Perverse sheaves

Gabriel Ribeiro

École Polytechnique

Summary

1. t-structures
2. Recollement
3. The perverse t-structure
4. Main properties

Why should I care?

"[...] But in my opinion, the most significant reason for the usefulness of perverse sheaves is the following secret known to experts: perverse sheaves are easy, in the sense that most arguments come down to a rather short list of tools, such as proper base change, smooth pullback, and open–closed distinguished triangles. In practice, one can reason and compute with perverse sheaves just using a list of these tools, much as calculus students might use a table of integrals. One does not have to dig into the details of flabby resolutions or sheafification any more than a calculus student needs to revisit Riemann sums to integrate a polynomial."

Pramod Achar

t-structures

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Can we generalize this construction to obtain other abelian subcategories of D(A)?

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(a) $\mathsf{D}^{\leq -1} \subset \mathsf{D}^{\leq 0}$ and $\mathsf{D}^{\geq 1} \subset \mathsf{D}^{\geq 0}$;

(b) $\mathrm{Hom}_{\mathsf{D}}(M, N) = 0$ for $M \in \mathsf{D}^{\leq 0}$ and $N \in \mathsf{D}^{\geq 1}$;

(c) For all $N \in \mathsf{D}$, there exists a distinguished triangle $M \to N \to P$, where $M \in \mathsf{D}^{\leq 0}$ and $P \in \mathsf{D}^{\geq 1}$.

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If $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ is a t-structure, then so is $(\mathsf{D}^{\leq n}, \mathsf{D}^{\geq n})$.

Canonical t-structure

If $\mathsf{D} := \mathsf{D}(\mathsf{A})$, we have a canonical t-structure given by

$$\mathsf{D}^{\leq 0} := \{M \in \mathsf{D} \mid \mathscr{H}^i(M) = 0 \text{ for } i > 0\}$$
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$$\tau^{\leq 0} N \to N \to \tau^{\geq 1} N \to \tau^{\leq 0} N[1].$$

Truncation functors

Theorem

Let $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ be a t-structure on D. Then,

(a) The inclusion $\mathsf{D}^{\leq n} \to \mathsf{D}$ has a right adjoint $\tau^{\leq n} : \mathsf{D} \to \mathsf{D}^{\leq n}$;

(b) The inclusion $\mathsf{D}^{\geq n} \to \mathsf{D}$ has a left adjoint $\tau^{\geq n} : \mathsf{D} \to \mathsf{D}^{\geq n}$;

(c) There's a unique natural transformation $\tau^{\geq n+1} \to \tau^{\leq n}[1]$ such that, for every $N \in \mathsf{D}$,

$$\tau^{\leq n} N \to N \to \tau^{\geq n+1} N \to \tau^{\leq n} N[1]$$

is a distinguished triangle.

The core

In particular, we may define cohomology functors.

Definition

Let $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ be a t-structure on D. We define the core $\mathsf{D}^{\heartsuit}$ as $\mathsf{D}^{\leq 0} \cap \mathsf{D}^{\geq 0}$ and the cohomology functor $\mathscr{H}^0 : \mathsf{D} \to \mathsf{D}^{\heartsuit}$ as $\tau^{\leq 0} \circ \tau^{\geq 0}$.

Of course, we also put $\mathscr{H}^n := \mathscr{H}^0(-[n]) = \tau^{\leq n} \circ \tau^{\geq n}[n]$.

The main theorem

Theorem

The core $\mathsf{D}^{\heartsuit}$ is an abelian category and the $\mathscr{H}^n : \mathsf{D} \to \mathsf{D}^{\heartsuit}$ are cohomological functors.

One lemma

We begin the proof of our theorem with a simple lemma.

Lemma

Let $M \to N \to P \to M[1]$ be a distinguished triangle in D. If M, P are in $\mathrm{D}^{\geq n}$, then so is N. Similarly, if M, P are in $\mathrm{D}^{\leq n}$, then so is N.

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$$\mathrm{Hom}_{\mathsf{D}}(\tau^{\geq n+1}N, \tau^{\geq n+1}N) \cong \mathrm{Hom}_{\mathsf{D}}(N, \tau^{\geq n+1}N).$$

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Finally, since $\mathrm{Hom}_{\mathsf{D}}(-, \tau^{\geq n+1} N)$ is a cohomological functor,

$$\underbrace{\mathrm{Hom}_{\mathsf{D}}(P, \tau^{\geq n+1} N)}_{=0} \to \mathrm{Hom}_{\mathsf{D}}(N, \tau^{\geq n+1} N) \to \underbrace{\mathrm{Hom}_{\mathsf{D}}(M, \tau^{\geq n+1} N)}_{=0}$$

is an exact sequence, finishing the proof.

Consequences of the lemma

This lemma implies two important facts:

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If $\mathsf{D} = \mathsf{D}(\mathsf{A})$, the cone above is simply the complex $C = [M \xrightarrow{\varphi} N]$ in degrees -1 and 0. In particular $\mathscr{H}^{-1}(C) = \ker \varphi$ and $\mathscr{H}^0(C) = \operatorname{coker} \varphi$.

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In general, we can use the axioms of a t-structure to show that $\mathscr{H}^{-1}(C)$ (resp. $\mathscr{H}^0(C)$) satisfies the universal property of the kernel (resp. cokernel) of φ.

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The proof that the $\mathscr{H}^n : \mathrm{D} \to \mathrm{D}^{\heartsuit}$ are cohomological functors is similar. (And also uses the octahedral axiom!)

Recollement

t-exact functors

Let $F : \mathsf{D}_1 \to \mathsf{D}_2$ be a triangulated functor and endow those categories with t-structures $(\mathsf{D}_i^{\leq 0}, \mathsf{D}_i^{\geq 0})$.

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Definition

We say that F is left t-exact if $F(\mathsf{D}_1^{\geq 0}) \subset \mathsf{D}_2^{\geq 0}$. It's right t-exact if $F(\mathsf{D}_1^{\leq 0}) \subset \mathsf{D}_2^{\leq 0}$. And it's t-exact if it's both left and right t-exact.

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If $F : \mathsf{A} \to \mathsf{B}$ is a left exact functor between abelian categories, then $\mathsf{R}F : \mathsf{D}(\mathsf{A}) \to \mathsf{D}(\mathsf{B})$ is left t-exact.

t-exact functors [BBD, Prop. 1.3.17]

Conversely, let F be a triangulated functor as above and put

$$\begin{tikzcd} \mathsf{D}_1^{\heartsuit} \arrow[r, hook] \arrow[rrr, bend right, "{}^p F"'] & \mathsf{D}_1 \arrow[r, "F"] & \mathsf{D}_2 \arrow[r, "\mathscr{H}^0"] & \mathsf{D}_2^{\heartsuit}. \end{tikzcd}$$

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If F is left t-exact, then ${}^pF$ is left exact. The same holds for right t-exact and t-exact.

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If F is left t-exact, then ${}^pF$ is left exact. The same holds for right t-exact and t-exact. Similarly, if $F \dashv G$ is a pair of adjoint functors, then F is right t-exact if and only if G is left t-exact. In this case, we have ${}^pF \dashv {}^pG$.

Recollement

Let's abstract a "gluing situation": consider a diagram of triangulated categories (which are not necessarily derived categories)

$$\mathrm{D}(Z) \xrightarrow{i_*} \mathrm{D}(X) \xrightarrow{j^*} \mathrm{D}(U).$$

Moreover, set $i_! = i_*$ and $j^! = j^*$.

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(d) For all $M \in \mathrm{D}(X)$, there are morphisms $i_* i^* M \to j_! j^! M[1]$ and $j_* j^* M \to i_! i^! M[1]$ making the triangles

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(e) The functors $j_!$, j_*, $i_! = i_*$ are fully faithful.

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(b) We can classify the simple objects of $\mathrm{D}(X)^\heartsuit$.

(c) The functor ${}^p i_*$ induces an equivalence between $\mathrm{D}(Z)^\heartsuit$ and the full subcategory of $\mathrm{D}(X)^\heartsuit$ whose objects M satisfy ${}^p j^* M = 0$.

Main theorem on recollements

Suppose that $\mathrm{D}(U)$ and $\mathrm{D}(Z)$ have t-structures. Then we define

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If X is a topological space with an open immersion $j : U \to X$, with complement $i : Z \to X$, and all of the above has the familiar meanings (along with the canonical t-structures), this procedure gives back the canonical t-structure on $\mathrm{D}(X)$.

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If X is a topological space with an open immersion $j : U \to X$, with complement $i : Z \to X$, and all of the above has the familiar meanings (along with the canonical t-structures), this procedure gives back the canonical t-structure on $\mathrm{D}(X)$. But we'll make perverse choices… 😈

The perverse t-structure

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Let X be a d-dimensional complex algebraic variety, along with its derived category of constructible sheaves $\mathrm{D}^b_c(X)$.

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If $\mathrm{D}^b_{\mathrm{loc}}(X)$ is the "derived category of local systems", we want to endow it with the following t-structure $(\mathrm{D}^b_{\mathrm{loc}}(X)^{\leq -d}, \mathrm{D}^b_{\mathrm{loc}}(X)^{\geq -d})$.

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As we saw, this is indeed a t-structure.

The perverse t-structure

A very clever observation is that some complex $M^\bullet$ in $\mathrm{D}^b_{\mathrm{loc}}(X)$ lies in $\mathrm{D}^b_{\mathrm{loc}}(X)^{\leq -d}$ precisely when

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In particular, we may define

$$\begin{aligned}
{}^p\mathrm{D}^b_{\mathrm{c}}(X)^{\leq 0} &:= \{M^\bullet \in \mathrm{D}^b_{\mathrm{c}}(X) \mid \dim \operatorname{supp} \mathscr{H}^i(M^\bullet) \leq -i \text{ for all } i \in \mathbb{Z}\} \\
{}^p\mathrm{D}^b_{\mathrm{c}}(X)^{\geq 0} &:= \{M^\bullet \in \mathrm{D}^b_{\mathrm{c}}(X) \mid \dim \operatorname{supp} \mathscr{H}^i(\mathrm{D}_X(M^\bullet)) \leq -i \text{ for all } i \in \mathbb{Z}\},
\end{aligned}$$

and then this induces the desired t-structure on $\mathrm{D}^b_{\mathrm{loc}}(X)$. This is the perverse t-structure on $\mathrm{D}^b_{\mathrm{c}}(X)$.

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A first observation is that we have a filtered colimit $\mathrm{D}^b_c(X) = 2\text{-colim}_S \, \mathrm{D}^b_{c,S}(X)$, where $\mathrm{D}^b_{c,S}(X)$ is the derived category of constructible sheaves for a fixed stratification S.

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Now, we can split X as $U \coprod Z$, where the restriction of every complex in $D_{c,S}^b(X)$ to U lies in $D_{loc}^b(U)$. (Modulo some small technicalities that I'm hiding) this gives the desired t-structure on $D_{c,S}^b(X)$ (and then on $D_c^b(X)$) by recollement.

Main properties

We stay with the same notations as in the previous sections.

Definition

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Important properties

(a) The category $\mathrm{Perv}(X)$ is both noetherian and artinian. I.e., every perverse sheaf has finite length.

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(f) The exterior tensor product $\boxtimes$ is t-exact.

Theorem (Weak Lefschetz)

Let X be a complex projective variety and $i : D \hookrightarrow X$ be the inclusion of a hyperplane section. Then, for $M \in \mathrm{Perv}(X)$, the restriction map $\mathrm{H}^i(X, M) \to \mathrm{H}^i(D, i^*M)$ is an isomorphism for $i < -1$ and injective for $i = -1$.

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The decomposition theorem (+ le théorème de Lefschetz vache)

Unfortunately, I don't have the time nor knowledge to give a proper introduction to the decomposition theorem. But the viewer should at least read something about it! (The whole chapter 6 of BBD is breathtaking!)

Questions?