpha - \beta|^p \leq \gamma_p(|\alpha|^p + |\beta|^p)$$

for arbitrary complex numbers α and β.

(b) Suppose μ is a positive measure on X, $0 < p < \infty, f \in L^p(\mu), f_n \in L^p(\mu), f_n(x) \to f(x)$ a.e., and $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$. Show that then $\lim\|f - f_n\|_p = 0$, by completing the two proofs that are sketched below.

(i) By Egoroff's theorem, $X = A \cup B$ in such a way that $\int_A |f|^p < \epsilon, \mu(B) < \infty$, and $f_n \to f$ uniformly on B. Fatou's lemma, applied to $\int_B |f_n|^p$, leads to

$$\limsup \int_A |f_n|^p \, d\mu \leq \epsilon.$$

(ii) Put $h_n = \gamma_p(|f|^p + |f_n|^p) - |f - f_n|^p$, and use Fatou's lemma as in the proof of Theorem 1.34.

(c) Show that the conclusion of (b) is false if the hypothesis $\|f_n\|_p \to \|f\|_p$ is omitted, even if $\mu(X) < \infty$.

■ SOLUTION

(a) If $1 \leq p < \infty$, then the function $f(x) = |x|^p$ is a convex function and

$$\left|\frac{\alpha+\beta}{2}\right|^p = f\left(\frac{\alpha+\beta}{2}\right) \leq \frac{f(\alpha)+f(\beta)}{2} = \frac{|\alpha|^p + |\beta|^p}{2}.$$

Therefore $|\alpha + \beta|^p \leq 2^{p-1}(|\alpha|^p + |\beta|^p)$ if $1 \leq p < \infty$. On the other hand, if $0 < p < 1$ and $|x| \leq 1$, then $|x| \leq |x|^p$. Hence

$$1 \leq \left|\frac{\alpha}{\alpha+\beta}\right| + \left|\frac{\beta}{\alpha+\beta}\right| \leq \left|\frac{\alpha}{\alpha+\beta}\right|^p + \left|\frac{\beta}{\alpha+\beta}\right|^p$$

and
$$|\alpha+\beta|^p \leq |\alpha|^p + |\beta|^p.$$
Since $|-\beta| = |\beta|$, the result follows.

(b)

(i) Applying Fatou's lemma as indicated,
$$\begin{aligned}\limsup \int_A |f_n|^p \, d\mu &= 1 - \liminf \int_B |f_n|^p \, d\mu \\ &\leq 1 - \int_B |f|^p \, d\mu \\ &= \int_A |f|^p \, d\mu \\ &\leq \epsilon.\end{aligned}$$

(ii)

(c) Let $0 < p < \infty$, $X = [-1,1]$ and $f_n = (\frac{n}{2})^{\frac{1}{p}} \chi_{[-\frac{1}{n},\frac{1}{n}]}$, then $\|f_n\|_p^p = \int_{-1}^{1} |f_n|^p \, dx = 1$, $f_n \in L^p([-1,1])$, $f_n(x) \to f(x) = 0$ for all $x \in [-1,1]$ and the function f is also in L^p. Nevertheless, $\|f_n\|_p \not\to \|f\|_p$ and, since $f \equiv 0$, $\|f - f_n\|_p = \|f_n\|_p \not\to 0$, as we desired to show.

If $p = 1$, the sequence f_n converges *in the sense of distributions* to the Dirac delta (generalized) function δ.

■

Remark. Using the fact that $|\alpha+\beta|^p \leq |\alpha|^p + |\beta|^p$, we can let α and β be functions, and integrate to conclude that the space of functions such that $\int_X |f| \, d\mu < \infty$ is a *translation-invariant* complete metric space (what we call an *F-space*) with the metric
$$d(f,g) := \int_X |f-g|^p \, d\mu.$$

EXERCISE 3.18 Let μ be a positive measure on X. A sequence of

■ SOLUTION M ■

EXERCISE 3.19 Define the *essential range* of a function $f \in L^\infty(\mu)$ to be the set R_f consisting of all complex numbers w such that
$$\mu(\{x : |f(x) - w| < \epsilon\}) > 0$$
for every $\epsilon > 0$. Prove that R_f is compact. What relation exists between the set R_f and the number $\|f\|_\infty$?

Let A_f be the set of all averages
$$\frac{1}{\mu(E)} \int_E f \, d\mu$$

where $E \in \mathfrak{M}$ and $\mu(E) > 0$. What relations exist between A_f and R_f? Is A_f always closed? Are there measures μ such that A_f is convex for every $f \in L^\infty(\mu)$? Are there measures μ such that A_f fails to be convex for some $f \in L^\infty(\mu)$?

How are theses results affected if $L^\infty(\mu)$ is replaced by $L^1(\mu)$, for instance?

■ SOLUTION We begin by proving that R_f is closed. Let $\{w_n\}$ be a sequence in R_f converging to w. We fix $\epsilon > 0$ and let N be such that

$$|w_n - w| < \frac{\epsilon}{2}$$

for all $n \geq N$. Now, if x is such that $|f(x) - w_N| < \epsilon/2$, then x also satisfies

$$|f(x) - w| \leq |f(x) - w_N| + |w_N - w| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

In other words,

$$\{x : |f(x) - w_N| < \epsilon/2\} \subset \{x : |f(x) - w| < \epsilon\}.$$

Since the latter contains a set of positive measure, its measure is also positive, meaning that $w \in R_f$.

The essential range R_f is also bounded as it is contained in the closed disk of radius $\|f\|_\infty$. In fact, if $|w| > \|f\|_\infty$, then the set

$$\{x : |f(x) - w| < |w| - \|f\|_\infty\}$$

has measure zero since

$$|w| - |f(x)| \leq |f(x) - w| < |w| - \|f\|_\infty$$

implies $|f(x)| > \|f\|_\infty$. Heine-Borel then implies that R_f is compact.

If μ is the Lebesgue measure on $[0, 1]$ and $f(x) = x$, we have that $A_f = (0, 1)$, which shows that A_f is not always closed. For μ equal to the unit mass centered at a point x_0, we have that $A_f = \{f(x_0)\}$, which is evidently convex for every $f \in L^\infty(\mu)$. Now, if μ is the counting measure on $X = \{a, b\}$,

$$A_f = \left\{ f(a), f(b), \frac{1}{2}(f(a) + f(b)) \right\},$$

which fails to be convex unless it is reduced to a point.

If $L^\infty(\mu)$ is replaced by $L^1(\mu)$, R_f continues to be closed but it is no longer necessarily bounded (just take a integrable non-bounded continuous function on the real line, for instance). ■

EXERCISE 3.20 Suppose φ is a real function of $\mathbb{R}^1$ such that

$$\varphi\left(\int_0^1 f(x)\, dx\right) \leq \int_0^1 \varphi(f) dx$$

for every real bounded measurable f. Prove that φ is then convex.

■ SOLUTION Let $\lambda \in [0,1]$, $t, s \in \mathbb{R}$ and $f\colon [0,1] \to \mathbb{R}$ defined by,

$$f(x) = \begin{cases} s, & \text{if } 0 \leq x \leq \lambda \\ t, & \text{if } \lambda < x \leq 1. \end{cases}$$

Then $\int_0^1 f(x)\,dx = (1-\lambda)t + \lambda s$, and $\int_0^1 \varphi(f)\,dx = (1-\lambda)\varphi(t) + \lambda\varphi(s)$, therefore the inequality becomes

$$\varphi((1-\lambda)t + \lambda s) \leq (1-\lambda)\varphi(t) + \lambda\varphi(s)$$

and φ is a convex function. ■

EXERCISE 3.21

■ SOLUTION Gu ■

EXERCISE 3.22

■ SOLUTION L ■

EXERCISE 3.23 Suppose μ is a positive measure on X, $\mu(X) < \infty$, $f \in L^\infty(\mu)$, $\|f\|_\infty > 0$, and

$$\alpha_n = \int_X \|f\|^n \,d\mu \qquad (n = 1, 2, 3, \ldots).$$

Prove that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty.$$

■ SOLUTION Since $n = \frac{n+1}{2} + \frac{n-1}{2}$, using Hölder inequality,

$$\int_X \|f\|^n \,d\mu \leq \left(\int_X \|f\|^{n+1} \,d\mu\right)^{\frac{1}{2}} \left(\int_X \|f\|^{n-1} \,d\mu\right)^{\frac{1}{2}},$$

therefore

$$\frac{\alpha_n}{\alpha_{n-1}} \leq \frac{\alpha_{n+1}}{\alpha_n}$$

and the sequence is monotone.

Since it is also bounded by $\|f\|_\infty$, the sequence converges and, by a theorem of Real Analysis,

To see this, take the log. Since the sequence converges, its Cesàro sum converges to the same limit.

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \lim_{n\to\infty} \sqrt[n]{\alpha_n} = \|f\|_\infty,$$

as we desired to prove. ■

EXERCISE 3.24

■ SOLUTION M ■

EXERCISE 3.25 Suppose μ is a positive measure on X and $f : X \to (0, \infty)$ satisfies $\int_X f \, d\mu = 1$. Prove, for every $E \subset X$ with $0 < \mu(E) < \infty$, that

$$\int_E (\log f) \, d\mu \leq \mu(E) \log \frac{1}{\mu(E)}$$

and, when $0 < p < 1$,

$$\int_E f^p \, d\mu \leq \mu(E)^{1-p}.$$

If $-f$ is convex, we usually say that f is *concave*.

■ SOLUTION Let $\nu = \mu/\mu(E)$. Since $\nu(E) = 1$ and $-\log$ is convex, Jensen's inequality implies that

$$\log \left(\frac{1}{\mu(E)} \int_E f \, d\mu \right) \geq \frac{1}{\mu(E)} \int_E f \, d\mu.$$

As log is increasing and

$$\int_E f \, d\mu \leq \int_X f \, d\mu = 1,$$

the first inequality follows.

Similarly, $-x^p$ is convex for $0 < p < 1$. By Jensen's inequality,

$$\left(\frac{1}{\mu(E)} \int_E f \, d\mu \right)^p \geq \frac{1}{\mu(E)} \int_E f^p \, d\mu.$$

The second inequality now follows in the same way as before. ■

EXERCISE 3.26 If f is a positive measurable function on $[0, 1]$, which is larger,

$$\int_0^1 f(x) \log f(x) \, dx \quad \text{or} \quad \int_0^1 f(s) \, ds \int_0^1 \log f(t) \, dt?$$

■ SOLUTION Let $\varphi(x) = x \log x$. Then φ is a convex function. Since $[0, 1]$ is a probability space with the Lebesgue measure and log is concave,

$$\begin{aligned}
\int_0^1 f(x) \log f(x) \, dx &= \int_0^1 \varphi(f(x)) \, dx \\
&\geq \varphi \left(\int_0^1 f(x) \, dx \right) \\
&= \int_0^1 f(s) \, ds \cdot \log \left(\int_0^1 f(t) \, dt \right) \\
&\geq \int_0^1 f(s) \, ds \int_0^1 \log f(t) \, dt.
\end{aligned}$$

Therefore the former is larger. ■

4

ELEMENTARY HILBERT SPACE THEORY

Exercise 4.1

■ **Solution** Gu ■

Exercise 4.2 Let $\{x_n : n = 1, 2, 3, \ldots\}$ be a linearly independent set of vectors in H. Show that the following construction yields an orthonormal set $\{u_n\}$ such that $\{x_1, \ldots, x_N\}$ and $\{u_1, \ldots, u_N\}$ have the same span for all N.

Put $u_1 = x_1/\|x_1\|$. Having $u_1, \ldots, u_{n-1}$ define

$$v_n = x_n - \sum_{i=1}^{n-1}(x_n, u_i)u_i, \qquad u_n = v_n/\|v_n\|.$$

This is called Gram-Schmidt process.

Note that this leads to a proof of existence of maximal orthonormal set in separable Hilbert spaces which makes no appeal to the Hausdorff maximality principle. (A space is *separable* if it contains a countable dense subset.)

■ **Solution** We'll prove the claim by induction. For $N = 1$, it is trivial. Suppose it is valid for N and define v_{N+1} and u_{N+1} as indicated. Then $v_{N+1} \neq 0$, since $\{x_n\}$ is linearly independent and $\|u_{N+1}\|$ is immediately 1. Furthermore

$$\|v_{N+1}\|(u_{N+1}, u_i) = (v_{N+1}, u_i) = (x_n, u_i) - (x_n, u_i)(u_i, u_i) = 0$$

for $i = 1, 2, \ldots, N$, and the set is still orthonormal. Finally, $u_{N+1} \in [\{x_1, \ldots, x_{N+1}\}]$ and reciprocally $x_{N+1} \in [\{u_1, \ldots, u_{N+1}\}]$, therefore the sets have the same span. ■

Exercise 4.3

■ **Solution** L ■

Exercise 4.4

■ **Solution** M ■

Exercise 4.5 If $M = \{x : Lx = 0\}$, where L is a continuous linear functional on H, prove that $M^\perp$ is a vector space of dimension 1 (unless $M = H$).

■ **Solution** Suppose $M \neq H$. By the first isomorphism theorem,

$$H/M \cong \operatorname{im} L = \mathbb{C}.$$

Therefore $\dim(H/M) = 1$. Now, consider the linear transformation $\bar{P}\colon M^\perp \to H/M$ defined by $\bar{P}(x) = \bar{x}$, where $\bar{x}$ is the element in H/M which contains x. Then P is injective since $\bar{x} = 0 \implies x \in M \cap M^\perp = \{0\}$. Moreover, $\bar{P}$ is surjective. In fact, let Q be as in Theorem 4.11 and $\bar{y} \in H/M$. Then $Qy \in M^\perp$ satisfies $\bar{P}(Qy) = \bar{y}$. Therefore $M^\perp$ is a vector space of dimension 1. ∎

Exercise 4.6 Let $\{u_n\}$ $(n = 1, 2, 3, \dots)$ be an orthonormal set in H. Show that this gives an example of a closed and bounded set which is not compact. Let Q be the set of all $x \in H$ of the form

$$x = \sum_{n=1}^{\infty} c_n u_n \qquad \left(\text{where } |c_n| \le \frac{1}{n}\right).$$

Prove that Q is compact. (Q is called the Hilbert cube.)

More generally, let $\{\delta_n\}$ be a sequence of positive numbers, and let S be the set of all $x \in H$ of the form

$$x = \sum_{n=1}^{\infty} c_n u_n \qquad (\text{where } |c_n| \le \delta_n)\,.$$

Prove that S is compact if and only if $\sum_1^\infty \delta_n^2 < \infty$.

Prove that H is not locally compact.

∎ Solution The set $\{u_n\}$ is clearly bounded as every element has unitary norm. Let $\{u_{n_k}\}$ be a convergent sequence in $\{u_n\}$. Since this sequence is Cauchy,

$$\|u_{n_i} - u_{n_j}\| < 1$$

for i, j sufficiently big. This implies that $\{u_{n_k}\}$ is eventually constant, since $\|u_n - u_m\|$ is 0 if $n = m$ and $\sqrt{2}$ otherwise. That is, $\{u_n\}$ is closed.

We recall that a metric space is compact if and only if every sequence has a convergent subsequence. Let $\{u_n\}$ be a sequence in itself. As we just saw, this sequence can't possibly be Cauchy (since taking $n_k = k$ would imply that $\{u_n\}$ is eventually constant) and thus it cannot have a convergent subsequence. In other words, $\{u_n\}$ is not compact.

What we've just shown implies that H is not locally compact since the closed unit ball contains the sequence $\{u_n\}$ which has no convergent subsequence. This shows that 0 has no neighborhood whose closure is compact.

We now prove that S is compact using a diagonal argument. Let $\{x^k\}$ be a sequence in S of the form

$$x^k = \sum_{n=1}^{\infty} c_n^k u_n.$$

Since the sequence $\{c_1^k\}$ is contained in the closed ball of radius δ_1, it has a convergent subsequence, namely $\{c_1^{\sigma_1(k)}\}$, for some (strictly) increasing function $\sigma_1 : \mathbb{N} \to \mathbb{N}$.

Now, the sequence $\{c_2^{\sigma_1(k)}\}$ is contained in the closed ball of radius δ_2. Thus, it has a convergent subsequence $\{c_2^{\sigma_2(k)}\}$. Observe that $\{c_1^{\sigma_2(k)}\}$ is a subsequence of $\{c_1^{\sigma_1(k)}\}$ and hence it is also convergent.

Similarly, we create (strictly) increasing functions σ_m such that $\{c_n^{\sigma_m(k)}\}$ converges for all $n \leq m$. For all n, we denote by c_n the limit of $\{c_n^{\sigma_n(k)}\}$ and by x the element

$$x = \sum_{n=1}^{\infty} c_n u_n.$$

We affirm that the "the diagonal sequence" $\{x^{\sigma_k(k)}\}$ converges to x.

Fix $\epsilon > 0$ and let N be such that

$$\sum_{n=N+1}^{\infty} \delta_n^2 < \epsilon.$$

Also, since $\{c_n^{\sigma_k(k)}\}$ is a subsequence of $\{c_n^{\sigma_n(k)}\}$ for $k \geq n$, there are numbers M_m such that

$$|c_n^{\sigma_k(k)} - c_n| < \sqrt{\frac{\epsilon}{N}}$$

for all $k \geq M_m$. We conclude that

$$\begin{aligned}\|x^{\sigma_k(k)} - x\|^2 &= \sum_{n=1}^{N} |c_n^{\sigma_k(k)} - c_n|^2 + \sum_{n=N+1}^{\infty} |c_n^{\sigma_k(k)} - c_n|^2 \\ &< \epsilon + 4\epsilon = 5\epsilon.\end{aligned}$$

The result follows.

Conversely, if $\sum_1^{\infty} \delta_n^2 = \infty$, we create a sequence $\{n_k\}$ such that

$$\sum_{n=n_k+1}^{n_{k+1}} \delta_n^2 \geq 1 \qquad \text{for all } k.$$

Let $\{x^k\}$ be a sequence defined by

$$x^k = \sum_{n=1}^{n_k} \delta_n u_n.$$

Then, if $k > k'$,

$$\|x^k - x^{k'}\| = \sum_{n=n_{k'}+1}^{n_k} \delta_n^2 \geq \sum_{n=n_{k'}+1}^{n_{k'+1}} \delta_n^2 \geq 1.$$

This implies that $\{x^k\}$ has no convergent subsequence. Thus, S is not compact. ∎

EXERCISE 4.7

■ SOLUTION Gu ■

EXERCISE 4.8 If H_1 and H_2 are two Hilbert spaces, prove that one of them is isomorphic to a subspace of the other. (Note that every closed subspace of a Hilbert space is a Hilbert space.)

■ SOLUTION Since there are sets A_1 and A_2 such that $H_1 \cong \ell^2(A_1)$ and $H_2 \cong \ell^2(A_2)$, it is sufficient prove the result for ℓ^2 spaces. By the Theorem of Comparability of Cardinals, there exists an injection $A_1 \to A_2$ or $A_2 \to A_1$. Suppose without loss of generality, $\varphi\colon A_1 \to A_2$ is an injection. Then it induces an isomorphism $\psi\colon \ell^2(A_1) \to \ell^2(\varphi(A_1)) \subset \ell^2(A_2)$ defined by $\psi(a_1, a_2, \dots) = (\varphi(a_1), \varphi(a_2), \dots)$. In fact, observe first that $\ell^2(\varphi(A_1))$ is a subspace of $\ell^2(A_2)$. Furthermore, ψ is an isomorphism ■

This result is equivalent to the *axiom of choice*.

EXERCISE 4.9

■ SOLUTION L ■

EXERCISE 4.10

■ SOLUTION M ■

EXERCISE 4.11 Find a nonempty closed set E in $L^2(T)$ that contains no element of smallest norm.

■ SOLUTION Let

$$E = \left\{ f_n(t) = \left(1 + \frac{1}{n}\right) e^{int} \colon n \in \mathbb{N} \right\}.$$

Then E contains no element of smallest norm, since $\|f_n\| = 1 + \frac{1}{n}$, and E is closed, since all of its elements are orthogonal (and the distance between them is at least 2). ■

EXERCISE 4.12 The constants c_k in Sec. 4.24 were shown to be such that $k^{-1}c_k$ is bounded. Estimate the relevant integral more precisely and show that

$$0 < \lim_{k\to\infty} k^{-1/2} c_k < \infty.$$

■ SOLUTION The constants c_k were chosen such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} c_k \left\{ \frac{1 + \cos t}{2} \right\}^k dt = 1.$$

Since $(1 + \cos t)/2 = \cos^2(t/2)$, this is equivalent to

$$\frac{c_k}{\pi} \int_{-\pi/2}^{\pi/2} \cos^{2k} t \, dt = 1.$$

Let A_k be the integral above, divided by π. Integrating by parts we observe that

$$A_k = \frac{2k-1}{2k} A_{k-1}.$$

We also know that $A_0 = 1$, so

$$A_k = \frac{1}{2^{2k-1}} \binom{2k+1}{k+1}.$$

Stirling's formula (Sec. 8.22 in *Principles of Mathematical Analysis*) implies that

$$\frac{n!}{(n/e)^n \sqrt{n}}$$

It actually implies that this limit is equal to $\sqrt{2\pi}$.

converges to a finite positive number. As $k^{-1/2} c_k = (\sqrt{k} A_k)^{-1}$, it suffices that $\sqrt{k} A_k$ also converges to a finite positive number as $k \to \infty$. Observe that

$$\sqrt{k} A_k = \frac{\sqrt{k}}{2^{2k-1}} \frac{(2k+1)!}{k!(k+1)!}$$

is equal to

$$\left[\frac{(2k+1)!}{\left(\frac{2k+1}{e}\right)^{2k+1} \sqrt{2k+1}} \right] \left[\frac{k!}{\left(\frac{k}{e}\right)^k \sqrt{k}} \right]^{-1} \left[\frac{(k+1)!}{\left(\frac{k+1}{e}\right)^{k+1} \sqrt{k+1}} \right]^{-1}$$

multiplied by

$$\frac{1}{2^{2k-1}} \sqrt{\frac{2k+1}{k+1}} \frac{(2k+1)^{2k+1}}{k^k (k+1)^{k+1}}.$$

Thus, it suffices to show that

$$\frac{1}{2^{2k-1}} \frac{(2k+1)^{2k+1}}{k^k (k+1)^{k+1}}$$

converges to a finite positive number as $k \to \infty$. Dividing above and below by $(2k)^{2k+1}$ we see that this limit turns out to be 4. The result follows. ■

This calculation shows that $k^{-1/2} c_k$ tends to $\sqrt{\pi}/4$.

EXERCISE 4.13

Gu

■ SOLUTION ■

EXERCISE 4.14 Compute

$$\min_{a,b,c} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 \, dx$$

and find

$$\max \int_{-1}^{1} x^3 g(x) \, dx,$$

where g is subject to the restrictions

$$\int_{-1}^{1} g(x)\,dx = \int_{-1}^{1} xg(x)\,dx = \int_{-1}^{1} x^2 g(x)\,dx = 0; \quad \int_{-1}^{1} |g(x)|^2\,dx = 1.$$

■ SOLUTION T ■

EXERCISE 4.15

■ SOLUTION L ■

EXERCISE 4.16

■ SOLUTION M ■

EXERCISE 4.17 Show that there is a continuous one-to-one mapping γ of $[0,1]$ into H such that $\gamma(b) - \gamma(a)$ is orthogonal to $\gamma(d) - \gamma(c)$ whenever $0 \le a \le b \le c \le d \le 1$. ($\gamma$ may be called a "curve with orthogonal increments.") *Hint:* Take $H = L^2$, and consider characteristic functions of certain subsets of $[0,1]$.

■ SOLUTION T ■

EXERCISE 4.18 Define $u_s(t) = e^{ist}$ for all $s \in \mathbb{R}^1$, $t \in \mathbb{R}^1$. Let X be the complex vector space consisting of all finite linear combinations of these functions u_s. If $f \in X$ and $g \in X$, show that

$$(f,g) = \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} f(t)\overline{g(t)}\,dt$$

exists. Show that this inner product makes X into a unitary space whose completion is a non-separable Hilbert space H. Show also that $\{u_s : s \in \mathbb{R}^1\}$ is a maximal orthonormal set in H.

■ SOLUTION To prove that the limit in the definition of (f,g) exists, it suffices to check the case $f = u_r$, $g = u_s$. We have that

$$(u_r, u_s) = \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} e^{i(r-s)t}\,dt = \begin{cases} 1 & \text{if } r = s \\ 0 & \text{otherwise} \end{cases}.$$

This also shows that $\{u_s\}$ is a orthonormal set.

Since X is dense in H, Theorem 4.18 implies that $\{u_s\}$ is a maximal orthonormal set in H. If H were separable, Exercise 4.2 would imply the existence of a countable orthonormal set, which contradicts the maximality of $\{u_s\}$. Hence, H is non-separable. ■

EXERCISE 4.19

■ SOLUTION Gu ■

5 EXAMPLES OF BANACH SPACE TECHNIQUES

EXERCISE 5.1

■ SOLUTION L ■

EXERCISE 5.2 Prove that the unit ball (open or closed) is convex in every normed linear space.

■ SOLUTION T ■

EXERCISE 5.3

■ SOLUTION M ■

EXERCISE 5.4 Let C be the space of all continuous functions of $[0,1]$, with the supremum norm. Let M consist of all $f \in C$ for which

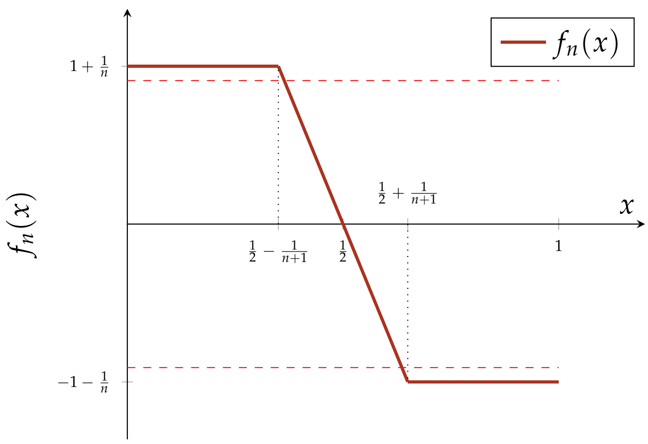
$$\int_0^{1/2} f(t)\,dt - \int_{1/2}^1 f(t)\,dt = 1.$$

Prove that M is a closed convex subset of C which contains no element of minimal norm.

■ SOLUTION Firstly, the triangular inequality implies that

$$\begin{aligned} 1 &\leq \left|\int_0^{1/2} f(t)\,dt\right| + \left|\int_{1/2}^1 f(t)\,dt\right| \\ &\leq \frac{1}{2}\|f\|_\infty + \frac{1}{2}\|f\|_\infty = \|f\|_\infty. \end{aligned}$$

Also, we can find functions in M with norms arbitrarily close to 1. For example, let f_n be the function with the graph below.



We see clearly that $f_n \in M$ for all n and that $\|f_n\|_\infty \to 1$. In other words,

$$\inf_{f \in M} \|f\|_\infty = 1.$$

Now, suppose there is a function $f \in M$ with unitary norm. This function satisfies

$$\int_0^{1/2} (f(t) - 1)\, dt + \int_{1/2}^{1} (-1 - f(t))\, dt = 0.$$

Since f is continuous and has unitary norm, both integrands are negative. This implies that

$$f(t) = 1 \text{ for } t \in (0, 1/2) \quad \text{and} \quad f(t) = -1 \text{ for } t \in (1/2, 1).$$

But then f is discontinuous. Absurd! ■

EXERCISE 5.5 Let M be the set of all $f \in L^1([0,1])$, relative to Lebesgue measure, such that

$$\int_0^1 f(t)\, dt = 1.$$

Show that M is a closed convex subset of $L^1([0,1])$ which contains infinitely many points of minimal norm. (Compare this and Exercise 4 with Theorem 4.10.)

■ SOLUTION T ■

Remark. There is a theorem which characterizes spaces satisfying the conclusion of Theorem 4.10. It is called *Day-James Theorem* and asserts the following.

For a normed space X, the following are equivalent:

- *X is strictly convex and reflexive (see Exercises 3 and 8).*
- *Every non-empty closed convex set in X has a unique point of minimal norm.*

EXERCISE 5.6

■ SOLUTION Gu ■

EXERCISE 5.7

■ SOLUTION L ■

EXERCISE 5.8 Let X be a normed linear space, and let X^* be its duas space, as defined in Sec. 1.21, with the norm

$$\|f\| = \sup\{\, |f(x)| \colon \|x\| \le 1 \,\}.$$

(a) Prove that X^* is a Banach space.

(b) Prove that the mapping $f \to f(x)$ is, for each $x \in X$, bounded linead functional on X^*, of norm $\|x\|$. (This gives a natural embedding of X in its "second dual" X^{**}, the dual space of X^*.)

(c) Prove that $\{\|x_n\|\}$ is bounded if $\{x_n\}$ is a sequence in X such that $\{f(x_n)\}$ is bounded for every $f \in X^*$.

When this embedding is, in fact, an isometric isomorphism, we say the space X is *reflexive*.

■ SOLUTION T ■

Remark. James found an example (called *James' space*) of a space X such that X is isometrically isomorphic to X^{**} but X is *not* reflexive (i.e. the embedding above is not an isomorphisms)! Therefore a reflexive space does not mean a space isomorphic to its second dual. Nonetheless, there are (a lot!) of theorems which characterize reflexive spaces, for example *James' Theorem* which asserts the following.

A Banach space X is reflexive if and only if every continuous linear functional on X attains its supremum on the closed unit ball.

Another important theorem is *Šmulian Theorem* (sometimes called *Šmulian-James Theorem*) which asserts the following.

A normed space X is reflexive if and only if for every nested sequence $C_1 \supset C_2 \supset C_3 \supset \cdots$ *of nonempty bounded closed convex subsets of X, their intersection is non-empty.*

EXERCISE 5.9

■ SOLUTION M ■

EXERCISE 5.10 If $\sum \alpha_i \xi_i$ converges for every sequence $\{\xi_i\}$ such that $\xi_i \to 0$ as $i \to \infty$, prove that $\sum |\alpha_i| < \infty$.

■ SOLUTION Firstly, lets write this using this chapter's formalism: let $\alpha = \{\alpha_i\}$ be a complex sequence such that $\sum \alpha_i \xi_i < \infty$ for every sequence $\xi = \{\xi_i\} \in c_0$. (Recall the definition of c_0 from the preceding exercise.) We want to prove that $\alpha \in \ell^1$.

Consider the following linear functional:

$$\begin{aligned} \Lambda_n : c_0 &\to \mathbb{C} \\ \xi &\mapsto \sum_{i=1}^{n} \alpha_i \xi_i. \end{aligned}$$

By the triangular inequality,

$$\|\Lambda_n\| = \sup_{\|\xi\| \le 1} \left| \sum_{i=1}^{n} \alpha_i \xi_i \right| \le \sup_{\|\xi\| \le 1} \sum_{i=1}^{n} |\alpha_i| \, |\xi_i| \le \sum_{i=1}^{n} |\alpha_i|.$$

Moreover, by taking $\xi_i = |\alpha_i|/\alpha_i$ for $1 \le i \le n$ (if $\alpha_i = 0$, just take $\xi_i = 0$) and $\xi_i = 0$ for $i > n$, we see that this bound is attained. Thus,

$$\|\Lambda_n\| = \sum_{i=1}^{n} |\alpha_i| < \infty.$$

Since $\sup_n |\Lambda_n \xi| < \infty$ for all ξ, the Banach-Steinhaus theorem implies that

$$\|\alpha\|_1 = \sum_{i=1}^{\infty} |\alpha_i| = \sup_n \|\Lambda_n\| < \infty.$$

This is what was to be shown. ■

Nowadays these functions are said to satisfy *α-Hölder condition*

Exercise 5.11 For $0 < \alpha \le 1$, let $\mathrm{Lip}\,\alpha$ denote the space of all complex functions f on $[a, b]$ for which

$$M_f = \sup_{s \neq t} \frac{|f(s) - f(t)|}{|s - t|^\alpha} < \infty.$$

Prove that $\mathrm{Lip}\,\alpha$ is a Banach space, if $\|f\| = |f(a)| + M_f$; also if

$$\|f\| = M_f + \sup_x |f(x)|.$$

(The members of $\mathrm{Lip}\,\alpha$ are sair to satisfy *Lipschitz condition of order α*.)

■ Solution T ■

Exercise 5.12

■ Solution Gu ■

Exercise 5.13

■ Solution L ■

Exercise 5.14 Let C be the space of all real continuous functions on $I = [0, 1]$ with the supremum norm. Let X_n be the subset of C consisting of those f for which there exists a $t \in I$ such that $|f(s) - f(t)| \le n|s - t|$ for all $s \in I$. Fix n and prove that each open set in X contains an open set which does not intersect X_n. (Each $f \in C$ can be uniformly approximated by a zigzag function g with very large slopes, and if $\|g - h\|$ is small, $h \notin X_n$.) Show that this implies the existence of a dense set G_δ in C which consists entirely of nowhere differentiable functions.

■ Solution T ■

Exercise 5.15

■ Solution M ■

EXERCISE 5.16 Suppose X and Y are Banach spaces, and suppose Λ is a linear mapping of X into Y, with the following property: For every sequence $\{x_n\}$ in X for which $x = \lim x_n$ and $y = \lim \Lambda x_n$ exist, it is true that $y = \Lambda x$. Prove that Λ is continuous.

This is the so-called "closed graph theorem". *Hint:* Let $X \oplus Y$ be the set of all ordered pairs (x, y), $x \in X$ and $y \in Y$, with addition and scalar multiplication defined componentwise. Prove that $X \oplus Y$ is a Banach space, if $\|(x,y)\| = \|x\| + \|y\|$. The graph G of Λ is the subset of $X \oplus Y$ formed by the pairs $(x, \Lambda x)$, $x \in X$. Note that our hypothesis says that G is closed; hence G is a Banach space. Note that $(x, \Lambda x) \to x$ is continuous, one-to-one, and linear and maps G onto X.

Observe that there exist *nonlinear* mappings (of $\mathbb{R}^1$ onto $\mathbb{R}^1$, for instance) whose graph is closed although they are not continuous: $f(x) = 1/x$ if $x \neq 0$, $f(0) = 0$.

■ SOLUTION The fact that the function

$$\begin{aligned} G &\to X \\ (x, \Lambda x) &\mapsto x \end{aligned}$$

is linear, surjective and maps Banach spaces into Banach spaces hints to the utilization of the Open Mapping Theorem or corollaries thereof. Let $\varphi : X \to G$ be its inverse. By Theorem 5.10, φ is continuous. The result now follows since

$$\Lambda = \pi_Y \circ \varphi,$$

where $\pi_Y : G \to Y$ is the (clearly continuous) projection from G to Y, is a composition of continuous functions.

The fact that $X \oplus Y$ is a Banach space is clear since if $\{(x_n, y_n)\}$ is a Cauchy sequence in $X \oplus Y$, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y, respectively. So, if $x_n \to x$ and $y_n \to y$, it follows that $(x_n, y_n) \to (x, y)$. ■

EXERCISE 5.17 If μ is a positive measure, each $f \in L^\infty(\mu)$ defines a multiplication operator M_f on $L^2(\mu)$ into $L^2(\mu)$, such that $M_f(g) = fg$. Prove that $\|M_f\| \leq \|f\|_\infty$. For which measures μ is it true that $\|M_f\| = \|f\|_\infty$ for all $f \in L^\infty(\mu)$? For which $f \in L^\infty(\mu)$ does M_f map $L^2(\mu)$ onto $L^2(\mu)$?

■ SOLUTION T ■

EXERCISE 5.18

■ SOLUTION Gu ■

EXERCISE 5.19

■ SOLUTION L ■

EXERCISE 5.20

(a) Does there exist a sequence of continuous positive functions f_n on $\mathbb{R}^1$ such that $\{f_n(x)\}$ is unbounded if and only if x is rational?

(b) Replace "rational" by "irrational" in (a) and answer the resulting question.

(c) Replace "$\{f_n(x)\}$ is unbounded" by "$f_n(x) \to \infty$ as $n \to \infty$" and answer the resulting analogues of (a) and (b).

■ SOLUTION T ■

EXERCISE 5.21

■ SOLUTION M ■

EXERCISE 5.22 Suppose $f \in C(T)$ and $f \in \operatorname{Lip} \alpha$ for some $\alpha > 0$. (See Exercise 5.11.) Prove that the Fourier series of f converges to $f(x)$, by completing the following outline: It is enough to consider the case $x = 0$, $f(0) = 0$. The difference between the partial sums $s_n(f; 0)$ and the integrals

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin nt}{t} \, dt$$

tends to 0 as $n \to \infty$. The function $f(t)/t$ is in $L^1(T)$. Apply the Riemann-Lebesgue lemma. More careful reasoning shows that the convergence is actually uniform on T.

■ SOLUTION Firstly, if $g(t) = f(t + x) - f(x)$, then by linearity we have that

$$s_n(f; x) - s_n(g; 0) = f(x).$$

In other words, if $s_n(g; 0) \to 0$, then $s_n(f; x) \to f(x)$ so it suffices to consider the case $x = 0$, $f(0) = 0$.

Now we use the usual formula for $\sin(x + y)$ to write

$$D_n(t) = \frac{\sin(n + \frac{1}{2})t}{\sin(t/2)} = 2 \cdot \frac{\sin nt}{t} + \left[\sin nt \left(\cot(t/2) - \frac{2}{t} \right) + \cos nt \right].$$

Since D_n is even, this implies that the difference in the exercise's statement is equal to

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left(\cot(t/2) - \frac{2}{t} \right) \sin nt \, dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt.$$

But $\cot(t/2) - 2/t \to 0$ as $t \to 0$, which implies that this function has only a removable singularity. Thus, by defining it to be 0 for $t = 0$ we see that both

$$f(t) \left(\cot(t/2) - \frac{2}{t} \right) \quad \text{and} \quad f(t)$$

are continuous functions and belong to $L^1(T)$. The Riemann-Lebesgue lemma then implies that both integrals tend to 0 as $n \to \infty$.

Since $f \in \operatorname{Lip}\alpha$,

$$\left|\frac{f(t)}{t}\right| = \frac{|f(t)-f(0)|}{|t-0|} \le M_f|t-0|^{\alpha-1} = M_f|t|^{\alpha-1}$$

for all $t \neq 0$. This implies that $|f(t)/t|$ is integrable and so $f(t)/t \in L^1(T)$. Finally, as we saw,

$$s_n(f;0) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin nt}{t}\,dt + x_n,$$

where $x_n \to 0$ as $n \to \infty$. A third application of the Riemann-Lebesgue lemma implies that $s_n(f;0) \to 0$.

In view of Arzelà-Ascoli's theorem (theorem 7.25 in *Principles of Mathematical Analysis*), it suffices to show that $\{s_n(f;x) - f(x)\}$ is equicontinuous to have uniform convergence. For that, we have to bound

$$|[s_n(f;x)-f(x)] - [s_n(f;y)-f(y)]|.$$

Since $\int_{-\pi}^{\pi} D_n(t)\,dt = 2\pi$, this is equal to

$$\left|\int_{-\pi}^{\pi} \{[f(x-t)-f(x)] - [f(y-t)-f(y)]\}\, D_n(t)\,dt\right|.$$

Using the triangular inequality twice we bound this integral by

$$\int_A \{|f(x-t)-f(x)| + |f(y-t)-f(y)|\}\,|D_n(t)|\,dt + \\ \int_B \{|f(x-t)-f(y-t)| + |f(x)-f(y)|\}\,|D_n(t)|\,dt,$$

where $A = \{t \in [-\pi,\pi] : |t| < |x-y|\}$ and $B = \{t \in [-\pi,\pi] : |t| > |x-y|\}$. As $f \in \operatorname{Lip}\alpha$, it follows that this is bounded by

$$\int_A \{2M_f|t|^\alpha\}\,|D_n(t)|\,dt + \int_B \{2M_f|x-y|^\alpha\}\,|D_n(t)|\,dt.$$

Now, since $(t/2)/\sin(t/2) \to 1$ as $t \to 0$, there exists $\delta > 0$ such that

$$|D_n(t)| < 4|t|^{-1}$$

for all $0 < |t| < \delta$. We conclude that, if $|x-y| < \min(\delta,\pi)$, the integral over A is bounded by

$$8M_f\int_A |t|^{\alpha-1}\,dt$$

and that the integral over B is bounded by

$$2M_f|x-y|^\alpha \underbrace{\int_B |D_n(t)|\,dt}_{\text{bounded}}.$$

This integral is bounded since $|D_n(t)| \to 0$ for all $t \neq 0$ as $n \to \infty$.

Finally, since $-1 < \alpha - 1 \leq 0$, we can estimate $\int_A |t|^{\alpha-1}\,dt$ in the following way: let $|x-y| = a$ and observe that

$$\begin{aligned}
\int_A |t|^{\alpha-1}\,dt &= 2\int_0^a t^{\alpha-1}\,dt \\
&= 2\left(\int_{a/2}^{a} t^{\alpha-1}\,dt + \int_{a/4}^{a/2} t^{\alpha-1}\,dt + \dots\right) \\
&\leq 2\left(\frac{a}{2}\left(\frac{a}{2}\right)^{\alpha-1} + \frac{a}{4}\left(\frac{a}{4}\right)^{\alpha-1} + \dots\right) \\
&= K|x-y|^{\alpha},
\end{aligned}$$

for some constant $K > 0$. Putting it all together, we get that if $|x-y| < \min(\delta, \pi)$, then

$$|[s_n(f;x) - f(x)] - [s_n(f;y) - f(y)]| < K'|x-y|^{\alpha},$$

for another constant $K' > 0$. This implies that $\{s_n(f;x) - f(x)\}$ is equicontinuous and the result follows. ∎

Remark. For $f \in \operatorname{Lip}\alpha$ we can have a explicit bound on $|\hat{f}(n)|$ in the following way: by periodicity we have that, for $n \neq 0$,

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(t + \frac{\pi}{n}\right)\underbrace{e^{-in(t+\pi/n)}}_{=-e^{-int}}\,dt.$$

Averaging the two expressions and utilising the definition of the Lipschitz condition,

$$\begin{aligned}
|\hat{f}(n)| &= \left|\frac{1}{4\pi}\int_{-\pi}^{\pi}\left[f(t) - f\left(t+\frac{\pi}{n}\right)\right]e^{-int}\,dt\right| \\
&\leq \frac{1}{4\pi}\int_{-\pi}^{\pi}\left|f(t) - f\left(t+\frac{\pi}{n}\right)\right|\,dt \\
&\leq \frac{2\pi}{4\pi}M_f\left|t - \left(t+\frac{\pi}{n}\right)\right|^{\alpha} = \frac{M_f}{2}\left|\frac{\pi}{n}\right|^{\alpha}.
\end{aligned}$$

Observe that letting $n \to \pm\infty$ we obtain another proof the the Riemann-Lebesgue lemma. Unfortunately, this bound is not strong enough to imply that $s_n(f;x) \to f(x)$ uniformly.

6 COMPLEX MEASURES

Exercise 6.1 If μ is a complex measure on a σ-algebra $\mathfrak{M}$, and if $E \in \mathfrak{M}$, define

$$\lambda(E) = \sup \sum |\mu(E_i)|,$$

the supremum being taken over all *finite* partitions $\{E_i\}$ of E. Does it follow that $\lambda = |\mu|$?

■ Solution Since $\{|\mu(E_i)| : \{E_i\} \text{ is a finite partition of } E\}$ is a subset of $\{|\mu(E_i)| : \{E_i\} \text{ is a partition of } E\}$, we clearly have that $\lambda \le |\mu|$.

Now let $\{E_i\}$ be a not necessarily finite partition of E and $\epsilon > 0$. Since $\sum_i |\mu(E_i)|$ converges to $|\mu|(E)$, there exists n such that

$$|\mu|(E) < \sum_{i=1}^{n} |\mu(E_i)| + \epsilon.$$

Since $\{E_i : i = 1, 2, \ldots, n\}$ is a finite partition of $E_1 \cup \ldots \cup E_n$,

By the same argument as in Theorem 6.2, λ is monotone.

$$\sum_{i=1}^{n} |\mu(E_i)| \le \lambda\left(\bigcup_{i=1}^{n} E_i\right) \le \lambda(E).$$

Using both inequalities we obtain that $|\mu| < \lambda + \epsilon$ for all $\epsilon > 0$. It follows that $|\mu| = \lambda$. ■

Exercise 6.2

■ Solution Gu ■

Exercise 6.3

■ Solution L ■

Exercise 6.4

■ Solution M ■

Exercise 6.5

■ Solution T ■

Exercise 6.6 Suppose $1 < p < \infty$ and prove that $L^q(\mu)$ is the dual space of $L^p(\mu)$ even if μ is not σ-finite. (As usual, $1/p + 1/q = 1$.)

■ Solution Ga ■

Exercise 6.7

■ Solution Gu ■

Exercise 6.8

■ Solution M ■

Exercise 6.9

■ Solution L ■

Exercise 6.10

■ Solution T ■

Exercise 6.11 Suppose μ is a positive measure on X, $\mu(X) < \infty$, $f_n \in L^1(\mu)$ for $n = 1, 2, 3, \dots$, $f_n(x) \to f(x)$ a.e., and there exists $p > 1$ and $C < \infty$ such that $\int_X |f_n|^p \, d\mu < C$ for all n. Prove that

$$\lim_{n \to \infty} \int_X |f - f_n| \, d\mu = 0.$$

Hint: $\{f_n\}$ is uniformly integrable.

■ Solution By Hölder's inequality,

$$\left| \int_E f_n \, d\mu \right| = \left| \int_X f_n \chi_E \, d\mu \right| \leq \left\{ \int_X |f_n|^p \, d\mu \right\}^{1/p} \mu(E)^{1/q},$$

with $1/p + 1/q = 1$. In other words, if $\mu(E) < \delta$ we have that

$$\left| \int_E f_n \, d\mu \right| < C^{1/p} \delta^{1/q}$$

which implies that $\{f_n\}$ is uniformly integrable. (Recall the definition in Exercise 6.10.) Since $f_n \in L^1(\mu)$ implies that f_n is finite almost everywhere, we satisfy all the conditions in Vitali's theorem (Exercise 6.10.(b)), which implies our result. ■

Exercise 6.12

■ Solution Gu ■

Exercise 6.13

■ Solution M ■

7 DIFFERENTIATION

The function Mf is called *Hardy-Littlewood maximal function*.

EXERCISE 7.1 Show that $|f(x)| \leq (Mf)(x)$ at every Lebesgue point of f if $f \in L^1(\mathbb{R}^k)$.

■ SOLUTION By the triangular inequality,

$$\frac{1}{m(B_r)} \left| \int_{B(x,r)} f \, dm \right| \leq \frac{1}{m(B_r)} \int_{B(x,r)} |f| \, dm.$$

Since x is a Lebesgue point of f, the left side tends to $|f(x)|$ as $r \to 0$. Also, the supremum of the right side, for $0 < r < \infty$, is $(Mf)(x)$. The result follows. ■

EXERCISE 7.2

■ SOLUTION Gu ■

EXERCISE 7.3

■ SOLUTION M ■

EXERCISE 7.4

■ SOLUTION L ■

EXERCISE 7.5

■ SOLUTION T ■

EXERCISE 7.6 Suppose G is a subgroup of $\mathbb{R}^1$ (relative to addition), $G \neq \mathbb{R}^1$, and G is Lebesgue measurable. Prove that then $m(G) = 0$.
Hint: Use Exercise 7.5.

■ SOLUTION Suppose that $m(G) > 0$. Then, by Exercise 7.5, $G + G \subset G$ contains an interval I. Let m be the midpoint of this interval. Since $m \in G$, $-m \in G$ and thus $I - m$ is an interval contained in G, centered at the origin. Since G is a subgroup, $kI \subset G$ for all $k = 1, 2, \ldots$. It follows that $G = \mathbb{R}^1$, which is absurd! We conclude that $m(G) = 0$. ■

EXERCISE 7.7

■ SOLUTION Gu ■

EXERCISE 7.8

■ SOLUTION M ■

Exercise 7.9

■ Solution L ■

Exercise 7.10

■ Solution T ■

Exercise 7.11 Assume that $1 < p < \infty$, f is absolutely continuous on $[a, b]$, $f' \in L^p$, and $\alpha = 1/q$, where q is the exponent conjugate to p. Prove that $f \in \operatorname{Lip} \alpha$.

■ Solution Let $x, y \in [a, b]$. Without loss of generality, suppose that $x \geq y$. By Hölder's inequality,

$$|f(x) - f(y)| = \left| \int_y^x f' \, dm \right| \leq \int_y^x |f'| \, dm \leq \|f'\|_p |x - y|^{1/q}.$$

The result follows. ■

Exercise 7.12

■ Solution Gu ■

Exercise 7.13

■ Solution M ■

Exercise 7.14

■ Solution L ■

Exercise 7.15

■ Solution T ■

Exercise 7.16 Suppose $E \subset [a, b]$, $m(E) = 0$. Construct an absolutely continuous monotonic function f on $[a, b]$ so that $f'(x) = \infty$ at every $x \in E$.

Hint: $E \subset \bigcap V_n$, V_n open, $m(V_n) < 2^{-n}$. Consider the sum of the characteristic functions of these sets.

■ Solution Since m is outer regular, there exists a countable collection of open sets $\{V_n\}$ containing E such that $V_{n+1} \subset V_n$ and $m(V_n) < 2^{-n}$ for all n. Let $g = \sum_n \chi_{V_n}$ and $f(x) = \int_a^x g \, dm$. The function f is clearly monotonic and is absolutely continuous by the following lemma, which is of independent interest.

Lemma. Let g be an integrable function on $[a, b]$. Then $f(x) = \int_a^x g \, dm$ is absolutely continuous.

Proof. Let $d\lambda = |g| \, dm$. By Theorem 6.11, to every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\int_A |g| \, dm < \epsilon$$

for all Lebesgue measurable sets A with $m(A) < \delta$. Now, if (α_1, β_1), ...,(α_n, β_n) is a disjoint collection of segments whose lengths satisfy

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) < \delta,$$

then $A = \bigcup_i (\alpha_i, \beta_i)$ satisfies $m(A) < \delta$ and so

$$\sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| = \sum_{i=1}^{n} \int_{\alpha_i}^{\beta_i} |g|\, dm = \int_A |g|\, dm < \epsilon.$$

The result follows. □

The only thing that we have to show now is that $f'(x) = \infty$ at every $x \in E$. Let $x \in V_n$. Since V_n is open, there is a $\delta > 0$ such that $x + h \in V_n$ for all h such that $|h| < \delta$. This implies that

$$|f(x+h) - f(x)| = \left| \int_x^{x+h} g\, dm \right| \leq \left| \int_x^{x+h} \sum_{i=1}^{n} \chi_{V_i}\, dm \right| = n|h|,$$

since all the points in the integration domain are in V_i for all $i = 1, 2, \ldots, n$. Making $h \to 0$, we have that if $x \in \bigcap_n V_n$, then $f'(x) \geq n$ for all n. In other words, $f'(x) = \infty$ for all $x \in E$. ■

EXERCISE 7.17

■ SOLUTION Gu ■

EXERCISE 7.18

■ SOLUTION M ■

EXERCISE 7.19

■ SOLUTION L ■

EXERCISE 7.20

■ SOLUTION T ■

EXERCISE 7.21 If f is a real function on $[0, 1]$ and

$$\gamma(t) = t + if(t),$$

the length of the graph of f is, by definition, the total variation of γ on $[0, 1]$. Show that this length is finite if and only if $f \in \mathrm{BV}$. (See Exercise 7.13.) Show that it is equal to

$$\int_0^1 \sqrt{1 + [f'(t)]^2}\, dt$$

if f is absolutely continuous.

■ SOLUTION If $f \in \mathrm{BV}$, let $\{t_i\}$ be as in the definition of total variation. We have that

$$\begin{aligned}\sum_{i=1}^{N} |\gamma(t_i) - \gamma(t_{i-1})| &= \sum_{i=1}^{N} |t_i - t_{i-1} + i(f(t_i) - f(t_{i-1}))| \\ &\leq \sum_{i=1}^{N} |t_i - t_{i-1}| + \sum_{i=1}^{N} |f(t_i) - f(t_{i-1})|.\end{aligned}$$

Taking the supremum of both sides, it follows that

$$\text{total variation of } \gamma \leq 1 + \text{total variation of } f < \infty.$$

Conversely, since

$$\begin{aligned}\sum_{i=1}^{N} |\gamma(t_i) - \gamma(t_{i-1})| &= \sum_{i=1}^{N} |t_i - t_{i-1} + i(f(t_i) - f(t_{i-1}))| \\ &= \sum_{i=1}^{N} \sqrt{(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2} \\ &\geq \sum_{i=1}^{N} |f(t_i) - f(t_{i-1})|,\end{aligned}$$

we have that $\gamma \in \mathrm{BV}$ implies $f \in \mathrm{BV}$.

Finally, if f is absolutely continuous, so is γ. Thus, it suffices to show that the total variation of γ is

$$\int_0^1 |\gamma'| \, dm.$$

Let G be the total variation function of γ. By the triangular inequality we have that

$$\sum_{i=1}^{N} |\gamma(t_i) - \gamma(t_{i-1})| = \sum_{i=1}^{N} \left| \int_{t_{i-1}}^{t_i} \gamma' \, dm \right| \leq \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} |\gamma'| \, dm = \int_0^1 |\gamma'| \, dm.$$

Taking the supremum we obtain

$$G(1) \leq \int_0^1 |\gamma'| \, dm.$$

For the other inequality, observe that $|\gamma(x) - \gamma(y)| \leq |G(x) - G(y)|$ for all $x, y \in [0, 1]$, which implies that, whenever γ and G are differentiable, $|\gamma'| \leq |G'| = G'$ (as G is nondecreasing). Now, since G is absolutely continuous,

$$\int_0^1 |\gamma'| \, dm \leq \int_0^1 G' \, dm = G(1) - G(0) = G(1).$$

The result follows. ■

Exercise 7.22

■ Solution Gu ■

Exercise 7.23

■ Solution M ■

8 INTEGRATION ON PRODUCT SPACES

EXERCISE 8.1 Find a monotone class $\mathfrak{M}$ in $\mathbb{R}^1$ which is not a σ-algebra, even though $\mathbb{R}^1 \in \mathfrak{M}$ and $\mathbb{R}^1 - A \in \mathfrak{M}$ for every $A \in \mathfrak{M}$.

■ SOLUTION Let $\mathfrak{M}$ consist of all the unbounded intervals in $\mathbb{R}^1$, together with the empty set. In other words, an element of $\mathfrak{M}$ is either the empty set, $\mathbb{R}^1$,

$$(-\infty, a), \quad (-\infty, a], \quad (a, \infty), \quad \text{or} \quad [a, \infty)$$

for some $a \in \mathbb{R}^1$. Clearly $\mathfrak{M}$ is a monotone class which is closed under complements. Nevertheless, it is *not* a σ-algebra since

$$(-\infty, 1) \cap (0, \infty) = (0, 1)$$

is a finite intersection of elements of $\mathfrak{M}$ which is not in $\mathfrak{M}$. ■

EXERCISE 8.2

■ SOLUTION Gu ■

EXERCISE 8.3

■ SOLUTION M ■

EXERCISE 8.4

■ SOLUTION L ■

EXERCISE 8.5

■ SOLUTION T ■

EXERCISE 8.6 — Polar coordinates in $\mathbb{R}^k$. Let S_{k-1} be the unit sphere in $\mathbb{R}^k$, i.e., the set of all $u \in \mathbb{R}^k$ whose distance from the origin 0 is 1. Show that every $x \in \mathbb{R}^k$, except for $x = 0$, has a unique representation of the form $x = ru$, where r is a positive real number and $u \in S_{k-1}$. Thus $\mathbb{R}^k - \{0\}$ bay be regarded as the cartesian product $(0, \infty) \times S_{k-1}$.

Let m_k be the Lebesgue measure on $\mathbb{R}^k$, and define a measure σ_{k-1} on S_{k-1} as follows: If $A \subset S_{k-1}$ and A is a Borel set, let $\widetilde{A}$ be the set of points ru, where $0 < r < 1$ and $u \in A$, and define

$$\sigma_{k-1}(A) = k \cdot m_k(\widetilde{A}).$$

Prove that the formula

$$\int_{\mathbb{R}^k} f\,dm_k = \int_0^\infty r^{k-1}\,dr \int_{S_{k-1}} f(ru)\,d\sigma_{k-1}(u)$$

is valid for every nonnegative Borel function f on $\mathbb{R}^k$. Check that this coincides with the familiar results when $k = 2$ and $k = 3$.

Suggestion: If $0 < r_1 < r_2$ and if A is an open subset of S_{k-1}, let E be the set of all ru with $r_1 < r < r_2$, $u \in A$, and verify that the formula holds for the characteristic function of E. Pass from these to characteristic functions of Borel sets in $\mathbb{R}^k$.

■ Solution Firstly, every non-zero $x \in \mathbb{R}^k$ can be written as

$$x = \|x\| \frac{x}{\|x\|},$$

where $\|x\|$ is a positive real number and $x/\|x\| \in S_{k-1}$. If we had two representations $x = r_1 u_1 = r_2 u_2$, then it would follow that

$$\frac{r_1}{r_2} u_1$$

has norm equal to 1 and so $r_1 = r_2$. This implies that $u_1 = u_2$.

In the rest of this exercise, we will use rA to denote the set $\{rx : x \in A\}$ for all $r > 0$ and $A \subset \mathbb{R}^k$. Also, $m_k(rA) = r^k m_k(A)$ holds for measurable set since it holds for k-cells. Now, by the regularity of the Lebesgue measure

$$m_k(E) = m_k(r_2\widetilde{A} - r_1\widetilde{A}) = (r_2^k - r_1^k) m_k(\widetilde{A}) = \frac{r_2^k - r_1^k}{k} \sigma_{k-1}(A).$$

In other words,

$$\begin{aligned}\int_{\mathbb{R}^k} \chi_E\,dm_k &= \int_{r_1}^{r_2} r^{k-1}\,dr \int_{S_{k-1}} \chi_A(u)\,d\sigma_{k-1}(u)\\ &= \int_0^\infty r^{k-1}\,dr \int_{S_{k-1}} \chi_{(r_1,r_2)}(r)\,\chi_A(u)\,d\sigma_{k-1}(u).\end{aligned}$$

Since $\chi_E(x) = 1$ if and only if $x = ru$ with $\chi_{(r_1,r_2)}(r)\chi_A(u) = 1$, this proves that the formula from the statement holds for the characteristic function of E.

CONTINUAR ■

Exercise 8.7

■ Solution Gu ■

Exercise 8.8

■ Solution M ■

Exercise 8.9

■ Solution L ■

Exercise 8.10

■ Solution T ■

Exercise 8.11 Let $\mathscr{B}_k$ be the σ-algebra of all Borel sets in $\mathbb{R}^k$. Prove that $\mathscr{B}_{m+n} = \mathscr{B}_m \times \mathscr{B}_n$. This is relevant in Theorem 8.14.

■ Solution If $A \in \mathscr{B}_m$ and $B \in \mathscr{B}_n$, both $A \times \mathbb{R}^n$ and $\mathbb{R}^m \times B$ are Borel sets in $\mathbb{R}^{m+n}$. Since

$$A \times B = (A \times \mathbb{R}^n) \cap (\mathbb{R}^m \times B),$$

this implies that $A \times B \in \mathscr{B}_{m+n}$. In other words, $\mathscr{B}_m \times \mathscr{B}_n \subset \mathscr{B}_{m+n}$.

Conversely, let $V \in \mathbb{R}^{m+n}$ be an open set. Since $\mathbb{R}^{m+n}$ is the topological product $\mathbb{R}^m \times \mathbb{R}^n$ and $\mathbb{R}^k$ is second countable for all k, V can be written as

$$V = \bigcup_{k=1}^{\infty} A_k \times B_k,$$

where $A_k \in \mathbb{R}^m$ and $B_k \in \mathbb{R}^n$ are open sets. This implies that $V \in \mathscr{B}_m \times \mathscr{B}_n$. As $\mathscr{B}_{m+n}$ is the smallest σ-algebra which contains the open sets, it follows that $\mathscr{B}_{m+n} \subset \mathscr{B}_m \times \mathscr{B}_n$. ■

Exercise 8.12

■ Solution Gu ■

Exercise 8.13

■ Solution M ■

Exercise 8.14

■ Solution L ■

Exercise 8.15

■ Solution T ■

Exercise 8.16 Prove the following analogue of Minkowski's inequality, for $f \geq 0$:

$$\left\{ \int \left[\int f(x,y)\, d\lambda(y) \right]^p d\mu(x) \right\}^{\frac{1}{p}} \leq \int \left[\int f^p(x,y)\, d\mu(x) \right]^{\frac{1}{p}} d\lambda(y).$$

Supply the required hypotheses. (Many further developments of this theme may be found in G. H. Hardy, J. E. Littlewood, and G. Pólya's book *Inequalities*.)

■ Solution Let $I(x) = \int f(x,y)\, d\lambda(y)$. Suppose that $\int I(x)^p\, d\mu(x)$ is finite so that Fubini's theorem is justified. Then, by Hölder's inequality,

$$\begin{aligned}
\int I(x)^p\, d\mu(x) &= \int I(x)^{p-1}\, d\mu(x) \int f(x,y)\, d\lambda(y) \\
&= \int d\lambda(y) \int I(x)^{p-1} f(x,y)\, d\mu(x) \\
&\leq \int d\lambda(y) \left\{ \left[\int f(x,y)^p\, d\mu(x) \right]^{\frac{1}{p}} \left[\int I(x)^p\, d\mu(x) \right]^{\frac{1}{q}} \right\} \\
&= \left[\int I(x)^p\, d\mu(x) \right]^{\frac{1}{q}} \int \left[\int f(x,y)^p\, d\mu(x) \right]^{\frac{1}{p}} d\lambda(y),
\end{aligned}$$

where p and q are conjugated exponents. Now, if $\int I(x)^p\, d\mu(x) > 0$, it follows that

$$\left\{ \int I(x)^p\, d\mu(x) \right\}^{\frac{1}{p}} \leq \int \left[\int f(x,y)^p\, d\mu(x) \right]^{\frac{1}{p}} d\lambda(y),$$

which is what we wanted to prove. If $\int I(x)^p\, d\mu(x) = 0$, $I(x) = 0$ for almost all x and then $f(x,y) = 0$ for almost all x and y. It follows that both sides of our desired inequality are zero, so that it holds trivially.

Actually, this inequality holds even if $\int I(x)^p\, d\mu(x) = \infty$. However, in this case the proof is a little more delicate. The reader is encouraged to check the details in the cited book. ■

9

FOURIER TRANSFORMS

Recall that in this chapter m denotes the Lebesgue measure divided by $\sqrt{2\pi}$.

EXERCISE 9.1 Suppose $f \in L^1$, $f > 0$. Prove that $|\hat{f}(y)| < \hat{f}(0)$ for every $y \neq 0$.

■ SOLUTION By the triangular inequality,

$$|\hat{f}(y)| = \left| \int_{-\infty}^{\infty} f(x) e^{-ixy}\, dm(x) \right| \leq \int_{-\infty}^{\infty} |f(x)|\, dm(x) = \hat{f}(0).$$

Now, lets suppose that there exists $y \neq 0$ such that $|\hat{f}(y)| = \hat{f}(0)$. Then Theorem 1.39(c) implies that there is a constant α such that $\alpha f(x) e^{-ixy} = f(x)$ holds for almost all x. Since $f > 0$, this implies that $\alpha e^{-ixy} = 1$ for all x (since the exponential function is continuous). In other words, $y = 0$. This contradiction implies the result. ■

EXERCISE 9.2

■ SOLUTION Gu ■

EXERCISE 9.3

■ SOLUTION M ■

EXERCISE 9.4

■ SOLUTION L ■

EXERCISE 9.5

■ SOLUTION T ■

EXERCISE 9.6 Suppose $f \in L^1$, f is differentiable almost everywhere, and $f' \in L^1$. Does it follow that the Fourier transform of f' is $ti\hat{f}(t)$?

■ SOLUTION As it is, the answer is *no*. Take $f = \chi_{[-1,1]}$, for example. Clearly $f \in L^1$ and $f' = 0$ almost everywhere so that $f' \in L^1$. However,

$$0 = \widehat{f'}(t) \neq ti\hat{f}(t) = i\sqrt{\frac{2}{\pi}} \sin t.$$

Nevertheless, if we suppose that f is continuously differentiable, then the result is true. (It suffices to integrate by parts.) ■

EXERCISE 9.7

■ SOLUTION Gu ■

EXERCISE 9.8

■ SOLUTION M ■

EXERCISE 9.9

■ SOLUTION L ■

EXERCISE 9.10

■ SOLUTION T ■

EXERCISE 9.11 Find conditions on f and/or $\hat{f}$ which ensure the correctness of the following formal argument: If

$$\varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-itx}\, dx$$

and

$$F(x) = \sum_{k=-\infty}^{\infty} f(x + 2k\pi)$$

then F is periodic, with period 2π, the nth Fourier coefficient of F is $\varphi(n)$, hence $F(x) = \sum \varphi(n) e^{inx}$. In particular,

$$\sum_{k=-\infty}^{\infty} f(2k\pi) = \sum_{n=-\infty}^{\infty} \varphi(n).$$

More generally,

$$\sum_{k=-\infty}^{\infty} f(k\beta) = \alpha \sum_{n=-\infty}^{\infty} \varphi(n\alpha) \qquad \text{if } \alpha > 0,\ \beta > 0,\ \alpha\beta = 2\pi. \qquad (*)$$

What does $(*)$ say about the limit, as $\alpha \to 0$, of the right-hand side (for "nice" functions, of course)? Is this in agreement with the inversion theorem?

[$(*)$ is known as the Poisson summation formula.]

■ SOLUTION Firstly, it is clear that we ought to have $f \in L^1(\mathbb{R}^1)$ so that φ is well-defined. Since

$$\begin{aligned}\int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} |f(x + 2k\pi)|\, dx &= \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} |f(x + 2k\pi)|\, dx \\ &= \int_{-\infty}^{\infty} |f(x)|\, dx < \infty,\end{aligned}$$

it follows that F is finite almost everywhere and so $F \in L^1(T)$. In order to conclude that

$$F(x) = \sum_{n=-\infty}^{\infty} \varphi(n) e^{inx}$$

we need to assure pointwise convergence of the Fourier series. The only result we have so far that implies pointwise convergence is Exercise 5.22. So, if $F \in \operatorname{Lip}\alpha$ for $0 < \alpha \leq 1$, it follows that

This is actually a condition on f since $f \in \operatorname{Lip}\alpha$ implies that F satisfies a Lipschitz condition of order $\alpha^2/(\alpha+1)$.

$$F(x) = \sum_{n=-\infty}^{\infty} \varphi(n) e^{inx}.$$

This exact same reasoning implies $(*)$. As $\alpha \to 0$, the right-hand side of $(*)$ tends to the Riemann integral of φ on the real line. Finally, the left-hand side tends to

$$\int_{-\infty}^{\infty} \varphi(t)\, dt = \int_{-\infty}^{\infty} \hat{f}(t)\, dm(t) = f(0),$$

which holds by the inversion formula. ■

Exercise 9.12

■ Solution Gu ■

Exercise 9.13

■ Solution M ■

Exercise 9.14

■ Solution L ■

Exercise 9.15

■ Solution T ■

Exercise 9.16 The *Laplacian* of a function f on $\mathbb{R}^k$ is

$$\Delta f = \sum_{j=1}^{k} \frac{\partial^2 f}{\partial x_j^2},$$

provided the partial derivatives exist. What is the relation between $\hat{f}$ and $\hat{g}$ is $g = \Delta f$ and all necessary integrability conditions are satisfied? It is clear that the Laplacian commutes with translations. Prove that it also commutes with rotations, i.e., that

$$\Delta(f \circ A) = (\Delta f) \circ A$$

whenever f has continuous second derivatives and A is a rotation of $\mathbb{R}^k$. (Show that it is enough to do this under the additional assumption that f has compact support.)

■ Solution Integrating by parts, we can prove that

$$\widehat{\frac{\partial f}{\partial x_j}}(t) = i t_j \hat{f}(t).$$

It follows that if $g = \Delta f$, then

$$\hat{g}(t) = -\|t\|^2 \hat{f}(t).$$

Now suppose that f has compact support and so it has a Fourier transform. By the previous exercise,

$$\Delta \widehat{(f \circ A)}(t) = -\|t\|^2 \widehat{(f \circ A)}(t) = -\|t\|^2 \hat{f}(At).$$

Since rotations preserve norms, this is also equal to

$$-\|At\|^2 \hat{f}(At) = (\widehat{\Delta f \circ A})(t).$$

The injectivity of the Fourier transform implies that

$$\Delta(f \circ A) = (\Delta f) \circ A$$

for all $f \in C_c^2(\mathbb{R}^k)$. For $f \in C^2(\mathbb{R}^k)$, let $t \in \mathbb{R}^k$ and B be a ball containing At. Similarly to Urysohn's lemma, there is $\varphi \in C_c^\infty(\mathbb{R}^k)$ such that $\varphi(x) = 1$ for all $x \in B$. Then

$$f\varphi = f$$

in B. This implies that

$$\Delta(f \circ A)(t) = \Delta(f\varphi \circ A)(t) = (\Delta f\varphi) \circ A(t) = (\Delta f) \circ A(t).$$

Since this holds for all t, the result follows. ■

EXERCISE 9.17

■ SOLUTION Gu ■

EXERCISE 9.18

■ SOLUTION M ■

EXERCISE 9.19

■ SOLUTION L ■

10

ELEMENTARY PROPERTIES OF HOLOMORPHIC FUNCTIONS

11 HARMONIC FUNCTIONS

12 THE MAXIMUM MODULUS PRINCIPLE

13 APPROXIMATION BY RATIONAL FUNCTIONS

14 CONFORMAL MAPPING

15

ZEROS OF HOLOMORPHIC FUNCTIONS

16

ANALYTIC CONTINUATION

17

H^p-SPACES

18

ELEMENTARY THEORY OF BANACH ALGEBRAS

19
HOLOMORPHIC FOURIER TRANSFORMS

20 UNIFORM APPROXIMATION BY POLYNOMIALS