Generic Vanishing for Holonomic $\mathcal{D}$-modules

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Abstract. We construct an algebraic space parametrizing multiplicative line bundles with flat connection, known as character sheaves, on commutative algebraic groups. We then prove a generic vanishing theorem: for each holonomic $\mathcal{D}$-module, there exists a dense open subset of this space over which the de Rham cohomology of twists by the corresponding character sheaves is concentrated in degree zero. As a key ingredient, we study extensions of abelian sheaves and various incarnations of Cartier duality.

Contents

1. Introduction

1.1. Main results

Let k be a field of characteristic zero and G be a connected commutative algebraic group over k, whose group operation is denoted by $m\colon G \times G \to G$. We will say that a line bundle with flat connection $(\mathscr{L}, \nabla)$ on G is a *character sheaf* if there exists an isomorphism $m^*(\mathscr{L}, \nabla) \simeq (\mathscr{L}, \nabla) \boxtimes (\mathscr{L}, \nabla)$. This notion of character sheaves, which coincide with Grothendieck's $\natural$-extensions of G by $\mathbb{G}_m$ [Del74, §10.2.7.1], is a de Rham analogue to the rank one ℓ-adic local systems $\mathscr{L}_\chi$ over a finite field $\mathbb{F}_q$ arising from a character $\chi\colon G(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ via the Lang isogeny.

The first objective of this paper is to construct a moduli space parametrizing character sheaves on G. A key ingredient of our approach is the so-called *de Rham space* G_{dR}, an fppf sheaf with the property that line bundles with flat connection on G correspond to $\mathbb{G}_m$-torsors over G_{dR}. Consequently, given a k-scheme S, the elements of

$$H^1_m(G_{dR} \times S, \mathbb{G}_m) := \ker\left(m^* - pr_1^* - pr_2^* \colon H^1(G_{dR} \times S, \mathbb{G}_m) \to H^1(G^2_{dR} \times S, \mathbb{G}_m)\right)$$

are isomorphism classes of line bundles $\mathscr{L}$ on G_S with flat connection ∇ relative to S satisfying $m^*(\mathscr{L}, \nabla) \simeq (\mathscr{L}, \nabla) \boxtimes (\mathscr{L}, \nabla)$. Using the constructions of Subsection 2.5, we prove the following result (see Corollary 2.41):

Theorem A. *There exists a connected group algebraic space* $G^\flat$, *smooth over* k, *satisfying* $\dim G \leqslant \dim G^\flat \leqslant 2 \dim G$ *and* $G^\flat(S) \simeq H^1_m(G_{dR} \times S, \mathbb{G}_m)$ *for all seminormal* k*-schemes* S.

In the particular case of an abelian variety, every line bundle with connection is a character sheaf. This moduli space of connections has a long history and has been studied by many authors in the interim. (See, for example, [MM74; Sim93; Sch15].) By taking into account the group structure, we are able to extend the construction of such a moduli space away from the proper case.

While a separated group algebraic space is necessarily a scheme, already the example of $\mathbb{G}_m^\flat \simeq \mathbb{A}^1/\mathbb{Z}$ shows that $G^\flat$ may not be quasi-separated. Regardless of this, we may consider dense open subspaces thereof; formalizing the idea that a result holds *for most* character sheaves. Furthermore, the dimension estimate $\dim G^\flat \geqslant \dim G$ indicates that not only does such a result hold for most character sheaves, but that there exists a large number of them.

This culminates in the main theorem of this paper.

Theorem B. *Suppose that* k *is algebraically closed and let* M *be a holonomic* $\mathcal{D}$*-module over* G. *There exists a dense open subspace* V *of* $G^\flat$ *such that*

$$H^i_{dR}(G, M \otimes_{\mathcal{O}_G} (\mathscr{L}, \nabla)) = H^i_{dR,c}(G, M \otimes_{\mathcal{O}_G} (\mathscr{L}, \nabla)) = 0 \text{ for } i \neq 0;$$
$$H^0_{dR}(G, M \otimes_{\mathcal{O}_G} (\mathscr{L}, \nabla)) \simeq H^0_{dR,c}(G, M \otimes_{\mathcal{O}_G} (\mathscr{L}, \nabla))$$

for every character sheaf $(\mathscr{L}, \nabla) \in V(k)$.

Theorem B extends results of Krämer [Krä14] and Schnell [Sch15]. Krämer's work focuses on the case where G is a semiabelian variety and M is regular, while Schnell's allows a general holonomic $\mathcal{D}$-module but restricts to abelian varieties. Both results, in turn, generalize the classical generic vanishing theorems of Green and Lazarsfeld [GL87], which have manifold applications in complex geometry. We refer the reader to [Sch13] for more.

1.2. Outline

Character sheaves and their moduli

According to the Barsotti-Chevalley theorem, the connected commutative algebraic group G can be expressed as an extension of an abelian variety by a linear group. However, it is not obvious from its definition that the functor $G \mapsto H^1_m(G_{dR}, \mathbb{G}_m)$ preserves exact sequences. We address this issue by establishing a comparison between $H^1_m(G_{dR}, \mathbb{G}_m)$ and an extension group, which has good functoriality properties.

More generally, let $\mathscr{G}$ and $\mathscr{A}$ be abelian sheaves on $(\mathsf{Sch}/k)_{fppf}$ (or even abelian groups on an arbitrary Grothendieck topos). For every k-scheme S, there are natural maps

$$\underline{\mathrm{Ext}}^1(\mathscr{G}, \mathscr{A})(S) \leftarrow \mathrm{Ext}^1_S(\mathscr{G}, \mathscr{A}) \to H^1(\mathscr{G}_S, \mathscr{A}_S),$$

and the Subsection 2.2 is devoted to their study.

Proposition 2.17 and Corollary 2.19 show that the arrow on the left becomes an isomorphism in many pertinent situations. The image of the arrow on the right lies in the subgroup $H^1_m(\mathscr{G}_S, \mathscr{A}_S) \subset H^1(\mathscr{G}_S, \mathscr{A}_S)$, composed of the isomorphism classes of *multiplicative* $\mathscr{A}_S$-torsors over $\mathscr{G}_S$. Further, the Corollary 2.15 and the Proposition 2.16 say that the restriction $\mathrm{Ext}^1_S(\mathscr{G}, \mathscr{A}) \to H^1_m(\mathscr{G}_S, \mathscr{A}_S)$ is also an isomorphism in the majority of cases that pique our interest.

Given the generality of these techniques, we believe that they could be useful in other contexts as well. For example, our methods reprove some of the results in [MM74, Chapter I] and in [Ser88, Chapter VII]. Frequently, our approach not only simplifies but also generalizes these results beyond their original formulations. We refer the reader to Remarks 2.14, 2.21, and 2.22 for more details.

Returning to our original goal, denote by $G^\natural$ the abelian sheaf $\underline{\mathrm{Ext}}^1(G_{dR}, \mathbb{G}_m)$. Given a reduced k-scheme S, the results of Subsection 2.2 yield an isomorphism $G^\natural(S) \simeq H^1_m(G_{dR} \times S, \mathbb{G}_m)$. The sheaf $G^\natural$ coincides with a stack-theoretic Cartier dual $\underline{\mathrm{Hom}}(G_{dR}, B\mathbb{G}_m)$. In the Subsection 2.4, we explain this operation and compare it with the Cartier duality for 1-motives, as studied by Deligne [Del74] and Laumon [Lau96]. Somewhat surprisingly, this comparison gives a rather explicit description for $G^\natural$ in Proposition 2.37.

The abelian sheaf $G^\natural$ usually fails to be representable. Nevertheless, in the Subsection 2.5, we use the description above to isolate precisely the problematic constituents of $G^\natural$ (which behave like formal schemes) and define a variant $G^\flat$ akin to a coarse moduli space. Finally, the Theorem A follows from Theorem 2.40 and its Corollary 2.41.

Generic vanishing

Here we suppose that k is algebraically closed and we switch to $\mathcal{D}$-module notations. Denote by $\mathscr{L}_\alpha$ the character sheaf corresponding to a point $\alpha \in G^\flat(k) \simeq G^\natural(k)$ in degree $\dim G$. We refer the reader to the beginning of Section 3 for an explanation of our choice of notation.

As before, the fact that G is an extension of an abelian variety by a linear group motivates us to consider a relative version of the generic vanishing theorem. Namely, we say that G *satisfies relative generic vanishing* if, for every smooth variety S over k and every object M of $D^b_h(\mathcal{D}_{G\times S})$, there exists a dense open subspace V of $G^\flat$ such that the forget-supports map

$$\mathrm{pr}_{S,!}(M\otimes_{G\times S}\mathrm{pr}_G^+\mathscr{L}_\alpha)\to\mathrm{pr}_{S,+}(M\otimes_{G\times S}\mathrm{pr}_G^+\mathscr{L}_\alpha),$$

where $\mathrm{pr}_S\colon G\times S\to S$ and $\mathrm{pr}_G\colon G\times S\to G$ are the projections, is an isomorphism for every $\alpha\in V(k)$.

When G is affine, the $\mathcal{D}$-module analogue of Artin vanishing says that $\mathrm{pr}_{S,!}$ is left t-exact and $\mathrm{pr}_{S,+}$ is right t-exact with respect to the standard t-structures. In particular, if M is concentrated in degree zero and the forget supports map above is an isomorphism, then

$$\mathrm{pr}_{S,!}(M\otimes_{G\times S}\mathrm{pr}_G^+\mathscr{L}_\alpha)\simeq\mathrm{pr}_{S,+}(M\otimes_{G\times S}\mathrm{pr}_G^+\mathscr{L}_\alpha)$$

is also concentrated in degree zero; generalizing the generic vanishing theorem.

The Subsection 3.1 *dévissages* this statement and proves, in Proposition 3.7, that relative generic vanishing for an affine group G follows from the particular case in which G has dimension one and $S=\mathrm{Spec}\,k$. This leaves the cases $G=\mathbb{G}_a$ and $G=\mathbb{G}_m$ to be treated.

As explained in Subsection 3.2, the unipotent case is intrinsically related to the Fourier transform for $\mathcal{D}$-modules. Indeed, given a unipotent group U and an object M of $D^b_h(\mathcal{D}_U)$, the Fourier transform $\mathrm{FT}_U(M)\in D^b_h(\mathcal{D}_{U^\flat})$ contains all the data of the cohomology groups $H^i(U,M\otimes_U\mathscr{L}_\alpha)$ and $H^i_c(U,M\otimes_U\mathscr{L}_\alpha)$ for every $i\in\mathbb{Z}$ and $\alpha\in U^\flat(k)$. The generic vanishing theorem then follows from the general properties of the Fourier transform.

Even though there exists a multiplicative analogue of the Fourier transform for tori, it is much less understood.[1] Consequently, we were inspired by [KL85] to compactify $\mathbb{G}_m$ and use monodromical arguments to obtain the generic vanishing theorem in Subsection 3.3. Our main tool is the V-filtration of Kashiwara, Malgrange, and Sabbah, which is a generalization of nearby and vanishing cycles to holonomic $\mathcal{D}$-modules.

Next, we describe Schnell's study of holonomic $\mathcal{D}$-modules on abelian varieties in Subsection 3.4. Similarly to the unipotent case, the key idea is to consider the Fourier-Mukai transform $\mathrm{FM}_A\colon D^b_h(\mathcal{D}_A)\to D^b_{coh}(\mathcal{O}_{A^\natural})$ defined by Laumon [Lau85; Lau96]. Here the generic vanishing theorem is encoded in the fact that FM_A is t-exact with respect to a perverse t-structure on the target [Sch15, Thm. 19.1].

Finally, the Subsection 3.5 combines the relative generic vanishing theorem for affine groups described above with Schnell's result for abelian varieties; finishing the proof of Theorem B.

[1] As Katz would say, *the miracle of the Fourier transform* is that it coincides with a variant with compact supports. This does not even make sense for the Mellin transform of $\mathcal{D}$-modules and it is false for its ℓ-adic variant [GL96].

1.3. Related and future work

Ever since Deligne's second paper on the Weil conjectures [Del80], it has been known that ℓ-adic methods can be used to understand a large class of exponential sums. A systematic study of these ideas was done in the recent book [FFK23] by Forey, Fresán and Kowalski, wherein the authors prove equidistribution results for exponential sums of the form

$$\sum_{x \in G(\mathbb{F}_q)} \mathrm{tr}_M(x)\, \chi(x),$$

where G is a commutative connected algebraic group over a finite field $\mathbb{F}_q$, χ is a character of $G(\mathbb{F}_q)$, and M is an ℓ-adic perverse sheaf over G whose trace function is tr_M.

As the Grothendieck trace formula relates the exponential sum above to traces of Frobenius acting on $R\Gamma(G, M \otimes \mathscr{L}_\chi)$, the fundamental theorem at the heart of the results presented in [FFK23] is a generic vanishing theorem for the ℓ-adic cohomology of perverse sheaves M on commutative algebraic groups [FFK23, Thm. 2.1], which is a finite-field analogue of our Theorem B.

In both settings, we say that a holonomic $\mathcal{D}$-module (resp. perverse sheaf) M is *negligible* if $H^0(G, M \otimes \mathscr{L})$ vanishes for most character sheaves $\mathscr{L}$. Those objects form a thick subcategory of $\mathsf{Hol}(\mathcal{D}_G)$ (resp. of $\mathsf{Perv}(G)$), and the generic vanishing theorems imply that the quotient is tannakian under convolution. (See [FFK23, Chapter 3] for more details.)

As a consequence of this, every holonomic $\mathcal{D}$-module (resp. perverse sheaf) has an associated tannakian group. In the ℓ-adic context, these groups dictate the distribution of the associated exponential sums. On the other hand, their de Rham analogues are rather explicit. In a future work, we will explain that they coincide with certain differential and difference Galois groups over unipotent groups and tori. Krämer also has an interesting description of those tannakian groups for abelian varieties [Krä22, Thm. 3.2].

The relations between these tannakian groups, when G is the additive group $\mathbb{G}_a$, constitute the crowning achievement of Katz's *tour de force* [Kat90]. We hope to extend these results to more general groups G and sheaves M in the future.

1.4. Notation and conventions

Throughout this paper, we will denote by k a base field and by G a connected commutative algebraic group (a group scheme of finite type) over k. By the Barsotti-Chevalley theorem [Mil17, Thm. 8.28], such a group fits into a short exact sequence

$$0 \to L \xrightarrow{\varphi} G \xrightarrow{\psi} A \to 0,$$

where L is a (not necessarily smooth) connected linear subgroup of G and A is an abelian variety. Since $\psi\colon G \to A$ is the Albanese map, this decomposition is unique up to a unique isomorphism. If k is perfect and G is smooth, L is a product $T \times U$ of a torus T and a unipotent group U [Mil17, Cor. 16.15].

Remark that, whenever k has characteristic zero, G is automatically smooth and U is necessarily a vector group. We will systematically denote by $m \colon G \times G \to G$ the group operation, by $p \colon G \to \operatorname{Spec} k$ the structure map, and by $e \in G(k)$ the identity element. Finally, every algebraic and formal group in this paper is supposed to be commutative.

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2. Character sheaves and their moduli

In this section we will study line bundles with integrable connection $(\mathscr{L}, \nabla)$ on G satisfying $m^*(\mathscr{L}, \nabla) \simeq (\mathscr{L}, \nabla) \boxtimes (\mathscr{L}, \nabla)$. We call them *character sheaves*. They are de Rham analogs to rank one ℓ-adic local systems $\mathscr{L}_\chi$ over a finite field $\mathbb{F}_q$ arising from a character $\chi \colon G(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ via the Lang isogeny.

Henceforth, we will see every algebraic group as an fppf sheaf on Sch/k. Given such a group G, we can consider its formal completion at the identity $\widehat{G}$, which is naturally a subsheaf of G. In characteristic zero, the quotient $G/\widehat{G}$ is the so-called *de Rham space* G_{dR} of G, and it has a remarkable property: line bundles over G_{dR} are the same as line bundles with integrable connection over G. (See the Appendix A for more on these objects.)

After the computation of some Cartier duals, we show in Subsection 2.2 that the abelian sheaf $G^\natural := \underline{\operatorname{Ext}}^1(G_{\mathrm{dR}}, \mathbb{G}_m)$ classifies character sheaves on G. Then, in Subsection 2.4, we define a stacky version of Cartier duality and we compare it with a dual defined by Laumon in [Lau96]. This comparison finally allows the construction in Subsection 2.5 of a variant $G^\flat$ of $G^\natural$, akin to a coarse moduli space, that happens to be representable by an algebraic space.

2.1. Cartier duality

Given an abelian sheaf $\mathscr{G}$, we denote its *Cartier dual* $\underline{\operatorname{Hom}}(\mathscr{G}, \mathbb{G}_m)$ by $\mathscr{G}^D$. If $G = \operatorname{Spec} R$ is an affine group scheme over k, its Cartier dual G^D is represented by the formal group $\operatorname{Spf} R^*$, where R^* is the dual Hopf algebra. Moreover, the double dual $(G^D)^D$ is naturally isomorphic to G. [SGA3.I, Exposé $\mathrm{VII_B}$]

As is customary, we will denote the Cartier dual of a torus T by X. It is representable by a group scheme étale-locally isomorphic to $\mathbb{Z}^r$ for some r [SGA3.II, Exposé X, Cor. 5.7.(i)]. Dually, the Cartier dual of X is T.

Unless explicitly stated otherwise, the base field k is supposed to have characteristic zero for the next two subsections. Recall that a unipotent group U over a characteristic zero field k is necessarily a vector group, and so we denote by U^* its vector space dual.

Proposition 2.1. *The Cartier dual of A vanishes and the Cartier dual of U is isomorphic to $\widehat{U^*}$. Dually, the Cartier dual of $\widehat{U}$ is isomorphic to U^*.*

Proof. Let S be a k-scheme. By the universal property of the global spectrum, a morphism of schemes $A_S \to \mathbb{G}_{m,S}$ over S is the same as a morphism of $\mathcal{O}_S$-algebras

$$\mathcal{O}_S[t, t^{-1}] \to p_*\mathcal{O}_{A_S} \simeq \mathcal{O}_S,$$

where $p\colon A_S \to S$ is the structure map [Stacks, Tag 0E0L]. In particular, such a morphism is constant. If it is a morphism of groups, it has to be trivial. This proves that A^D vanishes. Finally, the computation $U^D \simeq \widehat{U^*}$ follows from the fact that the dual of $\mathrm{Sym}(U^*)$ is the completion of $\mathrm{Sym}(U)$ at the ideal of degree one elements. (Upon a choice of basis, this is nothing but the isomorphism $k[x_1, \dots, x_n]^* \simeq k[\![x_1, \dots, x_n]\!]$.) □

Given a k-algebra R, we have that $\widehat{\mathbb{G}}_a(R)$ is the group of nilpotent elements in R and $\widehat{\mathbb{G}}_m(R)$ is that of unipotent elements. We remark that the formal groups $\widehat{\mathbb{G}}_a$ and $\widehat{\mathbb{G}}_m$ are isomorphic via the map

$$\begin{aligned} \widehat{\mathbb{G}}_m(R) &\to \widehat{\mathbb{G}}_a(R) \\ 1 + x &\mapsto \log(1 + x). \end{aligned}$$

This phenomenon is a general property of formal groups in characteristic zero, and it simplifies their study. (In positive characteristic, divided powers come on the scene, and give rise to the group scheme $\mathbb{G}_a^\sharp$[2]; the crystalline analog of $\widehat{\mathbb{G}}_a$ [Dri22].)

Proposition 2.2 (Cartier)**.** *Let $\mathfrak{g}$ be the Lie algebra of G, seen as a vector group. The formal completions of G and of $\mathfrak{g}$ along the identity coincide.*

Proof. Since an algebraic group and its formal completion share the same Lie algebra, the composition

$$\left\{ \begin{matrix} \text{Algebraic} \\ \text{groups over } k \end{matrix} \right\} \xrightarrow{\widehat{(-)}} \left\{ \begin{matrix} \text{Infinitesimal formal} \\ \text{groups over } k \end{matrix} \right\} \xrightarrow{\mathrm{Lie}(-)} \left\{ \begin{matrix} \text{Lie algebras} \\ \text{over } k \end{matrix} \right\}$$

is the functor associating an algebraic group to its Lie algebra. In particular, G and $\mathfrak{g}$, seen as a vector group, have the same image by the composition above. Now, by [SGA3.I, Exposé VIIB, Cor. 3.3.2], the functor on the right is an equivalence of categories. In particular, G and $\mathfrak{g}$ have isomorphic formal completions. □

[2]Not the same as the abelian sheaf $\mathbb{G}_a^\natural$ that appears in this paper.

The previous two propositions allow us to compute the Cartier dual of the formal completion $\widehat{G}$.

Corollary 2.3. *The Cartier dual of $\widehat{G}$ is naturally isomorphic to $\mathfrak{g}^*$. This also coincides with the* invariant *differentials Ω_G of G.*

We are now in position to compute the Cartier dual of the de Rham space G_{dR}.

Proposition 2.4. *The Cartier dual of G_{dR} vanishes.*

Proof. Since $\underline{\mathrm{Hom}}(-, \mathbb{G}_m)$ is left-exact, the vanishing of G_{dR}^D is equivalent to the morphism $\underline{\mathrm{Hom}}(G, \mathbb{G}_m) \to \underline{\mathrm{Hom}}(\widehat{G}, \mathbb{G}_m)$ being a monomorphism. We check this in some particular cases. For abelian varieties, this holds because their Cartier dual vanishes. For a unipotent group U, the morphism in question is isomorphic to $\widehat{U^*} \to U^*$, which is also monic. (Corollary A.5.) Finally, for a torus T, this morphism is isomorphic to $d\log \colon X \to \Omega_T$ and this is surely monic.

The de Rham functor $(-)_{dR}$ is exact, and so G_{dR} is an extension of $T_{dR} \times U_{dR}$ by A_{dR}. In particular, we have an induced exact sequence

$$0 \to \underline{\mathrm{Hom}}(A_{dR}, \mathbb{G}_m) \to \underline{\mathrm{Hom}}(G_{dR}, \mathbb{G}_m) \to \underline{\mathrm{Hom}}(T_{dR}, \mathbb{G}_m) \times \underline{\mathrm{Hom}}(U_{dR}, \mathbb{G}_m).$$

The cases considered above then imply the general result. □

2.2. Extensions by the multiplicative group

We will state our next circle of ideas, which follows ideas of Breen, Deligne, Clausen, and Scholze, in a higher level of generality than strictly needed in the hopes that it may be useful elsewhere.

The following proposition was hinted by Grothendieck in [SGA7.I, Exposé VII, Remarque 3.5.4] and a variant of it was constructed by Breen [Bre69]. The version below is a unpublished result of Deligne and a proof, by Clausen and Scholze, can be found in [Sch19, Thm. 4.10].

Proposition 2.5 (Breen-Deligne resolution)**.** *Let $\mathscr{G}$ be an abelian group in a Grothendieck topos E. There exists a functorial resolution of the form*

$$\cdots \to \bigoplus_{j=1}^{n_i} \mathbb{Z}[\mathscr{G}^{r_{i,j}}] \to \cdots \to \mathbb{Z}[\mathscr{G}^3] \oplus \mathbb{Z}[\mathscr{G}^2] \to \mathbb{Z}[\mathscr{G}^2] \to \mathbb{Z}[\mathscr{G}] \to \mathscr{G},$$

where the n_i and $r_{i,j}$ are all positive integers.

Clausen and Scholze's proof of the proposition above makes clear that the first terms of the resolution can be chosen as in [BBM82, §2.1.5]. In particular, this explicit description allows us to define two important invariants.

Definition 2.6. Let $\mathscr{G}$ and $\mathscr{A}$ be abelian groups in a Grothendieck topos E. Applying the functor $\mathrm{Hom}_{\mathsf{E}}(-,\mathscr{A})$ to the Breen-Deligne resolution of $\mathscr{G}$, we obtain the complex

$$\Gamma(\mathscr{G},\mathscr{A}) \to \Gamma(\mathscr{G}^2,\mathscr{A}) \to \Gamma(\mathscr{G}^3,\mathscr{A}) \oplus \Gamma(\mathscr{G}^2,\mathscr{A}) \to \Gamma(\mathscr{G}^4,\mathscr{A}) \oplus \Gamma(\mathscr{G}^3,\mathscr{A}) \oplus \Gamma(\mathscr{G}^3,\mathscr{A}) \oplus \Gamma(\mathscr{G}^2,\mathscr{A}) \oplus \Gamma(\mathscr{G},\mathscr{A}),$$

where $\Gamma(-,-) := \mathrm{Hom}_{\mathsf{E}}(-,-)$. We denote the first cohomology of this complex by $\mathrm{H}^2_s(\mathscr{G},\mathscr{A})$ and the second cohomology by $\mathrm{H}^3_s(\mathscr{G},\mathscr{A})$.

Remark 2.7. Even though $\mathscr{G}$ and $\mathscr{A}$ are abelian groups, the morphisms in $\mathrm{Hom}_{\mathsf{E}}(\mathscr{G},\mathscr{A})$ are those in the topos E. In other words, they need not be morphisms of groups. Moreover, $\mathrm{H}^2_s(\mathscr{G},\mathscr{A})$ is usually known as the symmetric subgroup of the second Hochschild cohomology group [Mil17, Chapter 15]. The (non-standard) notation $\mathrm{H}^3_s(\mathscr{G},\mathscr{A})$ indicates that this group is, in some sense, a variant of the third Hochschild cohomology which is better adapted to commutative groups.

Let us recall the definition of cohomology for some object X in a Grothendieck topos E. The cohomology groups $\mathrm{H}^i(X,-)$ are defined as the right derived functors of $\Gamma(X,-) := \mathrm{Hom}_{\mathsf{E}}(X,-)\colon \mathsf{Ab}(\mathsf{E}) \to \mathsf{Ab}$. As usual, given $\mathscr{A} \in \mathsf{Ab}(\mathsf{E})$, the group $\mathrm{H}^1(X,\mathscr{A})$ classifies $\mathscr{A}$-torsors over X, where the group operation on the latter is the contracted product [Gir71, §§III.2.4.2, III.2.4.5, III.3.5.4]. Moreover, a morphism $f\colon Y \to X$ in E induces a morphism of groups $f^*\colon \mathrm{H}^1(X,\mathscr{A}) \to \mathrm{H}^1(Y,\mathscr{A})$. This map sends a $\mathscr{A}$-torsor $P \to X$ to the pullback $f^*P \to Y$ [Gir71, §V.1.5.3].

Proposition 2.8 (Breen). *Let $\mathscr{G}$ and $\mathscr{A}$ be abelian groups in a Grothendieck topos E. Denote by $m\colon \mathscr{G}\times\mathscr{G} \to \mathscr{G}$ the group operation of $\mathscr{G}$. There exists a functorial exact sequence*

$$0 \to \mathrm{H}^2_s(\mathscr{G},\mathscr{A}) \to \mathrm{Ext}^1(\mathscr{G},\mathscr{A}) \to \mathrm{H}^1_m(\mathscr{G},\mathscr{A}) \to \mathrm{H}^3_s(\mathscr{G},\mathscr{A}) \to \mathrm{Ext}^2(\mathscr{G},\mathscr{A}),$$

where $\mathrm{H}^1_m(\mathscr{G},\mathscr{A})$ is the group of isomorphism classes of $\mathscr{A}$-torsors P over $\mathscr{G}$ satisfying $\mathrm{pr}_1^* P \wedge \mathrm{pr}_2^* P \simeq m^* P$.

Before delving into the proof, let us explain the morphism $\mathrm{Ext}^1(\mathscr{G},\mathscr{A}) \to \mathrm{H}^1_m(\mathscr{G},\mathscr{A})$. Since $\mathrm{Ext}^1(-,-)$ is an additive bifunctor, an extension $\mathscr{E}$ of $\mathscr{G}$ by $\mathscr{A}$ always satisfies

$$m^*\mathscr{E} = (\mathrm{pr}_1 + \mathrm{pr}_2)^*\mathscr{E} \simeq \mathrm{pr}_1^*\mathscr{E} + \mathrm{pr}_2^*\mathscr{E},$$

where the sum on the right is the Baer sum of extensions. Such an extension defines a $\mathscr{A}$-torsor over $\mathscr{G}$, which satisfies $\mathrm{pr}_1^*\mathscr{E} \wedge \mathrm{pr}_2^*\mathscr{E} \simeq m^*\mathscr{E}$.

Proof of Proposition 2.8. The universal property of free objects gives that $\mathrm{R}\Gamma(X,-) \simeq \mathrm{RHom}_{\mathsf{Ab}(\mathsf{E})}(\mathbb{Z}[X],-)$. In particular, the Breen-Deligne resolution yields an spectral sequence

$$E_1^{i,j}\colon \prod_{r=1}^{n_i} \mathrm{H}^j(\mathscr{G}^{r_{i,r}},\mathscr{A}) \implies \mathrm{Ext}^{i+j}(\mathscr{G},\mathscr{A}),$$

whose five-term exact sequence is precisely the one in the statement. □

Remark 2.9. In [SGA7.I, Exposé VII, §1.2], Grothendieck proved that the category of extensions of $\mathscr{G}$ by $\mathscr{A}$ is equivalent to a category of pairs (P, α), where P is a $\mathscr{A}$-torsor over $\mathscr{G}$ and $\alpha\colon \operatorname{pr}_1^* P \wedge \operatorname{pr}_2^* P \to m^*P$ is an isomorphism of $\mathscr{A}$-torsors over $\mathscr{G} \times \mathscr{G}$ making two diagrams (imposing that P admits an associative and commutative group law) commute. In particular, $\operatorname{Ext}^1(\mathscr{G}, \mathscr{A})$ is isomorphic to the group of isomorphism classes of such pairs, and our invariants $H^2_s(\mathscr{G}, \mathscr{A})$ and $H^3_s(\mathscr{G}, \mathscr{A})$ govern how far the map

$$\begin{aligned} \operatorname{Ext}^1(\mathscr{G}, \mathscr{A}) &\to H^1_m(\mathscr{G}, \mathscr{A}) \\ [P, \alpha] &\mapsto [P] \end{aligned}$$

is from being an isomorphism.

Even though the first terms of the Breen-Deligne resolution are explicit, the invariants $H^2_s(\mathscr{G}, \mathscr{A})$ and $H^3_s(\mathscr{G}, \mathscr{A})$ are usually quite hard to compute. Somewhat surprisingly, the following observation will suffice for their computations in many interesting cases.

Lemma 2.10. *If every morphism $\mathscr{G}^n \to \mathscr{A}$ is constant, then both $H^2_s(\mathscr{G}, \mathscr{A})$ and $H^3_s(\mathscr{G}, \mathscr{A})$ vanish. In particular, this implies that $\operatorname{Ext}^1(\mathscr{G}, \mathscr{A}) \simeq H^1_m(\mathscr{G}, \mathscr{A})$.*

Proof. Firstly, let us prove that $H^2_s(\mathscr{G}, \mathscr{A})$ vanishes. The kernel of

$$\Gamma(\mathscr{G}^2, \mathscr{A}) \to \Gamma(\mathscr{G}^3, \mathscr{A}) \oplus \Gamma(\mathscr{G}^2, \mathscr{A})$$

consists of maps $f\colon \mathscr{G}^2 \to \mathscr{A}$ satisfying $f(x+y, z) - f(y, z) = f(x, y+z) - f(x, y)$ and $f(x, y) = f(y, x)$. These conditions are tautological for constant morphisms. Now, the image of

$$\Gamma(\mathscr{G}, \mathscr{A}) \to \Gamma(\mathscr{G}^2, \mathscr{A})$$

consists of maps of the form $(x, y) \mapsto g(x+y) - g(x) - g(y)$, for some $g\colon \mathscr{G} \to \mathscr{A}$. And every constant map is also of this form. In particular, the cohomology vanishes.

Similarly, if every morphism $\mathscr{G}^n \to \mathscr{A}$ is constant, the map

$$\Gamma(\mathscr{G}^3, \mathscr{A}) \oplus \Gamma(\mathscr{G}^2, \mathscr{A}) \to \Gamma(\mathscr{G}^4, \mathscr{A}) \oplus \Gamma(\mathscr{G}^3, \mathscr{A}) \oplus \Gamma(\mathscr{G}^3, \mathscr{A}) \oplus \Gamma(\mathscr{G}^2, \mathscr{A}) \oplus \Gamma(\mathscr{G}, \mathscr{A})$$

acts as

$$(a, b) \mapsto (-a, -a-b, a-b, 2b, b).$$

We conclude that its kernel vanishes and so does $H^3_s(\mathscr{G}, \mathscr{A})$. □

In order to profit from this machinery, let us go back to the setting where E is the category of sheaves on $(\mathsf{Sch}/k)_{\mathrm{fppf}}$, or a localization thereof, and $\mathscr{A} = \mathbb{G}_m$. (Recall that we follow the notations from Subsection 1.4.) Under these hypotheses, we can simplify the preceding lemma even further. The result below is an application of the generalized Rosenlicht's lemma of González-Avilés [Gon17, Thm. 1.1].

Lemma 2.11 (Rosenlicht's lemma)**.** *Let G and H be connected algebraic groups over k, and let S be a reduced k-scheme. Any morphism of S-schemes $G_S \times_S H_S \to \mathbb{G}_{m,S}$ is a product of morphisms of S-schemes $G_S \to \mathbb{G}_{m,S}$ and $H_S \to \mathbb{G}_{m,S}$. Similarly, any morphism of sheaves $(G_{dR} \times H_{dR})_S \to \mathbb{G}_{m,S}$ over S is a product of morphisms $G_{dR} \times S \to \mathbb{G}_{m,S}$ and $H_{dR} \times S \to \mathbb{G}_{m,S}$ over S.*

Proof. The first statement follows from [Gon17, Thm. 1.1], so let us verify that the needed hypotheses hold. Since $G \to \operatorname{Spec} k$ is of finite type and k is noetherian, $G \to \operatorname{Spec} k$ is also of finite presentation. The morphism $G \to \operatorname{Spec} k$ is also smooth and clearly surjective. All these properties are stable under base change, so $G_S \to S$ is faithfully flat and of finite presentation.

By [Mil17, Cor. 1.32], G is geometrically connected. Also, by [Stacks, Tags 056T and 020I], G is geometrically reduced. In particular, if η is a generic point of an irreducible component of S, the scheme $G_S \times_S \operatorname{Spec} \overline{\kappa(\eta)} \simeq G \times_k \operatorname{Spec} \overline{\kappa(\eta)}$ is connected and reduced.

Of course $H_S \to S$ satisfies the same properties. Also, $G_S \times_S H_S \to H_S \to S$ is a composition of smooth surjective morphisms and so is smooth and surjective. (By [EGA IV.4, Cor. 17.16.3(ii)], this implies that it has an étale quasi-section.)

The same result holds for the de Rham spaces because the natural map $G \times H \to (G \times H)_{dR} \simeq G_{dR} \times H_{dR}$ is an epimorphism of sheaves. Indeed, since epimorphisms in topoi are stable under base change, the map $G_S \times_S H_S \simeq (G \times H)_S \to (G \times H)_{dR} \times S \simeq (G_{dR} \times H_{dR})_S$ is also epic. Then, given a map $(G_{dR} \times H_{dR})_S \to \mathbb{G}_{m,S}$, we apply the first result to the composition $G_S \times_S H_S \twoheadrightarrow (G_{dR} \times H_{dR})_S \to \mathbb{G}_{m,S}$. □

Proposition 2.12. *Let S be a reduced k-scheme. Then every morphism of sheaves $L^n_{dR} \times S \to \mathbb{G}_{m,S}$ over S is constant.*

Proof. Without loss of generality, we may assume that $S = \operatorname{Spec} R$ is affine and connected, and that L is a product of a unipotent group and a split torus. Moreover, by Rosenlicht's lemma, it suffices to consider L to be $\mathbb{G}_a$ or $\mathbb{G}_m$, and $n = 1$.

Recall that we have a short exact sequence $0 \to \widehat{L} \times S \to L_S \to L_{dR} \times S \to 0$ of abelian sheaves over S,[3] and so the universal property of quotients (of sets) says that the morphisms $L_{dR} \times S \to \mathbb{G}_{m,S}$ correspond to maps $L_S \to \mathbb{G}_{m,S}$ which are constant on the orbits of $\widehat{L} \times S$.

For $L = \mathbb{G}_a$, the reducedness of R gives that every morphism $\mathbb{G}_{a,R} \to \mathbb{G}_{m,R}$ is already constant, yielding the result. Now, a map $\mathbb{G}_{m,R} \to \mathbb{G}_{m,R}$ is necessarily of the form $x \mapsto ax^n$ for some $a \in R^\times$ and $n \in \mathbb{Z}$. The previous paragraph says that such a map descends to the quotient if and only if, for every R-algebra B, the morphism

$$\begin{aligned} f\colon B^\times &\to B^\times \\ x &\mapsto ax^n \end{aligned}$$

[3] We remind the reader that localization of topoi is exact [SGA4.I, Exposé IV, §5.2].

satisfies $f(xu) = f(x)$ for all $x \in B^\times$ and every unipotent $u \in B$. In particular, it has to send every such u to $f(1) = a$. Taking $r > |n|$, $B = R[z]/(z-1)^r$, and $u = z \in \mathrm{Uni}(B)$, we obtain that $n = 0$. This finishes the proof. □

Proposition 2.13. *Let S be a k-scheme. Then every morphism of S-schemes $A^n_S \to \mathbb{G}_{m,S}$ is constant and every morphism of sheaves $A^n_{dR} \times S \to \mathbb{G}_{m,S}$ over S is constant.*

Proof. The computation in the proof of Proposition 2.1 gives the first statement. Now, if $A^n_{dR} \times S \to \mathbb{G}_{m,S}$ is a morphism of sheaves over S, it follows that the composition $A^n_S \to A^n_{dR} \times S \to \mathbb{G}_{m,S}$ is constant. Since epimorphisms are stable under base change in topoi, the map $A^n_S \to A^n_{dR} \times S$ is an epimorphism; proving that $A^n_{dR} \times S \to \mathbb{G}_{m,S}$ has to be constant. □

Remark 2.14. The multiplicative group did not play an important part in the previous proposition. If A is an abelian scheme over S and B is a connected group scheme affine over S, the same proof shows that any morphism $A^n \to B$ is constant. This implies that the natural map

$$\mathrm{Ext}^1_S(A, B) \to H^1_m(A, B)$$

is an isomorphism. This reproves and generalizes [Ser88, Thm. VII.5]. We also observe that the local-to-global spectral sequence, as in the proof of Proposition 2.17, implies that the sheafification map

$$\mathrm{Ext}^1_T(A, B) \to \underline{\mathrm{Ext}}^1(A, B)(T)$$

is an isomorphism for every S-scheme T. Many other results in [Ser88, Chapter VII] can be generalized using these techniques.

Corollary 2.15. *Let S be a reduced k-scheme. Then every morphism of sheaves $G^n_{dR} \times S \to \mathbb{G}_{m,S}$ over S is constant.*

Proof. The group G is an extension of an abelian variety A by a linear group L. In particular, we obtain a short exact sequence of abelian sheaves over S

$$0 \to L^n_{dR} \times S \to G^n_{dR} \times S \to A^n_{dR} \times S \to 0.$$

Now, the universal property of quotients (of sets) says that a morphism $f\colon G^n_{dR} \times S \to \mathbb{G}_{m,S}$ factors through $G^n_{dR} \times S \to A^n_{dR} \times S$ if and only if it is constant on the orbits of $L^n_{dR} \times S$. In other words, it factors precisely if for every local section g of $G^n_{dR} \times S$ the map

$$\begin{aligned} L^n_{dR} \times S &\to \mathbb{G}_{m,S} \\ x &\mapsto f(x+g) \end{aligned}$$

is constant. By Proposition 2.12, this condition is tautological and so we can always factor $G^n_{dR} \times S \to \mathbb{G}_{m,S}$ as

$$G^n_{dR} \times S \to A^n_{dR} \times S \to \mathbb{G}_{m,S}.$$

Proposition 2.13 then finishes the proof. □

All in all, we obtained that the natural map $\mathrm{Ext}^1_S(G_{dR}, \mathbb{G}_m) \to H^1_m(G_{dR} \times S, \mathbb{G}_{m,S})$ is an isomorphism for reduced k-schemes S. The analogous result for G in the place of G_{dR} is a straightforward generalization of a result of Colliot-Thélène.

Proposition 2.16 (Colliot-Thélène). *Let S be a reduced k-scheme. The natural map*

$$\mathrm{Ext}^1_S(G, \mathbb{G}_m) \to H^1_m(G_S, \mathbb{G}_{m,S})$$

is an isomorphism.

Proof. The reader can find a proof, due to Gabber, of injectivity on [Col08, Prop. 3.2] and a proof of surjectivity on [Col08, Thm. 5.6]. In both cases the result was only proved for $S = \mathrm{Spec}\, k$, but the same arguments work if we replace the classical Rosenlicht's lemma by Lemma 2.11. □

Let $\mathscr{G}$ be an abelian sheaf on $(\mathsf{Sch}/k)_{\mathrm{fppf}}$. As $\underline{\mathrm{Ext}}^1(\mathscr{G}, \mathbb{G}_m)$ is the sheafification of the presheaf $S \mapsto \mathrm{Ext}^1_S(\mathscr{G}, \mathbb{G}_m)$, there is a natural morphism of groups

$$\mathrm{Ext}^1_S(\mathscr{G}, \mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(\mathscr{G}, \mathbb{G}_m)(S)$$

functorial on $\mathscr{G}$ and on k-schemes S. The next results establish that this map is an isomorphism in many interesting situations.

Proposition 2.17. *The sheafification map $\mathrm{Ext}^1_S(G_{dR}, \mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(G_{dR}, \mathbb{G}_m)(S)$ is always an isomorphism. Moreover, $\mathrm{Ext}^1_S(G, \mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(G, \mathbb{G}_m)(S)$ is an isomorphism for all S when G is an abelian variety, for reduced S when G is unipotent, and for irreducible geometrically unibranch S when G is a torus.*

Proof. Recall that the group of sections over S of $\mathscr{G}^D = \underline{\mathrm{Hom}}(\mathscr{G}, \mathbb{G}_m)$ is $\mathrm{Hom}_S(\mathscr{G}, \mathbb{G}_m)$. In particular there is a Grothendieck spectral sequence (usually called *local-to-global spectral sequence*) inducing the following exact sequence

$$0 \to H^1(S, \mathscr{G}^D) \to \mathrm{Ext}^1_S(\mathscr{G}, \mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(\mathscr{G}, \mathbb{G}_m)(S) \to H^2(S, \mathscr{G}^D).$$

In particular, the vanishing of the Cartier dual $\mathscr{G}^D$ (which holds for abelian varieties and de Rham spaces) implies that the map $\mathrm{Ext}^1_S(\mathscr{G}, \mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(\mathscr{G}, \mathbb{G}_m)(S)$ is an isomorphism for all S.

Let us consider the remaining cases. The unipotent case reduces to $G = \mathbb{G}_a$, and we claim that $H^i(S, \widehat{\mathbb{G}}_a)$ vanishes for all i when S is reduced. Firstly, recall that $\widehat{\mathbb{G}}_a(\mathrm{Spec}\, R)$ is the nilradical of R. In particular, the sheaf condition implies that $\widehat{\mathbb{G}}_a(S)$ vanishes for reduced S. Now, the cohomology $H^i_{\text{ét}}(S, \widehat{\mathbb{G}}_a)$ can be computed on the small étale site of S and [Stacks, Tag 03PC.(8)] implies that the restriction of $\widehat{\mathbb{G}}_a$ to this site vanishes.

As $\mathbb{G}_{a,dR}$ is the presheaf quotient of $\mathbb{G}_a$ by $\widehat{\mathbb{G}}_a$, we have the following morphism of long exact sequences.

$$\begin{array}{ccccccccccc} 0 & \to & \Gamma(S,\widehat{\mathbb{G}}_a) & \to & \Gamma(S,\mathbb{G}_a) & \to & \Gamma(S,\mathbb{G}_{a,dR}) & \to & H^1_{fppf}(S,\widehat{\mathbb{G}}_a) & \to & H^1_{fppf}(S,\mathbb{G}_a) & \to & \cdots \\ & & \| & & \| & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \Gamma(S,\widehat{\mathbb{G}}_a) & \to & \Gamma(S,\mathbb{G}_a) & \to & \Gamma(S,\mathbb{G}_{a,dR}) & \to & H^1_{\acute{e}t}(S,\widehat{\mathbb{G}}_a) & \to & H^1_{\acute{e}t}(S,\mathbb{G}_a) & \to & \cdots \end{array}$$

Given that S is reduced, we just showed that $H^i_{\acute{e}t}(S,\widehat{\mathbb{G}}_a) = 0$ for all i. Moreover, since $\mathbb{G}_a$ is smooth, the natural map $H^i_{fppf}(S,\mathbb{G}_a) \to H^i_{\acute{e}t}(S,\mathbb{G}_a)$ is an isomorphism. Using these facts, a diagram chase gives that $H^i_{fppf}(S,\widehat{\mathbb{G}}_a)$ vanishes for all i.

If $G = T$ is a torus, the sheaf $\underline{\mathrm{Ext}}^1(T,\mathbb{G}_m)$ vanishes [SGA7.I, Exposé VIII, Prop. 3.3.1] and so the result follows from the fact that $H^1(S,X) = 0$, where $X := T^D$ is the Cartier dual of T, for irreducible and geometrically unibranch S [SGA7.I, Exposé VIII, Prop. 5.1]. □

Even though it is not going to be needed, we remark that the sheafification map $\mathrm{Ext}^1_S(U,\mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(U,\mathbb{G}_m)(S)$ is also an isomorphism for a not-necessarily-reduced affine scheme S [Bha22, Remark 2.2.18].

In order to understand the sheafification map $\mathrm{Ext}^1_S(G,\mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(G,\mathbb{G}_m)(S)$ for an arbitrary connected algebraic group G, we will need the following vanishing result.

Proposition 2.18. *Let S be a seminormal k-scheme. Then $\underline{\mathrm{Ext}}^1(U,\mathbb{G}_m)(S)$ vanishes.*

Proof. As seminormal schemes are reduced [Stacks, Tag 0EUQ], Propositions 2.16 and 2.17 imply that $\underline{\mathrm{Ext}}^1(U,\mathbb{G}_m)(S) \simeq H^1_m(U_S,\mathbb{G}_{m,S})$. By Traverso's theorem [Sad21, Lemma 4.3], we have that

$$p^*\colon H^1(S,\mathbb{G}_{m,S}) \to H^1(U_S,\mathbb{G}_{m,S}),$$

where $p\colon U_S \to S$ is the structure map, is an isomorphism. In particular, $H^1_m(U_S,\mathbb{G}_{m,S})$ is isomorphic to the subgroup of $H^1(S,\mathbb{G}_{m,S})$ constituted of the elements $x \in H^1(S,\mathbb{G}_{m,S})$ satisfying $p^*x \in H^1_m(U_S,\mathbb{G}_{m,S})$. But p^*x lies in $H^1_m(U_S,\mathbb{G}_{m,S})$ if and only if $m^*p^*x = pr_1^*p^*x + pr_2^*p^*x$. However, the morphisms

$$p \circ m,\ p \circ pr_1,\ p \circ pr_2\colon U_S \times_S U_S \to S$$

are all equal to the structure map of U^2_S, which has a section $S \to U^2_S$. In particular, $m^*p^*x = pr_1^*p^*x + pr_2^*p^*x$ holds if and only if $x = 0$, finishing the proof. □

Corollary 2.19. *The sheafification map $\mathrm{Ext}^1_S(G,\mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(G,\mathbb{G}_m)(S)$ is an isomorphism for regular k-schemes S.*

Proof. Since G is an extension of an abelian variety A by a linear group L (which is a product of a torus and a unipotent group), we have a commutative diagram

$$\begin{array}{ccccccc} \mathrm{Hom}_S(L,\mathbb{G}_m) & \longrightarrow & \mathrm{Ext}^1_S(A,\mathbb{G}_m) & \longrightarrow & \mathrm{Ext}^1_S(G,\mathbb{G}_m) & \longrightarrow & \mathrm{Ext}^1_S(L,\mathbb{G}_m) \\ \| & & \downarrow\wr & & \downarrow & & \downarrow\wr \\ \underline{\mathrm{Hom}}(L,\mathbb{G}_m)(S) & \to & \underline{\mathrm{Ext}}^1(A,\mathbb{G}_m)(S) & \to & \underline{\mathrm{Ext}}^1(G,\mathbb{G}_m)(S) & \to & \underline{\mathrm{Ext}}^1(L,\mathbb{G}_m)(S), \end{array}$$

whose rows are exact. Proposition 2.18 implies that $\underline{\mathrm{Ext}}^1(L,\mathbb{G}_m)(S)$ vanishes and then Proposition 2.17 gives that $\mathrm{Ext}^1_S(L,\mathbb{G}_m) = 0$ as well. The result now follows from a diagram chase. □

In order to have a bird's-eye view of this subsection, consider the following definition.

Definition 2.20. We denote the abelian sheaf $\underline{\mathrm{Ext}}^1(G_{dR},\mathbb{G}_m)$ by $G^\natural$ and the abelian sheaf $\underline{\mathrm{Ext}}^1(G,\mathbb{G}_m)$ by G'.

Given a reduced k-scheme S, Corollary 2.15 and Proposition 2.17 give isomorphisms

$$G^\natural(S) := \underline{\mathrm{Ext}}^1(G_{dR},\mathbb{G}_m)(S) \xleftarrow{\sim} \mathrm{Ext}^1_S(G_{dR},\mathbb{G}_m) \xrightarrow{\sim} H^1_m(G_{dR}\times S,\mathbb{G}_{m,S})$$

that are functorial on G. In other words, $G^\natural(S)$ is isomorphic to the set of isomorphism classes of line bundles $(\mathscr{L},\nabla)$ on G_S with integrable connection relative to S satisfying $m^*(\mathscr{L},\nabla)\simeq(\mathscr{L},\nabla)\boxtimes(\mathscr{L},\nabla)$. Moreover, tensor products of connections give the group structure of $G^\natural(S)$ and this isomorphism preserves inverse images of connections.

Similarly, given a regular k-scheme S, Proposition 2.16 and Corollary 2.19 yield isomorphisms

$$G'(S) := \underline{\mathrm{Ext}}^1(G,\mathbb{G}_m)(S) \xleftarrow{\sim} \mathrm{Ext}^1_S(G,\mathbb{G}_m) \xrightarrow{\sim} H^1_m(G_S,\mathbb{G}_{m,S})$$

that are functorial on G. As above, this implies that $G'(S)$ is the group of isomorphism classes of line bundles $\mathscr{L}$ on G_S satisfying $m^*\mathscr{L}\simeq\mathscr{L}\boxtimes\mathscr{L}$.

The next two remarks will explain our choice of notation in Definition 2.20.

Remark 2.21 (The dual abelian variety). Let A be an abelian variety over k. The fact that our A' coincides with the dual abelian variety is usually called the *Barsotti-Weil formula*. As Harari remarks in [Jos09, Footnote to Thm. 1.2.2], there seems to be no published proof of this formula in the correct generality.

Given any scheme S (not necessarily over a characteristic zero field) and an abelian scheme A over S, our methods show that

$$\underline{\mathrm{Ext}}^1(A,\mathbb{G}_m)(T) \xleftarrow{\sim} \mathrm{Ext}^1_T(A,\mathbb{G}_m) \xrightarrow{\sim} H^1_m(A_T,\mathbb{G}_{m,T})$$

for all S-schemes T. (This holds for both the étale and the fppf topologies.)

Remark 2.22 (The universal vector extension). As above, let A be an abelian variety over k. Taking the long exact sequence in cohomology associated to the Cartier duality functor $(-)^D := \underline{\mathrm{Hom}}(-, \mathbb{G}_m)$ and the short exact sequence

$$0 \to \widehat{A} \to A \to A_{dR} \to 0,$$

we obtain that $A^\natural$ is an extension of A' by Ω_A. (In particular, $A^\natural$ is representable by an algebraic group.) We affirm that $A^\natural$ is the universal vector extension of A' in the sense of [MM74, §I.1].

Let S be any k-scheme. By Propositions 2.13 and 2.17, we have natural isomorphisms

$$A^\natural(S) := \underline{\mathrm{Ext}}^1(A_{dR}, \mathbb{G}_m)(S) \xleftarrow{\sim} \mathrm{Ext}^1_S(A_{dR}, \mathbb{G}_m) \xrightarrow{\sim} H^1_m(A_{dR} \times S, \mathbb{G}_{m,S}).$$

This implies that the presheaf $S \mapsto H^1_m(A_{dR} \times S, \mathbb{G}_{m,S})$ already satisfies fppf descent. It follows that the sheafification in the definition of $E^\natural$ [MM74, Def. I.4.1.6] is superfluous and we obtain that $A^\natural \simeq E^\natural$.

Mazur and Messing also define an abelian sheaf $\underline{\mathrm{Ext}}^\natural(A, \mathbb{G}_m)$ which, by Remark 2.9, is isomorphic to $A^\natural = \underline{\mathrm{Ext}}^1(A_{dR}, \mathbb{G}_m)$. In particular, our methods reprove their [MM74, Prop. I.4.2.1], that compares $\underline{\mathrm{Ext}}^\natural(A, \mathbb{G}_m)$ and $E^\natural$. Finally, by [MM74, Props. I.2.6.7 and I.3.2.3] we obtain that $A^\natural$ is the universal vector extension of A'.

2.3. Vanishing of extension sheaves

In this subsection we compile a number of vanishing results for extension sheaves that are going to be useful to us next. We recall the notations from Subsection 1.4: A is an abelian variety, U is an unipotent group, and T is a torus with character group X. All over a field k. Unless explicitly stated, the results of this subsection hold independently of the characteristic of k.

Proposition 2.23. *Both* $T' = \underline{\mathrm{Ext}}^1(T, \mathbb{G}_m)$ *and* $\underline{\mathrm{Ext}}^1(X, \mathbb{G}_m)$ *vanish.*

Proof. This follows from [SGA7.I, Exposé VIII, Prop. 3.3.1]. □

Proposition 2.24. *Suppose that* k *has characteristic zero, and let* S *be a seminormal* k*-scheme. Then* $U'(S)$ *vanishes.*

Proof. This is in here *pour mémoire*, since it was already proven in Proposition 2.18. □

Proposition 2.25. *Let* $\mathscr{G}$ *be a formal group over* k *whose Cartier dual is an algebraic group (i.e., separated and of finite type). Then* $\underline{\mathrm{Ext}}^1(\mathscr{G}, \mathbb{G}_m)$ *vanishes. In particular, if* G *is a connected algebraic group over* k*, we have that* $\underline{\mathrm{Ext}}^1(\widehat{G}, \mathbb{G}_m) = 0$.

Proof. The vanishing statement is [Rus13, Lemma 1.14]. □

For the reader's convenience, we remark that the Cartier dual of a formal group $\mathscr{G}$ is an algebraic group precisely if $\mathscr{G}(\bar{k})$ is of finite type [Rus13, Prop. 1.16].

Proposition 2.26. *The sheaf* $\underline{\mathrm{Ext}}^2(A, \mathbb{G}_m)$ *vanishes.*

Proof. This is [Bre75, Remarque 6 on page 340]. □

Corollary 2.27. *If* k *has characteristic zero, then* $\underline{\mathrm{Ext}}^2(A_{dR}, \mathbb{G}_m)$ *vanishes.*

Proof. The abelian sheaf $\underline{\mathrm{Ext}}^2(A_{\mathrm{dR}}, \mathbb{G}_m)$ fits into the exact sequence

$$\underline{\mathrm{Ext}}^1(\widehat{A}, \mathbb{G}_m) \to \underline{\mathrm{Ext}}^2(A_{\mathrm{dR}}, \mathbb{G}_m) \to \underline{\mathrm{Ext}}^2(A, \mathbb{G}_m),$$

and both extremities vanish by the previous propositions. □

Remark 2.28 (Schanuel's module). Recall from Proposition 2.24 that, over a characteristic zero field k, the abelian group $\underline{\mathrm{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m)(S)$ vanishes for seminormal k-schemes S. In [Ros23, Remark 2.2.16], Rosengarten constructs an example (due to Gabber) of an extension of $\mathbb{G}_a$ by $\mathbb{G}_m$ that does not split fppf-locally. Here we construct another non-zero section of $\underline{\mathrm{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m)$.

Let $S = \operatorname{Spec} R$, where $R = k[x, y]/(y^2 - x^3)$ is the coordinate ring of a cusp. This is the prototypical example of a scheme that is not seminormal. As S is reduced, Propositions 2.16 and 2.17 give that

$$\underline{\mathrm{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m)(S) \xleftarrow{\sim} \mathrm{Ext}^1_S(\mathbb{G}_a, \mathbb{G}_m) \xrightarrow{\sim} H^1_m(\mathbb{G}_{a,S}, \mathbb{G}_{m,S}) \subset \mathrm{Pic}(R[t]).$$

Now, consider the fractional ideals $I = (x, 1 + yt/x)$ and $J = (x, 1 - yt/x)$ of $R[t]$. Since

$$1 = x^2t^4 + (1 + xt^2)(1 - xt^2) \in IJ = (x^2, x + yt, x - yt, 1 - xt^2),$$

the ideal I is invertible and so defines an element of $\mathrm{Pic}(R[t])$. As one verifies directly, the fractional ideal $m^* J \, \mathrm{pr}_1^* I \, \mathrm{pr}_2^* I$ is equal to $R[t]$, proving that I is a non-zero element of $H^1_m(\mathbb{G}_{a,S}, \mathbb{G}_{m,S})$.

Remark 2.29 (Schanuel's module in positive characteristic). Let k be a field of characteristic $p > 0$. Proposition 2.16 still holds in positive characteristic, and the previous remark gives a k-scheme S such that $\mathrm{Ext}^1_S(\mathbb{G}_a, \mathbb{G}_m) \simeq H^1_m(\mathbb{G}_{a,S}, \mathbb{G}_{m,S}) \neq 0$. However, it is no longer true that $\underline{\mathrm{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m)(S) \simeq \mathrm{Ext}^1_S(\mathbb{G}_a, \mathbb{G}_m)$.

Given a k-algebra R, the colimit R_{perf} of the tower

$$R \xrightarrow{x \mapsto x^p} R \xrightarrow{x \mapsto x^p} R \xrightarrow{x \mapsto x^p} \cdots$$

is the so-called *colimit perfection* of R. It is always a perfect k-algebra and the natural map $R \to R_{\mathrm{perf}}$ is universal among morphisms from R to a perfect algebra. The kernel of $R \to R_{\mathrm{perf}}$ is the nilradical of R and $R \to R_{\mathrm{perf}}$ is surjective if R is semiperfect.[4]

[4] A $\mathbb{F}_p$-algebra R is said to be *semiperfect* if the Frobenius endomorphism is surjective.

Now, [Bha22, Remark 2.2.18] proves that $R\Gamma(R, \widehat{\mathbb{G}}_a) \simeq [R \to R_{\mathrm{perf}}]$, where the complex is in degrees zero and one. In particular, $H^2(R, \widehat{\mathbb{G}}_a) = 0$ but $H^1(R, \widehat{\mathbb{G}}_a)$ may not vanish. It follows that, for $S = \operatorname{Spec} k[x, y]/(y^2 - x^3)$, the sheafification map

$$\mathrm{Ext}^1_S(\mathbb{G}_a, \mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m)(S)$$

is surjective but not injective. More precisely, Rosengarten has recently proved that the sheaf $\underline{\mathrm{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m)$ vanishes [Ros23, Prop. 2.2.14].

2.4. Commutative group stacks and their duals

Let S be a scheme and $\mathcal{G}$ be a commutative group stack over S. These objects were originally defined by Deligne in [SGA4.III, Exposé XVIII] under the name *champs de Picard strictement commutatifs*. Morally, $\mathcal{G}$ is an fppf stack over S endowed with a morphism $m\colon \mathcal{G} \times_S \mathcal{G} \to \mathcal{G}$ satisfying some compatibilities akin to those satisfied by abelian groups.

Naturally, commutative group stacks form a 2-category and we denote by CGS its homotopy category. By a truncated version of the Dold-Kan correspondence, there is an equivalence between CGS and $\mathsf{D}^{[-1,0]}(\mathsf{Ab}(\mathsf{E}))$, where $\mathsf{E} = \mathsf{Sh}((\mathsf{Sch}/S)_{\mathrm{fppf}})$ [SGA4.III, Exposé XVIII, Prop. 1.4.15]. Every object of $\mathsf{D}^{[-1,0]}(\mathsf{Ab}(\mathsf{E}))$ is isomorphic to a complex $[\mathscr{H} \to \mathscr{G}]$ in degrees -1 and 0. Given such a complex, the equivalence is given by

$$[\mathscr{H} \to \mathscr{G}] \mapsto [\mathscr{G}/\mathscr{H}],$$

where $\mathscr{H}$ acts on $\mathscr{G}$ by translation. We will systematically identify commutative group stacks and the associated two-term complexes.

In this subsection, we will say that $\mathcal{G}^D = \underline{\mathrm{Hom}}(\mathcal{G}, \mathbb{G}_m)$, where $\underline{\mathrm{Hom}}$ denotes the inner Hom of commutative group stacks, is the 0-Cartier dual of $\mathcal{G}$. Similarly, we say that $\mathcal{G}^\vee = \underline{\mathrm{Hom}}(\mathcal{G}, B\mathbb{G}_m)$ is the 1-Cartier dual of $\mathcal{G}$. Under the equivalence above, we have that

$$[\mathscr{H} \to \mathscr{G}]^D \simeq \tau_{\leqslant 0} R\underline{\mathrm{Hom}}([\mathscr{H} \to \mathscr{G}], \mathbb{G}_m)$$
$$[\mathscr{H} \to \mathscr{G}]^\vee \simeq \tau_{\leqslant 0} R\underline{\mathrm{Hom}}([\mathscr{H} \to \mathscr{G}], \mathbb{G}_m[1]).$$

This description gives rise to some explicit computations, based on the simple observation that a complex M only having cohomology in degree i is isomorphic to $\mathscr{H}^i(M)[-i]$.

Proposition 2.30. *Let $\mathscr{G}$ be an abelian sheaf on $(\mathsf{Sch}/S)_{\mathit{fppf}}$. Then $B\mathscr{G}^\vee \simeq \mathscr{G}^D$. If $\underline{\mathrm{Ext}}^1(\mathscr{G}, \mathbb{G}_m) = 0$, then $\mathscr{G}^\vee \simeq B\mathscr{G}^D$. Similarly, if $\mathscr{G}^D = 0$, then $\mathscr{G}^\vee \simeq \underline{\mathrm{Ext}}^1(\mathscr{G}, \mathbb{G}_m)$.*

When S is the spectrum of a characteristic zero field k, the previous proposition, along with the computations done in the previous sections, give that

$$\begin{array}{llllll} BG_{dR}^\vee \simeq 0, & B\widehat{G}^\vee \simeq \Omega_G, & BA^\vee \simeq 0, & BT^\vee \simeq X, & BX^\vee \simeq T, & BU^\vee \simeq \widehat{U^*}, \\ G_{dR}^\vee \simeq G^\natural, & \widehat{G}^\vee \simeq B\Omega_G, & A^\vee \simeq A' & T^\vee \simeq BX, & X^\vee \simeq BT. & \end{array}$$

One would like the 1-Cartier dual of U to be $B\widehat{U^*}$, which is not true since $U' = \underline{\mathrm{Ext}}^1(U, \mathbb{G}_m) \neq 0$. Possible aiming to solve this issue, Laumon introduced the following class of commutative group stacks in [Lau96]. (See also [Rus13] for the generality on which we are working here.)

Definition 2.31. Let k be a field of arbitrary characteristic. A *generalized 1-motive* is a two-term complex of abelian fppf sheaves $[\mathscr{G} \to G]$, where G is a connected algebraic group over k and $\mathscr{G}$ is a formal group over k whose 0-Cartier dual is a connected algebraic group.

A usual 1-motive, as in [Del74, §10.1], is the particular case of the definition above in which k is algebraically closed, G is a semiabelian variety, and $\mathscr{G}$ is a finitely generated free $\mathbb{Z}$-module.

Let $[\mathscr{G} \to G]$ be a generalized 1-motive. Recall that G fits into a short exact sequence

$$0 \to L \xrightarrow{\varphi} G \xrightarrow{\psi} A \to 0,$$

uniquely defined up to isomorphism, where L is a connected linear group and A is an abelian variety. In the following lemma we will consider the composition $\mathscr{G} \to G \to A$, which we see as a generalized 1-motive $[\mathscr{G} \to A]$.

Lemma 2.32. *The complex* $R\underline{\mathrm{Hom}}([\mathscr{G} \to A], \mathbb{G}_m)$ *has no cohomology in degrees* 0 *and* 2. *Moreover,* $\underline{\mathrm{Ext}}^1([\mathscr{G} \to A], \mathbb{G}_m)$ *is representable by a connected algebraic group over* k.

Proof. We apply the functor $R\underline{\mathrm{Hom}}(-, \mathbb{G}_m)$ to the distinguished triangle $\mathscr{G} \to A \to [\mathscr{G} \to A]$ to obtain the long exact sequence below.

$$\begin{array}{l} 0 \longrightarrow \underline{\mathrm{Ext}}^0([\mathscr{G} \to A], \mathbb{G}_m) \longrightarrow 0 \longrightarrow \mathscr{G}^D \\ \longrightarrow \underline{\mathrm{Ext}}^1([\mathscr{G} \to A], \mathbb{G}_m) \longrightarrow A' \longrightarrow 0 \\ \longrightarrow \underline{\mathrm{Ext}}^2([\mathscr{G} \to A], \mathbb{G}_m) \longrightarrow 0 \end{array}$$

The vanishing results follow directly, and the representability of $\underline{\mathrm{Ext}}^1([\mathscr{G} \to A], \mathbb{G}_m)$ by a commutative connected algebraic group follows by descent and [Mil17, Prop. 5.59]. $\square$

Henceforth, we will denote the algebraic group representing $\underline{\mathrm{Ext}}^1([\mathscr{G} \to A], \mathbb{G}_m)$ by K. The following variant of 1-Cartier duality first appeared in [Del74] and was subsequently generalized in [Lau96] and [Rus13].

Definition 2.33. Let $[\mathscr{G} \to G]$ be a generalized 1-motive. We define its *Laumon dual* $[\mathscr{G} \to G]^L$ to be the generalized 1-motive $[L^D \to K]$, where $L^D \to K$ is the connecting morphism induced by the distinguished triangle $L \to [\mathscr{G} \to G] \to [\mathscr{G} \to A]$ via the 0-Cartier duality functor.

As the proposition below shows, the octahedral axiom gives rise to a comparison between the duality functor defined by Laumon and 1-Cartier duality.

Proposition 2.34. *There exists a natural map* $[\mathscr{G} \to G]^L \to [\mathscr{G} \to G]^\vee$*, whose cone is* $\underline{\mathrm{Ext}}^1(L, \mathbb{G}_m)$.

Proof. The distinguished triangle defining the Laumon dual induces the distinguished triangle below.

$$R\underline{\mathrm{Hom}}([\mathscr{G} \to A], \mathbb{G}_m[1]) \to R\underline{\mathrm{Hom}}([\mathscr{G} \to G], \mathbb{G}_m[1]) \to R\underline{\mathrm{Hom}}(L, \mathbb{G}_m[1])$$

By Lemma 2.32, $\underline{\mathrm{Ext}}^2([\mathscr{G} \to A], \mathbb{G}_m)$ vanishes and so $\underline{\mathrm{Ext}}^1([\mathscr{G} \to G], \mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(L, \mathbb{G}_m)$ is an epimorphism. Then, [Bro21, Lemma 3.10] implies that the triangle

$$\tau_{\leqslant 0} R\underline{\mathrm{Hom}}([\mathscr{G} \to A], \mathbb{G}_m[1]) \to \tau_{\leqslant 0} R\underline{\mathrm{Hom}}([\mathscr{G} \to G], \mathbb{G}_m[1]) \to \tau_{\leqslant 0} R\underline{\mathrm{Hom}}(L, \mathbb{G}_m[1])$$

is also distinguished. Yet another application of Lemma 2.32 gives that

$$\tau_{\leqslant 0} R\underline{\mathrm{Hom}}([\mathscr{G} \to A], \mathbb{G}_m[1]) \simeq K$$

and so, up to a shift, the distinguished triangle just obtained is $L^\vee[-1] \to K \to [\mathscr{G} \to G]^\vee$. Since $L^D \simeq \tau_{\leqslant 0}(L^\vee[-1])$, there is a natural map $L^D \to L^\vee[-1]$ making the square

$$\begin{array}{ccccc} L^D & \longrightarrow & K & \longrightarrow & [L^D \to K] \\ \downarrow & & \| & & \\ L^\vee[-1] & \longrightarrow & K & \longrightarrow & [\mathscr{G} \to G]^\vee \end{array}$$

commute and inducing a morphism of triangles. In this way we obtain the desired comparison map.

Now, by [Stacks, Tag 08J5], we have a distinguished triangle $L^D \to L^\vee[-1] \to \underline{\mathrm{Ext}}^1(L, \mathbb{G}_m)$. Finally, the octahedral axiom [Stacks, Tag 05R0] gives that the cone of $[\mathscr{G} \to G]^L \to [\mathscr{G} \to G]^\vee$ is isomorphic to $\underline{\mathrm{Ext}}^1(L, \mathbb{G}_m)$. □

This proposition implies that the comparison map $[\mathscr{G} \to G]^L \to [\mathscr{G} \to G]^\vee$ is an isomorphism if and only if $\underline{\mathrm{Ext}}^1(L, \mathbb{G}_m) = 0$. This holds whenever G is semiabelian, proving that the Cartier dual on 1-motives defined by Deligne [Del74, §§10.2.11] coincides with 1-Cartier duality. More generally, we have the corollary below.

Corollary 2.35. *The comparison morphism* $[\mathscr{G} \to G]^L \to [\mathscr{G} \to G]^\vee$ *is an isomorphism if* k *has positive characteristic. When* k *has characteristic zero, the comparison map is an isomorphism if and only if* G *is semiabelian.*

Proof. In positive characteristic the fppf sheaf $\underline{\mathrm{Ext}}^1(L, \mathbb{G}_m)$ always vanishes due to [Ros23, Prop. 2.2.17]. Now, if k has characteristic zero, L is a product of a torus and a vector group U. In particular, Proposition 2.23 implies that $\underline{\mathrm{Ext}}^1(L, \mathbb{G}_m) \simeq \underline{\mathrm{Ext}}^1(U, \mathbb{G}_m)$. By Remark 2.28, the latter vanishes precisely when U does. □

When the base field k has characteristic zero and $\mathscr{G} = \widehat{G}$, so that $[\widehat{G} \to G] \simeq G_{dR}$, one can give a more explicit description of $K = \underline{\mathrm{Ext}}^1([\widehat{G} \to A], \mathbb{G}_m)$. The long exact sequence associated to the extension

$$0 \to \widehat{A} \to A \to A_{dR} \to 0,$$

via the 0-Cartier duality functor, gives rise to the short exact sequence

$$0 \to \Omega_A \to A^\natural \to A' \to 0.$$

Now, the quotient map $\psi\colon G \to A$ induces to a pullback map $\psi^*\colon \Omega_A \to \Omega_G$, and we consider the pushout extension.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_A & \longrightarrow & A^\natural & \longrightarrow & A' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi^*} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Omega_G & \longrightarrow & (\Omega_G \times A^\natural)/\Omega_A & \longrightarrow & A' & \longrightarrow & 0 \end{array}$$

Proposition 2.36. *The fppf sheaf* $K = \underline{\mathrm{Ext}}^1([\widehat{G} \to A], \mathbb{G}_m)$ *is isomorphic to the quotient* $(\Omega_G \times A^\natural)/\Omega_A$.

Proof. Consider the following commutative diagram, whose rows are distinguished triangles.

$$\begin{array}{ccccc} \widehat{G} & \longrightarrow & A & \longrightarrow & [\widehat{G} \to A] \\ \downarrow & & \| & & \downarrow \\ \widehat{A} & \longrightarrow & A & \longrightarrow & [\widehat{A} \to A] \end{array}$$

After applying $R\underline{\mathrm{Hom}}(-, \mathbb{G}_m)$ and taking long exact sequences in cohomology we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_A & \longrightarrow & A^\natural & \longrightarrow & A' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi^*} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Omega_G & \longrightarrow & K & \longrightarrow & A' & \longrightarrow & 0, \end{array}$$

in which the map $\psi^*\colon \Omega_A \to \Omega_G$ is the same one appearing in the definition of $(\Omega_G \times A^\natural)/\Omega_A$. We affirm that the square on the left is cocartesian. In other words, we affirm that the complex

$$0 \to \Omega_A \to \Omega_G \times A^\natural \to K \to 0,$$

where we have the same action of Ω_A on $\Omega_G \times A^\natural$ as in the usual construction of the pushout, is exact. Now, this complex fits into the larger commutative diagram below.

$$\begin{CD}
@. @. 0 @. 0 \\
@. @. @VVV @VVV \\
@. 0 @>>> \Omega_G @= \Omega_G @>>> 0 \\
@. @VVV @VVV @VVV \\
0 @>>> \Omega_A @>>> \Omega_G \times A^\natural @>>> K @>>> 0 \\
@. @| @VVV @VVV \\
0 @>>> \Omega_A @>>> A^\natural @>>> A' @>>> 0 \\
@. @VVV @VVV @VVV \\
@. 0 @. 0 @. 0
\end{CD}$$

Here, every column is clearly exact, and both the top and the bottom row are exact. It follows that the middle row is exact as well. □

2.5. The moduli of character sheaves

Per the previous subsection, Laumon has defined a Cartier dual $[\widehat{G} \to G]^L$ that, in some sense, takes away the mysterious object $U' = \underline{\mathrm{Ext}}^1(U, \mathbb{G}_m)$ from inside of $G^\natural = G^\vee_{dR}$. In this subsection we will see that even $[\widehat{G} \to G]^L$ fails to be representable, due to the presence of a formal group in it. Taking this out as well we obtain an abelian sheaf $G^\flat$ that is representable by an algebraic space and satisfies $G^\flat(S) = G^\natural(S)$ for seminormal k-schemes S.

We begin by remarking that, since both formal completions and the de Rham functor are exact, we have a commutative diagram

$$\begin{CD}
@. 0 @. 0 @. 0 \\
@. @VVV @VVV @VVV \\
0 @>>> \widehat{T} \times \widehat{U} @>>> \widehat{G} @>>> \widehat{A} @>>> 0 \\
@. @VVV @VVV @VVV \\
0 @>>> T \times U @>>> G @>>> A @>>> 0 \\
@. @VVV @VVV @VVV \\
0 @>>> T_{dR} \times U_{dR} @>>> G_{dR} @>>> A_{dR} @>>> 0 \\
@. @VVV @VVV @VVV \\
@. 0 @. 0 @. 0,
\end{CD}$$

in which every column and row is exact. By applying the Cartier duality functor $(-)^D := \underline{\mathrm{Hom}}(-, \mathbb{G}_m)$ and passing to the long exact sequences in cohomology, we obtain

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & G^D & \xrightarrow{\varphi^*} & X \times \widehat{U^*} & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_A & \xrightarrow{\psi^*} & \Omega_G & \xrightarrow{\varphi^*} & \Omega_T \times \Omega_U & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^\natural & \xrightarrow{\psi^*} & G^\natural & \xrightarrow{\varphi^*} & T^\natural \times U^\natural & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & \longrightarrow & A' & \xrightarrow{\psi^*} & G' & \xrightarrow{\varphi^*} & U' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Once again, every row (including the snake-like line) and every column is exact. All the needed computations and vanishing results were already studied in the previous subsections. We remark that the morphisms in the columns all have a natural geometric interpretation; they are given by

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G^D & \longrightarrow & \Omega_G & \longrightarrow & G^\natural & \longrightarrow & G' \longrightarrow 0 \\
 & & \chi & \longmapsto & d\chi/\chi & & (\mathscr{L}, \nabla) & \longmapsto & \mathscr{L} \\
 & & & & \omega & \longmapsto & (\mathcal{O}_G, d + \omega). & &
\end{array}$$

Recall the algebraic group K characterized in the previous subsection as the following pushout extension.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_A & \longrightarrow & A^\natural & \longrightarrow & A' & \longrightarrow & 0 \\
 & & {\scriptstyle \psi^*}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega_G & \longrightarrow & K & \longrightarrow & A' & \longrightarrow & 0
\end{array}$$

Since K is a quotient of $\Omega_G \times A^\natural$, we will denote its sections as (equivalence classes of) pairs $(\omega, (\mathscr{L}, \nabla))$, where $\omega \in \Omega_G$ and $(\mathscr{L}, \nabla) \in A^\natural$.[5] The universal property defining K

[5]This is a little abuse of notation since the sheafification involved in defining the quotient sheaf may be non-trivial. That being said, Serre vanishing implies that $K(S)$ really is the quotient of $\Omega_G(S) \times A^\natural(S)$ by $\Omega_A(S)$ for affine S.

allows us to define the morphism of groups

$$\begin{aligned} \gamma\colon K &\to G^\natural \\ [\omega,(\mathscr{L},\nabla)] &\mapsto (\mathcal{O}_G, d+\omega)\otimes_{\mathcal{O}_G}\psi^*(\mathscr{L},\nabla). \end{aligned}$$

The usefulness of K comes from the fact that we understand this morphism $\gamma\colon K\to G^\natural$ relatively well. Indeed, γ is basically the comparison map between Laumon's dual and the 1-Cartier dual of G_{dR} as in Proposition 2.34.

Proposition 2.37. *The kernel of γ is $X\times\widehat{U^*}$ and its cokernel is U'.*

Proof. Recall that Proposition 2.36 gives an isomorphism between the group K defined as a pushout and the abelian sheaf $\underline{\mathrm{Ext}}^1([\widehat{G}\to A],\mathbb{G}_m)$. We affirm that the diagram

$$\begin{array}{ccc} K & \xrightarrow{\quad\gamma\quad} & G^\natural \\ \downarrow\wr & & \| \\ \underline{\mathrm{Ext}}^1([\widehat{G}\to A],\mathbb{G}_m) & \longrightarrow & \underline{\mathrm{Ext}}^1([\widehat{G}\to G],\mathbb{G}_m), \end{array}$$

on which the map $\underline{\mathrm{Ext}}^1([\widehat{G}\to A],\mathbb{G}_m)\to\underline{\mathrm{Ext}}^1([\widehat{G}\to G],\mathbb{G}_m)$ is induced by the natural morphism of complexes $[\widehat{G}\to G]\to[\widehat{G}\to A]$, commutes. This is the same as showing that the diagram

$$\begin{array}{ccccc} & & \underline{\mathrm{Ext}}^1([\widehat{A}\to A],\mathbb{G}_m) & & \\ & & \downarrow & \searrow & \\ \underline{\mathrm{Hom}}(\widehat{G},\mathbb{G}_m) & \longrightarrow & \underline{\mathrm{Ext}}^1([\widehat{G}\to A],\mathbb{G}_m) & & \\ & \searrow & & \searrow & \\ & & & & \underline{\mathrm{Ext}}^1([\widehat{G}\to G],\mathbb{G}_m) \end{array}$$

commutes. The upper triangle clearly commutes by functoriality and the lower triangle can be seen to commute by applying the functor $R\underline{\mathrm{Hom}}(-,\mathbb{G}_m)$ to the morphism of distinguished triangles

$$\begin{array}{ccccc} \widehat{G} & \longrightarrow & G & \longrightarrow & [\widehat{G}\to G] \\ \| & & \downarrow{\scriptstyle\psi} & & \downarrow \\ \widehat{G} & \longrightarrow & A & \longrightarrow & [\widehat{G}\to A] \end{array}$$

and taking long exact sequences in cohomology. Now, as in the proof of Proposition 2.34, there are two dashed morphisms making the diagram

$$\begin{array}{ccccc} K & \longrightarrow & [X\times\widehat{U^*}\to K] & \longrightarrow & X\times\widehat{U^*}[1] \\ \| & & \vdots\downarrow & & \downarrow \\ K & \longrightarrow & G^\natural = G_{\mathrm{dR}}^\vee & \longrightarrow & (X\times\widehat{U^*})^\vee \end{array}$$

commute: the comparison map of Proposition 2.34 and γ. The [Stacks, Tag 0FWZ] implies that they coincide and then the desired result follows from Proposition 2.34. $\square$

Inasmuch as U' has no k-points, this computation is very useful for obtaining concrete information about character sheaves. Indeed, it implies that $\gamma\colon K \to G^\natural$ induces a surjection on k-points.

Corollary 2.38. *Every character sheaf on G is of the form $(\mathcal{O}_G, d+\omega) \otimes_{\mathcal{O}_G} \psi^*(\mathscr{L}, \nabla)$, for some $\omega \in \Omega_G$ and $(\mathscr{L}, \nabla) \in A^\natural(k)$.*

The Proposition 2.37 yields a short exact sequence $0 \to K/(X \times \widehat{U^*}) \to G^\natural \to U' \to 0$ which, along with Proposition 2.24, implies that $G^\natural$ and $K/(X \times \widehat{U^*})$ have the same k-points. The sheaf $K/(X \times \widehat{U^*})$ also has no hope of being representable in general, due to the presence of this $\widehat{U^*}$ factor. However, since we have a short exact sequence

$$0 \to \widehat{U^*} \to K/X \to K/(X \times \widehat{U^*}) \to 0$$

and $H^1(k, \widehat{U^*})$ vanishes, the sheaf K/X also has the same k-points as $G^\natural$. (Even the same S-points for seminormal k-schemes S.) This turns out to be the correct "coarse moduli space".

Definition 2.39. We denote by $G^\flat$ the abelian sheaf K/X, where $X \hookrightarrow K$ is the morphism sending $\chi \in X$ to $[\omega, (\mathscr{L}, \nabla)]$, where ω is any element of Ω_G satisfying $\varphi^*\omega = d\chi/\chi$ and $(\mathscr{L}, \nabla)$ is the unique element of $A^\natural$ satisfying $\psi^*(\mathscr{L}, \nabla) \simeq (\mathcal{O}_G, d-\omega)$.

We remark that $U^\flat \simeq \Omega_U \simeq U^*$, $T^\flat \simeq T^\natural \simeq \Omega_T/X \simeq t^*/X$, and $A^\flat \simeq A^\natural$. The middle one is a (non-quasi-separated) group algebraic space, and the other two are algebraic groups. Just as $G^\natural$ is a $A^\natural$-torsor over $T^\natural \times U^\natural$, the same holds for the $\flat$-sheaves.

Theorem 2.40. *There exists a short exact sequence $0 \to A^\flat \to G^\flat \to T^\flat \times U^\flat \to 0$.*

Proof. Consider the map $K \to \Omega_T \times \Omega_U$ induced by $\varphi^*\colon \Omega_G \to \Omega_T \times \Omega_U$ and $0\colon A^\natural \to \Omega_T \times \Omega_U$. We affirm that the composition $K \to \Omega_T \times \Omega_U \to \Omega_T/X \times \Omega_U$ descends to the quotient K/X. By the universal property of the quotient, we need to verify that the composition

$$X \to X \times \widehat{U^*} \to K \to \Omega_T \times \Omega_U \to \Omega_T/X \times \Omega_U$$

is zero. Applying the functor $R\underline{\mathrm{Hom}}(-, \mathbb{G}_m)$ to the morphism of distinguished triangles

$$\begin{array}{ccccc} L & \longrightarrow & [\widehat{L} \to L] & \longrightarrow & \widehat{L}[1] \\ \| & & \downarrow & & \downarrow \\ L & \longrightarrow & [\widehat{G} \to G] & \longrightarrow & [\widehat{G} \to A] \end{array}$$

and taking long exact sequences in cohomology, we obtain that the composition $X \times \widehat{U^*} \to K \to \Omega_T \times \Omega_U$ is our well-known map which already appears on Page 23. In particular, this composition is the product of $X \to \Omega_T$ and $\widehat{U^*} \to \Omega_U$. It follows that our large composition vanishes and we obtain a map $K/X \to \Omega_T/X \times \Omega_U$.

Now, we have every morphism needed to consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & X & = & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^\natural & \longrightarrow & K & \longrightarrow & \Omega_T \times \Omega_U & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^\natural & \longrightarrow & K/X & \longrightarrow & \Omega_T/X \times \Omega_U & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array}$$

whose columns are clearly exact. As the top row is also exact, by the nine-lemma, it suffices to prove that the middle row is exact. This holds by an application of the snake lemma in the pushout extension defining K. □

The theorem above finally implies that $G^\flat$ is "a coarse moduli space" for $G^\natural$, in the sense that $G^\flat$ is represented by an algebraic space which has the same k-points as $G^\flat$.[6] We emphasize that, as in [Stacks, Tag 025Y] and contrarily to [LM00, Déf. 1.1], we *do not* suppose that algebraic spaces are quasi-separated.

Corollary 2.41. *The abelian sheaf $G^\flat$ is represented by a finite type smooth connected group algebraic space. Moreover, it satisfies $\dim G \leqslant \dim G^\flat \leqslant 2\dim G$ with equality on the left if and only if G is affine, and equality on the right if and only if G is proper.*

Proof. The previous theorem implies that $G^\flat \to T^\flat \times U^\flat$ is a $A^\flat$-torsor. In particular, fppf-locally on $T^\flat \times U^\flat$, the sheaf $G^\flat$ is isomorphic to the product $A^\flat \times T^\flat \times U^\flat$, which is an algebraic space. Then [Stacks, Tag 04SK] gives that $G^\flat$ is an algebraic space as well. Since $A^\flat \simeq A^\natural$ is an extension of A' by Ω_A, descent implies that $A^\flat \simeq A^\natural$ is smooth and of finite type. By [Mil17, Prop. 5.59], $A^\flat \simeq A^\natural$ is also connected. Using the exact sequence of Theorem 2.40, the same arguments show that $G^\flat$ is a finite type smooth connected group algebraic space.

Finally, since dimensions add on extensions, we have that $\dim G^\flat = \dim T^\flat + \dim U^\flat +$

[6] However, we do not know if there is a natural morphism $G^\natural \to G^\flat$, much less if this satisfies the universal property.

$\dim A^\flat$. Then,

$$\begin{aligned}
\dim T^\flat &= \dim \Omega_T/X = \dim \Omega_T = \dim T \\
\dim U^\flat &= \dim U^* = \dim U \\
\dim A^\flat &= \dim A^\natural = \dim \Omega_A + \dim A' = 2\dim A,
\end{aligned}$$

and so $\dim G^\flat = \dim T + \dim U + 2\dim A = \dim G + \dim A$. □

3. Generic vanishing

This section is devoted to the proof of our main result Theorem B. Motivated by the deep analogies between holonomic $\mathcal{D}$-modules and ℓ-adic perverse sheaves in positive characteristic, we will use notations concerning the six-functor formalism of holonomic $\mathcal{D}$-modules that highlights their similarities.

Our notations are mostly standard with the exception of two points. Firstly, we denote by $\otimes_X$ a dual version of the derived tensor product $\otimes^L_{\mathcal{O}_X}$. We focus on the former since it corresponds to (and has the same properties as) the usual tensor product of constructible sheaves. Moreover, just as shifted local systems are perverse sheaves, we systematically consider integrable connections in degree $\dim X$.[7]

We denote by $\mathscr{L}_\alpha$ the (shifted) character sheaf of G corresponding to a point $\alpha \in G^\flat(k) \simeq G^\natural(k)$. When G is the additive group $\mathbb{G}_a$, we have that $\mathbb{G}_a^\flat \simeq \mathbb{G}_a$ and so the character sheaf corresponding to $\alpha \in \mathbb{G}_a^\flat(k) = k$ is $\mathscr{L}_\alpha = (\mathcal{O}_{\mathbb{G}_a}, d + \alpha\, dt)[-1] \simeq \mathcal{D}_{\mathbb{G}_a}/\mathcal{D}_{\mathbb{G}_a}(\partial_t - \alpha)[-1]$. Similarly, when G is the multiplicative group $\mathbb{G}_m$, we have that $\mathbb{G}_m^\flat \simeq \mathbb{G}_a/\mathbb{Z}$ and so the character sheaf corresponding to $\alpha \in \mathbb{G}_m^\flat(k) = k/\mathbb{Z}$ is $\mathscr{L}_\alpha = (\mathcal{O}_{\mathbb{G}_m}, d + \alpha\, dt/t)[-1] \simeq \mathcal{D}_{\mathbb{G}_m}/\mathcal{D}_{\mathbb{G}_m}(t\partial_t - \alpha)[-1]$. We rewrite Theorem B using these notations.

Theorem 3.1. *Let M be a holonomic $\mathcal{D}$-module over G. There exists a dense open subspace V of $G^\flat$ such that*

$$\begin{gathered}
H^i(G, M \otimes_G \mathscr{L}_\alpha) = H^i_c(G, M \otimes_G \mathscr{L}_\alpha) = 0 \text{ for } i \neq 0; \\
H^0(G, M \otimes_G \mathscr{L}_\alpha) \simeq H^0_c(G, M \otimes_G \mathscr{L}_\alpha)
\end{gathered}$$

for every $\alpha \in V(k)$.

The equivalence between Theorems B and 3.1 follows from Proposition B.6. More generally, we refer the reader to the Appendix B, which contains a table comparing our notations with the standard references, for more information. Finally, throughout this section we suppose that k is an algebraically closed field of characteristic zero.

[7] Both ideas are already implicit in [KL85].

3.1. Relative generic vanishing

Given a connected algebraic group G over k, we will say that G *satisfies relative generic vanishing* if, for every smooth variety S over k and every object M of $D^b_h(\mathcal{D}_{G\times S})$, there exists a dense open subspace V of $G^\flat$ such that the forget-supports map

$$\mathrm{pr}_{S,!}(M \otimes_{G\times S} \mathrm{pr}_G^+ \mathscr{L}_\alpha) \to \mathrm{pr}_{S,+}(M \otimes_{G\times S} \mathrm{pr}_G^+ \mathscr{L}_\alpha),$$

where $\mathrm{pr}_S \colon G \times S \to S$ and $\mathrm{pr}_G \colon G \times S \to G$ are the projections, is an isomorphism for every $\alpha \in V(k)$. In this subsection we will establish some methods for proving such results.

Let X be a locally of finite type algebraic space over k. Our first lemma says that a dense open subset of $X(k)$, with its natural Zariski topology, gives rise to a dense open subset of X. Even though this result is surely well-known, we could not find a proof in the literature and so we provide one.

Lemma 3.2. *Let X be a locally of finite type algebraic space over k. The natural map $X(k) \to |X|$ is a bijection onto the finite type points of X, and we use this bijection to put a topology on $X(k)$. Then, the topological space $X(k)$ is very dense in $|X|$.*

Proof. By [Stacks, Tag 03E1], the map $X(k) \to |X|$ is injective. Now, let $\mathrm{Spec}\, K \to X$ be a finite type point of X. Since $X \to \mathrm{Spec}\, k$ is locally of finite type, so is $\mathrm{Spec}\, K \to X \to \mathrm{Spec}\, k$. Zariski's lemma then implies that K/k is a finite extension and so $K = k$, proving that the image of $X(k) \to |X|$ is the set of finite type points. The last statement is [Stacks, Tag 06EK]. ☐

For the reader's convenience, we remark that [GW10, Def. 3.34] has some different characterizations of very dense subsets. The first one implies that, given an open subset V' of $X(k)$, there exists an open subset V of X such that $V' = V \cap X(k)$. It is clear that if V' is dense, so is V.

Lemma 3.3. *Let G and H be two connected algebraic groups over k. If G and H satisfy relative generic vanishing, then so does $G \times H$.*

Proof. Consider the following commutative diagram, in which every morphism is a projection.

$$\begin{array}{ccccccc} G\times H\times S & & \xrightarrow{\quad q \quad} & & G\times H & \xrightarrow{q_1} & G \\ \downarrow{\scriptstyle p} & \searrow^{f} & & & \downarrow{\scriptstyle q_2} & & \\ S & \xleftarrow[p_2]{} & H\times S & \xrightarrow[p_1]{} & H & & \end{array}$$

We write a character sheaf on $G \times H$ as $\mathscr{L}_{(\alpha,\beta)} = \mathscr{L}_\alpha \boxtimes \mathscr{L}_\beta$ for some $(\alpha, \beta) \in G^\flat(k) \times H^\flat(k) \simeq (G \times H)^\flat(k)$. Let M be an object of $D^b_h(\mathcal{D}_{G\times H\times S})$. Our goal is to show that the forget-supports map

$$p_!(M \otimes_{G\times H\times S} q^+ \mathscr{L}_{(\alpha,\beta)}) \to p_+(M \otimes_{G\times H\times S} q^+ \mathscr{L}_{(\alpha,\beta)})$$

is an isomorphism *for most* α and β. By the commutativity of the diagram above, this is the same as

$$p_{2,!}f_!(M \otimes q^+q_1^+\mathscr{L}_\alpha \otimes q^+q_2^+\mathscr{L}_\beta) \to p_{2,+}f_+(M \otimes q^+q_1^+\mathscr{L}_\alpha \otimes q^+q_2^+\mathscr{L}_\beta),$$

where we ignore the subscripts in tensor products to simplify notation.

Using that $p_1 \circ f = q_2 \circ q$, the projection formula gives an isomorphism

$$p_{2,!}f_!(M \otimes q^+q_1^+\mathscr{L}_\alpha \otimes q^+q_2^+\mathscr{L}_\beta) \simeq p_{2,!}(f_!(M \otimes q^+q_1^+\mathscr{L}_\alpha) \otimes p_1^+\mathscr{L}_\beta).$$

Now, our hypothesis on G gives a dense open set $V \subset G^\flat$ such that

$$p_{2,!}(f_!(M \otimes q^+q_1^+\mathscr{L}_\alpha) \otimes p_1^+\mathscr{L}_\beta) \simeq p_{2,!}(f_+(M \otimes q^+q_1^+\mathscr{L}_\alpha) \otimes p_1^+\mathscr{L}_\beta)$$

for all $\alpha \in V(k)$. For each such α, our hypothesis on H gives a dense open set $U_\alpha \subset H^\flat$ such that

$$p_{2,!}(f_+(M \otimes q^+q_1^+\mathscr{L}_\alpha) \otimes p_1^+\mathscr{L}_\beta) \simeq p_{2,+}(f_+(M \otimes q^+q_1^+\mathscr{L}_\alpha) \otimes p_1^+\mathscr{L}_\beta)$$

holds for all $\beta \in U_\alpha(k)$. Another application of the projection formula finishes the proof. □

Given that a connected affine algebraic group L over k is necessarily a product of copies of $\mathbb{G}_a$ and $\mathbb{G}_m$, the preceding lemma implies that relative generic vanishing for L follows from relative generic vanishing for $\mathbb{G}_a$ and $\mathbb{G}_m$. The next lemmas will show that it even suffices to consider $S = \operatorname{Spec} k$.

Lemma 3.4. *Suppose that* G *satisfies relative generic vanishing for affine schemes* S *smooth over* k. *Then* G *satisfies relative generic vanishing in general.*

Proof. Let $j_l \colon U_l \to S$ be a finite open cover of S constituted of affine schemes U_l. Applying proper and smooth base change to the diagram

$$\begin{array}{ccc} G \times U_l & \xrightarrow{\mathrm{id}_G \times j_l} & G \times S \\ \downarrow{\scriptstyle \mathrm{pr}_{U_l}} & & \downarrow{\scriptstyle \mathrm{pr}_S} \\ U_l & \xrightarrow{j_l} & S, \end{array}$$

we obtain that $j_l^+ \mathrm{pr}_{S,!} \simeq \mathrm{pr}_{U_l,!}(\mathrm{id}_G \times j_l)^+$ and $j_l^+ \mathrm{pr}_{S,+} \simeq \mathrm{pr}_{U_l,+}(\mathrm{id}_G \times j_l)^+$. It follows that the restriction of the forget-supports map

$$\mathrm{pr}_{S,!}(M \otimes_X \mathrm{pr}_G^+ \mathscr{L}_\alpha) \to \mathrm{pr}_{S,+}(M \otimes_X \mathrm{pr}_G^+ \mathscr{L}_\alpha) \qquad (*)$$

to U_l is precisely the same morphism with S replaced by U_l. In particular, for each l there exists an open dense subset V_l of U^* making $j_l^+(*)$ an isomorphism for $\alpha \in V_l(k)$. All in all, we obtain that $*$ is an isomorphism for $\alpha \in (\bigcap_l V_l)(k) = \bigcap_l V_l(k)$. □

For the next lemma, recall that if $p\colon X \to Y$ is an affine morphism of schemes, then the underived direct image functor $p_*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$ is faithful. Indeed, under the equivalence $\mathsf{QCoh}(X) \simeq \mathsf{QCoh}(p_*\mathcal{O}_X)$ it becomes the forgetful functor.

Lemma 3.5. *Let $p\colon X \to Y$ be a smooth affine morphism between smooth varieties over k. Then $p_+\colon D^b_h(\mathcal{D}_X) \to D^b_h(\mathcal{D}_Y)$ is conservative and faithful.*

Proof. Let M be an object of $D^b_h(\mathcal{D}_X)$ and suppose that $p_+M = 0$. Since $p_+M \simeq p_*(\Omega^\bullet_{X/Y} \otimes^L_{\mathcal{O}_X} M)[d]$ [Dim+00, Prop. 1.4], where d is the relative dimension of p, the faithfulness of p_* implies that $\Omega^\bullet_{X/Y} \otimes^L_{\mathcal{O}_X} M$ vanishes. Now, if M is non-zero, let $i \in \mathbb{Z}$ be the smallest number such that $M^i \neq 0$. The left-most term in $\Omega^\bullet_{X/Y} \otimes^L_{\mathcal{O}_X} M$ is M^i, contradicting the hypothesis that M is non-zero. In other words, p_+ is conservative.

Similarly, if $\varphi\colon M \to N$ is a morphism in $D^b_h(\mathcal{D}_X)$ satisfying $p_+\varphi = 0$, the faithfulness of p_* implies that $\Omega^\bullet_{X/Y} \otimes^L_{\mathcal{O}_X} \varphi$ vanishes. Zariski-locally this morphism is nothing but the map $M^d \to N^d$ acting as φ on each component. It follows that φ has to vanish, proving that p_+ is faithful. □

Suppose that L is a one dimensional connected affine algebraic group over k. (Such a group is necessarily isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$, but both cases can be treated uniformly for now.) Consider a compactification

$$\begin{array}{ccccc} L & \xrightarrow{\ j\ } & \overline{L} & \xleftarrow{\ i\ } & Z \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle \overline{p}} & \swarrow{\scriptstyle q} & \\ & & \operatorname{Spec} k & & \end{array}$$

of L, and remark that we can suppose Z to be either a point or a disjoint union of two points. In any case, $q\colon Z \to \operatorname{Spec} k$ is a smooth affine morphism.

Lemma 3.6. *Let N be an object of $D^b_h(\mathcal{D}_L)$. Then the forget-supports map $j_!N \to j_+N$ is an isomorphism if and only if the forget-supports map $p_!N \to p_+N$ is.*

Proof. The cone of the forget-supports map $j_!N \to j_+N$ is $i_+i^+j_+N$ and, since i^+i_+ is isomorphic to the identity functor, it vanishes if and only if i^+j_+N does. Similarly, the cone of $p_!N \to p_+N$ is

$$\overline{p}_+i_+i^+j_+N \simeq q_+i^+j_+N,$$

which, by the previous lemma, vanishes precisely when i^+j_+N does. □

Putting together all of the preceding lemmata, we obtain the main result of this subsection.

Proposition 3.7. *Let L be a connected affine algebraic group over k. In order to prove that L satisfies relative generic vanishing, we may suppose that L has dimension one and that $S = \operatorname{Spec} k$.*

Proof. Suppose that L is one-dimensional and that, for all $N \in D^b_h(\mathcal{D}_L)$, there exists a dense open subset V of $L^\flat$ such that the forget-supports map

$$p_!(N \otimes_L \mathscr{L}_\alpha) \to p_+(N \otimes_L \mathscr{L}_\alpha)$$

is an isomorphism for $\alpha \in V(k)$. Finally, let S be an affine scheme smooth over k and let M be an object of $D^b_h(\mathcal{D}_{L\times S})$.

Consider the following diagram, whose squares are cartesian and whose columns are recollement sequences.

$$\begin{array}{ccc}
L \times S & \xrightarrow{\mathrm{pr}_L} & L \\
{\scriptstyle j\times \mathrm{id}_S}\downarrow & & \downarrow{\scriptstyle j} \\
\overline{L} \times S & \xrightarrow{\mathrm{pr}_{\overline{L}}} & \overline{L} \\
{\scriptstyle i\times \mathrm{id}_S}\uparrow & & \uparrow{\scriptstyle i} \\
Z \times S & \xrightarrow[\mathrm{pr}_Z]{} & Z
\end{array}$$

Our supposition, together with Lemma 3.6, gives a dense open subset $V \subset L^\flat$ such that $i^+ j_+(\mathrm{pr}_{L,+}(M) \otimes_L \mathscr{L}_\alpha) = 0$ for all $\alpha \in V(k)$. Fix some $\alpha \in V(k)$. By smooth base change and the projection formula,

$$\begin{aligned}
0 &= \mathrm{pr}_Z^+ \, i^+ j_+(\mathrm{pr}_{L,+}(M) \otimes_L \mathscr{L}_\alpha) \\
&= (i \times \mathrm{id}_S)^+ \, \mathrm{pr}_{\overline{L}}^+ \, j_+(\mathrm{pr}_{L,+}(M) \otimes_L \mathscr{L}_\alpha) \\
&= (i \times \mathrm{id}_S)^+ (j \times \mathrm{id}_S)_+ \, \mathrm{pr}_L^+(\mathrm{pr}_{L,+}(M) \otimes_L \mathscr{L}_\alpha) \\
&= (i \times \mathrm{id}_S)^+ (j \times \mathrm{id}_S)_+ \, \mathrm{pr}_L^+ \, \mathrm{pr}_{L,+}(M \otimes_{L\times S} \mathrm{pr}_L^+ \mathscr{L}_\alpha).
\end{aligned}$$

Since S is affine, Lemma 3.5 implies that the functor $\mathrm{pr}_{L,+}$ is faithful. This is equivalent to the counit $\mathrm{pr}_L^+ \, \mathrm{pr}_{L,+} \to \mathrm{id}$ being a point-wise epimorphism. Since epimorphisms in triangulated categories split, they are absolute (preserved by any functor) and we conclude that

$$(i\times \mathrm{id}_S)^+(j\times \mathrm{id}_S)_+ \, \mathrm{pr}_L^+ \, \mathrm{pr}_{L,+}(M \otimes_{L\times S} \mathrm{pr}_L^+ \mathscr{L}_\alpha) \to (i\times \mathrm{id}_S)^+(j\times \mathrm{id}_S)_+(M \otimes_{L\times S} \mathrm{pr}_L^+ \mathscr{L}_\alpha)$$

is an epimorphism. It follows that its codomain vanishes, and so does the cone of the forget-supports map $\mathrm{pr}_{S,!}(M \otimes_{L\times S} \mathrm{pr}_L^+ \mathscr{L}_\alpha) \to \mathrm{pr}_{S,+}(M \otimes_{L\times S} \mathrm{pr}_L^+ \mathscr{L}_\alpha)$. □

3.2. Unipotent groups

Let U be an n-dimensional unipotent group. We recall that $U^\flat$ is isomorphic to the vector space dual U^*. In particular, the latter parametrizes character sheaves on U. The key tool of this subsection is the Fourier transform for holonomic $\mathcal{D}$-modules, and so we give a quick sketch of its main properties. (We refer the reader to [Dai00] for proofs.)

Consider the evaluation map $\sigma\colon U \times U^* \to \mathbb{A}^1$ and the *exponential character sheaf* $\mathscr{E} := (\mathcal{O}_{\mathbb{A}^1}, d + dx)[-1]$ on $\mathbb{A}^1$. We define the *Fourier transform* functor $\mathrm{FT}_U\colon \mathsf{D}^b_h(\mathcal{D}_U) \to \mathsf{D}^b_h(\mathcal{D}_{U^*})$ as

$$\mathrm{FT}_U := \mathrm{pr}_{2,+}(\mathrm{pr}_1^+(-) \otimes_{U\times U^*} \sigma^+\mathscr{E}[n]),$$

where $\mathrm{pr}_1\colon U \times U^* \to U$ and $\mathrm{pr}_2\colon U \times U^* \to U^*$ are the canonical projections. This operation has a plethora of wonderful properties, but we will content ourselves with explaining those strictly needed for our purposes:

- The functor FT_U is t-exact with respect to the canonical t-structures on $\mathsf{D}^b_h(\mathcal{D}_U)$ and $\mathsf{D}^b_h(\mathcal{D}_{U^*})$;
- Let $\mathrm{FT}_{U,!} := \mathrm{pr}_{2,!}(\mathrm{pr}_1^+(-) \otimes_{U\times U^*} \sigma^+\mathscr{E}[n])$ be the "proper Fourier transform". The forget-supports map $\mathrm{FT}_{U,!} \to \mathrm{FT}_U$ is an isomorphism;
- We have an isomorphism of functors $\mathrm{D}_{U^*} \circ \mathrm{FT}_U \simeq \mathrm{inv}_{U^*}^+ \circ \mathrm{FT}_U \circ \mathrm{D}_U$, where D denotes the duality functor.

The reason for the importance of the Fourier transform on the generic vanishing theorem is the fact that $\mathrm{FT}_U(M)$ contains at once the data of all cohomology groups of every character twist of M.

Proposition 3.8. *Let M be an object of $\mathsf{D}^b_h(\mathcal{D}_U)$. Then,*

$$p_!(M \otimes_U \mathscr{L}_\alpha) \simeq \alpha^+ \mathrm{FT}_U(M)[-n] \quad \textit{and} \quad p_+(M \otimes_U \mathscr{L}_\alpha) \simeq \alpha^! \mathrm{FT}_U(M)[n]$$

hold for every $\alpha \in U^*(k)$.

Proof. An application of the proper base change theorem on the cartesian diagram

$$\begin{array}{ccc} U \times \operatorname{Spec} k & \xrightarrow{\mathrm{id}\times\alpha} & U \times U^* \\ \downarrow{\scriptstyle \overline{\mathrm{pr}}_1} & & \\ U & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ \downarrow{\scriptstyle p} & & \\ \operatorname{Spec} k & \xrightarrow{\alpha} & U^* \end{array}$$

gives that $\alpha^+ \circ \mathrm{pr}_{2,!} \simeq p_! \circ \overline{\mathrm{pr}}_{1,!} \circ (\mathrm{id}\times\alpha)^+$. Applying $\mathrm{pr}_1^+ M \otimes_{U\times U^*} \sigma^+\mathscr{E}$ to both sides, we obtain

$$\alpha^+ \mathrm{FT}_U(M)[-n] \simeq p_!\overline{\mathrm{pr}}_{1,!}N,$$

where $N = (\mathrm{id}\times\alpha)^+ \mathrm{pr}_1^+ M \otimes_{U\times\operatorname{Spec} k} (\mathrm{id}\times\alpha)^+\sigma^+\mathscr{E}$. Since (id, p) is the inverse of $\overline{\mathrm{pr}}_1$, the functors $\overline{\mathrm{pr}}_{1,!}$ and $(\mathrm{id}, p)^+$ are isomorphic. In particular,

$$\overline{\mathrm{pr}}_{1,!}N \simeq (\mathrm{id}, p)^+(\mathrm{id}\times\alpha)^+ \mathrm{pr}_1^+ M \otimes_U (\mathrm{id}, p)^+(\mathrm{id}\times\alpha)^+\sigma^+\mathscr{E} \simeq M \otimes_U \sigma(-, \alpha)^+\mathscr{E},$$

for $\mathrm{pr}_1 \circ (\mathrm{id} \times \alpha) \circ (\mathrm{id}, p) = \mathrm{id}$ and $\sigma \circ (\mathrm{id} \times \alpha) \circ (\mathrm{id}, p) = \sigma(-, \alpha)$.

Upon a choice of coordinates, both U and U^* become isomorphic to $\mathbb{G}_a^n$, and so the map $\sigma(-, \alpha)$ acts as

$$(x_1, \ldots, x_n) \mapsto \alpha_1 x_1 + \ldots + \alpha_n x_n.$$

It follows that $\sigma(-, \alpha)^+ \mathscr{E} \simeq \mathscr{L}_\alpha$ and we obtain the first desired isomorphism. The second then follows by duality. □

Now, the generic vanishing becomes a direct consequence.

Corollary 3.9. *Let M be an object of $D_h^b(\mathcal{D}_U)$. There exists a dense open subset $V \subset U^*$ such that the forget-supports map*

$$H_c^i(G, M \otimes_U \mathscr{L}_\alpha) \to H^i(G, M \otimes_U \mathscr{L}_\alpha)$$

is an isomorphism for all $\alpha \in V(k)$ and all $i \in \mathbb{Z}$. Moreover, if M is concentrated in degree zero, those cohomology groups vanish for $i \neq 0$.

Proof. As $\mathrm{FT}_U(M)$ is a bounded complex of holonomic $\mathcal{D}$-modules, there exists a dense open subset $V \subset U^*$ such that $\mathscr{H}^i(\mathrm{FT}_U(M)|_V)$ is locally free for all $i \in \mathbb{Z}$. Then, if $\alpha \in V(k)$, we have that [Bor+87, Rem. in §VI.4]

$$H^i(G, M \otimes_G \mathscr{L}_\alpha) \simeq \mathscr{H}^{i+n}(\alpha^! \,\mathrm{FT}_U(M)|_V) \simeq \alpha^* \mathscr{H}^i(\mathrm{FT}_U(M)|_V)$$

and

$$H_c^i(G, M \otimes_G \mathscr{L}_\alpha) \simeq \mathscr{H}^{i-n}(\alpha^+ \,\mathrm{FT}_U(M)|_V) \simeq \alpha^* \mathscr{H}^i(\mathrm{FT}_U(M)|_V).$$

The result is now clear. □

Combining the Proposition 3.7 with Corollary 3.9, we obtain the relative generic vanishing theorem for unipotent groups.

Proposition 3.10. *Let S be a smooth variety over k and let M be an object of $D_h^b(\mathcal{D}_X)$, where $X = U \times S$. There exists a dense open subspace V of $U^\flat$ such that the forget-supports map*

$$\mathrm{pr}_{S,!}(M \otimes_X \mathrm{pr}_U^+ \mathscr{L}_\alpha) \to \mathrm{pr}_{S,+}(M \otimes_X \mathrm{pr}_U^+ \mathscr{L}_\alpha),$$

where $\mathrm{pr}_S \colon X \to S$ and $\mathrm{pr}_U \colon X \to U$ are the projections, is an isomorphism for every $\alpha \in V(k)$.

We remark that one could define a relative Fourier transform $\mathrm{FT}_X \colon D_h^b(\mathcal{D}_{U\times S}) \to D_h^b(\mathcal{D}_{U^*\times S})$ and it would still be true that $\mathrm{pr}_{S,!}(M \otimes_X \mathrm{pr}_U^+ \mathscr{L}_\alpha) \simeq (\alpha, \mathrm{id}_S)^+ \,\mathrm{FT}_X(M)[-n]$. However, even though there exists an open dense subset of $U^* \times S$ over which $\mathrm{FT}_X(M)$ has locally free cohomology sheaves, this subset may not be of the form $V \times S$ for some open dense subset V of U^*.

3.3. Tori

Let T be a torus over k. Inspired by [KL85, Thm. 6.5], we will use monodromical arguments to prove relative generic vanishing for tori.

Proposition 3.11. *Let S be a smooth variety over k and let M be an object of $D^b_h(\mathcal{D}_X)$, where $X = T \times S$. There exists a dense open subspace V of $T^\flat$ such that the forget-supports map*

$$\mathrm{pr}_{S,!}(M \otimes_X \mathrm{pr}_T^+ \mathscr{L}_\alpha) \to \mathrm{pr}_{S,+}(M \otimes_X \mathrm{pr}_T^+ \mathscr{L}_\alpha),$$

where $\mathrm{pr}_S \colon X \to S$ and $\mathrm{pr}_T \colon X \to T$ are the projections, is an isomorphism for every $\alpha \in V(k)$.

Due to Proposition 3.7, one can suppose that $T = \mathbb{G}_m$ and that $S = \operatorname{Spec} k$. Also, by Lemmas 3.2 and 3.6, it suffices to obtain a finite subset $F \subset k/\mathbb{Z}$ such that

$$j_!(M \otimes_{\mathbb{G}_m} \mathscr{L}_\alpha) \to j_+(M \otimes_{\mathbb{G}_m} \mathscr{L}_\alpha)$$

is an isomorphism for $\alpha \in (k/\mathbb{Z}) \setminus F$, where $j \colon \mathbb{G}_m \to \mathbb{P}^1$ is the usual compactification of $\mathbb{G}_m$.

$$\begin{array}{ccccc} \mathbb{G}_m & \overset{j}{\hookrightarrow} & \mathbb{P}^1 & \overset{i}{\hookleftarrow} & \{0, \infty\} \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle \overline{p}} & & \\ & & \operatorname{Spec} k & & \end{array}$$

Moreover, as both $j_!$ and j_+ are exact functors, we may suppose that M is concentrated in degree zero.

The following lemma gives a general criterion for dealing with these kinds of problems using the V-filtration of M. Kashiwara, B. Malgrange and C. Sabbah.

Lemma 3.12. *Let $j \colon U \to X$ be an open immersion between smooth k-varieties, with complementary closed immersion $i \colon Z \to X$. Suppose that Z is smooth and of codimension 1. Given a holonomic $\mathcal{D}$-module N over U, the forget-supports map*

$$j_! N \to j_+ N$$

is an isomorphism if and only if $\mathrm{gr}_0^V(j_+ N)$ vanishes.

Since the V-filtration is the generalization of nearby and vanishing cycles to holonomic $\mathcal{D}$-modules, we will give a proof of this result focusing on standard properties of nearby and vanishing cycles. Once again, we refer the reader to Appendix B for more on the V-filtration.

Proof of Lemma 3.12. In this proof we will adopt the usual notations for the six-functors and nearby / vanishing cycles in either the analytic setting or in ℓ-adic cohomology. We recall that $\mathrm{gr}_0^V(j_+N)$ is the $\mathcal{D}$-module analog of the unipotent vanishing cycle functor $\phi_1(j_*N)$. Now, a first observation is that we have a *recollement* distinguished triangle

$$j_!N \to j_*N \to i_*i^*j_*N,$$

and so the forget-supports map is an isomorphism if and only if $i_*i^*j_*N = 0$. Moreover, i^*i_* is isomorphic to the identity functor and so this happens precisely when i^*j_*N vanishes.

Now, the usual theory of nearby and vanishing cycles gives two other distinguished triangles:

$$\begin{aligned} i^*j_*N &\to \psi_1(j_*N) \xrightarrow{\mathrm{can}} \phi_1(j_*N) \\ i^!j_*N &\to \phi_1(j_*N) \xrightarrow{\mathrm{var}} \psi_1(j_*N). \end{aligned}$$

The first triangle shows that the forget supports map is an isomorphism if and only if can is. Since $i^!j_* = 0$, the second triangle gives that $\psi_1(j_*N)$ and $\phi_1(j_*N)$ are isomorphic. In particular, if $\phi_1(j_*N)$ vanishes, then so does $\psi_1(j_*N)$ and can is an isomorphism.

Conversely, if the forget supports map is an isomorphism, our reasoning shows that can is an isomorphism as well. Now, var is also an isomorphism (since $i^!j_* = 0$) and then so is $\mathrm{can} \circ \mathrm{var}$. But this morphism is nilpotent, which implies that it is zero. It follows that $\phi_1(j_*N) = 0$. □

Consider the following commutative diagram, in which every row is a recollement sequence.

$$\begin{array}{ccccc} \mathbb{A}^1 & \xhookrightarrow{j_\infty} & \mathbb{P}^1 & \xhookleftarrow{i_\infty} & \{\infty\} \\ {\scriptstyle \bar{j}}\uparrow & & \| & & \downarrow{\scriptstyle h_\infty} \\ \mathbb{G}_m & \xhookrightarrow{j} & \mathbb{P}^1 & \xhookleftarrow{i} & \{0, \infty\} \\ {\scriptstyle \bar{j}}\downarrow & & \| & & \uparrow{\scriptstyle h_0} \\ \mathbb{A}^1 & \xhookrightarrow[j_0]{} & \mathbb{P}^1 & \xhookleftarrow[i_0]{} & \{0\} \end{array}$$

As the proof of Lemma 3.12 shows, the forget-supports map $j_!N \to j_+N$ is an isomorphism if and only if i^+j_+N vanishes. Now, by the Mayer-Vietoris distinguished triangle, we have that

$$i^+j_+N \simeq h_{0,+}\underbrace{h_0^+i^+}_{i_0^+}j_+N \oplus h_{\infty,+}\underbrace{h_\infty^+i^+}_{i_\infty^+}j_+N.$$

In particular, i^+j_+N vanishes precisely if both $h_{0,+}i_0^+j_+N$ and $h_{\infty,+}i_\infty^+j_+N$ do. But h_0 and h_∞ are closed immersions, and so this happens if and only if $i_0^+j_+N = i_\infty^+j_+N = 0$.

At this point, we can continue the proof of Lemma 3.12 as usual to conclude that $j_!N \to j_+N$ is an isomorphism if and only if $\mathrm{gr}_0^V(j_+N)$ vanishes for both V-filtrations. We are finally in position to finish the proof of Proposition 3.11.

Proof of Proposition 3.11. The discussion above shows that it suffices to prove the following statement: given a holonomic $\mathcal{D}$-module M over $\mathbb{G}_m$, there is a finite subset $F \subset k/\mathbb{Z}$ such that

$$\mathrm{gr}_0^V(j_+(M \otimes_{\mathbb{G}_m} \mathscr{L}_\alpha)) = 0$$

for all $\alpha \in (k/\mathbb{Z}) \setminus F$, and for both embeddings of $\mathbb{A}^1$ in $\mathbb{P}^1$.

Let us first analyse the embedding of $\mathbb{A}^1$ in $\mathbb{P}^1$ with complement $\{0\}$, along with its V-filtration. Using the notations of the diagram above, if t is the global coordinate of $\mathbb{P}^1 \setminus \{\infty\}$, we have that[8]

$$\begin{aligned}
[j_+(M \otimes_{\mathbb{G}_m} \mathscr{L}_\alpha)]_{\mathbb{P}^1\setminus\{\infty\}} &\simeq \bar{j}_+(M \otimes_{\mathbb{G}_m} \mathscr{L}_\alpha) \\
&\simeq \bar{j}_+(M \otimes^!_{\mathbb{G}_m} \mathscr{L}_\alpha)[2] \\
&\simeq \bar{j}_+ M \otimes^!_{\mathbb{A}^1} \bar{j}_+ \mathscr{L}_\alpha[2] \\
&\simeq \bar{j}_+ M \otimes^L_{\mathcal{O}_{\mathbb{A}^1}} \bar{j}_+ \mathscr{L}_\alpha[1] \\
&\simeq \bar{j}_+ M \otimes^L_{\mathcal{O}_{\mathbb{A}^1}} \bar{j}_+ [\mathcal{D}_{\mathbb{G}_m}/\mathcal{D}_{\mathbb{G}_m}(t\partial_t - \alpha)]\,.
\end{aligned}$$

Since $\bar{j}_+[\mathcal{D}_{\mathbb{G}_m}/\mathcal{D}_{\mathbb{G}_m}(t\partial_t - \alpha)] \simeq \bar{j}_*[\mathcal{D}_{\mathbb{G}_m}/\mathcal{D}_{\mathbb{G}_m}(t\partial_t - \alpha)]$ is a free $\mathcal{O}_{\mathbb{A}^1}(*\{0\})$-module, and tensoring over $\mathcal{O}_{\mathbb{A}^1}$ is the same as tensoring over $\mathcal{O}_{\mathbb{A}^1}(*\{0\})$, we conclude that

$$[j_+(M \otimes_{\mathbb{G}_m} \mathscr{L}_\alpha)]_{(\mathbb{P}^1\setminus\{\infty\})} \simeq \bar{j}_+ M \otimes_{\mathcal{O}_{\mathbb{A}^1}(*\{0\})} \bar{j}_+[\mathcal{D}_{\mathbb{G}_m}/\mathcal{D}_{\mathbb{G}_m}(t\partial_t - \alpha)].$$

In particular, the element $t\partial_t$ in the ring of differential operators of $\mathbb{P}^1 \setminus \{\infty\}$ acts as

$$m \otimes 1 \mapsto t\partial_t(m) \otimes 1 + m \otimes t\partial_t(1) = t\partial_t(m) \otimes 1 + m \otimes \alpha 1 = (t\partial_t + \alpha)(m \otimes 1).$$

Finally, [Sab87, Prop. 2.3.2] implies that $\mathrm{gr}_0^V(j_+(M \otimes_{\mathbb{G}_m} \mathscr{L}_\alpha)) \simeq \mathrm{gr}_\alpha^V(j_+M)$.

If we had chosen the embedding with complement $\{\infty\}$, the operator $t\partial_t$ would become $-t\partial_t$ and so we would have $\mathrm{gr}_0^V(j_+(M \otimes_{\mathbb{G}_m} \mathscr{L}_\alpha)) \simeq \mathrm{gr}_{-\alpha}^V(j_+M)$. In particular, both vanish unless $\pm\alpha$ is a zero of the Bernstein-Sato polynomials associated to i_0 and i_∞. This finishes the proof. □

3.4. Abelian varieties

Let A be an abelian variety over k. The generic vanishing theorem for A follows directly from the Fourier-Mukai transform defined by Laumon [Lau85; Lau96], together with Schnell's work on holonomic $\mathcal{D}$-modules on abelian varieties [Sch15]. We refer to Subsection A.2 for the needed facts on relative $\mathcal{D}$-modules.

The identity map $A^\natural \to A^\natural$ defines a section in $\underline{\mathrm{Ext}}^1(A_{dR}, \mathbb{G}_m)(A^\natural)$ that, according to our interpretation explained after Definition 2.20, is a line bundle $\mathcal{P}$ on $A \times A^\natural$ endowed

[8] Let $j\colon U \to X$ be an open immersion and let $M, N \in D^b_h(\mathcal{D}_U)$. By recollement and the projection formula we have that $j_! M \otimes_X j_! N \simeq j_!(M \otimes_U j^+ j_! N) \simeq j_!(M \otimes_U N)$. Dually, we have that $j_+ M \otimes^!_X j_+ N \simeq j_+(M \otimes^!_U N)$. This formula was used in the third isomorphism below.

with an integrable connection relative to $A^\natural$. In other words, $\mathcal{P}$ is a $\mathcal{D}$-module on $A \times A^\natural$ relative to $A^\natural$. This is the kernel of the so-called *Fourier-Mukai* transform

$$\begin{aligned} \mathrm{FM}_A \colon \mathrm{D}_{qc}(\mathcal{D}_A) &\to \mathrm{D}_{qc}(\mathcal{O}_{A^\natural}) \\ M &\mapsto \mathrm{pr}_{2,+}((\mathrm{pr}_1, q)^* M \otimes_{\mathcal{O}_{A \times A^\natural}} \mathcal{P}). \end{aligned}$$

All the morphisms involved in the definition above are the projections appearing in the cartesian square

$$\begin{array}{ccc} A \times A^\natural & \xrightarrow{\mathrm{pr}_1} & A \\ {\scriptstyle \mathrm{pr}_2}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ A^\natural & \xrightarrow[q]{} & \operatorname{Spec} k. \end{array}$$

The Fourier-Mukai transform restricts to a functor $\mathrm{D}^b_{\mathrm{coh}}(\mathcal{D}_A) \to \mathrm{D}^b_{\mathrm{coh}}(\mathcal{O}_{A^\natural})$ [Lau96, Cor. 3.1.3] that, similarly to the Fourier transform for unipotent groups in Subsection 3.2, satisfies the proposition below.

Proposition 3.13. *Let M be an object of $\mathrm{D}^b_{coh}(\mathcal{D}_A)$. Then,*

$$p_+(M \otimes_A \mathscr{L}_\alpha) \simeq L\alpha^* \, \mathrm{FM}_A(M)$$

hold for every $\alpha \in A^\natural(k)$.

Proof. The same exact proof as in Proposition 3.8 works here. The reader may find the needed base change and projection formula theorems in [Vig21]. □

In the unipotent case, the analogous of the preceding proposition implied the generic vanishing theorem via two key properties of the Fourier transform: the fact that it preserves holonomicity and that it is t-exact with respect to the standard t-structures. Here, the former does not make sense and the latter is false.

Schnell's main idea was to consider a different t-structure on $\mathrm{D}^b_{\mathrm{coh}}(\mathcal{O}_{A^\natural})$; the perverse t-structure defined by Kashiwara in [Kas04].

Theorem 3.14 (Kashiwara, Arinkin-Bezrukavnikov)**.** *Let X be a smooth variety over k. The following pair of full subcategories*

$$\begin{aligned} {}^m\mathrm{D}^b_{coh}(\mathcal{O}_X)^{\leqslant 0} &:= \{M \in \mathrm{D}^b_{coh}(\mathcal{O}_X) \mid \operatorname{codim} \operatorname{Supp} \mathscr{H}^i(M) \geqslant 2i \textit{ for all } i \in \mathbb{Z}\} \\ {}^m\mathrm{D}^b_{coh}(\mathcal{O}_X)^{\geqslant 0} &:= \{M \in \mathrm{D}^b_{coh}(\mathcal{O}_X) \mid \operatorname{codim} \operatorname{Supp} R^i\underline{\mathrm{Hom}}(M, \mathcal{O}_X) \geqslant 2i - 1 \textit{ for all } i \in \mathbb{Z}\} \end{aligned}$$

defines a bounded t-structure on $\mathrm{D}^b_{coh}(\mathcal{O}_X)$.

Proof. See [Kas04, Thm. 5.9], [AB10, Thm. 3.10] and [Sch15, Lemma 18.4]. □

The main result in [Sch15] is the fact that, when restricted to $\mathrm{D}^b_h(\mathcal{D}_A)$, the Fourier-Mukai transform is a t-exact functor with respect to the standard t-structure on $\mathrm{D}^b_h(\mathcal{D}_A)$ and the perverse t-structure on $\mathrm{D}^b_{\mathrm{coh}}(\mathcal{O}_{A^\natural})$. More precisely, [Sch15, Thm. 19.1] gives the description below.

Theorem 3.15 (Schnell)**.** *Let* M *be an object of* $D^b_h(\mathcal{D}_A)$*. Then, for an integer* i*,* M *lies in* $D^b_h(\mathcal{D}_A)^{\leqslant i}$ *if and only if* $\mathrm{FM}_A(M)$ *lies in* ${}^m D^b_{coh}(\mathcal{O}_{A^\natural})^{\leqslant i}$*. Similarly,* M *lies in* $D^b_h(\mathcal{D}_A)^{\geqslant i}$ *if and only if* $\mathrm{FM}_A(M)$ *lies in* ${}^m D^b_{coh}(\mathcal{O}_{A^\natural})^{\geqslant i}$*.*

This incredible theorem, along with the Proposition 3.13, subsumes the generic vanishing theorem for abelian varieties.

Corollary 3.16. *Let* M *be a holonomic* $\mathcal{D}$*-module over* A*. There exists a dense open subspace* V *of* $A^\natural$ *such that*

$$H^i(A, M \otimes_A \mathscr{L}_\alpha) = 0$$

for every $i \neq 0$ *and every* $\alpha \in V(k)$*.*

Proof. Just as in the proof of Corollary 3.9, it suffices to obtain a dense open subspace of $A^\natural$ over which $\mathrm{FM}_A(M)|_V$ is a locally free sheaf concentrated in degree zero. Now, Schnell's theorem says that $\mathrm{FM}_A(M)$ is a perverse coherent sheaf and so we have

$$\operatorname{codim} \operatorname{Supp} \mathscr{H}^i(\mathrm{FM}_A(M)) \geqslant 2i$$

for every integer i. This inequality, along with [Sch15, Lemma 18.5], implies that $\mathrm{FM}_A(M)$ is concentrated in degree zero over a dense open subspace. The result then follows by generic freeness [Stacks, Tag 051S]. □

We remark that a version of Corollary 3.16 for regular holonomic $\mathcal{D}$-modules over abelian varieties has also been proven by Krämer and Weissauer [KW15, Thm. 1.1].

3.5. The general case

We now go back to the general case in which G is a connected algebraic group over k. Recall that such a group necessarily fits into a short exact sequence

$$0 \to T \times U \xrightarrow{\varphi} G \xrightarrow{\psi} A \to 0,$$

in which T is a torus, U is a unipotent group, and A is an abelian variety. We denote by $\mathscr{L}_\omega$ the character sheaf $(\mathcal{O}_G, d + \omega)[-\dim G]$ defined by an invariant differential $\omega \in \Omega_G$. Using this notation, the main observation needed for this subsection comes from Corollary 2.38: every character sheaf on G is of the form

$$\mathscr{L}_\omega \otimes_G \psi^+ \mathscr{L}_\alpha,$$

for some $\omega \in \Omega_G$ and $\alpha \in A^\flat(k)$.

We begin the proof of the generic vanishing theorem by a lemma that generalizes Propositions 3.10 and 3.11.

Lemma 3.17. *Let M be an object of $D^b_h(\mathcal{D}_G)$. There exists a dense open subset V of $\Omega_T/X \times \Omega_U$ such that the forget-supports map*

$$\psi_!(M \otimes_G \mathscr{L}_\omega) \to \psi_+(M \otimes_G \mathscr{L}_\omega),$$

is an isomorphism for every $\omega \in \Omega_G$ such that $\varphi^*\omega \in V(k)$.

Proof. Let $\{S_i \to A\}_{i\in I}$ be an étale covering trivializing the $T \times U$-torsor $G \to A$. (By [SGA4.II, Exposé VI, 1.6.2] we may suppose that I is finite.) In particular, the following diagram

$$\begin{array}{ccccc} T \times U \times A \times S_i & \xrightarrow{\mathrm{pr}_{A\times S_i}} & A \times S_i & \xrightarrow{\mathrm{pr}_{S_i}} & S_i \\ \downarrow{\scriptstyle \mathrm{pr}_{T\times U}} & & & & \downarrow \\ T \times U & & & & \\ \downarrow{\scriptstyle \varphi} & & & & \\ G & & \xrightarrow{\quad\quad\psi\quad\quad} & & A \end{array}$$

is cartesian for all $i \in I$. Since checking whether or not the forget-supports map $\psi_!(M \otimes_G \mathscr{L}_\omega) \to \psi_+(M \otimes_G \mathscr{L}_\omega)$ is an isomorphism can be done étale-locally on A, it suffices to check that

$$\mathrm{pr}_{A\times S_i,!}(\mathrm{pr}^+_{T\times U}\,\varphi^+ M \otimes \mathrm{pr}^+_{T\times U}\,\varphi^+\mathscr{L}_\omega) \to \mathrm{pr}_{A\times S_i,+}(\mathrm{pr}^+_{T\times U}\,\varphi^+ M \otimes \mathrm{pr}^+_{T\times U}\,\varphi^+\mathscr{L}_\omega)$$

is an isomorphism for all $i \in I$. Combining Propositions 3.10 and 3.11 by means of the Lemma 3.3, we obtain dense open subsets V_i of $T^\flat \times U^\flat \simeq \Omega_T/X \times \Omega_U$ making

$$\mathrm{pr}_{A\times S_i,!}(\mathrm{pr}^+_{T\times U}\,\varphi^+ M \otimes \mathrm{pr}^+_{T\times U}\,\mathscr{L}_\alpha) \to \mathrm{pr}_{A\times S_i,+}(\mathrm{pr}^+_{T\times U}\,\varphi^+ M \otimes \mathrm{pr}^+_{T\times U}\,\mathscr{L}_\alpha)$$

an isomorphism for all $\alpha \in V_i(k)$. The result then follows by taking the intersection of the V_i, for $i \in I$. □

We are finally able to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Let M be a holonomic $\mathcal{D}$-module on G. Since G is a $T \times U$-torsor over A, descent implies that the map $\psi\colon G \to A$ is affine. Then, the analogue of Artin vanishing for $\mathcal{D}$-modules [Bor+87, Prop. VI.8.1] implies that whenever the forget-supports map

$$\psi_!(M \otimes_G \mathscr{L}_\omega) \to \psi_+(M \otimes_G \mathscr{L}_\omega)$$

is an isomorphism, both sides are concentrated in degree zero.

Lemma 3.17 gives a dense open subset V of $\Omega_T/X \times \Omega_U$ such that the map above is an isomorphism for all $\omega \in \Omega_G$ satisfying $\varphi^*\omega \in V(k)$. Fix one such ω and let $\alpha \in A^\flat(k)$. The projection formula implies that

$$\begin{aligned} \psi_!(M \otimes_G \mathscr{L}_\omega \otimes_G \psi^+\mathscr{L}_\alpha) &\simeq \psi_!(M \otimes_G \mathscr{L}_\omega) \otimes_A \mathscr{L}_\alpha \\ &\simeq \psi_+(M \otimes_G \mathscr{L}_\omega) \otimes_A \mathscr{L}_\alpha \\ &\simeq \psi_+(M \otimes_G \mathscr{L}_\omega \otimes_G \psi^+\mathscr{L}_\alpha), \end{aligned}$$

and so the generic vanishing theorem for abelian varieties, applied to $\psi_!(M \otimes_G \mathscr{L}_\omega) \simeq \psi_+(M \otimes_G \mathscr{L}_\omega)$, finishes the proof. $\square$

A. Crystals and de Rham spaces

As it was first observed by Simpson [Sim96], given a choice of cohomology theory H and a "space" X, there is often a stack X_H whose category of quasi-coherent sheaves coincides with the category of coefficients for H. Moreover, the association $X \mapsto X_H$ preserves the functoriality of the given cohomology theory.

In this appendix, we study the de Rham side of this story. Namely, given a variety X, the *de Rham space* X_{dR} has the marvellous property that quasi-coherent sheaves over it are the same as quasi-coherent $\mathcal{D}_X$-modules. Moreover, formal completions can also be understood in function of the de Rham spaces.

The author claims no originality for any result in this appendix: all the results in it are either available in the literature or are folklore. (See [GR17] and [Hen17] for more on this.) However, even the results that have published proofs are usually studied in the context of (derived) prestacks, so we thought that this appendix could be helpful to some readers.

A.1. Basic properties of the de Rham space

Let k be a field and consider the category Aff/k of affine schemes over k. In order to simplify notation, we will often denote an object $\operatorname{Spec} R$ of Aff/k as R.

Definition A.1 (de Rham space)**.** Given a presheaf X on Aff/k, its *de Rham space* X_{dR} is the presheaf defined by

$$X_{dR}(R) := \operatorname*{colim}_{I \subset R} X(R/I),$$

where the colimit runs through the filtered poset of nilpotent ideals of R. This presheaf comes equipped with a morphism $X \to X_{dR}$ induced by the trivial ideal $I = 0$.

We remark that this assignment is functorial: given a morphism $f\colon X \to Y$ of presheaves, there is an induced map $f_{dR}\colon X_{dR} \to Y_{dR}$ making the diagram

$$\begin{array}{ccc} X & \longrightarrow & X_{dR} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_{dR}} \\ Y & \longrightarrow & Y_{dR} \end{array}$$

commute. As it will be formalized in Corollary A.7, the geometric interpretation of X_{dR}, at least for smooth schemes X, is that it is a quotient of X where we identify infinitesimally close points.

We begin our study of the de Rham space by this following simple observation which is going to be useful later.

Proposition A.2. *The functor* $(-)_{\mathrm{dR}}\colon \mathsf{PSh}(\mathsf{Aff}/k) \to \mathsf{PSh}(\mathsf{Aff}/k)$ *preserves arbitrary colimits and finite limits.*

Proof. Since (co)limits of presheaves are computed pointwise, this is nothing but the fact that filtered colimits in the category of sets commute with arbitrary colimits and finite limits. □

Given a finite type k-algebra R (more generally, a noetherian k-algebra), its nilradical $\mathrm{Nil}(R)$ is nilpotent for it is generated by finitely many nilpotent elements. In particular, we have that $X_{\mathrm{dR}}(R) \simeq X(R/\mathrm{Nil}(R)) = X(R_{\mathrm{red}})$. This crucial property holds for every k-algebra as long as X is (represented by) a finite type scheme.

Proposition A.3. *Let* X *be a locally of finite type scheme over* k*. Then* $X_{\mathrm{dR}}(R) \simeq X(R_{\mathrm{red}})$ *for every* k*-algebra* R*.*

Proof. Let $S := \mathrm{colim}_{I \subset R} R/I$, where the colimit runs through the nilpotent ideals of R. As usual, we denote the elements of S as equivalence classes of the form $[I, x]$, for some nilpotent ideal $I \subset R$ and $x \in R$. Here, $[I, x] = [I', x']$ if there exists a nilpotent ideal J containing I and I' such that $x \equiv x' \mod J$.

The natural map $R \to S$, corresponding to the ideal $I = 0$, sends every nilpotent in R to zero. In other words, it factors through the nilradical yielding a map $R_{\mathrm{red}} \to S$. We affirm that this morphism is injective. Indeed, $[0, x] = 0$ means that there exists a nilpotent ideal J containing x. It follows that x is nilpotent and so vanishes on R_{red}. As $[I, x] \in S$ is the image of $x \in R$, we have that $R_{\mathrm{red}} \to S$ is an isomorphism.

Finally, since X is locally of finite type, [Stacks, Tag 01ZC] implies that $X_{\mathrm{dR}}(R) \simeq X(S) \simeq X(R_{\mathrm{red}})$, finishing the proof. □

Perhaps not surprisingly, given the aforementioned geometric interpretation of X_{dR}, formal completions of schemes can be written in terms of de Rham spaces.

Proposition A.4. *Let* X *be a* k*-scheme and let* Z *be a closed subscheme of* X*. The formal completion* $\widehat{X}_Z$ *of* X *along* Z *is isomorphic to* $X \times_{X_{\mathrm{dR}}} Z_{\mathrm{dR}}$*.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf defined by Z and let R be a k-algebra. Our goal is to obtain a functorial isomorphism

$$\operatorname*{colim}_{I \subset R} X(R) \times_{X(R/I)} Z(R/I) \simeq \operatorname*{colim}_{n \geqslant 0} \mathrm{Spec}_X(\mathcal{O}_X/\mathcal{I}^{n+1})(R),$$

where the colimit on the left runs through the nilpotent ideals of R. Given an ideal $I \subset R$, denote by i_I the closed immersion $\mathrm{Spec}\, R/I \to \mathrm{Spec}\, R$. Now, we have that

$$\begin{aligned}\operatorname*{colim}_{I \subset R} X(R) \times_{X(R/I)} Z(R/I) &\simeq \operatorname*{colim}_{I \subset R} \{x \in X(R) \mid i_I^* x^* \mathcal{I} = 0\} \\ &\simeq \operatorname*{colim}_{n \geqslant 0} \operatorname*{colim}_{I^{n+1} = 0} \{x \in X(R) \mid i_I^* x^* \mathcal{I} = 0\} \\ &\simeq \operatorname*{colim}_{n \geqslant 0} \{x \in X(R) \mid x^* \mathcal{I}^{n+1} = 0\} \simeq \widehat{X}_Z(R),\end{aligned}$$

where the last isomorphism is the universal property of the relative spectrum. $\square$

We remark that the expression $X \times_{X_{dR}} Z_{dR}$ makes sense even if $Z \to X$ is not a closed immersion. In these cases, it can be used to *define* formal completions. Moreover, this characterization makes it clear that the projection $\widehat{X}_Z \to X$ is a monomorphism whenever $Z \to X$ is.

Corollary A.5. *Given a closed subscheme Z of a locally of finite type scheme X over k, the projection $\widehat{X}_Z \to X$ is a monomorphism of presheaves.*

Proof. By [Stacks, Tag 01L7], the closed immersion $i\colon Z \to X$ is a monomorphism in the category of schemes. Now, the Yoneda embedding preserves limits and any functor that preserves limits preserves monomorphisms [Stacks, Tag 01L3]. In other words, i is a monomorphism of presheaves. Since the de Rham functor preserves finite limits, so is $Z_{dR} \to X_{dR}$. Finally, fibered products preserve monomorphisms and this finishes the proof. $\square$

Given a morphism of k-schemes $f\colon X \to S$, the universal product of fibered products induces a map $X \to X_{dR} \times_{S_{dR}} S$. As the proposition below shows, it faithfully encodes the differential information contained in f.

Proposition A.6. *Let $f\colon X \to S$ be a morphism of k-schemes. Then f is formally smooth (resp. formally unramified) if and only if $X \to X_{dR} \times_{S_{dR}} S$ is an epimorphism (resp. monomorphism) of presheaves.*

Proof. Recall that f is said to be formally smooth (resp. formally unramified) if, for every k-algebra R with a map $\operatorname{Spec} R \to S$ and for every nilpotent ideal $I \subset R$, the induced map

$$\operatorname{Hom}_S(\operatorname{Spec} R, X) \to \operatorname{Hom}_S(\operatorname{Spec} R/I, X)$$

is surjective (resp. injective). It is a quick exercise to translate this condition into the surjectivity (resp. injectivity) of $X(R) \to (X_{dR} \times_{S_{dR}} S)(R)$. $\square$

Let $Y \to X$ be an immersion of k-schemes that factors as $Y \to U \to X$, where $Y \to U$ is a closed immersion with ideal $\mathscr{I}$ and $U \to X$ is an open immersion. The formal completion of X along Y is usually defined as the colimit of $\operatorname{Spec}_U(\mathcal{O}_U/\mathcal{I}^{n+1})$, for $n \geqslant 0$. The previous proposition shows that $U \simeq U_{dR} \times_{X_{dR}} X$ and so

$$U \times_{U_{dR}} Z_{dR} \simeq X \times_{X_{dR}} U_{dR} \times_{U_{dR}} Z_{dR} \simeq X \times_{X_{dR}} Z_{dR},$$

proving that the Proposition A.4 also works for locally closed immersions.

Corollary A.7. *Let $X \to S$ be a formally smooth morphism of k-schemes. Then $X_{dR} \times_{S_{dR}} S$ is the coequalizer of*

$$\widehat{(X \times_S X)}_\Delta \rightrightarrows X,$$

where $\widehat{(X \times_S X)}_\Delta$ is the formal completion of $X \times_S X$ along the diagonal.

Proof. In order to simplify notation, let $Y = X_{dR} \times_{S_{dR}} S$. Since every epimorphism is effective in a topos, $X \to Y$ is the coequalizer of $X \times_Y X \rightrightarrows X$. Now, the result follows from general category theory: the pullback of

$$\begin{array}{ccc} & & X \times_S X \\ & & \downarrow \\ X_{dR} & \xrightarrow{\Delta_{dR}} & (X \times_S X)_{dR} \end{array}$$

is $X \times_Y X$. □

Given a scheme X, its functor of points is always a sheaf for the étale and fppf topologies. We will now study the descent properties of X_{dR}, and we begin with a lemma.

Lemma A.8. *Let R be a k-algebra and let $\{R \to R_i\}_{i \in I}$ be an étale covering. Then the reduction $\{R_{red} \to R_{i,red}\}_{i \in I}$ is also an étale covering. Moreover, any étale covering of R_{red} arises in this way.*

Proof. Since $R \to R_i$ is étale, so is its base change $R_{red} \to R_{red} \otimes_R R_i$. By [Stacks, Tag 033B], we have that $R_{red} \otimes_R R_i$ is reduced and then [EGA I, Cor. 5.1.8] gives that

$$R_{red} \otimes_R R_i = (R_{red} \otimes_R R_i)_{red} \simeq (R_{red} \otimes_{R_{red}} R_{i,red})_{red} \simeq R_{i,red}.$$

It follows that $\{R_{red} \to R_{i,red}\}_{i \in I}$ is an étale covering of R_{red}.

Now, consider an étale covering $\{R_{red} \to S_i\}_{i \in I}$ of R_{red}. By the topological invariance of the étale site, there exists a covering $\{R \to R_i\}_{i \in I}$ along with isomorphisms $R_{red} \otimes_R R_i \simeq S_i$ for all $i \in I$ [Stacks, Tag 04DZ]. The same argument as above shows that S_i is reduced, and then $S_i \simeq R_{i,red}$. □

Proposition A.9. *Let X be a locally of finite type scheme over k. The de Rham space X_{dR} is an étale sheaf on Aff/k.*

Proof. Let R be a k-algebra and let $\{R \to R_i\}_{i \in I}$ be an étale covering of R. We want to prove that the diagram

$$X(R_{red}) \longrightarrow \prod_i X(R_{i,red}) \rightrightarrows \prod_{i,j} X((R_i \otimes_R R_j)_{red})$$

is an equalizer. The lemma above says that $\{R_{red} \to R_{i,red}\}_{i \in I}$ is also an étale cover and then the fact that X is an étale sheaf implies that the diagram

$$X(R_{red}) \longrightarrow \prod_i X(R_{i,red}) \rightrightarrows \prod_{i,j} X(R_{i,red} \otimes_{R_{red}} R_{j,red})$$

is an equalizer. The same argument as in the proof of the previous lemma shows that $R_{i,red} \otimes_{R_{red}} R_{j,red}$ is reduced. Then [EGA I, Cor. 5.1.8] gives isomorphisms $R_{i,red} \otimes_{R_{red}} R_{j,red} \simeq (R_i \otimes_R R_j)_{red}$, finishing the proof. □

In particular, the preceding proposition implies that the de Rham space defines a functor from commutative algebraic groups over k to abelian étale sheaves on Aff/k.

Proposition A.10. *The functor* $(-)_{\mathrm{dR}}\colon \mathsf{AbAlgGrp}/k \to \mathsf{Ab}((\mathsf{Aff}/k)_{\mathrm{et}})$ *is exact.*

Proof. Let $0 \to A \to B \to C \to 0$ be an exact sequence of commutative algebraic groups over k. In particular, it is left-exact in the category of abelian *presheaves* on Aff/k. By Proposition A.2, the induced exact sequence $0 \to A_{\mathrm{dR}} \to B_{\mathrm{dR}} \to C_{\mathrm{dR}} \to 0$ is also left-exact in abelian presheaves. Since sheafification is exact, this sequence is left-exact in the category of abelian étale sheaves.

Let us verify that $B_{\mathrm{dR}} \to C_{\mathrm{dR}}$ is an epimorphism of abelian sheaves. Given a k-algebra R and an element $c \in C_{\mathrm{dR}}(R) = C(R_{\mathrm{red}})$, the fact that $B \to C$ is an epimorphism of étale sheaves implies that there exists a covering $\{R_{\mathrm{red}} \to S_i\}_{i\in I}$ such that $c|_{S_i}$ is in the image of $B(S_i) \to C(S_i)$ for all $i \in I$ [Stacks, Tag 00WN]. Lemma A.8 then gives a covering $\{R \to R_i\}_{i\in I}$ whose reduction is $\{R_{\mathrm{red}} \to S_i\}_{i\in I}$. It follows that $c|_{R_i} = c|_{S_i}$ is in the image of $B_{\mathrm{dR}}(R_i) \to C_{\mathrm{dR}}(R_i)$ for all $i \in I$, concluding the proof. □

We proved in Corollary A.7 that X_{dR} is a quotient of X in which we identify infinitesimally close points. When X is a commutative algebraic group G, the difference of two such points has to live in an infinitesimal neighborhood of the identity. This heuristic leads to the result below.

Proposition A.11. *Let G be a commutative algebraic group over k. Then G_{dR} is isomorphic to the presheaf quotient $G/\widehat{G}$, where $\widehat{G}$ is the formal completion of G along the identity. In particular, G_{dR} is also isomorphic to the sheaf quotient $G/\widehat{G}$.*

Proof. In this proof, let us consider every (co)limit to be taken inside the category of abelian presheaves on Aff/k. As the cokernel of the identity section $e\colon \operatorname{Spec} k \to G$ is G itself, a variant of Proposition A.2 for abelian presheaves shows that the cokernel of $e_{\mathrm{dR}}\colon \operatorname{Spec} k \to G_{\mathrm{dR}}$ is G_{dR}. The universal property of cokernels then induces the dashed map below.

$$\begin{array}{ccccc} \widehat{G} & \longrightarrow & G & \longrightarrow & G/\widehat{G} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{e_{\mathrm{dR}}} & G_{\mathrm{dR}} & = & G_{\mathrm{dR}} \end{array}$$

The square on the left is cartesian due to Proposition A.4, and $G \to G_{\mathrm{dR}}$ is an epimorphism since G is smooth. Then, [Stacks, Tag 08N4] implies that the square on the left is also cocartesian, and [Stacks, Tag 08N3] gives that $G/\widehat{G} \to G_{\mathrm{dR}}$ is an isomorphism. Since G_{dR} is already an étale sheaf, the presheaf and the sheaf quotients $G/\widehat{G}$ coincide. □

The following proposition is the unique result in this subsection that needs the base field k to have characteristic zero.

Proposition A.12. *Let* G *be a commutative algebraic group over a characteristic zero field* k*. Then* G_{dR} *is an fppf sheaf isomorphic to* $G/\widehat{G}$ *and the functor* $(-)_{dR}$ *from commutative algebraic groups over* k *to abelian fppf sheaves is exact.*

Proof. Recall that, by Proposition 2.3, the formal completion $\widehat{G}$ is a direct sum of copies of $\widehat{\mathbb{G}}_a$. Then, given a k-algebra R, [Bha22, Remark 2.2.18] says that $H^1_{fppf}(R, \widehat{\mathbb{G}}_a) = 0$ and so $(G/\widehat{G})(R) \simeq G(R)/\widehat{G}(R) \simeq G_{dR}(R)$, where the quotient on the left is taken on the fppf topology. In other words G_{dR} is an fppf sheaf isomorphic to $G/\widehat{G}$. The exactness of $(-)_{dR}$ here is a particular case of Proposition A.10. □

Remark A.13 (de Rham spaces in positive characteristic)*.* Let k be a field of characteristic $p > 0$. Given a k-algebra R, recall the construction of its colimit perfection R_{perf} as in Remark 2.29, and define a presheaf $\mathbb{G}_{a,perf}$ on Aff/k by $\mathbb{G}_{a,perf}(R) := R_{perf}$. As [Bha22, Remark 2.2.18] shows, we have an exact sequence of abelian fppf sheaves

$$0 \to \widehat{\mathbb{G}}_a \to \mathbb{G}_a \to \mathbb{G}_{a,perf} \to 0.$$

It follows that the natural map of abelian étale sheaves $\mathbb{G}_{a,dR} \to \mathbb{G}_{a,perf}$ identifies $\mathbb{G}_{a,perf}$ with the fppf sheafification of $\mathbb{G}_{a,dR}$.

Given a locally of finite type scheme X over k, we can restrict its functor of points to Aff/k, form its de Rham space X_{dR} and right Kan extend to obtain a presheaf on Sch/k. We also denote this extension as X_{dR}. It acts on a k-scheme S as $X_{dR}(S) \simeq X(S_{red})$ and, by Proposition A.9 and [Stacks, Tag 021E], X_{dR} it is always an étale sheaf.

We end this subsection by noting that we have an equivalence of topoi $\mathsf{Sh}((\mathsf{Aff}/k)_{ét}) \simeq \mathsf{Sh}((\mathsf{Sch}/k)_{ét})$, in which the functor $\mathsf{Sh}((\mathsf{Sch}/k)_{ét}) \to \mathsf{Sh}((\mathsf{Aff}/k)_{ét})$ is simply restriction and the functor $\mathsf{Sh}((\mathsf{Aff}/k)_{ét}) \to \mathsf{Sh}((\mathsf{Sch}/k)_{ét})$ is a right Kan extension [Stacks, Tag 021E]. The analogous result also holds for the fppf topoi [Stacks, Tag 021V]. In particular, every result in this section also hold with the extended definition of de Rham spaces.

A.2. Crystals and $\mathcal{D}$-modules

As it was said in the introduction of this appendix, quasi-coherent sheaves over X_{dR} are the same as quasi-coherent $\mathcal{D}_X$-modules. However, those de Rham spaces are usually far from being algebraic and so we begin this subsection by defining what do we mean by a quasi-coherent sheaf over a not-necessarily-representable sheaf.

Definition A.14 (Quasi-coherent sheaves)**.** Let X be an étale sheaf on Sch/k. We define the presheaf of categories $\mathsf{QCoh}\colon \mathsf{Sh}((\mathsf{Sch}/k)_{ét})^{op} \to \mathsf{Cat}$ as the right Kan extension of the usual functor $\mathsf{QCoh}\colon (\mathsf{Sch}/k)^{op} \to \mathsf{Cat}$ along the Yoneda embedding $\mathsf{Sch}/k \to \mathsf{Sh}((\mathsf{Sch}/k)_{ét})$.

Remark A.15*.* Equivalently, $\mathsf{QCoh}(X)$ is the 2-limit of $\mathsf{QCoh}(S)$ over the category of pairs (S, x), where S is a k-scheme and x is a S-point of X. This implies that a quasi-coherent sheaf $\mathscr{F}$ on X consists of the following data:

- for each k-scheme S and each $x \in X(S)$, a quasi-coherent sheaf $\mathscr{F}(x)$ on S;
- for each morphism of k-schemes $f\colon S \to S'$ and each $x \in X(S)$, an isomorphism $\alpha_{f,x}\colon f^*\mathscr{F}(f(x)) \xrightarrow{\sim} \mathscr{F}(x)$.

This data is supposed to satisfy a cocycle condition. Namely, if $f\colon S \to S'$ and $g\colon S' \to S''$ are morphisms of k-schemes and $x \in X(S)$, the diagram

$$\begin{array}{ccccc} f^*g^*\mathscr{F}(g(f(x))) & \xrightarrow{f^*\alpha_{g,f(x)}} & f^*\mathscr{F}(f(x)) & \xrightarrow{\alpha_{f,x}} & \mathscr{F}(x) \\ \downarrow\wr & & & & \| \\ (g\circ f)^*\mathscr{F}(g(f(x))) & & \xrightarrow{\alpha_{g\circ f,x}} & & \mathscr{F}(x) \end{array}$$

should commute.

As usual, we will denote by $\mathcal{O}_X$ the quasi-coherent sheaf on X which associates to each $x \in X(S)$ the trivial $\mathcal{O}_S$-module. (The isomorphisms $\alpha_{f,x}$ are nothing but $f^*\mathcal{O}_{S'} \xrightarrow{\sim} \mathcal{O}_S$.) We define the tensor product $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{F}'$ of two quasi-coherent sheaves $\mathscr{F}$ and $\mathscr{F}'$ simply via the tensor products of the point-wise quasi-coherent sheaves. Moreover, given a morphism $f\colon X \to Y$ of presheaves and a quasi-coherent sheaf $\mathscr{G}$ on Y, its inverse image $f^*\mathscr{G}$ associates each $x \in X(S)$ to $\mathscr{G}(f(x))$.

Remark A.16. The category $\mathsf{QCoh}(X)$ is automatically symmetric monoidal. One possible way of seeing this is by remarking that the presheaf $\mathsf{QCoh}\colon (\mathsf{Sch}/k)^{op} \to \mathsf{Cat}$ has values in $\mathsf{CAlg}(\mathsf{Pr}^L)$, the commutative algebra objects in the ∞-category of presentable categories with left adjoint functors. The result then follows from the fact that both forgetful functors $\mathsf{CAlg}(\mathsf{Pr}^L) \to \mathsf{Pr}^L \to \mathsf{Cat}_\infty$ preserve limits [Lur17, Cor. 3.2.2.5; Lur09, Prop. 5.5.3.13].

It is not true, however, that $\mathsf{QCoh}(X)$ is always an abelian category [Stacks, Tag 0ALF]. That being said, we prove below that $\mathsf{QCoh}(X_{dR})$, for a smooth k-scheme X, is equivalent to the category of quasi-coherent $\mathcal{D}_X$-modules and so it is ipso facto abelian.

We say that a quasi-coherent sheaf $\mathscr{F}$ on X *locally free of a given rank* if all the quasi-coherent sheaves $\mathscr{F}(x)$ are. In particular, one can consider the *Picard group* $\mathrm{Pic}(X)$ constituted of the isomorphism classes of rank one locally free sheaves over X, whose group operation is the tensor product. The proposition below gives another characterization of this object.

Proposition A.17. *Let X be an étale sheaf on Sch/k. We have a monoidal equivalence of categories*

$$\{\textit{Line bundles on } X\}^{\simeq} \xrightarrow{\sim} \{\mathbb{G}_m\textit{-torsors over } X\},$$

where $C^{\simeq}$ is the underlying groupoid of a category C. In particular, we have an isomorphism of groups $\mathrm{Pic}(X) \simeq H^1_{et}(X, \mathbb{G}_m)$.[9]

[9] We refer the reader to the discussion before Proposition 2.8 for the definition of $H^1_{et}(X, \mathbb{G}_m)$.

Proof. The usual proof also works here. □

We now present the *raison d'être* of this appendix: the category of *crystals*. This object first appeared in [Gro68] under the name infinitesimal topos.[10]

Definition A.18 (Crystals)**.** Let $X \to S$ be a morphism of k-schemes. Its category of *crystals* $\mathsf{Crys}(X/S)$ is defined as $\mathsf{QCoh}(X_{dR} \times_{S_{dR}} S)$.

In order to have a more concrete description of $\mathsf{Crys}(X/S)$, suppose that $X \to S$ is smooth and consider the following diagram, on which every map is a natural projection.

$$(\widehat{X \times_S X \times_S X})_\Delta \rightrightarrows (\widehat{X \times_S X})_\Delta \rightrightarrows X$$

We denote by pr_i (resp. pr_{ij}) the projection on the i-th factor (resp. on the i,j-th factors).

Lemma A.19. *Given a quasi-coherent sheaf M on X, the following data are equivalent:*

(i) An isomorphism of quasi-coherent sheaves $\varepsilon\colon \mathrm{pr}_1^* M \to \mathrm{pr}_2^* M$ satisfying the cocycle condition $\mathrm{pr}_{12}^*(\varepsilon) \circ \mathrm{pr}_{23}^*(\varepsilon) = \mathrm{pr}_{13}^*(\varepsilon)$;

(ii) A stratification on M as in [BO15, Def. 2.10];

(iii) A morphism of $\mathcal{O}_X$-algebras $\mathcal{D}_{X/S} \to \underline{\mathrm{End}}_{\mathcal{O}_S}(M)$.

Proof. Let $(X \times_S X)_\Delta^{(n)}$ be the n-th infinitesimal neighborhood of the diagonal in $X \times_S X$. Recall that the formal completion $(\widehat{X \times_S X})_\Delta$ is the filtered colimit of the $(X \times_S X)_\Delta^{(n)}$ in $\mathsf{PSh}(\mathsf{Sch}/k)$. If we consider the latter as a $(2,1)$-category, this is automatically a 2-colimit and so $\mathsf{QCoh}((\widehat{X \times_S X})_\Delta)$ is the 2-limit of the categories $\mathsf{QCoh}((X \times_S X)_\Delta^{(n)})$. It follows that an isomorphism $\varepsilon\colon \mathrm{pr}_1^* M \to \mathrm{pr}_2^* M$ amounts to a compatible system of isomorphisms $\varepsilon_n\colon (\mathrm{pr}_1^{(n)})^* M \to (\mathrm{pr}_2^{(n)})^* M$, where

$$(X \times_S X)_\Delta^{(n)} \underset{\mathrm{pr}_2^{(n)}}{\overset{\mathrm{pr}_1^{(n)}}{\rightrightarrows}} X$$

are the natural projections. This gives the equivalence between (i) and (ii). Finally, [BO15, Prop. 2.11] gives the equivalence between (ii) and (iii). The reader may want to see [BO15, Remark 2.13] as well. □

We remark that, given a morphism of quasi-coherent sheaves $\varphi\colon M \to N$ on X along with isomorphisms $\varepsilon_M\colon \mathrm{pr}_1^* M \to \mathrm{pr}_2^* M$ and $\varepsilon_N\colon \mathrm{pr}_1^* N \to \mathrm{pr}_2^* N$ as above, the diagram

$$\begin{array}{ccc} \mathrm{pr}_1^* M & \xrightarrow{\mathrm{pr}_1^* \varphi} & \mathrm{pr}_1^* N \\ {\scriptstyle \varepsilon_M}\downarrow & & \downarrow{\scriptstyle \varepsilon_N} \\ \mathrm{pr}_2^* M & \xrightarrow[\mathrm{pr}_2^* \varphi]{} & \mathrm{pr}_2^* N \end{array}$$

[10] Contrarily to what the name may indicate, $\mathsf{Crys}(X/S)$ is *not* the crystalline topos if k has positive characteristic. See [Gre19] for a similar approach to crystalline cohomology.

commutes if and only if φ is $\mathcal{D}_{X/S}$-linear. All in all, we obtain the proposition below.

Proposition A.20 (Grothendieck)**.** *Let $X \to S$ be a smooth morphism of k-schemes. Then $\mathsf{Crys}(X/S)$ is monoidally equivalent to the category $\mathsf{QCoh}(\mathcal{D}_{X/S})$ of quasi-coherent $\mathcal{D}_{X/S}$-modules.*

Proof. Consider the following cosimplicial diagram in $\mathsf{PSh}(\mathsf{Sch}/k)$, seen as a $(2,1)$-category.

$$\widehat{(X \times_S X \times_S X)}_\Delta \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \widehat{(X \times_S X)}_\Delta \;\substack{\longrightarrow\\ \longrightarrow}\; X$$

Since its 2-colimit can be computed by truncating [Car+17, Lemma 2.21], Corollary A.7 implies that it is $X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S$. It follows that $\mathsf{Crys}(X/S) = \mathsf{QCoh}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S)$ is the 2-limit of the diagram

$$\mathsf{QCoh}(X) \;\substack{\longrightarrow\\ \longrightarrow}\; \mathsf{QCoh}(\widehat{(X \times_S X)}_\Delta) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathsf{QCoh}(\widehat{(X \times_S X \times_S X)}_\Delta).$$

This is a category of descent data, whose objects are exactly quasi-coherent sheaves M on X endowed with the data of Lemma A.19.(i). The result then follows from the aforementioned lemma. □

Let us make some observations about this result. First, as the proof above shows, the equivalence $\mathsf{Crys}(X/S) \simeq \mathsf{QCoh}(\mathcal{D}_{X/S})$ preserves the natural forgetful functors to $\mathsf{QCoh}(X)$. Next, even though Proposition A.20 works over fields of arbitrary characteristic, we can give an even simpler description of $\mathsf{Crys}(X/S)$ when k has characteristic zero.

Recall that, given a quasi-coherent sheaf M on X, a connection relative to S is a $\mathcal{O}_S$-linear map $\nabla \colon M \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} M$ satisfying the Leibniz rule

$$\nabla(fm) = df \otimes m + f\nabla(m),$$

for local sections f of $\mathcal{O}_X$ and m of M. The connection ∇ is said to be *integrable* (or *flat*) if $\nabla_1 \circ \nabla = 0$, where $\nabla_1 \colon \Omega^1_{X/S} \otimes_{\mathcal{O}_X} M \to \Omega^2_{X/S} \otimes_{\mathcal{O}_X} M$ is given by

$$\nabla_1(\omega \otimes m) = d\omega \otimes m - \omega \wedge \nabla(m).$$

As is well-known, the data of an integrable connection ∇ relative to S on M is equivalent to the data of a morphism of $\mathcal{O}_X$-modules $\mathcal{D}^1_{X/S} \to \underline{\mathrm{End}}_{\mathcal{O}_S}(M)$, where $\mathcal{D}^1_{X/S} \subset \mathcal{D}_{X/S}$ is the sheaf of differential operators of order one [ABC20, Def. 4.2.1]. Now, in characteristic zero, the sheaf $\mathcal{D}^1_{X/S}$ generates $\mathcal{D}_{X/S}$ as a $\mathcal{O}_X$-algebra and so such a map $\mathcal{D}^1_{X/S} \to \underline{\mathrm{End}}_{\mathcal{O}_S}(M)$ extends uniquely to a morphism of $\mathcal{O}_X$-algebras $\mathcal{D}_{X/S} \to \underline{\mathrm{End}}_{\mathcal{O}_S}(M)$ [BO15, Thm. 2.15].

Corollary A.21. *Let k be a characteristic zero field and let $X \to S$ be a smooth morphism of k-schemes. Then $\mathsf{Crys}(X/S)$ is monoidally equivalent to the category $\mathsf{MIC}(X/S)$ of quasi-coherent sheaves on X endowed with an integrable connection relative to S.*

Since, once again, the equivalence $\mathsf{Crys}(X/S) \simeq \mathsf{MIC}(X/S)$ is compatible with the forgetful functor to $\mathsf{QCoh}(X)$, line bundles on $X_{dR} \times_{S_{dR}} S$ correspond to line bundles on X with integrable connection relative to S. In particular, $H^1_{fppf}(X_{dR} \times_{S_{dR}} S, \mathbb{G}_m)$ is the group of isomorphism classes of such objects.

Let us now consider functoriality from all these different points of view. Given a commutative (but not necessarily cartesian) square

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow[g]{} & S, \end{array}$$

where g is an arbitrary morphism of k-schemes, we obtain a map $(f, g)_{dR} \colon X'_{dR} \times_{S'_{dR}} S' \to X_{dR} \times_{S_{dR}} S$ giving rise to an inverse image functor $(f, g)^*_{dR} \colon \mathsf{Crys}(X/S) \to \mathsf{Crys}(X'/S')$.

Similarly, let M be a quasi-coherent sheaf on X endowed with a morphism of $\mathcal{O}_X$-algebras $\mathcal{D}_{X/S} \to \underline{\mathrm{End}}_{\mathcal{O}_S}(M)$. That is, let M be a quasi-coherent $\mathcal{D}_{X/S}$-module. The *transfer module* $f^*\mathcal{D}_{X/S} = \mathcal{O}_{X'} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}_{X/S}$ is naturally a $(\mathcal{D}_{X'/S'}, f^{-1}\mathcal{D}_{X/S})$-bimodule[11], and we define $(f, g)^*M$ as $f^*\mathcal{D}_{X/S} \otimes_{f^{-1}\mathcal{D}_{X/S}} f^{-1}M$, with its structure of $\mathcal{D}_{X'/S'}$-module.

Proposition A.22. *The following diagram*

$$\begin{array}{ccc} \mathsf{Crys}(X/S) & \xrightarrow{\sim} & \mathsf{QCoh}(\mathcal{D}_{X/S}) \\ {\scriptstyle (f,g)^*_{dR}}\downarrow & & \downarrow{\scriptstyle (f,g)^*} \\ \mathsf{Crys}(X'/S') & \xrightarrow{\sim} & \mathsf{QCoh}(\mathcal{D}_{X'/S'}) \end{array}$$

commutes up to isomorphism.

In characteristic zero, one can also define the inverse image of relative connections as in [ABC20, §5.1], and a quick verification shows that it coincides with the inverse image of $\mathcal{D}$-modules. Consequently, it also coincides with the inverse image functor for crystals defined above. Therefore, we will denote all the inverse image functors simply as $(f, g)^*$. Whenever g is the identity morphism of S, we shorten it further as f^*.

Finally, there also exists a direct image functor $(f, g)_+ \colon D_{qc}(\mathcal{D}_{X'/S'}) \to D_{qc}(\mathcal{D}_{X/S})$ for relative $\mathcal{D}$-modules [Vig21, Déf. 2.1.12]. (As above, when g is the identity morphism we denote $(f, g)_+$ simply as f_+.) However, defining it at the level of crystals requires a detour through the world of ind-coherent sheaves. We refer the reader to [GR17] for more on this.

B. The six-functor formalism of holonomic $\mathcal{D}$-modules

Even though it is well known that the derived category of holonomic $\mathcal{D}$-modules admits a six-functor formalism, the notations in the literature are not consistent, and very often

[11] Here $f^{-1}\mathcal{D}_{X/S}$ acts only on the second factor but $\mathcal{D}_{X'/S'}$ acts on both.

we have formulas which are similar, but not equal, to corresponding formulas in other formalisms.

The goal of this appendix is to gather all the needed results and to redefine the tensor product of $\mathcal{D}$-modules, shedding light on the striking similarity between holonomic $\mathcal{D}$-modules and (wildly ramified) ℓ-adic perverse sheaves in positive characteristic.

B.1. Definitions and simple calculations

Let X be an algebraic variety (separated scheme of finite-type) smooth over a field k of characteristic zero. (Henceforth, every k-variety is supposed to be smooth.) We denote by $\mathsf{Hol}(\mathcal{D}_X)$ the abelian category of holonomic $\mathcal{D}_X$-modules, and by $\mathsf{D}^b_h(\mathcal{D}_X)$ the full subcategory of $\mathsf{D}^b(\mathsf{Mod}(\mathcal{D}_X))$, the bounded derived category of the abelian category of left $\mathcal{D}_X$-modules, composed of the complexes M whose cohomology $\mathscr{H}^i(M)$ is holonomic for all i. The latter is a triangulated category [HT07, Cor. 3.1.4] and we endow it with its standard t-structure.

Integrable connections

As usual, we identify vector bundles (locally free sheaves) with integrable connections and $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules. We will hereafter address these objects simply as *connections*. Since connections correspond to local systems under the analogy between holonomic $\mathcal{D}$-modules and ℓ-adic perverse sheaves, we systematically put connections in degree $\dim X$. As it will soon become clear, under this convention, every functor works as it should. Finally, we denote by $\mathsf{D}^b_{\mathrm{int}}(\mathcal{D}_X)$ the full subcategory of $\mathsf{D}^b_h(\mathcal{D}_X)$ whose objects have connections as cohomologies.

Exceptional inverse image

Given a morphism $f\colon X \to S$, define $f^!$ to be $Lf^*[\dim X - \dim S]$ with its natural $\mathcal{D}$-module structure. (As in [HT07, §1.3].) This defines a triangulated functor $\mathsf{D}^b_h(\mathcal{D}_S) \to \mathsf{D}^b_h(\mathcal{D}_X)$ [HT07, Thm. 3.2.3] and, if $g\colon S \to S'$ is another morphism, we have $(g \circ f)^! \simeq f^! \circ g^!$ [HT07, Prop. 1.5.11].

Example B.1. If f is flat or if $M \in \mathsf{D}^b_{\mathrm{int}}(\mathcal{D}_S)$, then $\mathscr{H}^i(f^! M) \simeq f^* \mathscr{H}^{i+d}(M)$ holds for all i, where we pose $d = \dim X - \dim S$. In particular, $f^!$ restricts to $\mathsf{D}^b_{\mathrm{int}}(\mathcal{D}_S) \to \mathsf{D}^b_{\mathrm{int}}(\mathcal{D}_X)$. Moreover, the functor f^* coincides with the usual inverse image for connections.

Direct image

Given a morphism $f\colon X \to S$, we define f_+ as in [HT07, p. 40]. This defines a triangulated functor $\mathsf{D}^b_h(\mathcal{D}_X) \to \mathsf{D}^b_h(\mathcal{D}_S)$ [HT07, Thm. 3.2.3] and, if $g\colon S \to S'$ is another morphism, we have $(g \circ f)_+ \simeq g_+ \circ f_+$ [HT07, Prop. 1.5.21].

Duality functor

Given a variety X, we define the duality functor D_X as in [HT07, Def. 2.6.1]. This defines a t-exact triangulated functor $D^b_h(\mathcal{D}_X) \to D^b_h(\mathcal{D}_X)^{op}$ [HT07, Cor. 2.6.8] and there is a natural isomorphism of functors $\mathrm{id} \to D_X D_X$ [HT07, Prop. 2.6.5].

Example B.2. The duality functor restricts to $D_X \colon D^b_{int}(\mathcal{D}_X) \to D^b_{int}(\mathcal{D}_X)^{op}$, and we have that $\mathscr{H}^i(D_X(M)) \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathscr{H}^{-i}(M), \mathcal{O}_X)$ for $M \in D^b_{int}(\mathcal{D}_X)$ [HT07, Ex. 2.6.10].

Inverse image

Given a morphism $f \colon X \to S$, we define $f^+ \colon D^b_h(\mathcal{D}_S) \to D^b_h(\mathcal{D}_X)$ as $D_X \circ f^! \circ D_S$. This functor is left-adjoint to f_+ [HT07, Cor. 3.2.15].

Example B.3. If $f \colon X \to S$ is smooth and $d = \dim X - \dim S$, we have $f^+ \simeq f^![-2d] = f^*[-d]$ [HT07, Ex. 2.4.5 and Thm. 2.7.1]. In particular, if f is étale, then $f^+ \simeq f^!$ is t-exact.

Example B.4. As above, let $d = \dim X - \dim S$ for a (not-necessarily smooth) morphism $f \colon X \to S$ and let $M \in D^b_{int}(\mathcal{D}_S)$. By Example B.1, we have $\mathscr{H}^i(f^+M) \simeq f^*\mathscr{H}^{i-d}(M)$ for all i. (And so f^+ restricts to $D^b_{int}(\mathcal{D}_S) \to D^b_{int}(\mathcal{D}_X)$.) In particular, if $\mathscr{E}$ is a connection on S, we have that $f^+(\mathscr{E}[-\dim S]) \simeq f^*\mathscr{E}[-\dim X]$; showing that the functor f^+ is compatible with the usual inverse image of connections.

Proper direct image

Given a morphism $f \colon X \to S$, we define $f_! \colon D^b_h(\mathcal{D}_X) \to D^b_h(\mathcal{D}_S)$ as $D_S \circ f_+ \circ D_X$. This functor is left-adjoint to $f^!$ [HT07, Cor. 3.2.15]. Moreover, there exists a natural transformation $f_! \to f_+$, which is an isomorphism for proper f [HT07, Thm. 3.2.16].

Tensor product

Given two complexes M and N of $\mathcal{D}_X$-modules and $\mathcal{D}_Y$-modules, we define their *external tensor product* $M \boxtimes N$ as in [HT07, p. 39]. This defines a t-exact [Bei+18, §1.3.20] triangulated bifunctor $D^b_h(\mathcal{D}_X) \times D^b_h(\mathcal{D}_Y) \to D^b_h(\mathcal{D}_{X\times Y})$ [HT07, Prop. 3.2.2].

Let $\Delta_X \colon X \to X \times X$ be the diagonal embedding. We define the *exceptional tensor product* $\otimes^!_X \colon D^b_h(\mathcal{D}_X) \times D^b_h(\mathcal{D}_X) \to D^b_h(\mathcal{D}_X)$ as

$$M \otimes^!_X N := \Delta^!_X(M \boxtimes N).$$

This is isomorphic to $M \otimes^L_{\mathcal{O}_X} N[-\dim X]$, as defined in [HT07, p. 38], and its identity is $1^!_X := \mathcal{O}_X[\dim X]$. The main *tensor product* that we will use, denoted $\otimes_X \colon D^b_h(\mathcal{D}_X) \times D^b_h(\mathcal{D}_X) \to D^b_h(\mathcal{D}_X)$, is defined as

$$M \otimes_X N := \Delta^+_X(M \boxtimes N).$$

This is isomorphic to $D_X(D_X(M) \otimes^!_X D_X(N))$ and its identity is $1_X := \mathcal{O}_X[-\dim X]$.

Given a morphism $f\colon X \to S$, we remark that the functor f^+ is monoidal with respect to $\otimes$ and $f^!$ is monoidal with respect to $\otimes^!$ [HT07, Prop. 1.5.18.(ii)]. Moreover, given $M \in D^b_h(\mathcal{D}_X)$ and $N \in D^b_h(\mathcal{D}_Y)$, we have natural isomorphisms

$$M \boxtimes N \simeq \mathrm{pr}_X^+ M \otimes_{X\times Y} \mathrm{pr}_Y^+ N \simeq \mathrm{pr}_X^! M \otimes^!_{X\times Y} \mathrm{pr}_Y^! N,$$

where $\mathrm{pr}_X\colon X \times Y \to X$ and $\mathrm{pr}_Y\colon X \times Y \to Y$ are the projections.

Inner hom

Given two complexes $M, N \in D^b_h(\mathcal{D}_X)$, we define their *inner hom* $\underline{\mathrm{Hom}}_X(M, N) \in D^b_h(\mathcal{D}_X)$ as $\mathrm{D}_X(M \otimes_X \mathrm{D}_X(N))$. The functor $- \otimes_X M$ is left adjoint to $\underline{\mathrm{Hom}}_X(M, -)$ [HT07, Prop. 2.6.14]. We can also define the *dual* $M^\vee := \underline{\mathrm{Hom}}_X(M, 1_X)$ which, by smoothness of X, coincides with $\mathrm{D}_X(M)[-2\dim X]$.

Example B.5. Let $\mathscr{E}$ be a connection on X. The Example B.2 gives that $(\mathscr{E}[-\dim X])^\vee \simeq \mathscr{E}^*[-\dim X]$, where $\mathscr{E}^*$ is the usual dual of connections. In other words, under our convention of putting connections always in degree $\dim X$, the dual defined above coincides with the dual of connections.

Recall the definitions of reflexive and dualizable objects in a closed monoidal category: an object $M \in D^b_h(\mathcal{D}_X)$ is said to be *reflexive* if the natural map $M \to ((M)^\vee)^\vee$ is an isomorphism and it is said to be *dualizable* is the natural map $(M)^\vee \otimes_X M \to \underline{\mathrm{Hom}}_X(M, M)$ is an isomorphism. If M is dualizable, then we have an isomorphism $(M)^\vee \otimes_X N \simeq \underline{\mathrm{Hom}}_X(M, N)$ for every $N \in D^b_h(\mathcal{D}_X)$. The following proposition is basically [Stacks, Tag 0FPD].

Proposition B.6. *Every $M \in D^b_h(\mathcal{D}_X)$ is reflexive, and it is dualizable precisely when $M \in D^b_{\mathrm{int}}(\mathcal{D}_X)$. In particular, we have that*

$$M \otimes_X N \simeq M \otimes^!_X N[2\dim X] \simeq M \otimes_{\mathcal{O}_X} N[\dim X],$$

for $M \in D^b_{\mathrm{int}}(\mathcal{D}_X)$ and $N \in D^b_h(\mathcal{D}_X)$.

Example B.7. Let $M \in D^b_h(\mathcal{D}_X)$ and let $\mathscr{E}'$ be a connection on X. The isomorphisms above specialize to

$$M \otimes_X \mathscr{E}'[-\dim X] \simeq M \otimes_{\mathcal{O}_X} \mathscr{E}';$$

showing that, under our convention of putting connections in degree $\dim X$, the tensor product $\otimes_X$ is compatible with the usual tensor product $\otimes_{\mathcal{O}_X}$. If, moreover, M is also a connection $\mathscr{E}$ in degree $\dim X$, we have that

$$(\mathscr{E}[-\dim X]) \otimes_X (\mathscr{E}'[-\dim X]) \simeq \mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{E}'[-\dim X].$$

We also remark that the tensor product $\otimes_{\mathcal{O}_X}$ coincides with the one usually used for connections.

Comparison of notations

For the convenience of the reader, we provide a table comparing our notations to those of the most common references. As above, we consider a morphism $f\colon X \to S$ and put $d = \dim X - \dim S$.

Our notation	[Ber]	[Bor+87]	[HT07]	[Meb88]	[KL85, §7]
Lf^*	Lf^{Δ}	Lf°	Lf^*	$\mathbf{L}f^*$	$f^{\bullet}$
$f^!$	$f^!$	$f^!$	$f^{\dagger}$	$\mathbf{L}f^*[d]$	$f^!$
f_+	f_*	f_+	$\int_f$	$\int_{f*}$	f_*
D_X	D	D_X	$\mathbb{D}_X$	$(-)^*$	D
f^+	f^*	f^+	$f^{\star}$	$\mathbf{L}f^![-d]$	f^*
$f_!$	$f_!$	$f_!$	$\int_{f!}$	$\int_{f!}$	$f_!$
$\otimes^L_{\mathcal{O}_X}$	$\otimes^L_{O_X}$	$\otimes^L_{\mathcal{O}_X}$	$\otimes^L_{\mathcal{O}_X}$	$\otimes^{\mathbf{L}}_{\mathcal{O}_X}$	$\overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X}$
$\otimes^!_X$	$\triangle$	-	-	-	$\widetilde{\otimes}$
$\otimes_X$	-	-	-	-	-
$\underline{\mathrm{Hom}}_X$	Hom	-	-	-	$\widetilde{\mathscr{H}om}$

B.2. Main results

Recollement

Let $i\colon Z \hookrightarrow X$ be a closed immersion and let $j\colon U \hookrightarrow X$ be its complementary open immersion. Recall that i is proper (and so $i_! \simeq i_+$) and j is étale (and so $j^! \simeq j^+$). This data satisfies the *recollement conditions* [Bei+18, §1.4.3]. That is:

- $i^! j_+ = 0$ (so, by adjunction, $i^+ j_! = 0$ and $j^+ i_+ = 0$);
- for all $M \in D^b_h(\mathcal{D}_X)$, the adjunction maps $i_! i^! M \to M \to j_+ j^+ M$ and $j_! j^! M \to M \to i_+ i^+ M$ give rise to distinguished triangles;
- The adjunction maps $i^+ i_+ \to \mathrm{id} \to i^! i_!$ and $j^+ j_+ \to \mathrm{id} \to j^! j_!$ are all isomorphisms.

All of this is proven in [HT07, Cor. 1.6.2, Prop. 1.7.1; Bor+87, Prop. VI.8.2.(i)]. Since the counit $i^+ i_+ \to \mathrm{id}$ is an isomorphism, the functor i_+ is fully faithful and so the composition $i_+ i^! \to \mathrm{id} \to i_+ i^+$ yields a map $i^! \to i^+$. Section 1.4 in [Bei+18] shows that $i^+ j_+ \simeq i^! j_! [1]$ and that

$$
\begin{gathered}
j_! M \to j_+ M \to i_+ i^+ j_+ M \to j_! M[1] \\
i^! N \to i^+ N \to i^+ j_+ j^+ N \to i^! N[1],
\end{gathered}
$$

for $M \in D^b_h(\mathcal{D}_U)$ and $N \in D^b_h(\mathcal{D}_X)$, are distinguished triangles.

Proper base change

Given morphisms $f\colon X \to S$ and $g\colon S' \to S$, suppose that the fiber product

$$\begin{array}{ccc} X' & \xrightarrow{\tilde{f}} & S' \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S \end{array}$$

exists in our category of smooth algebraic varieties (i.e., that the fiber product of schemes $X \times_S S'$ is a smooth algebraic variety). Then, there exists an isomorphism $g^! \circ f_+ \simeq \tilde{f}_+ \circ \tilde{g}^!$ of functors $D^b_h(\mathcal{D}_X) \to D^b_h(\mathcal{D}_{S'})$ [HT07, Thm. 1.7.3; Dre13, Rem. 3.1.8]. By adjunction, there is also an isomorphism $f^+ \circ g_! \simeq \tilde{g}_! \circ \tilde{f}^+$ of functors $D^b_h(\mathcal{D}_{S'}) \to D^b_h(\mathcal{D}_X)$.

Smooth base change

Consider the same diagram as above and suppose, moreover, that $g\colon S' \to S$ is smooth. By the Example B.3, we have that $g^+ \simeq g^![2(\dim S - \dim S')]$ and so we obtain an isomorphism $g^+ \circ f_+ \simeq \tilde{f}^+ \circ \tilde{g}_+$ of functors $D^b_h(\mathcal{D}_X) \to D^b_h(\mathcal{D}_{S'})$.

Projection formula

Let $M \in D^b_h(\mathcal{D}_X)$ and $N \in D^b_h(\mathcal{D}_S)$. Then, the *projection formula* $f_! M \otimes_S N \simeq f_!(M \otimes_X f^+ N)$ holds. If f is proper or N is dualizable, we also have $f_+ M \otimes_S N \simeq f_+(M \otimes_X f^+ N)$ [HT07, Cor. 1.7.5].

Mayer-Vietoris

Let $i_1\colon Z_1 \to X$ and $i_2\colon Z_2 \to X$ be closed immersions covering X, and consider the cartesian diagram

$$\begin{array}{ccc} Z_1 \cap Z_2 & \longrightarrow & Z_2 \\ \downarrow & & \downarrow{\scriptstyle i_2} \\ Z_1 & \xrightarrow{i_1} & X. \end{array}$$

Denoting by $i_{12}\colon Z_1 \cap Z_2 \to X$ the diagonal map above, we have a distinguished triangle

$$M \to i_{1,!} i_1^+ M \oplus i_{2,!} i_2^+ M \to i_{12,!} i_{12}^+ M$$

for every M in $D^b_h(\mathcal{D}_X)$.

Vanishing and nearby cycles

Let $i\colon Z \hookrightarrow X$ be a closed immersion of codimension one. Given this data, one can endow a holonomic $\mathcal{D}$-module M on X with the so-called *V-filtration* as in [Meb88, Chap. III.4].

It is a filtration indexed by $\mathbb{C}/\mathbb{Z}$, whose graded pieces $\mathrm{gr}^V_\alpha(M)$ vanish unless $-\alpha$ is a root of the *Bernstein-Sato polynomial* associated to i [Meb88, Prop. 4.2.1].

The *unipotent nearby cycles* functor $\psi_1 : \mathsf{Hol}(\mathcal{D}_X) \to \mathsf{Hol}(\mathcal{D}_Z)$ is defined as gr^V_{-1} and the *unipotent vanishing cycles functor* $\phi_1 : \mathsf{Hol}(\mathcal{D}_X) \to \mathsf{Hol}(\mathcal{D}_Z)$ is defined as gr^V_0. They are endowed with natural transformations

$$\mathrm{can}\colon \psi_1 \to \phi_1 \quad \text{and} \quad \mathrm{var}\colon \phi_1 \to \psi_1$$

given locally by $-\partial_t$ and t, respectively, where t is a local equation for Z. Moreover, the compositions $\mathrm{can} \circ \mathrm{var}$ and $\mathrm{var} \circ \mathrm{can}$ are nilpotent [Meb88, §4.3.3]. Finally, by [Meb88, §§4.5.3 and 4.6.5], we have distinguished triangles

$$i^+M \to \psi_1(M) \xrightarrow{can} \phi_1(M)$$
$$i^!M \to \phi_1(M) \xrightarrow{var} \psi_1(M)$$

for every holonomic $\mathcal{D}$-module on X.

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