Equidistribution of Exponential Sums

Gabriel Ribeiro

École polytechnique

Summary

1. Exponential sums in nature
2. Cohomology to the rescue!
3. Deligne's equidistribution theorem
4. The general equidistribution result
5. Let's work out the case of Gauss' sums

Exponential sums in nature

Can we solve polynomial equations?

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Very often, these questions are way out of reach for our methods. This leads us to consider solutions mod p of the desired equations.

Let us then define a function $\mathrm{Sol}(f, p, t)$ which counts the number of solutions to $f(x) \equiv t \pmod{p}$.

Taking a Fourier transform

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$$\psi \mapsto \sum_{t \in \mathbb{F}_p} \psi(t)\,\mathrm{Sol}(f, p, t) = \sum_{x \in \mathbb{F}_p^n} \psi(f(x)).$$

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Since $\widehat{\mathbb{F}_p} = \mathbb{F}_p$, every character is of the form $\psi_a(x) := \exp(2\pi i a x / p)$. Via this identification, the function above is none other than

$$a \mapsto \sum_{x \in \mathbb{F}_p^n} \exp\left(\frac{2\pi i a f(x)}{p}\right);$$

an exponential sum!

Let’s properly define ES

In order to deal systematically with exponential sums, let us give a proper definition which encompasses the previous sum and many interesting others.

Definition - Exponential sum

Let k be a finite field and X be a finite-type scheme over k. An exponential sum is a sum of the form

$$S(f, E, \chi) := \sum_{x \in X(E)} \chi(f(x)),$$

where E/k is a finite extension, G is a commutative algebraic group, χ is a character of $G(E)$, and $f : X \to G$ is a morphism of schemes.

We're still counting solutions!

As before, we remark that

$$\begin{aligned}\widehat{G(E)} &\to \mathbb{C}\\ \chi &\mapsto \sum_{x\in X(E)} \chi(f(x))\end{aligned}$$

is the Fourier transform of

$$\begin{aligned}G(E) &\to \mathbb{C}\\ t &\mapsto \#\{x \in X(E) \mid f(x) = t\}.\end{aligned}$$

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The case where χ is the trivial character is already interesting and highly non-trivial.

Let's consider an example

Take X as the elliptic curve defined by $y^2 = 4x^3 - x - 1$. We denote by $N(X, q)$ the number of $\mathbb{F}_q$-points of X and wonder how the numbers $N(X, q)$ vary as a function of q.

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Take X as the elliptic curve defined by $y^2 = 4x^3 - x - 1$. We denote by $N(X, q)$ the number of $\mathbb{F}_q$-points of X and wonder how the numbers $N(X, q)$ vary as a function of q.

In analytic number theory, we usually divide the analysis into two cases: either we consider only the cases where q varies between the prime numbers (which are not 2 or 13), or we fix one such prime p and make q vary among the numbers of the form p^n, for some n.

Vertical distribution

We begin with the latter. Ever since Artin's thesis in the 1920's, it is known that there exist two complex numbers α_p and β_p, satisfying $\alpha_p \beta_p = p$, such that

$$N(X, p^n) = p^n + 1 - \alpha_p^n - \beta_p^n$$

for all $n \geq 1$.

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In particular, to determine $N(X, p^n)$ for all n, it suffices to know $N(X, p)$.

Horizontal distribution

The former case is much harder. By the Hasse bound, we know that

$$|N(X, p) - (p + 1)| \leq 2\sqrt{p}$$

and so there exists a unique ”angle” $\theta_p \in [0, \pi]$ such that

$$N(X, p) - (p + 1) = 2\sqrt{p}\cos(\theta_p).$$

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Our elliptic curve, however, does not have complex multiplication (its j-invariant is not an algebraic integer, for example).

Sato-Tate

The distribution of angles θ_p for elliptic curves without complex multiplication was the subject of a famous conjecture of Sato and Tate, which says that the sequence (θ_p) is equidistributed in $[0, \pi]$ for the Sato-Tate measure $\mu_{ST} := (2/\pi)\sin^2\theta \, \mathrm{d}\theta$.

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Several natural variants and generalizations remain wide-open.

Cohomology to the rescue!

Étale cohomology

The hero of our story is the theory of étale cohomology and, more precisely, Deligne's groundbreaking paper La Conjecture de Weil II. Since this is a huge machinery, we will begin by explaining its main features.

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- Let $k = \mathbb{F}_q$, where $q = p^n$;
- $\ell \neq p$ a prime number;
- X a smooth geometrically connected variety over k.

The étale fundamental group

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This led Grothendieck to define the étale fundamental group $\pi_1(X)$, a profinite group that classifies the finite étale covers of X.

Basic properties

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Moreover, if $X = \operatorname{Spec} k$, its fundamental group is nothing but the absolute Galois group of k.

Frobenii

In our case, where k is $\mathbb{F}_q$, this is the free profinite group $\widehat{\mathbb{Z}}$ on one canonical generator given by

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This is the so-called arithmetic Frobenius. As we'll see, its inverse, denoted by Frob_k and called geometric Frobenius, will play a key role in the theory.

Local systems

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Definition - Local system

An ℓ-adic local system $\mathscr{L}$ of rank r over X is a continuous representation $\rho : \pi_1(X) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$.

Trace functions

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This number is the image of x by $\mathrm{tr}_{\mathscr{L}}$.

The Lang isogeny

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$$\mathrm{id}_G - F_G : G \to G$$

is a finite étale cover, which is also Galois with group $G(k)$.

Character sheaves

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$$\pi_1(G) \to G(k) \to \overline{\mathbb{Q}}_\ell^\times,$$

corresponding to a rank one local system over G; denoted $\mathscr{L}_\chi$.

More generally, given a morphism $f : X \to G$ of k-schemes, we compose the morphism above with f_* to obtain a rank one local system $f^*\mathscr{L}_\chi$, commonly denoted $\mathscr{L}_{\chi(f)}$.

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Its trace at a point $x \in X(E)$ is given by $\chi(\mathrm{tr}^G_{E/k} f(x))$, where $\mathrm{tr}^G_{E/k} : G(E) \to G(k)$ sends $g \in G(E)$ to $g + \mathrm{Frob}_E(g) + \ldots + \mathrm{Frob}_E^{n-1}(g)$ for $n = [E : k]$.

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In topology, the constructible sheaves are those that restrict to local systems on a given stratification. Up to some minor technical details, the same definition works in the ℓ-adic setting.

The six-functors on étale cohomology

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For a morphism $f : X \to Y$,

- we have a direct image and a compactly supported direct image functor $\mathrm{R}f_*, \mathrm{R}f_! : \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}^b_c(Y, \overline{\mathbb{Q}}_\ell)$;

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These functors satisfy a large number of compatibility relations which are encapsulated in the designation six-functor formalism.

The trace formula

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In particular, our exponential sums may be written as

$$\begin{aligned} S(f, E, \chi) &= \sum_{x \in X(E)} \chi(f(x)) = \sum_{t \in G(E)} \chi(t) \, \#\{x \in X(E) \mid f(x) = t\} \\ &= \sum_{t \in G(E)} \chi(t) \, \mathrm{tr}_{Rf_! \overline{\mathbb{Q}}_\ell}(t). \end{aligned}$$

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Believe it or not, this is a tremendous simplification!

Deligne’s equidistribution theorem

Fourier-Deligne transform

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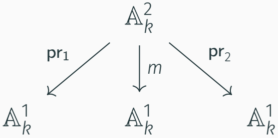
Consider the diagram on the right, where $m : (x, y) \mapsto xy$ is the multiplication map. The Fourier-Deligne transform is the functor

$$\mathsf{FT}_\chi : \mathrm{D}_c^b(\mathbb{A}_k^1, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}_c^b(\mathbb{A}_k^1, \overline{\mathbb{Q}}_\ell)$$
$$M \mapsto \mathrm{R}\mathsf{pr}_{2,!}(\mathsf{pr}_1^* M \otimes \mathscr{L}_{\chi(m)}).$$



ES as a *single* trace function

Recall that, if we fix a character $\tilde{\chi}$ of $k = \mathbb{A}^1(k)$, all $\chi \in \hat{E}$ are of the form $t \mapsto \tilde{\chi}(\mathrm{tr}_{E/k}(tx))$ for a unique $x \in E$.

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$$\{S(f, E, \chi)\}_{\chi \in \hat{E}} = \left\{\mathrm{tr}_{\mathsf{FT}_{\tilde{\chi}}(\mathrm{R}f_! \overline{\mathbb{Q}}_\ell)}(x)\right\}_{x \in E}.$$

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In particular, we may focus our study in the distribution of a single trace function.

Perverse sheaves

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Luckily, there is another abelian subcategory of $\mathrm{D}_c^b(\mathbb{A}_k^1, \overline{\mathbb{Q}}_\ell)$ which works much better; the category of perverse sheaves!

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- (Artin vanishing) If f is affine and quasi-finite, $\mathrm{R}f_!$ preserves perversity;
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In particular, for finite f, $M = \mathsf{FT}_{\tilde{\chi}}(\mathrm{R}f_!\overline{\mathbb{Q}}_\ell)$ is a perverse sheaf. Moreover, there's an open subscheme $U \hookrightarrow \mathbb{A}^1_k$ such that $M|_U$ is a local system $\mathscr{L}$.

Monodromy groups

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Clearly $G_{\mathrm{geom},\mathscr{L}}$ is a subgroup of $G_{\mathrm{arith},\mathscr{L}}$. Moreover, Deligne proved that, in our case, $G_{\mathrm{geom},\mathscr{L}}$ is reductive.

Deligne's equidistribution theorem

Fix an embedding $\overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$, and let K be a maximal compact subgroup of $G_{\mathrm{geom},\mathscr{L}}(\mathbb{C})$.

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Theorem (Deligne)

Suppose that $G_{\mathrm{geom},\mathscr{L}} = G_{\mathrm{arith},\mathscr{L}}$. Then the sums

$$\left\{ \frac{(-1)^d}{|E|^{d/2}} \sum_{x \in X(E)} \tilde{\chi}(\mathrm{tr}_{E/k}(tf(x))) \right\}_{t \in U(E)}$$

are distributed as traces of random matrices in K as the degree of E/k tends to infinity.

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More generally, we have equidistribution results for sums of the form

$$\sum_{x \in E} \tilde{\chi}(\mathrm{tr}_{E/k}(tx))\, \mathrm{tr}_M(x),$$

where M is a "nice" (= pure of weight 0) perverse sheaf.

The general equidistribution result

Why generalizing Deligne's result is hard

One crucial point in the discussion leading to Deligne's theorem is that, when $G = \mathbb{G}_a$, there's an algebraic variety ($\mathbb{A}^1$ itself) over k whose E-points parameterize the characters of $G(E)$.

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N. Katz had a brilliant idea: instead of considering a Fourier transform, we should consider a convolution of sheaves. If $m : G \times G \to G$ is the multiplication map, and M, N are objects of $\mathrm{D}_c^b(G, \overline{\mathbb{Q}}_\ell)$, the complex $M *_! N := \mathrm{R}m_!(\mathrm{pr}_1^* M \otimes \mathrm{pr}_2^* N)$ satisfies

$$\mathrm{tr}_{M*_!N}(x) = (\mathrm{tr}_M * \mathrm{tr}_N)(x) = \sum_{t \in G(E)} \mathrm{tr}_M(t)\, \mathrm{tr}_N(xt^{-1}).$$

Tannakian categories

The so-called formalism of *Tannakian categories* gives conditions on an abelian symmetric monoidal category that forces it to be equivalent to $\mathrm{Rep}(H)$, for some group scheme H.

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- Convolution defines a symmetric monoidal operation on $\mathrm{D}_c^b(G, \overline{\mathbb{Q}}_\ell)$, but this category is not abelian;
- $\mathrm{Perv}(G, \overline{\mathbb{Q}}_\ell)$ is abelian, but perverse sheaves are not preserved by convolution.

Negligible objects

Gabber and Loeser had the idea to quotient $\mathrm{Perv}(G, \overline{\mathbb{Q}}_\ell)$ by a Serre subcategory composed of *negligible objects*.

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When $G = \mathbb{G}_m$, the negligible objects are precisely those with zero Euler characteristic. This allowed Katz to prove an equidistribution theorem similar to the previous one.

Generic vanishing of cohomology

For higher-dimensional groups, the proof that a reasonable choice of negligible objects indeed yields a Tannakian category rests on a difficult cohomology vanishing theorem.

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Such a result was not known before the very recent preprint *Arithmetic Fourier Transforms over Finite Fields* by A. Forey, J. Fresán, and E. Kowalski, which uses as a fundamental tool the *Quantitative Sheaf Theory* of W. Sawin.

The general equidistribution theorem

Let M be a *semi*perverse sheaf on G, *mixed of weights* ≤ 0. The Tannakian formalism gives a "arithmetic monodromy group" G_{arith}.

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Theorem (Forey, Fresán, Kowalski)

The exponential sums $S(M, E, \chi) := \sum_{x \in G(E)} \chi(x) \, \mathrm{tr}_M(x)$, for $\chi \in \widehat{G(E)}$, become ν-equidistributed *on average* as the degree of E/k tends to infinity.

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$$\int_K f(\operatorname{tr}(x))\, \mathrm{d}\mu(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{[E:k] \leq n} \frac{1}{|G(E)|} \sum_{\chi \in \widehat{G(E)}} f(S(M, E, \chi)).$$

for every bounded continuous function $f : \mathbb{C} \to \mathbb{C}$.

Let's work out the case of Gauss' sums

The Gauss sum $g(\psi, \chi)$ is defined as

$$g(\psi, \chi) := \sum_{x \in \mathbb{F}_q^\times} \psi(x)\chi(x),$$

where ψ is an additive and χ is a multiplicative character of $\mathbb{F}_q$.

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where ψ is an additive and χ is a multiplicative character of $\mathbb{F}_q$.

Fix a nontrivial additive character ψ of $\mathbb{F}_p$ and denote by ψ_q the character of $\mathbb{F}_q$ obtained by composing with the trace.

Gauss sums

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If χ is trivial, $g(\psi_q, \chi)$ is simply -1. Else, its absolute value is $\sqrt{q}$ and we find $q-2$ points

$$\theta_{q,\chi} := \frac{g(\psi_q, \chi)}{\sqrt{q}} \in S^1,$$

one for each nontrivial multiplicative character.

How do these angles are distributed?

As in Sato-Tate's conjecture, we may wonder how do these "angles" are distributed on the unit circle as q tends *vertically* to infinity.

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Theorem (Deligne)

As q tends to infinity, the angles $\{\theta_{q,\chi}\}_{\chi\neq 1}$ become equidistributed on S^1 with respect to its normalized Haar measure. In other words, the equation

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,\mathrm{d}\theta = \lim_{q\to\infty}\frac{1}{q-2}\sum_{\chi\neq 1} f(\theta_{q,\chi})$$

is satisfied for all continuous functions $f : S^1 \to \mathbb{C}$.

Using our formalism

Our goal is to calculate

$$\lim_{q\to\infty} \frac{1}{q-2} \sum_{\chi\neq 1} f(\theta_{q,\chi}).$$

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Thank you!