Perverse sheaves

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- 1. t-structures
- 2. Recollement
- 3. The perverse t-structure
- 4. Main properties

"[...] But in my opinion, the most significant reason for the usefulness of perverse sheaves is the following secret known to experts: perverse sheaves are easy, in the sense that most arguments come down to a rather short list of tools, such as proper base change, smooth pullback, and open-closed distinguished triangles. In practice, one can reason and compute with perverse sheaves just using a list of these tools, much as calculus students might use a table of integrals. One does not have to dig into the details of flabby resolutions or sheafification any more than a calculus student needs to revisit Riemann sums to integrate a polynomial."

Pramod Achar

t-structures

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$$A \cong \{M^{\bullet} \in D(A) \mid \mathscr{H}^{i}(M^{\bullet}) = 0 \text{ for } i \neq 0\}.$$

Can we generalize this construction to obtain other abelian subcategories of D(A)?

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Definition

Let $(D^{\leq 0}, D^{\geq 0})$ be a pair of full subcategories of D and set $D^{\leq n} := D^{\leq 0}[-n]$, $D^{\geq n} := D^{\geq 0}[-n]$. Then $(D^{\leq 0}, D^{\geq 0})$ is said to be a **t-structure** if

- (a) $D^{\leq -1} \subset D^{\leq 0}$ and $D^{\geq 1} \subset D^{\geq 0}$;
- (b) $\operatorname{Hom}_{D}(M, N) = 0$ for $M \in D^{\leq 0}$ and $N \in D^{\geq 1}$;
- (c) For all $N \in D$, there exists a distinguished triangle $M \to N \to P$, where $M \in D^{\leq 0}$ and $P \in D^{\geq 1}$.

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If (D^{\leq 0}, D^{\geq 0}) is a t-structure, then so is (D^{\leq n}, D^{\geq n}).
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$$\tau^{\leq 0} N \to N \to \tau^{\geq 1} N \to \tau^{\leq 0} N[1].$$

Theorem

Let $(D^{\leq 0}, D^{\geq 0})$ be a t-structure on D. Then,

- (a) The inclusion $D^{\leq n} \rightarrow D$ has a right adjoint $\tau^{\leq n} : D \rightarrow D^{\leq n}$;
- (b) The inclusion $D^{\geq n} \rightarrow D$ has a left adjoint $\tau^{\geq n} : D \rightarrow D^{\geq n}$;
- (c) There's a unique natural transformation $\tau^{\geq n+1} \rightarrow \tau^{\leq n}$ [1] such that, for every $N \in D$,

$$\tau^{\leq n} N \to N \to \tau^{\geq n+1} N \to \tau^{\leq n} N[1]$$

is a distinguished triangle.

In particular, we may define cohomology functors.

Definition

Let $(D^{\leq 0}, D^{\geq 0})$ be a t-structure on D. We define the core D^{\heartsuit} as $D^{\leq 0} \cap D^{\geq 0}$ and the cohomology functor $\mathscr{H}^0 : D \to D^{\heartsuit}$ as $\tau^{\leq 0} \circ \tau^{\geq 0}$.

Of course, we also put $\mathscr{H}^n := \mathscr{H}^0(-[n]) = \tau^{\leq n} \circ \tau^{\geq n}[n]$.

Theorem

The core D^\heartsuit is an abelian category and the $\mathscr{H}^n: D\to D^\heartsuit$ are cohomological functors.

We begin the proof of our theorem with a simple lemma.

Lemma

Let $M \to N \to P \to M[1]$ be a distinguished triangle in D. If M, P are in $D^{\geq n}$, then so is N. Similarly, if M, P are in $D^{\leq n}$, then so is N.

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In order to prove that $N \in D^{\leq n}$, it suffices to check that $\tau^{\geq n+1}N = 0$. By adjunction,

$$\operatorname{Hom}_{\mathbb{D}}(\tau^{\geq n+1}N, \tau^{\geq n+1}N) \cong \operatorname{Hom}_{\mathbb{D}}(N, \tau^{\geq n+1}N).$$

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$$\operatorname{Hom}_{\mathbb{D}}(\tau^{\geq n+1}N, \tau^{\geq n+1}N) \cong \operatorname{Hom}_{\mathbb{D}}(N, \tau^{\geq n+1}N).$$

Finally, since $Hom_D(-, \tau^{\geq n+1}N)$ is a cohomological functor,

$$\underbrace{\operatorname{Hom}_{\mathbb{D}}(P,\tau^{\geq n+1}N)}_{=0} \to \operatorname{Hom}_{\mathbb{D}}(N,\tau^{\geq n+1}N) \to \underbrace{\operatorname{Hom}_{\mathbb{D}}(M,\tau^{\geq n+1}N)}_{=0}$$

is an exact sequence, finishing the proof.

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If D = D(A), the cone above is simply the complex $C = [M \xrightarrow{\varphi} N]$ in degrees -1 and 0. In particular $\mathscr{H}^{-1}(C) = \ker \varphi$ and $\mathscr{H}^{0}(C) = \operatorname{coker} \varphi$.

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In general, we can use the axioms of a t-structure to show that $\mathscr{H}^{-1}(C)$ (resp. $\mathscr{H}^{0}(C)$) satisfies the universal property of the kernel (resp. cokernel) of φ .

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The proof that the $\mathscr{H}^n : D \to D^{\heartsuit}$ are cohomological functors is similar. (And also uses the octahedral axiom!)

Recollement

Let $F : D_1 \to D_2$ be a triangulated functor and endow those categories with t-structures $(D_i^{\leq 0}, D_i^{\geq 0})$.

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Definition

We say that *F* is left t-exact if $F(D_1^{\geq 0}) \subset D_2^{\geq 0}$. It's right t-exact if $F(D_1^{\leq 0}) \subset D_2^{\leq 0}$. And it's t-exact if it's both left and right t-exact.

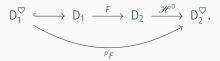
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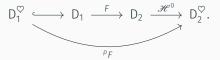
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If $F : A \to B$ is a left exact functor between abelian categories, then $RF : D(A) \to D(B)$ is left t-exact.

Conversely, let F be a triangulated functor as above and put

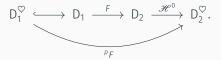


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If *F* is left t-exact, then ${}^{P}F$ is left exact. The same holds for right t-exact and t-exact. Similarly, if $F \dashv G$ is a pair of adjoint functors, then *F* is right t-exact if and only if *G* is left t-exact. In this case, we have ${}^{P}F \dashv {}^{P}G$.

Recollement

Let's abstract a "gluing situation": consider a diagram of triangulated categories (which are not necessarily derived categories)

 $D(Z) \xrightarrow{i_*} D(X) \xrightarrow{j^*} D(U).$

Moreover, set $i_! = i_*$ and $j^! = j^*$.

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- (b) j^* has a left adjoint $j_!$ and a right adjoint j_* ;

(c) $j^*i_* = 0;$

(d) For all $M \in D(X)$, there are morphisms $i_*i^*M \to j_!j^!M[1]$ and $j_*j^*M \to i_!i^!M[1]$ making the triangles

$$j_{!j}j^{!}M \to M \to i_{*}i^{*}M \to j_{!}j^{!}M[1]$$
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(d) For all $M \in D(X)$, there are morphisms $i_*i^*M \to j_!j^!M[1]$ and $j_*j^*M \to i_!i^!M[1]$ making the triangles

$$\begin{split} j_{!j}{}^{!}M &\to M \to i_{*}i^{*}M \to j_{!j}{}^{!}M[1] \\ i_{!}i^{!}M \to M \to j_{*}j^{*}M \to i_{!}i^{!}M[1] \end{split}$$

distinguished.

(e) The functors $j_{!}, j_{*}, i_{!} = i_{*}$ are fully faithful.

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- (a) There's a natural map $j_! \rightarrow j_*$ and we can define $j_{!*}$ to be $Im({}^pj_! \rightarrow {}^pj_*)$.
- (b) We can classify the simple objects of $D(X)^{\heartsuit}$.
- (c) The functor ${}^{p}i_{*}$ induces an equivalence between $D(Z)^{\heartsuit}$ and the full subcategory of $D(X)^{\heartsuit}$ whose objects M satisfy ${}^{p}j^{*}M = 0$.

$$D^{\leq 0}(X) := \{ M \in D(X) \mid j^*M \in D^{\leq 0}(U) \text{ and } i^*M \in D^{\leq 0}(Z) \}$$
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If X is a topological space with an open immersion $j : U \to X$, with complement $i : Z \to X$, and all of the above has the familiar meanings (along with the canonical t-structures), this procedure gives back the canonical t-structure on D(X).

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The perverse t-structure

Let X be a d-dimensional complex algebraic variety, along with its derived category of constructible sheaves $D_c^b(X)$.

In particular, $D_X(L[d]) \cong L^{\vee}[d]$.

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As we saw, this is indeed a t-structure.

A very clever observation is that some complex M^{\bullet} in $D^b_{loc}(X)$ lies in $D^b_{loc}(X)^{\leq -d}$ precisely when

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In particular, we may define

 ${}^{p}\mathsf{D}^{b}_{\mathsf{c}}(X)^{\leq 0} := \{M^{\bullet} \in \mathsf{D}^{b}_{\mathsf{c}}(X) \mid \operatorname{dim} \operatorname{supp} \mathscr{H}^{i}(M^{\bullet}) \leq -i \text{ for all } i \in \mathbb{Z} \}$ ${}^{p}\mathsf{D}^{b}_{\mathsf{c}}(X)^{\geq 0} := \{M^{\bullet} \in \mathsf{D}^{b}_{\mathsf{c}}(X) \mid \operatorname{dim} \operatorname{supp} \mathscr{H}^{i}(\mathsf{D}_{X}(M^{\bullet})) \leq -i \text{ for all } i \in \mathbb{Z} \},$

and then this induces the desired t-structure on $D_{loc}^{b}(X)$. This is the perverse t-structure on $D_{c}^{b}(X)$.

A first observation is that we have a filtered colimit $D_{c}^{b}(X) = 2$ -colim_S $D_{c,S}^{b}(X)$, where $D_{c,S}^{b}(X)$ is the derived category of constructible sheaves for a fixed stratification S.

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Now, we can split X as $U \coprod Z$, where the restriction of every complex in $D^b_{c,S}(X)$ to U lies in $D^b_{loc}(U)$. (Modulo some small technicalities that I'm hiding) this gives the desired t-structure on $D^b_{c,S}(X)$ (and then on $D^b_c(X)$) by recollement.

Main properties

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We already know that everything that follows formally from recollements is true for perverse sheaves. We also know that if L is a local system, then L[d] is a perverse sheaf. And we know that D_X is t-exact. Let's see what else can we do!

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- (e) Let $f : X \to S$ be a smooth morphism. Then $f^*[d] \cong f^![-d]$, for $d = \dim X \dim S$, is t-exact.
- (f) The exterior tensor product \boxtimes is t-exact.

Let X be a complex projective variety and $i : D \hookrightarrow X$ be the inclusion of a hyperplane section. Then, for $M \in Perv(X)$, the restriction map $\operatorname{H}^{i}(X, M) \to \operatorname{H}^{i}(D, i^{*}M)$ is an isomorphism for i < -1 and injective for i = -1.

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Let $j : U = X \setminus D \hookrightarrow X$ be the complementary open immersion. Recall the distinguished triangle

$$j_!j^!M \to M \to i_*i^*M \to j_!j^!M[1].$$

Let X be a complex projective variety and $i : D \hookrightarrow X$ be the inclusion of a hyperplane section. Then, for $M \in Perv(X)$, the restriction map $\operatorname{H}^{i}(X, M) \to \operatorname{H}^{i}(D, i^{*}M)$ is an isomorphism for i < -1 and injective for i = -1.

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As we saw, j^i is t-exact and $R_c\Gamma(U, -)$ is left t-exact (since U is affine). I.e., $H_c^i(U, j^iM) = 0$ for i < 0. The long exact sequence in cohomology then yields the result. Unfortunately, I don't have the time nor knowledge to give a proper introduction to the decomposition theorem. But the viewer should at least read something about it! (The whole chapter 6 of BBD is breathtaking!)

Questions?