Adjoints can be defined on objects

Gabriel Ribeiro

October 2022

1 Motivation

Let C be a complete category and I be a small category. The sole reasonable notion of a *limit functor* is a functor $\lim : \mathrm{Fun}(\mathsf{I}, \mathsf{C}) \to \mathsf{C}$, whose value at an object F is $\lim \mathsf{F}$. Of course, in order to define a functor it does not suffice to associate an isomorphism class of objects in the codomain to each object in the domain. In particular, this limit functor is not canonically defined; it requires an arbitrary choice of a limit for $\mathsf{F} : \mathsf{I} \to \mathsf{C}$.

Now, suppose that we have made all the choices above. If $\alpha : \mathsf{F} \to \mathsf{G}$ is a natural transformation between functors $\mathsf{I} \to \mathsf{C}$, what's the induced morphism $\lim \mathsf{F} \to \lim \mathsf{G}$ in C?

By the very definition of limit, we have maps $\lim \mathsf{F} \to \mathsf{F}(\mathsf{X})$ for all $\mathsf{X} \in \mathsf{I}$, and we have morphisms $\alpha_\mathsf{X} : \mathsf{F}(\mathsf{X}) \to \mathsf{G}(\mathsf{X})$. Since both maps are natural with respect to maps on I, we obtain morphisms $\lim \mathsf{F} \to \mathsf{G}(\mathsf{X})$ such that, for all morphisms $\mathsf{X} \to \mathsf{Y}$ in I, the diagram

$$\begin{tikzcd} & \lim \mathsf{F} \arrow[dl] \arrow[dr] & \\ \mathsf{G}(\mathsf{X}) \arrow[rr] & & \mathsf{G}(\mathsf{Y}) \end{tikzcd}$$

commutes. In other words, we have a cone on G. The universal property of $\lim \mathsf{G}$ then induces our desired map $\lim \mathsf{F} \to \lim \mathsf{G}$, proving that $\lim$ is indeed a functor.

In this case, we were lucky to have a universal property which allowed us to prove that $\lim$ is a functor. In other contexts, given two categories C and D, we may know how to associate an object of D to each object of C, but its construction may be so inexplicit that we cannot simply prove that this yields a functor.

This happens in the very foundations of homological algebra. If A is a Grothendieck abelian category, we have K-injective resolutions for objects of $\mathsf{D}(\mathsf{A})$. That is, to each object $\mathsf{M}^\bullet$ in $\mathsf{D}(\mathsf{A})$ we may associate a K-injective complex $\mathfrak{i}(\mathsf{M}^\bullet)$ in $\mathsf{K}(\mathsf{A})$. But is this a functor $\mathfrak{i} : \mathsf{D}(\mathsf{A}) \to \mathsf{K}_{\mathrm{inj}}(\mathsf{A})$?

Of course we have the same problem as before; we don't really have a *construction* of $\mathfrak{i}(\mathsf{M}^\bullet)$. We know that there exists such an object. We know that it's unique up to unique isomorphism. But we don't have an explicit representant. This can be solved by the axiom of choice.

But now we have a harder problem! Even if we choose some $i(M^\bullet)$ for each $M^\bullet$, it's not clear that we can make functorial choices. That is, we must also choose a morphism $i(M^\bullet) \to i(N^\bullet)$ for every morphism $M^\bullet \to N^\bullet$ in a way compatible with the compositions.

We remark that this is a *serious* problem! Indeed, if $F : A \to B$ is an additive functor between abelian categories, our functor i above is part of the construction of the derived functor RF:

$$\begin{tikzcd} D(A) \arrow[r, "i"] \arrow[rrr, bend right, "RF"'] & K_{\mathrm{inj}}(A) \arrow[r, "F"] & K(B) \arrow[r, "Q"] & D(B). \end{tikzcd}$$

(Just to be clear, here $K(A)$ is the homotopy category of A, $K_{\mathrm{inj}}(A)$ is the full subcategory composed of the K-injective complexes, $F : K_{\mathrm{inj}}(A) \to K(B)$ is the induced functor acting degree by degree, and $Q : K(B) \to D(B)$ is the localization functor.)

2 The solution

The goal of these notes is to show that there's an elegant way of getting around both of these problems.

Definition 2.1 Let $G : D \to C$ be a functor. Given objects N in C and M in D, and a morphism $\eta : N \to G(M)$ in C, we say that η witnesses M as a left-adjoint object to N under G if the composite $\mathrm{Hom}_D(M, -) \xrightarrow{G} \mathrm{Hom}_C(G(M), G(-)) \xrightarrow{\eta^*} \mathrm{Hom}_C(N, G(-))$ is a natural isomorphism of functors $D \to \mathsf{Set}$.

We recall that in order to check if a natural transformation is an isomorphism, it suffices to check it on objects. Also, we clearly have a natural dual notion for right-adjoints.

A surprisingly not-so-well-known consequence of the Yoneda lemma is the proposition below.

Proposition 2.1 A functor $G : D \to C$ admits a left-adjoint if and only if every N in C has a left-adjoint object under G.

Proof. If $F : C \to D$ is a left-adjoint of G, then it suffices to take $F(N)$ as a left-adjoint object to N witnessed by the unit. As for the converse, recall that the Yoneda lemma says that the natural functor

$$よ : D^{\mathrm{op}} \to \mathrm{Fun}(D, \mathsf{Set})$$

gives an equivalence of categories between D^{op} and the full subcategory of $\mathrm{Fun}(D, \mathsf{Set})$ composed of the representable functors. Now, the functor

$$\mathrm{Hom}_C(-, G(-)) : C^{\mathrm{op}} \times D \to \mathsf{Set}$$

gives rise via currying to a functor $H : C^{op} \to \mathrm{Fun}(D, \mathsf{Set})$, whose image is composed of representable functors. Indeed, if $N \in C$ we may let M be a left-adjoint object of N under G and then

$$H(N) = \mathrm{Hom}_C(N, G(-)) \cong \mathrm{Hom}_D(M, -).$$

In particular, we may compose it with the inverse of the Yoneda embedding to obtain a functor $C^{op} \to D^{op}$ and so a functor $F : C \to D$.

By construction, よ $\circ F^{op} \cong H$ as functors $C^{op} \to \mathrm{Fun}(D, \mathsf{Set})$. By currying, this equivalence means precisely that $\mathrm{Hom}_D(F^{op}(-), -) \cong \mathrm{Hom}_C(-, G(-))$ as functors $C^{op} \times D \to \mathsf{Set}$. $\square$

Now, let's see how this solves our problems. I affirm that in both examples, the desired functors can be defined as adjoints.

Let $\Delta : C \to \mathrm{Fun}(I, C)$ be the *diagonal functor*, which sends an object X of C to the functor $I \to C$ which sends every object of I to X and every morphism of I to id_X. For $X \in C$ and $F \in \mathrm{Fun}(I, C)$, the elements of the set

$$\mathrm{Hom}_{\mathrm{Fun}(I,C)}(F, \Delta(X))$$

are precisely the cocones over F with nadir X. It follows that a left-adjoint object to F under Δ is precisely the same as a colimit of F. In particular, our proposition implies that if C has all I-colimits, then we have a colimit functor $\mathrm{colim} : \mathrm{Fun}(I, C) \to C$. (And similarly for limits, of course!)

For the homological algebra part, we have to recall a couple of facts. Let A be an abelian category. A complex $I^\bullet$ in A is said to be *K-injective* if the map (induced by the localization functor)

$$\mathrm{Hom}_{K(A)}(-, I^\bullet) \to \mathrm{Hom}_{D(A)}(-, I^\bullet)$$

is a natural isomorphism of functors $K(A)^{op} \to \mathsf{Set}$. We'll need a different way to characterize K-injective complexes.

Lemma 2.2 A complex $I^\bullet \in K(A)$ is K-injective if and only if $\mathrm{Hom}_{K(A)}(N^\bullet, I^\bullet) = 0$ for all exact complexes $N^\bullet$.

Proof. A K-injective complex $I^\bullet \in K(A)$ certainly has $\mathrm{Hom}_{K(A)}(N^\bullet, I^\bullet) \cong \mathrm{Hom}_{D(A)}(N^\bullet, I^\bullet)$ equal to zero for all exact complexes $N^\bullet$. Now, suppose that this condition is satisfied. The universal property of the quotient implies that $\mathrm{Hom}_{K(A)}(-, I^\bullet) : K(A)^{op} \to \mathsf{Set}$ descends to a functor $D(A)^{op} \to \mathsf{Set}$, which we denote by the same name. We have a natural transformation

$$\mathrm{Hom}_{K(A)}(-, I^\bullet) \to \mathrm{Hom}_{D(A)}(-, I^\bullet),$$

and the Yoneda lemma says that natural transformations in the other direction coincide with elements of $\mathrm{Hom}_{\mathsf{K}(\mathsf{A})}(I^\bullet, I^\bullet)$. Finally, the identity on $I^\bullet$ induces an inverse to the natural transformation above. $\square$

A *K-injective resolution* of a complex $M^\bullet$ in A is an isomorphism $M^\bullet \to I^\bullet$ in $\mathsf{D}(\mathsf{A})$, where $I^\bullet$ is K-injective. Finally, the K-injective complexes form a thick subcategory of $\mathsf{K}(\mathsf{A})$, which we denote by $\mathsf{K}_{\mathrm{inj}}(\mathsf{A})$.

Proposition 2.3 Let $M^\bullet \in \mathsf{K}(\mathsf{A})$. The following are equivalent.

1. $M^\bullet$ has a K-injective resolution.
2. $M^\bullet$ has a right-adjoint object under the quotient functor $Q : \mathsf{K}(\mathsf{A}) \to \mathsf{D}(\mathsf{A})$.

Proof. Suppose that $M^\bullet \in \mathsf{K}(\mathsf{A})$ has a K-injective resolution $M^\bullet \to i(M^\bullet)$. Then, we have the following isomorphisms

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{A})}(-, M^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(\mathsf{A})}(-, i(M^\bullet)) \xleftarrow{\sim} \mathrm{Hom}_{\mathsf{K}(\mathsf{A})}(-, i(M^\bullet)),$$

where the first arrow is induced from the quasi-isomorphism $M^\bullet \to i(M^\bullet)$ and the second is from the definition of K-injective. This means that $i(M^\bullet)$ is a right-adjoint object to $M^\bullet$ under Q. (Witnessed by the inverse of $M^\bullet \to i(M^\bullet)$ in $\mathsf{D}(\mathsf{A})$.)

Conversely, suppose that $M^\bullet$ has a right-adjoint object $I^\bullet$ under Q, witnessed by a map $\varepsilon : I^\bullet \to M^\bullet$ in $\mathsf{D}(\mathsf{A})$. By supposition we know that the composition

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{A})}(-, I^\bullet) \xrightarrow{Q} \mathrm{Hom}_{\mathsf{D}(\mathsf{A})}(-, I^\bullet) \xrightarrow{\varepsilon_*} \mathrm{Hom}_{\mathsf{D}(\mathsf{A})}(-, M^\bullet)$$

is a natural isomorphism. We need to show that actually both arrows above are isomorphisms. Indeed, the one on the left encodes the fact that $I^\bullet$ is K-injective and the one on the right encodes the fact that ε is an isomorphism in $\mathsf{D}(\mathsf{A})$. Now, our isomorphism implies that

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{A})}(N^\bullet, I^\bullet) \cong \mathrm{Hom}_{\mathsf{D}(\mathsf{A})}(N^\bullet, M^\bullet) = 0$$

whenever $N^\bullet$ is exact. The previous lemma then implies that $I^\bullet$ is K-injective and so both arrows above are isomorphisms. $\square$

Combining both propositions, we conclude that if every complex $M^\bullet$ has a K-injective resolution $M \to i(M^\bullet)$, then we obtain a functor $i : \mathsf{D}(\mathsf{A}) \to \mathsf{K}(\mathsf{A})$, which is a right adjoint to the quotient functor $Q : \mathsf{K}(\mathsf{A}) \to \mathsf{D}(\mathsf{A})$, and which takes values in $\mathsf{K}_{\mathrm{inj}}(\mathsf{A})$.

Since the counit of the adjunction is an isomorphism, the general theory of reflective localizations [HK, Proposition 1.1.3] imply that i restricts to an isomorphism $\mathsf{D}(\mathsf{A}) \xrightarrow{\sim} \mathsf{K}_{\mathrm{inj}}(\mathsf{A})$.

References

The statement and the proof of proposition 2.1 comes the lecture notes for Algebraic and Hermitian K-Theory by Fabian Hebestreit. (They use ∞-categories there, but everything works in precisely the same way for 1-categories.) This is also Proposition 4.3.4 in E. Riehl's Category Theory in Context.

The statement of proposition 2.3 is adapted from Proposition 4.3.1 in H. Krause - Homological Theory of Representations, which we denote by [HK]. (The proof there takes our proposition 2.1 more or less for granted. And our proof differs in other ways as well.)