Contents

Preface

IT IS AN UNDERSTATEMENT to say that there is a huge number of excellent books for a first course in Analysis. Some of them marked generations of mathematicians and some of them are cited in more books than I can count. However, I think most are either too dense or doesn't cover properly some of the joys of this beautiful area of mathematics.

It is common to give Analysis books the task of developing *mathematical maturity* in the students. While this is one of the goals of this book, I don't think the best way to do it is by reading a terse and dense book. To become *mature*, one has to think a lot to be sure the theory is concise and complete, and my intent is to make this less painful.

Pierre Deligne once wrote the following in the Notices of the AMS[6]: "From [Grothendieck], I have learned not to take glory in the difficulty of a proof: difficulty means we have not understood. The ideal is to be able to paint a landscape in which the proof is obvious." This beautifully summarizes my approach with this book. It is the book I wish I had in the beginning of my studies in Analysis.

Whenever possible and not too troublesome, we'll generalize our results as much as possible. This usually makes clear what should be used to prove the theorems. For example, suppose we want to prove something related to a closed interval. A priori, we don't know what property of closed intervals makes our result true. It could be compactness, connectedness or almost anything. However if we state the theorem using only the fundamental property, it usually becomes clear what should be used in the proof.

It is expected that the reader knows the basics of set theory and linear algebra. While you could read this book without having studied calculus beforehand, it is highly recommended that the reader is familiar with the basic machinery of differential and integral calculus.

Some passages are denoted with a *dangerous bend symbol* to warn the reader of things that may not have been considered. This differs substantially from the original use of N. Bourbaki, in which it is used to forewarn the reader against serious errors.

By no means one should expect that all this work is original. It is basically my view on the things I learned with great books and great teachers. During the writing of this book, a couple of people helped me spot grammatical, theoretical and pedagogical errors. To them I owe my sincere gratitude. Last, but not least I would like to thank Umberto C. C. Malanga for showing me the beauty in mathematics.

Gabriel Ribeiro

CHAPTER 1

Metric Spaces

MAURICE FRÉCHET introduced in his 1906's work *Sur Quelques Points Du Calcul Fonctionnel* the concept of a metric space, which is basically a set endowed with a notion of distance between elements. As we'll see, that notion is enough to study many concepts in analysis such as sequences and continuity. Afterwards we'll understand normed spaces which, as the natural setting to series and derivatives, are metric spaces whose set of points is a vector space.

1.1 Metric Spaces

Definition 1.1.1 — Metric Space. Let M be a non-empty set whose elements we shall call points. A *metric* in M is a function $d : M \times M \to \mathbb{R}$ which satisfies the following properties for all $x, y, z \in M$.

- (Identity of Indiscernibles) $d(x,y) = 0$ if and only if $x = y$.
- (Triangle Inequality) $d(x,y) + d(x,z) \geq d(y,z)$.
- (Symmetry) $d(x,y) = d(y,x)$.
- (Positivity) $d(x,y) \geq 0$.

We call the pair (M,d) a *metric space*.

Notice that the first two axioms of a metric imply the other two. In fact, taking $x = z$ in the triangle inequality implies symmetry and taking $y = z$ implies positivity. You should convince yourself why this is true if it is not clear.

Whenever the context makes it clear, we'll talk about *the metric space M* without making reference to the metric. One should notice that every subset of a metric space is a metric space in its own right, with the same metric restricted to the subset.

Example 1.1 The prototypical example of metric spaces, from our standpoint, is the euclidean space $\mathbb{R}^n$. The metric in $\mathbb{R}^n$ is defined by

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}.$$

The Cauchy-Schwarz inequality assures that this is, in fact, a metric. ∎

Example 1.2 Another important example (particularly for counter-examples) of metric space is the *discrete* metric space. The discrete metric in any non-empty set is defined by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

∎

1.2 Open Sets

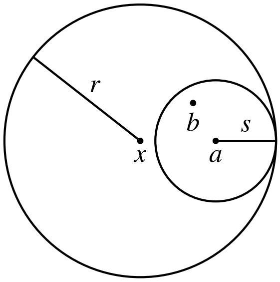
Definition 1.2.1 — Open Sets. Let M be a metric space and E be a subset of M. All points mentioned below are understood to be elements of M.

- An *open ball* centered at x with radius r is the set $B(x,r)$ consisting of all y such that $d(x,y) < r$.
- A point x is an *interior point* of E if there is an open ball centered at x contained in E. The set of all interior points of E is called $\operatorname{int} E$.
- E is open if every point of E is an interior point of E.

Since we defined an *open* ball, we ought to check that this is always an open set.

Theorem 1.2.1 In every metric space, an open ball is an open set.

Proof. *Let $a \in B(x,r)$. Then $d(x,a) < r$ and consequently $s = r - d(x,a)$ is a positive number. We affirm that $B(a,s) \subset B(x,r)$.*



In fact, if $b \in B(a,s)$, then $d(a,b) < s$ and so $d(x,b) \leq d(x,a) + d(a,b) < d(x,a) + s = r$. This implies $b \in B(x,r)$. □

Theorem 1.2.2 Consider a family of open subsets of a metric space. Then the following properties hold.

- Given a (not necessarily countable) family of indices Λ, $\bigcup_{\lambda\in\Lambda} E_\lambda$ is open.
- For every positive integer m, $\bigcap_{i=1}^{m} E_i$ is open.

Proof. *Let $E = \bigcup_\lambda E_\lambda$. If $x \in E$, then $x \in E_\lambda$ for some $\lambda \in \Lambda$. Since E_λ is open, then x is an interior point of E_λ and, consequently, of E. This proves the first part of the theorem.*
Next, put $F = \bigcap_{i=1}^{m} E_i$. For any $x \in F$, there exist open balls $B(x,r_i)$, such that $B(x,r_i) \subset E_i$ for all $i = 1,\ldots,m$. Define r to be the minimum of the set $\{r_1,r_2,\ldots,r_m\}$. Then $B(x,r) \subset E_i$ for $i = 1,\ldots,m$, so that $B(x,r) \subset F$, and F is open. □

It should be noted that the intersection of a infinite number of open sets need not be open. Can you think of an example?

Of all the open sets, the open ball is probably the most important since, in some sense, it provides a "basis" for all the open sets as is shown in the next corollary.

Corollary 1.2.3 A subset $E \subset M$ is open if, and only if, it is a union of open balls.

Proof. *The fact that if E is a union of open balls, then E is open follows readily from the preceding theorem. We shall then prove the converse.*
If E is open, then for all $x \in E$ we have that $\{x\} \subset B(x,r_x) \subset E$ for some $r_x > 0$. Taking unions, we see that

$$E = \bigcup_{x\in E}\{x\} \subset \bigcup_{x\in E} B(x,r_x) \subset E.$$

It follows that

$$E = \bigcup_{x\in E} B(x,r_x).$$

□

We already know how open sets behave under unions and intersections. The last set operation to be studied is the Cartesian product.

Theorem 1.2.4 Let E_1 be open in (M_1,d_1) and E_2 be open in (M_2,d_2). Then $E_1 \times E_2$ is open as a subset of $M_1 \times M_2$ endowed with the maximum metric.

$$d((x_1,x_2),(y_1,y_2)) = \max\{d_1(x_1,y_1),d_2(x_2,y_2)\}.$$

Proof. *Let $x = (x_1,x_2)$ be a point of $E_1 \times E_2$. Since E_1 is open in M_1, there is $r_1 > 0$ such that $B(x_1,r_1)$ is contained in E_1. As the same is valid for E_2, we have that $B(x_1,r_1) \times B(x_2,r_2)$ is contained in $E_1 \times E_2$. Let $r = \min\{r_1,r_2\}$. Then $B(x,r) \subset B(x_1,r_1) \times B(x_2,r_2) \subset E_1 \times E_2$ and the result follows.* □

Corollary 1.2.5 Let E_i be open in (M_i, d_i) for every i in $\{1,2,\ldots,n\}$. Then $E_1 \times \ldots \times E_n$ is open as a subset of $M_1 \times \ldots \times M_n$ endowed with the maximum metric.

$$d((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \max\{d_1(x_1,y_1),\ldots,d_n(x_n,y_n)\}.$$

Proof. *Since we have a finite number of sets in the Cartesian product, we can choose a smallest radius like we did before. The proof then follows similarly.* □

At first it seems like we proved a very particular case since the preceding corollary is only valid for one metric in the product space. However, we will see soon that *almost* every metric generates the same open sets, so that corollary is a pretty general one.

Definition 1.2.2 — Equivalence of Metrics. Let M be a set and d_1, d_2 be metrics on M. We say that d_1 and d_2 are *equivalent* (or $d_1 \sim d_2$) if there exist positive constants $c, C \in \mathbb{R}$ such that for all $x, y \in M$:

$$c\, d_2(x,y) \leq d_1(x,y) \leq C\, d_2(x,y).$$

You should check that this is, in fact, a equivalence relation. The motivation for defining this equivalence is the following theorem.

Theorem 1.2.6 Let (M, d_1) and (M, d_2) be metric spaces such that $d_1 \sim d_2$. A set $E \subset M$ is open in (M, d_1) if, and only if it is open in (M, d_2).

Proof. *If E is open in (M, d_1), then for all $x \in E$ there is some $r_1 > 0$ such that $\{y \in M \,|\, d_1(x,y) < r_1\} \subset E$. Since $d_1(x,y) \leq C\, d_2(x,y)$, taking $r_2 = r_1/C$ then we have that $d_2(x,y) < r_2$ implies $d_1(x,y) < r_1$ and hence E is open in (M, d_2). The converse is analogous.* □

To make Corollary 1.2.5 general we just need to prove that various metrics are equivalent to the maximum metric. This will be done on the section of normed spaces.

1.3 Closed Sets

Definition 1.3.1 — Closed Sets. All points mentioned below shall be understood as elements of M.

- A point x is a *limit point* of the set E if every open ball centered at x contains a point $y \neq x$ such that $y \in E$.
- We denote by E' the set of all limit points of E and by $\overline{E}$ the union of E and E'. This latter set $\overline{E}$ is called the *closure* of E. The set $\partial E = \overline{E} \setminus \operatorname{int} E$ is called the *boundary* of E.
- E is *closed* if every limit point of E is a point of E.

Notice that in order for E to be closed it is not necessary that every point of E is a limit point. A singleton is closed but it's sole point is not a limit point.

Theorem 1.3.1 A point $x \in E$ is a limit point if and only if every open ball centered at x contains infinitely many points from E.

Proof. *Suppose x is a limit point of E and $B(x,r)$ is a ball that contains finitely many points of E. Since $B(x,r) \cap E$ is finite, we can list its elements as $\{p_1, p_2, \ldots, p_n\}$. Let s be the minimum of all $d(x, p_i)$ for $i \in \{1, 2, \ldots, n\}$.*
Then $B(x,s)$ does not contain any points of E besides x, so that x is not a limit point of E. Absurd!
The converse is trivial. □

Based on the way we named some properties of subspaces, one can think that, in some way, closed sets are the "opposite" of open sets. This is not true, since there exist sets that are both open and closed. The empty set is a simple example. If that wasn't enough, there are sets that are neither closed nor open. The set $(0,1)$ considered as a subset of the real plane is neither open nor closed. However, there is a relation between these kinds of sets.

Theorem 1.3.2 A set E is open if, and only if its complement is closed.

Proof. *First suppose E^c is closed. Choose $x \in E$. Then $x \notin E^c$, and x is not a limit point of E^c. Hence there exists a ball B centered at x such that $E^c \cap B$ is empty, that is, $B \subset E$. Thus x is an interior point of E, and E is open.*

Next, suppose E is open. Let x be a limit point of E^c. Then every open ball centered at x contains a point of E^c, so that x is not an interior point of E. Since E is open, this means that $x \in E^c$. It follows that E^c is closed. □

Analogously to Theorem 1.2.2, we have the following result.

Theorem 1.3.3 Consider a family of closed subsets of a metric space. Then the following properties hold.

- Given a arbitrary family of indices Λ, $\bigcap_{\lambda \in \Lambda} E_\lambda$ is closed.
- For every positive integer m, $\bigcup_{i=1}^{m} E_i$ is closed.

Proof. *Taking complements of the equations in Theorem 1.2.2 and using De Morgan's relations the result is obvious.* □

Theorem 1.3.4 For any set E, its closure $\overline{E}$ is closed. Moreover, E is closed if, and only if $\overline{E} = E$.

Proof. *Note that if $x \in \overline{E}^c$, then x is neither a point of E not a limit point of E. Hence there exists an open ball centered at x which does not intersect $\overline{E}$ so that $\overline{E}^c$ is*

open. It follows that $\overline{E}$ is closed.
If E is closed, then $E' \subset E$ and hence $E = \overline{E}$. The converse is obvious. □

Theorem 1.3.5 Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \overline{E}$. Hence $y \in E$ if E is closed.

Proof. *If $y \in E$, then $y \in \overline{E}$. Assume $y \notin E$. For every $r > 0$ there exists then a point $x \in E$ such that $y - r < x < y$, for otherwise $y - r$ would be an upper bound of E. Thus y is a limit point of E and then $y \in \overline{E}$.* □

Corollary 1.3.6 Let E be a nonempty set of real numbers which is bounded below. Let $y = \inf E$. Then $y \in \overline{E}$. Hence $y \in E$ if E is closed.

1.4 Compact Sets

Since the beginning of the subject, it was noted that bounded closed sets had a special "something" that made them incredibly useful in analysis. Initially, people thought that it was the property that every infinite subset has a limit point. It took a long time for mathematicians to realize that the fundamental property was the one that we describe here as *compactness*.

Definition 1.4.1 Let $\{G_\lambda : \lambda \in \Lambda\}$ be a collection of open subset of M. We say that such collection is an *open cover* of $E \subset M$ if

$$E \subset \bigcup_{\lambda \in \Lambda} G_\lambda.$$

If Ω is a subset of Λ such that $\{G_\lambda : \lambda \in \Omega\}$ is still an open cover of E, we say that $\{G_\lambda : \lambda \in \Omega\}$ is a subcover of $\{G_\lambda : \lambda \in \Lambda\}$.

Definition 1.4.2 — Compact Sets. A subset E of a metric space M is called *compact* if every open cover of E contains a *finite* subcover.
More explicitly, the requirement is that if $\{G_\lambda : \lambda \in \Lambda\}$ is an open cover of E, then there are finitely many indices $\lambda_1, \ldots, \lambda_n$ such that

$$E \subset G_{\lambda_1} \cup \ldots \cup G_{\lambda_n}.$$

This property of a set is specially important because it allows the passage of some local properties to global properties.

As was thought before, the property that every infinite subset has a limit point is common to every set with this special "something". However, it is not the heart of the matter.

Theorem 1.4.1 — Bolzano-Weierstrass. If E is an infinite subset of a compact set K, then E has a limit point in K.

Proof. *If no point of K were a limit point of E, then each $x \in K$ would have a neighborhood N_x which contains at most one point of E (namely x, if $x \in E$). It is clear that no finite subcollection of $\{N_x\}$ can cover E; and the same is true of K, since $E \subset K$. This contradicts the compactness of K.* □

We'll see now that, in fact compact sets are quite similar to closed bounded sets.

Theorem 1.4.2 If E is a compact set, then E is closed and bounded.

Proof. *Let x_0 be a point in E and let $\{N_n(x_0) \mid n \in \mathbb{N}\}$ be an open covering of E. Since E is compact, for some $k \in \mathbb{N}$ the set $\{N_{n_1}(x_0), N_{n_2}(x_0), \ldots, N_{n_k}(x_0)\}$ is a finite subcover of E. Taking $m = \max\{n_1, n_2, \ldots, n_k\}$ and using the triangular inequality we see that for every $x, y \in E$*

$$d(x,y) \leq d(x,x_0) + d(x_0,y) < m + m = 2m,$$

so that E is bounded.
Suppose now $x \in E^c$. If $y \in E$, let V_y and W_y be neighborhoods of x and y, respectively, of radius less than $\frac{1}{2}d(x,y)$. Since E is compact, there are finitely many points $y_1, y_2, \ldots, y_n \in E$ such that

$$E \subset W_{y_1} \cup \ldots \cup W_{y_n} = W.$$

If we define V as $V_{y_1} \cap \ldots \cap V_{y_n}$, then V is a neighborhood of x which does not intersect W. Hence $V \subset E^c$, so that E^c is open. It follows that E is closed. □

It should be noted that, in general, the converse is not true! The set of all integers is closed and bounded when endowed with the discrete metric, however $\{\{x\} \subset \mathbb{Z} \mid x \in \mathbb{Z}\}$ is an open cover which has no finite subcover.

Theorem 1.4.3 Closed subsets of compact sets are compact.

Proof. *Suppose $E \subset K \subset M$, E is closed (relative to M) and K is compact. Let $\{G_\lambda\}$ be an open cover of E. If E^c is adjoined to $\{G_\lambda\}$ then this new set becomes an open cover of K. Since K is compact, there is a finite subcollection of $\{G_\lambda\} \cup E^c$ that covers K and hence E. If E^c is in this subcover we may remove it and obtain a finite subcollection of $\{G_\lambda\}$ that covers E.* □

Theorem 1.4.4 Let Ω be a collection of closed subsets of a compact set E such that the intersection of every finite subcollection of Ω is non-empty. Then the intersection of all the elements of Ω is not empty.

Proof. *Before we start, one should notice that the contrapositive of compactness is "Given any collection Φ of open sets, if no finite subcollection of Φ covers E, then Φ*

does not cover E."
Let $\Phi = \{E \setminus X \mid X \in \Omega\}$. Of course Φ is a collection of open sets. If a finite subcollection of Φ covered E, say $\{\phi_1, \ldots, \phi_n\} \subset \Phi$, we would have that

$$\bigcap_{i=1}^{n} (E \setminus \phi_i) = E \setminus \bigcup_{i=1}^{n} \phi_i = \varnothing.$$

Since $E \setminus \phi_i \in \Omega$ this is absurd! So we have that no finite subcollection of Φ covers E. Since E is compact, this means that Φ does not cover E. Then,

$$\bigcap_{X \in \Omega} X = E \setminus \bigcup_{Y \in \Phi} (Y) \neq \varnothing,$$

and the result follows. □

Corollary 1.4.5 If $\{K_n\}$ is a sequence of nonempty closed subsets of a compact metric space such that $K_{n+1} \subset K_n$ for all $n \in \{1, 2, \ldots\}$, then $\bigcap_{n=1}^{\infty} K_i$ is not empty.

It is a fact that the preceding corollary is valid for closed subsets of the real line. Once we prove that the closed interval $[a, b]$ is compact this becomes trivial. Since we cannot do this as of yet, we'll have to prove this result separately.

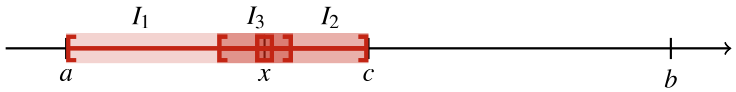
Theorem 1.4.6 — Nested Interval Theorem. If $\{I_n\}$ is a sequence of nonempty bounded closed intervals of the real line such that $I_{n+1} \subset I_n$ for all $n \in \{1, 2, \ldots\}$, then $\bigcap_{n=1}^{\infty} I_n$ is not empty.

Proof. *Suppose $I_n = [a_n, b_n]$ and let E be the set of all a_n. Then E is not empty and is bounded above by b_1. Let s be the supremum of E. Since s is the supremum of E, it is clear that $a_n \leq s$ for all n. Since every b_n is an upper bound of E, it follows that $s \in [a_n, b_n] = I_n$ for every $n \in \mathbb{N}$. The result follows.* □

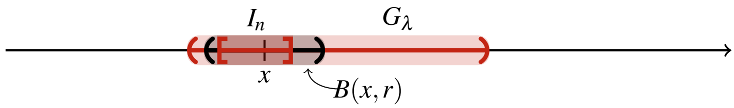
Although the following result is very important, we'll call it a lemma since it will be used to prove a much more general result.

Lemma 1.4.7 The closed interval $[a, b] \subset \mathbb{R}$ is compact.

Proof. *Suppose there is an open cover $\{G_\lambda\}$ of $[a, b]$ which contains no finite subcover. Let $c = (a + b)/2$ and consider the intervals $[a, c]$ and $[c, b]$. It follows that at least one of these intervals cannot be covered by a finite subcover of $\{G_\lambda\}$, otherwise $[a, b]$ itself would be covered. We then call this interval I_1, divide it in half and call the piece that cannot be covered I_2. Continuing this process we get a sequence $\{I_n\}$ of subsets of $[a, b]$ such that for all $n \in \mathbb{N}$: $I_{n+1} \subset I_n$, I_n is not covered by any finite subcover of $\{G_\lambda\}$ and* $\operatorname{diam} I_n = 2^{-n}(b - a)$.



By the nested interval theorem, there is a number $x \in [a,b]$ *such that* $x \in I_n$ *for all* n. *Since* $\{G_\lambda\}$ *covers* $[a,b]$, *there is a set* G_λ *such that* $x \in G_\lambda$. *Since* G_λ *is open, there is a neighborhood* $B(x,r)$ *of* x *that is entirely contained in* G_λ.



If we take n *large enough such that* $2^{-n}(b-a) < r$ *(which is possible since* $\mathbb{R}$ *is archimedean) then we have that* $I_n \subset B(x,r) \subset G_\lambda$. *This is absurd since no finite subset of* $\{G_\lambda\}$ *covers* I_n. *This establishes the proof.* □

Theorem 1.4.8 If E and F are compact sets, then $E \times F$ is compact.

Proof. *Let* $\{G_\lambda\}$ *be an open cover of* $E \times F$. *For each* $(a,b) \in E \times F$, *we can choose some* λ *such that* $(a,b) \in G_\lambda$. *Since* G_λ *is open, the point* (a,b) *is contained in some open box* $U_{(a,b)} \times V_{(a,b)} \subset G_\lambda$, *where* $U_{(a,b)} \subset E$ *and* $V_{(a,b)} \subset F$.

Suppose we fix a *and vary* b. *Then for every point* (a,b) *we find that the point is contained in an open box in the product* $E \times F$, *and that box is then itself the product of a subset of* E *with a subset of* F. *Proceeding in this manner, we observe that the collection of sets* $\{V_{(a,b)}\}_{b \in F}$ *is an open cover of* F. *Since by assumption* F *is compact, we can find a finite cover* $\{V_{(a,b_j(a))}\}$ *of* F *that consists of finitely many open sets containing points* $\{(a,b_j(a))\}$.

Now let $U_a = \bigcap_j U_{(a,b_j(a))}$. *Since* U_a *is the intersection of finitely many open sets, it is itself open. Since* E *is compact, there are finitely many* a_i *such that* $\{U_{a_i}\}$ *forms an open cover of* E. *Then it follows that the collection of sets* $\{U_{a_i} \times V_{(a_i,b_j(a_i))}\}$ *(for all* i *and* j*) is a finite subcover of* $E \times F$, *hence* $E \times F$ *is compact.* □

Corollary 1.4.9 If $E_1, E_2, \ldots, E_n$ are compact sets, then $E_1 \times \ldots \times E_n$ is compact.

In fact, a result much stronger than this is true: the arbitrary product of compact sets is compact. That is called Tychonoff's theorem and it is way out of our scope.

As a quick corollary we get our most powerful result yet: the Heine-Borel Theorem.

Theorem 1.4.10 — Heine-Borel. A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. *If* $E \subset \mathbb{R}^n$ *is bounded, then it is contained in a box of the form* $[a_1,b_1] \times \ldots \times [a_n,b_n]$ *for some suitable* a_i*'s and* b_i*'s. Since this box is compact, if* E *is closed then* E *is compact. The converse was already proved in a previous theorem.* □

1.5 Connected Sets

As we'll see, unlike compactness, connectedness is a very intuitive notion for metric spaces.

Definition 1.5.1 — Connected Set. Let E be a metric space. A separation of E is a pair U, V of disjoint nonempty open subsets of E whose union is E. The set E is said to be connected if there does not exist a separation of it.

In every metric space M, we have that M and the empty set are both open and closed (independent of the metric chosen). Connected metric spaces have the remarkable property that the only sets that are both open and closed are M and $\varnothing$.

Theorem 1.5.1 Let M be a connected metric space. Then the only sets that are both open and closed are M and $\varnothing$.

Proof. *Suppose A is a proper non-empty subset of M that is both open and closed. Since A is closed, $M \setminus A$ is open and since A is a proper subset of M, $M \setminus A \neq \varnothing$. Clearly $(M \setminus A) \cup A = M$ and $(M \setminus A) \cap A = \varnothing$, so M is disconnected. We conclude that the only closed and open sets on a connected metric space are the empty set and itself.* □

We now prove that every interval of the real line and the real line itself are connected.

Theorem 1.5.2 A subset E of the real line $\mathbb{R}$ is connected if and only if it is an interval (bounded or not).

Proof. *If E contains a single point, then it is connected. Suppose E contains two distinct points $a < b$. We prove that every x such that $a < x < b$ belongs to E. Otherwise, E would be the union of the nonempty disjoint sets $U = E \cap (-\infty, x)$ and $V = E \cap (x, +\infty)$, both of which are open in E. From this property, we deduce that E is necessarily an interval. Indeed, let $c \in E$, $p = \inf E$ and $q = \sup E$. If $p = -\infty$, then for every $x < c$, there is $y < x$ belonging to E and hence $x \in E$ so that $(-\infty, c)$ is contained in E. If p is finite and $p < c$, for every x such that $p < x < c$ there is $y \in E$ such that $p < y < x$, hence again $x \in E$, so that E contains the interval $(p, c]$. Similarly, one shows that E contains $[c, q)$ if $q > c$. In any case, it follows that E contains the interval (p, q).*

Conversely, suppose E is a nonempty interval of center a and radius b, both elements of $\overline{\mathbb{R}}$. Suppose U, V constitute a separation of E. Without loss of generality, suppose $x \in U$, $y \in V$ and $x < y$. Let $z = \sup U \cap [x, y]$. If $z \in U$, then $z < y$ and there is an interval $[z, z+h)$ contained in $[x, y]$ and in U, which contradicts the definition of z. On the other hand, if $z \in V$ then $x < z$, and there is an interval $(z-h, z] \subset V \cap [x, y]$, which again contradicts the definition of z. Hence z cannot belong to U nor to V, which is absurd since the closed set $[x, y]$ is contained in E. Hence E is connected. □

1.6 Continuous Functions

The concept of a continuous function is essential to almost every every branch of mathematics. It is truly remarkable that this property of functions generalizes so well to topological and metric spaces.

Definition 1.6.1 — Continuous Function. Let A and B be metric spaces. A function $f : A \to B$ is said to be *continuous* if for every open subset E of B, the set $f^{-1}(E)$ is an open subset of A. It should be noted that this property does not depend only on f but too on the metrics of A and B.

Alternatively, we could also define continuity as a property related to closed sets. If f is continuous and E is closed, $A \setminus f^{-1}(E) = f^{-1}(B \setminus E)$ is open, so that $f^{-1}(E)$ is closed. The converse is just as easy.

Definition 1.6.2 — Continuity at a Point. Let x be a point of A. We say that $f : A \to B$ is continuous at the point x if for each neighborhood V of $f(x)$ there is a neighborhood U of x such that $U \subset f^{-1}(V)$.

The definition given above is equivalent to the following:

"A function $f : A \to B$ is continuous at $p \in A$ if for every $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for all $x \in A$:

$$d_A(x,p) < \delta \implies d_B(f(x), f(p)) < \varepsilon."$$

I used indices on the metrics to indicate the sets where they are defined.

Theorem 1.6.1 A function is continuous if it is continuous at every point of its domain.

Proof. *Suppose f is continuous. Let $x \in A$ and let V be a neighborhood of $f(x)$. Then the set $f^{-1}(V)$ is open and so contains a neighborhood U of x such that $f(U) \subset f(f^{-1}(V)) \subset V$.*
Suppose now f is continuous at every point. Let E be a open subset of B. Is x is a point in $f^{-1}(E)$ then there is $y \in E$ such that $y = f(x)$. Since f is continuous in x, then there is a neighborhood U of x such that $f(U) \subset E$. This implies $U \subset f^{-1}(E)$ so that $f^{-1}(E)$ is open. The result follows. □

Theorem 1.6.2 Let $f : A \to B$ and $g : B \to C$ be continuous. Then the composition $g \circ f : A \to C$ is continuous.

Proof. *Let E be an open set in C. Since g is continuous, $g^{-1}(E)$ is open in B. Since f is continuous, $f^{-1}(g^{-1}(E)) = (g \circ f)^{-1}(E)$ is open in A. Hence $g \circ f$ is continuous.* □

We turn now to properties related to continuous functions with compact domain.

Theorem 1.6.3 The image of a compact set under a continuous function is compact.

Proof. *Let E be compact and f be continuous. Suppose $\{G_\lambda\}$ is an open cover of $f(E)$. Of course the set $\{f^{-1}(G_\lambda)\}$ is an cover of E. Since f is continuous this is an open cover. But because E is compact we have that*

$$E \subset f^{-1}(G_{\lambda_1}) \cup \ldots \cup f^{-1}(G_{\lambda_n}).$$

It's now clear that the set $\{G_{\lambda_1}, \ldots, G_{\lambda_n}\}$ is a finite subcover of $f(E)$. □

Theorem 1.6.4 — Extreme Value Theorem. Let A be a compact set and $f : A \to \mathbb{R}$ be a continuous function. Then there exist points $p, q \in A$ such that

$$f(p) \leq f(x) \leq f(q)$$

for all $x \in A$.

Proof. *Since A is compact and f is continuous, $f(A)$ is compact, hence closed and bounded. Theorem 2.3.9 then implies the result.* □

Last, but not least we have theorems that relates connected sets with continuous functions.

Theorem 1.6.5 The image of a connected set under a continuous function is connected.

Proof. *Let E be a connected set in the domain of f. Suppose $f(E)$ is not connected. Then there are sets U, V that make a separation of $f(E)$. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open, hence so are $E \cap f^{-1}(U)$ and $E \cap f^{-1}(V)$. A possible element of $f^{-1}(U) \cap f^{-1}(V)$ would its image would be a element of both U and V, which is impossible. The sets $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty too since U and V are non-empty. Lastly, its clear that $E \subset f^{-1}(U) \cup f^{-1}(V)$ since if $x \in E$, then $f(x) \in U$ or $f(x) \in V$ and that is the exact definition of $x \in f^{-1}(U) \cup f^{-1}(V)$. This implies $E = (E \cap f^{-1}(U)) \cup (E \cap f^{-1}(V))$, hence $E \cap f^{-1}(U)$ and $E \cap f^{-1}(V)$ constitute a separation of E, which is absurd!* □

Theorem 1.6.6 — Intermediate Value Theorem. Let A be a connected set and $f : A \to \mathbb{R}$ be a continuous function. If p and q are two points of A and if a is a point of $\mathbb{R}$ such that $f(p) < a < f(q)$, then there exists a point r of A such that $f(r) = a$.

Proof. *Since $f(A)$ is connected, Theorem 2.5.2 implies the result.* □

1.7 Normed Spaces

In general, there are no algebraic operations defined on a metric space, only a distance function. Most of the spaces that arise in analysis are vector spaces, and the metrics on them are usually derived from a norm, which gives the “length” of a vector.

Definition 1.7.1 — Normed Vector Spaces. A normed vector space is a (real or complex) vector space E together with a function $\|\cdot\| : E \to \mathbb{R}$, called a *norm* on E, such that for all $x, y \in E$ and $k \in \mathbb{R}$:

- (Positivity) $\|x\| > 0$ if $x \neq 0$ and $\|0\| = 0$.
- (Linearity) $\|kx\| = |k|\,\|x\|$.
- (Triangle Inequality) $\|x+y\| \leq \|x\| + \|y\|$.

If $(E, \|\cdot\|)$ is a normed vector space, then $d : E \times E \to \mathbb{R}$ defined by $d(x,y) = \|x-y\|$ is clearly a metric on E. Note that the metric associated with a norm has the additional properties that for all $x, y, z \in E$ and $k \in \mathbb{C}$:

$$d(x+z, y+z) = d(x,y), \quad d(kx, ky) = |k|\, d(x,y),$$

which are called *translation invariance* and *homogeneity*, respectively. These properties do not even make sense in a general metric space since we cannot add points or multiply them by scalars.

Theorem 1.7.1 Let $f : E \to F$ be a linear function between normed spaces and N be the unit neighborhood of 0. Then the following conditions are equivalent:

i) f is continuous;

ii) f is continuous at $0 \in E$;

iii) The restriction $f|_N$ of f to N is bounded. That is, $\sup_{x \in N} \|f(x)\| < \infty$.

Proof. *That* i) *implies* ii) *is clear.*
Supporting f is continuous at 0, □

Theorem 1.7.2 Let $f : X \to Y$, where X, Y are normed vector spaces and X is finite-dimensional, be a linear function. Then f is continuous.

Actually, we’ll prove a particular case of this result for now. We’ll assume that the norm defined on X is the 1-norm $\|\cdot\|_1$. (The metric without indices will be the one defined on Y.)

Proof. *Let $e_1, \ldots, e_n$ be a basis of X. Then, for $x = x_1 e_1 + \ldots + x_n e_n \in X$ and $y = y_1 e_1 + \ldots + y_n e_n \in X$ we have:*

$$\|f(x) - f(y)\| = \left\| \sum_{i=1}^{n} (x_i - y_i) f(e_i) \right\| \leq \sum_{i=1}^{n} |x_i - y_i| \, \|f(e_i)\|.$$

Let $\varepsilon > 0$ *be given and let* $M = \max_{1\le i\le n} \|f(e_i)\|$. *If we define* $\delta = \varepsilon/M$, *then for all* x *and* y *with* $\|x-y\|_1 < \delta$*:*

$$\|f(x)-f(y)\| \le \sum_{i=1}^{n} |x_i - y_i| \, \|f(e_i)\| \le M \sum_{i=1}^{n} |x_i - y_i| = M \, \|x-y\|_1 < \varepsilon.$$

Hence, f is continuous. □

Definition 1.7.2 Let X be a vector space and $\|\cdot\|_a$, $\|\cdot\|_b$ be any norms. We say that $\|\cdot\|_a$ and $\|\cdot\|_b$ are *equivalent* the metrics induced by them are equivalent. That is, if there exist positive constants c, C such that for all $x \in X$:

$$c\,\|x\|_a \le \|x\|_b \le C\,\|x\|_a .$$

We now prove that, in a finite dimensional vector space, the topology generated by every norm is unique. For that we'll need the following lemma.

Lemma 1.7.3 Let $f : X \to \mathbb{R}$, $f(x) = \|x\|$ is continuous under the metric induced by $\|\cdot\|_1$ on X if X is finite-dimensional.

Proof. *Let* $M = \max_{1\le i\le n} \|e_i\|$ *and* $\delta = \varepsilon/M$. *Then:*

$$\begin{aligned} \big|\|x\| - \|y\|\big| &\le \|x-y\| \\ &\le \sum_{i=1}^{n} |x_i - y_i| \, \|e_i\| \\ &\le M \sum_{i=1}^{n} |x_i - y_i| \\ &\le M \, \|x-y\|_1 \\ &< \varepsilon, \end{aligned}$$

if $\|x-y\|_1 < \delta$. □

Theorem 1.7.4 — Equivalence of Norms. Let X be a finite-dimensional vector space. Then all norms are equivalent.

Proof. *Since norm equivalence is transitive, it is sufficient to show that every norm is equivalent to some fixed norm. Let* $\|\cdot\|_1 = \sum_{i=1}^{n} |x_i|$ *be that fixed norm.*

If $x = 0$ *the result is trivial, so let's assume* $x \neq 0$ *and divide the inequality by* $\|x\|_1$. *We only need to prove that*

$$c \le \|u\| \le C$$

is true for all $u \in X$ *such that* $\|u\|_1 = 1$.

The unit sphere

$$S = \{x \in X \mid \|x\|_1 = 1\}$$

is closed and bounded, so it is compact. we have then that $f(x) = \|x\|$ must attain it's bounds on S.

Let

$$c = \min_{u \in S} \|u\| \text{ and } C = \max_{u \in S} \|u\|.$$

The result follows. □

Actually, it isn't obvious that S is compact since Heine-Borel was only proved on $\mathbb{R}^n$. However, note that the function $f : \mathbb{R}^n \to X$, $f(x_1, \ldots, x_n) = x_1 e_1 + \ldots + x_n e_n$ is a continuous (since it is linear) bijection. The set $S' = \{x \in \mathbb{R}^n \mid \|x\|_1 = 1\}$ is compact (Heine-Borel) and S is the image of S' by f. Since the continuous image of a compact set is compact, the result follows.

Exercise 1.1 Use the equivalence of norms to prove Theorem 2.7.2 for any norm defined on X.

Theorem 1.7.5 — Heine-Borel for Normed Spaces. Let X be a finite dimensional normed vector space and let E be a subset of X. Then E is compact if, and only if it is closed and bounded.

Proof. *We'll denote by $\|\cdot\|$ an arbitrary norm of either X or $\mathbb{R}^n$.*
Let E be a closed and bounded set. Consider the function $f : \mathbb{R}^n \to X$ given by $f(x_1, \ldots, x_n) = x_1 e_1 + \ldots + x_n e_n$. Since f is continuous (as it is linear), $f^{-1}(E)$ is closed. If $x, y \in f^{-1}(E)$, then $\|x - y\| \le c_1 \|x - y\|_1 = c_1 \|f(x) - f(y)\|_1$ for some $c_1 > 0$. Since E is bounded and $f(x), f(y) \in E$, $\|f(x) - f(y)\|_1 \le c_2 \|f(x) - f(y)\| \le c_2 M$ for some $M > 0$ and hence $f^{-1}(E)$ is bounded. As Heine-Borel holds in $\mathbb{R}^n$, $f^{-1}(E)$ is compact. Because f is bijective, $E = f(f^{-1}(E))$ is the continuous image of a compact set and hence compact.
The converse was already proved in an earlier result. □

1.8 Exercises

Exercise 1.2 If d is a metric, prove that so are $\rho_1(x,y) = d(x,y)/(1 + d(x,y))$ and $\rho_2(x,y) = \sqrt{d(x,y)}$.

Exercise 1.3 — $\mathbb{R}^n$ is Second-Countable. A collection $\mathfrak{B}$ of open subsets of a metric space M is said to be a *basis* if every open subset of M is the union of a subcollection of $\mathfrak{B}$. We say that a metric space is *second-countable* if it has a countable basis.
Show that $\mathbb{R}^n$ is second-countable.
Hint: Consider the neighborhoods centered at points with rational coordinates and with rational radius.

Exercise 1.4 — Urysohn's Lemma. Let A and B be non-empty disjoint closed subsets of a metric space M. Show that the function $f : M \to \mathbb{R}$,

$$f(x) = \frac{d(x,B)}{d(x,A) + d(x,B)}$$

is continuous, has value 1 for all $x \in A$, has value 0 for all $x \in B$ and satisfies $0 \leq f(x) \leq 1$ for all $x \in M$.
As metric spaces behave much better than topological spaces, it is useful to know when a topological space is also a metric space. A famous result, called *Urysohn Metrization Theorem* states that every normal, second-countable topological space is in fact a metric space. Urysohn's lemma is used in the proof of this result.

Exercise 1.5 — Metric Spaces are Normal. Let A and B be non-empty disjoint closed subsets of a metric space. Prove that there exist open sets E, F such that $A \subset E$, $B \subset F$ and $E \cap F = \varnothing$.

Exercise 1.6 Check the proof of Theorem 2.3.9 again and then read the following statement.
"Let E be a nonempty subset of $\mathbb{R}$. If $y = \sup E$ is finite and $\delta > 0$, then there exists some x in E such that $y - \delta < x < y$."
I affirm that it is wrong. Can you tell why?

Exercise 1.7 Let f and g be continuous functions from M_1 to M_2, where M_1 and M_2 are metric spaces. If E is dense in M_1, prove that $f(E)$ is dense in M_2. Moreover, show that if $f(x) = g(x)$ for all $x \in E$, then $f(x) = g(x)$ for all $x \in M_1$.

Exercise 1.8 Let M and N be a metric spaces and $f : M \to N$ be a function. Then f is continuous if, and only if $f(\overline{E}) \subset \overline{f(E)}$ for all subsets E of M.

Exercise 1.9 — Lebesgue Number. Let M be a compact metric space and let $\{G_\lambda\}$ be an open cover of M. Prove that there is a number $\delta > 0$, called *Lebesgue number* of M relative to $\{G_\lambda\}$, such that every subset of M with diameter less than δ is contained in some element of $\{G_\lambda\}$.

Exercise 1.10 — Lindelöf Covering Theorem. Let E be a subset of $\mathbb{R}^n$ and let $\{G_\lambda\}$ be an open cover of E. Prove that there exists a countable subcover $\{G_{\lambda_i}\}$ of A.

Exercise 1.11 Prove that the finite union of compact sets and the arbitrary intersection of compact sets is compact.

Exercise 1.12 — The Real Projective Space is Compact. Define an equivalence relation of $\mathbb{R}^{n+1} \setminus \{0\}$ by

$$x \sim y \iff x = ty, \text{ for some non-zero real number } t.$$

The *real projective space* $\mathbb{R}P^n$ is the set of all equivalence classes of this equivalence relation.
Geometrically, two points in $\mathbb{R}^{n+1} \setminus \{0\}$ are equivalent if they lie on the same line through the origin, so $\mathbb{R}P^n$ can be interpreted as the set of all lines through the origin in $\mathbb{R}^{n+1}$. As each line crosses the sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ exactly twice, this suggests that we define the following equivalence relation on S^n:

$$x \sim y \iff x = \pm y.$$

Let $S^n / \sim$ be the set of all equivalence classes of this equivalence relation. As suggested, there is a bijection from $\mathbb{R}P^n$ to $S^n / \sim$. Show that in fact there is a continuous bijection f from $\mathbb{R}P^n$ to $S^n / \sim$ such that f^{-1} is also continuous. Use this result to prove that $\mathbb{R}P^n$ is compact.

Exercise 1.13 Let A and B be subsets of the real line and define the set

$$A + B = \{a + b \in \mathbb{R} \mid a \in A, b \in B\}.$$

If A and B are open, $A + B$ is open too? What if only A is open?
If A and B are closed, $A + B$ is closed too?
If A and B are compact, $A + B$ is compact too?

Exercise 1.14 — Every Non-empty Perfect Set is Uncountable. A subset of a metric space that is closed and has no isolated points is denominated *perfect*. Show that every non-empty perfect set is uncountable. Conclude that $\mathbb{R}$ is uncountable.

Exercise 1.15 — $\mathbb{R} \setminus \mathbb{Q}$ is Uncountable. Prove that the set of all irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is uncountable.

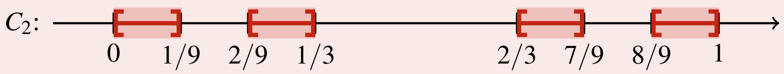
Exercise 1.16 — There Exist Transcendental Numbers. A complex number which is not algebraic is said to be *transcendental*. Show that there exist transcendental numbers. Moreover, show that the set of all transcendental numbers is uncountable.

It is curious that "almost" every complex number is transcendental and yet the only transcendental numbers that most people recognize are π and e. Even these numbers are hard to show that they are not algebraic!

Exercise 1.17 — Cantor Set. Let C_0 be the closed interval $[0,1]$.

C_1 will be the set that results when the open middle third is removed. That is, $C_1 = [0,1/3] \cup [2/3,1]$.

Now, construct C_2 in a similar way, removing the open middle third of each component of C_1. Then we have that $C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$.



If we continue this process inductively, then for each $n = 0,1,2,\ldots$ we get a set C_n consisting of 2^n closed intervals each having length $1/3^n$. Finally, we define the Cantor set C to be the intersection

$$C = \bigcap_{n=0}^{\infty} C_n.$$

Since C is an intersection of closed sets, C is closed and hence compact. Prove that C does not contain intervals and does not contain isolated points. Lastly, show that the Cantor set is uncountable.

Exercise 1.18 Let $X \subset \mathbb{R}^n$. Show that there exists a countable set $Y \subset \mathbb{R}^n$ such that $\overline{Y} = \overline{X}$.

Exercise 1.19 — Banach Fixed Point Theorem. Let X be a compact metric space and $\phi : X \to X$ be a continuous function such that $d(\phi(x),\phi(y)) < d(x,y)$ for all $x \neq y$. Then ϕ has a unique fixed point. That is, a point x_0 such that $\phi(x_0) = x_0$.

Exercise 1.20 — Every Convex Subset is Connected. A subset E of $\mathbb{R}^n$ is said to be convex if for all $p,q \in E$, the point $(1-t)p + tq$ exists in E for all $t \in (0,1)$.

(That is, every point in the segment that connects p to q is in E.)

Let A and B be separated subset of some $\mathbb{R}^n$, suppose $a \in A$, $b \in B$, and define

$$p(t) = (1-t)a + tb$$

for $t \in \mathbb{R}$. Prove that there exists $t_0 \in (0,1)$ such that $p(t_0) \notin A \cup B$. Conclude that every convex set is connected.

Exercise 1.21 — Operator Norm. Let $f : X \to Y$ be a linear function, where X and Y are normed vector spaces. Consider the set

$$F = \{c \geq 0 \mid \|f(x)\| \leq c\|x\| \text{ for all } x \in X\}.$$

Show that F is closed and non-empty. The infimum of this set is called the *operator norm* $\|f\|_{op}$ of f. Prove that this is, in fact, a norm.

Exercise 1.22 — Riesz's Lemma. Let E be a closed proper subspace of a normed vector space X. Given $0 < r < 1$, show that there is $x \in X$ with $\|x\| = 1$ such that

$$\|x - y\| \geq r$$

for all $y \in E$.

CHAPTER 2

Sequences and Series

As we'll see, most of the important concepts in analysis are well described in terms of sequences. As before, whenever it is possible and enlightening, we'll study sequences in it's more general setting possible. This will usually be metric or normed spaces.

In the first chapter, we saw (quite informally) that a sequence is a function $x : \mathbb{N} \to X$, where X is an arbitrary set. We denote the element $x(n)$ as x_n and write as (x_n) the sequence itself.

2.1 Convergent Sequences

Definition 2.1.1 — Convergent Sequence. A sequence (x_n) defined on a metric space M is said to *converge* if there exists a point $x \in M$ with the following property: for every neighborhood N_x of x, there is an integer n_0 such that $n > n_0$ implies $x_n \in N_x$. If such point does not exist, then (x_n) is said to *diverge*.

If (x_n) is convergent, we way that (x_n) converges to x or that x is the limit of (x_n) and write this as $x = \lim_{n\to\infty} x_n$ or $x_n \to x$.

The preceding definition can be restated in two very useful ways:

- (x_n) converges to x if every neighborhood of x contains every x_n with the possible exception of a finite number of points.
- (x_n) converges to x if for every $\varepsilon > 0$ there is an integer n_0 such that $n > n_0$ implies $d(x, x_n) < \varepsilon$.

A priori, nothing prevents a sequence to converging to more than one point. The following theorem shows that this is not the case.

Theorem 2.1.1 A sequence (x_n) can converge to at most one point.

Proof. *Suppose $x_n \to a$ and $x_n \to b$. Then, for every $\varepsilon > 0$, there are integers n_0 and m_0 such that, if $n > n_0$ we have that $d(a,x_n) < \varepsilon/2$ and if $n > m_0$ we have that $d(b,x_n) < \varepsilon/2$. Then, if $n > \max\{n_0,m_0\}$, the triangular inequality implies*

$$d(a,b) \leq d(a,x_n) + d(b,x_n) < \varepsilon.$$

It follows that $a = b$. □

Theorem 2.1.2 A point $x \in M$ is a limit point of M if, and only if there is a sequence (x_n) of points in $M \setminus \{x\}$ such that $x_n \to x$.

Proof. *Suppose x is a limit point of M and let N_k be a neighborhood of x with radius $1/k$. Since x is a limit point, $N_1 \setminus \{x\}$ is not empty. Let x_1 be one of its elements. Similarly, for all $n \in \mathbb{N}$, let x_n be a element of $N_n \setminus \{x\}$. It follows that (x_n) converges to x. The converse is trivial.* □

We say that the set of all points x_n is called the *range* of (x_n) and it is denoted $\{x_n\}$. When $\{x_n\}$ is bounded we'll typically abuse language and say that the sequence itself is bounded. You should notice the similarity between the range of a sequence and the range of a function.

Theorem 2.1.3 If (x_n) converges, then $\{x_n\}$ is bounded. Moreover, the converse does not hold.

Proof. *If $x_n \to x$, then there is an integer n_0 such that $n \geq n_0$ implies $d(x_n,x) < 1$. Let*

$$r = \max\{1, d(x_1,x), d(x_2,x), \ldots, d(x_{n_0},x)\}.$$

Then $d(x_n,x) \leq r$ for all $n \in \mathbb{N}$.

The sequence $x_n = (-1)^n$ is a counter-example to the converse since it is bounded and divergent. □

Theorem 2.1.4 A sequence of points $z_n = (x_n,y_n)$ defined on the Cartesian product $M_1 \times M_2$ endowed with the maximum metric converges to $z = (x,y) \in M_1 \times M_2$ if, and only if $x_n \to x$ and $y_n \to y$.

Proof. *Let d_1 be the metric defined on M_1, d_2 be the metric defined on M_2 and d be the maximum metric on $M_1 \times M_2$. Let $\varepsilon > 0$ and assume $z_n \to z = (x,y)$. By our assumption, there is a natural number n_0 such that $n > n_0$ implies*

$$d(z_n,z) < \varepsilon.$$

Since $d_1(x_n,x) \leq d(z_n,z)$ and $d_2(y_n,y) \leq d(z_n,z)$ it follows that

$$d_1(x_n,x) \leq d(z_n,z) < \varepsilon \text{ and } d_2(y_n,y) \leq d(z_n,z) < \varepsilon.$$

Hence $x_n \to x$ *and* $y_n \to y$.

Conversely, assume $x_n \to x$ *and* $y_n \to y$. *For every* $\varepsilon > 0$ *there are integers* n_1 *and* n_2 *such that* $n > n_1$ *implies* $d_1(x_n, x) < \varepsilon$ *and* $n > n_2$ *implies* $d_2(y_n, y) < \varepsilon$. *Taking* $n_0 = \max\{n_1, n_2\}$ *we have that* $n > n_0$ *implies* $d(z_n, z) < \varepsilon$ *and hence* $z_n \to z$. □

Corollary 2.1.5 A sequence of points $z_n = (x_n, y_n)$ defined on the Cartesian product $X_1 \times X_2$ of normed vector spaces of finite dimension converges to $z = (x, y) \in X_1 \times X_2$ if, and only if $x_n \to x$ and $y_n \to y$.

Proof. *Since all norms are equivalent on finite dimensional normed spaces, the particular result of Theorem* 3.1.4 *becomes a general result in finite dimensional normed spaces.* □

The two preceding results clearly generalize to products of any finite number of spaces.

Besides understanding the topological aspects of sequences, we can also study some algebraic properties.

Theorem 2.1.6 Let (x_n) and (y_n) be sequences defined on a normed vector space X and let (p_n) be a sequence in X's field. If $x_n \to x$, $y_n \to y$ and $p_n \to p$, then the following holds.

$$\lim_{n\to\infty}(x_n + y_n) = x + y, \qquad \lim_{n\to\infty}(p_n x_n) = px \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{p_n} = \frac{1}{p}.$$

The last property is true provided that $p_n \neq 0$ for all $n \in \mathbb{N}$.

Proof. *For every* $\varepsilon > 0$ *there are integers* n_1 *and* n_2 *such that*

$$n > n_1 \implies \|x_n - x\| < \varepsilon/2 \quad \textit{and} \quad n > n_2 \implies \|y_n - y\| < \varepsilon/2.$$

The triangular inequality then implies

$$\|(x_n + y_n) - (x + y)\| \leq \|x_n - x\| + \|y_n - y\| < \varepsilon$$

for all $n > \max\{n_1, n_2\}$.

For the second result note that

$$\|p_n x_n - px\| = \|p_n(x_n - x) + (p_n - p)x\| \leq |p_n|\,\|x_n - x\| + |p_n - p|\,\|x\|.$$

Since $x_n \to x$ *and* $p_n \to p$, *for every* $\varepsilon > 0$ *there are integers* n_1, n_2 *such that*

$$n > n_1 \implies \|x_n - x\| < \varepsilon/2(\max|p_n|) \quad \textit{and} \quad n > n_2 \implies |p_n - p| < \varepsilon/2\,\|x\|.$$

Then, if $n > \max\{n_1, n_2\}$:

$$\|p_n x_n - px\| < \frac{\varepsilon|p_n|}{2(\max|p_n|)} + \frac{\varepsilon}{2} < \varepsilon.$$

Lastly, choosing n_0 such that $|p_n - p| < |p|/2$ if $n > n_0$, we see that[*]

$$|p_n| > \frac{1}{2}|p| \text{ for all } n > n_0.$$

Given $\varepsilon > 0$, there is an integer n_1 such that $n > n_1$ implies

$$|p_n - p| < \frac{1}{2}|p|^2\varepsilon.$$

Hence, for $n > \max\{n_0, n_1\}$,

$$\left|\frac{1}{p_n} - \frac{1}{p}\right| = \left|\frac{p_n - p}{p_n p}\right| < \frac{2}{|p|^2}|p_n - p| < \varepsilon.$$

□

Theorem 2.1.7 Let (p_n) be a sequence of real numbers such that $p_n \geq 0$ for all $n > n_0$, where n_0 is a fixed integer. If (p_n) converges to p, then $p \geq 0$.

Proof. *If $p < 0$, then there is an integer m_0 such that*

$$|p_n - p| < -\frac{p}{2}$$

for all $n > \max\{n_0, m_0\}$. Hence $p_n - p < -p/2$ and then $p_n < p/2 < 0$, which is absurd! The result follows. □

Corollary 2.1.8 Let (p_n) and (q_n) be sequences of real numbers such that $p_n \geq q_n$ for all $n > n_0$, where n_0 is a fixed integer. If (p_n) converges to p and (q_n) converges to q, then $p \geq q$.

Proof. *Apply Theorem 3.1.7 to the sequence $(p_n - q_n)$.* □

Definition 2.1.2 — Monotonicity. A sequence (p_n) of real numbers is said to be *increasing* if $p_{n+1} \geq p_n$ for all $n \in \mathbb{N}$. Similarly, we say that (p_n) is *decreasing* if $p_{n+1} \leq p_n$. If the strict inequality holds we say that (p_n) is *strictly* increasing or decreasing. In every case we say that (p_n) is *monotonic*.

Suppose (p_n) is an increasing sequence. Then for every $n \in \mathbb{N}$ we have that $p_n \geq n$. The proof follows readily by induction. This fact is a useful property of increasing sequences. Another useful property is the following.

[*] Combining $|p_n - p| \geq |p| - |p_n|$ and $\frac{1}{2}|p| > |p_n - p|$ we get the desired result.

Theorem 2.1.9 — Monotone Convergence Theorem. Let (p_n) be a monotonic sequence of real numbers. Then (p_n) is convergent if, and only if it is bounded.

Proof. *I'll suppose $p_{n+1} \geq p_n$ since all the other cases follow readily or are analogous to it. If (p_n) is bounded, then $\{p_n\}$ has a least-upper-bound. Let $p = \sup\{p_n\}$. For all $\varepsilon > 0$ there is an integer n_0 such that*

$$p - \varepsilon < p_{n_0} \leq p.$$

Otherwise the leftmost inequality would imply that $p - \varepsilon$ is an upper bound of $\{p_n\}$ smaller than p. The rightmost inequality follows from the fact that p is an upper bound of $\{p_n\}$. Since (p_n) is monotonic, for all $n > n_0$ the inequality $p_n \geq p_{n_0}$ holds. Hence, for all $n > n_0$

$$p - \varepsilon < p_n \leq p < p + \varepsilon.$$

It follows that (p_n) converges (to p). The converse was proved in Theorem 3.1.3. □

2.2 Subsequences and Sequential Compactness

Definition 2.2.1 — Subsequence. Let (n_k) be a strictly increasing sequence of positive integers. Given a arbitrary sequence (x_n), the sequence (x_{n_k}) is said to be a *subsequence* of (x_n). If (x_{n_k}) converges, its limit is called a *subsequential limit* of (x_n).

Theorem 2.2.1 A sequence (x_n) converges to x if, and only if every subsequence (x_{n_k}) of (x_n) converges to x.

Proof. *If there was a subsequence that* ***didn't*** *converge to x, then there will be a neighborhood of x that has infinite terms of the subsequence* ***outside*** *of it. So this neighborhood has infinite terms of the sequence outside of it, and then the sequence does not converge to x.*

Since every sequence is a subsequence of itself, the other direction is trivial. □

Some books say that a topological space is *sequentially compact* if every sequence has a convergent subsequence. The following theorem shows that every compact metric space is sequentially compact.

Theorem 2.2.2 Let (x_n) be a sequence in a compact metric space M. Then some subsequence (x_{n_k}) converges.

Proof. *If $\{x_n\}$ is finite, there is a point $x \in \{x_n\}$ and a strictly increasing sequence of positive integers (n_k) such that*

$$x_{n_1} = x_{n_2} = x_{n_3} = \ldots = x.$$

The sequence (x_{n_k}) then converges to x.

If $\{x_n\}$ is infinite, then Bolzano-Weierstrass implies $\{x_n\}$ has a limit point x. Theorem 3.1.2 then shows that there is a sequence of points of $\{x_n\}$ that converges to x. Let n_1 be the smallest index among the elements of this sequence. Having chosen $n_1, \dots, n_{k-1}$ let n_k be the smallest index among the elements of this sequence that is bigger than $n_1, \dots, n_{k-1}$. The sequence (x_{n_k}) then converges to x. □

Corollary 2.2.3 Let (x_n) be a bounded sequence in $\mathbb{R}^n$. Then some subsequence (x_{n_k}) converges.[a]

[a] Here the n in (x_n) is an index while the n in $\mathbb{R}^n$ is a fixed number.

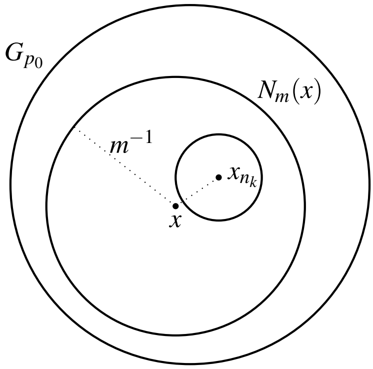
Proof. *Since $\overline{\{x_n\}}$ is a closed and bounded subset of $\mathbb{R}^n$, $\overline{\{x_n\}}$ is compact. The previous theorem implies the result.* □

Actually, in metric spaces the notions of compactness and sequential compactness are equivalent. We just need to show that every sequentially compact metric space is also compact. The following lemmata will be of great aid in this task. You should notice the resemblance of the first lemma to Exercise 2.10.

Lemma 2.2.4 Let M be a sequentially compact metric space and let $\{G_p\}$ be an open cover of M. Then, there exists a real number $\delta > 0$ such that every neighborhood with radius less than δ is contained in some element of $\{G_p\}$.

Proof. *Let $N_n(x)$ be a neighborhood of x with radius $1/n$. If the theorem is not true, for every $n \in \mathbb{N}$ some neighborhood $N_n(x_n)$ is not contained in any element of the cover. Since M is sequentially compact, we can find a subsequence (x_{n_k}) which converges to $x \in M$.*

Now, x is contained in some G_{p_0}. Since each G_p is open, there is an positive integer m such that $N_m(x) \subset G_{p_0}$. Since $x_{n_k} \to x$, for all $k > k_0$ $d(x, x_{n_k}) < 1/2m$. Since $n_k \geq k$, taking $k > \max\{k_0, 2m\}$ We have $N_{n_k}(x_{n_k}) \subset N_m(x) \subset G_{p_0}$.



This contradicts our choice of (x_n). The result follows. □

Lemma 2.2.5 Let M be a sequentially compact metric space. For all $\varepsilon > 0$ there is a finite open cover of M which consists only of neighborhoods with radii ε.

Proof. *Consider $N(x)$ to be a neighborhood of x with radius ε. If M is finite, then the result is obvious. Suppose M is infinite and the result does not hold. Construct a sequence (x_n) as follows. Let x_1 be any element of M. Having chosen $x_1,\dots,x_n$, pick any member of $M \setminus \bigcup_{k=1}^n N(x_k)$ to be x_{n+1}. If for some n the set $M \setminus \bigcup_{k=1}^n N(x_k)$ is empty, them the result is proved. Otherwise, note that $d(x_n,x_m) \geq \varepsilon$ for every $n \neq m$. Hence (x_n) has no convergent subsequence which is absurd!* □

Why does $d(x_n,x_m) \geq \varepsilon$ imply (x_n) has no convergent subsequence? Try to figure this out by yourself. In every case, this should be clear to you by the end of the next section.

Theorem 2.2.6 Let M be a sequentially compact metric space. Then M is compact.

Proof. *Let $\{G_p\}$ be an open cover of M. Let δ be the number of Lemma 3.2.4. Lemma 3.2.5 then implies that M can be covered by a finite number of neighborhoods with radii $\varepsilon = \delta$. Since each neighborhood N_i is contained in some G_{p_i}, the set of all G_{p_i} is a finite subcover of M.* □

Let (p_n) be a sequence of real numbers. If for all real numbers M there is an integer n_0 such that $p_n > M$ whenever $n > n_0$ we say that $p_n \to +\infty$. Analogously if $p_n < M$ whenever $n > n_0$ we say that $p_n \to -\infty$.

We're actually abusing a little of the notation here since the symbol $\to$ was used before to denote converging sequences and the kinds of sequences we just defined clearly diverge. This is in no way a change to the concept of convergence.

Definition 2.2.2 — Limit Superior/Inferior. Let (p_n) be a sequence of real numbers and $E \subset \overline{\mathbb{R}}$ be the set of all subsequential limits of (p_n). We define the limit superior and the limit inferior of (p_n) to be:

$$\limsup_{n\to\infty} p_n = \sup E \quad \text{and} \liminf_{n\to\infty} p_n = \inf E.$$

Of course there are sequences (p_n) for which $\limsup_{n\to\infty} p_n \neq \liminf_{n\to\infty} p_n$. A simple example is the sequence $p_n = (-1)^n$. In this case we have that $\limsup_{n\to\infty}(-1)^n = 1$ and $\liminf_{n\to\infty}(-1)^n = -1$.

Another way to state Theorem 3.2.1 for sequences of real numbers is as follows: (p_n) converges if, and only if

$$\limsup_{n\to\infty} p_n = \liminf_{n\to\infty} p_n = p,$$

where p is finite. In this case we have that $p_n \to p$.

Exercise 2.1 Prove that the set E that was just defined is closed in $\overline{\mathbb{R}}$. That is, there are subsequences (p_{n_k}) and (p_{m_k}) such that

$$\limsup_{n\to\infty} p_n = \lim_{k\to\infty} p_{n_k} \quad \text{and} \quad \liminf_{n\to\infty} p_n = \lim_{k\to\infty} p_{m_k}.$$

We now prove an analogous result to Corollary 3.1.8 to the limit superior and inferior.

Theorem 2.2.7 Let (p_n) and (q_n) be sequences of real numbers such that $p_n \geq q_n$ for all $n > n_0$ where n_0 is a fixed positive integer. Then

$$\limsup_{n\to\infty} p_n \geq \limsup_{n\to\infty} q_n \quad \text{and} \quad \liminf_{n\to\infty} p_n \geq \liminf_{n\to\infty} q_n.$$

Proof. *Assume the theorem is false. As you proved in Exercise* 2.1 *(if you didn't it may be helpful to check the solution in the end of the book), there is a subsequence* (q_{n_k}) *such that*

$$\lim_{k\to\infty} q_{n_k} = \limsup_{n\to\infty} q_n.$$

Hence,

$$\limsup_{n\to\infty} p_n < \lim_{k\to\infty} q_{n_k}.$$

Since $\limsup_{n\to\infty} p_n$ *is the biggest subsequential limit of* (p_n)*, we have that:*

$$\lim_{k\to\infty} p_{n_k} < \lim_{k\to\infty} q_{n_k}.$$

Taking $\varepsilon = \frac{1}{2}(\lim_{k\to\infty} q_{n_k} - \lim_{k\to\infty} p_{n_k})$ *we see that there is an positive integer* k_0 *for which*

$$p_{n_k} < q_{n_k}$$

whenever $k > k_0$*. This contradiction estabilishes the theorem. The proof for the limit inferior is analogous.* □

The following theorem provides a very useful method of calculating limits.

Theorem 2.2.8 Let (p_n) and (q_n) be sequences of real numbers. If $0 \leq p_n \leq q_n$ for all $n > n_0$, where n_0 is an positive integer, and $q_n \to 0$ then $p_n \to 0$.

Proof. *This result is a quick corollary to the preceding theorem since*

$$0 \leq \liminf_{n\to\infty} p_n \leq \limsup_{n\to\infty} p_n \leq \lim_{n\to\infty} q_n = 0.$$

We conclude that $\liminf_{n\to\infty} p_n = \limsup_{n\to\infty} p_n = 0$ *and hence* $p_n \to 0$. □

The following lemma will be useful in proving later results.

Lemma 2.2.9 Let (x_n) be a sequence of real numbers. If a is a real number such that $a > \limsup_{n\to\infty} x_n$, then there is an integer n_0 such that $x_n < a$ whenever $n > n_0$.

Proof. *If we had that $x_n \geq a$ for infinitely many values of n, then the subsequence (x_{n_k}) of all x_n such that $x_n \geq a$ converges to a value $x \geq a > \limsup_{n\to\infty} x_n$. This contradicts the definition of the limit superior.* □

We shall now present some examples of convergent sequences. Unless explicitly stated, it will be assumed that the metric space is the real field.

■ **Example 2.1** If $p > 0$, then $\lim 1/n^p = 0$. In fact, for all $\varepsilon > 0$ we can take any integer n_0 that is bigger than $1/\varepsilon^{1/p}$ (which exists since $\mathbb{R}$ is archimedean). Then, if $n > n_0$:

$$\frac{1}{n^p} < \frac{1}{n_0^p} < \frac{1}{1/\varepsilon} = \varepsilon.$$

It follows that $1/n^p \to 0$. ■

■ **Example 2.2** The sequence $(\sqrt[n]{n})$ converges to 1. Take $x_n = \sqrt[n]{n} - 1$. The binomial theorem then implies

$$n = (1+x_n)^n = \sum_{k=0}^{n} \binom{n}{k} x_n^k \geq \binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2,$$

since $x_n \geq 0$. Hence, for all $n \geq 2$:

$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}}.$$

Since $1/\sqrt{n-1} \to 0$ (from the previous example), we conclude that $x_n \to 0$. ■

■ **Example 2.3** If $p > 0$ and $q \in \mathbb{R}$, then $\lim n^q/(1+p)^n = 0$. Let k be an integer such that $k > q$ and $k > 0$. As one can verify, the following inequality holds for binomial coefficients: $\binom{n}{k} \geq n^k/k^k$. Hence,

$$(1+p)^n > \binom{n}{k} p^k \geq \frac{n^k}{k^k} p^k.$$

And then,

$$0 < \frac{n^q}{(1+p)^n} < \frac{k^k}{p^k} n^{q-k}.$$

Since $q - k < 0$, the result follows from Example 3.1. ■

■ **Example 2.4** Taking $q = 0$ in the previous example we get that

$$\lim_{n\to\infty} x^n = 0$$

whenever $|x| < 1$. ■

2.3 Cauchy Sequences

Unfortunately, it is not always easy to find the point x in the definition of a convergent sequence. So we must find better methods to tell if a sequence converges or not. Augustin-Louis Cauchy found an ingenious way.

Definition 2.3.1 — Cauchy Sequence. A sequence (x_n) in a metric space X is said to be a *Cauchy sequence* if for every $\varepsilon > 0$ there is an integer n_0 such that $d(x_n, x_m) < \varepsilon$ whenever $n, m > n_0$.

Note that, although the limit point is explicitly involved in the definition of a convergent sequence, it is not in the definition of a Cauchy sequence. As of now, convergent sequences and Cauchy sequences are two completely different things, but it would be wonderful if they shared properties. Fortunately, they do.

Theorem 2.3.1 Every convergent sequence in a metric space is Cauchy.

Proof. *Let (x_n) be a sequence convergent to x. Then, for all $\varepsilon > 0$ there is an integer n_0 such that $d(x_n, x) < \varepsilon/2$ whenever $n > n_0$. Hence,*

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \varepsilon,$$

whenever $n, m > n_0$. □

As one can easily verify, the sequence $(1/n)$ defined on $(0, 1]$ is Cauchy but diverges. However, there exist some important cases where the converse of the preceding theorem holds.

Theorem 2.3.2 Let M be a compact metric space. Then every Cauchy sequence in M converges.

Proof. *Let (x_n) be a Cauchy sequence in M. For any $\varepsilon > 0$ let n_0 be an positive integer such that*

$$d(x_n, x_m) < \frac{\varepsilon}{2} \quad \text{for all} \quad n, m > n_0.$$

Theorem 3.2.2 implies that there is a subsequence (x_{n_k}) that converges to some $x \in M$. That is, for every $\varepsilon > 0$ there is an positive integer k_0 such that

$$d(x_{n_k}, x) < \frac{\varepsilon}{2} \quad \text{whenever} \quad k > k_0.$$

Since $n_k \geq k$, $d(x_{n_k}, x) < \varepsilon/2$ whenever $n_k > k_0$. Let j be an integer such that $j > n_0$, $j > k_0$ and $j = n_k$ for some k. We have then that

$$d(x_n, x) \leq d(x_n, x_j) + d(x_j, x) < \varepsilon.$$

Hence (x_n) converges (to x). □

Lemma 2.3.3 Every Cauchy sequence is bounded.

Proof. *Let (x_n) be a Cauchy sequence in a metric space M. For $\varepsilon = 1$ there is a positive integer n_0 such that $d(x_n, x_m) < 1$ whenever $n, m > n_0$. That is, the set*

$$\{x_{n_0+1}, x_{n_0+2}, x_{n_0+3}, \dots\}$$

is bounded, since its diameter is less than 1. It follows that the set

$$\{x_n\} = \{x_1, \dots, x_{n_0}\} \cup \{x_{n_0+1}, x_{n_0+2}, x_{n_0+3}, \dots\}$$

is bounded. □

Theorem 2.3.4 In $\mathbb{R}^n$ every Cauchy sequence converges.

Proof. *Let (x_n) be a Cauchy sequence in $\mathbb{R}^n$. It follows from the previous lemma that $\overline{\{x_n\}}$ is a bounded closed subset of $\mathbb{R}^n$. Heine-Borel implies compacticity and hence this theorem follows from the previous one.* □

We noted before that the sequence $(1/n)$ defined on $(0,1)$ is Cauchy but diverges. Our intuition may tell that the problem here is that $(0,1)$ lacks the point to which $(1/n)$ "should" converge. The next theorem shows roughly that this is always the case.

Theorem 2.3.5 Let (x_n) be a Cauchy sequence in a metric space M. If there is a subsequence (x_{n_k}) which converges, then (x_n) converges.

Proof. *Let $x = \lim_{k\to\infty} x_{n_k}$. We'll prove that $x_n \to x$. In fact, let $\varepsilon > 0$ be an arbitrary real number. Since (x_{n_k}) converges to x, there exists a positive integer k_0 such that $d(x_{n_k}, x) < \varepsilon/2$ whenever $k > k_0$. Similarly, since (x_n) is a Cauchy sequence, there exists a positive integer m_0 such that $d(x_n, x_m) < \varepsilon/2$ whenever $n, m > m_0$. Let $n_0 = \{k_0, m_0\}$. Taking k big enough such that $n_k > n_0$ we have that*

$$d(x_n, x) \le d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

whenever $n > n_0$. The result follows. □

2.4 Complete Metric Spaces

The metric spaces in which every Cauchy sequence converges are so special that they are given a particular name.

Definition 2.4.1 — Complete Metric Spaces. A metric space in which every Cauchy sequence converges is said to be *complete*. A complete normed vector space is said to be a *Banach space*.

Thus, Theorem 3.3.2 and Theorem 3.3.3 say that *all compact metric spaces and all Euclidean spaces are complete*. It also implies that every closed subset E of a complete metric space M is complete.† A example of a metric space which is *not* complete is the space of all rational numbers, with $d(x,y) = |x-y|$ as a metric.

Theorem 2.4.1 Let M be a complete metric space. Let $\{F_n\}$ be a collection of non-empty closed subsets of M such that $F_{n+1} \subset F_n$ for all $n \in \mathbb{N}$. If $\operatorname{diam} F_n \to 0$, then

$$F = \bigcap_{n=1}^{\infty} F_n$$

consists of a single point.

Proof. *For all $n \in \mathbb{N}$ pick a point $x_n \in F_n$. The sequence (x_n) so defined has the property that $x_n, x_m \in F_{n_0}$ whenever $n,m > n_0$. Since $\operatorname{diam} F_n \to 0$, for all $\varepsilon > 0$ there is a positive integer n_0 such that $\operatorname{diam} F_{n_0} < \varepsilon$ and hence $d(x_n, x_n) < \varepsilon$ whenever $n,m > n_0$. That is, (x_n) is a Cauchy sequence. Since M is complete, let x be the point to which (x_n) converges.*

For any positive integer k we have that $x_n \in F_k$ whenever $n \geq k$. That implies $x \in F_k$ for all $k \in \mathbb{N}$. We conclude that F is not empty.

If F had more than one point, then it would follow that $\operatorname{diam} F > 0$. For each $n \in \mathbb{N}$, $F \subset F_n$ means that $\operatorname{diam} F \leq \operatorname{diam} F_n$ and hence $\operatorname{diam} F_n$ does not converge to 0. □

Note the resemblance of Theorem 3.4.1 and Corollary 2.4.5. The former is about complete spaces and the latter is about compact spaces. One may ask if the condition $\operatorname{diam} F_n \to 0$ is really necessary. I affirm it is. Let $F_n = [n, +\infty)$ be closed subsets of the real line (which is a complete metric space as we just proved). You may note that $\bigcap_{n=1}^{\infty} F_n = \varnothing$.

Theorem 2.4.2 — Baire's Theorem. Let M be a complete metric space and $\{E_n\}$ be a collection of sets where each E_n is open and dense in M. Then the set

$$E = \bigcap_{n=1}^{\infty} E_n$$

is a dense subset of M.

Proof. *Let N be **any** neighborhood. We'll prove that $E \cap N$ is not empty. Let $N_1 = N$. Since E_1 is open and dense, $E_1 \cap N_1$ is open and not empty. Hence it contains a neighborhood N_2 which we can suppose so little that $\overline{N_2} \subset E_1 \cap N_1$ and $\operatorname{diam} N_2 < 1/2$. Similarly, since E_2 is open and dense there is a neighborhood N_3 such that $\overline{N_3} \subset E_2 \cap N_2$ and $\operatorname{diam} N_3 < 1/3$. We then obtain a sequence of neighborhoods N_k such that $\overline{N_{k+1}} \subset \overline{N_k}$, $\overline{N_{k+1}} \subset E_k \cap N_k$ and $\operatorname{diam} N_k \to 0$. Theorem 2.4.1 then implies $\bigcap_{k=1}^{\infty} \overline{N_k}$ consists of a*

†Every Cauchy sequence in E is a Cauchy sequence in M, hence it converges to some $x \in M$, and actually $x \in E$ since E is closed.

single point x. Since $\overline{N_{k+1}} \subset E_k \cap N_k$, $x \in E_n$ *for all* $n \in \mathbb{N}$ *and* $x \in N_1 = N$. *Hence* $x \in E \cap N$. □

We could have proved the following corollary before but, since we didn't needed it before and this proof is so simple, this is a better place for it.

Corollary 2.4.3 The set of all real numbers $\mathbb{R}$ is uncountable.

Proof. *If* $\mathbb{R}$ *was countable we could list its elements as* $\{r_1, r_2, r_3, \dots\}$. *Since each set with only one element is closed, for each* r_n *the set* $E_n = \mathbb{R} \setminus \{r_n\}$ *is open and dense in* $\mathbb{R}$. *However note that*

$$\bigcap_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty} \mathbb{R} \setminus \{r_n\} = \mathbb{R} \setminus \bigcup_{n=1}^{\infty} \{r_n\} = \mathbb{R} \setminus \mathbb{R} = \varnothing.$$

This contradicts Baire's Theorem. □

In the first chapter we defined the real field as a ordered field with the least-upper-bound property. However, what if such a specific field does not exists? It would be a shame as I've spent months writing more than fifty pages of numerous results about the real field that are now *vacuously*[‡] true since there is no real field. Fortunately I now possess the machinery to present an example of a ordered field with the least-upper-bound property!

Definition 2.4.2 — Equivalent Sequences. Two sequences (x_n) and (y_n) defined on a same metric space M are said to be *equivalent* if

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

If (x_n) is equivalent to (y_n) we denote this fact as $(x_n) \sim (y_n)$.

It should be clear that this is an equivalence relation.

Theorem 2.4.4 Let Q be the set of all Cauchy sequences of rational numbers. The set $Q/\sim$ of all equivalence classes of such sequences is an ordered field with the least-upper-bound property.

Denoting by $\overline{x}$ and $\overline{y}$ the arbitrary elements $[(x_n)]$ and $[(y_n)]$ of $Q/\sim$ we'll define the sum of two elements of $Q/\sim$ as $\overline{x}+\overline{y} = \overline{x+y} = [(x_n+y_n)]$ and the product as $\overline{x}\cdot\overline{y} = \overline{xy} = [(x_n y_n)]$. The elements $\overline{0} = [(0)]$ and $\overline{1} = [(1)]$ are the additive and multiplicative identities and the element $\overline{-x} = [(-x_n)]$ is the additive inverse of $\overline{x} = [(x_n)]$. Last, but not least, we say that $\overline{x} > \overline{y}$ if $\overline{x-y} = [(x_n - y_n)]$ for some Cauchy sequences (x_n) and (y_n) such that $x_n > y_n$ for all $n > n_0$, where n_0 is an positive integer.

[‡] A logic statement is vacuously true if it asserts that all members of the empty set have a certain property. It's like a child saying to his or her parent: "I ate every vegetable in the plate!" when there was not a single vegetable there to begin with.

We can also embed $\mathbb{Q}$ into $Q/\sim$ using the function $x \mapsto [(x)]$. This function is injective since $[(x)] = [(y)]$ imply $\lim_{n\to\infty} |x-y| = |x-y| = 0$ and hence $x = y$.

Since the proof of this theorem is not really useful for the rest of the book but only acts as a motivation for you to believe that this book is not entirely useless, you can work it by yourself. (I'm not even sorry.)

Exercise 2.2 Prove Theorem 3.4.4.

2.5 Series in Normed Vector Spaces

Definition 2.5.1 — Series. Let X be a normed vector space. Given a sequence (x_n) in X we construct another sequence (s_n) as follows. Let s_1 be x_1. Having chosen $s_1, \ldots, s_n$, pick s_{n+1} to be $s_n + x_{n+1}$. It is said that the sequence (s_n) is an *series*. If (s_n) converges to s we denote this fact as

$$s = \sum_{n=1}^{\infty} x_n$$

and say that s is the sum of the series. It will be common to abuse language and denote by $\sum x_n$ the series itself. It is said that s_n is a *partial sum* of the series.

It should be noticed that, while we use the term "sum" and the symbol $\sum$, a series is no ordinary sum at all. A priori it is not even clear that series share properties with sums. Usually it is too hard to calculate the limit of a series. Until we develop a sufficiently robust machinery to deal with this task, we shall focus on convergence and divergence.

■ **Example 2.5 — Geometric Series.** Let x be any complex number such that $|x| < 1$. Using induction it is simple to verify that

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$

and hence

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

■

The next few theorems are simple consequences of important facts about sequences.

Theorem 2.5.1 — Cauchy Criterion. Let $\sum x_n$ be a series in a normed space X. If $\sum x_n$ converges, then for all $\varepsilon > 0$ there is an positive integer n_0 such that

$$\left\| \sum_{k=n}^{m} x_k \right\| < \varepsilon$$

whenever $m \geq n > n_0$. Moreover, in Banach spaces the converse holds.

Proof. *This is merely a restatement of Theorem* 2.3.1 *applied to the sequence* (s_n) *of partial sums. The converse is basically the definition of a complete space.* □

Corollary 2.5.2 If $\sum x_n$ converges, then $x_n \to 0$.

Proof. *Take* $m = n$ *in the preceding theorem.* □

The converse of this theorem does not hold even in complete spaces.

Example 2.6 — Harmonic Series. Take $x_n = 1/n$. The corresponding series is said to be the *harmonic series*. Clearly $x_n \to 0$. However, consider the sequence (y_n) such that

$$y_n = \sum_{k=n+1}^{2n} \frac{1}{k}.$$

Since $y_{n+1} - y_n = 1/(2k+1) - 1/(2k+2) > 0$, the sequence (y_n) is strictly increasing and hence $y_n > y_1 = 1/2$. It follows that the Cauchy criterion does not hold for $x_n = 1/n$ and the harmonic series diverges. ■

Theorem 2.5.3 If $\sum x_n$ and $\sum y_n$ are series which converge to s_1 and s_2 respectively and p is an element of X's field, the following holds.

$$\sum_{n=1}^{\infty} (x_n + y_n) = s_1 + s_2 \quad \text{and} \quad \sum_{n=1}^{\infty} p x_n = p s_1.$$

Proof. *Follows readily from Theorem* 2.1.6. □

Theorem 2.5.4 If $\sum x_n$ and $\sum y_n$ are converging series of real numbers such that $x_n \geq y_n$ for all $n \in \mathbb{N}$ then

$$\sum_{n=1}^{\infty} x_n \geq \sum_{n=1}^{\infty} y_n.$$

Proof. *Follows from Corollary* 2.1.8. □

2.6 Absolute Convergence

We shall meet now a particular family of series which behave much like ordinary sums.

Definition 2.6.1 — Absolute Convergence. A series $\sum x_n$ is said to be *absolutely convergent* if the series $\sum \|x_n\|$ converges.
If $\sum x_n$ converges but not absolutely, it is said to be *conditionally convergent*.

Theorem 2.6.1 In a Banach space X, an absolutely convergent series converges. Moreover we have that

$$\left\| \sum_{n=1}^{\infty} x_n \right\| \leq \sum_{n=1}^{\infty} \|x_n\| .$$

Proof. *The Cauchy criterion readily implies convergence. For $m \geq 1$:*

$$\left\| \sum_{n=1}^{m} x_n \right\| \leq \sum_{n=1}^{m} \|x_n\| \leq \sum_{n=1}^{\infty} \|x_n\| .$$

Then Corollary 3.1.8 *implies the inequality.* □

A permutation of a set S is a function $\sigma : S \to S$ which is bijective. Of course, since the sum of elements in a field or vector space is commutative, taking $S = \{1, 2, \ldots, n\}$ we see that

$$\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} x_{\sigma(k)}$$

for any sequence (x_k), any $n \in \mathbb{N}$ and any permutation σ. After all, we just rearranged the elements of the sum. One would expect that series share this property. As we'll see soon, Bernhard Riemann showed that we couldn't be more wrong!

■ **Example 2.7 — Alternating Harmonic Series.** For now, you'll trust me that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges to a value $s \neq 0$. Consider the following permutation in the set of all positive integers:

$$\sigma(n) = \begin{cases} 4n/3 & \text{if } n \text{ is divisible by } 3 \\ (2n+1)/3 & \text{if } n-1 \text{ is divisible by } 3 \\ (4n-2)/3 & \text{if } n-2 \text{ is divisible by } 3 \end{cases} .$$

If $x_n = (-1)^{n+1}/n$, the series $\sum x_{\sigma(n)}$ can be decomposed as

$$\sum_{n=1}^{\infty} x_{\sigma(n)} = \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2(2n-1)} - \frac{1}{4n} \right) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{s}{2}.$$

Hence, a rearrangement of the terms can modify the value to which a series converges. ■

Fortunately, rearranging series that converge absolutely does not change its value.

Theorem 2.6.2 Let $\sum x_n$ be a series in a Banach space X which converges absolutely. Then every rearrangement $\sum x_{\sigma(n)}$ converges, and they all converge to the same value.

Proof. *Let (s'_n) be the sequence of the partial sums of $\sum x_{\sigma(n)}$. Since $\sum x_n$ is absolutely convergent, given $\varepsilon > 0$ there is an integer n_0 such that*

$$\sum_{k=n_0}^{m} \|x_k\| < \varepsilon$$

for all $m \geq n_0$. Let

$$p = \max_{1 \leq i < n_0} \sigma^{-1}(i).$$

If $n > p$, we have that $\{1,2,\ldots,n_0-1\}$ is a subset of $\{\sigma(1),\sigma(2),\ldots,\sigma(n)\}$. Hence all the x_i for $i = 1,2,\ldots,n_0-1$ are cancelled in $s_n - s'_n$. So,

$$\left\|s_n - s'_n\right\| \leq \sum_{k=n_0}^{m} \|x_k\| < \varepsilon.$$

We conclude that (s'_n) converges to the same value as (s_n). □

The following theorem is Riemann's way of saying: "Thou shall be careful when dealing with such outlandish objects as series!"

Theorem 2.6.3 — Riemann Rearrangement Theorem. Let $\sum x_n$ be a conditionally convergent series of real numbers. Suppose

$$-\infty \leq \alpha \leq \beta \leq +\infty.$$

Then there exists a rearrangement $\sum x_{\sigma(n)}$ with partial sums s'_n such that

$$\liminf_{n\to\infty} s'_n = \alpha, \quad \text{and} \quad \limsup_{n\to\infty} s'_n = \beta.$$

Proof. *For all $n \in \mathbb{N}$, define $p_n = (|x_n| + x_n)/2$ and $q_n = (|x_n| - x_n)/2$. If both $\sum p_n$ and $\sum q_n$ were convergent, then so would be $\sum |x_n| = \sum(p_n + q_n)$. Since*

$$\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} (p_k - q_k) = \sum_{k=1}^{n} p_k - \sum_{k=1}^{n} q_k,$$

the divergence of just one of $\sum p_n$ or $\sum q_n$ implies in the divergence of $\sum x_n$. We conclude that the series $\sum p_n$ and $\sum q_n$ must both diverge.

Now, let $P_1, P_2, P_3, \ldots$ denote the non-negative terms of $\sum x_n$, in the order in which they occur, and let $Q_1, Q_2, Q_3, \ldots$ be the absolute values of the negative terms of $\sum x_n$, also in their original order.

The series $\sum P_n$ and $\sum Q_n$ differ from $\sum p_n$ and $\sum q_n$ only by zero terms, and are therefore divergent.

We shall construct sequences (m_n), (k_n), such that the series

$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} + \cdots,$$

which clearly is a rearrangement of $\sum x_n$*, satisfies the statement of the theorem.*

Choose real-valued sequences (α_n)*, and* (β_n) *such that* $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\alpha_n < \beta_n$ *and* $\beta_1 > 0$.

Let m_1, k_1 *be the smallest integers such that*

$$P_1 + \cdots + P_{m_1} > \beta_1,$$
$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} < \alpha_1.$$

Let m_2, k_2 *be the smallest integers such that*

$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} > \beta_2,$$
$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} < \alpha_2,$$

and continue in this way. This is possible since $\sum P_n$ *and* $\sum Q_n$ *diverge.*

If b_n *and* a_n *are the partial sums whose last terms are* P_{m_n} *and* $-Q_{k_n}$*, then*

$$|b_n - \beta_n| \leq P_{m_n}, \qquad |a_n - \alpha_n| \leq Q_{k_n}.$$

Since $P_n \to 0$ *and* $Q_n \to 0$*, we see that* $b_n \to \beta$ *and* $a_n \to \alpha$.

Finally, it is clear that no number less than α *or greater than* β *can be a subsequential limit of the partial sums of our rearrangement.* □

The preceding proof and slightly generalization of the original result by Bernhard Riemann appears to be due to W. Rudin[13].

2.7 Series of Non-Negative Real Numbers

Series of non-negative real numbers share a particularly simple behaviour, so we'll focus on them for a while.

Theorem 2.7.1 Let $\sum x_n$ be a series of non-negative real numbers. Then $\sum x_n$ converges if, and only if its sequence of partial sums (s_n) is bounded.

Proof. *Since* $x_n \geq 0$ *for all* $n \in \mathbb{N}$, (s_n) *is increasing. Hence the Monotone Convergence Theorem applies.* □

Theorem 2.7.2 — Comparison Test. Let $\sum x_n$ and $\sum y_n$ be series of real numbers such that $\sum y_n$ converges and $|x_n| \leq y_n$ for all $n > n_0$. Then $\sum x_n$ is absolutely convergent, and hence convergent (since $\mathbb{R}$ is Banach).

Proof. *Since* $\sum y_n$ *converges, for all* $\varepsilon > 0$ *there exists* $n_1 \geq n_0$ *such that*

$$\left| \sum_{k=n}^{m} y_k \right| < \varepsilon,$$

whenever $m \geq n > n_1$. *So,*

$$\sum_{k=n}^{m} |x_k| \leq \sum_{k=n}^{m} y_k < \varepsilon.$$

The result follows from the Cauchy Criterion. □

Corollary 2.7.3 Let $\sum x_n$ and $\sum y_n$ be series of non-negative real numbers such that $x_n \geq y_n$ for all $n > n_0$. If $\sum y_n$ diverges, then so do $\sum x_n$.

Proof. *If* $\sum x_n$ *converges, then the previous theorem implies that* $\sum y_n$ *converges too. Which is an absurd!* □

Corollary 2.7.4 — Limit Comparison Test. Let $\sum x_n$ and $\sum y_n$ be series of non-negative real numbers. If

$$0 < \lim_{n\to\infty} \frac{x_n}{y_n} < +\infty,$$

then either both series converge or both series diverge.

Proof. *If* $x_n/y_n \to c$, *let* $\varepsilon = c/2$. *The limit then implies that for all* $n > n_0$:

$$\frac{c}{2} y_n < x_n < \frac{3c}{2} y_n.$$

If $\sum x_n$ *converges, then so does* $\sum c y_n/2$ *and hence* $\sum y_n$ *converges. Similarly, if* $\sum y_n$ *converges, then so does* $\sum 2x_n/3c$ *and hence* $\sum x_n$ *converges.* □

Although the next example is wonderful, it assumes the knowledge of the fundamental theorem of arithmetic. So, don't be afraid to skip it. The following proof is due to J. A. Clarkson[3].

■ **Example 2.8 — The Series of Primes Reciprocals.** Let (p_n) be the sequence of all the prime numbers. I affirm that the series of prime reciprocals,

$$\sum_{n=1}^{\infty} \frac{1}{p_n}$$

diverges. If it converged, then there would exist an integer k such that

$$\left| \sum_{n=1}^{\infty} \frac{1}{p_n} - \sum_{n=1}^{k} \frac{1}{p_n} \right| = \sum_{n=k+1}^{\infty} \frac{1}{p_n} < \frac{1}{2}.$$

Let $Q = p_1 \cdots p_k$, and consider the numbers $1 + nQ$ for $n \in \{1, \ldots, r\}$. Since these numbers are not divisible by any of the $p_1, \ldots, p_k$, the prime factors of all these numbers are members of a finite set $\{p_{k+1}, p_{k+2}, \ldots, p_{m(r)}\}$. If $\tau(n)$ is the number of (not necessarily distinct) prime factors in $1+nQ$ we have that

$$\begin{aligned}\sum_{n=1}^{r}\frac{1}{1+nQ} &= \sum_{t=1}^{\infty}\sum_{\substack{n=1\\ \tau(n)=t}}^{r}\frac{1}{1+nQ}\\ &< \sum_{t=1}^{\infty}\left(\sum_{n=k+1}^{m(r)}\frac{1}{p_n}\right)^t\\ &< \sum_{t=1}^{\infty}\left(\sum_{n=k+1}^{\infty}\frac{1}{p_n}\right)^t\\ &< \sum_{t=1}^{\infty}\left(\frac{1}{2}\right)^t = 1.\end{aligned}$$

The inequality

$$\sum_{\substack{n=1\\ \tau(n)=t}}^{r}\frac{1}{1+nQ} < \left(\sum_{n=k+1}^{m(r)}\frac{1}{p_n}\right)^t$$

is justified by the fact that every term in the leftmost sum appears at least once in the sum on the right. We conclude that $\sum 1/(1+nQ)$ converges, since its sequence of partial sums is bounded. However the limit comparison test says otherwise (when compared to the harmonic series). ∎

Leonhard Euler was the first one to prove this result and to notice that it implies Euclid's theorem on the infinitude of prime numbers.

Theorem 2.7.5 — Cauchy Condensation Test. Let (x_n) be a decreasing sequence of non-negative real numbers. Then the series $\sum x_n$ converges if, and only if the series

$$\sum_{k=0}^{\infty} 2^k x_{2^k} = x_1 + 2x_2 + 4x_4 + 8x_8 + \cdots$$

converges.

Proof. *Let*

$$s_n = \sum_{i=1}^{n} x_n, \qquad t_k = \sum_{i=0} 2^i x_{2^i}.$$

For $n < 2^k$,

$$\begin{aligned} s_n &\le x_1 + (x_2 + x_3) + \cdots + (x^{2^k} + \cdots + x_{2^{k+1}-1})\\ &\le x_1 + 2x_2 + \cdots + 2^k x_{2^k}\\ &= t_k.\end{aligned}$$

Similarly, for $n > 2^k$,

$$\begin{aligned} s_n &\geq x_1 + x_2 + (x_3 + x_4) = \cdots + (x_{2^{k-1}+1} + \cdots + x_{2^k}) \\ &\geq \frac{1}{2}x_1 + x_2 + 2x_4 + \cdots + 2^{k-1}x_{2^k} \\ &= \frac{1}{2}t_k. \end{aligned}$$

We conclude that (s_n) and (t_k) are both bounded or both unbounded. Theorem 3.7.1 implies the result. □

■ **Example 2.9** The series $\sum 1/n^p$ converges if, and only $p > 1$. In fact, if $p \leq 0$ then $1/n^p$ does not tend to 0 as $n \to \infty$ and hence $\sum 1/n^p$ diverges. For $p > 0$, the preceding theorem says that $\sum 1/n^p$ converges or diverges together with

$$\sum_{k=0}^{\infty} 2^k \cdot \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{(1-p)k}.$$

We conclude that $\sum 1/n^p$ converges if, and only if $p > 1$. ■

The following theorem shows that an even stronger result than Corollary 2.5.2 holds for series of decreasing non-negative real numbers.

Theorem 2.7.6 Let (x_n) be a decreasing sequence of non-negative real numbers. Then the series $\sum x_n$ converges only if

$$\lim_{n\to\infty} nx_n = 0.$$

Proof. *If $\sum x_n$ converges, then for all $\varepsilon > 0$ we have that*

$$\sum_{k=m}^{n} x_n < \frac{\varepsilon}{2}$$

whenever $n \geq m > n_0$. Since (x_n) is decreasing,

$$\frac{n}{2}x_n \leq (n-m+1)x_n \leq \sum_{k=m}^{n} x_n < \frac{\varepsilon}{2}$$

whenever $n > 2m$. It follows that $nx_n \to 0$. □

The next theorem allow us to determine the convergence of a different family of series. The series to which Leibniz test applies are called *alternating series*. A quick corollary of this test is that the alternating harmonic series converges.

Theorem 2.7.7 — Leibniz Test. Let (x_n) be a decreasing sequence of non-negative real numbers such that $x_n \to 0$. Then, the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} x_n$$

converges.

Proof. *Let s_n be the partial sums of the series and define $I_n = [s_{2n}, s_{2n-1}]$.*

Note that $s_{2n+2} - s_{2n} = -x_{2n+2} + x_{2n+1} \geq 0$. Similarly, $s_{2n+1} - s_{2n-1} \leq 0$ and hence $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$. Since $x_n \to 0$ it follows that $\operatorname{diam} I_n \to 0$ and then

$$\bigcap_{n=1}^{\infty} I_n = \{s\}.$$

Pick n_0 such that $\operatorname{diam} I_n < \varepsilon$ for $n > n_0$. Then $|s_n - s| < \varepsilon$ and the result follows. □

Exercise 2.3 Give another proof of the Leibniz Test using only the Cauchy criterion.

2.8 Two Fundamental Tests and Power Series

The following two theorems are consequences of the comparison test and constitute our most powerful machinery.

Theorem 2.8.1 — Root Test. Let $\sum x_n$ be a series in a normed vector space X. If

$$\limsup_{n\to\infty} \sqrt[n]{\|x_n\|} < 1,$$

$\sum x_n$ is absolutely convergent. Hence $\sum x_n$ converges if X is Banach.

Proof. *Let k be a real number such that $\limsup_{n\to\infty} \sqrt[n]{\|x_n\|} < k < 1$. Lemma 3.2.9 then implies that there is an integer n_0 such that $\sqrt[n]{\|x_n\|} < k$ whenever $n > n_0$. Hence, $\|x_n\| < k^n$. Using the comparison test with the geometric series we conclude that $\sum \|x_n\|$ converges.* □

Note that while $\sum x_n$ is a series in a normed vector space, $\sum \|x_n\|$ is a series of real numbers and hence the comparison test applies.

Theorem 2.8.2 — Ratio Test. Let $\sum x_n$ be a series in a normed vector space X. If

$$\limsup_{n\to\infty} \frac{\|x_{n+1}\|}{\|x_n\|} < 1,$$

then $\sum x_n$ is absolutely convergent. Hence $\sum x_n$ converges if X is Banach.

Proof. *Let k be a real number such that $\limsup_{n\to\infty} \sqrt[n]{\|x_n\|} < k < 1$. Lemma 2.2.9 then implies that there is an integer n_0 such that $\|x_{n+1}\| < k\|x_n\|$ whenever $n > n_0$. Hence,*

$$\left\|x_{n_0+p}\right\| < k\left\|x_{n_0+p-1}\right\| < k^2\left\|x_{n_0+p-2}\right\| < \cdots < k^{p-1}\left\|x_{n_0+1}\right\|$$

and then

$$\|x_n\| < \frac{\left\|x_{n_0+1}\right\|}{k^{n_0+1}}k^n$$

whenever $n > n_0$. The result now follows from the comparison test. □

Whenever the ratio test implies convergence, the root test does too but the converse does not hold. However it is usually easier to apply the ratio test. It should be noted that we only have two criteria for determining divergence. Namely Corollary 2.5.2 and Theorem 2.7.6.

Definition 2.8.1 — Power Series. Given a sequence (a_n) of complex numbers, the series

$$\sum_{n=0}^{\infty} a_n z^n,$$

defined for complex z, is said to be a *power series*. The numbers a_n are the *coefficients* of the series.

The convergence of a power series depends on the value of z, so we'll study which values of z makes a given power series convergent or not. In general, there is a number $R \in \overline{\mathbb{R}}$, called the *radius of convergence* such that $\sum a_n z^n$ converges for $|z| < R$. This is the soul of the following theorem.

Theorem 2.8.3 Given a power series $\sum a_n z^n$, let

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

Then, $\sum a_n z^n$ converges for $|z| < R$.

Proof. *If $|z| < R$, then*

$$\limsup_{n\to\infty} \sqrt[n]{|a_n z^n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{|z|}{R} < 1.$$

The root test then implies convergence. □

In this proof we used the fact that $\mathbb{C}$ is complete, a result that we did not explicitly proved. However, the only difference between $\mathbb{C}$ and $\mathbb{R}^2$ is the definition of an multiplication of elements. As normed vector spaces they are exactly the same thing.

The following series defines the most important function in mathematics.

■ **Example 2.10** The series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges (absolutely) everywhere. As

$$\frac{|z^{n+1}|n!}{|z^n|(n+1)!} = \frac{|z|}{n+1},$$

the ratio test implies convergence for every $z \in \mathbb{C}$. (Except for $z = 0$, but in this case the result is trivial.) ■

Definition 2.8.2 — Exponential Function. For all $z \in \mathbb{C}$ we define $\exp(z)$ to be

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

This function is said to be the *exponential function*. The number $\exp(1)$ is so important that it will be denoted e and called *Euler's number*, in honor of Leonhard Euler.

Example 2.10 shows that this is a well defined function. Unfortunately, we cannot prove some familiar properties of this function yet. However we can prove two known facts about e.

Theorem 2.8.4 Euler's Number e is irrational.

Proof. *Let s_n be the partial sums of $\sum 1/n!$. Note that*

$$\begin{aligned} e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots \\ &< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right) = \frac{1}{n!n}. \end{aligned}$$

Hence,

$$0 < n!(e - s_n) < \frac{1}{n},$$

for all $n \in \mathbb{N}$. The number e is clearly a positive real number. If e was rational, then we would have that $e = m/n$ for some $m, n \in \mathbb{N}$. Since ne is an integer, so is $n!(e - s_n)$. That is, we found an integer between 0 and 1. Absurd! □

Theorem 2.8.5 The sequences

$$x_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad y_n = \sum_{k=0}^{n} \frac{1}{k!}$$

converge to the same limit. Namely, Euler's number e.

Proof. *For $n \geq 2$, the binomial theorem implies*

$$x_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \prod_{j=1}^{n-1} \left(1-\frac{j}{n}\right) \leq \sum_{k=0}^{n} \frac{1}{k!} = y_n.$$

Hence, $\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n = e$*. Now, if $n \geq m$ we have that*

$$x_n = \sum_{k=0}^{n} \frac{1}{k!} \prod_{j=1}^{n-1} \left(1-\frac{j}{n}\right) \geq \sum_{k=0}^{m} \frac{1}{k!} \prod_{j=1}^{m-1} \left(1-\frac{j}{n}\right).$$

Let $n \to \infty$ keeping m fixed. We get

$$\liminf_{n\to\infty} x_n \geq \liminf_{n\to\infty} \left[\sum_{k=0}^{m} \frac{1}{k!} \prod_{j=1}^{m-1} \left(1-\frac{j}{n}\right)\right] \stackrel{*}{=} \sum_{k=0}^{m} \frac{1}{k!} \prod_{j=1}^{m-1} (1-0) = \sum_{k=0}^{m} \frac{1}{k!} = y_m.$$

Hence, $\liminf_{n\to\infty} x_n \geq \liminf_{n\to\infty} y_n = e$*. The result follows.* □

In this proof we assumed that the properties of Theorem 3.1.6 *were also valid for the limit inferior. However this is not true as suggested by Exercises* 3.13 *and* 3.14*. Show that the proof of Theorem* 3.8.5 *is still valid provided that we switch the starred equality sign by an $\geq$ sign.*

We now define the trigonometric functions without any aid of geometrical interpretations.

Definition 2.8.3 — Trigonometric Functions. If $z \in \mathbb{C}$, we define the trigonometric functions *sine* and *cosine* as

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \quad \cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}.$$

The trigonometric function *tangent* is defined in the usual way as

$$\tan(z) = \frac{\sin(z)}{\cos(z)}.$$

For trigonometric functions it is usual to let $\sin^n(z)$ mean $(\sin(z))^n$.

In the present state we aren't even able to prove that $\sin^2(z) + \cos^2(z) = 1$. However, Euler's Identity follows readily from the definition.

Theorem 2.8.6 — Euler's Identity. For all $t \in \mathbb{R}$, the following equation holds:

$$\exp(it) = \cos(t) + i\sin(t).$$

Proof. *This is merely a restatement of the definition.* □

2.9 The Cauchy Product of Series

Given two finite sums $\sum_{k=1}^{n} a_k$ and $\sum_{k=1}^{n} b_k$, it is clear that their product is given by the sum of all $a_i b_j$, such that $1 \leq i, j \leq n$. However, when dealing with series, the order in which we sum the $a_i b_j$ may change the limit. History has shown that, for the purposes of analysis, there is one best way of dealing with this problem. Is this section we'll deal exclusively with series of real or complex numbers.

Definition 2.9.1 — Cauchy Product. Given two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, we define a sequence (c_n) as

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

and call $\sum_{n=0}^{\infty} c_n$ the *Cauchy product* of the two series.

This definition can be motivated by observing that when we multiply two polynomials of the same degree and collect the terms which contain the same power of z we get the sequence (c_n):

$$\left(\sum_{k=0}^{n} a_k z^k\right)\left(\sum_{k=0}^{n} b_k z^k\right) = \sum_{k=0}^{2n} c_k z^k,$$

provided that we consider $a_k = b_k = 0$ for $k \geq n$.

If $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$, it is natural to ponder whether $\sum_{n=0}^{\infty} c_n$ converges to AB or not. Mertens proved that if at least one of the series converges absolutely, then the result holds.

Theorem 2.9.1 Let $A = \sum_{n=0}^{\infty} a_n$ be an absolutely convergent series and let $B = \sum_{n=0}^{\infty} b_n$ be a convergent series. Then $\sum_{n=0}^{\infty} c_n$ converges to AB.

Proof.

□

2.10 Exercises

Exercise 2.4 Let (x_n) be an sequence of real numbers such that

$$\lim_{n\to\infty} (2x_{n+1} - x_n) = x.$$

Prove that $x_n \to x$.

Exercise 2.5 Let (x_n) and (y_n) be sequences of real numbers. Assume that $y_n \to 0$ and that there is a real number $k \in (0,1)$ such that $x_{n+1} \leq kx_n + y_n$ for every $n \in \mathbb{N}$. Prove that $x_n \to 0$.

Exercise 2.6 If $p > 0$, prove that

$$\lim_{n\to\infty} \sqrt[n]{p} = 1.$$

Exercise 2.7 — The Arithmetic-Geometric Mean. Let x, y be two positive real numbers. Define the sequences (x_n) and (y_n) by $x_1 = \sqrt{xy}$, $y_1 = (x+y)/2$, and

$$x_{n+1} = \sqrt{x_n y_n}, \quad y_{n+1} = \frac{x_n + y_n}{2},$$

for all $n \in \mathbb{N}$. Prove that the two sequences are convergent and have the same limit.

Exercise 2.8 — Cesàro-Stolz Theorem. Let (x_n) and (y_n) be two sequences of real numbers with (y_n) strictly positive, increasing and unbounded. If

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - x_n} = L,$$

then the limit

$$\lim_{n\to\infty} \frac{x_n}{y_n}$$

exists and is equal to L

Exercise 2.9 — Cesàro Means. Let (x_n) be an convergent sequence of real numbers. Prove that the sequence

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

Exercise 2.10 Let (x_n) and (y_n) be Cauchy sequences in a metric space X. Show that the sequence $(d(x_n, y_n))$ converges.

Exercise 2.11 Let (x_n) be a sequence of real numbers. Decide whether each of the following propositions are true or false. In each case, prove it or show a counter-example.

a) If every subsequence of (x_n) that is not itself converges, then (x_n) converges as well.

b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

Exercise 2.12 Let (x_n) be a sequence of real numbers. Show that

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_m \mid m \geq n\} = \lim_{n\to\infty} \sup_{m\geq n} x_m,$$

where the rightmost equality is a common abuse of notation. Moreover, show that this is also valid for the limit inferior.

Exercise 2.13 For any two real sequences (x_n) and (y_n), prove that

$$\liminf_{n\to\infty}(x_n + y_n) \geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n,$$

provided the sum on the right is not of the form $\infty - \infty$. Moreover show that this is also valid for the limit superior, provided that we switch the side of the inequality.

Exercise 2.14 For any two real sequences (x_n) and (y_n) such that x_n and y_n non-negative, prove that

$$\liminf_{n\to\infty}(x_n y_n) \geq \left(\liminf_{n\to\infty} x_n\right)\left(\liminf_{n\to\infty} y_n\right),$$

provided the sum on the right is not of the form $0 \cdot \infty$. Moreover show that this is also valid for the limit superior, provided that we switch the side of the inequality.

Exercise 2.15 Let (x_n) be a sequence of positive real numbers. Then

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \leq \liminf_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

In particular this result implies that if $\lim_{n\to\infty} x_{n+1}/x_n = L$, with $0 \leq L \leq \infty$, then $\lim_{n\to\infty} \sqrt[n]{x_n} = L$.

Exercise 2.16 For which real numbers p does the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n^p}$$

converge?

Exercise 2.17 Let $S = \{x_1, x_2, \dots\}$ be the set of all integers that do not contain the digit 9 in their decimal representation. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{x_n}$$

converges.

Exercise 2.18 Find the radius of convergence of each of the following power series:

a) $\sum_{n=0}^{\infty} n^3 z^n$,

b) $\sum_{n=0}^{\infty} \frac{2^n}{n!} z^n$,

c) $\sum_{n=1}^{\infty} \frac{2^n}{n^2} z^n$,

d) $\sum_{n=0}^{\infty} \frac{n^3}{3^n} z^n$.

Exercise 2.19 Let (x_n) be a sequence of non-negative real numbers such that $\sum x_n$ diverges. Consider the following series:

$$\sum_{n=1}^{\infty} \frac{x_n}{1+x_n}, \quad \sum_{n=1}^{\infty} \frac{x_n}{1+nx_n}, \quad \sum_{n=1}^{\infty} \frac{x_n}{1+n^2 x_n}, \quad \sum_{n=1}^{\infty} \frac{x_n}{1+x_n^2}.$$

What can be said about their convergence?

Exercise 2.20 Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

Exercise 2.21 Associate to each sequence $a = (\alpha_n)$, in which α_n is 0 or 2, the real number

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

Prove that the set of all $x(a)$ is exactly the Cantor set described in Exercise 2.19. Since there is an uncountable number of such sequences (Theorem 1.5.6), this gives another proof of the fact that the Cantor set is uncountable.

Exercise 2.22 Let (n_k) be a strictly increasing sequence of positive integers such that

$$\lim_{k\to\infty} \frac{n_k}{n_1 n_2 n_3 \cdots n_{k-1}} = +\infty.$$

Prove that the series $\sum 1/n_k$ is convergent and its limit is an irrational number.

Exercise 2.23 — Summation by Parts. Let (x_n) and (y_n) be sequences in a normed vector space X. Let $s_n = x_1 + x_2 + \cdots + x_n$ and set $s_0 = 0$. Use the observation that $x_n = s_n - s_{n-1}$ to verify the formula

$$\sum_{i=n}^{m} x_i y_i = s_m y_{m+1} - s_{n-1} y_n + \sum_{i=n}^{m} s_i (y_i - y_{i+1}).$$

Exercise 2.24 — Dirichlet's Test. Let (x_n) and (y_n) be sequences of real numbers such that $\sum x_n$ has bounded partial sums, $y_n \to 0$ and (y_n) satisfies

$$y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0.$$

Prove that the series

$$\sum_{n=1}^{\infty} x_n y_n$$

converges. Use this result to give one more proof of the Leibniz test.

Exercise 2.25 — Abel's Test. In the view of the preceding exercise, show that if $\sum x_n$ converges, then we can drop the restriction $y_n \to 0$ while keeping the result.

Exercise 2.26 Let (x_{nm}) be a doubly indexed sequence of points in a normed space X. Suppose that:

- For each $m \in \mathbb{N}$, the series $\displaystyle\sum_{n=1}^{\infty} x_{nm}$ is convergent. Denote by y_m its limit.
- For each $n \in \mathbb{N}$, the series $\displaystyle\sum_{j=1}^{\infty} \sum_{i=n}^{\infty} x_{ij}$ is convergent. Denote by t_n its limit.

Show that, for each $n \in \mathbb{N}$, the series $\displaystyle\sum_{m=1}^{\infty} x_{nm}$ is convergent. Denote by z_n its limit.

Moreover, prove that $\displaystyle\sum_{m=1}^{\infty} y_m = \sum_{n=1}^{\infty} z_n$ if, and only if $t_n \to 0$.

Exercise 2.27 Let (x_{nm}) be a doubly indexed sequence of real numbers such that $x_{nm} \geq 0$ for all $n, m \in \mathbb{N}$. Prove that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} x_{nm} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{nm},$$

whenever one of the double sums converges. Moreover, if one diverges then so does the other.

CHAPTER 3

Functional Limits and Continuity

Since most of this book is about the things we can do to functions, its about time we generalize the notion of a limit to a function. This will be the basis of our future studies. As usual, we'll stick to metric spaces whenever possible to clarify the concepts and proofs.

3.1 Functional Limits

Unlike in the limit of a sequence, the definition of a limit becomes clearer when we step a little away from the topological aspect.

Definition 3.1.1 — Limit of a Function. Let M and N be metric spaces, $f : M \to N$ be a function and p be a limit point of M. We say that

$$\lim_{x \to p} f(x) = L$$

if for all $\varepsilon > 0$ there is a real number $\delta > 0$ such that

$$d(f(x), L) < \varepsilon \quad \text{whenever} \quad 0 < d(x, p) < \delta.$$

In this case we say that L is the limit of $f(x)$ as x approaches p or, more concisely, $f(x) \to L$ as $x \to p$.

In this definition and in the entirety of this book I'll be sloppy and use the same letter d to denote two distinct metric functions defined on M and on N whenever the context makes it clear. We should note that p need not be a point of M but only a point of M'. Similarly, L need not be a point of $f(M)$.

If f is a function from $E \subset M$ to N, then the x which satisfies $0 < d(x, p) < \delta$ has to be an element of E. Otherwise $f(x)$ would not be defined. The point p has also to be a limit point of E.

Since the statement $d(f(x),L) < \varepsilon$ is equivalent to $f(x) \in U$, where U is a neighborhood of L in N, and the statement $0 < d(x,p) < \delta$ is equivalent to $x \in V \setminus \{p\}$,[*] where V is a neighborhood of p in M, we can define the limit of a function "topologically" as follows.

Definition 3.1.2 — Limit of a Function. Let M and N be metric spaces, $f : M \to N$ be a function and p be a limit point of M. We say that

$$\lim_{x \to p} f(x) = L,$$

if for every neighborhood U of L there is a neighborhood V of p such that $f(x) \in U$ for all $x \in V \setminus \{p\}$.

As suggested by the dangerous bend sign in the previous page, if f is a function from $E \subset M$ to N, then we have that $V \cap E$ has to be non-empty and $f(x) \in U$ for all $x \in (V \cap E) \setminus \{p\}$.

It is remarkable how similar are the definitions of the limit of a sequence and that of a function. The definition of $(x_n) \to p$ conveys the idea that x_n can be made arbitrarily close to p if n is big enough. Analogously, $f(x) \to L$ conveys the idea that $f(x)$ can be made arbitrarily close to L if x is close enough to p. In fact, the following theorem shows that the two concepts are closely related.

Theorem 3.1.1 Let M, N, $E \subset M$ be metric spaces, $f : E \to N$ be a function and p be an limit point of E. Then,

$$\lim_{x \to p} f(x) = L$$

if, and only if,

$$\lim_{n \to \infty} f(x_n) = L$$

for every sequence (x_n) of points in $E \setminus \{p\}$ which converges to p.

Proof. *Let (x_n) be any sequence in $M \setminus \{p\}$ which converges to p. Since $f(x) \to L$, for all $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$d(f(x),L) < \varepsilon \quad \textit{whenever} \quad 0 < d(x,p) < \delta.$$

For this δ, there is a positive integer n_0 such that

$$d(x_n,p) < \delta \quad \textit{whenever} \quad n > n_0.$$

Hence $d(f(x_n),L) < \varepsilon$ for $n > n_0$ which implies $f(x_n) \to L$.

We'll now suppose that $f(x_n) \to L$ for every sequence (x_n) and that $f(x)$ does not converge to L. Since $f(x) \nrightarrow L$, there is a $\varepsilon > 0$ such that for all $\delta > 0$ there is a point $x \in M$ which satisfies

$$0 < d(x,p) < \delta \quad \textit{but} \quad d(f(x),L) \geq \varepsilon.$$

[*] It is usual to say that $V \setminus \{p\}$ is a *punctured neighborhood* of p.

Taking $\delta = 1/n$ *we obtain a sequence of points* (x_n) *in* $M \setminus \{p\}$ *such that*

$$0 < d(x_n, p) < 1/n \quad \textit{but} \quad d(f(x_n), L) \geq \varepsilon.$$

This sequence clearly converges to p but the sequence $(f(x_n))$ *does not converge to L, contradicting our original assumption.* □

This theorem makes an simple way to show that a limit does not exist. If we find sequences (x_n) and (y_n) in $E \setminus \{p\}$ such that $\lim x_n = \lim y_n = p$ and $\lim f(x_n) \neq \lim f(y_n)$ then we can conclude that $\lim f(x)$ does not exist.

Now, almost every important result about limits of functions follows from the preceding theorem and the fact that a point $p \in M$ is a limit point if, and only if there is a sequence (p_n) of points in $M \setminus \{p\}$ which converges to p (Theorem 3.1.2).

Corollary 3.1.2 — A function can converge to at most one point. Let M and N be metric spaces and p be a limit point of M. If $f(x) \to L_1$ as $x \to p$ and $f(x) \to L_2$ as $x \to p$, then $L_1 = L_2$.

Proof. *Follows from Theorems* 4.1.1*,* 3.1.2 *and* 3.1.1. □

Corollary 3.1.3 Suppose X_1, X_2 are normed vector spaces of finite dimension and M is a metric space. If $f = (f_1, f_2)$ is a function from M to $X_1 \times X_2$, then

$$\lim_{x \to p} f(x) = (L_1, L_2)$$

if, and only if $f_1(x) \to L_1$ and $f_2(x) \to L_2$.

Proof. *Follows from Theorems* 4.1.1*,* 3.1.2*,* 3.1.4 *and* 2.7.4. □

Corollary 3.1.4 Suppose X is a normed space, M is a metric space and $f, g : M \to X$ are functions. If $f(x) \to L_1$ and $g(x) \to L_2$ as $x \to p$, then

$$\lim_{x \to p} (f(x) + g(x)) = L_1 + L_2.$$

If the codomain of f is $\mathbb{R}$ or $\mathbb{C}$, then

$$\lim_{x \to p} f(x)g(x) = L_1 L_2.$$

Moreover, if $L_1 \neq 0$, then

$$\lim_{x \to p} \frac{1}{f(x)} = \frac{1}{L_1}.$$

Proof. *Follows from Theorems* 4.1.1*,* 3.1.2 *and* 3.1.6. □

Corollary 3.1.5 Let M be a metric space and $f : M \to \mathbb{R}$, $g : M \to \mathbb{R}$ be functions. If $f(x) \geq g(x)$ for all $x \in M$, then

$$\lim_{x \to p} f(x) \geq \lim_{x \to p} g(x).$$

Proof. *Follows from Theorems* 4.1.1, 3.1.2 *and Corollary* 3.1.8. □

Corollary 3.1.6 Let M be a metric space and $f : M \to \mathbb{R}$, $g : M \to \mathbb{R}$ be functions. If $0 \leq f(x) \leq g(x)$ for all $x \in M$, and if $g(x) \to 0$ as $x \to p$, then

$$\lim_{x \to p} f(x) = 0.$$

Proof. *Follows from Theorems* 4.1.1, 3.1.2 *and* 3.2.8. □

■ **Example 3.1** Consider the function $f : E \to \mathbb{R}$ such that $f(x) = x/|x|$. If $E = (0,1)$, then

$$\lim_{x \to 0} f(x) = 1.$$

And if $E = (-1,0)$, then

$$\lim_{x \to 0} f(x) = -1.$$

This shows that it is important to consider the domain of a function when evaluating its limit. ■

■ **Example 3.2** Consider the function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

We'll show that $f(x)$ has no limit when $x \to 0$. If $f(x)$ converged to L, then we would have that $f(x_n) \to L$ for every sequence of non-zero real numbers. However, if $x_n = 1/n$:

$$\lim_{x \to 0} f(x_n) = 1.$$

Hence $L = 1$. But taking $x_n = \sqrt{2}/n$ we see that

$$\lim_{x \to 0} f(x_n) = 0.$$

And then $L = 0$. This contradiction implies the result. ■

3.2 Limits on the Extended Real Line

As it is, the extended real line is not a metric space as there is no obvious way to assign a real number to the expression $d(x, \infty)$, where x is a real number. However, it is almost tempting to write things like

$$\lim_{x\to 0}\frac{1}{x^2}=\infty \quad \text{and} \quad \lim_{x\to\infty}\frac{1}{x}=0.$$

While we shall not define a metric on $\overline{\mathbb{R}}$, we can quite naturally define the neighborhoods of $\pm\infty$.

Definition 3.2.1 If c is a real number, the sets $(c,+\infty)$ and $(-\infty,c)$ are defined to be neighborhoods of $+\infty$ and $-\infty$, respectively.

We now define limits in the extended real line using the topological definition given before.

Definition 3.2.2 — Limit of a Function. Let E be a subset of the real line and $f: E\to\mathbb{R}$ be a function. We say that

$$\lim_{x\to p} f(x)=L,$$

where $p,L\in\overline{\mathbb{R}}$, if for every neighborhood U of L there is a neighborhood V of p such that $V\cap E$ is not empty, and such that $f(x)\in U$ for all $x\in(V\cap E)\setminus\{p\}$.

We can now prove that the results in the beginning of this section are valid. For all $\varepsilon>0$, we can take $c=1/\varepsilon$ so that $1/x\in(-\varepsilon,\varepsilon)$ whenever $x\in(c,+\infty)$. This implies

$$\lim_{x\to\infty}\frac{1}{x}=0.$$

The other limit is analogous. We now define the limit superior and inferior for functions on the real line.

Definition 3.2.3 — Limit Superior/Inferior of Functions. Let E be a subset of $\mathbb{R}$, p be a limit point of E and $f: E\to\mathbb{R}$ be a function. Consider the set

$$F=\{y\in\overline{\mathbb{R}}\mid y=\lim_{n\to\infty} f(x_n),\ \{x_n\}\in E\setminus\{p\}\}.$$

We define the limit superior and the limit inferior of f as

$$\limsup_{x\to p} f(x)=\sup F, \quad \liminf_{x\to p} f(x)=\inf F.$$

Since Theorem 4.1.1 holds, the limit superior and inferior of functions satisfies most of the properties of the limit superior and inferior of sequences. We'll leave the reader with the task of checking these properties.

3.3 Asymptotic Comparison

We're often interested in the behaviour of a function in a normed space when it is restricted to some neighborhood, as it is the case with the limits. Based on this fact, the German mathematician Edmund Landau invented the following notation.

Definition 3.3.1 — Little-o Notation. Let M be a metric space, X be a normed space and p be a limit point of M. Suppose $g : M \to X$ is a function. We then define $o_p(g(x))$ to be the set of all functions $f : M \to X$ with the following property.
For every $\varepsilon > 0$, there is a neighborhood V of p such that

$$\|f(x)\| < \varepsilon \|g(x)\|$$

whenever $x \in V \setminus \{p\}$.

Whenever the context makes it clear, we'll omit the index p.

It is usual in literature to abuse notation[†] and denote $f(x) \in o(g(x))$ as $f(x) = o(g(x))$. In this case we define the expression $f(x) = h(x) + o(g(x))$ as $f(x) - h(x) = o(g(x))$.

If $g(x)$ is not 0 in a punctured neighborhood of p, then $f(x) = o(g(x))$ is equivalent to

$$\lim_{x \to p} \frac{f(x)}{g(x)} = 0.$$

Theorem 3.3.1 — Properties of Little-o. Let $f, f_1, f_2, g, g_1, g_2 : M \to X$ be functions. The following properties hold.

i) If $f_1(x) = o(g(x))$ and $f_2(x) = o(g(x))$, then $f_1(x) + f_2(x) = o(g(x))$.

ii) If $f(x) = o(g(x))$, then $cf(x) = o(g(x))$ for every non-zero real number c.

iii) If $X = \mathbb{R}$ or $X = \mathbb{C}$, $f_1(x) = o(g_1(x))$ and $f_2(x) = o(g_2(x))$, then $f_1(x)f_2(x) = o(g_1(x)g_2(x))$.

iv) If $f(x) = o(g_1(x))$ and $g_1(x) = o(g_2(x))$, then $f(x) = o(g_2(x))$.

Proof. *In every item, $\varepsilon > 0$ will be a arbitrary positive real number.*

i) *There is a neighborhood V of p such that $\|f_1(x)\| < \varepsilon \|g(x)\|/2$ and a neighborhood U of p such that $\|f_2(x)\| < \varepsilon \|g(x)\|/2$ whenever $x \in V \setminus \{p\}$ and $x \in U \setminus \{p\}$ respectively. Hence,*

$$\|f_1(x) + f_2(x)\| \le \|f_1(x)\| + \|f_2(x)\| < \varepsilon \|g(x)\|$$

whenever $x \in (V \cap U) \setminus \{p\}$.

ii) *If c is positive, there is a neighborhood V of p such that $\|f(x)\| < \varepsilon \|g(x)\|/c$ whenever $x \in V \setminus \{p\}$. Hence,*

$$\|cf(x)\| = c\|f(x)\| < \varepsilon \|g(x)\|$$

whenever $x \in V \setminus \{p\}$. If c is negative just take a neighborhood U of p such that $\|f(x)\| < \varepsilon \|g(x)\|/(-c)$ whenever $x \in U \setminus \{p\}$.

[†] As Donald Knuth pointed out: "mathematicians customarily use the $=$ sign as they use the word 'is' in English: Aristotle is a man, but a man isn't necessarily Aristotle."

iii) *There is a neighborhood V of p such that $|f_1(x)| < \sqrt{\varepsilon}|g_1(x)|$ and a neighborhood U of p such that $|f_2(x)| < \sqrt{\varepsilon}|g_2(x)|$ whenever $x \in V \setminus \{p\}$ and $x \in U \setminus \{p\}$ respectively. Hence,*

$$|f_1(x)f_2(x)| = |f_1(x)||f_2(x)| < \varepsilon|g_1(x)||g_2(x)| = \varepsilon|g_1(x)g_2(x)|$$

whenever $x \in (V \cap U) \setminus \{p\}$.

iv) *There is a neighborhood V of p such that $\|f(x)\| < \sqrt{\varepsilon}\,\|g_1(x)\|$ and a neighborhood U of p such that $\|g_1(x)\| < \sqrt{\varepsilon}\,\|g_2(x)\|$ whenever $x \in V \setminus \{p\}$ and $x \in U \setminus \{p\}$ respectively. Hence,*

$$\|f(x)\| < \sqrt{\varepsilon}\,\|g_1(x)\| < \sqrt{\varepsilon}(\sqrt{\varepsilon}\,\|g_2(x)\|) = \varepsilon\,\|g_2(x)\|$$

whenever $x \in (V \cap U) \setminus \{p\}$.

This completes the proof. □

Definition 3.3.2 Suppose $f, g : M \to X$ are functions. We say that $f(x) \sim_p g(x)$ if $f(x) - g(x) = o_p(g(x))$.

As you should check, this is an equivalence relation. If $g(x)$ is not 0 in a punctured neighborhood of p, then $f(x) \sim_p g(x)$ is equivalent to

$$\lim_{x \to p} \frac{f(x)}{g(x)} = 1.$$

As before, we'll usually omit the subscript when the context makes it clear.

Example 3.3 — Fundamental Trigonometric Limit. I affirm that $\sin(x) \sim x$ (as $x \to 0$) holds. In fact, since

$$\frac{\sin(x) - x}{x} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} = x \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n-1},$$

we have that

$$\frac{|\sin(x) - x|}{|x|} \leq |x| \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} |x|^{2n-1}.$$

As $x \to 0$, $|x| \to 0$ and $\sum (-1)^n |x|^{2n-1}/(2n+1)! \to 0$.[‡] Hence, for all $\varepsilon > 0$,

$$|\sin(x) - x| < \varepsilon |x|$$

if x is close enough to 0. A quick corollary of this result is the following limit:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

■

[‡] Are you sure you can prove this rigorously?

3.4 Continuity

As we saw in chapter 2, if M and N are metric spaces, a function $f: M \to N$ is said to be continuous at $p \in M$ if for all $\varepsilon > 0$ there is a real number $\delta > 0$ such that

$$d(f(x), f(p)) < \varepsilon \quad \text{whenever} \quad d(x,p) < \delta.$$

Obviously this definition is very similar to Definition 4.1.1. However it has a few notable differences. Firstly, for a function to be continuous at p, p has to be an element of M, while in the limit it can just be a limit point. If p is an isolated point of M, f is surely continuous at p since there is a neighborhood N of p such that $N = \{p\}$ and hence $d(f(x), f(p)) = 0$ for all $x \in N$.

If p is a limit point of M, then f is continuous at p if, and only if

$$\lim_{x \to p} f(x) = f(p).$$

We can also characterize continuity with sequences.

Theorem 3.4.1 Let M, N be metric spaces, $f: M \to N$ be a function and p be a point of M. Then f is continuous at p if and only if

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(p)$$

for every sequence (x_n) of points in M which converges to p.

Proof. *The only change from this proof to that of Theorem* 4.1.1 *is the lack of requirement that $x_n \neq p$ for all $n \in \mathbb{N}$.* □

Analogously to Corollary 4.1.4, the following properties hold.

Theorem 3.4.2 Suppose X is a normed space, M is a metric space and $f, g: M \to X$ are functions.
If f, g are continuous at $p \in M$, then so is $f(x) + g(x)$ and $\alpha f(x)$ for all complex α.
Moreover, if the codomain of f is $\mathbb{R}$ or $\mathbb{C}$ and $f(p) \neq 0$, then $1/f(x)$ and $f(x)g(x)$ are also continuous at p.

Proof. *If p is an isolated point of M, then the result is trivial. Otherwise the result follows from Corollary* 4.1.4. □

■ **Example 3.4** We'll now consider rational functions of the form $f: \mathbb{C} \to \mathbb{C}$ such that

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}$$

for some suitable complex constants $a_n, \ldots, a_0, b_m, \cdots, b_0$ and positive integers n, m.

Of course the identity function $f(x) = x$ is continuous everywhere (just take $\delta = \varepsilon$). Theorem 4.4.2 implies that every polynomial function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

is continuous everywhere.

If $b_m p^m + b_{m-1} p^{m-1} + \ldots + b_1 p + b_0 \neq 0$ then

$$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0}$$

is also continuous at p. ■

From this example it follows that $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^n - a$, where a is a positive real number and n is a positive integer, is continuous. We construct a new function $\tilde{f} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ such that

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R} \\ \lim\limits_{t \to x} f(t) & \text{if } x = \pm\infty \end{cases}.$$

Our new function $\tilde{f}$ is continuous at $\pm\infty$ by the definition of limit, hence it is continuous everywhere. Since $f(0) = -a$ and $f(+\infty) = +\infty$, the Intermediate Value Theorem implies the existence of a positive real number y such that $y^n = a$. If there were two such y, say y_1 and y_2 with $y_1 > y_2$, we would have that $a = y_1^n > y_2^n = a$, which is absurd. We say that this y is the n-th root of a and denote it by $\sqrt[n]{a}$. For $a = 0$ we define $\sqrt[n]{0} = 0$ for every $n \in \mathbb{N}$.

We can now define rational powers of real numbers as follows. If $b = p/q$ is a rational number ($q > 0$), we define a^b as $\sqrt[q]{a^p}$. You should check that if $m/n = p/q$, with $n, q > 0$, then $= \sqrt[n]{a^m} = \sqrt[q]{a^p}$. This implies that a^b is uniquely determined.

For real exponents b, we define a^b as the supremum of

$$\{a^t \in \mathbb{R} \mid t \leq b,\ t \in \mathbb{Q}\}.$$

The following exercise shows that this is well defined.

Exercise 3.1 If $b = p/q$ is a rational number and a is a non-negative real number, show that

$$\sup\{a^t \in \mathbb{R} \mid t \leq b,\ t \in \mathbb{Q}\} = \sqrt[q]{a^p}.$$

Moreover, show that if r, s are two real numbers, then $a^{r+s} = a^r a^s$.

A continuous function $f : E \subset \mathbb{R} \to \mathbb{R}$ is sometimes described, intuitively, as one whose graph can be drawn without lifting your pencil from the paper. The following example shows that this is not quite true.

■ **Example 3.5** Consider the function $f : \mathbb{N} \to \mathbb{R}$ such that $f(n) = n$.
If p is a fixed positive integer, taking $\delta = 1/2$ we see that

$$|x - p| < \delta \implies |f(x) - f(p)| = 0 < \varepsilon$$

for any $\varepsilon > 0$. Hence f is continuous. This also follows readily from the fact that every point of $\mathbb{N}$ is isolated. ■

3.5 Uniform Continuity

Lets prove explicitly (that is, not using Theorem 4.4.2) that two simple functions are continuous and contrast their proofs.

■ **Example 3.6** Consider the identity function on $\mathbb{R}$. That is, $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)=x$. Suppose p is any point of $\mathbb{R}$ and $\varepsilon>0$ is an arbitrary real number. Then, since $|f(x)-f(p)|=|x-p|$, we can take $\delta=\varepsilon$ and

$$|f(x)-f(p)|<\varepsilon \quad \text{whenever} \quad |x-p|<\delta$$

holds. It follows that f is continuous. ■

■ **Example 3.7** Consider the function $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)=x^2$. Suppose p is any point of $\mathbb{R}$ and $\varepsilon>0$ is an arbitrary real number. Note that $|f(x)-f(p)|=|x-p||x+p|$. To prove that f is continuous, we would like to bound the term $|x+p|$. We can use the triangular inequality in the following way:

$$|x+p|=|x-p+2p|\leq|x-p|+2|p|.$$

Hence, if $\delta<\min\{\sqrt{\varepsilon/2},\varepsilon/4|p|\}$, it follows that

$$|f(x)-f(p)|\leq|x-p|^2+2|p||x-p|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

whenever $|x-p|<\delta$. ■

Note that in Example 4.6, we choose a δ that works for whatever p we picked before. That was not possible in Example 4.7. This property possessed by the identity function is called *uniform continuity*.

Definition 3.5.1 — Uniform Continuity. Let M,N be metric spaces. A function $f:M\to N$ is said to be *uniformly continuous* if for every $\varepsilon>0$, there is a $\delta>0$ such that for all $x,y\in M$:

$$d(f(x),f(y))<\varepsilon \quad \text{whenever} \quad d(x,y)<\delta.$$

Compare the definitions of continuity and uniform continuity. A function $f:M\to N$ is continuous if:

$$(\forall y\in M)(\forall\varepsilon>0)(\exists\delta>0)(\forall x\in M)(d(x,y)<\delta \implies d(f(x),f(y))<\varepsilon).$$

And it is uniformly continuous if:

$$(\forall\varepsilon>0)(\exists\delta>0)(\forall x\in M)(\forall y\in M)(d(x,y)<\delta \implies d(f(x),f(y))<\varepsilon).$$

The only difference between these definitions is the placement of the quantifier $(\forall y\in M)$. This means that for a function to be continuous, the δ chosen *can* depend on y. However, for it to be uniformly continuous we have to find a δ that works for whichever y is available in the domain.

It should be clear that every uniformly continuous function is also continuous but not the other way around. The next theorem shows when the converse is valid.

Theorem 3.5.1 Let M,N be metric spaces such that M is compact. If $f: M \to N$ is continuous, then it is uniformly continuous.

Proof. *If f is not uniformly continuous, there exists a $\varepsilon > 0$ such that for any $\delta > 0$ there exists two points $x,y \in M$ such that $d(x,y) < \delta$ but $d(f(x),f(y)) \geq \varepsilon$. Taking $\delta = 1/n$ we get two sequences $(x_n),(y_n)$ such that $d(x_n,y_n) < 1/n$ but $d(f(x_n),f(y_n)) \geq \varepsilon$ for each $n \in \mathbb{N}$.*
Since M is compact, (x_n) has a convergent subsequence (x_{n_k}). Similarly, (y_{n_k}) has a convergent subsequence $(y_{n_{k_j}})$. As (x_{n_k}) converges, so does $(x_{n_{k_j}})$. Using the fact that $d(x_{n_{k_j}},y_{n_{k_j}}) < 1/n_{k_j}$ we see that $(x_{n_{k_j}})$ and $(y_{n_{k_j}})$ converge to the same value, namely $a \in M$. But f is continuous, so it follows that

$$\lim_{j\to\infty} f(x_{n_{k_j}}) = f(a) = \lim_{j\to\infty} f(y_{n_{k_j}}).$$

Hence, there is a positive integer j_0 such that for all $j > j_0$:

$$d(f(x_{n_{k_j}}),f(a)) < \varepsilon/2 \quad \text{and} \quad d(f(x_{n_{k_j}}),f(a)) < \varepsilon/2.$$

So,

$$d(f(x_{n_{k_j}}),f(y_{n_{k_j}})) \leq d(f(x_{n_{k_j}}),f(a)) + d(f(y_{n_{k_j}}),f(a)) < \varepsilon.$$

This absurd establishes the result. □

■ **Example 3.8** In exercise 4.15 you'll show that if f is uniformly continuous and $\lim(x_n - y_n) = 0$ then

$$\lim_{n\to\infty} f(x_n) - f(y_n) = 0.$$

Take $f(x) = x^2$, $x_n = n + 1/n$ and $y_n = n$. Clearly $\lim(x_n - y_n) = 0$ but

$$\lim_{n\to\infty} f(x_n) - f(y_n) = \lim_{n\to\infty} n^2 + 2 + \frac{1}{n^2} - n^2 = 2.$$

Hence f is not uniformly continuous in any domain that contains $\{x_n\}$ and $\{y_n\}$. Since f is continuous, this implies $\mathbb{R}$ is not compact.[§] ■

■ **Example 3.9** While $f(x) = x^2$ is not uniformly continuous in $\mathbb{R}$, it is uniformly continuous in any bounded set. Suppose $E \subset \mathbb{R}$ is bounded. That is, there are constants a,b such that $a < x < b$ for all $x \in E$. We conclude that $2a < x + y < 2b$ and then $|x+y| < \max|2a|,|2b|$ for all $x,y \in E$. Hence,

$$|f(x) - f(y)| = |x+y||x-y| < (\max\{|2a|,|2b|\})|x-y| < \varepsilon$$

whenever $|x-y| < \varepsilon/(\max\{|2a|,|2b|\})$. Notice that $\delta = \varepsilon/(\max\{|2a|,|2b|\})$ works for whichever points x,y in E we happen to choose. Hence f is uniformly continuous in E. ■

[§]As if we didn't already knew that...

3.6 Exercises

Exercise 3.2 Show that

$$\lim_{x\to\infty} \sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}=\frac{1}{2}.$$

Exercise 3.3 Let $f:(0,\infty)\to(0,\infty)$ be an increasing function with

$$\lim_{t\to\infty}\frac{f(2t)}{f(t)}=1.$$

Prove that

$$\lim_{t\to\infty}\frac{f(mt)}{f(t)}=1$$

for any $m>0$.

Exercise 3.4 Suppose $f:\mathbb{R}\to\mathbb{R}$ function which satisfies

$$\lim_{h\to 0}[f(x+h)-f(x-h)]=0$$

for every $x\in\mathbb{R}$. Does this imply that f is continuous?

Exercise 3.5 — Thomae's Function. Consider the function $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x)=\begin{cases}1 & \text{if } x=0\\ 1/q & \text{if } x=p/q\in\mathbb{Q}\setminus\{0\} \text{ with } q>0 \text{ and } \gcd(p,q)=1\\ 0 & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}.$$

Prove that f is continuous at all $x\in\mathbb{R}\setminus\mathbb{Q}$ and discontinuous at all $x\in\mathbb{Q}$.

Exercise 3.6 Prove that there is not a function $g:\mathbb{R}\to\mathbb{R}$ that is continuous at all $x\in\mathbb{Q}$ and discontinuous at all $x\in\mathbb{R}\setminus\mathbb{Q}$.
Hint: for each $n\in\mathbb{N}$, consider the set

$$A_n=\{x\in\mathbb{R}\,|\,\exists\delta>0 \text{ such that } |x-a|<\delta \text{ and } |x-b|<\delta \implies |g(a)-g(b)|<1/n\}.$$

Prove that A_n is open for each n and that g is continuous at x if, and only if $x\in\bigcap_{n=1}^{\infty}A_n$. Suppose that $\mathbb{Q}$ can be written as a countable intersection of open sets and show that this contradicts Baire's theorem.

Exercise 3.7 — Norms are Continuous. If X is a normed space, show that the function $f : X \to \mathbb{R}$ such that $f(x) = \|x\|$ is continuous.

Exercise 3.8 Let M,N be metric spaces such that M is compact. If $f : M \to N$ is a continuous bijection, show that f^{-1} is also continuous.

Exercise 3.9 Let $f,g : [0,1] \to [0,\infty)$ be continuous functions satisfying

$$\sup_{0 \le x \le 1} f(x) = \sup_{0 \le x \le 1} g(x).$$

Prove that there exists $t \in [0,1]$ with $f(t)^2 + 3f(t) = g(t)^2 + 3g(t)$.

Exercise 3.10 Prove that a continuous function from $\mathbb{R}$ to $\mathbb{R}$ which maps open sets to open sets must be monotonic.

Exercise 3.11 Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that $|f(x) - f(y)| \ge |x - y|$ for all x and y. Prove that the range of f is all of $\mathbb{R}$.

Exercise 3.12 — Croft's Lemma. Let $f : (0,\infty) \to \mathbb{R}$ be a continuous function with the property that for any $x > 0$, $\lim_{n\to\infty} f(nx) = 0$. Prove that $\lim_{x\to\infty} f(x) = 0$.

Exercise 3.13 — Lipschitz Functions. Let E be a subset of $\mathbb{R}$. A function $f : E \to \mathbb{R}$ is said to be *Lipschitz* if there exists a real number $M > 0$ such that

$$|f(x) - f(x)| \le M|x - y|$$

for all $x,y \in E$. Prove that if a function is Lipschitz, then it is uniformly continuous. Show that the converse does not hold.

Exercise 3.14 Prove that a function $f : (a,b) \to \mathbb{R}$ is uniformly continuous if, and only if there exists a continuous function $\tilde{f} : [a,b] \to \mathbb{R}$ such that $\tilde{f}(x) = f(x)$ for all $x \in (a,b)$.

Exercise 3.15 Recall the definition of equivalent sequences given in Chapter 3 (Definition 3.4.2). Let M,N be metric spaces and $(x_n),(y_n)$ be two arbitrary equivalent sequences in M. Prove that a function $f : M \to N$ is uniformly continuous if and only if $(f(x_n))$ is equivalent to $(f(y_n))$.

Exercise 3.16 Let M,N be metric spaces. If $f : M \to N$ is uniformly continuous and (x_n) is a Cauchy sequence in M, show that $(f(x_n))$ is a Cauchy sequence in N.

Exercise 3.17 Let M,N be metric spaces and p be a limit point of M. If $f : M \to N$ is uniformly continuous, show that the limit $\lim_{x\to p} f(x)$ exists.

Exercise 3.18 Let A,B,C be metric spaces. If $f : A \to B$ and $g : B \to C$ are uniformly continuous, show that $g \circ f : A \to C$ is uniformly continuous.

Exercise 3.19 Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function such that

$$\lim_{x\to\infty} f(x)$$

exists and is finite. Prove that f is uniformly continuous.

Exercise 3.20 Let $M,N,E \subset M$ be metric spaces such that E is bounded. If $f : M \to N$ is uniformly continuous, show that $f(E)$ is bounded.

Exercise 3.21 Show that the requirement in the definition of uniform continuity can be rephrased as follows, in terms of diameters of sets: To every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\operatorname{diam} f(E) < \varepsilon$ for all $E \subset X$ with $\operatorname{diam} E < \delta$.

CHAPTER 4

The Derivative

The motivation for the derivative comes from geometrical investigations. That perspective, together with countless applications, are the bulk of any calculus book. In analysis our outlook will be focused on the properties and generalizations of such object.

Unlike the previous chapters, we present firstly the usual derivative of real functions and then consider generalizations. This is due the fact that there is not a single possible generalization and it is not obviously clear how one should generalize the derivative to broader spaces than the real line.

4.1 The Derivative of a Real Function

Definition 4.1.1 — Derivative of a Real Function. Let $f : E \subset \mathbb{R} \to \mathbb{R}$ be a function and p be a point of E. It is said that f is *differentiable* at p if the limit

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p}$$

exists. In this case its value is said to be the *derivative* of f at p and it is denoted $f'(p)$ or $\frac{\mathrm{d}f}{\mathrm{d}x}(p)$. We then associate with f a function f', defined in the points where f is differentiable, such that $f'(x)$ is the derivative of f at x. This function is, with a slightly abuse of language, also called the *derivative* of f.

Leibniz's notation $\frac{\mathrm{d}f}{\mathrm{d}x}$ makes the derivative appear to be the quotient of two quantities. Although this interpretation is possible (at least in the case of a real function), it is not simple at all. The reader is safer thinking about $\frac{\mathrm{d}f}{\mathrm{d}x}$ as a single expression.

It may be possible that the derivative of f is also differentiable. In this case its derivative is the *second derivative* of f and it is denoted as f'', $f^{(2)}$ or even $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$. In general, we denote the n-th derivative of f as $f^{(n)}$ or $\frac{\mathrm{d}^n f}{\mathrm{d}x^n}$. If f has derivatives of all orders, then it is said to be *smooth*.

Theorem 4.1.1 If $f : E \subset \mathbb{R} \to \mathbb{R}$ is differentiable at p, then it is continuous at p.

Proof. *Since* $x - p \to 0$ *and* $(f(x) - f(p))/(x-p) \to f'(p)$ *as* $x \to p$,

$$\lim_{x\to p} f(x) - f(p) = \lim_{x\to p} \frac{f(x)-f(p)}{x-p} \cdot (x-p) = f'(p) \cdot 0 = 0.$$

Hence, f *is continuous at* p. □

The converse does not hold, as it can be seen in the following example.

■ **Example 4.1** Consider $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = |x|$. If $p > 0$, then

$$f'(p) = \lim_{x\to p} \frac{f(x)-f(p)}{x-p} = 1.$$

Similarly, if $p < 0$ we have that $f'(p) = -1$. However, f is not differentiable at 0 as

$$\lim_{x\to 0} \frac{|x|}{x}$$

does not exist. Hence f is continuous but not differentiable at 0. ■

As we'll see in the seventh chapter, there are continuous functions on the real line that are nowhere differentiable.*

Theorem 4.1.2 Suppose f and g are differentiable at p. Then so are $f+g$, fg and f/g (provided $g(p) \neq 0$ in the last case). Moreover,

$$(f+g)'(p) = f'(p) + g'(p), \quad (fg)'(p) = f'(p)g(p) + f(p)g'(p)$$

and

$$(f/g)'(p) = \frac{f'(p)g(p) - f(p)g'(p)}{g(p)^2},$$

if $g(p) \neq 0$.

Proof. *The proof of this theorem is based on two algebraic identities. Letting* $x \to p$ *in*

$$\frac{f(x)g(x) - f(p)g(p)}{x-p} = f(x) \cdot \frac{g(x)-g(p)}{x-p} + g(p) \cdot \frac{f(x)-f(p)}{x-p}$$

results in $(fg)'(p) = f'(p)g(p) + f(p)g'(p)$. *Similarly,*

$$\frac{f(x)/g(x) - f(p)/g(p)}{x-p} = \frac{1}{g(x)g(p)} \left(g(p) \cdot \frac{f(x)-f(p)}{x-p} - f(p) \cdot \frac{g(x)-g(p)}{x-p} \right)$$

results in the corresponding identity for the derivative of the quotient. The derivative of the sum is trivial. □

*Actually, in some sense, most continuous functions are nowhere differentiable.

■ **Example 4.2** Consider $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)=1/x$. Since

$$\frac{f(x)-f(p)}{x-p}=-\frac{1}{xp}$$

and f is continuous at every $p\neq 0$ it follows that $f'(p)=-1/p^2$. From this example and the previous theorem it follows that if $f(x)=x^n$ then $f'(p)=np^{n-1}$. (If $n\leq 0$ it is needed that $p\neq 0$.) ■

Arguably, the following theorem is the most important result about derivatives of real functions. It is usually denoted by the name of "chain rule" in calculus.

Theorem 4.1.3 — Chain Rule. Suppose $f,g:\mathbb{R}\to\mathbb{R}$ are functions such that f is differentiable at $p\in\mathbb{R}$ and g is differentiable at $f(p)$. Then $g\circ f$ is differentiable at p and its derivative is

$$(g\circ f)'(p)=g'(f(p))f'(p).$$

Proof. *The naive approach would be to write*

$$\frac{g(f(x))-g(f(p))}{x-p}=\frac{g(f(x))-g(f(p))}{f(x)-f(p)}\cdot\frac{f(x)-f(p)}{x-p}$$

and let $x\to p$ in both sides. This does not work since it may happen that $f(x)=f(p)$ for every x in a neighborhood of p. To avoid this detail we define a function $h:\mathbb{R}\to\mathbb{R}$ such that

$$h(y)=\begin{cases}\dfrac{g(y)-g(f(p))}{y-f(p)} & \text{if } y\neq f(p)\\ g'(f(p)) & \text{if } y=f(p)\end{cases}.$$

Now, if $f(x)\neq f(p)$ we have that

$$\frac{g(f(x))-g(f(p))}{x-p}=h(f(x))\cdot\frac{f(x)-f(p)}{x-p}.$$

If $f(x)=f(p)$ while $x\neq p$, then both sides of the previous equation equal zero and hence this equation holds for all $x\in\mathbb{R}\setminus\{p\}$.
Our function h is continuous at $f(p)$ since g is differentiable at $f(p)$. Hence, letting $x\to p$ we get the desired result. □

4.2 Mean Value Theorems

4.3 L'Hôpital's Rule

4.4 Fréchet Derivative

4.5 Partial Derivatives

4.6 The Implicit / Inverse Function Theorem

4.7 Exercises

APPENDIX A

Logic and Set Theory

As David Hilbert predicted in the beginning of the 90's, "no one will drive us from the paradise which Cantor created for us". Set theory is the basis of almost all areas in mathematics and analysis is no exception.

We will adopt the naive point of view to set theory, which assumes that the concept of a set of objects is intuitively clear. The only remark that will be made is that we will only allow sets to be defined by a property of its elements if it is a subset of a known set. Of course this is not formal, but I will leave the task of studying sets for a set theory book.

The only reason for that remark is to avoid contradictions such as Russell's paradox*. An formal axiomatization is well beyond our scope. Luckily, naive set theory works wonderfully for all the sets we need to use in analysis.

A.1 Equivalence Relations

Definition A.1.1 An *relation* between two sets A and B is a subset R of the Cartesian product $A \times B$.

Since most of the relations we encounter on analysis are denoted by $x = y$, $x > y$, $x \sim y$ or other similar notations, we denote a general relation by xRy instead of $(x,y) \in R$.

A function $f : A \to B$ is a pretty particular case of relation: specifically, one which every $x \in A$ appears exactly once as the first element of $(x, f(x)) \in A \times B$.

Definition A.1.2 An *equivalence relation* in a set A is a relation $\sim$ in $A \times A$ (There is no special reason to denote a relation by a letter. The symbol $\sim$ works just fine.)

*Consider the set R of all the sets X that are elements of itself. That is, $R = \{X \mid X \notin X\}$. What would happen if $R \in R$? And if $R \notin R$?

that satisfies the following properties:

- (Reflexitivity) For all $x \in A$, $x \sim x$.
- (Symmetry) If $x \sim y$, then $y \sim x$.
- (Transitivity) If $x \sim y$ and $y \sim z$, then $x \sim z$.

If A and B are sets, it will be implicit that $A \sim B$ if, and only if, there is a bijection between the elements of A and B. You should verify that this is, in fact, an equivalence relation. This allows us to define the cardinality of a set. If A is a non-empty set, we say that $|A| = n$ if $A \sim \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$. By definition, $|\varnothing| = 0$.

We can then extend the concept of cardinality to infinite sets. In this case, we say that $|A| \leq |B|$ if there is an injection from A to B. The equality occurs only when $A \sim B$. In 1878 Georg Cantor conjectured that there was not a set S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$. This became known by the name of "Continuum Hypothesis". In 1940 Kurt Gödel showed it was impossible to disprove the continuum hypothesis using ZFC. Then Paul Cohen showed in 1963 that it cannot be proved either (this means that we should take the continuum hypothesis or it's negation as an axiom). That was all assuming that ZFC is consistent, which we still don't know for sure. This is still an active area of research.

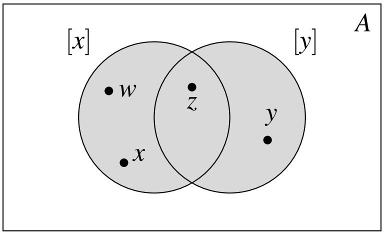
Definition A.1.3 Let $\sim$ be an equivalence relation of a set A and x be an element of A. An *equivalence class* is a subset $[x] \subset A$ defined by

$$[x] = \{p \in A \mid p \sim x\}.$$

Obviously, equivalence classes are non-empty, since equivalence relations are reflexive.

Theorem A.1.1 Two equivalence classes $[x]$ and $[y]$ are either disjoint or equal.

Proof. *Let $[x]$ be the equivalence class determined by x, and let $[y]$ be the equivalence class determined by y. Suppose that $[x] \cap [y]$ is not empty. So, let $z \in [x] \cap [y]$.*



By definition, we have $z \sim x$ and $z \sim y$. Symmetry allows us to conclude that $x \sim z$ and $z \sim y$. From transitivity, it follows that $x \sim y$. If w is any point of $[x]$, we have $w \sim x$ by definition, and $w \sim y$ by transitivity. We conclude that $[x] \subset [y]$.

The symmetry of the situation allows us to conclude that $[y] \subset [x]$ as well, so that $[x] = [y]$. □

A *partition* $\mathscr{P}$ on a set A is a collection of subsets of A with the following property: every element of A belongs to exactly one of the sets in $\mathscr{P}$. Notice that the set of all equivalence classes for a given equivalence relation satisfies exactly this! We have then that every partition *induces* an equivalence relation in A and every equivalence relation induces a partition. We denote the set of all equivalence classes of a equivalence relation $\sim$ defined of a set A by $A/\sim$.

A.2 Countability

Definition A.2.1 For any set A, we say:

- A is *finite* if $|A| = n$ for some n in $\{0, 1, 2, \ldots\}$.
- A is *infinite* if A is not finite.
- A is *countable* if $A \sim \mathbb{N}$.
- A is *uncountable* if A is neither finite nor countable.
- A is *at most countable* if A is finite or countable.

It should be clear now that, if C is countable, U is uncountable and M is at most countable, then $|M| \leq |C| \leq |U|$ and $|C| < |U|$. Countable sets are sometimes called *enumerable*, or *denumerable*. While finite sets cannot be equivalent to one of its proper subsets, this is possible for infinite sets, like the example shown below:

Example A.1 Let $\mathbb{Z}$ be the set of all integers. Consider the following mapping:

$$f : \mathbb{N} \to \mathbb{Z}, \quad f(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ -\dfrac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

f is a bijective map from $\mathbb{N}$ to $\mathbb{Z}$. So, $\mathbb{Z}$ is countable. ■

This suggests that we might characterize infinite sets as the ones which are equivalent to some proper subset of itself. It is true and you should try to prove it for countable sets.

The proof that every infinite (not necessarily countable) set is equivalent to some proper subset of itself is not as easy and will be omitted.

Theorem A.2.1 Every infinite subset of a countable set A is countable.

Proof. *Suppose $E \subset A$, and E is infinite. Since A is countable, we can write it as $\{a_1, a_2, \ldots\}$, where all the a_n are distinct. Construct then a sequence (n_k) as follows:*

Let n_1 be the smallest positive integer such that $a_{n_1} \in E$. Having chosen $n_1, \ldots, n_{k-1}$, let n_k be the smallest integer greater than n_{k-1} such that $a_{n_k} \in E$.

Putting $f(k) = a_{n_k}$, we have a bijection between E and $\mathbb{N}$. □

This theorem shows that, roughly speaking, countable sets represent the "smallest" infinity. That is, no uncountable set can be a subset of a countable set.

The following lemma will be very useful to prove some important theorems.

Lemma A.2.2 Let (E_n), $n = 1, 2, \dots$, be a sequence[a] of countable sets and put

$$S = \bigcup_{n=1}^{\infty} E_n.$$

Then S is countable.

[a] Formally, a sequence is a function from $\mathbb{N}$ to any set. It is usual to denote the element $f(i)$ as x_i and to denote the whole function as (x_n).

Proof. *Let every set E_n be arranged in a sequence (e_{nk}), $k = 1, 2, \dots$, and consider the infinite array*

$$\begin{array}{ccccc}
e_{11} & e_{12} & e_{13} & e_{14} & \cdots \\
e_{21} & e_{22} & e_{23} & e_{24} & \cdots \\
e_{31} & e_{32} & e_{33} & e_{34} & \cdots \\
e_{41} & e_{42} & e_{43} & e_{44} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

in which the elements of E_n form the nth row. The array contains all elements of S. As indicated by the arrows, these elements can be arranged in a sequence

$$e_{11}, e_{21}, e_{12}, e_{31}, e_{22}, e_{13}, \dots .$$

If any two of the sets E_n have elements in common, these will appear more than once in our sequence. Hence there is a subset T of the set of all positive integers such that $S \sim T$, which shows that S is at most countable. Since $E_1 \subset S$, and E_1 is infinite, S is infinite, and thus countable. □

Theorem A.2.3 Let A and B be countable sets. Then the cartesian product $A \times B$ is countable.

Proof. *Name the elements of A and B as follows:*

$$A = \{a_1, a_2, a_3, \dots\}, \quad B = \{b_1, b_2, b_3, \dots\}.$$

We can then use the array of Lemma 1.5.2

$$\begin{array}{ccccc}
(a_1, b_1) & (a_1, b_2) & (a_1, b_3) & (a_1, b_4) & \cdots \\
(a_2, b_1) & (a_2, b_2) & (a_2, b_3) & (a_2, b_4) & \cdots \\
(a_3, b_1) & (a_3, b_2) & (a_3, b_3) & (a_3, b_4) & \cdots \\
(a_4, b_1) & (a_4, b_2) & (a_4, b_3) & (a_4, b_4) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

to create the sequence

$$(a_1,b_1),(a_2,b_1),(a_1,b_2),(a_3,b_1),(a_2,b_2),(a_1,b_3),\dots.$$

The result follows with the same argument we used in the end of Lemma 1.5.2. □

Corollary A.2.4 Let A be a countable set, and let A^n be the set of all n-tuples of elements of A. Then, A^n is countable.

Proof. *We approach by induction.*

A^1 is obviously countable, since $A^1 = A$. Suppose A^{k-1} is countable. The set A^k is equivalent to $A \times A^{k-1}$. Theorem 1.5.3 *then implies that A^k is countable. The result follows by induction.* □

Cartesian product isn't associative. An element of $(A\times B)\times C$ is of the form $((a,b),c)$, while an element of $A\times B\times C$ is of the form (a,b,c). However these two sets are clearly equivalent as there is an obvious bijection between them. In the preceding proof, $A\times A^{k-1} \neq A^k$ but $A\times A^{k-1} \sim A^k$.

Corollary A.2.5 The set of all rational numbers is countable.

Proof. *There is a bijection from the set of all fractions b/a (while $\frac{-1}{-2}$ and $\frac{1}{2}$ are certainly the same number, we will consider them to be different fractions) to the set $\mathbb{Z}^2$ of ordered pairs (b,a). Of course there is an injection from $\mathbb{Q}$ to the set of all fractions, so $\mathbb{Q}$ is at most countable. But since $\mathbb{N}\subset\mathbb{Q}$ and $\mathbb{N}$ is infinite, $\mathbb{Q}$ is countable.* □

We shall now present our first uncountable set.

Theorem A.2.6 Let A be the set of all sequences whose elements are the digits 0 and 1. Then, A is uncountable.

Proof. *Of course A is infinite, so let E be a countable subset of A. Then arrange the elements of $E = \{e_1, e_2, \dots\}$ in a array similar to that of lemma* 1.5.2. *(Here we used indices to denote the elements of E and arguments to denote the elements of each sequence.)*

$$\begin{array}{llllll}
e_1: & e_1(1) & e_1(2) & e_1(3) & e_1(4) & \cdots \\
e_2: & e_2(1) & e_2(2) & e_2(3) & e_2(4) & \cdots \\
e_3: & e_3(1) & e_3(2) & e_3(3) & e_3(4) & \cdots \\
e_4: & e_4(1) & e_4(2) & e_4(3) & e_4(4) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Define a new sequence b such that $b(n) = 1 - e_n(n)$. Notice that b is an element of A and that b is not a element of E, since it is different from each one of its elements in at least one place.

So, it's been shown that every countable subset of A is a proper subset of A. It follows that A is uncountable (for otherwise A would be a proper subset of itself). □

The idea of the above proof was first used by Georg Cantor, and is called Cantor's diagonal argument.

Theorem A.2.7 A complex number z is said to be *algebraic* if there are integers $a_0, \dots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = 0.$$

Then, the set of all algebraic numbers is countable.

Proof. *Let*

$$A_m = \{z \in \mathbb{C} \mid a_0 z^n + \ldots + a_n = 0 \text{ and } n + |a_0| + \ldots + |a_n| = m\}.$$

The set A_m is finite, since each equation has only a finite set of solutions and there are only finitely many equations satisfying $n + |a_0| + \ldots + |a_n| = m$ for each m.

The set of all algebraic numbers, i.e. the union $\bigcup_{m=2}^{\infty} A_m$, is then at most countable. Since every rational number is algebraic and there are infinite rationals, the set of all algebraic numbers is exactly countable. □

Notice how fortunate the preceding result is: at the present stage we can't even prove that the sum of two algebraic numbers is algebraic, let alone that the set of all algebraic numbers is a field (it is). Yet we can prove that it is countable. To most people, the only two non-algebraic (i.e. transcendental) numbers they'll find in their lives are π and e even though almost all complex numbers are transcendental.

A.3 Exercises

Exercise A.1 Let A be a finite set. Prove that cardinality is a well defined function. That is, if $|A| = n$ and $|A| = m$, then $n = m$.

Exercise A.2 — Inclusion-Exclusion Principle. If A and B are finite sets, prove that $|A \cup B| = |A| + |B| - |A \cap B|$.
Prove a similar formula for 3 sets.

Exercise A.3 — Dirichlet's Box Principle. Let A and B be finite sets such that $|A| > |B|$. Prove that there is no injection from A to B.
It is usual to write this principle in the following way:

"If you have m objects to put in $n < m$ boxes, then at least one box will have more than one object."
Hint: It may be useful to prove the result of Exercise 9 for finite sets.

Exercise A.4 — Cantor-Schröder-Bernstein. If you are feeling really brave today, try to prove the following theorem:
Suppose there exist functions $f : A \to B$ and $g : B \to A$ such that f and g are injective. Prove that there is a bijection from A to B.
It may be useful to partition the sets A and B in four parts A_1, A_2, B_1, B_2 such that

- $A_1 \cup A_2 = A$
- $B_1 \cup B_2 = B$
- $f(A_1) = B_1$
- $A_1 \cap A_2 = \varnothing$
- $B_1 \cap B_2 = \varnothing$
- $g(B_2) = B_2$.

Exercise A.5 Prove that no order can be defined in the complex field that turns it into an ordered field.
Hint: -1 is a square.

Exercise A.6 Prove that every ordered field contains a copy of the rational field. (Actually it may contain a set that is *isomorphic* to the rational field. That is, a set that is equivalent to $\mathbb{Q}$ and preserves all the ordered field operations.)

Exercise A.7 Let p be a fixed prime number. If $x, y \in \mathbb{Z}$ define the equivalence relation $x \sim y$ to hold if, and only if p divides $x - y$. Check that this is, in fact, a equivalence relation.
We denote by $\mathbb{Z}/p\mathbb{Z}$ the set of all equivalence classes of $\sim$. In this set we'll define two operations $+$ and $\cdot$. If a is the equivalence class determined by α and b is the equivalence class determined by β, then $a + b = [\alpha + \beta]$ and $a \cdot b = [\alpha\beta]$.
Prove that $\mathbb{Z}/p\mathbb{Z}$ is a field.

Exercise A.8 In the proof of Lemma 1.5.2 we ordered the elements e_{nk} in a sequence $e_{11}, e_{21}, e_{12}, e_{31}, e_{22}, e_{13}, \dots$. That is, we implied that there is a bijection $f : \mathbb{N} \to \mathbb{N}^2$ such that $f(1) = (1,1)$, $f(2) = (2,1)$, $f(3) = (1,2)$, $f(4) = (3,1)$ and so on.
Find this function explicitly and prove that it is a bijection.

Exercise A.9 — The Real Field is Unique up to a Isomorphism. Let $\mathbb{F}$ be an ordered field with the least-upper-bound property endowed with sum $\oplus$, product $\otimes$ and order $\prec$. Denote by $\overline{0}$ and $\overline{1}$ the additive and multiplicative identities of $\mathbb{F}$ and by $\overline{n} = \overline{1} \oplus \overline{1} \oplus \dots \oplus \overline{1}$ (n times). For negative numbers we define $\overline{-n} = -\overline{n}$.

We'll define a function $f : \mathbb{R} \to \mathbb{F}$ as

$$f(p/q) = \overline{p}/\overline{q} \quad \text{for all} \quad p/q \in \mathbb{Q}$$

and, for irrational x,

$$f(x) = \sup\{\overline{p}/\overline{q} \in \mathbb{F} \mid p/q < x\}.$$

Prove that f is an isomorphism. That is, f is an bijection and for all $x, y \in \mathbb{R}$ it holds that $f(x+y) = f(x) \oplus f(y)$ and $f(xy) = f(x) \otimes f(y)$. It also holds that $x < y$ implies $f(x) \prec f(y)$. (This is another hard exercise.)

Exercise A.10 Let α be a irrational number. We denote by $\{x\}$ the *fractional part* of x. That is, the only real number that satisfies $0 \leq \{x\} < 1$ and $x = m + \{x\}$ for some integer m.
Prove that the set $A = \{\,\{n\alpha\} \mid n \in \mathbb{N}\}$ is dense in $[0,1)$. That is, for all non-empty subsets $(a,b) \subset [0,1)$ there is a real number $x \in A$ such that $a < x < b$.
You may want to partition the interval $[0,1)$ in the following way

$$[0,1) = \left[0, \frac{1}{n}\right) \cup \left[\frac{1}{n}, \frac{2}{n}\right) \cup \ldots \cup \left[\frac{n-1}{n}, 1\right)$$

and use Dirichlet's box principle.

Exercise A.11 — Dirichlet's Approximation Theorem. Use the preceding result to show that, given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $n_0 \in \mathbb{N}$, there exists $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qn_0}.$$

Exercise A.12 — Pythagorean Theorem. Prove the following theorem:
If $x \cdot y = 0$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.
Show that the converse does not necessarily hold.

Exercise A.13 — Parallelogram Identity. Prove the following identity:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

What does this identity mean geometrically for vectors of $\mathbb{R}^3$?

Exercise A.14 Let X be an infinite set. Prove that X contains a countable subset.
This exercise is meant to be easy. However, your solution is surely wrong if we do not assume the Axiom of Choice. Now would be a good time to delight yourself with

a bit of math.
Take a look at the precise statements of the Axiom of Choice and the Well Ordering Theorem. It should be almost clear that the Axiom of Choice is "obviously" right and the Well Ordering Theorem is "obviously" wrong. Yet they are equivalent...

Exercise A.15 Let B be the set of all bijections from $\mathbb{N}$ to itself. Prove that B is uncountable.
Hint: use a variation of Cantor's diagonal argument here.

Exercise A.16 — Cantor's Theorem. Let X be a set and $\mathscr{P}(X)$ be the set of its subsets. We will prove that there is no bijection from X to $\mathscr{P}(X)$.

Suppose there is a bijection $f : X \to \mathscr{P}(X)$ and consider the set $A = \{x \in X \mid x \notin f(x)\}$. Since A is a subset of X, there is an element $a \in X$ such that $A = f(a)$. What can we conclude from here?

Exercise A.17 At first it seems like we could use Cantor's diagonal argument to prove that $(0,1)$ is uncountable in the following way.
Suppose $(0,1)$ is countable, then list the decimal expansions of all its elements. We create a new number that is in $(0,1)$ but was not counted by switching the i-th digit of the i-th number in our sequence from k to $k+1$ if $k \neq 9$ and to 0 if $k = 9$.
This is not a valid proof. Why?

Exercise A.18 Let B be a family of disjoint intervals. Prove that B is at most countable. Notice the paramount importance of the word "disjoint" here. If the intervals were not disjoint, then B could be the set of all intervals, which is equivalent to $\mathbb{R}^2$. (Why?)

Appendix B
The Construction Of The Real Field

B.1 Introduction

The real line has exceptional importance in analysis basically because it is *continuous*. In some sense, it means that it has no "gaps". We will see that, while the rational line has lots of numbers, it has even more gaps than numbers.

Theorem B.1.1 There is no rational x such that $x^2 = 2$. Moreover, the set $A = \{x \in \mathbb{Q}_+ \mid x^2 < 2\}$ has no largest element and the set $B = \{x \in \mathbb{Q}_+ \mid x^2 > 2\}$ has no smallest element.

Proof. *Suppose there was a rational x such that*

$$x^2 = 2.$$

Since x is rational, we can write x as p/q, where p and q are integers not both even. We get then that

$$p^2 = 2q^2,$$

which implies that p^2 and hence p is even. Since p is even, we can write p as $2k$, for some integer k, and conclude that q is even too. This contradicts our assumption that p and q are not both even.

Suppose now that B has a smallest element b and consider $a = 2/b$. Of course $a \neq b$, as that would imply a rational solution to $x^2 = 2$. Since $a^2 = 4/b^2 < 4/2 = 2$, we conclude that $a \in A$ and hence $a < b$. However $(a-b)^2 > 0$ implies $(\frac{a+b}{2})^2 > 2$, which contradicts the minimality of b as $(a+b)/2 \in B$ and $(a+b)/2 < b$.

Suppose A had a largest element a and consider the numbers $b = 2/a$ and $c = (a+b)/2$. Similarly to what we just did, it follows that $a < c < b$. Writing b and c in terms of a, we get that $(\frac{a+c}{2})^2 - 2 = 9(a^2-2)(a^2-2/9)/16a^2 < 0$ for $2/9 < a^2 < 2$. This is a contradiction to the maximality of a. □

As the set of all real numbers is so important in analysis, now would be a good time to construct such set. However, I find it's hard for most students to appreciate such construction and to notice *why* it is important at this point. My approach will be to show some standard set-theoretic definitions and we'll define the set of all real numbers as the set that satisfies some desirable properties. As soon as we have the machinery of Cauchy sequences I shall present one of the possible constructions. I also recommend the reading of Rudin's[13] first chapter for an algebraic approach to the real number set.

B.2 Ordered Fields

To characterize the real number system, we will define a set of axioms for an algebraic structure called *field* and then say that the real set is a "special" kind of field.

Definition B.2.1 A *field* is a set $\mathbb{F}$ with two operations, $+$ and $\cdot$ (both functions from $\mathbb{F}\times\mathbb{F}$ to $\mathbb{F}$) that satisfies the following axioms.

- (Closure) For all x,y in $\mathbb{F}$, both $x+y$ and $x\cdot y$ are elements of $\mathbb{F}$.
- (Associativity) For all x,y,z in $\mathbb{F}$ the following holds: $(x+y)+z = x+(y+z)$ and $(x\cdot y)\cdot z = x\cdot(y\cdot z)$.
- (Commutativity) For all x,y in $\mathbb{F}$ the following holds: $x+y = y+x$ and $x\cdot y = y\cdot x$.
- (Existence of identity) There exists two distinct elements of $\mathbb{F}$ denoted 0 and 1 such that for all x in $\mathbb{F}$, $0+x = x$ and $1\cdot x = x$.
- (Existence of inverses) For all x in $\mathbb{F}$ there exists an element denoted $-x$ such that $x+(-x) = 0$. Likewise, for all $x \neq 0$ in $\mathbb{F}$ there exists an element denoted x^{-1} such that $x\cdot x^{-1} = 1$. The elements $a+(-b)$ and $a\cdot b^{-1}$ are also denoted $a-b$ and a/b respectively.
- (Distributivity) For all x,y,z in $\mathbb{F}$, the following holds: $x\cdot(y+z) = x\cdot y + x\cdot z$.

It is usual to drop the dot and write products as xy instead of $x\cdot y$.

It is not hard to memorize the axioms for the frequently used algebraic structures because they are essential to some important sets. For example, both $\mathbb{R}$ and $\mathbb{Q}$ are fields. However, $\mathbb{R}$ and $\mathbb{Q}$ are not the only fields that exists. We need further characterization to know what really is $\mathbb{R}$.

Definition B.2.2 — Ordered Sets. An *ordered set* is a set S, together with a relation $>$ such that:

- (Trichotomy) For any $x,y \in S$, exactly one of $x > y$, $x = y$, or $y > x$ holds.
- (Transitivity) If $x > y$ and $y > z$, then $x > z$.

We say that $x < y$ if and only if $y > x$.

The set of all rational numbers becomes an ordered set when we define $x > y$ if $x - y = p/q$ where $p, q \in \mathbb{N}$. Similarly, $\mathbb{Z}$ and $\mathbb{N}$ are ordered sets with the usual order relation.

It is customary to define $x \geq y$ to mean $x = y$ or $x > y$, without specifying which one applies. The notation $x \leq y$ is defined in the obvious way.

It is not obvious at all that the order we defined earlier for the cardinality of sets is in fact a order relation. Trichotomy holds if, and only if $|A| \geq |B|$ and $|B| \geq |A|$ implies $|A| = |B|$, which is a result called Cantor-Schröder-Bernstein Theorem.

Definition B.2.3 Let S be an ordered set and E be a subset of S.

- If there exists an element u of S such that $x \leq u$ for all $x \in E$, we say that E is *bounded above* and u is an *upper bound* of E. The least upper bound of a set (if it exists) is called the *supremum* of E and is denoted $\sup E$.

- If there exists an element l of S such that $x \geq l$ for all $x \in E$, we say that E is *bounded below* and l is an *lower bound* of E. The greatest lower bound of a set (if it exists) is called the *infimum* of E and is denoted $\inf E$.

Notice that a supremum or infimum of E need not be in E. For example, the set $E = \{x \in \mathbb{Q} \mid x > 0\}$ has infimum 0 and 0 is not an element of E. The set $\mathbb{Z}$ has no infimum and no supremum since it is not bounded above or below.

Definition B.2.4 An ordered set S is said to possess the *least-upper-bound property* if every nonempty subset $E \subset S$ that is bounded above has a least upper bound in S. That is, for all non-empty set $E \subset S$ that is bounded above, $\sup E$ exists in S.

This property is sometimes called the *Dedekind completeness property*. Theorem 1.1.1 showed that the set of all rational numbers does not possess the least-upper-bound property.

Definition B.2.5 An *ordered field* is a field $\mathbb{F}$ which is also an ordered set and satisfies the following axioms.

- For all x, y, z in $\mathbb{F}$ such that $x > y$ we have that $x + z > y + z$.

- For all x, y in $\mathbb{F}$ such that $x > 0$ and $y > 0$ we have that $xy > 0$.

If $x > 0$ we say that x is *positive* and if $x < 0$ we say that is it *negative*.

You should prove that some familiar properties of $\mathbb{Q}$ follow directly from the ordered field axioms. Once we know that every field satisfies these properties we won't need to prove them again for $\mathbb{R}$ and for $\mathbb{C}$.

Exercise B.1 Prove the following statements using only the field axioms. All letters are meant to denote elements from a field.

a) If $x + y = x + z$ then $y = z$.

b) If $x \neq 0$ and $xy = xz$ then $y = z$.

c) $0x = 0$.

d) If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

The following properties hold for ordered fields. Notice that, while the complex number set is a field, it is *not* an ordered field.

Exercise B.2 Prove the following statements using only the ordered field axioms.[a] (That means you can only use definitions 1.3.1, 1.3.2 and 1.3.5.)
All letters are meant to denote elements from an ordered field.

a) If x is positive then $-x$ is negative.

b) If x is positive and $y > z$ then $xy > xz$.

c) If x is negative and $y > z$ then $xy < xz$.

d) If $x \neq 0$ then $x^2 > 0$.

[a] Even if you managed to do all the items in Exercise 1.1 and 1.2 you should check the solutions in the end of the book.

B.3 The Real and Complex Fields

We are now ready to define the set of all real numbers.

Definition B.3.1 — The Real Field. The *real field* is the only ordered field with the least-upper-bound property. Moreover, $\mathbb{Q}$ is a subfield of $\mathbb{R}$. That is, $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{Q}$ inherits its order relation and its field operations from $\mathbb{R}$.

Well, we can always rename the elements of $\mathbb{R}$ to create another set that satisfies definition 1.4.1. Technically we say that the real number set is the only ordered field with the least-upper-bound property up to a isomorphism. And that is a theorem that I will not prove.

One of the most important consequences of the least-upper-bound property in the real line is the Archimedean property. It roughly says that the set $\{x \in \mathbb{R} \mid x > 0\}$ has no smallest element.

Theorem B.3.1 — Archimedean Property. For all $\varepsilon > 0$ in $\mathbb{R}$ there exists a natural n such that $1/n < \varepsilon$.

Proof. *Suppose the theorem is false. Then there exists $x_0 > 0$ in $\mathbb{R}$ such that no natural n makes $1/n < x_0$ true. That is, for all $n \in \mathbb{N}$ we have that $nx_0 \leq 1$. Let*

$$S = \{y \in \mathbb{R} \mid y = nx_0,\, n \in \mathbb{N}\}.$$

Since S is bounded above by 1 and S is a subset of $\mathbb{R}$, S has a least upper bound. We *shall call it k.*

Of course we have that $k - x_0 < k$ since x_0 is positive. But then $k - x_0$ is not an upper bound of S, so $mx_0 > k - x_0$ for some $m \in \mathbb{N}$. That means $k < (m+1)x_0$, which is absurd since k is an upper bound of S. □

For now, we'll say that a set $E \subset \mathbb{R}$ is dense in $\mathbb{R}$ if, and only if for all a, b in $\mathbb{R}$ such that $b > a$ there exists c in E such that $a < c < b$. (We will generalize this definition to metric spaces in the following chapter.)

Theorem B.3.2 The set of all rational numbers is dense in the set of all real numbers.

Proof. *I will assume $a > 0$ without loss of generality.*
By the Archimedean property, there exists an integer n_0 such that $1/n_0 < b - a$. Let

$$M = \{m \in \mathbb{Q} \,|\, m = \frac{k}{n_0},\ k \in \mathbb{N} \text{ and } m \leq a\}.$$

Let $s = \max M$. Of course $s \leq a$. Since s is the maximum of M, $s + 1/n_0 \notin M$. That is, $s + 1/n_0 > a$. Since $1/n_0 < b - a$, we have that

$$a < \underbrace{s + 1/n_0}_{c \in \mathbb{Q}} < b.$$

The result follows. □

Corollary B.3.3 The set of all irrational numbers is dense in the set of all real numbers.

Proof. *Substitute M in the proof of Theorem 1.4.2 by*

$$M' = \{m \in \mathbb{R} \setminus \mathbb{Q} \,|\, m = \sqrt{2}\,\frac{k}{n_0},\ k \in \mathbb{N} \text{ and } m \leq a\}$$

and $c = \max M + 1/n_0$ by $c' = \max M' + \sqrt{2}/n_0$. □

We now define another set called *the extended real line*.

Definition B.3.2 — Extended Real Line. The *extended real line* is the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ which inherits $\mathbb{R}$ order, operations and satisfies

$$-\infty < x < +\infty$$

for all $x \in \mathbb{R}$.
We also define the following operations, which are valid for all $x \in \overline{\mathbb{R}}$:

- $x+\infty = +\infty$ if $x \neq -\infty$
- $x-\infty = -\infty$ if $x \neq +\infty$
- $x \cdot (+\infty) = +\infty$ if $x > 0$
- $x \cdot (+\infty) = -\infty$ if $x < 0$
- $x \cdot (-\infty) = -\infty$ if $x > 0$
- $x \cdot (-\infty) = +\infty$ if $x < 0$
- $x/(+\infty) = 0$ if $x \neq \pm\infty$
- $x/(-\infty) = 0$ if $x \neq \pm\infty$
- $+\infty/x = +\infty$ if $x > 0$ and $x \neq \pm\infty$
- $+\infty/x = -\infty$ if $x < 0$ and $x \neq \pm\infty$
- $-\infty/x = -\infty$ if $x > 0$ and $x \neq \pm\infty$
- $-\infty/x = +\infty$ if $x < 0$ and $x \neq \pm\infty$

The operations $+\infty-\infty$, $0\cdot(\pm\infty)$ and $\pm\infty/\pm\infty$ are left undefined.

Notice that $x+\infty$ means both $x+(+\infty)$ and $x-(-\infty)$, while $x-\infty$ means both $x+(-\infty)$ and $x-(+\infty)$. If the context makes it clear, we'll usually write ∞ instead of $+\infty$. It should be clear that $\overline{\mathbb{R}}$ is *not* a field.

A very important property of $\overline{\mathbb{R}}$, which $\mathbb{R}$ lacks, is that every subset has a supremum and a infimum. In particular, the extended real line has the least-upper-bound property.

To define the set of all complex numbers rigorously, we'll characterize them as ordered pairs of real numbers and then see how the real line "sits" inside this set.

Definition B.3.3 — The Complex Field. The *complex field* is the set of all ordered pairs (a,b) of real numbers such that

- For all a,b,c,d in $\mathbb{R}$: $(a,b)+(c,d) = (a+c, b+d)$.
- For all a,b,c,d in $\mathbb{R}$: $(a,b)\cdot(c,d) = (ac-bd, ad+bc)$.
- $(a,b) = (c,d)$ implies $a = c$ and $b = d$.

You should prove that this is, in fact, a field. For every complex number $z = (a,b)$, we define a to be the real part $\mathrm{Re}(z)$ and b to be the imaginary part $\mathrm{Im}(z)$ of z.

Exercise B.3 Prove that the complex field, with the operations defined above and with $(0,0)$ and $(1,0)$ as the sum and product identities respectively, satisfies all the field axioms. You should define $-(a,b)$ as $(-a,-b)$ and $(a,b)^{-1}$ as $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$.

Since $(a,0)+(b,0) = (a+b,0)$ and $(a,0)\cdot(b,0) = (ab,0)$ we will write $(a,0)$ as simply a from now on. Provided that we define i as $(0,1)$ we can write every complex number as $a+bi$ since $a+bi = (a,b)$.

We shall then define the conjugate of the number $a+bi$ as $a-bi$ and denote it by $\overline{a+bi}$. We define the absolute value of a complex number $a+bi$ to be $\sqrt{a^2+b^2}$ and denote it by $|a+bi|$.* While square roots are still undefined, this will soon be fixed.

*We had a few notations popping in different places here. They usually aren't a coincidence. If A is a set, $|A|$ measures roughly how "big" a set is and if z is a complex number, $|z|$ measures roughly how "big" it is. We'll see in the next chapter that $\overline{E}$ denotes the closure of a set E. If (a,b) is a subset of $\mathbb{R}$, its closure is $[a,b]$. It is based on this fact that we denote the extended real line as $\overline{\mathbb{R}}$. However, I don't know a good reason for the conjugate of a complex number to be defined with the same symbol as the closure of a set. Neither do I know

You should prove now the usual properties of these operations.

Exercise B.4 Prove the following properties about complex conjugation and absolute value. All letters are meant to be complex numbers.

a) $\overline{z+w} = \overline{z} + \overline{w}$

b) $\overline{zw} = \overline{z} \cdot \overline{w}$

c) $z\overline{z} = |z|^2$

d) $|\overline{z}| = |z|$

e) $|zw| = |z||w|$

f) $z + \overline{z} = 2\,\mathrm{Re}(z)$

g) $\mathrm{Re}(z) \leq |z|$

h) $|z+w| \leq |z| + |w|$

Definition B.3.4 — Real and Complex Vector Spaces. If n is a positive integer, we define $\mathbb{C}^n$ to be the set of all n-tuples of complex numbers. That is, the elements of $\mathbb{C}^n$ are numbers of the form

$$x = (x_1, x_2, \ldots, x_n),$$

where $x_i \in \mathbb{C}$ for all $1 \leq i \leq n$.
The elements of $\mathbb{C}^n$ are called vectors.
We define the following operations on $\mathbb{C}^n$:

- $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$.
- If a is a complex number, $a(x_1, x_2, \ldots, x_n) = (ax_1, ax_2, \ldots, ax_n)$.
- $(x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n) = \sum_{i=1}^{n} x_i \overline{y_i}$
- $\|(x_1, x_2, \ldots, x_n)\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$.

The set $\mathbb{R}^n$ is defined in the same way.

It will be usual to call the identity element of the sum $(0, 0, \ldots, 0)$ as simply 0. We'll call the set $\mathbb{R}^n$ an *euclidean space*. As usual, you should prove the following properties.

Exercise B.5 Prove the following properties about norms and inner products (the product of vectors we just defined). All letters are meant to be elements of a vector space ($\mathbb{R}^n$ or $\mathbb{C}^n$).

a) $x \cdot y = \overline{y \cdot x}$.

b) If $a \in \mathbb{C}$, $(ax) \cdot y = a(x \cdot y)$.

c) If $a \in \mathbb{C}$, $x \cdot (ay) = \overline{a}(x \cdot y)$.

d) $(x+y) \cdot z = x \cdot z + y \cdot z$.

e) $\|x\|^2 = x \cdot x$.

f) $\|x\| = 0$ implies $x = 0$.

One of the most important properties of this vector space (or any vector space in general) is the Cauchy-Schwarz inequality.

why we denote ordered pairs the same way as we denote open intervals of the real line.

Theorem B.3.4 — Cauchy-Schwarz Inequality. Let x and y be vectors of $\mathbb{C}^n$. Then the following inequality holds.

$$|x \cdot y| \le \|x\| \, \|y\| \, .$$

Proof. *If $x \cdot y = 0$, then the theorem holds trivially. Otherwise, let λ be the complex number $x \cdot y / \|y\|^2$. Since the square of every real number is positive (or zero),*

$$\begin{aligned}
0 &\le \|x - \lambda y\|^2 \\
&= \|x\|^2 - \overline{\lambda}(x \cdot y) - \lambda(y \cdot x) + |\lambda|^2 \|y\|^2 \\
&= \|x\|^2 - \frac{|x \cdot y|}{\|y\|^2} .
\end{aligned}$$

This implies then that, $|x \cdot y| \le \|x\| \, \|y\|$. □

A quick corollary of the Cauchy-Schwarz inequality is a generalization of the property proved in exercise 1.4.*h*.

Corollary B.3.5 — Triangle Inequality. Let x and y be vectors of $\mathbb{C}^n$. Then the following inequality holds.

$$\|x + y\| \le \|x\| + \|y\| \, .$$

Proof.

$$\begin{aligned}
\|x + y\|^2 &= \|x\|^2 + x \cdot y + y \cdot x + \|y\|^2 \\
&= \|x\|^2 + 2\,\mathrm{Re}(x \cdot y) + \|y\|^2 \\
&\le \|x\|^2 + 2|x \cdot y| + \|y\|^2 \\
&\le \|x\|^2 + 2\,\|x\| \, \|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2 .
\end{aligned}$$

The Cauchy-Schwarz inequality was used to justify the fourth line. □

APPENDIX C

Linear Algebra

Solutions

1- Naive Set Theory

Solution 1 Although this might seem obvious, these propositions need to be proved and it is important to understand each step in order to obtain a solid knowledge about fields

a) Since x belongs to a field, by the property of existence of inverses, there exists -x such that

$$\begin{aligned} x+y=x+z &\implies (-x)+(x+y)=(-x)+(x+z) \\ &\implies ((-x)+x)+y=((-x)+x)+z \quad \text{(By associativity)} \\ &\implies 0+y=0+z \\ &\implies y=z. \end{aligned}$$

b) By the existence of inverses, for all $x \neq 0$

$$\begin{aligned} xy=xz &\implies (x^{-1})\cdot(xy)=(x^{-1})\cdot(xz) \\ &\implies (x^{-1}x)\cdot y=(x^{-1}x)\cdot z \quad \text{(By associativity)} \\ &\implies 1\cdot y=1\cdot z \\ &\implies y=z. \end{aligned}$$

c) Distributivity implies

$$0x=(a+(-a))\cdot x=ax+(-ax)=0.$$

d) Suppose $xy=0$ then, since $y \neq 0$, there exists y^{-1} such that $yy^{-1}=1$. From (c),

$$\begin{aligned} &xy(y^{-1})=0(y^{-1})=0 \\ &\implies x(yy^{-1})=0 \quad \text{(By associativity)} \\ &\implies x\cdot 1=x=0. \end{aligned}$$

Which is a contradiction.

Remark: You might ask why we can add or multiply a number in both sides of an equation. In fact, this is basically the formal definition of equality. Given two objects x and y, we say that $x=y$ if and only if, $P(x)=P(y)$ for any predicate P.

Solution 2

a) From definition 1.3.5, $0 < x$ implies $(-x) < x + (-x) = 0$.

b) By the associativity and the second part of definition 1.3.5 it follows that $y > z \implies y - z > 0 \implies x(y-z) > 0 \implies xy - xz > 0 \implies xy > xz$.

c) From (a), $(-x)(y-z) > 0$. Hence, $xy < xz$.

d) By trichotomy, $x > 0$, $x = 0$ or $x < 0$. If $x > 0$, then by second part of definition 1.3.5 $x^2 > 0$. If $x < 0$ however, from (b), $x < 0$ implies $x^2 > 0 \cdot x$, then by the result from exercise 1.1(c), $x^2 > 0$.

Solution 3 The closure property is assured by the definition given. Let x, y, z be (a,b),(c,d) and (e,f), respectively.

Associativity: $(x+y)+z = (a+c, b+d) + (e,f) = (a+c+e, b+d+f) = (a,b) + (c+e, d+f) = x + (y+z)$.

Commutativity: $x + y = (a+c, b+d) = (c+a, d+b) = y + x$.

Existence of identity: $0 + x = (0+a, 0+b) = (a,b) = x$; $1 \cdot x = (a,b)\cdot(1,0) = (a\cdot 1 - b\cdot 0, a \cdot 0 + b \cdot 1) = (a,b) = x$.

Existence of inverses: $x + (-x) = (a,b) + (-a,-b) = (a-a, b-b) = (0,0) = 0$; $xx^{-1} = (a,b)\cdot\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = \left(\frac{a^2+b^2}{a^2+b^2}, \frac{a(-b)+ab}{a^2+b^2}\right) = (1,0)$.

Distributivity: $x\cdot(y+z) = (a,b)\cdot(c+e, d+f) = (a(c+e) - b(d+f), a(d+f) + b(c+e)) = (ac + ae - bd - bf, ad + af + bc + be) = ((ac-bd) + (ae-bf), (ad+bc) + (af+be)) = xy + xz$.

Solution 4 Let z and w be (a,b) and (c,d), respectively.

a) $\overline{z+w} = (a+c, -(b+d)) = (a,-b) + (c,-d) = \overline{z} + \overline{w}$

b) $\overline{zw} = (ac - bd, -(ad+bc)) = (ac - (-b)(-d), a(-d) + (-b)c) = (a,-b)\cdot(c,-d) = \overline{z}\cdot\overline{w}$.

c) $z\overline{z} = (a^2 - b(-b), a(-b) + ba) = (a^2+b^2, 0) = |z|^2$.

d) $|\overline{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2+b^2} = |z|$.

e) $|zw| = |(a+bi)(c+di)| = |ac + bd + (ad+bc)i| = \sqrt{(ac-bd)^2 + (ad+bc)^2} = \sqrt{a^2(c^2+d^2) + b^2(c^2+d^2) - 2abcd + 2abcd} = \sqrt{(a^2+b^2)(c^2+d^2)} = \sqrt{(a^2+b^2)}\sqrt{(c^2+d^2)} = |z||w|$.

f) $z + \overline{z} = a + bi + (a - bi) = 2a = 2Re(z)$.

g) $Re(z) = a \le \sqrt{a^2} \le \sqrt{a^2+b^2} = |z|$.

h) First we will prove that $x^2 \le y^2 \implies x \le y$. If $x, y \ge 0$. $x^2 \le y^2 \implies x^2 - y^2 \le 0 \implies (x+y)(x-y) \le 0 \implies x - y \le 0 \implies x \le y$. So, $|z+w|^2 = (z+w)\overline{(z+w)} = z\overline{z} + w\overline{w} + z\overline{w} + \overline{z}w = |z|^2 + |w|^2 + z\overline{w} + \overline{z\overline{w}} = |z|^2|w|^2 + 2Re(z\overline{w}) \le |z|^2 + |z|^2 + 2|z\overline{w}| = (|z| + |w|)^2$. The result follows.

Solution 5 a) $x \cdot y = \sum_{i=1}^{n} x_i\overline{y_i} = \sum_{i=1}^{n} \overline{\overline{x_i}y_i} = \overline{\sum_{i=1}^{n} y_i\overline{x_i}} = \overline{y \cdot x}$.

b) $(ax) \cdot y = \sum_{i=1}^{n} (ax_i)\overline{y_i} = \sum_{i=1}^{n} a(x_i\overline{y_i}) = a(x \cdot y)$.

c) $x \cdot (ay) = \sum_{i=1}^{n} x_i\overline{ay_i} = \sum_{i=1}^{n} \overline{a}(x_i\overline{y_i}) = \overline{a}(x \cdot y)$.

d) $(x+y) \cdot z = \sum_{i=1}^{n} (x_i + y_i)\overline{z_i} = \sum_{i=1}^{n} x_i\overline{z_i} + y_i\overline{z_i} = \sum_{i=1}^{n} x_i\overline{z_i} + \sum_{i=1}^{n} y_i\overline{z_i} = x \cdot z + y \cdot z$.

e) $||x||^2 = \sum_{i=1}^{n} |x_i|^2 = \sum_{i=1}^{n} x_i\overline{x_i} = x \cdot x$.

f) $||x|| = 0 \implies \sum_{i=1}^{n} |x_i|^2 = 0$. But $|x_i|^2 > 0$ if $x_i \neq 0$, so $x_i = 0$ for $i = 1, 2, \ldots n$ and hence $x = 0$.

Solution 6 Suppose $n > m$. It follows that $A \sim \{1, \ldots, n\}$ and $A \sim \{1, \ldots, m\}$ implies $\{1, \ldots, n\} \sim \{1, \ldots, m\}$, by the transitivity of the equivalence. But it is an absurd since no set can be equivalent to its own proper subset.

Solution 7 Let's prove by induction on $n = |A| + |B|$.

If $n = 0$, then $|A| = |B| = 0$ and hence the result is trivial.

Now, suppose the proposition is valid for $n = k$. If we add a element x to A or B, either x belongs to $A \cap B$ or not, so let's study both cases.

If x is added to A and belongs to B, it follows that

$$\begin{aligned}
|A \cup \{x\}| + |B| - |(A \cup \{x\}) \cap B| &= |A| + 1 + |B| - |(A \cap B) \cup (\{x\} \cap B)| \\
&= |A| + 1 + |B| - (|A \cap B| + 1) \\
&= |A \cup B| \\
&= |A \cup (B \cup \{x\})| \\
&= |(A \cup \{x\}) \cup B|.
\end{aligned}$$

If x do not belong to B,

$$\begin{aligned}
|A \cup \{x\}| + |B| - |(A \cup \{x\}) \cap B| &= |A| + 1 + |B| + |(A \cap B) \cup (\{x\} \cap B)| \\
&= |A| + 1 + |B| + |(A \cap B)| \\
&= |A \cup B| + 1 \\
&= |(A \cup \{x\}) \cup B|.
\end{aligned}$$

Hence, the proposition holds for any non-negative integer n.

Now, in possession of this result, for three sets:

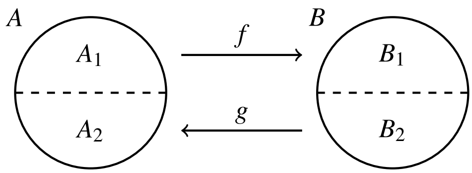
$$\begin{aligned}
|A \cup B \cup C| &= |(A \cup B) \cup C| \\
&= |A \cup B| + |C| - |(A \cup B) \cap C| \\
&= |A \cup B| + |C| - |(A \cap C) \cup (B \cap C)| \\
&= |A \cup B| + |C| - (|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|) \\
&= |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|.
\end{aligned}$$

Solution 8 Firstly, I'll prove the result of Exercise 1.9 for finite sets. That is, if there is an injection from A to B and an injection from B to A then there is a bijection from A to B.

Let $f : A \to B$ and $g : B \to A$ be the aforementioned functions. Since f is an injection, $|A| \leq |B|$. Since g is an injection, $|A| \geq |B|$. Hence $|A| = |B|$ and there is an bijection from A to B. (Note that this proof does not involves the fact that $|\cdot|$ is an order relation. It just involves the fact that $|A|$ is an integer if A is a finite set.)

To prove our exercise we just need to know that $|A| > |B|$ means that there is an injection from B to A but there is not a bijection. If there was an injection from A to B, the result we just proved would imply in such bijection, which is absurd!

Solution 9 First lets see that, if the partition described exists, then there is a bijection from A to B. In fact, since $f(A_1) = B_1$ and $g(B_2) = A_2$, the restrictions of f and g to A_1 and B_2 respectively are bijections.



Then the function

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_1 \\ g^{-1}(x) & \text{if } x \in A_2 \end{cases}$$

is a bijection. (With a little of notational abuse, since the functions f and g above are the restrictions of f and g.) We now show that there exist such partition.

Let $\rho : \mathscr{P}(A) \to \mathscr{P}(A)$ be such that $\rho(X) = A \setminus g(B \setminus f(X))$. Note that, if ρ has a fixed point, that is, a set $F \subset A$ such that $\rho(F) = F$, then we could take $A_1 = F$, $A_2 = A \setminus F$, $B_1 = f(F)$ and $B_2 = B \setminus f(F)$. It is the existence of such fixed point that I will prove.

Consider the set $E = \{X \subset A \,|\, X \subset \rho(X)\}$. Since $\varnothing \in E$, E is not empty. Let

$$F = \bigcup_{X \in E} X.$$

I affirm that F is a fixed point of ρ.

Note that

$$\begin{aligned}
\rho(F) &= A \setminus g\left(B \setminus f\left(\bigcup X\right)\right) \\
&= A \setminus g\left(B \setminus \bigcup f(X)\right) \\
&= A \setminus g\left(\bigcap (B \setminus f(X))\right) && \text{(De Morgan)} \\
&= A \setminus \bigcap g(B \setminus f(X)) && (g \text{ is injective}) \\
&= \bigcup A \setminus g(B \setminus f(X)) && \text{(De Morgan)} \\
&= \bigcup \rho(X).
\end{aligned}$$

Then, $F = \bigcup X \subset \bigcup \rho(X) = \rho(F)$. We only have to prove now that $\rho(F) \subset F$.
Indeed, if $x \in \rho(F) = A \setminus g(B \setminus f(F))$, then $x \notin g(B \setminus f(F))$.
Since $F \subset A \setminus g(B \setminus f(F))$, we have that $g(B \setminus f(F)) \subset A \setminus F$ and hence $g(B \setminus f(F)) \subset A \setminus (F \cup \{x\})$.
As $f(F) \subset f(F \cup \{x\})$, it follows that

$$g(B \setminus f(F \cup \{x\})) \subset g(B \setminus f(F)) \subset A \setminus (F \cup \{x\})$$

and then $F \cup \{x\} \subset \rho(F \cup \{x\})$. That is, $F \cup \{x\} \in E$.
We conclude that $F \cup \{x\} \subset F$ and hence $x \in F$. The result follows.
(I told you it was hard.)

Solution 10 — From (13). From Exercise 1.2d we know that if $\mathbb{C}$ is an ordered field, then $x^2 > 0$ for all $x \neq 0$. However note that $i^2 = (0,1) \cdot (0,1) = (-1,0) = -1$. Taking $x = 1$ in the inequality $x^2 > 0$ we see that $1 > 0$. From Exercise 1.2a if follows that -1 is negative from which it follows that $\mathbb{C}$ cannot be an ordered field.

Solution 11 We say that the *characteristic* of a field is the least positive integer n such that

$$\underbrace{1+1+1+\cdots+1}_{n \text{ ones}} = 0.$$

If such a number does not exist, we say that the characteristic of the field is 0.
I affirm that every ordered field has characteristic 0.
In fact, if the field had characteristic n, then we would have that

$$0 < 1 < 1+1 < \cdots < \underbrace{1+1+1+\cdots+1}_{n \text{ ones}} = 0,$$

which is absurd! This automatically implies that every field contains a subfield isomorphic to $\mathbb{N}$.
Since fields have additive inverses, if x is in the field, so is $-x$. This means that every field contains a subfield isomorphic to $\mathbb{Z}$. The existence of inverses and the closure by multiplication then implies the result.

Solution 12 We'll prove some results about $\mathbb{Z}/p\mathbb{Z}$.

First result: $\sim$ is an equivalence relation.

(Reflexitivity) Since every integer divides 0, $x \sim x$ for all $x \in \mathbb{Z}$.

(Simmetry) If $x \sim y$, then $x - y = np$ for some integer n. That is, $y - x = (-n)p$ and hence $y \sim x$.

(Transitivity) If $x \sim y$ and $y \sim z$, then $x - y = np$ and $y - z = mp$. Summing these equations we get that $x - z = (n+m)p$ and hence $x \sim z$.

Second result: $\mathbb{Z}/p\mathbb{Z}$ has exactly p elements.

I affirm that if $x = y + kp$ for some integer k, then $[x] = [y]$. In fact, if $z \in [x]$, then $z - x = np$ for some integer n and hence $z - y = (k+n)p$. That is, $[x] \subset [y]$. The other inclusion is analogous.

Third result: If p does not divide neither x nor y, then p does not divide xy.

The converse of this result "If p divides xy then either p divides x or p divides y is a basic fact in number theory. So, I'll not prove it. In our context it says that if $[x]$ and $[y]$ are both different from $[0]$, then $[x] \cdot [y] \neq [0]$.

The only relevant part of the proof that $\mathbb{Z}/p\mathbb{Z}$ is a field is the existence of multiplicative inverses. If $[a] \in \mathbb{Z}/p\mathbb{Z}$ is not equal to $[0]$, then the set

$$\{[0] \cdot [a], [1] \cdot [a], [2] \cdot [a], \dots .[p-1] \cdot [a]\}$$

has exactly p distinct elements and hence it is equal to $\mathbb{Z}/p\mathbb{Z}$. Since $[1] \in \mathbb{Z}/p\mathbb{Z}$, there is an integer $b \in \{0, 1, \dots, p-1\}$ such that $[b] \cdot [a] = [1]$.

Solution 13 Let n be a positive integer. The fundamental theorem of arithmetic implies that every positive integer is the product of an power of 2 and an odd integer. That is,

$$n = 2^{u_n - 1}(2v_n - 1),$$

for some positive integers u_n and v_n.

The function $f : \mathbb{N} \to \mathbb{N}^2$ such that

$$f(n) = (u_n, v_n)$$

is clearly onto. If we had that $f(n) = f(m)$, then it would follow $u_n = u_m$ and $v_n = v_m$, hence $n = m$.

Solution 14 — From (14). Let A_x be the set $\{\overline{p}/\overline{q} \in \mathbb{F} \mid p/q < x\}$.

It is easy to check that

$$f(m+n) = f(m) \oplus f(n), \quad f(mn) = f(m) \otimes f(n),$$

for all integers m and n. Note that the definition of f for rational numbers makes sense because if $m/n = k/l$, then $ml = nk$, so $\overline{m} \cdot \overline{l} = \overline{k} \cdot \overline{n}$. Hence, $\overline{m}/\overline{n} = \overline{k}/\overline{l}$.

It is also easy to check that

$$f(r_1 + r_2) = f(r_1) \oplus f(r_2), \quad f(r_1 r_2) = f(r_1) \otimes f(r_2),$$

for all rational numbers r_1 and r_2, and that $f(r_1) \prec f(r_2)$ if $r_1 < r_2$.

The set A_x is certainly not empty, and it is also bounded above, for if r_0 is a rational number with $r_0 > x$, then $f(r_0) > f(r)$ for all $f(r) \in A_x$. Since $\mathbb{F}$ has the least-upper-bound property, the set A_x has a least upper bound so that $\sup A_x$ is well defined. We now shall show that the definition of f for irrational x is actually a general definition. In other words, if x is a rational number, we want to show that $\sup A_x = f(x)$, where $f(x)$ here denotes $\overline{m}/\overline{n}$, for $x = m/n$. This is not automatic, but depends on the least-upper-bound property of $\mathbb{F}$; a slight digression is thus required.

Since $\mathbb{F}$ has the least-upper-bound property, $\mathbb{F}$ is archimedean. The consequences of this fact for $\mathbb{R}$ have exact analogues in $\mathbb{F}$: in particular, if a and b are elements of $\mathbb{F}$ with $a \prec b$, then there is a rational number r such that $a \prec f(r) \prec b$. Having made this observation, we return to the proof that the two definitions of $f(x)$ agree for rational x. If y is a rational number with $y < x$, then we have already seen that $f(y) \prec f(x)$. Thus every element of A_x is $\prec f(x)$. Consequently,

$$\sup A_x \preceq f(x).$$

On the other hand, suppose that we had $\sup A_x \prec f(x)$. Then there would be a rational number r such that

$$\sup A_x \prec f(r) \prec f(x).$$

But the condition $f(r) \prec f(x)$ means that $r < x$, which means that $f(r)$ is in the set A_x; this clearly contradicts the condition $\sup A_x \prec f(r)$. This shows that the original assumption is false, so $\sup A_x = f(x)$. It follows that $f(x) = \sup A_x$ holds for all $x \in \mathbb{R}$.

We'll now prove that f is an isomorphism of fields.

1. If $x < y$, then $f(x) \prec f(y)$.

   If x and y are real numbers with x < y, then clearly A_x is contained in A_y. Thus

   $$f(x) = \sup A_x \preceq \sup A_y = f(y).$$

   To rule out the possibility of equality, notice that there are rational numbers r and s with $x < r < s < y$. We know that $f(r) \prec f(s)$. It follows that

   $$f(x) \preceq f(r) < f(s) \preceq f(y).$$

   This proves this item.

2. f is injective.

   If $x \neq y$, then either $x < y$ or $y < x$; in the first case $f(x) \prec f(y)$, and in the second case $f(y) \prec f(x)$; in either case $f(x) \neq f(y)$.

3. f is onto.

   Let a be an element of $\mathbb{F}$, and let B be the set of all rational numbers r with $f(r) \prec a$. The set B is not empty, and it is also bounded above, because there is a rational number s with $a \prec f(s)$, so that $f(r) \prec f(s)$ for r in B, which implies that $r < s$. Let x be the least upper bound of B; we claim that $f(x) = a$. In order to prove this it suffices to eliminate the alternatives $f(x) \prec a$ and $a \prec f(x)$.

In the first case there would be a rational number r with $f(x) \prec f(r) \prec a$. But this means that $x < r$ and that r is in B, which contradicts the fact that $x = \sup B$. In the second case there would be a rational number r with $a \prec f(r) \prec f(x)$. This implies that $r < x$. Since $x = \sup B$, this means that $r < s$ for some s in B. Hence $f(r) \prec f(s) \prec a$, again a contradiction. Thus $f(x) = a$.

4. $f(x+y) = f(x) \oplus f(y)$.

   Suppose that $f(x+y) \neq f(x) \oplus f(y)$. Then either $f(x+y) \prec f(x) \oplus f(y)$ or $f(x) \oplus f(y) \prec f(x+y)$. In the first case there would be a rational number r such that $f(x+y) \prec f(r) \prec f(x) \oplus f(y)$. But this would mean that $x+y < r$. Therefore r could be written as the sum of two rational numbers $r = r_1 + r_2$, where $x < r_1$ and $y < r_2$. Then, using the facts checked about f for rational numbers, it would follow that $f(r) = f(r_1 + r_2) = f(r_1) \oplus f(r_2) \succ f(x) + f(y)$, a contradiction. The other case is handled similarly.

5. $f(xy) = f(x) \otimes f(y)$.

   The same reasoning of the item 4 proves this for positive real numbers. The general case is then a simple consequence.

(Solution from [14])

Solution 15 The Dirichlet's box principle implies that there exists two distinct integers k and j such that $\{k\alpha\}$ and $\{j\alpha\}$ lie in the same element of the partition. Without loss of generality, suppose that $k > j$. It follows that

$$\{(k-j)\alpha\} \in \left[0, \frac{1}{n}\right) \quad \text{or} \quad \{(k-j)\alpha\} \in \left[\frac{n-1}{n}, 1\right),$$

since

$$\{(k-j)\alpha\} = \begin{cases} \{k\alpha\} - \{j\alpha\} & \text{if } \{k\alpha\} \geq \{j\alpha\} \\ 1 + \{k\alpha\} - \{j\alpha\} & \text{if } \{k\alpha\} < \{j\alpha\} \end{cases}.$$

Then we have that there is an element of the set

$$\{\{m(k-j)\alpha\} \mid m \in \mathbb{N}\}$$

in each of the n elements of the partition. The result follows since for every subset $(a,b) \subset [0,1)$ there are integers m, n such that

$$\left[\frac{m}{n}, \frac{m+1}{n}\right) \subset (a,b).$$

Solution 16 The result is equivalent to

$$|q\alpha - p| < \frac{1}{n_0}.$$

Taking $p = \lfloor q\alpha \rfloor$, we just need to find some $q \in \mathbb{N}$ such that

$$\{q\alpha\} < \frac{1}{n_0}.$$

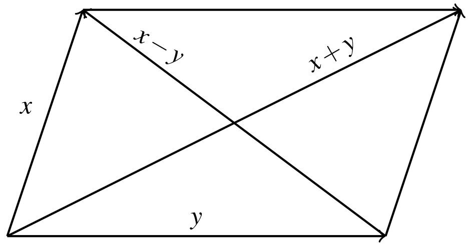
But this follows readily from the previous exercise.

Solution 17 $||x+y||^2 = ||x||^2 + x\cdot y + y\cdot x + ||y||^2 = ||x||^2 + 2\,\mathrm{Re}(x\cdot y) + ||y||^2 = ||x||^2 + ||y||^2$.

Notice that $||x+y||^2 = ||x||^2 + ||y||^2$ does not imply $x\cdot y = 0$, since $\mathrm{Re}(x\cdot y)$ can be 0 while $x\cdot y \neq 0$. Take $x = (1, 1+i)$ and $y = (1, 1-2i)$, for instance.

Solution 18 — From (13). $||x+y||^2 + ||x-y||^2 = (||x||^2 + 2\,\mathrm{Re}(x\cdot y) + ||y||^2) + (||x||^2 - 2\,\mathrm{Re}(x\cdot y) + ||y||^2) = 2||x||^2 + 2||y||^2$.

You may interpret $||x||$, $||y||$ as the lengths of the edges and $||x+y||$, $||x-y||$ as the lengths of the diagonals.



Solution 19 — Adapted from (2). Let x_1 be a element of X. Having chosen $x_1, \dots, x_{n-1}$, define x_n to be any element of X that was not already chosen. Then the set $\{x_1, x_2, x_3, \dots\}$ is a countable subset of X.

However, the axiom of choice basically means that given a set we can always choose a arbitrary element of it. Even if this seems to be obvious, it is not possible to prove it from the usual axioms of set theory.

Solution 20 Let $E = \{f_1, f_2, f_3, \cdots\}$ be a countable subset of B (of course B is infinite since we can permute the elements of $\mathbb{N}$ in an infinitely many ways). Let P be the set of all prime numbers (P could be any infinite subset of $\mathbb{N}$ such that $\mathbb{N}\setminus P$ is infinite too). We'll construct a bijection $g : \mathbb{N} \to \mathbb{N}$ in the following way:

- $g(2)$ is the smallest element of P which is not an element of $\{f_1(2)\}$.
- After defining the values of $g(2)$, $g(4)$, $\cdots$, $g(2k-2)$, define $g(2k)$ to be the smallest element of P which is not an element of $\{g(2), g(4), \cdots, g(2k-2), f_k(2k)\}$.

List now all the elements of $A = \mathbb{N}\setminus P = \{a_1, a_2, \cdots\}$ and define $g(2k-1)$ as a_k for all $k \in \mathbb{N}$.
We have then that g is an element of B which is not an element of E, since $g(2k) \neq f_k(2k)$ for all $k \in \mathbb{N}$. We conclude that every countable subset of B is proper. It follows that B is uncountable. (Otherwise B would be a proper subset of B.)

Solution 21 If $a \in A$, then $a \in f(a)$ which is a contradiction. But if $a \notin A$, then $a \notin f(a)$, which is a contradiction too. We conclude that no such function exists.

Solution 22 There are two problems in this proof. Firstly, real numbers have more than one decimal representation. For example,

$$\frac{1}{2} = 0,5 = 0,49999\dots.$$

Another flaw lies in the fact that is possible that the process described generates the number $1 = 0,9999\dots$, which is not an element of $(0,1)$.

Solution 23 Suppose B is uncountable. As $\mathbb{Q}$ is dense in $\mathbb{R}$, each element x of B contains a distinct rational number $q(x)$. The function $q : B \to q(B) \subset \mathbb{Q}$ is bijective, meaning that $q(B)$ is an uncountable subset of $\mathbb{Q}$. Since $\mathbb{Q}$ is countable, this is absurd! The result follows.

To write the bijection from the set of all (open) intervals to $\mathbb{R}^2$ just map the interval (a,b) to the element (a,b) of $\mathbb{R}^2$. (Horrible notation issues...)

2- Elements of Topology

Solution 1

Solution 2

Solution 3

Solution 4

Solution 5

Solution 6

Solution 7

Solution 8

Solution 9

Solution 10

Solution 11

Solution 12

Solution 13

Solution 14

Solution 15

Solution 16

Solution 17

Solution 18

Solution 19

Solution 20

Solution 21 Let $f(x) = d(x, \phi(x))$. Note that, by the triangular inequality

$$d(x, \phi(x)) \leq d(x, y) + d(y, \phi(y)) + d(\phi(y), \phi(x)),$$

for all $x, y \in X$. That is,

$$d(x, \phi(x)) - d(y, \phi(y)) \leq d(x, y) + d(\phi(y), \phi(x)).$$

Reversing the roles of x and y we get

$$|d(x, \phi(x)) - d(y, \phi(y))| \leq d(x, y) + d(\phi(x), \phi(y)) < 2\delta$$

whenever $d(x, y) < \delta$. This means that f is continuous.

Let $\alpha = \inf_{x \in X} f(x)$. By the extreme value theorem, we know that there is some $x_0 \in X$ such that $f(x_0) = \alpha$. If $\alpha > 0$, then

$$f(\phi(x_0)) < f(x_0) = \alpha,$$

which is absurd since α is the least value f can take. We conclude that $\alpha = 0$ and $\phi(x_0) = x_0$.

Suppose now that there is some other fixed point x_0'. Then $d(\phi(x_0), \phi(x_0')) = d(x_0, x_0')$, which implies $x_0 = x_0'$.

Solution 22 — From (13).

Solution 23

Solution 24 Let $x_1 \in X \setminus E$. Since E is a closed set, the distance from x_1 to the elements of E possesses a positive minimum d. Since $d/r > d$, there exists $p \in E$ for which $\|x_1 - p\| \leq d/r$. Define now $x = (x_1 - p)/\|x_1 - p\|$. Then,

$$\|x - y\| = \frac{\|x_1 - (p + y\|x_1 - p\|)\|}{\|x_1 - p\|} \geq \frac{r}{d}d = r.$$

Since $p + y\|x_1 - p\|$ is an element of E, the result follows.

3- Sequences and Series

Solution 1

Solution 2

Solution 3 Let s_n be the partial sums of the series. If m is even and $m < n$ we have that

$$\begin{aligned} s_n - s_m &= x_{m+1} - x_{m+2} + x_{m+3} - x_{m+4} + x_{m+5} - \cdots + (-1)^{n+1} x_n \\ &= x_{m+1} - (x_{m+2} - x_{m+3}) - (x_{m+4} - x_{m+5}) - \cdots + (-1)^{n+1} x_n \\ &\leq x_{m+1} \leq x_m. \end{aligned}$$

Similarly,

$$\begin{aligned} s_m - s_n &= -x_{m+1} + x_{m+2} - x_{m+3} + x_{m+4} - x_{m+5} + \cdots - (-1)^{n+1} x_n \\ &= -x_{m+1} + x_{m+2} - (x_{m+3} - x_{m+4}) - x_{m+5} + \cdots - (-1)^{n+1} x_n \\ &\leq -x_{m+2} + x_{m+2} \leq x_{m+2} \leq x_{m+1} \leq x_m. \end{aligned}$$

Hence $|s_n - s_m| < x_m$. Since $x_m \to 0$, the Cauchy criterion implies that (s_n) converges. The proof for m odd is analogous.

Solution 4 — **From (4).**

Solution 5 — **From (4).**

Solution 6

Solution 7

Solution 8

Solution 9

Solution 10 — **From (13).**

Solution 11 — **From (1).**

Solution 12

Solution 13

Solution 14

Solution 15

Solution 16 — **From (4).**

Solution 17 — **From (4).**

Solution 18 — **From (13).**

Solution 19 — **From (13).**

Solution 20 — **From (13).**

Solution 21 — **From (13).**

Solution 22 — **From P. Erdös in the American Mathematical Monthly.**

Solution 23

Solution 24

Solution 25

Solution 26 — **From (7).**

Solution 27

4- Functional Limits and Continuity

Solution 1

Solution 2 — From (8).

Solution 3 — From (8).

Solution 4 — From (13).

Solution 5

Solution 6

Solution 7

Solution 8

Solution 9 — From (4).

Solution 10 — From (4).

Solution 11 — From (4).

Solution 12 — From (8).

Solution 13

Solution 14

Solution 15

Solution 16

Solution 17

Solution 18

Solution 19

Solution 20

Solution 21

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