Six-functor formalisms

And all that *jazz*

Gabriel Ribeiro

École Polytechnique

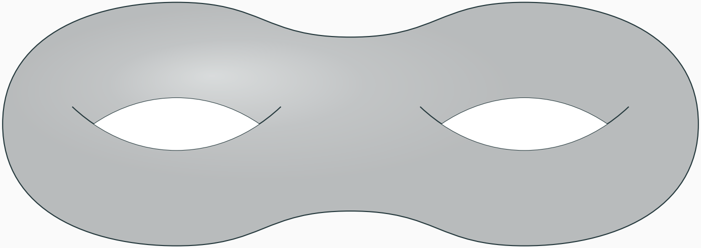
Summary

1. Why should we care?
2. What **is** a six-functor formalism
3. What follows formally
4. How this encodes cohomology
5. Examples

Why should we care?

Noether's point of view

If we wish to study a topological space X, a useful collection of invariants are the *Betti numbers* $b_n(X)$, measuring the number of n-dimensional holes of X.



Noether's point of view

Emmy Noether famously emphasized that the Betti numbers $b_n(X)$ are mere shadows of the more fundamental homology groups $H_n(X)$.

$$\begin{array}{c} H_n(X) \\ \text{rank}\, \Big\downarrow \quad \not\Big\uparrow \\ b_n(X) \end{array}$$

The arithmetic side

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This information can also be obtained from more fundamental groups

$$\#X(\mathbb{F}_q) = \sum_{i=0}^{2n} (-1)^i \operatorname{tr}(\mathrm{Frob}_q \mid H^i_{\text{ét}}(\overline{X})).$$

Going one step further

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$$\begin{array}{c} \mathrm{D}_c^b(\overline{X};\mathbb{Q}_\ell) \\ \downarrow\!\!\wr \;\; \not\uparrow\!\!\wr \\ H^\bullet_{\text{ét}}(\overline{X}) \\ \downarrow\!\!\wr \;\; \not\uparrow\!\!\wr \\ \#X(\mathbb{F}_q) \end{array}$$

What is a six-functor formalism

The objects

Let C be the category of the *spaces* in consideration. For each $X \in \mathsf{C}$, we define triangulated categories $\mathsf{D}(X)$, standing for a certain kind of derived category of sheaves over X. We suppose that, for each $X \in \mathsf{C}$:

(SF1) $\mathsf{D}(X)$ is a closed symmetric monoidal category with identity $\mathscr{O}_X$.

Functoriality

Given a morphism $f : X \to S$ in C, we suppose that

(SF2) There exist adjoint (triangulated) functors

$$f^* : \mathrm{D}(S) \leftrightarrows \mathrm{D}(X) : f_* \qquad \text{and} \qquad f_! : \mathrm{D}(X) \leftrightarrows \mathrm{D}(S) : f^!.$$

Moreover, there exists a natural morphism $f_! \to f_*$, which is an isomorphism when f is proper, and f^* is monoidal.

Base change

These functors should behave well with respect to base change. So, given a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\overline{g}} & X \\ {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S, \end{array}$$

the *proper base change* axiom then imposes that

(PBC) There is an isomorphism of functors $g^* \circ f_! \cong \bar{f}_! \circ \overline{g}^*$.

Locality

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(LOC) $\mathrm{D}(Z) \xrightarrow{i_*} \mathrm{D}(X) \xrightarrow{j^*} \mathrm{D}(U)$ is a localization sequence. (BBD §1.4.3)

What follows formally

Local forms

Proposition

Let $f : X \to S$ be a morphism in C. Then,

$$\underline{\mathrm{Hom}}(- \otimes -, -) \cong \underline{\mathrm{Hom}}(-, \underline{\mathrm{Hom}}(-, -))$$

$$\underline{\mathrm{Hom}}(-, f_* -) \cong f_* \underline{\mathrm{Hom}}(f^* -, -).$$

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$$\mathrm{Hom}(Q, \underline{\mathrm{Hom}}(M \otimes N, P)) \cong \mathrm{Hom}(Q \otimes (M \otimes N), P)$$

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\end{aligned}$$

naturally in M, N, P, Q. The fully-faithfullness of the Yoneda embedding then implies the first isomorphism.

Local forms

We observe that, since

$$\mathrm{Hom}(\mathscr{O}_X, \underline{\mathrm{Hom}}(M, N)) \cong \mathrm{Hom}(\mathscr{O}_X \otimes M, N) = \mathrm{Hom}(M, N),$$

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What about $f_! \dashv f^!$? One could certainly imagine that there exists an isomorphism

$$f_* \underline{\mathrm{Hom}}(M, f^! N) \xrightarrow{\sim} \underline{\mathrm{Hom}}(f_! M, N)$$

recovering the global adjunction.

Local forms

Proposition

Let $f : X \to S$ be a morphism. The existence of one of the morphisms

$$\gamma : f_* \underline{\mathrm{Hom}}(-, f^! -) \to \underline{\mathrm{Hom}}(f_! -, -), \;\; \delta : \underline{\mathrm{Hom}}(f^* -, f^! -) \to f^! \underline{\mathrm{Hom}}(-, -),$$
$$\text{and} \quad \pi : - \otimes f_! - \to f_!(f^* - \otimes -)$$

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We say that we're in the *Verdier-Grothendieck context* if there exists one (hence all) of the morphisms above, and it's an isomorphism. We'll suppose this from now on.

Purity

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Proposition - Relative purity

Let $f : X \to S$ be a morphism and $M \in \mathrm{D}(S)$ be a dualizable object. Then the map

$$\varphi : f^*M \otimes f^!N \to f^!(M \otimes N)$$

is an isomorphism for all $N \in \mathrm{D}(S)$. In particular, $f^!M \cong f^*M \otimes f^!\mathscr{O}_S$.

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The calculation of $f^! \mathscr{O}_S$ is said to be a result of *absolute purity*.

Dualizing complexes

Suppose that C has a terminal object S.

Definition

Let $p : X \to S$ be the natural morphism. We define the *duality functor* $\mathrm{D}_X : \mathsf{D}(X) \to \mathsf{D}(X)^{\mathrm{op}}$ as $\underline{\mathrm{Hom}}(-, p^! \mathscr{O}_S)$.

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If $f : X \to Y$ is a morphism of spaces over S, the morphisms γ and δ specialize to

$$f_* \mathrm{D}_X \cong \mathrm{D}_Y f_! \qquad \text{and} \qquad \mathrm{D}_X f^* \cong f^! \mathrm{D}_Y.$$

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$$f_*\mathrm{D}_X \cong \mathrm{D}_Y f_! \qquad \text{and} \qquad \mathrm{D}_X f^* \cong f^!\mathrm{D}_Y.$$

If the natural map $\mathrm{id} \to \mathrm{D}_X\mathrm{D}_X$ is an isomorphism, then

$$f_! \cong \mathrm{D}_Y f_* \mathrm{D}_X \qquad \text{and} \qquad f^! \cong \mathrm{D}_X f^* \mathrm{D}_Y,$$

simplifying their study.

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If, moreover, a morphism $f : X \to Y$ is such that $f_*\mathrm{D}_c(X) \subset \mathrm{D}_c(Y)$ and $f^*\mathrm{D}_c(Y) \subset \mathrm{D}_c(X)$, then the strategy above works for expressing $f_!$ and $f^!$ in terms of f_* and f^*.

How this encodes cohomology

Motivation

For now, suppose that C is the category of ringed spaces, that $\mathrm{D}(X)$ is the derived category of $\mathscr{O}_X$-Mod, with the usual functors.

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Since $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{O}_X, -) = \mathrm{id}$, we have that $\Gamma(X, -) = \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, -)$ and so

$$H^i(X, -) = \mathrm{Ext}^i_{\mathscr{O}_X}(\mathscr{O}_X, -) = \mathrm{Hom}_{\mathsf{D}(X)}(\mathscr{O}_X, -[i]).$$

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If $p : X \to S$ is the unique map from X to a point (with $\underline{\mathbb{Z}}$ as structure sheaf), we can use the adjunction to write this as

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Similarly, as the module of sections with quasi-compact support $\Gamma_c(X, -)$ is defined as $\Gamma(S, p_! -)$, it follows that

$$H^i_c(X, -) = \mathrm{Hom}_{\mathsf{D}(S)}(\mathscr{O}_S, p_! -[i]).$$

Definition of cohomology

Going back to our abstract setting, we define cohomology as

$$H^i(X,M) = \mathrm{Hom}_{\mathrm{D}(S)}(\mathscr{O}_S, p_*M[i]) \quad \text{and} \quad H^i_c(X,M) = \mathrm{Hom}_{\mathrm{D}(S)}(\mathscr{O}_S, p_!M[i]),$$

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Basically all cohomological constructions seem to work very simply in this setting.

Basic constructions

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4. A distinguished triangle in $\mathrm{D}(X)$ gives rise to a long exact sequence in A-Mod.

Künneth formula

Let $p : X \to S$ and $q : Y \to S$ be the natural maps in C and consider the cartesian diagram

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\overline{p}} & Y \\ {\scriptstyle \overline{q}}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{p} & S. \end{array}$$

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If $M \in \mathrm{D}(X)$ and $N \in \mathrm{D}(Y)$ we define their *exterior tensor product* as

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By the projection formula and proper base change, we have

$$\overline{q}_!(M \boxtimes N) = \overline{q}_!(\overline{q}^* M \otimes \overline{p}^* N) = M \otimes \overline{q}_! \overline{p}^* N = M \otimes p^* q_! N.$$

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Let $f = q\overline{p} = p\overline{q}$. We apply $p_!$ to the equation above and use the projection formula once again to obtain

$$f_!(M \boxtimes N) = p_!(M \otimes p^* q_! N) = p_! M \otimes q_! N.$$

Künneth formula (*)

If every element of $\mathrm{D}(S)$ is a sum of shifts of $\mathscr{O}_S$ (that's the case if $S = \operatorname{Spec} k$), then

$$H^i_c(X \times_S Y, M \boxtimes N) = \bigoplus_{p+q=i} H^p_c(X, M) \otimes_A H^q_c(Y, N).$$

Cup product

The adjoint of the map

$$p^*(p_*M \otimes p_*N) \cong p^*p_*M \otimes p^*p_*N \to M \otimes N,$$

defined using the counit of the adjunction, is a natural morphism

$$\mu : p_*M \otimes p_*N \to p_*(M \otimes N).$$

Cup product

We then define the cup product as the composition

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H^i(X,M)\times H^j(X,N) = & \mathrm{Hom}_{\mathrm{D}(S)}(\mathscr{O}_S, p_*M[i]) \times \mathrm{Hom}_{\mathrm{D}(S)}(\mathscr{O}_S, p_*N[j]) \\
\Big\downarrow \smile & \Big\downarrow \otimes \\
& \mathrm{Hom}_{\mathrm{D}(S)}(\mathscr{O}_S, p_*M[i]) \times \mathrm{Hom}_{\mathrm{D}(S)}(p_*M[i], p_*M \otimes p_*N[i+j]) \\
& \Big\downarrow \circ \\
& \mathrm{Hom}_{\mathrm{D}(S)}(\mathscr{O}_S, p_*M \otimes p_*N[i+j]) \\
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As usual, if $M = N = \mathscr{O}_X$, this defines a structure of graded ring on $H^\bullet(X)$.

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If M is dualizable, this allows us to define a perfect pairing

$$H^i(X, M^\vee \otimes p^! \mathscr{O}_S) \times H^{-i}_c(X, M) \xrightarrow{\smile} H^0_c(X, p^! \mathscr{O}_S) \xrightarrow{\varepsilon} H^0(S, \mathscr{O}_S) = A.$$

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More generally, we have a perfect pairing

$$\mathrm{Ext}^i(M, p^! \mathscr{O}_S) \times H^{-i}_c(X, M) \to A.$$

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1. Mayer-Vietoris
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5. Alexander and Lefschetz dualities
6. More?

Examples

Verdier duality

Let C be the category of "nice" topological spaces and D(X) be the derived category of abelian sheaves over X. In this case,

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If $f : X \to S$ is smooth of relative dimension n, $f^!\underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}[n]$.In particular, if L is a local system and X is a manifold of dimension n,

$$H^{d-i}(X, L^\vee) \cong H^{-i}(X, L^\vee \otimes \underline{\mathbb{Z}}[d]) \cong H^i_c(X, L)^\vee.$$

Verdier duality

All the functors restrict to the full subcategory $\mathsf{D}^b_c(X)$ of complexes with bounded and constructible cohomology. This makes the natural maps

$$\mathsf{id} \to \mathrm{D}_X \mathrm{D}_X$$

isomorphisms.

Étale cohomology

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Extending these results to ℓ-adic cohomology and to algebraic stacks was a difficult problem.

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Let C be the category of ”nice” schemes with ”nice” morphisms, and let $\mathrm{D}(X)$ be the full subcategory of $\mathrm{D}(\mathscr{O}_X\text{-Mod})$ whose objects have quasi-coherent cohomology.

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$$H^{n-i}(X, \mathcal{E}^\vee \otimes \Omega^n_{X/S}) \cong H^{-i}(X, \mathcal{E}^\vee \otimes \Omega^n_{X/S}[n]) \cong H^i(X, \mathcal{E})^\vee.$$

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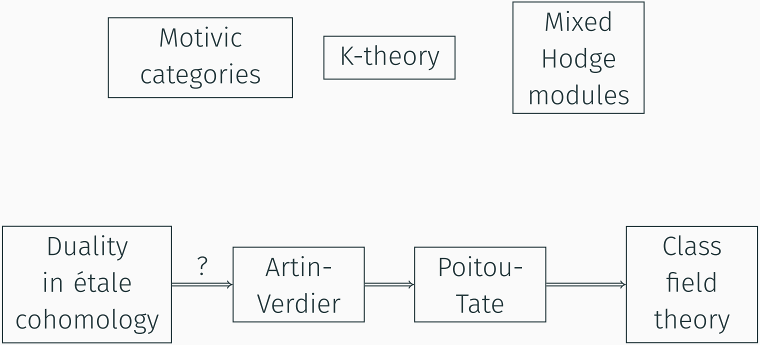
- f_* preserves "finiteness" $\implies H^i_{\mathrm{dR}}(X)$ is finite-dimensional
- $\mathrm{D}(X) \cong D_c(X^{\mathrm{an}}) \implies H^i_{\mathrm{dR}}(X) \cong H^i(X^{\mathrm{an}}, \mathbb{C})$

Others (**)


Motivic categories
K-theory
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?
Artin-Verdier
Poitou-Tate
Class field theory

Questions?