Descent and sheaves on the étale site

With a descent into absolute bullshit

Gabriel Ribeiro

École Polytechnique

Summary

1. Why étale morphisms?
2. Sites and sheaves
3. Stalks and topoi
4. Stalks of the structure sheaf
5. Descent theory
6. At long last, some calculations

Why étale morphisms?

Local inversion theorem

Let $f : M \to N$ be a smooth map between smooth manifolds. The local inversion theorem says that if $\mathrm{d}f_p : T_pM \to T_{f(p)}N$ is an isomorphism, then there exists neighborhoods U of p and V of $f(p)$ such that $U \to V$ is an isomorphism.

Local inversion theorem

Let $f : M \to N$ be a smooth map between smooth manifolds. The local inversion theorem says that if $\mathrm{d}f_p : T_pM \to T_{f(p)}N$ is an isomorphism, then there exists neighborhoods U of p and V of $f(p)$ such that $U \to V$ is an isomorphism.

Let $f : X \to S$ a map between smooth varieties over $\bar{k}$. If $\mathrm{d}f_x : T_xX \to T_{f(x)}S$ is an isomorphism, then there exists étale neighborhoods U of x and V of $f(x)$ such that $U \to V$ is an isomorphism.

Sites and sheaves

Grothendieck pretopology

We want to generalize the category Open_X with its notion of *open cover*.

Grothendieck pretopology

We want to generalize the category Open_X with its notion of *open cover*.

Definition - Grothendieck pretopology

Let C be a small category with fibered products. A *Grothendieck pretopology* on C is the data, for each object $U \in \mathsf{C}$, of a set $\mathsf{Cov}(U)$ of *coverings*. The elements of $\mathsf{Cov}(U)$ are collections of morphisms $\{U_i \to U\}_{i \in I}$ which satisfy

- If $f : V \to U$ is an isomorphism, then $\{f\} \in \mathsf{Cov}(U)$.

Grothendieck pretopology

We want to generalize the category Open_X with its notion of *open cover*.

Definition - Grothendieck pretopology

Let C be a small category with fibered products. A *Grothendieck pretopology* on C is the data, for each object $U \in \mathsf{C}$, of a set $\mathsf{Cov}(U)$ of *coverings*. The elements of $\mathsf{Cov}(U)$ are collections of morphisms $\{U_i \to U\}_{i \in I}$ which satisfy

- If $f : V \to U$ is an isomorphism, then $\{f\} \in \mathsf{Cov}(U)$.
- If $\{U_i \to U\}_{i \in I} \in \mathsf{Cov}(U)$, and $g : V \to U$ is any morphism, then $\{V \times_U U_i \to V\}_{i \in I} \in \mathsf{Cov}(V)$.

Grothendieck pretopology

We want to generalize the category Open_X with its notion of *open cover*.

Definition - Grothendieck pretopology

Let C be a small category with fibered products. A *Grothendieck pretopology* on C is the data, for each object $U \in C$, of a set $\mathsf{Cov}(U)$ of *coverings*. The elements of $\mathsf{Cov}(U)$ are collections of morphisms $\{U_i \to U\}_{i\in I}$ which satisfy

- If $f : V \to U$ is an isomorphism, then $\{f\} \in \mathsf{Cov}(U)$.
- If $\{U_i \to U\}_{i\in I} \in \mathsf{Cov}(U)$, and $g : V \to U$ is any morphism, then $\{V \times_U U_i \to V\}_{i\in I} \in \mathsf{Cov}(V)$.
- If $\{U_i \to U\}_{i\in I} \in \mathsf{Cov}(U)$ and, for every $i \in I$, $(U_{ij} \to U_i)_{j\in J} \in \mathsf{Cov}(U_i)$, then $(U_{ij} \to U_i \to U)_{i,j} \in \mathsf{Cov}(U)$.

Bullshit remarks ([Stacks, Tag 020K] & [Vistoli, §2.3.5])

Naturally, we'll want to consider presheaves as functors $C^{op} \to \mathrm{Set}$. But if C is not small, the objects of $\mathsf{Fun}(C^{op}, \mathrm{Set})$ doesn't even form a class!

Bullshit remarks ([Stacks, Tag 020K] & [Vistoli, §2.3.5])

Naturally, we'll want to consider presheaves as functors $C^{op} \to Set$. But if C is not small, the objects of $\mathsf{Fun}(C^{op}, Set)$ doesn't even form a class! That's why C *has* to be small. In practice, it often suffices to replace it by an equivalent category or to consider all spaces contained in a fixed universe. (Keyword: comparison lemma.)

Bullshit remarks ([Stacks, Tag 020K] & [Vistoli, §2.3.5])

Naturally, we'll want to consider presheaves as functors $C^{op} \to Set$. But if C is not small, the objects of $\mathsf{Fun}(C^{op}, Set)$ doesn't even form a class! That's why C *has* to be small. In practice, it often suffices to replace it by an equivalent category or to consider all spaces contained in a fixed universe. (Keyword: comparison lemma.)

There's also a notion of Grothendieck *topology*, defined using *sieves*.

Bullshit remarks ([Stacks, Tag 020K] & [Vistoli, §2.3.5])

Naturally, we'll want to consider presheaves as functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. But if C is not small, the objects of $\mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ doesn't even form a class! That's why C *has* to be small. In practice, it often suffices to replace it by an equivalent category or to consider all spaces contained in a fixed universe. (Keyword: comparison lemma.)

There's also a notion of Grothendieck *topology*, defined using *sieves*. We'll see why this may be important later, but for now we remark that a pretopology always gives rise to a, not necessarily unique, topology.

Bullshit remarks ([Stacks, Tag 020K] & [Vistoli, §2.3.5])

Naturally, we'll want to consider presheaves as functors $C^{op} \to \text{Set}$. But if C is not small, the objects of $\mathsf{Fun}(C^{op}, \text{Set})$ doesn't even form a class! That's why C *has* to be small. In practice, it often suffices to replace it by an equivalent category or to consider all spaces contained in a fixed universe. (Keyword: comparison lemma.)

There's also a notion of Grothendieck *topology*, defined using *sieves*. We'll see why this may be important later, but for now we remark that a pretopology always gives rise to a, not necessarily unique, topology. As it has became usual, we'll now forget about this and use the word topology to mean pretopology.

Sites

A *site* is a category equipped with a Grothendieck topology.

Sites

A *site* is a category equipped with a Grothendieck topology. There's a simple way to obtain sites.

Site construction lemma

Let S be a scheme and C/S be a full subcategory of Sch/S closed under fiber products. Moreover, suppose that P is a property of morphisms that's

- true for isomorphisms
- stable under base change
- stable under composition.

Define $\mathrm{Cov}(U)$ to be the set of all families $\{f_i : U_i \to U\}_{i \in I}$ such that f_i satisfies P and $U = \bigcup_{i \in I} f_i(U_i)$. This defines a topology on C/S.

Sites (see `bit.ly/3FSSBv6` for set-theoretical bullshit)

Examples - Small sites

If we let P be open immersions / étale morphisms and C/S consist of those morphisms $X \to S$ which satisfy P, we obtain the *small sites* S_{zar} and $S_{\text{ét}}$.

Sites (see `bit.ly/3FSSBv6` for set-theoretical bullshit)

Examples - Small sites

If we let P be open immersions / étale morphisms and C/S consist of those morphisms $X \to S$ which satisfy P, we obtain the *small sites* S_{zar} and $S_{\text{ét}}$.

Examples - Big sites

If we let P be open immersions / étale morphisms / faithfully flat morphisms locally of finite presentation and $\mathrm{C}/S = \mathrm{Sch}/S$, we obtain the *big sites* $(\mathrm{Sch}/S)_{\mathrm{zar}}$, $(\mathrm{Sch}/S)_{\text{ét}}$, and $(\mathrm{Sch}/S)_{\mathrm{fppf}}$.

The fpqc site ([Vistoli, §2.3.2])

Definition - fpqc morphism

A faithfully flat morphism of schemes $f : X \to S$ is said to be *fpqc* is every quasi-compact open subset of S is the image of a quasi-compact open subset of X.

The fpqc site ([Vistoli, §2.3.2])

Definition - fpqc morphism

A faithfully flat morphism of schemes $f : X \to S$ is said to be *fpqc* is every quasi-compact open subset of S is the image of a quasi-compact open subset of X.

Main facts:

- open immersion $\implies$ étale $\implies$ fppf $\implies$ fpqc

The fpqc site ([Vistoli, §2.3.2])

Definition - fpqc morphism

A faithfully flat morphism of schemes $f : X \to S$ is said to be *fpqc* is every quasi-compact open subset of S is the image of a quasi-compact open subset of X.

Main facts:

- open immersion $\implies$ étale $\implies$ fppf $\implies$ fpqc
- fpqc morphisms are stable under base change and composition, yielding a ”site” $(\mathrm{Sch}/S)_{\mathrm{fpqc}}$

The fpqc site ([Vistoli, §2.3.2])

Definition - fpqc morphism

A faithfully flat morphism of schemes $f : X \to S$ is said to be *fpqc* is every quasi-compact open subset of S is the image of a quasi-compact open subset of X.

Main facts:

- open immersion $\implies$ étale $\implies$ fppf $\implies$ fpqc
- fpqc morphisms are stable under base change and composition, yielding a "site" $(\mathrm{Sch}/S)_{\mathrm{fpqc}}$
- if $X \to S$ is fpqc, S has the quotient topology

Definition - Sheaf

Let C be a site and A be an algebraic category. A *sheaf* on C with values in A is a presheaf $\mathscr{F} : \mathsf{C}^{\mathrm{op}} \to \mathsf{A}$ such that, for every $U \in \mathsf{C}$ and every covering $\{U_i \to U\}_i$ of U, the diagram

$$\mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \times_U U_j)$$

is an equalizer. If only the left arrow is monic, we say that $\mathscr{F}$ is *separated*.

Čech stuff

Let $\mathscr{U} = \{U_i \to U\}_{i\in I}$ be a covering. Recall the usual construction of the 0-th Čech cohomology group:

$$\check{H}^0(\mathscr{U}, \mathscr{F}) := \left\{ (s_i)_{i\in I} \in \prod_{i\in I} \mathscr{F}(U_i) \;\middle|\; s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \text{ for all } i, j \in I \right\}.$$

Čech stuff

Let $\mathscr{U} = \{U_i \to U\}_{i \in I}$ be a covering. Recall the usual construction of the 0-th Čech cohomology group:

$$\check{H}^0(\mathscr{U}, \mathscr{F}) := \left\{ (s_i)_{i \in I} \in \prod_{i \in I} \mathscr{F}(U_i) \;\middle|\; s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \text{ for all } i, j \in I \right\}.$$

Since any two coverings admit a common refinement,

$$\mathscr{F}^+(U) := \check{H}^0(U, \mathscr{F}) := \underset{\mathscr{U} \in \mathsf{Cov}(U)}{\operatorname{colim}} \check{H}^0(\mathscr{U}, \mathscr{F})$$

is a filtered colimit.

Čech stuff

Let $\mathscr{U} = \{U_i \to U\}_{i \in I}$ be a covering. Recall the usual construction of the 0-th Čech cohomology group:

$$\check{H}^0(\mathscr{U}, \mathscr{F}) := \left\{ (s_i)_{i\in I} \in \prod_{i \in I} \mathscr{F}(U_i) \;\middle|\; s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \text{ for all } i, j \in I \right\}.$$

Since any two coverings admit a common refinement,

$$\mathscr{F}^+(U) := \check{H}^0(U, \mathscr{F}) := \operatorname*{colim}_{\mathscr{U} \in \mathsf{Cov}(U)} \check{H}^0(\mathscr{U}, \mathscr{F})$$

is a filtered colimit. This defines a presheaf $\mathscr{F}^+$, along with a canonical map $\mathscr{F} \to \mathscr{F}^+$ given by

$$\mathscr{F}(U) = \check{H}^0(\{\mathrm{id}_U\}, \mathscr{F}) \to \operatorname*{colim}_{\mathscr{U} \in \mathsf{Cov}(U)} \check{H}^0(\mathscr{U}, \mathscr{F}) = \mathscr{F}^+(U).$$

Sheafification [LM, Thm 11.25]

Proposition - Grothendieck's ++ construction

- The presheaf $\mathscr{F}^+$ is separated.

Sheafification [LM, Thm 11.25]

Proposition - Grothendieck's ++ construction

- The presheaf $\mathscr{F}^+$ is separated.
- If $\mathscr{F}$ is separated, $\mathscr{F}^+$ is a sheaf.

Sheafification [LM, Thm 11.25]

Proposition - Grothendieck's ++ construction

- The presheaf $\mathscr{F}^+$ is separated.
- If $\mathscr{F}$ is separated, $\mathscr{F}^+$ is a sheaf.
- The map $\mathscr{F} \to \mathscr{F}^{++}$ satisfies the universal property of sheafification.

Sheafification [LM, Thm 11.25]

Proposition - Grothendieck's ++ construction

- The presheaf $\mathscr{F}^+$ is separated.
- If $\mathscr{F}$ is separated, $\mathscr{F}^+$ is a sheaf.
- The map $\mathscr{F} \to \mathscr{F}^{++}$ satisfies the universal property of sheafification.

Sheafification [LM, Thm 11.25]

Proposition - Grothendieck's ++ construction

- The presheaf $\mathscr{F}^+$ is separated.
- If $\mathscr{F}$ is separated, $\mathscr{F}^+$ is a sheaf.
- The map $\mathscr{F} \to \mathscr{F}^{++}$ satisfies the universal property of sheafification.

In particular, $\Gamma(X, \mathscr{F}) = \check{H}^0(X, \mathscr{F})$ if $\mathscr{F}$ is a sheaf. We didn't define cohomology yet, but it's always true that $H^1(X, \mathscr{F}) = \check{H}^1(X, \mathscr{F})$ and $H^i_{\text{ét}}(X, \mathscr{F}) = \check{H}^i(X, \mathscr{F})$ holds for X quasiprojective over an affine scheme. [Milne, Thm 2.17]

Categorical remarks

The universal property of sheafification says that the inclusion functor ι from presheaves to sheaves is right adjoint to sheafification. This gives many things for free:

Categorical remarks

The universal property of sheafification says that the inclusion functor ι from presheaves to sheaves is right adjoint to sheafification. This gives many things for free:

- Since ι is right adjoint and fully faithful, it not only preserves limits but creates them.

Categorical remarks

The universal property of sheafification says that the inclusion functor ι from presheaves to sheaves is right adjoint to sheafification. This gives many things for free:

- Since ι is right adjoint and fully faithful, it not only preserves limits but creates them.
- Since $(-)^{++}$ is left adjoint, it preserves colimits.

Categorical remarks

The universal property of sheafification says that the inclusion functor ι from presheaves to sheaves is right adjoint to sheafification. This gives many things for free:

- Since ι is right adjoint and fully faithful, it not only preserves limits but creates them.
- Since $(-)^{++}$ is left adjoint, it preserves colimits.
- But $(-)^{++}$ is a filtered colimit, so it preserves finite limits as well!

Categorical remarks

The universal property of sheafification says that the inclusion functor ι from presheaves to sheaves is right adjoint to sheafification. This gives many things for free:

- Since ι is right adjoint and fully faithful, it not only preserves limits but creates them.
- Since $(-)^{++}$ is left adjoint, it preserves colimits.
- But $(-)^{++}$ is a filtered colimit, so it preserves finite limits as well!

Categorical remarks

The universal property of sheafification says that the inclusion functor ι from presheaves to sheaves is right adjoint to sheafification. This gives many things for free:

- Since ι is right adjoint and fully faithful, it not only preserves limits but creates them.
- Since $(-)^{++}$ is left adjoint, it preserves colimits.
- But $(-)^{++}$ is a filtered colimit, so it preserves finite limits as well!

In particular, sheaves with values in an abelian category form an abelian category.

Direct and inverse images

Let C/X and C/S be sites constructed using our *site construction lemma* relative to a property P.

Direct and inverse images

Let C/X and C/S be sites constructed using our *site construction lemma* relative to a property P.

Definition - Direct image

Let $f : X \to S$ be a morphism of schemes, and let $\mathscr{F}$ be a presheaf on X. The *direct image* $f_*\mathscr{F}$ is the presheaf on S defined by

$$\Gamma(V, f_*\mathscr{F}) := \Gamma(V \times_S X, \mathscr{F}),$$

where $V \to S$ is an element of C/S.

Direct and inverse images

Let C/X and C/S be sites constructed using our *site construction lemma* relative to a property P.

Definition - Direct image

Let $f : X \to S$ be a morphism of schemes, and let $\mathscr{F}$ be a presheaf on X. The *direct image* $f_*\mathscr{F}$ is the presheaf on S defined by

$$\Gamma(V, f_*\mathscr{F}) := \Gamma(V \times_S X, \mathscr{F}),$$

where $V \to S$ is an element of C/S.

As usual, $f_*\mathscr{F}$ is a sheaf if $\mathscr{F}$ is.

Direct and inverse images

The same construction as in the topological case works for inverse images. Namely, let $\mathscr{G}$ be a sheaf on S and $U \to X$ be an element of C/X and consider commutative squares of the form

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S, \end{array}$$

where $V \to S$ is in C/S.

Direct and inverse images

The same construction as in the topological case works for inverse images. Namely, let $\mathscr{G}$ be a sheaf on S and $U \to X$ be an element of C/X and consider commutative squares of the form

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S, \end{array}$$

where $V \to S$ is in C/S.

Definition - Inverse image image

The *inverse image* $f^*\mathscr{G}$ is the presheaf on X defined by

$$\Gamma(U, f^*\mathscr{G}) := \operatorname{colim} \Gamma(V, \mathscr{G}),$$

where the colimit is taken over all possible commutative diagrams as above.

Direct and inverse images

The inverse image $f^*\mathscr{G}$ need not be a sheaf even if $\mathscr{G}$ is (as usual). In this case, we redefine it to be the sheafification of the presheaf inverse image.

Direct and inverse images

The inverse image $f^*\mathscr{G}$ need not be a sheaf even if $\mathscr{G}$ is (as usual). In this case, we redefine it to be the sheafification of the presheaf inverse image.

This satisfies all the properties one should expect!

- The colimit in the definition of f^* is filtered if C/S has finite limits.

Direct and inverse images

The inverse image $f^*\mathscr{G}$ need not be a sheaf even if $\mathscr{G}$ is (as usual). In this case, we redefine it to be the sheafification of the presheaf inverse image.

This satisfies all the properties one should expect!

- The colimit in the definition of f^* is filtered if C/S has finite limits.
- $f^* \dashv f_*$. Moreover, f^* is exact if C/S has finite limits.

Direct and inverse images

The inverse image $f^*\mathscr{G}$ need not be a sheaf even if $\mathscr{G}$ is (as usual). In this case, we redefine it to be the sheafification of the presheaf inverse image.

This satisfies all the properties one should expect!

- The colimit in the definition of f^* is filtered if C/S has finite limits.
- $f^* \dashv f_*$. Moreover, f^* is exact if C/S has finite limits.
- If the categories of sheaves are monoidal, then so is f^* when C/S has finite limits.

Direct and inverse images

The inverse image $f^*\mathscr{G}$ need not be a sheaf even if $\mathscr{G}$ is (as usual). In this case, we redefine it to be the sheafification of the presheaf inverse image.

This satisfies all the properties one should expect!

- The colimit in the definition of f^* is filtered if C/S has finite limits.
- $f^* \dashv f_*$. Moreover, f^* is exact if C/S has finite limits.
- If the categories of sheaves are monoidal, then so is f^* when C/S has finite limits.

Direct and inverse images

The inverse image $f^*\mathscr{G}$ need not be a sheaf even if $\mathscr{G}$ is (as usual). In this case, we redefine it to be the sheafification of the presheaf inverse image.

This satisfies all the properties one should expect!

- The colimit in the definition of f^* is filtered if C/S has finite limits.
- $f^* \dashv f_*$. Moreover, f^* is exact if C/S has finite limits.
- If the categories of sheaves are monoidal, then so is f^* when C/S has finite limits.

The condition about finite limits is satisfied for all the sites under consideration.

Recollement

Let $j : U \to X$ be an open immersion, $i : Z \to X$ be the complementary closed immersion, and endow all schemes with the étale topology.

Recollement

Let $j : U \to X$ be an open immersion, $i : Z \to X$ be the complementary closed immersion, and endow all schemes with the étale topology.

Let $\mathscr{F}$ be a sheaf on X. We define $\mathscr{F}_1 := i^*\mathscr{F}$, $\mathscr{F}_2 := j^*\mathscr{F}$, and $\varphi_{\mathscr{F}} : \mathscr{F}_1 \to i^*j_*\mathscr{F}_2$ as the image under i^* of the unit for $j^* \dashv j_*$.

Recollement

Let $j : U \to X$ be an open immersion, $i : Z \to X$ be the complementary closed immersion, and endow all schemes with the étale topology.

Let $\mathscr{F}$ be a sheaf on X. We define $\mathscr{F}_1 := i^*\mathscr{F}$, $\mathscr{F}_2 := j^*\mathscr{F}$, and $\varphi_{\mathscr{F}} : \mathscr{F}_1 \to i^*j_*\mathscr{F}_2$ as the image under i^* of the unit for $j^* \dashv j_*$. This defines a functor

$$\mathrm{Ab}(X_{\text{ét}}) \to \mathrm{T}(X),$$

where $\mathrm{T}(X)$ is the category of such triples.

Proposition

This functor is an equivalence of categories.

Recollement

Proposition

This functor is an equivalence of categories.

Using this, it's easy to define the following functors:

$$\begin{aligned} i^* &: \mathscr{F}_1 \leftarrow\!\shortmid (\mathscr{F}_1, \mathscr{F}_2, \varphi) & j^* &: (\mathscr{F}_1, \mathscr{F}_2, \varphi) \mapsto \mathscr{F}_2 \\ i_* &: \mathscr{F}_1 \mapsto (\mathscr{F}_1, 0, 0) & j_* &: (i^* j_* \mathscr{F}_2, \mathscr{F}_2, \mathrm{id}) \leftarrow\!\shortmid \mathscr{F}_2 \\ i^! &: \ker \varphi \leftarrow\!\shortmid (\mathscr{F}_1, \mathscr{F}_2, \varphi) & j^! &:= j^* \\ i_! &:= i_* & j_! &: (0, \mathscr{F}_2, 0) \leftarrow\!\shortmid \mathscr{F}_2, \end{aligned}$$

Proposition

This functor is an equivalence of categories.

Using this, it's easy to define the following functors:

$$
\begin{aligned}
i^* &: \mathscr{F}_1 \hookleftarrow (\mathscr{F}_1, \mathscr{F}_2, \varphi) & j^* &: (\mathscr{F}_1, \mathscr{F}_2, \varphi) \mapsto \mathscr{F}_2 \\
i_* &: \mathscr{F}_1 \mapsto (\mathscr{F}_1, 0, 0) & j_* &: (i^* j_* \mathscr{F}_2, \mathscr{F}_2, \mathrm{id}) \hookleftarrow \mathscr{F}_2 \\
i^! &: \ker \varphi \hookleftarrow (\mathscr{F}_1, \mathscr{F}_2, \varphi) & j^! &:= j^* \\
i_! &:= i_* & j_! &: (0, \mathscr{F}_2, 0) \hookleftarrow \mathscr{F}_2,
\end{aligned}
$$

which, of course, behave as expected.

Recollement

- Not only $i^* \dashv i_*$ and $j^* \dashv j_*$, but also $i_! \dashv i^!$ and $j_! \dashv j^!$.

Recollement

- Not only $i^* \dashv i_*$ and $j^* \dashv j_*$, but also $i_! \dashv i^!$ and $j_! \dashv j^!$.
- Not only i^* and j^* are exact, but also $i_!$ and $j_!$.

Recollement

- Not only $i^* \dashv i_*$ and $j^* \dashv j_*$, but also $i_! \dashv i^!$ and $j_! \dashv j^!$.
- Not only i^* and j^* are exact, but also $i_!$ and $j_!$.
- $i^! i_! \cong i^* i_* \cong \mathrm{id}$ and $j^! j_! \cong j^* j_* \cong \mathrm{id}$.

Recollement

- Not only $i^* \dashv i_*$ and $j^* \dashv j_*$, but also $i_! \dashv i^!$ and $j_! \dashv j^!$.
- Not only i^* and j^* are exact, but also $i_!$ and $j_!$.
- $i^! i_! \cong i^* i_* \cong \mathrm{id}$ and $j^! j_! \cong j^* j_* \cong \mathrm{id}$.
- $i^* j_! \cong i^! j_! \cong i^! j_* \cong 0$ and $j^* i_* \cong 0$.

Recollement

- Not only $i^* \dashv i_*$ and $j^* \dashv j_*$, but also $i_! \dashv i^!$ and $j_! \dashv j^!$.
- Not only i^* and j^* are exact, but also $i_!$ and $j_!$.
- $i^! i_! \cong i^* i_* \cong \mathrm{id}$ and $j^! j_! \cong j^* j_* \cong \mathrm{id}$.
- $i^* j_! \cong i^! j_! \cong i^! j_* \cong 0$ and $j^* i_* \cong 0$.
- The sequence $0 \to j_! j^! \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \to 0$ is exact.

Recollement

- Not only $i^* \dashv i_*$ and $j^* \dashv j_*$, but also $i_! \dashv i^!$ and $j_! \dashv j^!$.
- Not only i^* and j^* are exact, but also $i_!$ and $j_!$.
- $i^! i_! \cong i^* i_* \cong \mathrm{id}$ and $j^! j_! \cong j^* j_* \cong \mathrm{id}$.
- $i^* j_! \cong i^! j_! \cong i^! j_* \cong 0$ and $j^* i_* \cong 0$.
- The sequence $0 \to j_! j^! \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \to 0$ is exact.

Recollement

- Not only $i^* \dashv i_*$ and $j^* \dashv j_*$, but also $i_! \dashv i^!$ and $j_! \dashv j^!$.
- Not only i^* and j^* are exact, but also $i_!$ and $j_!$.
- $i^! i_! \cong i^* i_* \cong \mathrm{id}$ and $j^! j_! \cong j^* j_* \cong \mathrm{id}$.
- $i^* j_! \cong i^! j_! \cong i^! j_* \cong 0$ and $j^* i_* \cong 0$.
- The sequence $0 \to j_! j^! \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \to 0$ is exact.

The functors $f_!$ and $f^!$, for a general morphism f, weren't yet defined. But they will generalize (the derived functors of) the functors above.

Derived functors [Categories and Sheaves, Chap 18]

Let C be a site.

Derived functors [Categories and Sheaves, Chap 18]

Let C be a site. From our work on sheafification, it's clear that Ab(C) is an abelian category with coproducts where filtered colimits are exact. That is, it's an AB5 category.

Derived functors [Categories and Sheaves, Chap 18]

Let C be a site. From our work on sheafification, it's clear that Ab(C) is an abelian category with coproducts where filtered colimits are exact. That is, it's an AB5 category.

It's, moreover, Grothendieck. In particular, we have K-injective and K-flat resolutions. So, for a morphism of schemes $f : X \to S$, we define the derived functors

$$\begin{aligned} \mathrm{R}f_* &: \mathrm{D}(X) \to \mathrm{D}(S) & \mathrm{R}\,\mathbf{Hom} &: \mathrm{D}(X) \times \mathrm{D}(X)^{\mathrm{op}} \to \mathrm{D}(\mathrm{Ab}) \\ f^* &: \mathrm{D}(S) \to \mathrm{D}(X) & \mathrm{R}\underline{\mathrm{Hom}} &: \mathrm{D}(X) \times \mathrm{D}(X)^{\mathrm{op}} \to \mathrm{D}(X) \\ \mathrm{R}\Gamma &: \mathrm{D}(X) \to \mathrm{D}(\mathrm{Ab}) & - \otimes^{\mathrm{L}} - &: \mathrm{D}(X) \times \mathrm{D}(X) \to \mathrm{D}(X). \end{aligned}$$

Local cohomology

Also, if we're in the small étale site and $i : Z \to X$ is a closed immersion, we define

$$\Gamma_Z := \Gamma \circ i^! = \Gamma \circ i_! \circ i^!.$$

Local cohomology

Also, if we're in the small étale site and $i : Z \to X$ is a closed immersion, we define

$$\Gamma_Z := \Gamma \circ i^! = \Gamma \circ i_! \circ i^!.$$

As usual, all the expected properties follow formally. I can talk a little about this later, if someone wants.

Stalks and topoi

Topoi

Definition - Topoi

A *topos* $\mathscr{X}$ is a category equivalent to $\widetilde{\mathrm{C}} := \mathrm{Sh}(\mathrm{C})$, for some site C.

Topoi

Definition - Topoi

A *topos* $\mathscr{X}$ is a category equivalent to $\widetilde{\mathsf{C}} := \mathrm{Sh}(\mathsf{C})$, for some site C.

The topos, instead of the site, should be thought as the fundamental object.

Topoi

Definition - Topoi

A *topos* $\mathscr{X}$ is a category equivalent to $\widetilde{\mathrm{C}} := \mathrm{Sh}(\mathrm{C})$, for some site C.

The topos, instead of the site, should be thought as the fundamental object. Indeed,

- $\mathscr{X}$ has arbitrary limits and colimits (and finite limits commute with filtered colimits!)

Topoi

Definition - Topoi

A *topos* $\mathscr{X}$ is a category equivalent to $\widetilde{\mathrm{C}} := \mathrm{Sh}(\mathrm{C})$, for some site C.

The topos, instead of the site, should be thought as the fundamental object. Indeed,

- $\mathscr{X}$ has arbitrary limits and colimits (and finite limits commute with filtered colimits!)
- $\mathscr{X}$ has internal homs (i.e. exponential objects)

Topoi

Definition - Topoi

A *topos* $\mathscr{X}$ is a category equivalent to $\widetilde{\mathsf{C}} := \mathrm{Sh}(\mathsf{C})$, for some site C.

The topos, instead of the site, should be thought as the fundamental object. Indeed,

- $\mathscr{X}$ has arbitrary limits and colimits (and finite limits commute with filtered colimits!)
- $\mathscr{X}$ has internal homs (i.e. exponential objects)
- $\mathscr{X}$ has a sub-object classifier

Examples

Examples

- If $C = \mathsf{open}_X$, where $X = \mathsf{pt}$, the associated topos is Set.

Examples

Examples

- If $C = \mathsf{open}_X$, where $X = \mathsf{pt}$, the associated topos is Set.
- If C is any small category with the trivial Grothendieck topology, the associated topos is $\mathsf{PSh}(C)$.

Examples

Examples

- If $\mathsf{C} = \mathsf{open}_X$, where $X = \mathsf{pt}$, the associated topos is Set.
- If C is any small category with the trivial Grothendieck topology, the associated topos is $\mathsf{PSh}(\mathsf{C})$.
- Let G be a topological group. The category BG, whose objects are discrete sets with a continuous G-action, is a topos.

Examples

Examples

- If $\mathsf{C} = \mathsf{open}_X$, where $X = \mathsf{pt}$, the associated topos is Set.
- If C is any small category with the trivial Grothendieck topology, the associated topos is $\mathsf{PSh}(\mathsf{C})$.
- Let G be a topological group. The category BG, whose objects are discrete sets with a continuous G-action, is a topos.

Examples

Examples

- If $\mathsf{C} = \mathsf{open}_X$, where $X = \mathsf{pt}$, the associated topos is Set.
- If C is any small category with the trivial Grothendieck topology, the associated topos is $\mathsf{PSh}(\mathsf{C})$.
- Let G be a topological group. The category BG, whose objects are discrete sets with a continuous G-action, is a topos.

Remark: Different sites may generate the same topos!

Examples

- If $\mathsf{C} = \mathbf{open}_X$, where $X = \mathbf{pt}$, the associated topos is Set.
- If C is any small category with the trivial Grothendieck topology, the associated topos is $\mathbf{PSh}(\mathsf{C})$.
- Let G be a topological group. The category BG, whose objects are discrete sets with a continuous G-action, is a topos.

Remark: Different sites may generate the same topos! For example, $\widetilde{(\mathsf{Sch}/S)_{\mathrm{zar}}} \cong \widetilde{S_{\mathrm{zar}}}$ and $\widetilde{(\mathsf{Sch}/S)_{\mathrm{\acute{e}t}}} \cong \widetilde{(\mathsf{Sch}/S)_{\mathrm{smooth}}}$. [Stacks, Tag 055V]

Geometric morphism

There's a natural notion of maps between topoi.

Geometric morphism

There's a natural notion of maps between topoi.

Definition - Geometric morphism

Let $\mathscr{X}$ and $\mathscr{Y}$ be two topoi. A *geometric morphism* $f : \mathscr{X} \to \mathscr{Y}$ is a pair of functors $f^* : \mathscr{Y} \to \mathscr{X}$ and $f_* : \mathscr{X} \to \mathscr{Y}$, such that f^* is left adjoint to f_* and f^* preserves finite limits.

Geometric morphism

There's a natural notion of maps between topoi.

Definition - Geometric morphism

Let $\mathscr{X}$ and $\mathscr{Y}$ be two topoi. A *geometric morphism* $f : \mathscr{X} \to \mathscr{Y}$ is a pair of functors $f^* : \mathscr{Y} \to \mathscr{X}$ and $f_* : \mathscr{X} \to \mathscr{Y}$, such that f^* is left adjoint to f_* and f^* preserves finite limits.

- If X, Y are topological spaces and Y is sober, every geometric morphism $\mathsf{Sh}(X) \to \mathsf{Sh}(Y)$ comes from a continuous map $X \to Y$. [SGA4, §IV.4.2]

Geometric morphism

There's a natural notion of maps between topoi.

Definition - Geometric morphism

Let $\mathscr{X}$ and $\mathscr{Y}$ be two topoi. A *geometric morphism* $f : \mathscr{X} \to \mathscr{Y}$ is a pair of functors $f^* : \mathscr{Y} \to \mathscr{X}$ and $f_* : \mathscr{X} \to \mathscr{Y}$, such that f^* is left adjoint to f_* and f^* preserves finite limits.

- If X, Y are topological spaces and Y is sober, every geometric morphism $\mathsf{Sh}(X) \to \mathsf{Sh}(Y)$ comes from a continuous map $X \to Y$. [SGA4, §IV.4.2]
- If $\mathscr{X}$ is a topos, there's a unique geometric morphism $f : \mathscr{X} \to \mathsf{Set}$. Namely, $f_*(\mathscr{F}) = \mathsf{Hom}(F, \mathscr{F})$ and $f^*(A) = \coprod_A F$, where F is the final object of $\mathscr{X}$. [SGA4, §IV.4.3]

Topos-theoretic points

A point x of a topological space X determines a geometric morphism $\mathrm{Set} \to \mathsf{Sh}(X)$. Indeed, we have a pair of adjunct functors ”skyscraper sheaf at x” and ”stalk at x”.

Definition - Point of a topos

Let $\mathscr{X}$ be a topos. A *point* of $\mathscr{X}$ is a geometric morphism $x : \mathrm{Set} \to \mathscr{X}$.

Topos-theoretic points

A point x of a topological space X determines a geometric morphism $\mathrm{Set} \to \mathsf{Sh}(X)$. Indeed, we have a pair of adjunct functors ”skyscraper sheaf at x” and ”stalk at x”.

Definition - Point of a topos

Let $\mathscr{X}$ be a topos. A *point* of $\mathscr{X}$ is a geometric morphism $x : \mathrm{Set} \to \mathscr{X}$. The *stalk* at a point x of some object $\mathscr{F} \in \mathscr{X}$ is the set $\mathscr{F}_x := x^* \mathscr{F}$.

Topos-theoretic points

When dealing with sheaves over topological spaces, basically everything can be checked in the stalks. But a non-trivial topos may have no points!

Definition - Enough points

Let $\mathscr{X}$ be a topos. We say that $\mathscr{X}$ *has enough points* if the inverse image functors are jointly conservative. That is, if for every morphism $\varphi : \mathscr{F} \to \mathscr{G}$ in $\mathscr{X}$, the stalk $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ being an isomorphism for all points x implies that φ is also an isomorphism.

Topos-theoretic points

In this case, everything works as with sheaves in a topological space!

Proposition - [SGA4, Corollaire I.6.3]

Let $\mathscr{X}$ be a topos. The following are equivalent:

- $\mathscr{X}$ has enough points;

Topos-theoretic points

In this case, everything works as with sheaves in a topological space!

Proposition - [SGA4, Corollaire I.6.3]

Let $\mathscr{X}$ be a topos. The following are equivalent:

- $\mathscr{X}$ has enough points;
- For every morphism φ in $\mathscr{X}$, if φ_x is a monomorphism for every point x, then so is φ;

Topos-theoretic points

In this case, everything works as with sheaves in a topological space!

Proposition - [SGA4, Corollaire I.6.3]

Let $\mathscr{X}$ be a topos. The following are equivalent:

- $\mathscr{X}$ has enough points;
- For every morphism φ in $\mathscr{X}$, if φ_x is a monomorphism for every point x, then so is φ;
- For every morphism φ in $\mathscr{X}$, if φ_x is an epimorphism for every point x, then so is φ;

Topos-theoretic points

In this case, everything works as with sheaves in a topological space!

Proposition - [SGA4, Corollaire I.6.3]

Let $\mathscr{X}$ be a topos. The following are equivalent:

- $\mathscr{X}$ has enough points;
- For every morphism φ in $\mathscr{X}$, if φ_x is a monomorphism for every point x, then so is φ;
- For every morphism φ in $\mathscr{X}$, if φ_x is an epimorphism for every point x, then so is φ;
- For every pair of morphisms $\varphi, \psi : \mathscr{F} \to \mathscr{G}$ in $\mathscr{X}$, if $\varphi_x = \psi_x$ for every point x, then $\varphi = \psi$.

Examples

- If X is a sober topological space, the topos-theoretic points of $\mathbf{Sh}(X)$ correspond precisely to the points of X. This holds, in particular, for $\widetilde{S_{\mathrm{zar}}}$, where S is a scheme.

Examples

- If X is a sober topological space, the topos-theoretic points of $\mathsf{Sh}(X)$ correspond precisely to the points of X. This holds, in particular, for $\widetilde{S_{\mathrm{zar}}}$, where S is a scheme.
- It seems that there's no simple description of the topos-theoretic points of the fppf topos. (Also there's no fpqc topos!)

Examples

Examples

- If X is a sober topological space, the topos-theoretic points of $\mathsf{Sh}(X)$ correspond precisely to the points of X. This holds, in particular, for $\widetilde{S_{\text{zar}}}$, where S is a scheme.
- It seems that there's no simple description of the topos-theoretic points of the fppf topos. (Also there's no fpqc topos!)

Examples

Examples

- If X is a sober topological space, the topos-theoretic points of $\mathsf{Sh}(X)$ correspond precisely to the points of X. This holds, in particular, for $\widetilde{S_{\text{zar}}}$, where S is a scheme.
- It seems that there's no simple description of the topos-theoretic points of the fppf topos. (Also there's no fpqc topos!)

These all have enough points.

Points of the étale topos

Definition / Proposition - [Stacks, Tag 04HU]

Let S be a scheme. A *geometric point* of S is a morphism $\operatorname{Spec}\Omega \to S$, where Ω is a separably closed field. We denote a geometric point by $\bar{s}$ and its set-theoretic image by s.

Points of the étale topos

Definition / Proposition - [Stacks, Tag 04HU]

Let S be a scheme. A *geometric point* of S is a morphism $\operatorname{Spec}\Omega \to S$, where Ω is a separably closed field. We denote a geometric point by $\bar{s}$ and its set-theoretic image by s. These correspond to the points of the étale topos, which has enough points.

Points of the étale topos

Definition / Proposition - [Stacks, Tag 04HU]

Let S be a scheme. A *geometric point* of S is a morphism $\operatorname{Spec}\Omega \to S$, where Ω is a separably closed field. We denote a geometric point by $\bar{s}$ and its set-theoretic image by s. These correspond to the points of the étale topos, which has enough points.

The proof of this result is not so simple, but this has to be the answer!

Points of the étale topos

Definition / Proposition - [Stacks, Tag 04HU]

Let S be a scheme. A *geometric point* of S is a morphism $\operatorname{Spec}\Omega \to S$, where Ω is a separably closed field. We denote a geometric point by $\bar{s}$ and its set-theoretic image by s. These correspond to the points of the étale topos, which has enough points.

The proof of this result is not so simple, but this has to be the answer! Since the Zariski topology is coarser, we may expect a point of the étale topos to also be a point of the Zariski topos. That is, a morphism $\operatorname{Spec} k \to S$.

Points of the étale topos

Definition / Proposition - [Stacks, Tag 04HU]

Let S be a scheme. A *geometric point* of S is a morphism $\operatorname{Spec}\Omega \to S$, where Ω is a separably closed field. We denote a geometric point by $\bar{s}$ and its set-theoretic image by s. These correspond to the points of the étale topos, which has enough points.

The proof of this result is not so simple, but this has to be the answer! Since the Zariski topology is coarser, we may expect a point of the étale topos to also be a point of the Zariski topos. That is, a morphism $\operatorname{Spec} k \to S$.

Such a point has trivial topological fundamental group, but may have non-trivial étale fundamental group.

Points of the étale topos

Definition / Proposition - [Stacks, Tag 04HU]

Let S be a scheme. A *geometric point* of S is a morphism $\operatorname{Spec}\Omega \to S$, where Ω is a separably closed field. We denote a geometric point by $\bar{s}$ and its set-theoretic image by s. These correspond to the points of the étale topos, which has enough points.

The proof of this result is not so simple, but this has to be the answer! Since the Zariski topology is coarser, we may expect a point of the étale topos to also be a point of the Zariski topos. That is, a morphism $\operatorname{Spec} k \to S$.

Such a point has trivial topological fundamental group, but may have non-trivial étale fundamental group. It is trivial precisely when k is separably closed.

Stalks of the structure sheaf

Digression into henselian rings [LM, §13.3]

Proposition

Let $(A, \mathfrak{m}, \kappa)$ be a local ring. If $f \in A[x]$, we denote by $\bar{f}$ its reduction modulo $\mathfrak{m}$.

Digression into henselian rings [LM, §13.3]

Proposition

Let $(A, \mathfrak{m}, \kappa)$ be a local ring. If $f \in A[x]$, we denote by $\bar{f}$ its reduction modulo $\mathfrak{m}$. The following are equivalent:

- Let $f \in A[x]$ be monic, and $a_0 \in \kappa$ be such that $\bar{f}(a_0) = 0$ and $\bar{f}'(a_0) \neq 0$. Then there exists a unique $a \in A$ such that $f(a) = 0$ and $a \equiv a_0 \pmod{\mathfrak{m}}$.

Proposition

Let $(A, \mathfrak{m}, \kappa)$ be a local ring. If $f \in A[x]$, we denote by $\bar{f}$ its reduction modulo $\mathfrak{m}$. The following are equivalent:

- Let $f \in A[x]$ be monic, and $a_0 \in \kappa$ be such that $\bar{f}(a_0) = 0$ and $\bar{f}'(a_0) \neq 0$. Then there exists a unique $a \in A$ such that $f(a) = 0$ and $a \equiv a_0 \pmod{\mathfrak{m}}$.
- Let f be monic and $\bar{g}, \bar{h} \in \kappa[x]$ be coprime monic polynomials such that $\bar{f} = \bar{g}\bar{h}$. Then there exists $g, h \in A[x]$ such that $f = gh$, and whose reductions are $\bar{g}$ and $\bar{h}$.

Digression into henselian rings [LM, §13.3]

Proposition

Let $(A, \mathfrak{m}, \kappa)$ be a local ring. If $f \in A[x]$, we denote by $\bar{f}$ its reduction modulo $\mathfrak{m}$. The following are equivalent:

- Let $f \in A[x]$ be monic, and $a_0 \in \kappa$ be such that $\bar{f}(a_0) = 0$ and $\bar{f}'(a_0) \neq 0$. Then there exists a unique $a \in A$ such that $f(a) = 0$ and $a \equiv a_0 \pmod{\mathfrak{m}}$.
- Let f be monic and $\bar{g}, \bar{h} \in \kappa[x]$ be coprime monic polynomials such that $\bar{f} = \bar{g}\bar{h}$. Then there exists $g, h \in A[x]$ such that $f = gh$, and whose reductions are $\bar{g}$ and $\bar{h}$.
- Every finite A-algebra B is a finite product of local rings.

Proposition

Let $(A, \mathfrak{m}, \kappa)$ be a local ring. If $f \in A[x]$, we denote by $\bar{f}$ its reduction modulo $\mathfrak{m}$. The following are equivalent:

- Let $f \in A[x]$ be monic, and $a_0 \in \kappa$ be such that $\bar{f}(a_0) = 0$ and $\bar{f}'(a_0) \neq 0$. Then there exists a unique $a \in A$ such that $f(a) = 0$ and $a \equiv a_0 \pmod{\mathfrak{m}}$.
- Let f be monic and $\bar{g}, \bar{h} \in \kappa[x]$ be coprime monic polynomials such that $\bar{f} = \bar{g}\bar{h}$. Then there exists $g, h \in A[x]$ such that $f = gh$, and whose reductions are $\bar{g}$ and $\bar{h}$.
- Every finite A-algebra B is a finite product of local rings.
- Let X be an étale scheme over $S = \operatorname{Spec} A$, s be the closed point of S such that X_s contains a point x with $\kappa(x) = \kappa(s)$. Then there exists a unique section g of $X \to S$ such that $g(s) = x$.

Digression into henselian rings [LM, §13.3]

Definition

Let $(A, \mathfrak{m}, \kappa)$ be a local ring. We say that A is *henselian* if it satisfies the conditions of the preceding proposition.

Digression into henselian rings [LM, §13.3]

Definition

Let $(A, \mathfrak{m}, \kappa)$ be a local ring. We say that A is *henselian* if it satisfies the conditions of the preceding proposition. If, moreover, κ is separably closed, we say that A is *strictly henselian*.

Digression into henselian rings [LM, §13.3]

Definition

Let $(A, \mathfrak{m}, \kappa)$ be a local ring. We say that A is *henselian* if it satisfies the conditions of the preceding proposition. If, moreover, κ is separably closed, we say that A is *strictly henselian*.

Newton's method implies that a complete local ring is henselian.

Digression into henselian rings [LM, §13.3]

Definition

Let $(A, \mathfrak{m}, \kappa)$ be a local ring. We say that A is *henselian* if it satisfies the conditions of the preceding proposition. If, moreover, κ is separably closed, we say that A is *strictly henselian*.

Newton's method implies that a complete local ring is henselian.

Our ring A is strictly henselian iff:

Let X be an étale scheme over $S = \operatorname{Spec} A$, s be the closed point of S, and $x \in X_s$. Then there exists a unique section g of $X \to S$ such that $g(s) = x$.

Digression into henselian rings [LM, §13.3]

We shall need two results about henselian rings.

Proposition

Let $(A, \mathfrak{m}, \kappa)$ be a henselian ring. Tensoring by κ yields an equivalence of categories $\mathsf{FEt}(A) \xrightarrow{\sim} \mathsf{FEt}(\kappa)$.

Digression into henselian rings [LM, §13.3]

We shall need two results about henselian rings.

Proposition

Let $(A, \mathfrak{m}, \kappa)$ be a henselian ring. Tensoring by κ yields an equivalence of categories $\mathsf{FEt}(A) \xrightarrow{\sim} \mathsf{FEt}(\kappa)$.

Proposition

Let $(A, \mathfrak{m}, \kappa)$ be a strictly henselian ring, $S = \operatorname{Spec} A$, and $\bar{s} : \operatorname{Spec} \kappa \to S$. Then $\Gamma(S, \mathscr{F}) = \mathscr{F}_{\bar{s}}$ for every abelian sheaf $\mathscr{F}$ on $S_{\text{ét}}$.

Digression into henselian rings [LM, §13.3]

We shall need two results about henselian rings.

Proposition

Let $(A, \mathfrak{m}, \kappa)$ be a henselian ring. Tensoring by κ yields an equivalence of categories $\mathsf{FEt}(A) \xrightarrow{\sim} \mathsf{FEt}(\kappa)$.

Proposition

Let $(A, \mathfrak{m}, \kappa)$ be a strictly henselian ring, $S = \operatorname{Spec} A$, and $\bar{s} : \operatorname{Spec} \kappa \to S$. Then $\Gamma(S, \mathscr{F}) = \mathscr{F}_{\bar{s}}$ for every abelian sheaf $\mathscr{F}$ on $S_{\text{ét}}$.

Indeed, our characterization of strictly henselian rings implies that the identity map $S \to S$ is cofinal in the category of all étale neighborhoods of s.

Stalks of the structure sheaf

Just for now, believe me that the structure sheaf $\mathscr{O}_S$ of a scheme S is a sheaf for the étale topology.

Stalks of the structure sheaf

Just for now, believe me that the structure sheaf $\mathscr{O}_S$ of a scheme S is a sheaf for the étale topology.

Proposition

Let $\bar{s}$ be a geometric point of S. Then the stalk $\mathscr{O}_{S,\bar{s}}$ is strictly henselian.

Stalks of the structure sheaf

Just for now, believe me that the structure sheaf $\mathscr{O}_S$ of a scheme S is a sheaf for the étale topology.

Proposition

Let $\bar{s}$ be a geometric point of S. Then the stalk $\mathscr{O}_{S,\bar{s}}$ is strictly henselian.

Let's see how this can be proven!

(Strict) henselisation

Definition

Let $(A, \mathfrak{m}, \kappa)$ be a local ring. We say that a local morphism $i : A \to A^{\text{sh}}$ is the *strict henselisation* of A if whenever $j : A \to H$ is a local morphism and H is strict henselian, there exists a local morphism $k : A^{\text{sh}} \to H$ such that $j = k \circ i$.

(Strict) henselisation

Definition

Let $(A, \mathfrak{m}, \kappa)$ be a local ring. We say that a local morphism $i : A \to A^{\mathrm{sh}}$ is the *strict henselisation* of A if whenever $j : A \to H$ is a local morphism and H is strict henselian, there exists a local morphism $k : A^{\mathrm{sh}} \to H$ such that $j = k \circ i$.

A somewhat long verification shows that this always exists. After fixing a separable closure κ^{sep} of κ, it can be constructed as

$$A^{\mathrm{sh}} := \operatorname{colim} B,$$

where the (filtered) colimit runs over the diagrams of the form

$$\begin{array}{ccc} A & \xrightarrow{\text{étale}} & B \\ & \searrow \quad \swarrow & \\ & \kappa^{\mathrm{sep}}. & \end{array}$$

Stalks of the structure sheaf

By definition, the stalk $\mathscr{O}_{S,\bar{s}}$ is the colimit of $\Gamma(U, \mathscr{O}_U)$, where (U, u) is an étale neighborhood of $\bar{s}$.

Stalks of the structure sheaf

By definition, the stalk $\mathscr{O}_{S,\bar{s}}$ is the colimit of $\Gamma(U, \mathscr{O}_U)$, where (U, u) is an étale neighborhood of $\bar{s}$. The diagram defining a neighborhood restricts to

$$\begin{array}{ccc} \mathscr{O}_{S,s} & \xrightarrow{\text{étale}} & \mathscr{O}_{U,u} \\ & \searrow \quad \swarrow & \\ & \kappa^{\text{sep}}. & \end{array}$$

Also, $\mathscr{O}_{U,u}$ is the colimit of $\Gamma(V, \mathscr{O}_V)$, where $V \subset U$ is a Zariski-neighborhood of u.

Stalks of the structure sheaf

By definition, the stalk $\mathscr{O}_{S,\bar{s}}$ is the colimit of $\Gamma(U, \mathscr{O}_U)$, where (U, u) is an étale neighborhood of $\bar{s}$. The diagram defining a neighborhood restricts to

$$\begin{array}{ccc} \mathscr{O}_{S,s} & \xrightarrow{\text{étale}} & \mathscr{O}_{U,u} \\ & \searrow \quad \swarrow & \\ & \kappa^{\text{sep}}. & \end{array}$$

Also, $\mathscr{O}_{U,u}$ is the colimit of $\Gamma(V, \mathscr{O}_V)$, where $V \subset U$ is a Zariski-neighborhood of u. Those neighborhoods are, in particular, étale neighborhoods of $\bar{s}$; proving that

$$\mathscr{O}_{S,\bar{s}} = \operatorname{colim} \mathscr{O}_{U,u} = \mathscr{O}^{\text{sh}}_{S,s}.$$

Properties of the strict henselisation

Let A be a local ring.

- A^{sh} is faithfully flat over A;

Properties of the strict henselisation

Let A be a local ring.

- A^{sh} is faithfully flat over A;
- If A is noetherian, then so is A^{sh}.

Hensel's lemma vs inverse function theorem

Let $f : X \to S$ be an étale map, and $\bar{x}$ be a geometric point of X.

Hensel's lemma vs inverse function theorem

Let $f : X \to S$ be an étale map, and $\bar{x}$ be a geometric point of X. By functoriality of the strict henselisation, we obtain a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & S \\ \uparrow & & \uparrow \\ \operatorname{Spec} \mathscr{O}_{X,\bar{x}} & \longrightarrow & \operatorname{Spec} \mathscr{O}_{S,\overline{f(x)}}. \end{array}$$

Hensel's lemma vs inverse function theorem

Let $f : X \to S$ be an étale map, and $\bar{x}$ be a geometric point of X. By functoriality of the strict henselisation, we obtain a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & S \\ \uparrow & & \uparrow \\ \operatorname{Spec} \mathscr{O}_{X,\bar{x}} & \longrightarrow & \operatorname{Spec} \mathscr{O}_{S,\overline{f(x)}}. \end{array}$$

Clearly, every étale neighborhood of $\bar{x}$ is also an étale neighborhood of $\overline{f(x)}$. Our characterization of strictly henselian rings implies that such neighborhoods are cofinal.

Hensel's lemma vs inverse function theorem

Let $f : X \to S$ be an étale map, and $\bar{x}$ be a geometric point of X. By functoriality of the strict henselisation, we obtain a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & S \\ \uparrow & & \uparrow \\ \operatorname{Spec} \mathcal{O}_{X,\bar{x}} & \longrightarrow & \operatorname{Spec} \mathcal{O}_{S,\overline{f(x)}}. \end{array}$$

Clearly, every étale neighborhood of $\bar{x}$ is also an étale neighborhood of $\overline{f(x)}$. Our characterization of strictly henselian rings implies that such neighborhoods are cofinal. It follows that $\operatorname{Spec} \mathcal{O}_{X,\bar{x}} \to \operatorname{Spec} \mathcal{O}_{S,\overline{f(x)}}$ is an isomorphism.

Descent theory

The fundamental results

After almost 40 slides of abstract nonsense, we know lots of stuff about $\widetilde{S_{\text{ét}}}$, but we don't know a single element of it!

The fundamental results

After almost 40 slides of abstract nonsense, we know lots of stuff about $\widetilde{S_{\text{ét}}}$, but we don't know a single element of it!

Of course, we should hope for quasi-coherent sheaves to be étale sheaves… and this is our first theorem!

The fundamental results

After almost 40 slides of abstract nonsense, we know lots of stuff about $\widetilde{S_{\text{ét}}}$, but we don't know a single element of it!

Of course, we should hope for quasi-coherent sheaves to be étale sheaves... and this is our first theorem!

Theorem A

Let S be a scheme and $\mathscr{F}$ a quasi-coherent sheaf on S. Then the presheaf (which we'll still denote by $\mathscr{F}$)

$$\begin{aligned} \mathrm{Sch}/S &\to \mathrm{Set} \\ (f : X \to S) &\mapsto \Gamma(X, f^* \mathscr{F}) \end{aligned}$$

is a sheaf for the fpqc topology. In particular, it's an étale sheaf.

The fundamental results

Another large source of sheaves is our Theorem B.

Theorem B

Let S be a scheme and $X \in \mathrm{Sch}/S$. Then h_X is a sheaf for the fpqc topology. In particular, it's an étale sheaf.

Another large source of sheaves is our Theorem B.

Theorem B

Let S be a scheme and $X \in \mathrm{Sch}/S$. Then h_X is a sheaf for the fpqc topology. In particular, it's an étale sheaf.

This means precisely the following. Let U be a scheme over S and let $(V_i \to U)$ be a fpqc cover of U.

The fundamental results

Another large source of sheaves is our Theorem B.

Theorem B

Let S be a scheme and $X \in \mathrm{Sch}/S$. Then h_X is a sheaf for the fpqc topology. In particular, it's an étale sheaf.

This means precisely the following. Let U be a scheme over S and let $(V_i \to U)$ be a fpqc cover of U. If we have morphisms $f_i : V_i \to X$ such that

$$f_i|_{V_i \times_U V_j} = f_j|_{V_i \times_U V_j}$$

for all i, j, then there exists a unique morphism $f : U \to X$ such that $f|_{V_i} = f_i$ for all i.

The first fundamental lemma

The proof of both theorems will follow rather quickly once we establish two fundamental lemmas.

The first fundamental lemma

The proof of both theorems will follow rather quickly once we establish two fundamental lemmas.

First fundamental lemma

Let S be a scheme and $\mathscr{F}$ be a presheaf on Sch/S. If

- $\mathscr{F}$ is a sheaf in the Zariski topology;

The first fundamental lemma

The proof of both theorems will follow rather quickly once we establish two fundamental lemmas.

First fundamental lemma

Let S be a scheme and $\mathscr{F}$ be a presheaf on Sch$/S$. If

- $\mathscr{F}$ is a sheaf in the Zariski topology;
- whenever $V \to U$ is a faithfully flat morphism of affine S-schemes, the diagram

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(V) \rightrightarrows \mathscr{F}(V \times_U V)$$

  is an equalizer;

The first fundamental lemma

The proof of both theorems will follow rather quickly once we establish two fundamental lemmas.

First fundamental lemma

Let S be a scheme and $\mathscr{F}$ be a presheaf on Sch/S. If

- $\mathscr{F}$ is a sheaf in the Zariski topology;
- whenever $V \to U$ is a faithfully flat morphism of affine S-schemes, the diagram

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(V) \rightrightarrows \mathscr{F}(V \times_U V)$$

  is an equalizer;

The first fundamental lemma

The proof of both theorems will follow rather quickly once we establish two fundamental lemmas.

First fundamental lemma

Let S be a scheme and $\mathscr{F}$ be a presheaf on Sch/S. If

- $\mathscr{F}$ is a sheaf in the Zariski topology;
- whenever $V \to U$ is a faithfully flat morphism of affine S-schemes, the diagram

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(V) \rightrightarrows \mathscr{F}(V \times_U V)$$

  is an equalizer;

then $\mathscr{F}$ is a sheaf in the fpqc topology.

The second fundamental lemma

We’ll also need another fundamental lemma.

The second fundamental lemma

We'll also need another fundamental lemma.

Second fundamental lemma

Let $\varphi : A \to B$ be a faithfully flat morphism of rings, and M a A-module. Then

$$0 \longrightarrow M \xrightarrow{\varphi} M \otimes_A B \xrightarrow{\delta} M \otimes_A B \otimes_A B$$

is an exact sequence of A-modules, where $\varphi(m) = m \otimes 1$ and $\delta(m \otimes b) = m \otimes (b \otimes 1 - 1 \otimes b)$.

The second fundamental lemma

We'll also need another fundamental lemma.

Second fundamental lemma

Let $\varphi : A \to B$ be a faithfully flat morphism of rings, and M a A-module. Then

$$0 \longrightarrow M \xrightarrow{\varphi} M \otimes_A B \xrightarrow{\delta} M \otimes_A B \otimes_A B$$

is an exact sequence of A-modules, where $\varphi(m) = m \otimes 1$ and $\delta(m \otimes b) = m \otimes (b \otimes 1 - 1 \otimes b)$.

In great grothendieckian fashion, we dévissage until this becomes obvious...

Proof of theorem A

For now, write F for the presheaf $(f : X \to S) \mapsto \Gamma(X, f^*\mathscr{F})$ on Sch/S associated to the quasi-coherent sheaf $\mathscr{F}$.

Proof of theorem A

For now, write F for the presheaf $(f : X \to S) \mapsto \Gamma(X, f^*\mathscr{F})$ on Sch/S associated to the quasi-coherent sheaf $\mathscr{F}$. It's a sheaf for the Zariski topology since $f^*\mathscr{F}$ always is.

Proof of theorem A

For now, write F for the presheaf $(f : X \to S) \mapsto \Gamma(X, f^*\mathscr{F})$ on Sch/S associated to the quasi-coherent sheaf $\mathscr{F}$. It's a sheaf for the Zariski topology since $f^*\mathscr{F}$ always is.

We then show that F satisfies the sheaf condition for faithfully flat morphisms of affine schemes.

Proof of theorem A

For now, write F for the presheaf $(f : X \to S) \mapsto \Gamma(X, f^*\mathscr{F})$ on Sch/S associated to the quasi-coherent sheaf $\mathscr{F}$. It's a sheaf for the Zariski topology since $f^*\mathscr{F}$ always is.

We then show that F satisfies the sheaf condition for faithfully flat morphisms of affine schemes. If $U = \operatorname{Spec} A$, $V = \operatorname{Spec} B$ and $\mathscr{F} = \widetilde{M}$, this means precisely that

$$M \longrightarrow M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B$$

is an equalizer.

Proof of theorem A

For now, write F for the presheaf $(f : X \to S) \mapsto \Gamma(X, f^*\mathscr{F})$ on Sch$/S$ associated to the quasi-coherent sheaf $\mathscr{F}$. It's a sheaf for the Zariski topology since $f^*\mathscr{F}$ always is.

We then show that F satisfies the sheaf condition for faithfully flat morphisms of affine schemes. If $U = \operatorname{Spec} A$, $V = \operatorname{Spec} B$ and $\mathscr{F} = \widetilde{M}$, this means precisely that

$$M \longrightarrow M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B$$

is an equalizer. That is, the sequence

$$0 \longrightarrow M \xrightarrow{\varphi} M \otimes_A B \xrightarrow{\delta} M \otimes_A B \otimes_A B$$

is exact. But that's precisely our second fundamental lemma.

Proof of theorem B

We don't have time to see all the details, but you should do it! The clearest reference probably is [Vistoli, Theorem 2.55].

Descent of quasi-coherent sheaves

Let S be a scheme and $(U_i \to S)$ be a fpqc covering of S.

Descent of quasi-coherent sheaves

Let S be a scheme and $(U_i \to S)$ be a fpqc covering of S. A *descent datum* for quasi-coherent sheaves with respect to this covering amounts to objects $\mathscr{F}_i \in \mathrm{QCoh}(U_i)$, along with isomorphisms $\varphi_{ij} : \mathscr{F}_i|_{U_i \times_S U_j} \to \mathscr{F}_j|_{U_i \times_S U_j}$ that satisfy the cocycle condition.

Descent of quasi-coherent sheaves

Let S be a scheme and $(U_i \to S)$ be a fpqc covering of S. A *descent datum* for quasi-coherent sheaves with respect to this covering amounts to objects $\mathscr{F}_i \in \mathrm{QCoh}(U_i)$, along with isomorphisms $\varphi_{ij} : \mathscr{F}_i|_{U_i \times_S U_j} \to \mathscr{F}_j|_{U_i \times_S U_j}$ that satisfy the cocycle condition.

We say that a descent datum is *effective* if it comes from a quasi-coherent sheaf on S.

Descent of quasi-coherent sheaves

Let S be a scheme and $(U_i \to S)$ be a fpqc covering of S. A *descent datum* for quasi-coherent sheaves with respect to this covering amounts to objects $\mathscr{F}_i \in \mathrm{QCoh}(U_i)$, along with isomorphisms $\varphi_{ij} : \mathscr{F}_i|_{U_i \times_S U_j} \to \mathscr{F}_j|_{U_i \times_S U_j}$ that satisfy the cocycle condition.

We say that a descent datum is *effective* if it comes from a quasi-coherent sheaf on S.

Theorem

Every descent datum is effective.

Descent of quasi-coherent sheaves

Let S be a scheme and $(U_i \to S)$ be a fpqc covering of S. A *descent datum* for quasi-coherent sheaves with respect to this covering amounts to objects $\mathscr{F}_i \in \mathrm{QCoh}(U_i)$, along with isomorphisms $\varphi_{ij} : \mathscr{F}_i|_{U_i \times_S U_j} \to \mathscr{F}_j|_{U_i \times_S U_j}$ that satisfy the cocycle condition.

We say that a descent datum is *effective* if it comes from a quasi-coherent sheaf on S.

Theorem

Every descent datum is effective.

For the fancy reader, this means that the fibered category $\mathrm{QCoh}/S \to (\mathrm{Sch}/S)_{fpqc}$ is a *stack*.

Examples of sheaves

If G is a commutative group scheme over S, then $(X \to S) \mapsto G(X)$ is an abelian sheaf for the fpqc topology.

Examples of sheaves

If G is a commutative group scheme over S, then $(X \to S) \mapsto G(X)$ is an abelian sheaf for the fpqc topology. In particular:

- for an abelian group C, the contant group scheme $\underline{C}$ $(\underline{C}(X) = C^{\pi_0(X)})$

are fpqc sheaves.

Examples of sheaves

If G is a commutative group scheme over S, then $(X \to S) \mapsto G(X)$ is an abelian sheaf for the fpqc topology. In particular:

- for an abelian group C, the contant group scheme $\underline{C}$ ($\underline{C}(X) = C^{\pi_0(X)}$)
- $\mathbb{G}_{a,S}(X) = \mathrm{Hom}_S(X, \operatorname{Spec} \mathbb{Z}[x] \times_{\mathbb{Z}} S) = \mathrm{Hom}(X, \operatorname{Spec} \mathbb{Z}[x]) = \mathrm{Hom}(\mathbb{Z}[x], \Gamma(X, \mathscr{O}_X)) = \Gamma(X, \mathscr{O}_X)$

are fpqc sheaves.

Examples of sheaves

If G is a commutative group scheme over S, then $(X \to S) \mapsto G(X)$ is an abelian sheaf for the fpqc topology. In particular:

- for an abelian group C, the contant group scheme $\underline{C}$ $(\underline{C}(X) = C^{\pi_0(X)})$
- $\mathbb{G}_{a,S}(X) = \mathrm{Hom}_S(X, \operatorname{Spec} \mathbb{Z}[x] \times_{\mathbb{Z}} S) = \mathrm{Hom}(X, \operatorname{Spec} \mathbb{Z}[x]) = \mathrm{Hom}(\mathbb{Z}[x], \Gamma(X, \mathscr{O}_X)) = \Gamma(X, \mathscr{O}_X)$
- $\mathbb{G}_{m,S}(X) = \mathrm{Hom}_S(X, \operatorname{Spec} \mathbb{Z}[x, x^{-1}] \times_{\mathbb{Z}} S) = \mathrm{Hom}(X, \operatorname{Spec} \mathbb{Z}[x, x^{-1}]) = \mathrm{Hom}(\mathbb{Z}[x, x^{-1}], \Gamma(X, \mathscr{O}_X)) = \Gamma(X, \mathscr{O}_X^{\times})$

are fpqc sheaves.

Examples of sheaves

If G is a commutative group scheme over S, then $(X \to S) \mapsto G(X)$ is an abelian sheaf for the fpqc topology. In particular:

- for an abelian group C, the contant group scheme $\underline{C}$ $(\underline{C}(X) = C^{\pi_0(X)})$
- $\mathbb{G}_{a,S}(X) = \mathrm{Hom}_S(X, \operatorname{Spec} \mathbb{Z}[x] \times_{\mathbb{Z}} S) = \mathrm{Hom}(X, \operatorname{Spec} \mathbb{Z}[x]) = \mathrm{Hom}(\mathbb{Z}[x], \Gamma(X, \mathscr{O}_X)) = \Gamma(X, \mathscr{O}_X)$
- $\mathbb{G}_{m,S}(X) = \mathrm{Hom}_S(X, \operatorname{Spec} \mathbb{Z}[x, x^{-1}] \times_{\mathbb{Z}} S) = \mathrm{Hom}(X, \operatorname{Spec} \mathbb{Z}[x, x^{-1}]) = \mathrm{Hom}(\mathbb{Z}[x, x^{-1}], \Gamma(X, \mathscr{O}_X)) = \Gamma(X, \mathscr{O}_X^\times)$
- $\mu_{n,S} = \operatorname{Spec} \mathbb{Z}[x]/(x^n - 1) \times_{\mathbb{Z}} S = \ker \left(\mathbb{G}_{m,S} \xrightarrow{\times n} \mathbb{G}_{m,S} \right)$

are fpqc sheaves.

Examples of sheaves

If G is a commutative group scheme over S, then $(X \to S) \mapsto G(X)$ is an abelian sheaf for the fpqc topology. In particular:

- for an abelian group C, the contant group scheme $\underline{C}$ $(\underline{C}(X) = C^{\pi_0(X)})$
- $\mathbb{G}_{a,S}(X) = \mathrm{Hom}_S(X, \operatorname{Spec}\mathbb{Z}[x] \times_{\mathbb{Z}} S) = \mathrm{Hom}(X, \operatorname{Spec}\mathbb{Z}[x]) = \mathrm{Hom}(\mathbb{Z}[x], \Gamma(X, \mathscr{O}_X)) = \Gamma(X, \mathscr{O}_X)$
- $\mathbb{G}_{m,S}(X) = \mathrm{Hom}_S(X, \operatorname{Spec}\mathbb{Z}[x, x^{-1}] \times_{\mathbb{Z}} S) = \mathrm{Hom}(X, \operatorname{Spec}\mathbb{Z}[x, x^{-1}]) = \mathrm{Hom}(\mathbb{Z}[x, x^{-1}], \Gamma(X, \mathscr{O}_X)) = \Gamma(X, \mathscr{O}_X^\times)$
- $\mu_{n,S} = \operatorname{Spec}\mathbb{Z}[x]/(x^n - 1) \times_{\mathbb{Z}} S = \ker\left(\mathbb{G}_{m,S} \xrightarrow{\times n} \mathbb{G}_{m,S}\right)$
- if S is a scheme over $\mathbb{F}_p$, $\alpha_{p,S} = \operatorname{Spec}\mathbb{F}_p[x]/(x^p) \times_{\mathbb{F}_p} S = \ker\left(\mathbb{G}_{a,S} \xrightarrow{\mathrm{Frob}_p} \mathbb{G}_{a,S}\right)$

are fpqc sheaves.

At long last, some calculations

Proposition - [LM, Thm 15.9]

Let $\mathscr{F}$ be a quasi-coherent sheaf on S. Then $H^i(S, \mathscr{F}) = H^i_{\text{Zar}}(S, \mathscr{F})$ for all i.

Quasi-coherent sheaves

Proposition - [LM, Thm 15.9]

Let $\mathscr{F}$ be a quasi-coherent sheaf on S. Then $H^i(S, \mathscr{F}) = H^i_{\text{Zar}}(S, \mathscr{F})$ for all i.

The proof is basically an application of the Čech-to-cohomology spectral sequence, together with our second fundamental lemma for the affine case.

Artin-Schreier theory

Let S be a scheme over $\mathbb{F}_p$.

Artin-Schreier theory

Let S be a scheme over $\mathbb{F}_p$. Since

$$x^p - x = \prod_{k=0}^{p-1} (x - k)$$

in $\mathbb{F}_p[x]$, $\mathbb{F}_p[x]/(x^p - x) \cong \mathbb{F}_p^{\oplus p}$

Artin-Schreier theory

Let S be a scheme over $\mathbb{F}_p$. Since

$$x^p - x = \prod_{k=0}^{p-1}(x - k)$$

in $\mathbb{F}_p[x]$, $\mathbb{F}_p[x]/(x^p - x) \cong \mathbb{F}_p^{\oplus p}$ and so

$$\operatorname{Spec} \mathbb{F}_p[x]/(x^p - x) \cong \underline{\mathbb{F}_p}.$$

Artin-Schreier theory

Let S be a scheme over $\mathbb{F}_p$. Since

$$x^p - x = \prod_{k=0}^{p-1}(x-k)$$

in $\mathbb{F}_p[x]$, $\mathbb{F}_p[x]/(x^p - x) \cong \mathbb{F}_p^{\oplus p}$ and so

$$\operatorname{Spec} \mathbb{F}_p[x]/(x^p - x) \cong \underline{\mathbb{F}_p}.$$

Thus, $\underline{\mathbb{F}_p}(X) = \{a \in \Gamma(X, \mathscr{O}_X) \mid a^p = a\}$.

Artin-Schreier theory

Let S be a scheme over $\mathbb{F}_p$. Since

$$x^p - x = \prod_{k=0}^{p-1}(x-k)$$

in $\mathbb{F}_p[x]$, $\mathbb{F}_p[x]/(x^p - x) \cong \mathbb{F}_p^{\oplus p}$ and so

$$\operatorname{Spec} \mathbb{F}_p[x]/(x^p - x) \cong \underline{\mathbb{F}_p}.$$

Thus, $\underline{\mathbb{F}_p}(X) = \{a \in \Gamma(X, \mathscr{O}_X) \mid a^p = a\}$. In particular,

$$0 \to \underline{\mathbb{F}_p} \to \mathbb{G}_{a,S} \xrightarrow{\mathsf{Frob}_p - \mathsf{id}} \mathbb{G}_{a,S} \to 0$$

is exact on the left.

Artin-Schreier theory

We check that $\mathsf{Frob}_p - \mathsf{id}$ is surjective on stalks.

Artin-Schreier theory

We check that $\mathrm{Frob}_p - \mathrm{id}$ is surjective on stalks. Indeed, the map induced on stalks is of the form

$$\begin{aligned} A &\to A \\ a &\mapsto a^p - a, \end{aligned}$$

where $(A, \mathfrak{m}, \kappa)$ is a strictly henselian ring and $\mathrm{char}\,\kappa = p$.

Artin-Schreier theory

We check that $\mathrm{Frob}_p - \mathrm{id}$ is surjective on stalks. Indeed, the map induced on stalks is of the form

$$\begin{aligned} A &\to A \\ a &\mapsto a^p - a, \end{aligned}$$

where $(A, \mathfrak{m}, \kappa)$ is a strictly henselian ring and $\mathrm{char}\, \kappa = p$. If $a \in A$, the polynomial $f(x) = x^p - x - a$ has a root and derivative $-1 \neq 0$ in $\kappa[x]$.

Artin-Schreier theory

We check that $\mathrm{Frob}_p - \mathrm{id}$ is surjective on stalks. Indeed, the map induced on stalks is of the form

$$\begin{aligned} A &\to A \\ a &\mapsto a^p - a, \end{aligned}$$

where $(A, \mathfrak{m}, \kappa)$ is a strictly henselian ring and $\mathrm{char}\,\kappa = p$. If $a \in A$, the polynomial $f(x) = x^p - x - a$ has a root and derivative $-1 \neq 0$ in $\kappa[x]$. Hensel's lemma then proves that $a \mapsto a^p - a$ is surjective.

Artin-Schreier theory

Let K/k be a cyclic extension of characteristic p and $S = \operatorname{Spec} k$.

Artin-Schreier theory

Let K/k be a cyclic extension of characteristic p and $S = \operatorname{Spec} k$. Our exact sequence yields

$$0 \longrightarrow H^0(S, \mathbb{F}_p) \longrightarrow H^0(S, \mathbb{G}_a) \longrightarrow H^0(S, \mathbb{G}_a) \longrightarrow H^1(S, \mathbb{F}_p) \longrightarrow H^1(S, \mathbb{G}_a) \longrightarrow H^1(S, \mathbb{G}_a) \longrightarrow \cdots,$$

where $H^1(S, \mathbb{G}_a) = H^1_{\mathrm{Zar}}(S, \mathcal{O}_S) = 0$ due to Grothendieck's vanishing.

Let K/k be a cyclic extension of characteristic p and $S = \operatorname{Spec} k$. Our exact sequence yields

$$0 \longrightarrow H^0(S, \mathbb{F}_p) \longrightarrow H^0(S, \mathbb{G}_a) \longrightarrow H^0(S, \mathbb{G}_a) \longrightarrow H^1(S, \mathbb{F}_p) \longrightarrow H^1(S, \mathbb{G}_a) \longrightarrow H^1(S, \mathbb{G}_a) \longrightarrow \cdots,$$

where $H^1(S, \mathbb{G}_a) = H^1_{\mathrm{Zar}}(S, \mathcal{O}_S) = 0$ due to Grothendieck's vanishing. In particular,

$$0 \to \mathbb{F}_p \to k \xrightarrow{x \mapsto x^p - x} k \to H^1(S, \mathbb{F}_p) \to 0$$

is exact.

Artin-Schreier theory

Let K/k be a cyclic extension of characteristic p and $S = \operatorname{Spec} k$. Our exact sequence yields

$$0 \longrightarrow H^0(S, \mathbb{F}_p) \longrightarrow H^0(S, \mathbb{G}_a) \longrightarrow H^0(S, \mathbb{G}_a) \longrightarrow H^1(S, \mathbb{F}_p) \longrightarrow H^1(S, \mathbb{G}_a) \longrightarrow H^1(S, \mathbb{G}_a) \longrightarrow \cdots,$$

where $H^1(S, \mathbb{G}_a) = H^1_{\mathrm{Zar}}(S, \mathcal{O}_S) = 0$ due to Grothendieck's vanishing. In particular,

$$0 \to \mathbb{F}_p \to k \xrightarrow{x \mapsto x^p - x} k \to H^1(S, \mathbb{F}_p) \to 0$$

is exact. We'll see next week that this gives precisely that K is the splitting field of $f(x) = x^p - x + a$ for some $a \in k$.

Kummer theory

Let S be any scheme such that n be inversible in S. (That is, $n \in \Gamma(S, \mathscr{O}_S)^\times$.)

Kummer theory

Let S be any scheme such that n be inversible in S. (That is, $n \in \Gamma(S, \mathscr{O}_S)^\times$.) As before, the sequence

$$0 \to \mu_{n,S} \to \mathbb{G}_{m,S} \xrightarrow{\times n} \mathbb{G}_{m,S} \to 0$$

is exact.

Kummer theory

Let S be any scheme such that n be inversible in S. (That is, $n \in \Gamma(S, \mathscr{O}_S)^\times$.) As before, the sequence

$$0 \to \mu_{n,S} \to \mathbb{G}_{m,S} \xrightarrow{\times n} \mathbb{G}_{m,S} \to 0$$

is exact. And it yields classical Kummer theory.

Kummer theory

Let S be any scheme such that n be inversible in S. (That is, $n \in \Gamma(S, \mathscr{O}_S)^\times$.) As before, the sequence

$$0 \to \mu_{n,S} \to \mathbb{G}_{m,S} \xrightarrow{\times n} \mathbb{G}_{m,S} \to 0$$

is exact. And it yields classical Kummer theory.

The hard part is that $\mathbb{G}_{m,S}$ is not a quasi-coherent sheaf on S, so that it's not so clear that $H^1(S, \mathbb{G}_{m,S}) = 0$.

Kummer theory

Let S be any scheme such that n be inversible in S. (That is, $n \in \Gamma(S, \mathscr{O}_S)^\times$.) As before, the sequence

$$0 \to \mu_{n,S} \to \mathbb{G}_{m,S} \xrightarrow{\times n} \mathbb{G}_{m,S} \to 0$$

is exact. And it yields classical Kummer theory.

The hard part is that $\mathbb{G}_{m,S}$ is not a quasi-coherent sheaf on S, so that it's not so clear that $H^1(S, \mathbb{G}_{m,S}) = 0$.

This is a result whose name shan't be explicitly written.

Questions?