Part I.

Complex Geometry

1. Complex analysis in one variable

1.1. Basic definitions

Let $U \subset \mathbb{C}$ be a connected open set and $z = x + iy$ be the complex variable, where $x, y \in \mathbb{R}$. If f is a function of class C^1 on U, then

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z},$$

where $dz := dx + idy$, $d\bar{z} := dx - idy$ (both elements of $\Gamma(T^\vee\mathbb{C})$ so that, for all $p \in \mathbb{C}$ $dz_p, d\bar{z}_p \in \mathrm{Hom}_{\mathbb{R}}(T_p^\vee\mathbb{C}, \mathbb{C})$)[1], and

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

as vector fields on U. (Note that the signs in the vector fields are opposite to the signals in the differential forms!) We say that f is *holomorphic at* $p \in U$ if df_p is $\mathbb{C}$-linear. In other words, if $\partial f/\partial \bar{z}(p) = 0$. This gives the famous Cauchy-Riemann equations. By the same token, this implies that the differential of a holomorphic function $f = u + iv$ is

$$df = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \text{where} \quad a = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ b = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Since a *conformal* function is one such that its differential is a positive scalar multiple of a rotation, this implies that holomorphic functions with non-zero derivative are conformal.

Finally, we denote by $\mathscr{O}(U)$ the rings of holomorphic functions defined on U. If $f \in C^1(U)$ and $g \in C^1(V)$ is such that $g(V) \subset U$, then

$$\begin{aligned}\frac{\partial}{\partial z}(f \circ g) &= \left(\frac{\partial f}{\partial z} \circ g\right)\frac{\partial g}{\partial z} + \left(\frac{\partial f}{\partial \bar{z}} \circ g\right)\frac{\partial \bar{g}}{\partial z} \\ \frac{\partial}{\partial \bar{z}}(f \circ g) &= \left(\frac{\partial f}{\partial z} \circ g\right)\frac{\partial g}{\partial \bar{z}} + \left(\frac{\partial f}{\partial \bar{z}} \circ g\right)\frac{\partial \bar{g}}{\partial \bar{z}}.\end{aligned}$$

In particular, if f and g are holomorphic, then so is $f \circ g$.

[1] Note that while dz is $\mathbb{C}$-linear, $d\bar{z}$ is anti-$\mathbb{C}$-linear.

1.2. Cauchy integral formula

In this section we prove the Cauchy integral formula and its many striking consequences.

Theorem 1.2.1 — Cauchy integral formula. Let $K \subset \mathbb{C}$ a compact set with piecewise C^1 boundary ∂K. If $f : K \to \mathbb{C}$ is a C^1 function, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial K} \frac{f(w)}{w - z} dw + \frac{1}{2\pi i} \int_K \frac{\partial f}{\partial \overline{w}}(w) \frac{dw \wedge d\overline{w}}{w - z}$$

for every $z \in K^\circ$.

Even though dz_p *and* $d\bar{z}_p$ *span* $T_p^\vee \mathbb{C}$ *for all* $p \in \mathbb{C}$, $dz \wedge d\bar{z}$ *is not the Lebesgue measure on* $\mathbb{C}$. *The Lebesgue measure is* $dx \wedge dy = \frac{1}{2}(dz + d\bar{z}) \wedge \frac{1}{2i}(dz - d\bar{z}) = \frac{i}{2} dz \wedge d\bar{z}$.

Proof. We fix $z \in K^\circ$ and consider the 1-form

$$\eta = \frac{1}{2\pi i} \frac{f(w)}{w - z} dw$$

defined on $K \setminus \{z\}$. Note that

$$d\eta = -\frac{1}{2\pi i} \frac{\partial f}{\partial \overline{w}}(w) \frac{dw \wedge d\overline{w}}{w - z}.$$

Let $r > 0$ be sufficiently small so that $D_r(z) \subset K^\circ$. By Stokes' theorem,

$$\int_{\partial D_r(z)} \eta = \int_{\partial K} \eta - \int_K d\eta + \int_{D_r(z)} d\eta.$$

It suffices then to show that

$$\lim_{r \to 0} \int_{\partial D_r(z)} \eta = f(z) \qquad \text{and} \qquad \lim_{r \to 0} \int_{D_r(z)} d\eta = 0.$$

Both of them follow quickly from the parametrization $w = z + re^{it}$. □

Many of the fundamental theorems in complex analysis follow rather quickly from the Cauchy integral formula. We now present some of them.

Corollary 1.2.2 — Particular case of Cauchy theorem. Let $f : U \to \mathbb{C}$ be a holomorphic function. Then, if D is a disc whose closure is contained in U,

$$\int_{\partial D} f(w)\, dw = 0.$$

Proof. Fix $z \in D$. This result is just a particular case of the Cauchy integral formula applied to the function $w \mapsto (w-z)f(w)$. □

The general form of this result says that if γ is a closed piecewise C^1 curve which is contractible, then the integral of f over γ is zero. This version can be proven by noticing that the 1-form $\omega := f(w)\, dw$ is closed[2] and then using Stokes' theorem.

As one can easily verify, analytic functions are holomorphic. The next result shows that the converse is also true.

Corollary 1.2.3 Let $f : U \to \mathbb{C}$ be a holomorphic function. Then f is analytic. That is, if D is a disc centered at z_0 whose closure is contained in U, then f has a power series expansion at z_0 which converges for all $z \in D$.

Proof. Fix $z \in D$. Using the geometric series we have that

$$\frac{1}{w-z} = \frac{1}{w-z_0} \cdot \frac{1}{1-(z-z_0)/(w-z_0)} = \frac{1}{w-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n.$$

This series converges uniformly for $z \in D$ and $w \in \partial D$. The Cauchy formula then implies that

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z_0)^{n+1}} dw \right) (z-z_0)^n,$$

which concludes the proof. □

This implies that a holomorphic function has derivatives of all orders, which was not clear from the definition. Moreover, the proof shows that the power series expansion converges in any disc, no matter how big, as long as its closure is contained in the domain. In particular, if f is holomorphic on all of $\mathbb{C}$ (we then say that it is *entire*), then f has a power series expansion with infinite radius of convergence.

Corollary 1.2.4 Let $f : U \to \mathbb{C}$ be a holomorphic function. Then, if D is a disc centered at z_0 contained in U,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z_0)^{n+1}} dw.$$

Proof. Its clear from the fact that the n-th derivative at z_0 of a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

is given by $n! a_n$. □

[2]Since $d\omega = -\frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z}$, ω is closed if and only if f is holomorphic.

Corollary 1.2.5 — Cauchy inequalities. Let $f : U \to \mathbb{C}$ be a holomorphic function. Then, if D is a disc centered at z_0 and of radius r whose closure is contained in U,

$$|f^{(n)}(z_0)| \leqslant \frac{n!\,\|f\|_{\partial D}}{r^n},$$

where $\|f\|_{\partial D}$ is the supremum of $|f(z)|$ for all $z \in \partial D$.

Proof. Its just the triangular inequality applied at our previous corollary:

$$|f^{(n)}(z_0)| \leqslant \frac{n!}{2\pi}\left|\int_{\partial D} \frac{f(w)}{(w-z_0)^{n+1}}dw\right| = \frac{n!}{2\pi}\left|\int_0^{2\pi} \frac{f(z_0+re^{it})}{(re^{it})^n}\,dt\right| \leqslant \frac{n!\,\|f\|_{\partial D}}{r^n},$$

where $w = z_0 + re^{it}$. □

We endow the ring $\mathscr{O}(U)$ with the compact-open topology. In other words, a sequence of functions (f_n) converges to f in $\mathscr{O}(U)$ if it converges uniformly on every compact subset of U. (Since U is σ-compact, this topology is metrizable.) The Cauchy inequalities imply that the differentiation operator $\mathscr{O}(U) \to \mathscr{O}(U)$ is continuous. Moreover, we'll soon show that it is a closed subset of $C^0(U, \mathbb{C})$ and so it is a Fréchet space.

The next result shows that $\mathscr{O}(U)$ has the Heine-Borel property: a subset is compact if and only if it is closed and bounded.

Corollary 1.2.6 — Montel's theorem. Let $A \subset \mathscr{O}(U)$ a family of holomorphic functions which is locally uniformly bounded. Then there is a sequence of elements of A which converges uniformly on every compact subset of U.

Proof. The Cauchy integral formula implies that any family in $\mathscr{O}(U)$ which is locally uniformly bounded is locally equicontinuous. For this it suffices to observe that if the elements of A are bounded by M on a disk of radius M, then they are equicontinuous on every smaller disk. Let $r < R$, and choose ρ with $r < \rho < R$. If $f \in A$ is holomorphic on $D_R(0)$, bounded in absolute value by M, and if $z, z_0 \in D_r(0)$, then

$$\begin{aligned}
|f(z) - f(z_0)| &= \left|\frac{1}{2\pi i}\int_{|w|=\rho} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{|w|=\rho} \frac{f(w)}{w-z_0}dw\right| \\
&= \left|\frac{z-z_0}{2\pi i}\int_{|w|=\rho} \frac{f(w)}{(w-z)(w-z_0)}dw\right| \\
&\leqslant \frac{|z-z_0|M}{2\pi}\int_0^{2\pi} \frac{\rho}{|\rho e^{it}-z|\,|\rho e^{it}-z_0|}dt \\
&\leqslant \frac{|z-z_0|M\rho}{(\rho-r)^2}.
\end{aligned}$$

The result then follows by an application of the Arzelà-Ascoli theorem. □

Corollary 1.2.7 — Liouville's theorem. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function which is bounded. Then f is constant.

Proof. Since $\mathbb{C}$ is connected, if suffices to prove that $f' = 0$. If $f(z) \leqslant M$ for all $z \in \mathbb{C}$, the Cauchy inequalities imply that

$$|f'(z_0)| \leqslant \frac{M}{r}$$

for all $z_0 \in \mathbb{C}$ and $r > 0$. The result follows by taking the limit $r \to \infty$. □

The same argument also proves a more general result: if there exist $a, b > 0$ and a positive integer m such that $|f(z)| \leqslant a + b|z|^m$, then f is a polynomial of degree $\leqslant m$.

Liouville's theorem provides an incredibly simple proof to the fundamental theorem of algebra which we now present.

Corollary 1.2.8 — Fundamental theorem of algebra. Let $P : \mathbb{C} \to \mathbb{C}$ be a non-constant polynomial. Then there exists $z \in \mathbb{C}$ such that $P(z) = 0$. In other words, $\mathbb{C}$ is algebraically closed.

Proof. Suppose $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then $1/P$ is a entire function. Since $|P(z)| \to \infty$ as $|z| \to \infty$, there exists a compact set K such that

$$\left| \frac{1}{P(z)} \right| < 1$$

for all $z \in \mathbb{C} \setminus K$. Moreover, $1/P$ is bounded on K by compactness. Then Liouville's theorem implies that P is constant, which is absurd. □

We end this section by interpreting the Cauchy integral formula in the language of distributions.

Corollary 1.2.9 The function $z \mapsto 1/\pi z \in L^1_{\text{loc}}(\mathbb{C})$ is a fundamental solution of the operator $\partial/\partial\bar{z}$ on $\mathbb{C}$. In other words, if T is the distribution defined by this function, then $\partial T/\partial\bar{z} = \delta_0$. As a consequence, if ν is a distribution with compact support on $\mathbb{C}$, then the convolution $u = T * \nu$ is a solution of $\partial u/\partial\bar{z} = \nu$.

For the proof, recall that the Lebesgue measure on $\mathbb{C}$ is given by $\frac{i}{2} dz \wedge d\bar{z}$.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{C})$ be a test function and K a compact containing its support. Then,

$$\frac{\partial T}{\partial \bar{z}}(\varphi) = -T\left(\frac{\partial \varphi}{\partial \bar{z}}\right) = \frac{1}{2\pi i} \underbrace{\int_{\partial K} \frac{\varphi(z)}{z} dz}_{=0} + \frac{1}{2\pi i} \int_K \frac{1}{z} \frac{\partial \varphi}{\partial \bar{z}} dz \wedge d\bar{z} = \varphi(0)$$

by the Cauchy integral formula. □

Note that, even though $\operatorname{supp} \nu$ is compact, there's no reason for $\operatorname{supp} u$ to be.

1.3. The normal form of a holomorphic function

Locally, every holomorphic function is basically $z \mapsto z^k$ for some positive integer k. This is the so-called *normal form* of a holomorphic function, which has another myriad of striking consequences.

Definition 1.3.1 Let $f : U \to \mathbb{C}$ be a holomorphic function and $z_0 \in U$. The *order* of z_0, denoted $\operatorname{ord}_{z_0}(f)$, is the smallest non-negative integer n such that $f^{(n)}(z_0) \neq 0$.

Equivalently, the order of z_0 is the index of the smallest non-zero coefficient in the power series expansion of f in a neighborhood of z_0. If z_0 has order n, then $f(z) = (z - z_0)^n h(z)$, where h is a holomorphic function such that $h(z_0) \neq 0$.

Theorem 1.3.1 — Normal form. Let $f : U \to \mathbb{C}$ be a holomorphic function and $z_0 \in U$ be a point of order n. Then there exists a neighborhood V of p and a injective holomorphic function $g : V \to \mathbb{C}$ such that

$$f \circ g^{-1}(z) = z^n$$

for all $z \in g(V)$.

Proof. Without loss of generality, we may suppose $z_0 = 0$. Let h be a holomorphic function such that $f(z) = z^n h(z)$ and $h(0) \neq 0$. Since $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is the universal over of $\mathbb{C} \setminus \{0\}$ and the exponential function is a local biholomorphism (that is, locally $\exp$ has holomorphic inverses), there exists a holomorphic function H such that $H(z)^n = h(z)$. As the derivative of $z \mapsto zH(z)$ at 0 is $H(0) \neq 0$, the inverse function theorem[3] implies that this function is our desired g. □

Corollary 1.3.2 — Open mapping theorem. Let $f : U \to \mathbb{C}$ be a non-constant holomorphic function. Then $f(U)$ is an open subset of $\mathbb{C}$.

Proof. Since openness is a local condition, this follows from the preceding theorem and the fact that $z \mapsto z^k$ is an open map for $k \geqslant 1$. □

Corollary 1.3.3 — Maximum modulus principle. Let $f : U \to \mathbb{C}$ be a non-constant holomorphic function. Then $|f|$ does not attain a maximum on U. In particular, if U is bounded and f is continuous on $\overline{U}$, then $|f|$ attains a maximum on ∂U.

Proof. Suppose that $|f|$ attains a maximum on $z_0 \in U$. Since f is a non-constant holomorphic function, the open mapping theorem implies that there exists a disk

[3] The local inverse ψ of a holomorphic function φ given by the standard inverse function theorem is also holomorphic since $d\psi = (d\varphi)^{-1}$.

$D \subset f(U)$ centered at $f(z_0)$. But then some of the points in D have a larger absolute value than $f(z_0)$, which is a contradiction. □

Corollary 1.3.4 — Analytic continuation. Let $f, g : U \to \mathbb{C}$ be holomorphic functions such that $f(z) = g(z)$ for all z in a set with a limit point. Then $f = g$.

Proof. Without loss of generality we assume $g = 0$. Since $z \mapsto z^k$ has discrete zeros if $k > 0$, the result follows from the normal form. □

1.4. Morera's theorem

In this section we present a converse to the Cauchy theorem which allows us to pass the stability of integrals under uniform limits to holomorphic functions.

Proposition 1.4.1 — Morera's theorem. Let $f : U \to \mathbb{C}$ be a continuous function such that for every triangle $\Delta \subset U$ we have that

$$\int_\Delta f(z)\, dz = 0.$$

Then f is holomorphic.

Proof. We fix $z_0 \in U$ and $r > 0$ such that $D_r(z_0) \subset U$. We denote by F the function defined by

$$F(z) = \int_{[z_0, z]} f(w)\, dw, \qquad \text{for all } z \in D_r(z_0).$$

By the Lebesgue differentiation theorem, we have that

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = \lim_{h \to 0} \frac{1}{h} \int_{[z, z+h]} f(w)\, dw = f(z).$$

This implies that F is holomorphic at z_0. Since $F' = f$, so is f. □

Note that we needed the hypothesis that f integrates to 0 over triangles to obtain that $F(z+h) - F(z)$ is the integral of f over $[z, z+h]$.

As promised, we now see that $\mathcal{O}(U)$ is a closed subset of $C^0(U, \mathbb{C})$.

Corollary 1.4.2 Let $f_k : U \to \mathbb{C}$ be a sequence of holomorphic functions which converge uniformly on every compact subset of U to $f : U \to \mathbb{C}$. Then f is holomorphic. Moreover, the sequence of derivatives $f_k^{(n)}$ converges uniformly to $f^{(n)}$ on every compact subset of U.

Proof. Let D be a disc whose closure is contained in U and Δ a triangle in D. By the Cauchy theorem,

$$\int_\Delta f_k(z)\, dz = 0.$$

But integrals are stable over uniform limits so

$$\int_\Delta f(z)\, dz = 0$$

and then Morera's theorem implies that f is holomorphic on U. The demonstration of the second statement is not so enlightening and can be seen in [**?**]. □

1.5. Meromorphic functions

In this section we study the behavior of functions which may have singularities. As we shall see, those singularities tell us much about the function itself.

Proposition 1.5.1 — Laurent series. Let $R > r \geqslant 0$ be real numbers and consider the annulus $A_{R,r} = \{z \in \mathbb{C} \mid r < |z| < R\}$. If $f : A_{R,r} \to \mathbb{C}$ is a holomorphic function, then f can be expanded as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

This series is called the *Laurent series* of f. The same reasoning can be applied for annuli centered around other points.

Proof. Let R', r' be such that $0 \leqslant r < r' < R' < R$. The Cauchy integral formula implies that

$$f(z) = \frac{1}{2\pi i}\int_{|w|=R'} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i}\int_{|w|=r'} \frac{f(w)}{w-z} dw$$

for all z such that $r' < |z| < R'$. Since $|z| < |w| = R'$ in the first integral and $|z| > |w| = r'$ in the second, we expand $(w-z)^{-1}$ as

$$\frac{1}{w-z} = \frac{1}{w}\frac{1}{\left(1-\frac{z}{w}\right)} = \frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n \quad \text{and} \quad \frac{1}{w-z} = -\frac{1}{z}\frac{1}{\left(1-\frac{w}{z}\right)} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n.$$

This gives our desired expansion with

$$a_n = \frac{1}{2\pi i}\int_{|w|=\rho} \frac{f(w)}{w^{n+1}} dw.$$

By the Cauchy theorem this value is independent of the choice of $\rho \in (r, R)$. □

We now classify the different kinds of singularities.

Definition 1.5.1 Let f be a holomorphic function defined on a annulus centered at $z_0 \in \mathbb{C}$. Also, let

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

be its Laurent series. We say that

- f has a *removable singularity at* z_0 if $a_n = 0$ for all $n < 0$;
- f has a *pole of order* k *at* z_0 if $a_{-k} \neq 0$ but $a_{-n} = 0$ for all $n > k$;
- f has an *essential singularity at* z_0 if its Laurent series has infinitely many negative coefficients.

Removable singularities are the tamest of all such beasts. The next results shows that it is very easy to recognize these kinds of singularities.

Proposition 1.5.2 — Riemann removable singularities theorem. Let U be a neighborhood of a point $z_0 \in \mathbb{C}$ and $f : U \setminus \{z_0\} \to \mathbb{C}$ be a holomorphic function. If f is bounded on $U \setminus \{z_0\}$ then z_0 is a removable singularity.

Proof. Without loss of generality we assume $z_0 = 0$. Let M be an upper bound on $|f|$ on $U \setminus \{z_0\}$. Proceeding as in the proof of the Cauchy estimates, we have that the n-th coefficient of the Laurent series of f satisfies

$$|a_n| \leqslant \frac{M}{\rho^n}$$

for all sufficiently small ρ. Taking $\rho \to 0$ we get that $a_n = 0$ for all $n < 0$. □

We now begin our study of the most interesting of all singularities: poles.

Definition 1.5.2 — Residue. Let f be a holomorphic function defined on punctured neighborhood of $z_0 \in \mathbb{C}$ where it has a pole. The coefficient a_{-1} of its Laurent series centered at z_0 is the *residue* of f at z_0, denoted $\mathrm{res}_{z_0}(f)$.

If f has a pole of order k at z_0, we can use a combination of derivatives and limits to isolate the coefficient a_{-1} of its Laurent series, getting an explicit formula for the residue:

$$\mathrm{res}_{z_0}(f) = \lim_{z \to z_0} \frac{1}{(k-1)!} \left(\frac{d}{dz}\right)^{k-1} (z - z_0)^k f(z).$$

Surprisingly, the integral of a function which is holomorphic except at a finite number of poles is determined by its residues. This is the content of our next theorem, which is very useful both theoretically and in concrete calculations.

Theorem 1.5.3 — Residue theorem. Let $z_1, \dots, z_n \in U$ be a finite collection of points and let $f : U \setminus \{z_1, \dots, z_n\}$ be a holomorphic function. If γ is closed piecewise C^1 curve contained in U and whose interior contains all these points, then

$$\int_\gamma f(w)\, dw = 2\pi i \sum_{k=1}^{n} \operatorname{res}_{z_k}(f).$$

Proof. Without loss of generality, we assume that we have a single point $z_1 = 0$. Also, by the Cauchy theorem, it suffices to prove the result when γ is the boundary of a disk centered at 0. In this case, the theorem follows from the fact that

$$\int_{|w|=1} \frac{1}{w^k} dw = \begin{cases} 2\pi i & \text{if } k = -1 \\ 0 & \text{otherwise} \end{cases}$$

by integrating the Laurent series of f centered at 0. □

Definition 1.5.3 — Meromorphic function. We say that a function $f : U \to \mathbb{C}$ is *meromorphic* if there exists a subset $A \subset U$ without limit points (necessarily at most countable) such that the restriction of f to $U \setminus A$ is holomorphic and f has poles at the points of A. We denote the set of meromorphic functions on U by $\mathcal{M}(U)$.

It is clear that the set of meromorphic functions is a field. A non-trivial fact which we'll see in the next section is that $\mathcal{M}(U)$ is precisely the fraction field of the ring of holomorphic functions $\mathcal{O}(U)$.

The notion of order can be extended a meromorphic function f in the following way. If f is holomorphic at z_0, then the our notion of order still applies. Else, if f has a pole of order k at z_0, then $\operatorname{ord}_{z_0}(f) := -k$.

Proposition 1.5.4 — Argument principle. Let $f : U \to \mathbb{C}$ be a meromorphic function and D a disk whose closure is contained in U. If ∂D does not contain any zeros or poles of f, then

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f'(w)}{f(w)} dw = \sum_{z \in D} \operatorname{ord}_z(f).$$

Since an infinite sequence in a compact set as necessarily a limit point, this sum is finite. Moreover, it is usually interpreted as being the number of zeros of f minus the number of poles, counted with their respective orders.

Proof. The proof is based on the clever observation that if f has order k at z_0 then $f(z) = (z - z_0)^k g(z)$, where g is holomorphic and non-zero in a neighborhood of z_0, and so

$$\frac{f'(z)}{f(z)} = \frac{k}{z - z_0} + \frac{g'(z)}{g(z)}.$$

In other words, f'/f has residue k at z_0. The result then follows by the residue formula. $\square$

We finish this section with Rouché's theorem which is, in some sense, a continuity statement. It says that slight perturbations of meromorphic functions does not change the number of zeros and poles.

Corollary 1.5.5 — Rouché's theorem. Let $f, g : U \to \mathbb{C}$ be meromorphic functions and D a disk whose closure is contained in U. If $|f(z) - g(z)| < |g(z)|$ for all $z \in \partial D$, then

$$\sum_{z \in D} \operatorname{ord}_z(f) = \sum_{z \in D} \operatorname{ord}_z(g).$$

Proof. Firstly, we note that the condition of f and g implies that ∂D does not contain any zeros or poles of these functions. Let $h = f/g$. Since $|h(z) - 1| < 1$ on ∂D, the image of ∂D by h is contained in $D_1(1)$ and so

$$0 = \int_{h(\partial D)} \frac{1}{w} dw = \int_{\partial D} \frac{h'(w)}{h(w)} dw = \int_{\partial D} \frac{f'(w)}{f(w)} dw - \int_{\partial D} \frac{g'(w)}{g(w)} dw.$$

The result then follows by the argument principle. $\square$

Just for fun, we give another quick proof of the fundamental theorem of algebra. Let $f(z) = a_n z^n + \ldots + a_0$ be a non-constant polynomial and $g(z) = a_n z^n$. Since $f - g$ is a polynomial of degree at most $n - 1$, we have that $|f(z) - g(z)| < |g(z)|$ for all z in the border of a sufficiently large disk centered at 0. Rouché's theorem then implies that f has precisely n zeros inside this disk.

1.6. Harder results

In this last section we gather, mostly without proof, a few difficult theorems which are nonetheless very important in complex analysis. We begin our quest with a simple theorem which admits a massive generalization.

Proposition 1.6.1 — Casorati-Weierstrass. Let U be a connected open set, $z_0 \in U$ and $f : U \setminus \{z_0\}$ a holomorphic function with an essential singularity at z_0. Then, if V is any neighborhood of z_0 contained in U, $f(V \setminus \{z_0\})$ is dense in $\mathbb{C}$.

Proof. Suppose that there is a disc D centered at $a \in \mathbb{C}$ which is not in $f(V \setminus \{z_0\})$. Then

$$g(z) = \frac{1}{f(z) - b}$$

defines a bounded holomorphic function on $V \setminus \{z_0\}$. Proposition 1.5.2 implies that g can be holomorphically extended to all of V. Observe that

$$f(z) = \frac{1}{g(z)} + b.$$

Depending on whether $g(z_0)$ is zero or not, $1/g$ has either a pole or is holomorphic at z_0. Both cases contradict the hypothesis that z_0 is an essential singularity of f. □

Actually, much more is true!

Theorem 1.6.2 — Great Picard's theorem. In the hypothesis of the preceding proposition, the function f restricted to $V \setminus \{z_0\}$ takes every complex value, with at most a single exception, infinitely often.

The reader interested in its (hard) proof may check the beautiful book [?]. With a similar spirit, we also have the following generalization of Liouville's theorem.

Theorem 1.6.3 — Little Picard theorem. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire non-constant function. Then the complement of its image $\mathbb{C} \setminus f(\mathbb{C})$ has at most one point.

One can prove this result by using that the unit disk $D_1(0)$ is the universal cover of $\mathbb{C}$ minus two points, which can be proven either by using Riemann's uniformization theorem or by proving that the modular lambda function is such a cover. Then, if $f : \mathbb{C} \to \mathbb{C}$ omits two points, we lift it a function $\tilde{f} : \mathbb{C} \to D_1(0)$, which is necessarily constant by Liouville's theorem. This concludes the proof.

The little theorem also follows from the big one by using that an entire function f is either a polynomial or has an essential singularity at infinity (that is, $z \mapsto f(1/z)$ has an essential singularity at 0). In fact, if f is not a polynomial, we can write

$$f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{a_n}{z^n},$$

where infinitely many a_n are non-zero. This implies that $z \mapsto f(1/z)$ has an essential singularity at 0.

Riemann mapping theorem (uma função é biholomorfa se e somente se ela é conforme e injetiva. Então o teorema de Riemann diz que dois domínios são conformemente equivalentes e isso nos dá um biholomorfismo entre eles)

Weierstrass factorization theorem (falar que o corpo de funções meromorfas é o corpo de frações do anel de funções holomorfas. Thm 15.12 rudin. Falar que isso não vale para superfícies de Riemann compactas mas vale para não-compactas. Forster Thm. 26.5.)

Mittag-Leffler (provar usando feixes: corolário 10.24 do Wedhorn de variedades)

Runge's approximation theory

2. Complex analysis in several variables

In this chapter we study holomorphic functions which are defined on a open set of $\mathbb{C}^n$. Contrarily to what one could expect, there are a lot of new and interesting phenomena that arise when $n > 1$. Notably, the fact that the zeros of a holomorphic function are never isolated! This leads to Hartogs' extension theorem and the notion of domains of holomorphy.

Nevertheless, the study of some of these particular phenomena is best done using tools which naturally belong to the context of complex manifolds, which is the content of the next chapter. Accordingly, in this chapter we will be content with the generalization of some classic results and with the Hartogs' extension theorem.

2.1. Basic definitions

Just as before, let $U \subset \mathbb{C}^n$ be a connected open set and $z_j = x_j + iy_j$, for $j = 1, \ldots, n$, be the complex variables, where $x_j, y_j \in \mathbb{R}$. If f is a function of class C^1 on U, then

$$df = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j \right) = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \right),$$

where $dz_j = dx_j + idy_j$, $d\bar{z}_j = dx_j - idy_j$, and

$$\frac{\partial}{\partial z_j} := \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \qquad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right)$$

as vector fields on U. We say that f is *holomorphic at* $p \in U$ if df_p is $\mathbb{C}$-linear. In other words, if $\partial f / \partial \bar{z}_j(p) = 0$ for all j. This gives the analogous Cauchy-Riemann equations. A C^1 function from U to $\mathbb{C}^m$ is said to be holomorphic at $p \in U$ if all its components are. Just as in the 1-dimensional case, the composition of holomorphic functions is still holomorphic.

If f is holomorphic, then it is clear that f is holomorphic individually on each variable. Remarkably, the converse, a deep theorem of Hartogs, is also true.[1] For the rest of these notes we will assume this result. The reader should compare this with

[1] The original proof of this result is deep and intricate. As far as the writer knows, there are no other known proofs.

the real case, where a function which has partial derivatives in each variable need not even be continuous.

As before, we denote by $\mathscr{O}(U)$ the ring of holomorphic functions defined on U with values on $\mathbb{C}$. If $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, we denote by $|\alpha|$ the sum $\alpha_1 + \ldots + \alpha_n$, by $\alpha!$ the number $\alpha_1! \cdots \alpha_n!$, by $\alpha+1$ the multi-index $(\alpha_1+1, \cdots, \alpha_n+1)$, and by z^α the monomial $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, which is an element of $\mathbb{C}[z_1, \ldots, z_n]$. A product of discs will be called a *polydisk*. If $z_0 \in \mathbb{C}^n$, $r \in \mathbb{N}^n$ is a multi-index, and

$$D_r(z) := D_{r_1}(z_1) \times \cdots \times D_{r_n}(z_n)$$

is a polydisk, we will denote by $\Gamma_r(z)$ the *distinguished boundary*

$$\Gamma_r(z) := \partial D_{r_1}(z_1) \times \cdots \times \partial D_{r_n}(z_n).$$

Notice that this is *not* the topological boundary of $D_r(z)$.

2.2. Cauchy integral formula

Just as in the $n = 1$ case, the Cauchy integral formula is the main ingredient in many important results.

Theorem 2.2.1 — Cauchy integral formula. Let $K_1, \ldots, K_n$ be compact sets in $\mathbb{C}$ with piecewise C^1 boundary. If f is C^1 on $K_1 \times \cdots \times K_n$ and holomorphic on $K_1^\circ \times \cdots \times K_n^\circ$, then

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial K_1} \cdots \int_{\partial K_n} \frac{f(w)}{(w_1 - z_1) \cdots (w_n - z_n)} dw_1 \wedge \cdots \wedge dw_n$$

for every $z \in K_1^\circ \times \cdots \times K_n^\circ$.

Proof. This is just an iteration of Theorem 1.2.1. □

We now describe the results of section 1.2 which generalize nicely to $n > 1$. Since the proofs are identical, they will be omitted.

Corollary 2.2.2 — Cauchy theorem. Let $f : U \to \mathbb{C}$ be a holomorphic function. Then, if D is a polydisk whose closure is contained in U and Γ is its distinguished boundary,

$$\int_\Gamma f(w)\, dw = 0,$$

where we denote $dw_1 \wedge \cdots \wedge dw_n$ by dw.

As before, holomorphic and analytic functions are one and the same.

Corollary 2.2.3 Let $f : U \to \mathbb{C}$ be a holomorphic function. Then f is analytic. That is, if D is a polydisc centered at z_0 whose closure is contained in U, then f has a power series expansion at z_0

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (z - z_0)^\alpha,$$

which converges for all $z \in D$.

Corollary 2.2.4 Let $f : U \to \mathbb{C}$ be a holomorphic function. Then, if D is a polydisc centered at z_0 contained in U and Γ is its distinguished boundary,

$$f^{(\alpha)}(z_0) = \frac{\alpha!}{(2\pi i)^n} \int_\Gamma \frac{f(w)}{(w - z_0)^{\alpha+1}} dw.$$

Corollary 2.2.5 — Cauchy inequalities. Let $f : U \to \mathbb{C}$ be a holomorphic function. Then, if D is a polydisc centered at z_0 and of radius r whose closure is contained in U and Γ is its distinguished boundary,

$$|f^{(\alpha)}(z_0)| \leqslant \frac{\alpha! \, \|f\|_\Gamma}{r^\alpha},$$

where $\|f\|_\Gamma$ is the supremum of $|f(z)|$ for all $z \in \Gamma$.

Just as before, we endow the ring $\mathcal{O}(U)$ with the compact-open topology. This is a Fréchet space (since we can prove in a similar fashion as before that $\mathcal{O}(U)$ is closed on $C^0(U, \mathbb{C})$) on which the differentiation operator acts continuously.

Corollary 2.2.6 — Montel's theorem. Let $A \subset \mathcal{O}(U)$ a family of holomorphic functions which is locally uniformly bounded. Then there is a sequence of elements of A which converges uniformly on every compact subset of U.

Following the obvious generalization, we say that a holomorphic function defined on all of $\mathbb{C}^n$ is *entire*.

Corollary 2.2.7 — Liouville's theorem. Let $f : \mathbb{C}^n \to \mathbb{C}$ be an entire function which is bounded. Then f is constant.

By the same token, the polynomial variant of Liouville's theorem still holds.

2.3. Other generalizations

Unlike in section 1.3, there's no normal form for a holomorphic function of several variables. Notwithstanding, we can still prove the same corollaria. Analytic continuation still holds by a simple argument, the open mapping theorem holds by reducing to the one dimensional case and the maximum modulus principle is proved exactly in the same way as before.

Corollary 2.3.1 — Analytic continuation. Let $f, g : U \to \mathbb{C}$ be holomorphic functions such that $f(z) = g(z)$ for all z in an open subset of U. Then $f = g$.

Proof. Without loss of generality, we assume $g = 0$. Let Z be the subset of U where all the derivatives of f vanish. By supposition, Z is non-empty. Since f has a power series expansion centered at every point of Z, Z is open. By continuity, Z is closed. The result now follows by the connectedness of U. □

In order to prove the open mapping theorem, we'll use the following lemma.

Lemma 2.3.2 Let $f : U \to \mathbb{C}$ be a holomorphic function, $a \in U$ and $b \in \mathbb{C}^n$. Then the set $V := \{t \in \mathbb{C} \mid a + tb \in U\}$ is open, contains 0 and the function

$$f_{a,b} : V \to \mathbb{C}, \quad t \mapsto f(a + tb)$$

is holomorphic.

Proof. The openness of U clearly implies that of V and 0 is in V since a is in U. The function $f_{a,b} : t \mapsto f(a+tb)$ is holomorphic since it's the composition of a holomorphic function with an affine mapping. □

Corollary 2.3.3 — Open mapping theorem. Let $f : U \to \mathbb{C}$ be a non-constant holomorphic function. Then $f(U)$ is an open subset of $\mathbb{C}$.

Proof. We must show that for any point $z_0 \in U$ and any ball B centered at z_0 whose closure is contained in U, the image $f(B)$ contains a neighborhood of $f(z_0)$. By translation we may assume that $z_0 = 0$ and $f(z_0) = 0$.

By analytic continuation, there exists a point $b \in B$ such that $f \neq 0$ in a neighborhood of b. Consider the line $V := \{t \in \mathbb{C} \mid tb \in U\}$. The image $f(V)$ is precisely the image of $f_{0,b}$, which is open by the previous lemma and the open mapping theorem in one dimension. In particular, $f(B)$ contains a neighborhood of $f(0)$. □

Finally, the maximum modulus principle follows in the exact same way by using that $f(U)$ is open on $\mathbb{C}$.

Corollary 2.3.4 — Maximum modulus principle. Let $f : U \to \mathbb{C}$ be a non-constant holomorphic function. Then $|f|$ does not attain a maximum on U. In particular, if U is bounded and f is continuous on $\overline{U}$, then $|f|$ attains a maximum on ∂U.

2.4. The theorems of Riemann and Hartogs

In this last section we touch the subject of extending holomorphic functions beyond their original domain. In the one-dimensional case, this can only be done around a removable singularity. As we shall see, in several complex variables the situation is much richer. Our main result is the theorem below.

Theorem 2.4.1 — Hartogs' extension theorem. Let K be a compact subset of $U \subset \mathbb{C}^n$, with $n > 1$, and suppose that $U \setminus K$ is connected. Then, if $f : U \setminus K \to \mathbb{C}$ is a holomorphic function, there exists $F \in \mathscr{O}(U)$ which coincides with f on $U \setminus K$.

Before we prove this marvelous result, we need a lemma which says that we can find a solution with compact support for the differential equation in corollary 1.2.9, under a certain integrability condition when $n > 1$.

Lemma 2.4.2 Let $f_1, \dots, f_n : \mathbb{C}^n \to \mathbb{C}$, be smooth functions with compact support satisfying the *integrability condition*

$$\frac{\partial f_j}{\partial \bar{z}_k} = \frac{\partial f_k}{\partial \bar{z}_j}, \qquad \text{for all } j, k \in \{1, \dots, n\}.$$

Then, there exists a smooth function $u : \mathbb{C}^n \to \mathbb{C}$ such that

$$\frac{\partial u}{\partial \bar{z}_j} = f_j$$

for all $j \in \{1, \dots, n\}$. If $n > 1$, u has compact support. Moreover, if K is the support of $(f_1, \dots, f_n)$ and $\mathbb{C}^n \setminus K$ is connected, then $\operatorname{supp} u = K$.

Proof. We affirm that the function defined by

$$u(z) := \frac{1}{2\pi i} \int_{\mathbb{C}} f_1(w, z_2, \dots, z_n) \frac{dw \wedge d\bar{w}}{w - z_1} = \frac{1}{2\pi i} \int_{\mathbb{C}} f_1(w + z_1, z_2, \dots, z_n) \frac{dw \wedge d\bar{w}}{w}$$

is the desired solution. Let K be a compact in $\mathbb{C}$ large enough so that $f_j(w, z_2, \dots, w_n)$ is zero when $w \notin K$ for all j. Then the Cauchy integral formula implies that

$$f_j(z) = \frac{1}{2\pi i} \int_K \frac{\partial f_j}{\partial \bar{z}_1}(w, z_2, \dots, z_n) \frac{dw \wedge d\bar{w}}{w - z_1}.$$

Then, by the integrability condition,

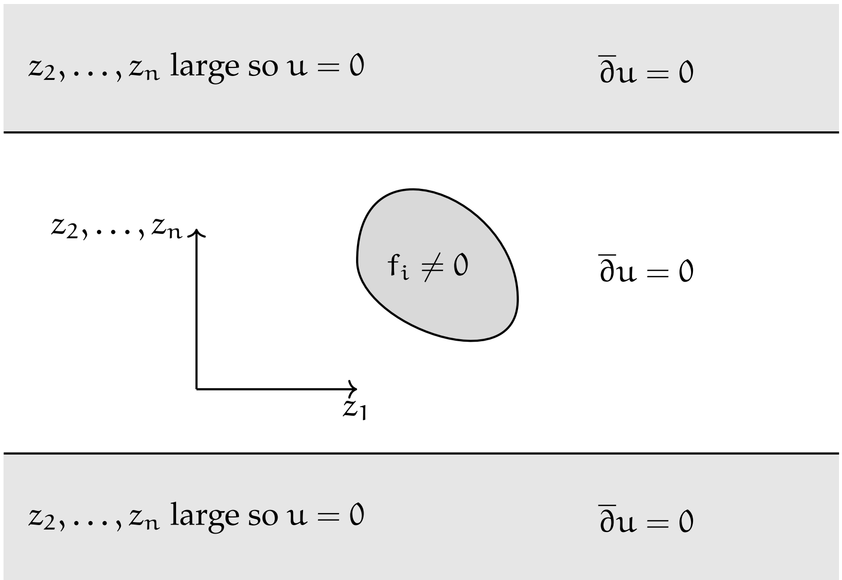
$$\begin{aligned}
\frac{\partial u}{\partial \bar{z}_j}(z) &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial f_1}{\partial \bar{z}_j}(w + z_1, z_2, \dots, z_n) \frac{dw \wedge d\bar{w}}{w} \\
&= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial f_j}{\partial \bar{z}_1}(w + z_1, z_2, \dots, z_n) \frac{dw \wedge d\bar{w}}{w} \\
&= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial f_j}{\partial \bar{z}_1}(w, z_2, \dots, z_n) \frac{dw \wedge d\bar{w}}{w - z_1} = f_j(z),
\end{aligned}$$

which means that u is a solution of our system of differential equations.

We now show that u has compact support when $n > 1$. When one of the $z_2, \dots, z_n$ is large, $u(z_1, \dots, z_n)$ is zero since the function f_1 in its definition is zero. In particular, u is zero on an open set. Now, note that away from the support of the f_j,

$$\frac{\partial u}{\partial \bar{z}_j} = f_j = 0$$

and so u is holomorphic there.



By analytic continuation, u is zero away from the support of the f_j. In particular, it has compact support. In particular, it is clear that if K is the compact of the f_i and $\mathbb{C}^n \setminus K$ is connected, then K is also the support of u. □

Now that the hard part was over, the proof of the theorem follows by extending f in an arbitrary way to a (not necessarily holomorphic) function $\tilde{f}$ and then "correcting" it with the preceding lemma so that it becomes holomorphic.

Proof of the Hartogs' extension theorem. Let $\varphi \in \mathcal{D}(U)$ be a smooth function which is 1 in a neighborhood of K and is compactly supported on U. We consider the function $\tilde{f} := (1 - \varphi) f$, which is smooth on U, identically zero on K and coincides with f near the boundary of U, where φ is zero. This condition implies that the functions $f_1, \dots, f_n : \mathbb{C}^n \to \mathbb{C}$ defined by

$$f_j := \frac{\partial \tilde{f}}{\partial \bar{z}_j} \text{ on } U \quad \text{and} \quad f_j := 0 \text{ outside } U,$$

for all $j \in \{1, \dots, n\}$, are smooth. They are also compactly supported. In fact, they are zero outside of $U \setminus K$ and equal to

$$f_j = \frac{\partial}{\partial \bar{z}_j}((1-\varphi)f) = \frac{\partial f}{\partial \bar{z}_j} - \frac{\partial \varphi}{\partial \bar{z}_j} f - \varphi \frac{\partial f}{\partial \bar{z}_j} = -\frac{\partial \varphi}{\partial \bar{z}_j} f$$

on $U \setminus K$, where they are compactly supported since φ is. Since the partial derivatives commute, the f_j satisfy the integrability condition of the preceding lemma. We then let $u \in \mathcal{D}(U)$ be the function given by this result. We affirm that $F := \tilde{f} - u$ is the desired extension. It is holomorphic:

$$\frac{\partial F}{\partial \bar{z}_j} = \frac{\partial \tilde{f}}{\partial \bar{z}_j} - \frac{\partial u}{\partial \bar{z}_j} = f_j - f_j = 0.$$

Since u is compactly supported on U, F agrees with f near the boundary of U. The connectedness of $U \setminus K$ implies that $F = f$ on $U \setminus K$. □

By analytic continuation, the zero set of a holomorphic function in one variable is always discrete. Moreover, by the Weierstrass factorization theorem, every discrete set is the zero set of a non-constant holomorphic function. Hartogs' extension theorem shows that the situation is drastically different in higher dimensions.

Corollary 2.4.3 Suppose $U \subset \mathbb{C}^n$, $n > 1$, and let $f : U \to \mathbb{C}$ be a holomorphic function. Then f has no isolated zeros.

Proof. If f had a isolated zero at z_0, then $1/f$ would be holomorphic in a neighborhood of z_0 by Hartogs' extension theorem, which is absurd. □

In the context of several complex variables we still have an analog of the Riemann extension theorem. For that we begin with a definition.

Definition 2.4.1 — Local boundedness. Let X be a subset of U and $f : U \setminus X \to \mathbb{C}$ be a holomorphic function. We say that f is *locally bounded on* U if for every $z \in U$ there exists a neighborhood V of p such that f is bounded on $V \cap (U \setminus X)$.

In one dimension, if a function is holomorphic and bounded on $U \setminus \{z_0\}$, then the function extends holomorphically to U (proposition 1.5.2). In several variables the same theorem holds, provided that we change boundedness to local boundedness and the single point to the zero set of a holomorphic function.

Theorem 2.4.4 — Riemann extension theorem. Let $g : U \to \mathbb{C}$ be a non-zero holomorphic function and $N = g^{-1}(0)$. If $f \in \mathcal{O}(U \setminus N)$ is locally bounded on U, then there exists exactly one $F \in \mathcal{O}(U)$ which coincides with f on $U \setminus N$.

Similarly to the case of the open mapping theorem, this result follows from the one-dimensional case by considering complex lines.

Proof. Bearing in mind the principle of analytic continuation, we know that N has empty interior and therefore the solution is unique. In particular, if we can solve this locally, we can glue these to form a global solution, so we can suppose that $U = D_1(0)$ is the unit polydisk.

After perhaps another change of variables we may suppose that $z_n \mapsto f(0, \dots, 0, z_n)$ has an isolated zero at 0. We may achieve this by considering a line passing connecting the origin and a non-zero point of f; after rotating this line to be the z_n-axis, then $f(0, \dots, 0, z_n)$ is not identically zero, so the zero is isolated.

Breaking up $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$ and writing $z = (w, z_n)$ where $w \in \mathbb{C}^{n-1}, z_n \in \mathbb{C}$ we can see that there is a small enough $r > 0$ such that f is not zero for $0 < |z_n| \leqslant r$, and, by continuity, there is a small enough $\delta > 0$ such that

$$|f(0, z_n) - f(w, z_n)| < \inf_{|z_n|=r} |f(0, z_n)| \quad (|w| < \delta),$$

so by Rouché's Theorem, $f(w, z_n)$ has precisely k (necessarily isolated) zeros as a function of z_n, where k is the order of the zero at $z_n = 0$ of $f(0, z_n)$.

Now, we may finally switch our focus to g, noticing that for each slice w is constant, g, as a funcion of z_n, has k singularities which are removable from the well known $n = 1$ case. Cauchy's integral formula will then tell us that

$$g(w, z_n) = \frac{1}{(2\pi i)^n} \int_{|\zeta_n|=r} \frac{g(w, \zeta_n)}{(\zeta_n - z_n)} d\zeta_n$$

is an expression for g, which is not only continuous and holomorphic in z_n for each fixed w, but also, differentiating at an integral sign, holomorphic in w for each fixed z_n, and by Osgood's lemma we're done. □

2.5. Domains of holomorphy

As we just saw, the world of several complex variables has a myriad of pairs of connected open sets $V \subsetneq U \subset \mathbb{C}^n$ such that *every* function that is holomorphic on V necessarily extends to a function holomorphic on the strictly larger set U. In this section we try to understand the sets which does *not* admit such a property.

Definition 2.5.1 — Domain of holomorphy. Let $U \subset \mathbb{C}^n$ be a connected open set. We say that U is a *domain of holomorphy* if, for every connected open set $V \subset \mathbb{C}^n$ which meets ∂V and every connected component W of $U \cap V$ there exists $f \in \mathscr{O}(U)$ such that $f|_W$ has no holomorphic extension to V.

Under these hypotheses we have that $\varnothing \neq \partial W \cap V \subset \partial U$. Thus, in order to show that a connected open set U is a domain of holomorphy, it suffices to find, for every $z_0 \in \partial U$, a function $f \in \mathscr{O}(U)$ which is unbounded near z_0.

■ **Example 2.5.1** Every connected open set in $\mathbb{C}$ is a domain of holomorphy. In fact, if z_0 is in its boundary, then the function $z \mapsto (z - z_0)^{-1}$ is unbounded near z_0.

Using the supporting hyperplane theorem this result generalizes to *convex* connected open sets in higher dimensions. In fact, such a supporting hyperplane containing a point z_0 of the boundary is necessarily of the form $\{z \in \mathbb{C}^n \mid \operatorname{Re}\langle z - z_0, w\rangle = 0\}$ for some $w \in \mathbb{C}^n$. Then the function $1 \mapsto \langle z - z_0, w\rangle^{-1}$ is unbounded near z_0. ■

This illustrates the fact that *convexity* is a good notion to understand domains of holomorphy. We know that often domains of holomorphy are not convex (in the case $n = 1$, for example). But we are heading in the right direction.

Actually, the right notion *has* to be more than simply geometrical. For example, the set $\mathbb{C}^2 \setminus \mathbb{R}^2$ is not a domain of holomorphy while $\mathbb{C}^2 \setminus \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 = 0\}$ is. Yet they are geometrically the same: both are 4-dimensional real vector spaces minus a 2-dimensional subspace.

Definition 2.5.2 — Holomorphical convexity. Let $K \subset U \subset \mathbb{C}^n$ be a compact set. The *holomorphic hull* of K in U is defined to be

$$\hat{K}_U := \left\{ z \in U \;\middle|\; |f(z)| \leqslant \sup_{w \in K} |f(w)|, \text{ for all } f \in \mathscr{O}(U) \right\}.$$

We say that U is *holomorphically convex* if the holomorphic hull $\hat{K}_U$ of every compact $K \subset U$ is compact.

In fact, this completely characterizes domains of holomorphy.

Theorem 2.5.1 — Cartan–Thullen. Let $U \subset \mathbb{C}^n$ be a connected open set. The following are equivalent:

(a) U is a domain of holomorphy;

(b) for all compact $K \subset U$, $\operatorname{dist}(K, \partial U) = \operatorname{dist}(\hat{K}_U, \partial U)$;

(c) U is holomorphically convex.

Proof. aaaaaaaa □

We can prove that holomorphic convexity is a biholomorphic invariant. Thus, being a domain of holomorphy is also a biholomorphic invariant. This fact is not easy to prove from the definition of a domain of holomorphy, as the biholomorphism is defined only on the interior of our domains.

Holomorphic convexity is an intrinsic notion; it does not require knowing anything about points outside of U. It is a much better way to think about domains of holomorphy. Holomorphic convexity generalizes easily to complex manifolds, while the notion of a domain of holomorphy only makes sense for connected open sets in $\mathbb{C}^n$.

3. Complex manifolds

3.1. Basic definitions

Most basic facts about smooth manifolds carry over perfectly to the context of complex manifolds. In this section we explain them in order to fix some notations.

Definition 3.1.1 — Complex manifold. A *complex manifold* is a topological manifold X of dimension $2n$ endowed with a maximal atlas whose transition functions are holomorphic under the identification $\mathbb{R}^{2m} \cong \mathbb{C}^n$. We say that its (complex) dimension is n.

One dimensional complex manifolds are called *Riemann surfaces*. If $\varphi : U \to \mathbb{C}^n$ is a chart, we let $z_j = \mathrm{pr}_j \circ \varphi$ be its j-th component. The functions $z_1, \ldots, z_n$ are called *local coordinates* on U.

Definition 3.1.2 — Holomorphic function. Let X and Y be complex manifolds of dimension n and m, respectively. A continuous map $f : X \to Y$ is said to be *holomorphic at a point* $p \in X$ if there are charts (V, ϕ) about $f(p)$ in Y and (U, φ) about p in X such that

$$\psi \circ f \circ \varphi^{-1} : \varphi(f^{-1}(V) \cap U) \to \mathbb{C}^m$$

is holomorphic at $\varphi(p)$. A *biholomorphism* is a bijective holomorphic map f whose inverse f^{-1} is also holomorphic.

Since the transition functions are holomorphic, if $f : X \to Y$ is holomorphic at p, if (V, ϕ) is any chart about $f(p)$ in Y and if (U, φ) is any chart about p in X, the function $\psi \circ f \circ \varphi^{-1}$ is holomorphic at $\varphi(p)$. Clearly the composition of holomorphic maps is still holomorphic. It is also clear that we can check if a map is holomorphic in its components.

We say that f is a holomorphic function on a open set U of X if $f : U \to \mathbb{C}$ is holomorphic, where both U and $\mathbb{C}$ have their obvious atlases. We denote it by $f \in \mathscr{O}_X(U)$. Clearly $\mathscr{O}_X$ is a sheaf of $\mathbb{C}$-algebras, which we will call the *structure sheaf* of X.

Lets see, for a moment, a complex manifold X as a smooth manifold of real dimension $2n$. Let $\varphi : U \to \mathbb{C}^n \cong \mathbb{R}^{2n}$ be a chart with (real) coordinates $x_1, y_1, \ldots, x_n, y_n$ and $f : X \to \mathbb{C}$ a smooth function. We denote by $r_1, s_1, \ldots, r_n, s_n$ the standard coordinates on $\mathbb{R}^{2n}$. Recall that, for $p \in U$, we define the *partial derivative* $\partial f / \partial x_j$ at p to be

$$\left.\frac{\partial}{\partial x_j}\right|_p f := \frac{\partial f}{\partial x_j}(p) := \frac{\partial (f \circ \varphi^{-1})}{\partial r_j}(\varphi(p)).$$

Similarly, the partial derivative $\partial f/\partial y_j$ at p is defined to be

$$\left.\frac{\partial}{\partial y_j}\right|_p f := \frac{\partial f}{\partial y_j}(p) := \frac{\partial (f \circ \varphi^{-1})}{\partial s_j}(\varphi(p)).$$

Now, as before, we define the operators

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Contrarily to the real case, we have multiple natural notions of tangent space on a complex manifold, which we'll now study. Recall that if $C_p^\infty(X, \mathbb{R})$ is the stalk at p of the sheaf (of $\mathbb{R}$-algebras) of real-valued C^∞ functions on X, a *derivation at* p in X is a linear map $D : C_p^\infty(X, \mathbb{R}) \to \mathbb{R}$ such that

$$D(fg) = D(f)g(p) + f(p)D(g)$$

for all $f, g \in C_p^\infty(X, \mathbb{R})$. Similarly we define derivations on $C_p^\infty(X, \mathbb{C})$ and on $\mathscr{O}_{X,p}$, where now the derivation is $\mathbb{C}$-linear.

Definition 3.1.3 — Tangent spaces. Let X be a complex manifold. We define the *real tangent space* to X at $p \in X$ to be $\mathbb{R}$-vector space $T_{\mathbb{R},p}X$, which consists of all the $C_p^\infty(X, \mathbb{R})$ derivations at p. The *complexified tangent space* to X at $p \in X$ is the $\mathbb{C}$ vector space $T_{\mathbb{C},p}X$, which consists of all the $C_p^\infty(X, \mathbb{C})$ derivations at p. Finally, the *holomorphic tangent space* to X at $p \in X$ is the $\mathbb{C}$ vector space T_pX, which consists of all the $\mathscr{O}_{X,p}$ derivations at p.

It is also usual in the real case to define the tangent space as the vector space of the derivations at p on $C^\infty(X, \mathbb{R})$. This does not *work in the complex setting since a compact complex manifold has no non-constant global functions. The equivalence between these definitions in smooth manifolds uses the existence of bump functions, which are unavailable in analytic manifolds.*

We begin by observing some relations between these vector spaces. Since $\mathscr{O}_{X,p}$ is naturally a subspace of $C_p^\infty(X, \mathbb{C})$, T_pX is naturally a subspace of $T_{\mathbb{C},p}X$. Its clear that $T_{\mathbb{C},p}X = T_{\mathbb{R},p}X \otimes_{\mathbb{R}} \mathbb{C}$. Moreover, the subspace of $T_{\mathbb{C},p}X$ consisting of derivations that vanish on antiholomorphic functions (i.e., those f such that $\bar{f}$ is holomorphic) determines another $\mathbb{C}$-vector space $\overline{T_pX}$ such that

$$T_{\mathbb{C},p}X = T_pX \oplus \overline{T_pX}.$$

This is often called the *antiholomorphic tangent space*. If $z_j = x_j + iy_j$ are local coordinates on a neighborhood of p, we have that

$$T_{\mathbb{R},p}X = \mathbb{R}\left\{ \frac{\partial}{\partial x_j}\bigg|_p, \frac{\partial}{\partial y_j}\bigg|_p \right\}, \qquad T_{\mathbb{C},p}X = \mathbb{C}\left\{ \frac{\partial}{\partial x_j}\bigg|_p, \frac{\partial}{\partial y_j}\bigg|_p \right\} = \mathbb{C}\left\{ \frac{\partial}{\partial z_j}\bigg|_p, \frac{\partial}{\partial \bar{z}_j}\bigg|_p \right\}$$

$$T_pX = \mathbb{C}\left\{ \frac{\partial}{\partial z_j}\bigg|_p \right\} \qquad \text{and} \qquad \overline{T_pX} = \mathbb{C}\left\{ \frac{\partial}{\partial \bar{z}_j}\bigg|_p \right\}.$$

Taking disjoint unions of the tangent spaces we obtain the correspondent tangent bundles $T_{\mathbb{R}}X$, $T_{\mathbb{C}}X$, TX and $\overline{TX}$. Similarly to the real case, these bundles are all smooth manifolds. The holomorphic bundle TX is even a complex manifold. The decomposition $T_{\mathbb{C},p}X = T_pX \oplus \overline{T_pX}$ yields a decomposition of vector bundles

$$T_{\mathbb{C}}X = TX \oplus \overline{TX}.$$

As usual, a C^∞ map between complex manifolds induces a linear map between tangent spaces, called its differential.

Definition 3.1.4 — Differential of a map. Let $f : X \to Y$ be a C^∞ map between complex manifolds and $p \in X$. We define a map

$$df_p : T_{\mathbb{C},p}X \to T_{\mathbb{C},f(p)}Y$$

as follows. If $v \in T_{\mathbb{C},p}X$, $df_p(v)$ is the tangent vector in $T_{\mathbb{C},f(p)}Y$ defined by

$$df_p(v)g := v(g \circ f) \in \mathbb{C}$$

for all $g \in C^\infty_{f(p)}(X, \mathbb{C})$. The reader can verify that this map is linear and that f is holomorphic if and only if $df_p(T_pX) \subset T_{f(p)}X$. We denote the restriction $df_p : T_pX \to T_{f(p)}X$ in the same way.

In practice we'll can often be cavalier about the distinction between a germ and a representative function for the germ since we're only interested in the behavior of a function in a sufficiently small neighborhood of a point.

As before, the chain rule holds in the form $d(g \circ f)_p = dg_{f(p)} \circ df_p$, which implies that if f is a diffeomorphism then $T_{\mathbb{C},p}X$ and $T_{\mathbb{C},f(p)}Y$ are isomorphic and if f is a biholomorphism then T_pX and $T_{f(p)}Y$ are isomorphic.

Let $f : X \to Y$ be a C^∞ map and $p \in X$. If $z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n$ are local coordinates on a neighborhood of p and $w_1, \overline{w}_1, \ldots, w_m, \overline{w}_m$ are local coordinates on a neighborhood of $f(p)$, the differential df_p is written locally as

$$df_p\left(\frac{\partial}{\partial z_j}\bigg|_p \right) = \sum_{k=1}^{m} \left(\frac{\partial f_k}{\partial z_j} \frac{\partial}{\partial w_k}\bigg|_p + \frac{\partial \bar{f}_k}{\partial z_j} \frac{\partial}{\partial \overline{w}_k}\bigg|_p \right),$$

where $f_k := w_k \circ f$ and $\bar{f}_k := \overline{w}_k \circ f$. Similarly,

$$df_p\left(\left.\frac{\partial}{\partial \bar{z}_j}\right|_p\right) = \sum_{k=1}^{m}\left(\frac{\partial f_k}{\partial \bar{z}_j}\left.\frac{\partial}{\partial w_k}\right|_p + \frac{\partial \bar{f}_k}{\partial \bar{z}_j}\left.\frac{\partial}{\partial \overline{w}_k}\right|_p\right).$$

Since the conjugate of $\partial f_k/\partial z_j$ is $\partial \bar{f}_k/\partial \bar{z}_j$, if f is holomorphic its Jacobian matrix relative to these bases is

$$\begin{pmatrix} J & 0 \\ 0 & \bar{J} \end{pmatrix},$$

where $J = (\partial f_k/\partial z_j)$. In particular, if $n = m$, the determinant of the Jacobian matrix of f is equal to $\det J \cdot \det \bar{J} = |\det J|^2 \geqslant 0$. I.e., holomorphic maps are orientation-preserving. The same calculation shows that the transition functions have positive determinant and so complex manifolds are always oriented.

3.2. Differential forms

The decomposition $T_{\mathbb{C}}X = TX \oplus \overline{TX}$ on the tangent bundles induces a corresponding decomposition on the cotangent bundles and thus

$$\bigwedge^k T^\vee_{\mathbb{C}}X = \bigoplus_{p+q=k} \bigwedge^p T^\vee X \otimes \bigwedge^q \overline{T^\vee X}.$$

Passing to the sheaf of C^∞ sections, we have a decomposition

$$\Omega^k_{X,\mathbb{C}} = \bigoplus_{p+q=k} \Omega^{p,q}_X,$$

where, in local coordinates $z_1, \ldots, z_n$, a local section of $\Omega^{p,q}_X$ (a form *of type / bidegree* (p, q)) is written as

$$\omega = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} a_{\alpha,\beta}\, dz_\alpha \wedge d\bar{z}_\beta.$$

We can also write such a form using the real basis $dx_\alpha \wedge dy_\beta$, *for* $|\alpha| + |\beta| = k$, *but this writing is* not *compatible with the splitting of* $\Omega^k_{X,\mathbb{C}}$ *in its* (p, q) *components.*

Taking the exterior derivative of such a form we see that

$$d\omega = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} da_{\alpha,\beta} \wedge dz_\alpha \wedge d\bar{z}_\beta,$$

which is the sum of a form of type $(p, q+1)$ and a form of type $(p+1, q)$. This motivates the next definition.

Definition 3.2.1 Let ω be a form of type (p,q) on a complex manifold X. We define $\partial\omega$ to be the component of type $(p+1,q)$ and $\bar\partial\omega$ to be the component of type $(p,q+1)$ of $d\omega$.

In local coordinates $z_1,\dots,z_n$, the operators ∂ and $\bar\partial$ act as

$$\partial\left(\sum_{\alpha,\beta} a_{\alpha,\beta}\, dz_\alpha \wedge d\bar z_\beta\right) = \sum_{j=1}^{n}\sum_{\alpha,\beta}\frac{\partial a_{\alpha,\beta}}{\partial z_j} dz_j \wedge dz_\alpha \wedge d\bar z_\beta$$
$$\bar\partial\left(\sum_{\alpha,\beta} a_{\alpha,\beta}\, dz_\alpha \wedge d\bar z_\beta\right) = \sum_{j=1}^{n}\sum_{\alpha,\beta}\frac{\partial a_{\alpha,\beta}}{\partial \bar z_j} d\bar z_j \wedge dz_\alpha \wedge d\bar z_\beta.$$

In particular, since the differential of a function f is given by

$$df = \sum_{j=1}^{\infty}\left(\frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \bar z_j} d\bar z_j\right),$$

the operators ∂ and $\bar\partial$ act on functions as

$$\partial f = \sum_{j=1}^{n}\frac{\partial f}{\partial z_j} dz_j \qquad \text{and} \qquad \bar\partial f = \sum_{j=1}^{n}\frac{\partial f}{\partial \bar z_j} d\bar z_j.$$

The next two propositions describe the essential properties of these operators.

Proposition 3.2.1 — Leibniz' rule. Let ω and η be forms of type (p,q) and (p',q'), respectively. Then

$$\partial(\omega\wedge\eta) = \partial\omega\wedge\eta + (-1)^p\omega\wedge\partial\eta$$
$$\bar\partial(\omega\wedge\eta) = \bar\partial\omega\wedge\eta + (-1)^p\omega\wedge\bar\partial\eta.$$

Proof. The first relation follows by taking the component of type $(p+p'+1, q+q')$ of $d(\omega\wedge\eta)$. The second follows from the first by noting that $\bar\partial\omega = \overline{\partial\overline{\omega}}$. □

Proposition 3.2.2 The operators ∂ and $\bar\partial$ satisfy the following relations:

$$\partial^2 = 0, \quad \bar\partial^2 = 0, \quad \partial\bar\partial + \bar\partial\partial = 0.$$

Proof. This follows from the fact that $d^2 = 0$ and $d = \partial + \bar\partial$. Indeed, we have that

$$0 = d^2 = (\partial+\bar\partial)^2 = \partial^2 + \partial\bar\partial + \bar\partial\partial + \bar\partial^2.$$

Now, if ω is a form of type (p,q), then $\partial^2\omega$ is of type $(p+2,q)$, $(\partial\bar\partial+\bar\partial\partial)\omega$ is of type $(p+1,q+1)$, and $\bar\partial\omega$ is of type $(p,q+2)$. Ergo, $d^2\omega = 0$ implies that

$$\partial^2\omega = 0, \quad \bar\partial^2\omega = 0, \quad (\partial\bar\partial+\bar\partial\partial)\omega = 0.$$

The result follows. □

There's yet another natural sheaf of differential forms: the sheaf of holomorphic sections of $\bigwedge^p T^\vee X$, which we denote by Ω_X^p. Since TX is naturally a subbundle of $T_{\mathbb{C}}X$, we have that Ω_X^p is naturally a subsheaf of $\Omega_X^{p,0}$. A form ω of type $(p,0)$ is in Ω_X^p if its coefficients in local coordinates are holomorphic. That is, if $\overline{\partial}\omega = 0$.

3.3. Dolbeault cohomology

Just like the fact that $d^2 = 0$ allows us to define the de Rham cohomology, the fact that $\overline{\partial}^2 = 0$ implies that

$$\operatorname{im}\left\{\overline{\partial} : \Omega_X^{p,q-1}(X) \to \Omega_X^{p,q}(X)\right\} \subset \ker\left\{\overline{\partial} : \Omega_X^{p,q}(X) \to \Omega_X^{p,q+1}(X)\right\},$$

where we adopt the convention that $\Omega_X^{p,-1}(X) = 0$. This motivates our next definition.

Definition 3.3.1 — Dolbeault cohomology. Let X be a complex manifold. The *Dolbeault cohomology* groups ($\mathbb{C}$-vector spaces, in fact) are defined as

$$H_{\overline{\partial}}^{p,q}(X) := \frac{\ker\left\{\overline{\partial} : \Omega_X^{p,q}(X) \to \Omega_X^{p,q+1}(X)\right\}}{\operatorname{im}\left\{\overline{\partial} : \Omega_X^{p,q-1}(X) \to \Omega_X^{p,q}(X)\right\}},$$

where $p, q \geqslant 0$ are integers.

If $f : X \to Y$ is a holomorphic map between complex manifolds, the pullback of a form of type (p,q) on Y is again a form of type (p,q) on X, since the components f_k of f on any chart are holomorphic and so $f^* dz_k = df_k$ is $\mathbb{C}$-linear. In particular, the equality $df^*\omega = f^* d\omega$ implies that

$$\partial f^*\omega = f^*\partial\omega \qquad \text{and} \qquad \overline{\partial} f^*\omega = f^*\overline{\partial}.$$

In particular, the pullback by a holomorphic function sends $\overline{\partial}$-closed forms to $\overline{\partial}$-closed forms and $\overline{\partial}$-exact forms to $\overline{\partial}$-exact forms. It follows that f^* induces a map in cohomology:

$$f^* : H_{\overline{\partial}}^{p,q}(Y) \to H_{\overline{\partial}}^{p,q}(X).$$

Just as in the real case, the Leibniz rule implies that the operation

$$\begin{aligned} H_{\overline{\partial}}^{p,q}(X) \times H_{\overline{\partial}}^{p',q'}(X) &\to H_{\overline{\partial}}^{p+p',q+q'}(X) \\ ([\omega],[\eta]) &\mapsto [\omega \wedge \eta] \end{aligned}$$

is well-defined (in other words, $\omega \wedge \eta$ is $\overline{\partial}$-closed and its cohomology class is independent of any choice of representatives). This defines a product on

$$H_{\overline{\partial}}^{*}(X) := \bigoplus_{p,q=0}^{\infty} H_{\overline{\partial}}^{p,q}(X),$$

which turns $H^*_{\bar\partial}(X)$ into a anticommutative graded algebra over $\mathbb{C}$. Since $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$, we obtain a contravariant functor from the category of complex manifolds to the category of anticommutative graded algebras over $\mathbb{C}$.

Recall that the de Rham theorem says that if M is a smooth manifold, the pairing of differential forms and chains, via integration, gives an isomorphism $H^k_{dR}(M) \to H^k(M, \mathbb{R})$ between the de Rham cohomology group and the singular cohomology group. In a similar fashion, the de Rham cohomology group $H^k_{dR}(M)$ is isomorphic to the Čech cohomology group $\check{H}^k(M, \underline{\mathbb{R}})$ with values on the constant presheaf $\underline{\mathbb{R}}$. (Proposition 10.6 in [**?**].)

An analogous result has not the least possibility to exist in the context of this chapter as the Dolbeault cohomology groups generally depend upon not just the topological data, but upon the complex structure of the manifold as well. Nevertheless, we shall see that the Dolbeault cohomology group $H^{p,q}_{\bar\partial}(X)$ is isomorphic to the Čech cohomology of the sheaf of holomorphic p-forms $\check{H}^q(X, \Omega^p_X)$.

For that we need a result analogous to the Poincaré lemma.

Proposition 3.3.1 — Dolbeault-Grothendieck lemma. Let $D \subset \mathbb{C}^n$ be the unitary polydisk and U a neighborhood of $\overline{D}$. If $\omega \in \Omega^{p,q}_{\mathbb{C}^n}(U)$, with $q > 0$, satisfies $\bar\partial\omega = 0$, then there exists $\eta \in \Omega^{p,q-1}_{\mathbb{C}^n}(D)$ such that $\bar\partial\eta = \omega$ on D.

Proof. First, we'll show that the case $p = 0$ implies the general one: indeed, we can write $\omega = \sum_\alpha \omega_\alpha \wedge dz_\alpha$ where $\omega_\alpha \in \Omega^{0,q}(U)$, and since $\bar\partial\omega = 0$, a simple bidegree analysis yields $\bar\partial\omega_\alpha = 0$; so if the lemma is proven for $p = 0$ we can write $\omega_\alpha = \bar\partial\eta_\alpha$ in D and therefore

$$\omega = \sum_\alpha \omega_\alpha \wedge dz_\alpha = \sum_\alpha \bar\partial\eta_\alpha \wedge dz_\alpha = \bar\partial\left(\sum_\alpha \eta_\alpha \wedge dz_\alpha\right)$$

is $\bar\partial$-exact.

Now, let k be the greatest index $d\bar z_k$ which appears on the decomposition of ω, that is, we write

$$\omega = d\bar z_k \wedge \alpha + \beta,$$

where α and β involve only $d\bar z_j$ with $j < k$. If we can integrate ω *modulo* $d\bar z_1, \dots, d\bar z_{k-1}$, that is, find η such that $\omega - \bar\partial\eta$ is a (necessarily closed) form involving only the differentials $d\bar z_j$ in the first $k-1$ variables, then the result follows by induction (the case $k = 0$ being trivial, since $q > 0$ implies that $\omega = 0$).

So we write

$$\alpha = \sum_{k \in \beta} a_\beta \, d\bar z_{\beta \setminus \{k\}}$$

(where we are summing only over indices β which include k, and $\beta \setminus k$ stands for β with the index k removed) and notice that, because $\bar\partial\omega = 0$ we have $d\bar z_k \wedge \bar\partial\alpha = -\bar\partial\beta$. Doing some bidegree analysis, we must have a_β holomorphic on z_l for $l > k$.

Now, we need some analytical arguments: multiplying by an appropriate bump function we can suppose that each a_β has compact support contained in U (while not changing their values on D), so we can apply lemma 2.4.2 for $n = 1$, that is, we can find a smooth functions u_β that satisfy

$$\frac{\partial u_\beta}{\partial \bar{z}^k} = a_\beta.$$

We note that, as a corollary of the *proof* of lemma 2.4.2, that each u_β will also be holomorphic on z_j for $j > k$ (by differentiation at an integral sign, since each a_β is). Now we define $\eta = \sum_{k\in\beta} u_\beta \, d\bar{z}_{\beta\setminus\{k\}}$ and note that

$$\bar{\partial}\eta = d\bar{z}_k \wedge \alpha + \sum_{\beta, l>k} \underbrace{\frac{\partial u_\beta}{\partial \bar{z}_l}}_{=0} d\bar{z}_l \wedge d\bar{z}_{\beta\setminus\{k\}} + \delta \quad (\text{in } D),$$

where δ only involves $d\bar{z}_j$ for $j < k$, finishing the proof. □

Corollary 3.3.2 Let X be a complex manifold. Then the sequence of sheaves

$$0 \to \Omega^p_X \to \Omega^{p,0}_X \xrightarrow{\bar{\partial}} \Omega^{p,1}_X \xrightarrow{\bar{\partial}} \dots$$

is exact.

Proof. From the fact that exactness of a sequence of sheaves is a local property, in view of the lemma of Dolbeaut-Grothendieck, it follows that this sequence is exact from the term $\Omega^{p,1}_X$ onward, and we must only compute the kernel of $\bar{\partial} : \Omega^{p,0}_X \to \Omega^{p,1}_X$. But a simple bidegree analysis yields that all coefficients of a $\bar{\partial}$-closed form involving only dz_j must be holomorphic. □

Notice that this entails $\Omega^p_X(X) = H^{p,0}_{\bar{\partial}}(X)$, connecting holomorphic forms and Dolbeaut cohomology. This is the reason we choose $\bar{\partial}$ in its definition: this way we can properly extract information about the complex structure of our manifold.

Corollary 3.3.3 Let D be a polydisk on $\mathbb{C}^n$ (not necessarily with compact closure). Then $H^{p,q}(D) = 0$ for $q > 0$.

Proof. Let ω be a $\bar{\partial}$-closed form of type (p, q), $q > 0$. We can write D as a union of concentric disks D_n where each of those are compactly contained in the next, that is, $\overline{D}_n \subset D_{n+1}$. Applying Dolbeaut-Grothendieck's lemma to each D_n relative to D_{n+1} and multiplying by an appropriate bump function this gives us a sequence $\eta_n \in \Omega^{p,q}(D)$ such that $\bar{\partial}\eta_n|D_n = \varphi|D_n$.

If η_n converges uniformly on compact sets to a form η, then we are done, since any such limit ψ satisfies the equation $\bar{\partial}\eta = \omega$. We must show that we can find some

$\tilde{\eta}_n$ with this property, each cohomologous to the respective η_n. We'll proceed by induction on q (followed by an induction on n to construct $\tilde{\eta}_n$):

For $q = 1$, $\eta_n = \psi_n$ are, in fact, smooth functions and by hypothesis $\psi_{n+1} - \psi_n$ is holomorphic on D_n. Let $\tilde{\psi}_0 = \psi_0$, and suppose that we have constructed $\tilde{\psi}_0, \dots, \tilde{\psi}_n$ with

$$\sup_{D_{k-1}} |\tilde{\psi}_{k+1} - \tilde{\psi}_k| < \frac{1}{2^k}$$

for all appropriate k. Then, since $\psi_{n+1} - \tilde{\psi}_n$ is holomorphic on D_n, its power series expansion converging uniformily on D_{n-1}, we can find a polynomial p_{n+1} (which is holomorphic on D!) such that

$$\sup_{D_{n-1}} |\tilde{\psi}_{n+1} - \tilde{\psi}_n - p_{n+1}| < \frac{1}{2^n}$$

and then $\tilde{\psi}_{n+1} = \psi_{n+1} - p_{n+1}$ will do.

Suppose the claim is true for $q > 1$, then we can do even better: we can find such sequence $\tilde{\eta}_n$ that is eventually constant on compact subsets. Indeed, suppose we have found such $\tilde{\eta}_n$: then $\eta_{n+1} - \tilde{\eta}_n$ is a closed form on D_n, so, by induction hypothesis, since $q - 1 > 0$, we have

$$\tilde{\eta}_n|D_n - \eta_{n+1}|D_n = \bar{\partial}\alpha|D_n, \quad (\alpha \in \Omega^{p,q-2}(D)),$$

and therefore we can choose $\tilde{\eta}_{n+1} = \eta_{n+1} + \bar{\partial}\alpha$, which is cohomologous to η_{n+1} and equals to $\tilde{\eta}_n$ on D_n. □

Theorem 3.3.4 — Dolbeault. Let X be a complex manifold. Then

$$H^{p,q}_{\bar{\partial}}(X) \cong \check{H}^q(X, \Omega^p_X)$$

for all $p, q \geqslant 0$.

Proof. It all comes down to seeing that the sequence in corollary 3.3.2 is an acyclic resolution for Ω^p_X, for then sheaf cohomology yields

$$H^q(X, \Omega^p) = H^q(\Omega^{p,\bullet}(X)) = H^{p,q}(X).$$

□

A simple corollary of this result is the fact that $\check{H}^q(X, \Omega^p_X)$ vanishes for sufficiently big q.

Corollary 3.3.5 Let X be a complex manifold of dimension n. Then

$$\check{H}^q(X, \Omega_X^p) = 0$$

for all $p \geqslant 0$ and $q > n$.

3.4. Currents

In order to understand better the relations between the de Rham and the singular cohomologies, Georges de Rham developed in 1955 the formalism of *currents*, which puts differential forms and chains in the same footing. This is the subject of this section. For the sake of generality, we'll deal with smooth manifolds and comment on the specificities of the complex world.

Let M be a smooth oriented manifold of dimension n and U an open set of M. We denote by $\mathcal{D}^k(U)$ the $\mathbb{R}$-vector space composed by the forms in Ω_M^k with compact support on U. This space is endowed with a locally convex topology on which a sequence (ω_j) converges to ω if

1. there exists a compact set $K \subset U$ such that $\operatorname{supp} \omega_j \subset K$ for all j;
2. for every chart $V \to \mathbb{R}^n$ the derivatives of all orders of the component functions of ω_j converge to those of ω uniformly on every compact of V.

The reader should ponder the resemblance of this definition with that of the topology of $C_c^\infty(\mathbb{R}^n)$ in distribution theory. In fact, both notions coincide when $k = 0$.

Definition 3.4.1 — Current. The *space of currents of dimension* k on M is the topological dual of $\mathcal{D}^k(M)$. We denote it by $\mathcal{D}'_k(M)$. We say that an element of this space has *degree* $n - k$. The *support* of a current $T \in \mathcal{D}'_k(M)$ is the largest open set $U \subset M$ such that $T(\omega) = 0$ whenever $\omega \in \mathcal{D}_k(U)$. The subspace of all the currents with compact support is denoted by $\mathcal{E}'_k(M)$.

The terminology used for the dimension and degree of a current is justified by the following two fundamental examples.

■ **Example 3.4.1** Let η is a differential form of degree k with L^1_{loc} coefficients. Associated with η there is a current $T_\eta \in \mathcal{D}'_{n-k}(M)$ defined by

$$T_\eta(\omega) := \int_M \eta \wedge \omega, \qquad \text{for } \omega \in \mathcal{D}^{n-k}(M).$$

The current T_η evidently has degree k. The correspondence $\eta \mapsto T_\eta$ is injective, so will identify η with its image in $\mathcal{D}'_{n-k}$. ■

■ **Example 3.4.2** Let Γ be a piecewise smooth, oriented k-chain in M. Then Γ defines a current $T_\Gamma \in \mathcal{D}'_k(M)$ by

$$T_\Gamma(\omega) := \int_\Gamma \omega, \qquad \text{for } \omega \in \mathcal{D}^k(M).$$

Similarly, a closed oriented submanifold $Z \subset M$ of dimension k defines a current T_Z of dimension k with $\operatorname{supp} T_Z = Z$. ■

When X is a complex manifold and $U \subset X$ is an open set, we define similarly $\mathcal{D}^{p,q}(U)$ (resp. $\mathcal{D}^k(U)$) to be the $\mathbb{C}$-locally convex space composed by the forms in $\Omega_X^{p,q}(U)$ (resp. $\Omega^k_{X,\mathbb{C}}(U)$) with compact support on U and $\mathcal{D}'_{p,q}(U)$ (resp. $\mathcal{D}'_k(U)$) to be its topological dual. By duality we have that

$$\mathcal{D}'_k(U) = \bigoplus_{p+q=k} \mathcal{D}'_{p,q}(U).$$

Many of the operations available for differential forms can be extended to currents. We now describe a couple of them. We'll leave to the reader the task of verifying that they actually are continuous forms.

Definition 3.4.2 — Exterior derivative. Let $T \in \mathcal{D}'_k(M)$. Its *exterior derivative* $dT \in \mathcal{D}'_{k-1}(M)$ is defined by $dT(\omega) := (-1)^{k+1}T(d\omega)$.

One can motivate this definition by noting that the Stokes' theorem implies that

$$T_{d\eta}(\omega) = \int_M d\eta \wedge \omega = \underbrace{\int_M d(\eta\wedge\omega)}_{=0} + (-1)^{k+1}\int_M \eta \wedge d\omega = (-1)^{k+1}T_\eta(d\omega)$$

for all $\eta \in \mathcal{D}^k(M)$. In other words, $dT_\eta = T_{d\eta}$. Similarly, the Stokes' theorem says precisely that $T_\Gamma(d\omega) = T_{\partial\Gamma}(\omega)$ so that $dT_\Gamma = (-1)^{k+1}T_{\partial\Gamma}$. In the complex case we define analogously the operators $\partial : \mathcal{D}'_{p,q}(X) \to \mathcal{D}'_{p-1,q}(X)$ and $\bar\partial : \mathcal{D}'_{p,q}(X) \to \mathcal{D}'_{p,q-1}(X)$.

Definition 3.4.3 — Wedge product. Let $T \in \mathcal{D}'_k(M)$ and $\eta \in \Omega^r_M$ be a differential form with compact support. The *wedge product* $T \wedge \eta \in \mathcal{D}'_{k-r}(M)$ is defined by $(T\wedge\eta)(\omega) := T(\eta\wedge\omega)$.

Using Leibniz' rule we readily verify that $d(T\wedge\eta) = dT\wedge\eta + (-1)^{n-k}T\wedge d\eta$. Moreover $\operatorname{supp}(T\wedge\eta) \subset (\operatorname{supp} T) \cap (\operatorname{supp}\eta)$.

Let $f : M \to N$ be a smooth map between oriented manifolds. We may wish to use the pull-back morphism to define the direct image of a current T by f. The problem is that even if $\omega \in \Omega^k_N$ has compact support, there is no reason for $f^*\omega \in \Omega^k_M$ to have. Since $\operatorname{supp} f^*\omega \subset f^{-1}(\operatorname{supp}\omega)$, this construction becomes well defined if the restriction of f to $\operatorname{supp} T$ is proper. We summarize this discussion in the following definition.

Definition 3.4.4 — Direct image. Let $T \in \mathcal{D}'_k(M)$ be a current and $f : M \to N$ be a map between oriented manifolds such that its restriction to $\operatorname{supp} T$ is proper. The *direct image* $f_*T \in \mathcal{D}'_k(N)$ of T by f is defined by $f_*T(\omega) := T(f^*\omega)$.

Under the conditions of this definition, it is clear that $\operatorname{supp} f_*T \subset f(\operatorname{supp} T)$ and $d(f_*T) = f_*(dT)$. Furthermore, if η is a differential form with compact support on N and $g : N \to P$ is a smooth map such that the restriction of $g \circ f$ to $\operatorname{supp} T$ is proper, we have that $f_*(T \wedge f_*\eta) = (f_*T) \wedge \eta$ and $g_*(f_*T) = (g \circ f)_*T$.