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Komplexe Analysis – Differential and Algebraic methods in Kähler spaces

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Abstract. Our workshop focused on recent results in our main field (complex geometry) and its connection with other branches of mathematics. The main theme of an important proportion of the talks was Hodge theory, combined with differential-geometric methods in the study of singular spaces. One special lecture was a very comprehensive introduction in Scholze-Clausen's theory of condensed mathematics.

Mathematics Subject Classification (2020): 14-XX, 32-XX.

Introduction by the Organizers

The workshop *Komplexe Analysis – Differential and Algebraic methods in Kähler spaces*, organized by Philippe Eyssidieux (Grenoble), Jun-Muk Hwang (Daejeon), Stefan Kebekus (Freiburg) and Mihai Păun (Bayreuth), was held the week stating from the 9th of April 2023. It was attended by over 50 participants from around the world, ranging from doctoral students to senior researchers. The program featured twenty lectures, and allowed ample time for discussion and interaction; the very well-equipped and inspiring discussion rooms were constantly occupied.

Now concerning the talks, the organisers aimed at a balanced meeting, reflecting the current generation change in the subject: the program included many talks by younger colleagues, as well as talks by seniors, whose lectures were often full of interesting ideas and promising paths to follow.

The list of talks mentioned below is not exhaustive, but illustrates the diversity and the importance of recent contributions to the field.

• **Hodge theory and its applications.** This was one of the dominant themes of the lectures in our workshop. The main reason is that a few important conjectures concerning hyperkähler manifolds have been solved recently by using methods from Hodge theory.

B. Bakker presented his solution to the following beautiful problem proposed by D. Matsushita: *any Lagrangian fibration of an irreducible hyperkähler manifold is either isotrivial or has maximal variation.* C. Schnell explained in a very clear and convincing manner the proof of two conjectures proposed by Shen, Yin and Maulik, about the singular fibers of Labrangian fibrations on holomorphic symplectic Kähler manifolds. Kobayashi hyperbolicity of manifolds is a very old and important topic: C. Lehn presented his joint work with L. Kamenova, in which they show that the Kobayashy pseudometric vanishes identically for a large, natural class of compact hyperkähler manifolds.

C. Li's lecture was centred to one of his very recent contributions, showing that a compact complex manifold is Kähler if it admits a closed, real 2-form α together with a non-singular fibration such that the base in Kähler and the restriction of α to each fiber contains a Kähler form.

The powerful L^2 methods are appearing more and more often in recent Hodge theory. A lecture in this spirit was given by J. Shentu.

• **Geometric and complex analysis** T. Peternell presented his recent joint work with A. Höring concerning the characterisation of the nefness of the tangent bundle of a compact Kähler manifold in terms of the extension defined by a Kähler metric. He took this opportunity to recall some other classical, still unsolved problems. H.-J. Hein gave a very beautiful talk about the construction and asymptotic of interesting metrics on a class of singular surfaces. In a very dynamic lecture, V. Tosatti explained his results concerning the behaviour of the volume function near the boundary of the pseudo-effective cone of a smooth projective manifold. Singular Monge-Ampère equations were equally present in our workshop, thanks to the talk by E. Di Nezza, and recent contributions to the field of CSCK metrics were presented by S. T. Paul. A couple of very well-presented and interesting talks concerning applications of the minimal model program were given by S. Schreieder and J. Moraga.

Finally, we would like to mention two evening sessions, in which the rule was that very young participants were encouraged to present their research topics in 10 minutes' talks. In one of these short presentations, M. Villadsen managed to present a very clever proof of a deep result due to M. Popa and C. Schnell!

Workshop: Komplexe Analysis – Differential and Algebraic methods in Kähler spaces

Table of Contents

Abstracts

Nef tangent bundles and canonical extensions

Thomas Peternell
(joint work with Andreas Höring)

This is a report on joint work with Andreas Höring, [5] and [6].

Given a compact Kähler manifold (M, ω), the class $[\omega] \in H^1(M, \Omega_M)$ defines a non-split extension

$$0 \to \mathcal{O}_M \to V \to T_M \to 0,$$

where T_M denotes the tangent bundle of M. Then we may consider the manifold

$$Z_M = \mathbb{P}(V) \setminus \mathbb{P}(T_M)$$

which is an affine bundle over M. In this context, we have the following conjecture [3], see also [5, Conj.1.1]:

Conjecture 1. *Let M be a compact Kähler manifold.*

- *If Z_M is Stein, then T_M is nef.*
- *If M is projective and Z_M is affine, then M is rational homogeneous, i.e., $M = G/P$ with G a semi-simple complex Lie group and P a parabolic subgroup.*

Recall that a vector bundle V is nef, if the tautological line bundle $\mathcal{O}_V(1)$ on $\mathbb{P}(V)$ is nef, i.e., $c_1(\mathcal{O}_V(1))$ is in the closure of the Kähler cone of $\mathbb{P}(V)$.

To put things into perspective, recall the following structure theorem, proved in [2].

Theorem 2. *Let M be a compact Kähler manifold such that T_M is nef. Then - up to finite étale cover - the Albanese map $\alpha : M \to \mathrm{Alb}(M)$ is a flat fiber bundle whose fiber is a Fano manifold F with T_F nef.*

Concerning Fano manifolds with nef tangent bundle, wer have the following conjecture, [1]

Conjecture 3. *A Fano manifold with nef tangent bundle is rational homogeneous.*

This conjecture is known to be true in dimension at most five, but in general it is wide open.

Coming back to the original situation, Greb and Wong showed that Z_M never contains any compact analytic set of positive dimension. Thus Z_M is Stein if and only if Z_M is holomorphically comvex. Further they show that if M is homogeneous, then Z_M is Stein, and if M is rational homogeneous, then Z_M is affine.

More generally, the paper [5] proves, using the base point free theorem,

Theorem 4. *If Z_M is projective and T_M is big and nef, then Z_M is affine.*

Note that, as a classical fact, the tangent bundle of a rational homogeneous manifold is indeed big, i.e., $\mathcal{O}_V(1)$ is big.

Towards Conjecture 1, [5] shows

Theorem 5. *Let M be a smooth projective surface. If Z_M is affine, then $M = \mathbb{P}_2$ or $M = \mathbb{P}_1 \times \mathbb{P}_1$.*
If Z_M is Stein, then

(1) M is an étale quotient of a torus;
(2) $M = \mathbb{P}_2$ or $M = \mathbb{P}_1 \times \mathbb{P}_1$;
(3) $M = \mathbb{P}(\mathcal{E})$ with $\mathcal{E}$ of rank two over an elliptic curve;
(4) $M = \mathbb{P}(\mathcal{E})$ with $\mathcal{E}$ semi-stable of rank two over a curve B of genus $g(B) \geq 2$.

The last case has recently been ruled out by N. Müller, [7]. Thus, it remains to rule out the case that $M = \mathbb{P}(\mathcal{E})$ with $\mathcal{E}$ unstable of rank two over an elliptic curve. Notice in all remaining cases Z_M is indeed Stein (resp. affine if $M = \mathbb{P}_2$ or $M = \mathbb{P}_1 \times \mathbb{P}_1$).

In particular, if $M = \mathbb{P}(\mathcal{E})$ with $\mathcal{E}$ semi-stable or rank two over an elliptic curve B, then Z_M is Stein. This includes the famous example of Serre, where $\mathcal{E}$ shows up as non-split extension

$$0 \to \mathcal{O}_B \to \mathcal{E} \to \mathcal{O}_B \to 0.$$

In that case Z_M is biholomorphic to $\mathbb{C}^* \times \mathbb{C}^*$, but Z_M is not affine.

In dimension three we restrict ourselves to Fano manifolds:

Theorem 6. *([6]) Let M be a (smooth) Fano threefold with Z_M affine. Then M is rational homogeneous.*

The case $b_2(M) \geq 2$ is treated via Mori theory. As to $b_2(M) = 1$, we use the general fact that T_M must be big, as observed by Greb-Wong. Now a remarkable theorem of Höring-Liu [4] states in particular that a Fano threefold M with $b_2(M) = 1$ and T_M big is either $\mathbb{P}_3$, the threedimensional quadric, or the threefold V_5, obtained by intersection the Grassmannian $G(2,5) \subset \mathbb{P}_9$ with three general hyperplanes.

Ruling out $M = V_5$ is rather delicate. In fact, we show, [6]:

Theorem 7. *Let $M = V_5$. Then there exists a sequence of antiflips (i.e., a sequence of inverses of flips))*

$$\psi : \mathbb{P}(V) \dashrightarrow \mathbb{P}(V)^-$$

such that

(1) ψ induces a biholomorphic map

$$Z_M \to (\mathbb{P}(V)^- \setminus (\mathbb{P}(T_M)^- \cup A))$$

with an analytic set $A \subset \mathbb{P}(V)^-$ of codimension at least two and $\mathbb{P}(T_M)^-$ denoting the strict transform of $\mathbb{P}(T_M)$;

(2) $\mathbb{P}(V)^-$ and $\mathbb{P}(T_M)^-$ are $\mathbb{Q}$-factorial with terminal singularities and big and nef anticanonical bundles;

(3) $\mathbb{P}(V)^- \setminus \mathbb{P}(T_M)^-$ *is affine.*

Note that T_M is not nef; in fact, there are lines $C \subset M = V_5$ such that

$$T_{M|C} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-1).$$

As to higher dimensions, things are reduced to study the uniruled case:

Theorem 8. *([5]) Let M be a projective manifold with Z_M Stein. Then either M is uniruled or an étale quotient of an abelian variety.*

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The Matsushita alternative

Benjamin Bakker

Let X be an irreducible compact hyperkähler manifold, that is, a simply-connected compact Kähler manifold X for which $H^0(X, \Omega_X^2) = \mathbb{C}\sigma$ for a nowhere-degenerate holomorphic two-form σ. Such X are one of the three building blocks of K-trivial Kähler manifolds as per the Beauville–Bogomolov decomposition theorem.

By a theorem of Matsushita [6], X only has one possible nontrivial fibration structure—a fibration by Lagrangian tori. Precisely, a Lagrangian fibration of X is a proper morphism $f : X \to B$ to a normal compact analytic variety B whose generic fiber is smooth, connected, and Lagrangian (see [5] for a recent survey). It follows that every smooth fiber is in fact an abelian variety. We let $B^\circ \subset B$ be a dense Zariski open smooth subset over which the base-change $f^\circ : X^\circ \to B^\circ$ is smooth. By the period map of f we mean the period map $\phi : B^\circ \to M$ to an appropriate moduli space M of polarized abelian varieties associated to the natural variation of (polarized) weight one integral Hodge structures on B° with underlying local system $R^1 f^\circ_* \mathbb{Z}_{X^\circ}$. We say f is *isotrivial* if the period map is trivial (equivalently if $R^1 f^\circ_* \mathbb{Z}_{X^\circ}$ has finite monodromy) and *of maximal variation* if the period map is generically finite.

Our main result is to resolve a conjecture of Matsushita:

Theorem 1. *Let X be an irreducible hyperkähler manifold (or more generally a primitive symplectic variety in the sense of* [1]*). Then any Lagrangian fibration $f : X \to B$ is either isotrivial or of maximal variation.*

Both possibilities in Theorem 1 occur for K3 surfaces S—see for example [4, Chapter 11]—and therefore also for their Hilbert schemes $S^{[g]}$ in each (even) dimension. Primitive symplectic varieties are the natural singular analog (as far as deformation theory is concerned) of irreducible hyperkähler manifolds; for example, the irreducible holomorphic symplectic varieties that appear in the singular Beauville–Bogomolov decomposition theorem are primitive symplectic varieties.

Theorem 1 has previously been treated in two main contexts. First, the Beauville–Mukai system of an ample divisor on a K3 surface has been shown to be of maximal variation in many cases by Ciliberto–Dedieu–Sernesi [2] by studying the extendability of a canonically embedded curve to a K3 surface (where in fact the period map is shown to be quasifinite) and by Dutta–Huybrechts [3] by understanding the derivative of the period map. In particular, Dutta–Huybrechts show that Theorem 1 implies a complete answer:

Corollary 1. *Let H be a basepoint-free ample divisor on a K3 surface S. Then the complete linear system $|H|$ is of maximal variation.*

Proof. The genus 2 case is proven unconditionally in [3, Prop. 5.4], and the genus $g \geq 3$ case in [3, Prop. 5.2] assuming Theorem 1. □

Second, van Geemen and Voisin have proven Theorem 1 generically for $b_2 \geq 7$. More precisely, let $T_0 \subset H^2(X, \mathbb{Q})$ be the rational transcendental lattice, namely, the smallest rational Hodge substructure containing $[\sigma] \in H^{2,0}(X)$. Assuming that X is projective, T_0 has generic (special) Mumford–Tate group (namely $\mathbf{SO}(T_0, q_X)$, where q_X is the Beauville–Bogomolov–Fujiki form), and $\mathrm{rk} T_0 \geq 5$, van Geemen and Voisin [7, Theorem 5] show that any fiber of a Lagrangian fibration that is not of maximal variation must be a factor of the Kuga–Satake variety of T_0, hence locally constant. Their result in particular applies to the very general projective deformation of $f : X \to B$ assuming $b_2(X) \geq 7$, which includes all known deformation types.

The argument of van Geemen–Voisin therefore eventually relies on the largeness of the generic Mumford–Tate group. We adapt their proof to prove Theorem 1 by replacing this input with the near simplicity of the complex variation of Hodge structures on $R^1 f^\circ_* \mathbb{C}_{X^\circ}$ which holds without any genericity assumption. Simply put, an argument of Voisin shows that the *real* variation of Hodge structures $R^1 f^\circ_* \mathbb{R}_{X^\circ}$ is irreducible. On the one hand, this puts severe restrictions on the splitting type of the variation over $\mathbb{C}$. On the other hand, as in van Geemen–Voisin, a nontrivial generic fiber of the period map produces an unexpected splitting of $T^\vee \otimes R^1 f^\circ_* \mathbb{R}_{X^\circ}$ where T is the real transcendental lattice, and this forces the variation to be isotrivial. We note in passing that the irreducibility of $R^1 f^\circ_* \mathbb{R}_{X^\circ}$ has a number of other implications, for example to classifying monodromy representations in the isotrivial case.

We further deduce from Theorem 1 a result on torsion points of sections:

Corollary 2. *Let X be a primitive symplectic variety and $f : X \to B$ a Lagrangian fibration. Let L be a line bundle whose restriction to the smooth fibers is topologically trivial. Then the set of points $b \in B^\circ$ for which $L|_{X_b}$ is torsion is analytically dense in B.*

Corollary 2 was proven by Voisin [8, Theorem 1.3] assuming either f is of maximal variation and $\dim X \leq 8$ or isotrivial with no restriction on the dimension.

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Constant Scalar Curvature metrics on Algebraic Manifolds

Sean Timothy Paul

Let $(X, \mathbb{L})$ be a polarized complex manifold. Let h be a Hermitian metric on $\mathbb{L}$ with positive curvature $(1,1)$ form $\omega := -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(h)$. As usual, $\mathcal{H}_\omega$ denotes the Kähler metrics in the class $[\omega]$, and

$$\nu_\omega : \mathcal{H}_\omega \to \mathbb{R}$$

denotes Mabuchi's *K-energy map.* Recall that the Mabuchi is *proper* iff there are positive constants δ, C such that

$$\nu_\omega(\varphi) \geq \delta J_\omega(\varphi) - C$$

for all $\varphi \in \mathcal{H}_\omega$.

Theorem A. [6] $(X, \mathbb{L})$ *is asymptotically semi-stable iff ν_ω is bounded below.*

Theorem B. [6] $(X, \mathbb{L})$ *is asymptotically stable (with stability exponent m) iff ν_ω is proper with coefficient $\delta = \frac{1}{m}$.*

The following (difficult) result is due to Tian in the Fano case (1997) and Chen-Cheng in general.

Theorem. [4],[5] *Assume that $Aut(X)$ is finite. Then there is a cscK metric ω_φ in $[\omega]$ iff $\nu_\omega : \mathcal{H}_\omega \to \mathbb{R}$ is proper.*

Corollary. *Let $(X,\mathbb{L})$ be a polarized complex manifold with finite (reduced) automorphism group. Let h be a Hermitian metric on $\mathbb{L}$ with positive curvature form ω. Then there is a cscK metric $\omega_\varphi \in \mathsf{c}_1(\mathbb{L})$ iff $(X,\mathbb{L})$ is asymptotically stable.*

The assumption that $(X,\mathbb{L})$ is polarized allows us to exploit the *Kodaira embeddings ι_k* furnished by (unitary) bases $\{S_0, S_1, \ldots, S_{N_k}\}$ of $H^0(X,\mathbb{L}^k)$ for large k.

$$X \ni p \to \iota_k(p) := [S_0(p) : \cdots : S_{N_k}(p)] \in \mathbb{P}(H^0(X,\mathbb{L}^k)^\vee)$$

For $\sigma \in SL(N_k+1,\mathbb{C})$ we define $\frac{1}{k}\Psi_{\sigma,k} \in \mathcal{H}_\omega$ by

$$\Psi_{\sigma,k} := \log \sum_{i=0}^{N_k} |\sigma \cdot S_i|^2_{h^k} \text{ where } \sigma \cdot S_i := \sum_j \sigma_{ji} S_j \ .$$

The punch line is that for **each** $k >> 0$ we have a map

$$SL(N_k+1,\mathbb{C}) \to \mathcal{H}_\omega \quad , \ \sigma \to \frac{1}{k}\Psi_{\sigma,k} \ .$$

The image of this map (usually denoted by $\mathcal{B}_k \subset \mathcal{H}_\omega$) serves as a concrete supply of "test potentials". These finite dimensional spaces of metrics are called *Bergman metrics* of "level" k.

Theorem. [1]

$$\overline{\bigcup_{k>>0} \mathcal{B}_k} = \mathcal{H}_\omega \quad \textit{closure in } C^2 \textit{ topology.}$$

This shows that the $\mathcal{B}_k$ are quite plentiful.

Key Corollary.

$$\inf_{\varphi \in \mathcal{H}_\omega} \nu_\omega(\varphi) = \inf_{k>0} \ \inf_{\sigma \in \ G} \nu_\omega\left(\frac{1}{k}\Psi_{\sigma,k}\right)$$

Tian's 1990 proposal (see [1] pg. 101 top paragraph) was to relate the properness of the *Calabi functional* restricted to $\mathcal{B}_k$ to the stability of

$$X \subset \mathbb{P}^{N_k}$$

in the Chow variety of degree d dimension n algebraic cycles in $\mathbb{P}^{N_k}$. This is essentially correct, but we will use the Mabuchi energy instead of the Calabi energy, and most importantly, the Hilbert-Mumford theory of (semi)stability must be extended to ***two*** representations.

Let $X \subset \mathbb{P}^N$ be smooth & linearly normal with degree d. Recall that for any $p \in X$ that the embedded tangent space to X at p is the n dimensional *projective* linear subspace

$$\mathbb{T}_p(X) \in \mathbb{G}(n \ , \ \mathbb{P}^N)$$

obtained by projectivizing the tangent space the the cone over X at any point $v \in \mathbb{C}^{N+1} \setminus \{0\}$ lying over p.

Given any $0 \le j \le n$ we define the following subvariety $Z_j(X)$ of the Grassmannian $\mathbb{G}_j := \mathbb{G}(N-(j+1), \mathbb{P}^N)$

$$Z_j(X) := \{E \in \mathbb{G}_j \ | \exists \ p \in X \cap E \ \& \dim(E \cap \mathbb{T}_p(X)) \ge n-j\} \ .$$

Generally $Z_j(X)$ has ***codimension one*** in $\mathbb{G}_j$ for all $0 \leq j \leq n$ and is therefore cut out by a (homogeneous) polynomial in the Plucker coordinates on $\mathbb{G}_j$. To make the defining polynomial of $Z_j(X)$ concrete we view $\mathbb{G}_j$ in Stiefel coordinates by observing that there is a dominant map

$$M^o_{(j+1)\times(N+1)} \ni A \to \pi(\texttt{ker}(A)) \in \mathbb{G}_j \ .$$

The superscript o denotes matrices of maximal rank.

We may then consider the divisor

$$Z^j(X) := \overline{\pi^{-1}(Z_j(X))} \subset W_j := M_{(j+1)\times(N+1)} \ .$$

$Z^j(X)$ is an irreducible algebraic hypersurface of degree d_j in the affine space W_j with defining $SL(j+1)$ invariant polynomial

$$\Delta_j(X) = \Delta_j \in \mathbb{C}_{d_j}[W_j] \quad , \quad \mathrm{div}(\Delta_j) = Z^j(X).$$

Note that Δ_j is given only up to $\mathbb{C}^*$.

We associated two divisors $Z^n(X)$ and $Z^{n-1}(X)$ cut out by irreducible polynomials Δ_n and Δ_{n-1} respectively. Following conventional notation we write

$$R_X := \Delta_n \quad \& \quad \Delta_X := \Delta_{n-1} \ .$$

It is not hard to show that the degrees are given as follows

$$\deg Z^n(X) = (n+1)d$$

$$\deg Z^{n-1}(X) = n(n+1)d - d\mu$$

Therefore we have

$$R_X \in \mathbb{C}_{d(n+1)}\left[M_{(n+1)\times(N+1)}\right]^{SL(n+1,\mathbb{C})}$$

$$\Delta_X \in \mathbb{C}_{n(n+1)d-d\mu}\left[M_{n\times(N+1)}\right]^{SL(n,\mathbb{C})} \ .$$

We must "normalize the degrees" of these polynomials.

$$(R,\Delta) := \left({R_X}^{\deg(\Delta_X)} \ , \ {\Delta_X}^{\deg(R_X)}\right) \ .$$

Definition. $X \subset \mathbb{P}^N$ *is* ***semistable*** *if and only if the orbit closures are disjoint inside the projectivization of the sum of the two obvious G representations*

$$\overline{\mathcal{O}}_{R\Delta} \cap \overline{\mathcal{O}}_R = \emptyset \ .$$

We have defined

$$\mathcal{O}_{R\Delta} := G \cdot \left[\left(R_X^{\deg(\Delta_X)} \ , \ \Delta_X^{\deg(R_X)}\right)\right]$$

$$\mathcal{O}_R := G \cdot \left[\left(R_X^{\deg(\Delta_X)} \ , \ 0\right)\right] \ .$$

Definition. *$X \subset \mathbb{P}^N$ is **stable** if and only if the pair (R, Δ) is stable for the action of G. Explicitly, there is an integer $m \geq 2$ such that the pair*

$$\left(I^q \otimes R_X^{(m-1)\deg(\Delta_X)}\ ,\ \Delta_X^{m\deg(R_X)}\right)$$

is semistable for the action of G and $q = \deg(R_X)\deg(\Delta_X)$.

Definition. *A polarized manifold $(X, \mathbb{L})$ is **asymptotically semistable** if and only if the log of the minimal distance between the orbit closures is finite. Precisely*

$$\inf_{k>0} \frac{1}{k^{2n}} \log\tan \mathtt{dist}_0(\overline{\mathcal{O}}_{R\Delta}, \overline{\mathcal{O}}_R) > -\infty\ .$$

Definition. *A polarized manifold $(X, \mathbb{L})$ is **asymptotically stable** with exponent m iff the log of the minimal distance between the m-perturbed orbit closures is finite. Precisely*

$$\inf_{k>0} \frac{1}{k^{2n+1}} \log\tan \mathtt{dist}_0(\overline{\mathcal{O}}_{vw}, \overline{\mathcal{O}}_v) > -\infty\ .$$

$$(v, w) := \left(I^q \otimes R_X^{(km-1)\deg(\Delta_X)}\ ,\ \Delta_X^{km\deg(R_X)}\right)$$

We refer the reader to [6] for a complete account of semi-stability/stability.

At last, the connection between the Mabuchi energy and (semi)stability is established through the following two propositions, which imply Theorems A & B.

Proposition 1. [6] *For any polarized manifold $(X, \mathbb{L})$ and any large k embedding $\iota_k(X) \subset \mathbb{P}^{N_k}$ the infimum of the Mabuchi energy restricted to $\mathcal{B}_k$ is given by*

$$\inf_{\sigma\in G} \nu_\omega\left(\frac{1}{k}\Psi_{\sigma,k}\right) = \frac{k^{-2n}}{(n+1)} \log\tan \mathtt{dist}_0(\overline{\mathcal{O}}_{R\Delta}, \overline{\mathcal{O}}_R) + o(1)\ .$$

In particular $\iota_k(X)$ is semistable iff the Mabuchi energy is bounded below on $\mathcal{B}_k$.

Proposition 2. [6] *Let m be a positive integer. For any polarized manifold $(X, \mathbb{L})$ and any large k embedding we have*

$$\inf_{\sigma\in G}\left(m\nu_\omega\left(\frac{\Psi_{\sigma,k}}{k}\right) - \frac{\deg(\Delta_X)}{d} J_\omega\left(\frac{\Psi_{\sigma,k}}{k}\right)\right) = \frac{k^{-(2n+1)}}{(n+1)} \log\tan \mathtt{dist}_0(\overline{\mathcal{O}}_{vw}, \overline{\mathcal{O}}_v) + O(1).$$

We have defined the pair (v, w) and q as follows

$$(v, w) := \left(I^q \otimes R_X^{(km-1)\deg(\Delta_X)}\ ,\ \Delta_X^{km\deg(R_X)}\right)$$

$$q := \deg(R_X)\deg(\Delta_X)\ .$$

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Abelian differentials and their periods

Bruno Klingler

(joint work with Leonardo Lerer)

An abelian differential (or translation surface) is a pair (C, ω), where C denotes a smooth irreducible complex projective curve and $\omega \in H^0(C, \Omega^1_C) \setminus \{0\}$ is a non-zero algebraic one-form on C. Its periods are the complex numbers $\int_\gamma \omega$, for γ an element in the relative homology group $H_1(C^{\mathrm{an}}, Z(\omega); \mathbb{Z})$, where $Z(\omega)$ denotes the finite set of zeroes of ω. The moduli space $\mathbb{P}\Omega\mathcal{M}_g$ of abelian differentials $(C, [\omega])$ of genus g (up to scaling) admits a natural algebraic stratification $\{\mathrm{S}_\alpha\}_\alpha$, where the partition α of $2g-2$ describes the multiplicity of the zeroes of ω.

According to a remarkable theorem of Veech, the complex manifold $\mathrm{S}_\alpha^{\mathrm{an}}$ associated with S_α admits a natural *uniformization* in terms of periods. Let $x_0 := (C_0, [\omega_0]) \in \mathrm{S}_\alpha^{\mathrm{an}}$, let $\mathrm{V}_{\alpha,\mathbb{Z}}$ be the relative cohomology group $H^1(C_0^{\mathrm{an}}, Z([\omega_0]); \mathbb{Z})$ and $\mathrm{V}_\alpha := \mathrm{V}_{\alpha,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ the associated complex vector space. If U is a sufficiently small simply connected neighborhood of x_0 in $\mathrm{S}_\alpha^{\mathrm{an}}$, the holomorphic map $D_U : U \to \mathbb{P}\mathrm{V}_\alpha$ which to $x = (C, [\omega]) \in U$ associates the flat transport (along any path in U joining x to x_0) of the Betti cohomology class $[\omega]_{\mathrm{Betti}} \in \mathbb{P}H^1(C^{\mathrm{an}}, Z([\omega]); \mathbb{C})$ of $[\omega]$ is a bi-holomorphism from U onto an open subset of $\mathbb{P}\mathrm{V}_\alpha$, called a *period chart* for $\mathrm{S}_\alpha^{\mathrm{an}}$.

This defines an *integral projective structure* on $\mathrm{S}_\alpha^{\mathrm{an}}$, namely an atlas of charts with value in $\mathbb{P}\mathrm{V}_\alpha$ whose transition functions are locally constant elements of the integral projective linear group $\mathbf{PGL}(\mathrm{V}_{\alpha,\mathbb{Z}})$. These period charts of $\mathrm{S}_\alpha^{\mathrm{an}}$ are highly transcendental with respect to the algebraic structure of S_α: the maps are defined via the non-algebraic operations of parallel transport and integration. We would like to understand their transcendence properties, in the context of bi-algebraic structures as defined in [K17]:

- A bi-algebraic subvariety $W \subset \mathrm{S}_\alpha$ is by definition an irreducible closed algebraic subvariety W of S_α such that W^{an} is algebraic in the period charts: the relative periods of ω satisfy (up to scaling) exactly $\mathrm{codim}_{\mathrm{S}_\alpha} W$ independent algebraic relations over $\mathbb{C}$ when $(C,[\omega])$ ranges through W.

• A $\overline{\mathbb{Q}}$-bi-algebraic subvariety of S_α is a bi-algebraic subvariety $W \subset \mathrm{S}_\alpha$ such that both W^{an} (in the period charts) and W are defined over $\overline{\mathbb{Q}}$.

For instance, a $\overline{\mathbb{Q}}$-bi-algebraic point of S_α (also called an *arithmetic point*) is an abelian differential $(C,[\omega]) \in \mathrm{S}_\alpha(\overline{\mathbb{Q}})$ such that for one basis $(\gamma_i)_{0\leq i\leq d_\alpha}$ of the relative homology $H_1(C^{\mathrm{an}}, Z([\omega]);\mathbb{Z})$ (and then for any) the relative period line $\left[\int_{\gamma_0}\omega,\ldots,\int_{\gamma_{d_\alpha}}\omega\right]$ lies in $\mathbb{P}^{d_\alpha}(\overline{\mathbb{Q}})$.

A priori, the simplest bi-algebraic subvarieties of S_α are the *linear ones*: A linear subvariety $W \subset \mathrm{S}_\alpha$ is an irreducible closed algebraic subvariety such that W^{an} is (projectively) linear in the period charts. It is $\overline{\mathbb{Q}}$-linear if moreover both W and W^{an} (in the period charts) are defined over $\overline{\mathbb{Q}}$. A particular class of linear subvarieties have been studied in great depth by dynamicists in the last twenty years: the so-called *invariant* ones. The group $\mathbf{GL}^+(2,\mathbb{R})$ acts naturally on $\mathrm{V}_{\alpha,\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{C}$ by identifying $\mathbb{C}$ with $\mathbb{R}^2$ and extending the natural action of $\mathbf{GL}^+(2,\mathbb{R})$ on $\mathbb{R}^2$ coordinate-wise on $\mathrm{V}_{\alpha,\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{R}^2$. This action in period charts commutes with the one of $\mathbf{GL}(\mathrm{V}_{\alpha,\mathbb{Z}})$, hence descends to the natural principal $\mathbb{G}_{\mathrm{m}}$-bundle $\mathrm{H}_\alpha^{\mathrm{an}}$ over $\mathrm{S}_\alpha^{\mathrm{an}}$. A linear subvariety W of S_α which is the projection of a $\mathbf{GL}^+(2,\mathbb{R})$-invariant subvariety of $\mathrm{H}_\alpha^{\mathrm{an}}$ is said to be *invariant*. One easily shows that a linear subvariety W is invariant if and only if W^{an} is defined over $\mathbb{R}$ in the period charts. Prominent examples are the famous *Teichmüller curves*, which are $\overline{\mathbb{Q}}$-linear, see for instance [McM03a], [Mö06], [McM07].

In [Ler21] and [KL22], the authors develop for S_α the first steps of the bi-algebraic heuristic, which has been very useful for studying the diophantine properties of Shimura varieties. They completely characterize geometrically the arithmetic points of S_α. They prove that arithmetic points of rank 1 are dense in $\mathrm{S}_\alpha^{\mathrm{an}}$; but that, on the other hand, any algebraically primitive Teichmüller curve contains only finitely many arithmetic points as soon as $g \geq 3$. They prove in many cases that bi-algebraic subvarieties of S_α linear. Contrary to the hope expressed in [KL22], Deroin-Mattheus proved recently [DM] that there exist bi-algebraic subvarieties of S_α which are not linear.

A tantalizing open question is to prove the Ax-Schanuel conjecture for S_α and attack the following Zilber-Pink conjecture for S_α: The stratum S_α contains only finitely many arithmetic points of rank at least 3.

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ODD (Riemannian) metrics

Lukas Braun

The concept of Riemannian metrics on manifolds M has seen several generalizations. Apart from the so-called *Finsler metrics*, which are indeed norms on the tangent space T_pM, these are given by 2-tensors with 'relaxed properties'. The most important ones among them are probably the *pseudo- (or semi-)Riemannian* and in particular *Lorentz metrics*, which may be negative-definite in some direction but are everywhere nondegenerate. All these generalizations have motivations from certain fields, such as physics, control theory, etc..

A motivation from birational geometry: singular Kähler metrics. In the talk, we presented a new generalization of Riemannian metrics, called *ODD metrics*, introduced in [5]. Here, ODD stands for '**O**rthogonally **D**egenerating on a **D**ivisor', which stems from the initial motivation for defining these metrics: Kähler cone metrics with *cone angle* $\beta > 2\pi$.

Kähler metrics with cone angle $\beta < 2\pi$ along a *smooth divisor* D are far better understood and in fact important in the proof of the Yau-Tian-Donaldson conjecture as is evident from the titles of the work [3] and its successors.

A natural direction in complex birational geometry is to generalize statements about complex manifolds to well-behaved singular varieties, so called *klt pairs* (X,Δ). Kähler-Einstein metrics for example generalize to *singular Kähler-Einstein metrics* introduced in [2], where also the existence for $c_1(X,\Delta) \leq 0$ was proven. The equivalence of existence to *log K-polystability* in the case $c_1(X,\Delta) > 0$ and thus the singular version of the Yau-Tian-Donaldson conjecture was recently proven in [4].

While the existence is thus now settled, the *singularities* of these metrics are still not fully understood. The approach goes via solving a complex Monge-Ampère equation on a log-resolution $Y \to X$ with a possibly *degenerate and singular* right hand side *along the exceptional divisor*. This exceptional divisor is not smooth, but at least has simple normal crossings and the zeros and poles in the MA-equation correspond to cone angles that may be greater than 2π. Thus in contrary to the smooth case, in the singular case one has to deal with intersections of the prime divisors *and* with cone angles $> 2\pi$. In addition, further singularity may be introduced where the MA-equation degenerates.

One particular situation of interest is the case of $-(K_X + \Delta)$ nef on a klt pair. In this case, an MA-equation on the resolution provides us with a degenerate and singular Kähler metric with Ricci curvature bounded from below. We expect this metric to be an ODD metric at least in certain cases of interest. Developing Riemannian Geometry for ODD metrics thus would us enable to transfer statements about e.g. the fundamental group from the smooth case (i.e. manifolds M with $-K_M$ nef, see [1]) to the klt case.

The differential-geometric approach: ODD metrics. In order to do so, it makes sense to start with the most general definition and look how far we can get with it:

Definition 1 (temporary, experimental)**.** Let M be a real analytic manifold of dimension n. An *ODD metric* $\mathfrak{g}$ on M is an analytic section of $\mathrm{Sym}^2(T^*M)$, such that:

(1) $\mathfrak{g}_p$ is positive semidefinite for all $p \in M$.
(2) There is a finite collection $N_1, \dots, N_m$, $m \in \mathbb{N}$ of closed analytic submanifolds of M of *strictly smaller dimension*, such that $\mathfrak{g}_p$ is nondegenerate for $p \in M \setminus \bigcup N_j$.
(3) For each N_j, $1 \leq j \leq m$, the symmetric bilinear form $\mathfrak{g}_p^{N_j}$ induced by $\mathfrak{g}_p$ on TN_j is again an ODD Riemannian metric.

Here, note that we cannot demand the restrictions $\mathfrak{g}_p^{N_j}$ to be classical Riemannian since in general, they will at least degenerate at intersections with other N_i. Also, the experimental flavor of this introduction becomes apparent if we think about *how* the submanifolds N_j intersect. Definition 1 makes no assumptions here and in this generality, we still get an induced metric space structure on M. However, we expect a 'normal crossings property' to become important for statements from advanced Riemannian Geometry.

ODD vector fields and forms. In [5], we defined ODD orthonormal frames, vector fields, forms, raising and lowering of indices, etc.. Here, we try to give an intuitive account by considering the maybe simplest possible example.

The ODD metric $\mathfrak{g} = x^2 \mathrm{dx}^2$ on $M = \mathbb{R}$ degenerates at the origin. For this metric, starting with the frame $\partial_x = \frac{\partial}{\partial x}$, we obtain an *ODD orthonormal frame* E just by *normalizing* ∂_x *with respect to* $\mathfrak{g}$, which yields

$$E := \frac{\partial_x}{|\partial_x|_{\mathfrak{g}}} = \frac{1}{\sqrt{x^2}} \partial_x = \frac{1}{|x|} \partial_x.$$

This is not even a real-meromorphic, but an *algebroid* vector field. Thus, we define *ODD vector fields* X with respect to $\mathfrak{g}$ to be those algebroid vector fields on M that are of the form $X = f\,E$ with analytic f. Consequently, *ODD one-forms* are those that can be written as $\omega = g\varepsilon$ with analytic g in the *ODD conormal frame* $\varepsilon = |x|\,\mathrm{dx}$ defined by $\varepsilon(E) = 1$. In the standard frame, the raising and lowering of indices are thus given by

$$f\,\partial_x \mapsto x^2 f\,\mathrm{dx}, \quad g\,\mathrm{dx} \mapsto \frac{1}{x^2} g\,\partial_x$$

and map ODD vector fields to ODD one-forms and vice versa. We can further define higher differential forms, in particular an *ODD volume form*

$$dV_{\mathfrak{g}} = \varepsilon_1 \wedge \ldots \wedge \varepsilon_n = \sqrt{\det(\mathfrak{g}_{ij}}\, \mathrm{dx}_1 \wedge \ldots \wedge \mathrm{dx}_n$$

for an orthonormal coframe $(\varepsilon_i)_i$, which in our example amounts to $dV_{\mathfrak{g}} = \varepsilon = |x|\mathrm{dx}$. In particular, we see that compact regular domains will have nonzero volume with respect to this measure. Moreover, we are able to define ODD versions of the gradient, divergence, and Laplacian.

ODD curves and the ODD distance. An *ODD regular curve* should be a continuous curve $\gamma[a,b] \to M$, such that for some partition $a < a_1 < \ldots < b$, γ is analytic on each (a_j, a_{j+1}), and

$$\lim_{t \nearrow a_j} \frac{\dot\gamma(t)}{|\dot\gamma(t)|_{\mathfrak{g}}} = \lim_{t \searrow a_j} \frac{\dot\gamma(t)}{|\dot\gamma(t)|_{\mathfrak{g}}}.$$

An ODD regular curve through the origin in our example case is given by

$$\gamma\colon [-1,1] \to M = \mathbb{R}; \quad t \mapsto \operatorname{sgn}(t)\sqrt{|x|}.$$

It is easy to see in this one-dimensional example, but indeed true in full generality, that there is always a reparametrization of an ODD regular curve which is a classical regular curve.

It is now straightforward to use ODD regular curves to define an *ODD distance function* $d_{\mathfrak{g}}$ on M and we can finally prove:

Theorem 2. *Let $(M, \mathfrak{g})$ be a connected ODD Riemannian manifold and $d_{\mathfrak{g}}$ its distance function. Then $(M, d_{\mathfrak{g}})$ is a metric space and the metric topology associated to $d_{\mathfrak{g}}$ is the same as the topology of M as a manifold.*

Integrability of ODD vector fields and existence of ODD geodesics. We continue in [5] with defining the *ODD Levi-Civita connection*, Christoffel symbols etc. in analogy to the classical case. In our example case from above, we compute $\Gamma^1_{11} = \frac{1}{x}$. In particular, $\nabla_E E = 0$ but $\nabla_{\partial_x}\partial_x = \frac{1}{x}\partial_x$ is not even an ODD analytic vector field. We define ODD vector fields along curves, where we have to be careful in the case of curves inside the degeneracy locus. We see that at least a covariant derivative D_t along an ODD curve always gives a well-defined vector field along the curve.

Then we turn to investigate integrability of *ODD vector fields.* In [5], we give examples of *nonunique* integral curves through a point $p \in M$ of an ODD vector field X. However, we prove the following:

Theorem 3. *Let $(M, \mathfrak{g})$ be an ODD Riemannian manifold and $p \in M$. Let X be an ODD vector field. If either*

(1) p is a general point of M, or
(2) p is a general point of some $N_j \subseteq \mathcal{D}$, which is maximal in the sense that it is not contained in any other N_k,

then, there exists $\epsilon > 0$ and a unique ODD integral curve $\gamma\colon (-\epsilon, \epsilon) \to M$ with $p(0) = p$.

We also conjecture that the *existence* of integral curves holds for every $p \in M$ and *even uniqueness holds* in the sense that if γ and μ are two integral curves through p, then

$$\lim_{t\to 0} \frac{\dot\gamma(t)}{|\dot\gamma(t)|_{\mathfrak{g}}} \neq \lim_{t\to 0} \frac{\dot\mu(t)}{|\dot\mu(t)|_{\mathfrak{g}}}.$$

Along very similar lines we can prove the existence of *ODD geodesics* at general points of the degeneracy locus:

Theorem 4. *Let $(M, \mathfrak{g})$ be an ODD Riemannian manifold and $p \in M$. Let $v \in T_pM$ be a tangent vector at p. If either*

(1) p is a general point of M, or

(2) p is a general point of some $N_j \subseteq \mathcal{D}$, which is maximal in the sense that it is not contained in any other N_k,

then, there exists a unique ODD geodesic γ through p in direction v.

We finally conjecture that unique geodesics exist for all points $p \in M$ and tangent directions $v \in T_pM$.

Further directions. Since the ultimate application of ODD metrics we have in mind is in Kähler geometry of klt spaces, the following directions hopefully can be addressed in further works on ODD metrics:

(1) Give rigorous proofs of the mentioned conjectures about integrability of vector fields and existence of geodesics and develop large parts of Riemannian geometry for ODD metrics, as far as e.g. the *Bishop-Gromov volume comparison* or the *Margulis Lemma*.

(2) Carry over the concept of ODD metrics from manifolds to *orbifolds*. An envisioned ODD orbifold metric should be given in orbifold charts by ODD Riemannian metrics compatible with each other *and* the orbifold structure. In particular, the degeneracy locus of the metric and the codimension-one ramification locus of the orbifold can agree, which yields *arbitrary cone angles* $\beta \in \mathbb{Q}_{>0}$.

(3) Finally, show that solutions to complex Monge-Ampère equations with degenerate and singular right hand side are – maybe at least under additional assumptions – of ODD type.

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Liquid vector spaces for complex geometers

Johan Commelin

1. Introduction

In this talk I will give an exposition of condensed mathematics and liquid vector spaces developed by Dustin Clausen and Peter Scholze. I will not present work of my own; but all mistakes are mine. This talk will not give many details or precise definitions. But I hope that it will be an appetizer that provides the motivation sit down for an elaborate main course.

In this section I give a axiomatic introduction to condensed sets.

Fact 1.1. The category CHaus of compact Hausdorff spaces is a full subcategory of the category Cond of condensed sets. The category Cond has all limits and colimits.

These facts allow us to single out the "Hausdorff" condensed sets, which admit a rather elementary (and in my opinion, intuitive) description.

Definition 1.2. A monomorphism $X \to Y$ of condensed sets is *closed* if for every $K \in$ CHaus mapping to Y the pullback $X \times_Y K$ is compact Hausdorff.

Definition 1.3. A condensed set X is *separated* if the diagonal $X \to X \times X$ is closed.

1.4. A related notion in category theory is that of *quasiseparated* objects. For condensed sets, these notions turn out to be the same. The terminology *quasiseparated* prevails in the literature. Quasiseparated condensed sets are the same as *compactological spaces*, a notion introduced by Waelbroeck in the '70s.

Definition 1.5 ([4, Ch. III])**.** A *compactological space* is a set X equiped with a *compactology*, which consists of a topology and a bornology that are compatible in the way prescribed below. Recall that a bornology endows X with a collection of "small" subsets that satisfy the following conditions:

- every finite subset of X is small;
- finite unions of small subsets are small;
- subsets of small sets are small.

The topology and bornology form a compactology if they satisfy the following axioms:

- the closure of a small set is small;
- the closed small subsets are compact Hausdorff;
- the topology on X is the colimit topology of the closed small subsets.

A morphism of compactological spaces $X \to Y$ is a function that is continuous and sends small subsets of X to small subsets of Y.

Fact 1.6 ([3, Prop. 1.2])**.** The category of quasiseparated condensed sets is equivalent to the category of compactological spaces.

1.7. What about the condensed sets that are not quasiseparated? Their existence is very important for the whole theory: they are the reason that Cond has nice categorical properties, which in turn is the reason that we can unlock the tools from sheaf theory and homological algebra.

All such condensed sets are quotients of compactological spaces. If X is a compactological space and $E \subset X \times X$ is compactological subspace that is an equivalence relation, then we can form X/E as condensed set. If E is a closed equivalence relation, then X/E is again compactological. If it is not closed, then we get one of the "mystery" objects.

2. Liquid vector spaces

Fact 2.1 ([1, Thm 2.14])**.** Fix $0 < p \leq 1$. A qs condensed $\mathbb{R}$-vector space is *p-liquid* if for all $q < p$ and every compact $K \subset V$ there exists a compact *q-convex* subset of V containing K.

Definition 2.2. A compact $K \subset V$ is *q-convex* if for all $x_1, \dots, x_n \in K$ and all $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ with $\sum |\lambda_i|^q \leq 1$ we have $\sum \lambda_i x_i \in K$.

The general definition of a (non-qs) p-liquid vector space looks a bit different. I will not give that definition in this talk, but I strongly recommend taking a look a the first four lectures of [1], which contain a detailed account. Once again, the non-qs objects are quotients of the qs liquid vector spaces.

2.3. All complete locally convex topological vector spaces are p-liquid. In particular, all Banach spaces and Frechet spaces[1] are p-liquid. But there are many more p-liquid vector spaces. The category has very nice properties (which relies crucially on the fact that non-qs objects exist). Continuing our axiomatic approach, we list some of these properties below.

Fact 2.4 ([3, §VI])**.**

- The category of p-liquid vector spaces is an abelian category.
- It is a full subcategory of Cond($\mathbb{R}$), stable under all limits, colimits, and extensions.
- It has an internal Hom, and a tensor product, that are adjoint in the expected manner.
- The tensor product agrees with the tensor product of nuclear Frechet spaces. (Nuclear spaces are the objects in functional analysis where all "37" different topological tensor products agree.)
- There is a *liquidification* functor Cond($\mathbb{R}$) $\to \mathrm{Liq}_p$ which is left adjoint to the inclusion $\mathrm{Liq}_p \subset$ Cond($\mathbb{R}$).

[1] Recall that a Frechet space is a metrizable locally convex complete TVS, alternatively, it is a locally convex complete TVS whose topology is induced by a countable family of seminorms.

3. Quasicoherent liquid sheaves

3.1. Let U be some open subset of $\mathbb{C}^n$ (for the analytic topology). Then we can consider the ring of holomorphic function $\mathcal{O}(U)$ as condensed ring. It is liquid because it is a Frechet space. Thus it makes sense to speak of liquid $\mathcal{O}(U)$-modules, and hence of liquid $\mathcal{O}$-module sheaves.

Fact 3.2 ([1, Exc. 1 of §VI]). Consider an open subset $U \subset \mathbb{C}^n$ (for the analytic topology). Let $\mathcal{O}_U$ denote the structure sheaf (of holomorphic functions). A *quasicoherent* liquid sheaf, is a liquid $\mathcal{O}$-module sheaf $\mathcal{M}$ such that for every open polydisk $D \subset U$ the natural map

$$\mathcal{M}(D) \otimes_{\mathcal{O}(D)} \mathcal{O}|_D \to \mathcal{M}|_D$$

is an isomorphism.

3.3. In fact, one should really do all of this in the derived setting. And here it pays off to use the machinery of ∞-categories. One big benefit of working with ∞-categories, is that they can be glued. Indeed, the construction $U \mapsto C(U)$ is a sheaf of stable ∞-categories. (Being *stable* is the ∞-analogue of being an abelian category.) By gluing, we obtain a category C_X for any complex analytic space X. It can be viewed as the derived (∞-)category of *quasicoherent* liquid sheaves on X.

Another missing piece of the puzzle, that can now be filled in, is the exceptional pushforward functor. This functor turns out to be part of a six functor formalism. We omit a discussion of six functor formalisms from these notes. Instead we give some applications.

Theorem 3.4 (Serre duality, [1, Thm 13.6]). *Let $f\colon X \to Y$ be a smooth morphism of dimension d between complex analytic spaces. Then there is a natural isomorphism*

$$f^! M \cong f^* M \otimes_{\mathcal{O}_X} \Omega^d_{X/Y}$$

for $M \in C_Y$.

3.5. The proof consists of two components:

(1) The computation that $f^! \mathcal{O}_Y = \Omega^d_{X/Y}[d]$. This is done by deformation to the normal cone. The argument fits on one page.
(2) Abstract manipulations in the six functor formalism.

3.6. Several other fundamental results in complex geometry can have their proofs simplified by using the liquid machinery. Clausen and Scholze [1] reprove:

- Serre duality (as we saw above), generalized from coherent to quasicoherent sheaves.
- GAGA. Again, in the quasicoherent setting, generalizing the coherent case.
- Finiteness of coherent cohomology.
- Hirzebruch–Riemann–Roch.

Clausen also showed that the comparison isomorphism between algebraic and analytic de Rham cohomology can be established by formally reducing to the 1-dimensional cases of the disk and the punctured disk.

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Hodge-to-singular correspondence

Mirko Mauri
(joint work with Luca Migliorini, Roberto Pagaria)

1. Smooth vs singular character varieties

Let C be a compact Riemann surface of genus $g \geq 2$ with canonical bundle ω_C. The character variety of C, or *Betti moduli space*, is the affine quotient

$$\begin{aligned} M_{\mathrm{B}} &= \mathrm{Hom}(\pi_1(C), \mathrm{GL}_n) /\!\!/ \mathrm{GL}_n \\ &= \Big\{(A_1, B_1, \dots, A_g, B_g) \in \mathrm{GL}_n^{2g} \mid \prod_{j=1}^{g} [A_j, B_j] = 1_{\mathrm{GL}_n}\Big\} /\!\!/ \mathrm{GL}_n\,. \end{aligned}$$

It parametrizes isomorphism classes of semi-simple representations of the fundamental group of C. The entire study of nonabelian Hodge theory may be thought of as the study of the geometry of M_{B}. Character varieties are also central objects in various fields of mathematics like Teichmüller or knot theory, and play an essential role in the study of topological 3-manifolds or in the geometric Langlands program.

One of the main technical problems with working with character varieties is that M_{B} is often singular. Historically, to avoid dealing with singular moduli spaces, the moduli problem has been slightly modified or twisted to

$$M_{\mathrm{B}}(n, e) = \Big\{(A_1, B_1, \dots, A_g, B_g) \in \mathrm{GL}_n^{2g} \mid \prod_{j=1}^{g} [A_j, B_j] = e^{2\pi i \frac{e}{n}} 1_{\mathrm{GL}_n}\Big\} /\!\!/ \mathrm{GL}_n,$$

for some e coprime to n. Hausel and Thaddeus commented in [11, §1] that "the bait-and-switch is perhaps regrettable, but it is standard practice in the subject". Under this coprimality assumption, the cohomology of $M_{\mathrm{B}}(n, e)$ has been extensively studied in the last decades; see for instance [12, 9, 11, 10, 18, 7, 2, 18, 3, 15, 8]. However only recently studies about the (intersection) cohomology of M_{B}, denoted $IH(M_{\mathrm{B}}, \mathbb{Q})$, have started to emerge, and the attention of the mathematical community was driven back to the original moduli problem of M_{B}; see [5, 6, 13, 16, 17, 14].

The Hodge-to-singular correspondence shows how the cohomology of the smooth moduli spaces $M_{\mathrm{B}}(n, e)$, with $\gcd(n, e) = 1$, can be expressed in terms of the intersection cohomology $IH(M_{\mathrm{B}}(n', 0), \mathbb{Q})$ with $n' \leq n$; see also [1] for further advances in the field. In particular, this viewpoint suggests a unifying approach to the study of character varieties: $IH(M_{\mathrm{B}}(n,d),\mathbb{Q})$ should be studied simultaneously for all degree d, with no coprimality assumption. It is conceivable that the conjectures concerning the symmetry of their cohomology, like the P=W or the topological mirror symmetry conjecture, in degree $d = 0$ should imply the conjectures for arbitrary d.

In a different context, this viewpoint has already been successfully employed to compute the Hodge numbers of the exceptional O'Grady 10 example of compact hyperkähler manifolds, out of the Betti numbers of Hilbert schemes of five points on K3 surfaces; see [4]. Indeed, both spaces can be interpreted as special moduli spaces of sheaves on K3 surfaces corresponding to different degrees d (or rather Euler characteristics), and the computation of their Betti numbers can be reduced to determining how their cohomology depends on d.

2. Effective decomposition theorem for the Hitchin system

Remarkably, the Betti moduli space $M_{\mathrm{B}}(n,d)$, for any integer d, is homeomorphic to the *Dolbeault moduli space* $M_{\mathrm{Dol}}(n,d)$, parametrizing isomorphism classes of semistable Higgs bundles on C of degree d, i.e. pairs (E,ϕ) with E vector bundle of rank n and degree d, and $\phi \in H^0(C,\mathrm{End}(E)\otimes K)$; see [20]. The Dolbeault moduli space is equipped with a projective fibration called *Hitchin fibration*

$$\chi(n,d)\colon M_{\mathrm{Dol}}(n,d) \to A_n = \bigoplus_{i=1}^{n} H^0(C,\omega_C^{\otimes i}),$$

which assigns to (E,ϕ) the characteristic polynomial $\mathrm{char}(\phi)$ of the Higgs field ϕ. Since the pioneering work of Hitchin [12], the (non-algebraic) homeomorphism between $M_{\mathrm{B}}(n,d)$ and $M_{\mathrm{Dol}}(n,d)$ has been exploited to study the topology of character varieties. The decomposition theorem is a key tool to investigate the cohomology of $M_{\mathrm{Dol}}(n,d)$, alias $M_{\mathrm{B}}(n,d)$: it allows to decompose

$$IH(M_{\mathrm{B}}(n,d),\mathbb{Q}) = IH(M_{\mathrm{Dol}}(n,d),\mathbb{Q}) = H(A_n, R\chi(n,d)_*\,\mathrm{IC}_{M_{\mathrm{Dol}}(n,d)})$$

into building blocks, which are given by the cohomology of some perverse sheaves on A_n. Identifying the perserve sheaves that appear in this decomposition is a challenging task that we achieved on the locus A_n^{red} of reduced characteristic polynomials. In the following, we write that a partition $\underline{n} = \{n_i\}$ of n is d-integral if $n_i d/n \in \mathbb{Z}$.

Main Theorem (Hodge-to-singular correspondence)**.** *For any partition $\underline{n} = \{n_i\}$ of n, let $S_{\underline{n}} \subseteq A_n^{\mathrm{red}}$ be the closure of the polynomials whose irreducible factors have degree n_i, and $g_{\underline{n}}\colon P_{\underline{n}} \to S'_{\underline{n}}$ be the relative Jacobian of the simultaneous normalization of the spectral curves of equation* $\mathrm{char}(\phi) = 0$ *for all polynomials on a dense open set $S'_{\underline{n}} \subseteq S^\circ_{\underline{n}} = S_{\underline{n}} \setminus \bigcup_{\underline{m}\leq\underline{n}} S_{\underline{m}}$.*

Then there is an isomorphism in the derived category $D^b(A_n^{\mathrm{red}})$ of $\mathbb{Q}$-constructible complexes on A_n^{red}:

$$(1)\quad R\chi(n,d)_* \operatorname{IC}_{M(n,d)}|_{A_n^{\mathrm{red}}} \simeq \bigoplus_{\substack{\underline{n}=(n_i)\dashv n:\\ n_i d/n\in\mathbb{Z}}} \operatorname{IC}(S_{\underline{n}}, Rg_{\underline{n},*}\mathbb{Q}_{P_{\underline{n}}} \otimes \mathcal{L}_{\underline{n}}(d))[-2\mathrm{codim} S_n],$$

where $\mathcal{L}_{\underline{n}}(d)$ is a local system on $S_{\underline{n}}^{\circ}$ which coincides as $\pi_1(S_{\underline{n}}^{\circ})$-representation with the top homology of the geometric realization of the poset of non d-integral partitions $\underline{m} \geq \underline{n}$, up to a twist by the sign representation of $\pi_1(S_{\underline{n}}^{\circ}) = \mathrm{Aut}(\underline{n})$.

Further, if $d = e$ is coprime with n, $W_{\underline{n}}$ is a general normal slice through $M_{\underline{n}}(e)$, and $\widetilde{W_{\underline{n}}}$ a symplectic resolution of $W_{\underline{n}}$, then $\mathcal{L}_{\underline{n}}(e)$ is a local system with stalk $H^{\dim W_{\underline{n}}}(\widetilde{W_{\underline{n}}}, \mathbb{Q})$.

The result is a refinement of the celebrated Ngô support theorem; see [19, §7]. The main issue is to determine the support $S_{\underline{n}}$ of the decomposition theorem for $R\chi(n,d)_* \operatorname{IC}_{M(n,d)}$ and the mysterious local systems $\mathcal{L}_{\underline{n}}(d)$. The previous theorem shows that the following dichotomy holds: either the subvarieties $S_{\underline{n}}$ have a Hodge-theoretic nature as summands of $R\chi(n,d)_* \operatorname{IC}_{M(n,d)}$, which happens when $\underline{n}$ is not d-integral; or they are image of the stratum of a canonical Whitney stratification of $M_{\mathrm{Dol}}(n,d)$, which happens when $\underline{n}$ is d-integral.

This means that we have two extreme cases: the space $M_{\mathrm{Dol}}(n,0)$, i.e. $d=0$, is the most singular moduli space, but $R\chi(n,0)_* \operatorname{IC}_{M_{\mathrm{Dol}}(n,0)}|_{A_n^{\mathrm{red}}}$ has no proper summand; while when $\gcd(n,e)=1$, $M_{\mathrm{Dol}}(n,e)$ is smooth with the maximal number of summands of $R\chi(n,e)_* \operatorname{IC}_{M_{\mathrm{Dol}}(n,e)}|_{A_n^{\mathrm{red}}}$. Remarkably, the local systems $\mathcal{L}_{\underline{n}}(e)$ that appear in these new summands of $R\chi(n,e)_* \operatorname{IC}_{M_{\mathrm{Dol}}(n,e)}$ can be expressed in terms of the topology of the symplectic singularities $W_{\underline{n}}$. We could informally say that there exists a conservation law between Hodge-theoretic summands of the decomposition theorem and singular strata as the degree d varies. For these reasons we called our effective version of the decomposition theorem for Hitchin systems Hodge-to-singular correspondence.

It would be ideal and very useful for applications if we could drop the restriction to the locus of reduced spectral curve in (1). This would follow in particular from the following conjecture.

Conjecture (Full support)**.** *The complex $R\chi(n,0)_* \operatorname{IC}_{M_{\mathrm{Dol}}(n,0)}$ has full support on the whole basis A_n of the Hitchin system, i.e. no direct summand of it is properly supported on a subvariety of A_n.*

In other words, we ask whether the complex $R\chi(n,0)_* \operatorname{IC}_{M_{\mathrm{Dol}}(n,0)}$ is determined by the restriction to any open subset of A_n, in particular the locus over which the Hitchin fibration is smooth. It suggests that $IH(M_{\mathrm{Dol}}(n,0))$ should be regarded as a primitive indecomposable building block of the cohomology of any other Dolbeault moduli space $M_{\mathrm{Dol}}(n,d)$.

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Regularity of the volume function

Valentino Tosatti
(joint work with Simion Filip and John Lesieutre)

This talk was a report on ongoing work with Simion Filip and John Lesieutre, aimed at clarifying the regularity properties of the volume function near the boundary of the pseudoeffective cone.

Let X^n be a smooth projective variety over $\mathbb{C}$, and let $\mathcal{B}_X \subset N^1(X, \mathbb{R})$ be the cone of big classes inside the real Néron-Severi group. Its closure $\overline{\mathcal{B}_X}$ is the cone of pseudoeffective classes. The volume function

$$\mathrm{Vol} : \overline{\mathcal{B}_X} \to \mathbb{R}_{\geq 0}$$

is known to be continuous, zero precisely on $\partial\mathcal{B}_X$, and C^1 on $\mathcal{B}_X$ [1, 2, 3]. However, the regularity of Vol at points on $\partial\mathcal{B}_X$ remained mysterious.

Theorem 1. *There is a Calabi-Yau 3-fold X with a very ample divisor A and a pseudoeffective $\mathbb{R}$-divisor class $D \in \partial\mathcal{B}_X$ such that the function*

$$t \mapsto \mathrm{Vol}(D + tA)$$

is C^1 on $[0, +\infty)$ but not $C^{1,\alpha}$ on $[0, \varepsilon)$ for any small $\varepsilon, \alpha > 0$.

This answers negatively a question of Lazarsfeld [2].

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A gluing construction for surfaces with hyperbolic cusps

Hans-Joachim Hein

(joint work with Xin Fu, Xumin Jiang)

Consider the family of affine sextics

$$X_\sigma = \{(z_1^2 + z_1)^3 + (z_2^2 + z_2)^3 + (z_3^2 + z_3)^3 = \sigma\} \subset \mathbb{C}^3.$$

The projective closures $\overline{X}_\sigma \subset \mathbb{CP}^3$ are smooth for $0 < |\sigma| \ll 1$ but $\overline{X}_0$ is singular. Each singularity of $\overline{X}_0$ is locally isomorphic to the ordinary triple point

$$(*) \qquad (0,0,0) \in \{z_1^3 + z_2^3 + z_3^3 = 0\} \subset \mathbb{C}^3.$$

There are 8 such singularities in total, and they are equivalent under the action of $\mathrm{Aut}(\overline{X}_0)$. We have the following canonical Kähler metrics in this setting:

- On $\overline{X}_\sigma$, $0 < |\sigma| \ll 1$, we have a unique negative Kähler-Einstein metric $\omega_{KE,\sigma}$ provided by the Aubin-Yau theorem [1, 10] because $c_1(\overline{X}_\sigma) < 0$. This metric is fundamentally non-explicit.
- On the regular part of $\overline{X}_0$, i.e., away from the 8 singularities, we have a unique complete Kähler-Einstein metric $\omega_{KE,0}$ provided by a theorem of Kobayashi [5]. This metric is non-explicit deep in the interior of $(\overline{X}_0)^{reg}$ but was recently [2, 3] proved to be asymptotic to the model metric

  $$\omega_{cusp} = -3i\partial\overline{\partial}\log(-\log h), \;\; h = e^{-\varphi}(|z_1|^2 + |z_2|^2 + |z_3|^2),$$

  on the ordinary triple point singularity $(*)$. Here, φ is a smooth function on the elliptic curve $E \subset \mathbb{CP}^2$ over which $(*)$ is an affine cone, and $i\partial\overline{\partial}\varphi$ is the difference between $\omega_{FS}|_E$ and the unique flat Kähler metric representing the hyperplane class on E. The theorems on existence and asymptotics of [2, 3, 5] hold more generally on any variety of general type whose only singularities are cones over abelian varieties. The most obvious examples of such varieties are ball quotients of finite volume with their hyperbolic metrics, and indeed the model metric ω_{cusp} is hyperbolic. In fact, the key property of ω_{cusp} required in almost all of the proofs of [2, 3, 5] is that it has bounded holomorphic sectional curvature. However, ball quotients are never smoothable, so $\overline{X}_0$ is certainly not a ball quotient.

- Locally analytically near each singularity of $\overline{X}_0$, the smoothing family $\overline{X}_\sigma$ is isomorphic to the affine family
$$Y_\sigma = \{z_1^3 + z_2^3 + z_3^3 = \sigma\} \subset \mathbb{C}^3.$$
The projective closure $\overline{Y}_\sigma$ of this family in $\mathbb{CP}^3$ is a family of del Pezzo surfaces sharing the elliptic curve E as a common anticanonical hyperplane section and degenerating to the normal cone of this hyperplane section as $\sigma \to 0$. For any smooth Fano manifold $\overline{Y}$ and smooth divisor $D \subset \overline{Y}$ with $-K_{\overline{Y}} = [D]$, Tian-Yau [9] proved that the affine variety $Y = \overline{Y} \setminus D$ admits a complete (but not a priori unique or canonical) Calabi-Yau metric ω_{TY}. This was again based on the existence of a suitable model metric on the punctured normal bundle $N_{D/\overline{Y}} \setminus D$ given by a Calabi-type ansatz. In our setting, this Ricci-flat Calabi-type model metric has the explicit form
$$\omega_{Calabi} = i\partial\overline{\partial}(\log h)^{\frac{3}{2}}, \ \ h \text{ as above.}$$
It was proved in [4, 6] that ω_{TY}, while non-explicit in the interior of Y, is asymptotic to ω_{Calabi} near the compactifying divisor D in $\overline{Y} = Y \cup D$.

Let $m_\sigma : Y_\sigma \to Y_1$ denote the scaling map $m_\sigma(z) = \sigma^{-1/3}z$ and let ω_{TY} denote the Tian-Yau metric on the fixed affine cubic Y_1. The main result of our work is that $\omega_{KE,\sigma}$ is well-approximated by $|\log|\sigma||^{-3/2}m_\sigma^*\omega_{TY}$ near the vanishing cycles of the smoothing and by $\omega_{KE,0}$ on fixed compact subsets of $(\overline{X}_0)^{reg}$. The complete statement describes $\omega_{KE,\sigma}$ asymptotically as $\sigma \to 0$ everywhere on X_σ (there are 7 relevant regions). In particular, near the vanishing cycles we have that:

$$\exists\sigma_0 > 0 : \forall K \subset \mathbb{C}^3 \text{ compact}, \ \forall 0 < \alpha < \frac{1}{3}, \ \forall\varepsilon > 0 : \exists C < \infty : \forall 0 < |\sigma| \leq \sigma_0 :$$
$$\left\|\omega_{KE,\sigma} - |\log|\sigma||^{-\frac{3}{2}}m_\sigma^*\omega_{TY}\right\|_{C^{1,\alpha}\left(m_\sigma^{-1}(K),|\log|\sigma||^{-\frac{3}{2}}m_\sigma^*\omega_{TY}\right)} \leq C|\log|\sigma||^{-\frac{3}{4}(\frac{1}{3}-\alpha)+\varepsilon}.$$

The main ingredients of the proof of this result are:

- $|\log|\sigma||^{-3/2}m_\sigma^*\omega_{Calabi}$ is a reasonably good match to ω_{cusp} in regions of the form $|z| \sim |\sigma|^\beta$ ($z \in \mathbb{C}^3$) for any fixed $\beta \in (0, 1/3)$.
- While this matching is not sufficient for a gluing construction, it can be improved by introducing a new radial Kähler-Einstein model metric on $(*)$ that interpolates between ω_{Calabi} and ω_{cusp}.
- Standard machinery of the inverse function theorem in weighted Hölder spaces, similar to the machinery that was used e.g. in [8] for smoothings of ordinary *double* point singularities.

- Realizing that this machinery is obstructed due to the existence of an "approximate kernel" of the linearized Monge-Ampère operator near each singularity of $\overline{X}_0$, which consists of functions that interpolate between a nonzero constant on the Tian-Yau end and zero on the cusp.

Our way of dealing with this obstruction is to vary the Ricci potential of the glued metric by a constant before solving the Monge-Ampère equation. This forces us to assume that all the singularities of $\overline{X}_0$ are equivalent under $\mathrm{Aut}(\overline{X}_0)$ and it introduces the artificial restriction $\alpha < 1/3$ in the statement of the final estimate, where $\alpha < 1$ would be natural ($\omega_{KE,\sigma}$ cannot be $C^{1,1}$ close to a scaled pullback of ω_{TY} because they are Einstein metrics with different sign of the Einstein constant). We hope to remove these artificial restrictions in future work.

Regarding generalizations, cones over abelian varieties in higher dimensions are not smoothable [7], and cones over non-flat Calabi-Yau manifolds do not admit Kähler-Einstein model metrics of bounded curvature. However, there are other log-canonical surface singularities with model metrics of bounded curvature, see [2, 5], to which one could hope to apply a similar gluing construction.

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One forms without zeros

Stefan Schreieder

(joint work with Feng Hao and Ruijie Yang)

Tischler's theorem [Ti70] states that on a closed connected differentiable manifold X, the following conditions are equivalent:

(1) there is a closed real 1-form $\alpha \in A^1(X)_{\mathbb{R}}$ without zeros;
(2) there is a fibration over the circle $f : X \to S^1$.

Item (2) means that the manifold X is diffeomorphic to a quotient $F \times [0,1]/\sim$, where F denotes a fibre of f and $F \times 0$ is identified with $F \times 1$ via some diffeomorphism of F. It is clear that (2)⇒(1). To see the converse, let $\alpha \in A^1(X)_{\mathbb{R}}$ be a real closed 1-form without zeros. Up to perturbing α a little bit we may assume that the cohomology class $[\alpha] \in H^1(X,\mathbb{R})$ is rational. Up to multiplying α with a nonzero integer, we may then assume that $[\alpha] \in H^1(X,\mathbb{Z})$ is integral. But then for any base point $x_0 \in X$, the map

$$f : X \longrightarrow \mathbb{R}/\mathbb{Z}, \qquad x \mapsto \int_{x_0}^{x} \alpha$$

satisfies $df = \alpha$ and so it is a submersion because α has no zeros. That is, f yields a fibration over the circle, as we want.

It is natural to wonder if Tischler's theorem admits an analogue in the complex analytic setting. We are then asking for geometric characterizations of compact complex (or compact Kähler) manifolds that admit a holomorphic one-form without zeros. The naive analogue of Tischler's theorem fails in this context even for smooth complex projective varieties: there are threefolds X (e.g. a blow-up of the product $E_1 \times E_2 \times \mathbb{P}^1$ along $E_1 \times 0 \times 0 \cup 0 \times E_2 \times \infty$, where E_1, E_2 are general elliptic curves), that admit a holomorphic one-form without zeros but such that there is no smooth morphism from X to a complex torus of positive dimension.

Even though Tischler's theorem does not admit a direct analogue in complex algebraic geometry, Kotschick [Kot22] makes the following remarkable conjecture, which predicts that the question whether a compact Kähler manifold carries a holomorphic one-form without zeros depends only on the underlying differentiable manifold, hence is in some sense a topological property:

Conjecture 1 (Kotschick). *Let X be a compact Kähler manifold. Then the following are equivalent:*

(1) there is a holomorphic one-form without zeros;
(2) there is a closed real 1-form $\alpha \in A^1(X)_{\mathbb{R}}$ without zeros, or by Tischler's theorem equivalently, the underlying differentiable manifold admits a fibration over the circle $f : X \to S^1$.

If ω is a holomorphic one form without zeros, then its real and imaginary part cannot have any zeros, because $Re(\omega) \wedge Im(\omega)$ is a nonzero multiple of $\omega \wedge \bar{\omega}$ and the latter is nonzero at each point, because ω has no zero (indeed, ω and $\bar{\omega}$ have different types, hence must both vanish at a point where they are linearly dependent). In particular, (1) in Kotschick's conjecture implies (2) and the nontrivial part is the converse.

Kotschick observes that item (2) implies $\chi(X, \Omega_X^p) = 0$ for all p. This implies the conjecture in dimension one and one can use classification of surfaces to deduce it also in dimension two, see e.g. [Kot22, Sch21].

In [Sch21], it had been shown that (2) implies the following stronger condition, which generalizes the vanishing of the Euler characteristics $\chi(X, \Omega_X^p)$ mentioned above: there is a holomorphic one-form $\omega \in H^0(X, \Omega_X^1)$ such that the following complex is exact:

$$(1) \qquad \ldots \xrightarrow{\wedge\omega} H^{i-1}(X,\mathbb{C}) \xrightarrow{\wedge\omega} H^{i}(X,\mathbb{C}) \xrightarrow{\wedge\omega} H^{i+1}(X,\mathbb{C}) \xrightarrow{\wedge\omega} \ldots .$$

Moreover, the above complex remains exact if we replace X by a finite étale cover $\tau : X' \to X$ and ω by $\tau^*\omega$.

This implication of condition (2) comes from the following observations: the cohomology of the above complex agrees with the cohomology of the local system $L(\omega) := \ker(d + \wedge\omega) \subset \mathcal{O}_X$. To prove that the above complex is exact one then has to see that for some ω, $L(\omega)$ has no cohomology. By generic vanishing [GL87], one can reduce this to the statement that there is some complex rank one local system with trivial Chern class that has no cohomology. The latter can be produced explicitly by pulling back a general local system on S^1 via the fibration $f : X \to S^1$. (The fact that this has no cohomology follows from a Leray spectral sequence argument.)

The main result in [HS21] uses the exactness of (1) to settle Kotschick's conjecture for smooth projective threefolds.

Theorem 2 ([HS21])**.** *Let X be a smooth complex projective threefold. Then Kotschick's conjecture holds for X. More precisely, assume that the differentiable manifold that underlies X admits a fibration over the circle. Then up to (inverses of) blow-ups of elliptic curves that are not contracted via the Albanese morphism, one of the following holds:*

(1) there is a finite étale cover $X' \to X$ such that $X' \cong A \times Z$ admits a positive dimensional complex torus A as a factor such that A is not contracted to a point in $Alb(X)$.
(2) X is a smooth del Pezzo fibration $g : X \to E$ over an elliptic curve E.
(3) X is a conic bundle $g : X \to S$ over a surface S such that there is a one form $\omega \in H^0(S, \Omega^1_S)$ without zeros and such that $g^*\omega$ has no zeros on X.

The proof of this theorem relies on the minimal model program. It is natural to conjecture that the above result holds also true for compact Kähler threefolds.

We are also led to the expectation that a smooth complex projective variety of non-negative Kodaira dimension with a one form without zeros admits a birational model with an étale cover which admits a positive dimensional abelian variety as a factor. Both conjectures remain open for the moment.

Instead of concentrating on low dimensions, it is natural to consider the case where some other invariants of X are small. For instance, the smallest nontrivial value for $b_1(X)$ is 2, in which case Kotschick's conjecture predicts the following: Let X be a compact Kähler manifold whose underlying differentiable manifold admits a fibration over the circle. If $b_1(X) = 2$, then the Albanese map $X \to Alb(X)$ is smooth.

If $b_1(X) = 2$, then $Alb(X)$ is an elliptic curve and hence it is a simple complex torus. The main result in [SY22] yields then the following evidence in favour of Kotschick's conjecture (in loc. cit. the statement assumes that X is Kähler but the proof works more generally for an arbitrary compact complex manifold X).

Theorem 3 ([SY22]). *Let X be a compact complex manifold with a morphism $f : X \to A$ to a simple complex torus. Assume that there is a closed real 1-form $\alpha \in A^1(X)_{\mathbb{R}}$ without zeros such that $[\alpha] \in f^*H^1(A,\mathbb{R})$. Then $R^if_*\mathbb{Z}$ is a local system for all i.*

The above theorem says that the fibres of f have locally constant integral cohomology; the analogous result for rational cohomology has previously been proven in [DHL21]. The proof of the above theorem differs from the strategy in [DHL21] and relies on a characterization of simple perverse sheaves with vanishing Euler characteristic in [KW15] together with the generic vanishing results from Bhatt–Schnell–Scholze in [BSS18].

The example of elliptic fibrations with multiple fibres shows that there are non-smooth proper morphisms such that $R^if_*\mathbb{Q}$ is a local system for all i. Similar examples where $R^if_*\mathbb{Z}$ is a local system for all i are not known. In fact, the following conjecture of Bobadilla-Kollár [BK12] would imply smoothness of f in the above situation:

Conjecture 4 (Bobadilla–Kollár). *Let $f : X \to Y$ be a proper morphism between complex manifolds. Then f is smooth if and only if $R^if_*\mathbb{Z}$ is a local system for all i.*

The above theorem from [SY22] thus shows that Kotschick's conjecture in the case where $Alb(X)$ is simple follows from the Bobadilla–Kollár conjecture.

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Moduli of Foliations

Calum Spicer
(joint work with Michael McQuillan, Roberto Svaldi)

The main conjectures of the Minimal Model Program (MMP) imply that any variety may be decomposed into fibrations in atomic varieties with simple curvature properties, namely, Fano varieties ($c_1(X)$ is positive), Calabi-Yau varieties ($c_1(X) = 0$) and canonically polarised varieties ($-c_1(X)$ is positive). Even better, we expect these atomic varieties to fit into nice families of varieties which may be parametrised by corresponding moduli spaces. Realising this expectation for varieties of general type has been a major program of research since the late 1980s and is now essentially complete. We refer to [2] for a detailed treatment of the topic, but we remark here that the MMP in dimension $n + 1$ plays a central role in the construction of the moduli space for varieties of dimension n.

In recent years there has been an increasing body of work on the applications of the MMP to the study of foliations. In the case of foliations on surfaces, by work of Brunella, McQuillan and Mendes, foliated analogues of the main conjectures of the MMP have been proven. We are therefore interested in realising the second expectation of the MMP, namely, producing moduli spaces of foliations on surfaces, especially in the case of the foliations of general type. In light of recent developments on the MMP for foliations on threefolds, see [7, 8, 9, 10], many of the technical tools required for this are in place.

Our aim here is to describe some important examples which demosntrate some of the novel features of the moduli of foliations, survey some recent progress on moduli of foliations and to report on some work in progress with R. Svaldi and M. McQuillan.

1. Moduli of varieties

Before explaining the difficulties arising in producing a reasonable moduli theory for foliations, we briefly recall the moduli theory for varieties of general type. To define a moduli functor we need to specify the objects being parametrised as well as specifying families of these objects. The objects we are interested in parametrising are stable varieties, namely projective varieties X such that X is semi-log canonical and $K_X := -c_1(X)$ is ample. A family of stable varieties is given by a flat morphism $f\colon \mathcal{X} \to T$ such that $K_{\mathcal{X}/T}$ is $\mathbb{Q}$-Cartier and f-ample, and the fibres are all semi-log canonical.

As a result of the work of many mathematicians, e.g., [11], [12], [2], we have the following statement on the existence of moduli spaces for stable varieties.

Theorem 1.1. *Fix $n \in \mathbb{Z}_{>0}$ and $v \in \mathbb{Q}_{>0}$. The moduli functor of stable varieties of dimension n with $K_X^n = v$ is coarsely represented by a projective variety $M_{n,v}$.*

We remark that thanks to [1] any smooth variety X of general type admits a birational model $X \dashrightarrow X'$ such that X' is stable.

2. Examples

By a foliation $\mathcal{F}$ on a normal variety X we mean the data of a coherent subsheaf $T_{\mathcal{F}} \subset T_X$ such that

(1) $T_X/T_{\mathcal{F}}$ is torsion free; and
(2) $T_{\mathcal{F}}$ is closed under Lie bracket.

In general we will need to consider foliations on non-normal varieties, but the correct definition is technical and so we will not give a precise definition here.

We now present several examples which show how a moduli theory of foliations must diverge from the account of moduli of varieties provided above. To be precise, these examples will explain why defining a stable foliation as being one such that $K_{\mathcal{F}}$ is ample and $\mathcal{F}$ is semi-log canonical will not yield a good moduli theory.

Example 2.1. Let $(X, \mathcal{F})$ be a normal foliated surface such that $K_{\mathcal{F}}$ is big and nef. Suppose that there exists a cycle of rational curves $E \subset X$ which is invariant by $\mathcal{F}$ and such that $K_{\mathcal{F}} \cdot E = 0$. Such a confinguration of curves is called an elliptic Gorenstein leaf (e.g.l.). By a theorem of McQuillan, [3], it is known that $K_{\mathcal{F}}|_E \not\sim_{\mathbb{Q}} 0$ and therefore $K_{\mathcal{F}}$ is not semi-ample.

Such examples do in fact exist (take for example an appropriate ramified cover of a minimal resolution of the Bailey-Borel compactification of a Hilbert modular surface equipped with one of the tautological foliatons).

This example shows if given a foliation $\mathcal{F}$ of general type on a surface X, then finding a nice model on which $K_{\mathcal{F}}$ becomes ample is too much to hope for. Rather one has the following result due to [3].

Theorem 2.2. *Let X be a smooth projective surface and $\mathcal{F}$ a foliation of general type on X with canonical singularities. Then there exist a birational contraction $p\colon X \to Y$ to an algebraic surface (not necessarily projective) such that if $\mathcal{G} = p_*\mathcal{F}$ then*

(1) $K_{\mathcal{G}}$ is not necessarily $\mathbb{Q}$-Cartier, but it is ample in the sense of Mumford intersection theory, [13, §4.1]*;*
(2) $\mathcal{G}$ has canonical singularities (in a numerical sense); and
(3) Y has log canonical singularities (in particular K_Y is $\mathbb{Q}$-Cartier).

We call the model $(Y, \mathcal{G})$ guaranteed by Theorem 2.2 the canonical model of $(X, \mathcal{F})$.

When trying to compactify the moduli space of normal varieties, it is necessary to allow normal varieties to degenerate to non-normal varieties. Similarly, to compactify the moduli space of foliations it is necessary to consider foliations on non-normal surfaces. Here we see that it even too much to expect that $K_{\mathcal{F}}$ is big on every component of a degeneration.

Example 2.3. We believe it is reasonable to assert that for $g \geq 2$ the foliation induced by the fibration $\overline{M}_{g,1} \to \overline{M}_g$ between the moduli space of pointed stable curves and the moduli space of curves is a natural model of an algebraically integrable foliation, and that a Hilbert scheme of curves in $\overline{M}_g$, or some subscheme thereof, should represent some component of the moduli space of such foliated surfaces.

We now recall an example originally noticed by Keel. Assume $g \geq 3$ and fix a curve C of genus equal to 2 and another curve B with genus equal to $g-2$ and fix a point $b \in B$. We produce a family of stable curves $S \to C$ as follows: we glue $C \times B$ to $C \times C$ so that $C \times \{b\}$ is glued to the diagonal $\Delta \subset C \times C$. If $\mathcal{F}$ is the foliation induced by $S \to C$ then, on one hand, this *should* be a stable foliation. On the other hand, $K_{\mathcal{F}}|_{C\times C} = p^*K_C + \Delta$ where p is one of the projections, which is big and nef but not semi-ample, and, $K_{\mathcal{F}}|_{C\times B} = q^*K_B + C \times \{b\}$ where q is the projection onto B, which is nef but not big.

As observed before, fixing $K_X^{\dim X}$ suffices to get a bounded moduli problem, namely, the moduli functor can be represented by union finitely many quasi-projective varieties. However, for foliations more invariants need to be fixed as the following shows.

Example 2.4. There are examples due to Xiao, [6], of smooth morphisms $f_i : X_i \to C_i$, $i = 1, 2, \ldots$, where X_i is a surface, C_i is a curve and the fibres of f_i are curves of a fixed genus $g \geq 2$ such that $g(C_i)$ goes to infinity, but $K^2_{X_i/C_i} = v$ for some fixed $v \in \mathbb{Z}_{>0}$.

Finally, we record one other interesting behaviour of the moduli of foliations. If $X \to T$ is a flat family and $K_{X/T}$ is $\mathbb{Q}$-Cartier then the set

$$\{t \in T : X_t \text{ has canonical singularities.}\} \subset T$$

is open, [14]. This is not the case for foliations.

Example 2.5. Consider the family of foliations $\mathbb{A}^2_{x,y} \times \mathbb{A}^1_t \to \mathbb{A}^1_t$ defined by the vector field $x\partial_x + ty\partial_y$. For $t \in \mathbb{Q} \geq 0$ the foliation on the fibre over t has log canoincal but not canonical singularities, and for $tin\mathbb{C} \setminus \mathbb{Q}$ the foliation on the fibre over t has canonical singularities.

3. Stability and ϵ-adjoint stability

We now explain an idea, initially considered in work of J. V. Pereira and R. Svaldi, [5], which provides a potential way of addressing these idiosyncratic features of the moduli of foliations and which points the way to a good definition of the moduli functor for foliations. The main observation is that for $0 < \epsilon \ll 1$ the divisors $K_{\mathcal{F}} + \epsilon K_X$ where $K_{\mathcal{F}} := -c_1(T_{\mathcal{F}})$ are better behaved from the perspective of birational geometry.

We say a foliated surface $(X, \mathcal{F})$ is ϵ-stable provided $K_{\mathcal{F}} + \epsilon K_X$ is a $\mathbb{Q}$-Cartier ample divisor and $(X, \mathcal{F})$ is ϵ-adjoint log canonical. We do not define ϵ-adjoint log canonicity here, but only note that it depends on a natural generalisation of the discrepancy function to divisors of the form $K_{\mathcal{F}} + \epsilon K_X$. In [4] we prove two main results regarding ϵ-stable foliations on normal surfaces, namely existence and boundedness, cf. Examples 2.1 and 2.4.

Theorem 3.1. *There exists a universal $\tau > 0$ such that for all $0 < \epsilon < \tau$ the following holds. Let X be a smooth surface and $\mathcal{F}$ a foliation of general type then here exists an ϵ-stable model of $(X, \mathcal{F})$.*

Theorem 3.2. *Fix $0 < \epsilon < \tau$ and $v > 0$. The set of ϵ-stable models $(X, \mathcal{F})$ where X is normal and where $(K_{\mathcal{F}} + \epsilon K_X)^2 = v$ is bounded.*

4. Properness and separatedness

We now explain how to use these ideas to study the properness and separatedness of the moduli space of foliations.

Let $C = \mathrm{Spec}(A)$ where A is a DVR, let $c \in C$ be the unique closed point, let $\eta \in C$ be the generic point, and let $C^\circ = C \setminus \{c\}$. Let $(X^\circ, \mathcal{F}^\circ) \to C^\circ$ be a family of foliations of general type. Since we are particularly interested in compactifying the component corresponding to canonical models of foliations of general type on surfaces, we will assume that X° is normal.

Let us assume that on the generic fibre we have $K_{\mathcal{F}^\circ}$ is a big and nef $\mathbb{Q}$-Cartier divisor and that $\mathcal{F}^\circ$ has canonical singularities. Building off [4], for $0 < \epsilon \ll 1$ we are able to compactify $(X^\circ, \mathcal{F}^\circ)$ to a family of ϵ-stable models. Usually this compactification depends on the choice of ϵ, but we are able to show as $\epsilon \to 0$ that this compactification stabilises to a unique limit compactification $(X_{\mathrm{lim}}, \mathcal{F}_{\mathrm{lim}})$.

However, owing to the possible existence of e.g.l.s, we cannot in general assume that $K_{\mathcal{F}^\circ}$ is ample over the generic point of C. One of the key points is therefore understand this obstruction in families. Ideally, we would like to show that on the limit model $(X_{\mathrm{lim}}, \mathcal{F}_{\mathrm{lim}})$ that e.g.l.s move in a family.

Theorem 4.1 (Work in progress)**.** *Notation as above. Then there exists a divisor $E \subset X_{\mathrm{lim}}$ such that $E_\eta \subset X_\eta$ and $E_c \subset X_c$ are elliptic Gorenstein leaves, and any elliptic Gorenstein leaf in X_η or X_c is contained in E.*

Moreover, there exists a contraction $\pi \colon X_{\mathrm{lim}} \to Y/C$ which contracts E to a section of $Y \to C$.

We remark that owing to the failure of the basepoint free theorem the existence of the contraction $\pi \colon X_{\mathrm{lim}} \to Y$ in the category of algebraic spaces is a delicate problem, and must proceed by a very careful analysis of the structure of the degeneration.

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Gedodesic distance

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(joint work with Tamas Darvas and Chinh Lu)

Let X be a compact Kähler manifold of complex dimension n and fix a Kähler metric ω normalized such that $\int_X \omega^n = 1$. We then consider the set of ω-plurisubharmonic functions (ω-psh for short) denoted by $\mathrm{PSH(X,\omega)}$. We say that a function $u : X \to \mathbb{R}\cup\{-\infty\}$ is quasi-plurisubharmonic (qpsh) if locally $u = \rho+\varphi$ where φ is plurisubharmonic (psh) and ρ is smooth. A qpsh function u is ω-psh if $\omega + i\partial\bar{\partial}u \geq 0$ in the weak sense of currents.

Given a ω-psh function u, one can define the so called *non-pluripolar Monge-Ampère measure* $\omega_u^n := (\omega + i\partial\bar{\partial}u)^n$. When u is smooth, the latter is simply the wedge product n-times of the smooth positive $(1,1)$-form $\omega_u := \omega + +i\partial\bar{\partial}u$. When u has singularities, one is not allowed to do the wedge product of currents and the non-pluripolar Monge-Ampère measure was defined by Guedj and Zeriahi [3] as an increasing limit of positive Radon measures. It has to be emphasized that by construction we have

$$\int_X \omega_u^n \leq \int_X \omega^n = 1.$$

Also, still by construction the resulting measure $\omega_u^n := (\omega+i\partial\bar{\partial}u)^n$ does not charge pluripolar sets, i.e. $\omega_u^n(P) = 0$ for any P pluripolar set.

One can then wonder about the range of the Monge-Ampére operator. More precisely, given a positive measure μ on X can we find a ω-psh function φ such that $\mu = (\omega+i\partial\bar{\partial}\varphi)^n$? The answer is given by the following result we prove in [1]:

Theorem 1. *Let μ be a non-pluripolar positive measure with $\mu(X) = m, m \in (0,1]$. Then there exists a unique $\varphi \in \mathrm{PSH(X,\omega)}$ (normalized with $\sup_X \varphi = 0$) such that $\mu = (\omega + i\partial\bar{\partial}\varphi)^n$. Moreover $\varphi \in \mathcal{E}(X,\omega,\phi)$ where ϕ is a model potential* *with mass* $\int_X \omega_\phi^n = m > 0$ *and*

$$\mathcal{E}(X,\omega,\phi) := \left\{ u \in \mathrm{PSH}(X,\omega),\ u \le \phi + C, C > 0 \text{ and } \int_X \omega_u^n = \int_X \omega_\phi^n \right\}.$$

We recall that ϕ is called a *model potential* if it is the least singular function among those having fixed mass equal to m.

One can then wonder about the regularity of the solution φ in terms of the regularity of μ. The next result in [2] takes care of this:

Theorem 2. *The following are equivalent.*

(i) $\mu = (\omega + dd^c\varphi)^n$ *for some* $\varphi \in \mathcal{E}_\chi(X,\omega,\phi)$, *with* $\sup_X \varphi = 0$.

(ii) $\chi(|\phi - u|) \in L^1(\mu)$, *for all* $u \in \mathcal{E}_\chi(X,\omega,\phi)$.

Here $\mathcal{E}_\chi(X,\omega,\phi)$ are weighted energy classes defined as follows. A weight is a continuous increasing function $\chi : [0,\infty) \to [0,\infty)$ such that $\chi(0) = 0$ and $\chi(\infty) = \infty$. We also assume that the weight χ satisfies the following growth condition:

$$\forall t \ge 0,\ \forall \lambda \ge 1,\ \chi(\lambda t) \le \lambda^M \chi(t),$$

where $M \ge 1$ is a fixed constant. We then let $\mathcal{E}_\chi(X,\omega,\phi)$ denote the set of all $u \in \mathcal{E}(X,\omega,\phi)$ such that

$$E_\chi(u,\phi) := \int_X \chi(|u-\phi|)\omega_u^n < \infty.$$

In the particular case $\phi = 0$, the above result was proved by Guedj and Zeriahi [3] for a particular class of weights. They asked the question whether the same result would hold for a generic weight. Theorem 2 answers that.

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Kähler structures for holomorphic submersions

Chi Li

Let X, B be compact complex manifolds. Assume that there is a holomorphic submersion $\pi : X \to B$. We prove the following criterion for the existence of Kähler structures on X.

Theorem 1. *There is a Kähler metric on* X *if and only if the following two conditions are satisfied:*

(**Condition I**) *There is a class* $[Q] \in H^2(X,\mathbb{R})$ *that restricts to be a Kähler class on each fiber* $X_b = \pi^{-1}(b)$, $b \in B$.

(**Condition II**) B *is a Kähler manifold.*

As a corollary, we can answer positively a question by Harvey-Lawson ([7, Note D])

Theorem 2. *Assume that the fiber of the submersion has dimension 1. Then there exists a Kähler metric on X if and only if the homology class of any fiber of X is not zero.*

There is a more refined version of Theorem 1. To state it, recall that there is the Leray spectral sequence $\{E_r^{p,q}, d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}\}$ that converges to $H^{p+q}(X,\mathbb{R})$ (see [6]). There are isomorphisms:

$$E_2^{p,q} = H^p(B, R^k\pi_*\mathbb{R})$$

which is the sheaf cohomology with coefficient the local system $R^k\pi_*\mathbb{R}$.

Theorem 3. *There is a Kähler metric on X if and only if the following conditions are all satisfied:*
(**Condition Ia**) *There is an element $[\omega] \in H^0(B, R^2\pi_*\mathbb{R})$ restricts to be a Kähler class on X_b for any $b \in B$.*
(**Condition Ib**) $d_2[\omega] = 0$ *in* $E_2^{2,1} = H^2(B, R^1\pi_*\mathbb{R})$.
(**Condition II**) *B is Kähler.*

Theorem 3 generalizes an old result of Blanchard in [1] who considered a special class of isotrivial holomorphic submersions. It can also be thought as a converse to Blanchard-Deligne's result about the E_2-degeneration of the Leray spectral sequence of Kähler fibrations ([1, 3]). The proof of Theorem 3 generalizes Blanchard's construction of closed $(1,1)$-form guided by the Leray spectral sequence. The main new ingredient is a new $D'D''$-lemma, where D denotes the Gauss-Manin connection, based on Deligne's Hodge theory corresponding to the group $H^p(B, R^q\pi_*\mathbb{R})$.

Our motivation for proving Theorem 1 comes from a question of Li-Zhang [8] and Streets-Tian [10] about the existence of Hermitian-Symplectic (HS) structures on non-Kähler complex manifolds. Let X be a complex manifold and Q be a closed 2-form on X. By definition, Q is a Hermitian-Symplectic (HS) structure if Q satisfies the following two conditions.

(1) Q is a symplectic form. In other words, $Q^{\dim X}$ is non-vanishing and $dQ = 0$.
(2) If $Q = Q^{2,0} + Q^{1,1} + Q^{0,2}$ is the decomposition of Q into differential forms of type $(2,0)$, $(1,1)$ and $(0,2)$ respectively, then $Q^{1,1}$ is a positive definite $(1,1)$-form.

By adapting Blanchard-Deligne's argument for E_2-degeneration of Kähler fibrations, we can show that HS submersions always satisfy (**Condition Ib**). By applying Theorem 3 we get the following general result.

Proposition 4. *Let $\pi : X \to B$ be a holomorphic submersion with Kähler fibers and a Kähler base. Assume that X admits a Hermitian-Symplectic structure. Then the following conditions are equivalent:*

(1) X is Kähler.

(2) There exists $[\omega] \in H^0(B, R^2\pi_*\mathbb{R})$ that restricts to a Kähler class on each fiber F.
(3) The variation of Hodge structure $R^2\pi_*\mathbb{R}$ is polarizable.
(4) X satisfies the $\partial\bar{\partial}$-lemma.

Li-Zhang and Streets-Tian asked whether there are examples of HS structure on non-Kähler complex manifolds. The question is still open in general, though there are negative results which say that there are no such examples among complex surfaces ([8, 10]), nilmanifolds with invariant complex structures ([5]), twistor spaces ([11]), Moishezon manifolds and complex manifolds of Fujiki class ([9, 2]). See also [4, 12] for some analytic approach to the general problem. We use the above result (Proposition 4) to rule out HS structures for some classes of non-Kähler complex manifolds that admit structures of holomorphic submersions.

Theorem 5. *Let $\pi : X \to B$ be a holomorphic submersion with Kähler fibers and a Kähler base. Assume that there is a Hermitian-Symplectic structure on X. Let F denote a fiber of π. Then X must be Kähler if one of the following conditions is satisfied:*
(1) The holomorphic submersion is isotrivial.
(2) The fibers of π are complex tori (of possibly varying complex structures).
(3) The monodromy action of $\pi_1(B)$ on $H^2(F)$ is trivial.
(4) $H^{2,0}(F) = 0$.

We end with an open problem.

Problem: Extend the main results in this paper to more general holomorphic maps. In the case of Lefschetz fibrations, We expect that Zucker's generalization of Deligne's result in [13] and a careful treatment near the singular fibers should lead to a generalization of the Kählerian criteria in this paper.

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Hyperbolicity in presence of a large local system

Yohan Brunebarbe

Let X be a (non-necessarily smooth nor irreducible, but reduced) proper complex algebraic variety. Following Lang, we define the *special* subsets $\mathrm{Sp}_{alg}(X)$, $\mathrm{Sp}_{ab}(X)$, and $\mathrm{Sp}_h(X)$ of X as the union respectively of

- all (positive-dimensional) integral closed subvarieties not of general type;
- the images of all non-constant rational maps $A \dashrightarrow X$ with source an abelian variety A;
- all the entire curves of X, i.e. the image of all non-constant holomorphic maps $\mathbb{C} \to X$.

It is not clear from their definition whether these subsets are Zariski-closed in X. One easily check that the inclusions

$$\mathrm{Sp}_{ab}(X) \subset \mathrm{Sp}_{alg}(X) \tag{1}$$

and

$$\mathrm{Sp}_{ab}(X) \subset \mathrm{Sp}_h(X) \tag{2}$$

always hold.

In my talk, I discussed the following result.

Theorem 1 ([Bru22])**.** *Let X be a projective complex algebraic variety. Assume that X admits a large complex local system, i.e. a complex local system $\mathcal{L}$ such that for every normal projective complex algebraic variety Y equipped with a non-constant morphism $Y \to X$ the pull-back of $\mathcal{L}$ to Y is not trivial. Then,*

(1) The inclusions in (1) and (2) are equalities;
In such a case, the special subset is denoted $\mathrm{Sp}(X)$ without risk of confusion.
(2) $\mathrm{Sp}(X)$ is Zariski-closed in X;
(3) $\mathrm{Sp}(X) \neq X$ if and only if X is of general type. (A non-necessarily irreducible projective variety is said of general type if at least one of its irreducible component is.)

This result shows that a strong version of conjectures of Green-Griffiths [GG80] and Lang [Lan86] holds for projective varieties admitting a large complex local systems. Examples of such varieties include:

(1) Projective complex algebraic varieties admitting a finite morphism to an abelian variety. In that case, Theorem 1 follows from works of Bloch [Blo26], Ueno [Uen75], Ochiai [Och77], Kawamata [Kaw80] and Yamanoi [Yam15a].
(2) Projective complex algebraic varieties admitting a (graded-polarizable) variation of $\mathbb{Z}$-mixed Hodge structure with a finite period map. When in addition the Hodge structures are pure, then it follows from works of Griffiths and Schmid [GS69] that $\mathrm{Sp}_{alg}(X) = \mathrm{Sp}_{ab}(X) = \mathrm{Sp}_h(X) = \emptyset$.

In a nutshell, the proof of Theorem 1 consists in reducing the general case to these two special cases by using general structure results from non-abelian Hodge theory. A crucial input is also given by the following result, whose proof relies on Nevanlinna's theory.

Theorem 2 ([Bru23]). *Let $X \to S$ be a proper morphism between two complex algebraic varieties. If there exists an abelian scheme $A \to S$ and a finite S-morphism $X \to A$, then, for every $* \in \{alg, ab, h\}$, the set*

$$\mathrm{Sp}_*(X/S) := \bigcup_{s \in S} \mathrm{Sp}_*(X_s)$$

is Zariski-closed in X. In particular, the set of $s \in S$ such that $\mathrm{Sp}(X_s) = \emptyset$ (respectively $\mathrm{Sp}(X_s) \neq X_s$) is Zariski open in S.

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Fundamental groups of algebraic singularities

Joaquín Moraga

We work over the field of complex numbers $\mathbb{C}$. The goal of this note is to discuss some known results regarding fundamental groups of algebraic singularities and explain some recent developments in the topic. We are mostly interested in the following question.

Question 1. *Let $\mathcal{C}_n$ be a class of n-dimensional algebraic singularities. What fundamental groups can we find among the singularities in $\mathcal{C}_n$?*

In what follows, we write $\pi_1(X;x)$ for the fundamental group of the intersection of the smooth locus of X and a small analytic ball centered at x. If $(X;x)$ is an isolate singularity, then this group agrees with the fundamental group of the link $\mathrm{Link}(X;x)$. By the work of Milnor [11] and Durfee [5], the fundamental group stabilizes if the radius of the ball is small enough. Furthermore, we have the following statement.

Theorem 1. *Let $(X;x)$ be a normal algebraic singularity. The fundamental group $\pi_1(X;x)$ is finitely presented.*

Normal singularities. In [14], Mumford proved that the fundamental group of a normal surface singularity detects smoothness. Indeed, the following theorem holds.

Theorem 2. *Let $(X;x)$ be a normal surface singularity. The fundamental group $\pi_1(X;x)$ is trivial if and only if X is smooth at x.*

However, in [7, Chapter X], Grothendieck showed that this is far from being the case for higher dimensional algebraic singularities. This is a consequence of the so-called Grothendieck-Lefschetz hyperplane Theorem.

Theorem 3. *Let $(X;x)$ be an isolated local complete intersection singularity of dimension at least* 3. *The fundamental group $\pi_1(X;x)$ is trivial.*

The following statement, due to Kollár and Kapovich, gives a complete answer to Question 1 in the class of normal algebraic singularities.

Theorem 4. *Let G be a finitely presented group. There exists a normal isolated 3-fold singularity $(X_G;x)$ for which $\pi_1(X_G;x) \simeq G$.*

In [9, Theorem 2], Kollár and Kapovich construct a projective simple normal crossing surface S_G with fundamental group G. Then, by taking a cone over this surface and deforming it (to make the singularity isolated), they obtain the singularity $(X_G;x)$.

Quotient and toric singularities. Among normal singularities, finite quotient singularities (orbifold singularities) and toric singularities are the most common ones in the literature. In this direction, we have the following theorems that characterizes the fundamental groups of orbifold singularities. This theorem dates back to the work of Camille Jordan in the 1870's [8].

Theorem 5. *There exists a constant $c_0(n)$, only depending on n, satisfying the following. Let $(X;x)$ be an orbifold singularity of dimension n. Then $\pi_1(X;x)$ admits a finite normal abelian subgroup A of rank at most n and index at most $c_0(n)$.*

Furthermore, by recent work of Collins [3], we can take $c_0(n) = (n+1)!$ whenever $n \geq 71$. On the other hand, the fundamental groups of toric singularities have been understood by Cox, Little, and Schenck [4, Theorem 2.1.10].

Theorem 6. *Let $(X;x)$ be a n-dimensional toric singularity. Then the fundamental group $\pi_1(X;x)$ is finite abelian of rank at most n.*

The two previous theorems give a complete answer to Question 1 in the case of quotient and toric singularities.

Rational and Cohen-Macaulay singularities. In the case of rational singularities, we do not have a definitive answer for Question 1. However, due to the work of Kollár and Kapovich, we can understand the fundamental group of their dual complexes. The dual complex, denoted by $\mathcal{D}(X;x)$, is a combinatorial object that encodes the intersection of the exceptional divisors on a resolution.

Theorem 7. *Let $(X;x)$ be a rational singularity. Then $\pi_1(\mathcal{D}(X;x))$ is a $\mathbb{Q}$-superperfect group. Furthermore, for every finitely presented $\mathbb{Q}$-superperfect group G, there exists a 6-dimensional rational singularity $(X_G;x)$ for which $\pi_1(X_G;x) \simeq G$.*

On the other hand, in [10, Theorem 4], Kollár gives a complete answer for Question 1 in the case of Cohen-Macaulay singularities.

Theorem 8. *Let G be a finitely presented group. Then, the following statements are equivalent:*

- *The group G is $\mathbb{Q}$-perfect, and*
- *The group G is the fundamental group of a Cohen-Macaulay singularity.*

Log terminal singularities. Log terminal singularities are the singularities that appear in the minimal model program. The following theorem was proved by Braun, Filipazzi, Svaldi, and the author (see [2, Theorem 2]). It essentially says that the fundamental group of a log terminal singularity is similar to such of a quotient singularity.

Theorem 9. *There exists a constant $c_1(n)$, only depending on n, satisfying the following. Let $(X;x)$ be a n-dimensional log terminal singularity. Then $\pi_1(X;x)$ admits a finite normal abelian subgroup A of rank at most n and index at most $c_1(n)$.*

The proof of the previous theorem used local-to-global techniques from the Minimal Model Program and the boundedness of singular Fano varieties due to Birkar [1, Theorem 1.1]. In general, the constant $c_1(n)$ is not the same as $c_0(n)$ in Theorem 5. It is not clear yet how to obtain sharp bounds of $c_1(n)$. In [13, Theorem 7], the author shows that rank(A) is bounded above by the *regularity* of

$(X;x)$, i.e., the maximum dimension of $\mathcal{D}(X,\Gamma;x)$ where (X,Γ) is log canonical around x. In [12], the author studies how the existence of a large group A affects the geometry of the klt singularity $(X;x)$. [12, Theorem 1] states that whenever A is large (isomorphic to $\mathbb{Z}_m^n$ with m large compared with n) the singularity $(X;x)$ is similar to the deformation of a toric quotient singularity.

Log canonical singularities. Log canonical singularities appear often as limits of log terminal singularities. Hence, these singularities emerge naturally in moduli contexts. Although, some simple examples (for instance, the cone over an elliptic curve) show that the fundamental groups of lc singularities may not be finite. In [6], Figueroa and the author started a systematic study of the fundamental groups of log canonical singularities. In dimension 2, a complete description of the possible groups is given. More precisely, we have the following theorem (see [6, Theorem 2]).

Theorem 10. *Let $(X;x)$ be a log canonical surface singularity. Then, $\pi_1(X;x)$ is the extension of a solvable group of length at most 2 and a finite group of order at most 6.*

In the case of 3-dimensional lc singularities the group can be far from solvable. Indeed, every surface group appears as the fundamental group of a lc 3-fold singularity:

Theorem 11. *Let S be a Riemann surface without boundary and $G_S = \pi_1(S)$. There exists a 3-dimensional isolated lc singularity $(X_S;x)$ for which $\pi_1(X_S;x) \simeq G_S$.*

However, by studying the dual complexes of lc 3-fold singularities, Figueroa and the author proved that not every finitely presented group appears as the fundamental group of a lc 3-fold singularity (see [6, Theorem 7]). More precisely, we have the following statement about free groups.

Theorem 12. *Let $r \geq 2$. The free group F_r is not the fundamental group of an isolated lc 3-fold singularity.*

In [6, Section 4], Figueroa and the author develop a technique to construct lc singularities with interesting fundamental groups. This technique associates a $(n+1)$-dimensional lc singularity $(X;x)$ with $\pi_1(X;x) \simeq \pi_1(M)$ to each n-dimensional smooth manifold M. This technique is quite involved and requires tools from different topics in mathematics: polyhedral combinatorics, polyhedral complexes, toric geometry, and algebraic geometry. However, this technique can be proved to work for 3-manifolds that admit smooth embeddings into $\mathbb{R}^4$. Taking $M_r = \#_{i=1}^r (S^2 \times S^1)$, we obtain the following theorem.

Theorem 13. *Let $r \geq 1$. There exists a 4-dimensional isolated lc singularity $(X;x)$ for which $\pi_1(X;x) \simeq F_r$.*

This naturally leads to the following question.

Question 2. *Does any finitely presented group appear as the fundamental group of a 5-dimensional lc singularity?*

In a similar vein, we propose the following question.

Question 3. *Let M be a n-dimensional smooth manifold. When is M homotopic to the dual complex of a $(n+1)$-dimensional log canonical singularity?*

Finally, it is expected that log canonical rational singularities have better-behaved fundamental groups.

Question 4. *Let $(X;x)$ be an lc rational singularity. Is $\pi_1(X;x)$ virtually nilpotent?*

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Hodge theory and Lagrangian fibrations

Christian Schnell

The topic of my talk was the Hodge theory of Lagrangian fibrations on holomorphic symplectic Kähler manifolds. This setting includes both compact hyperkähler manifolds (such as elliptic K3 surfaces) and noncompact examples (such as the Hitchin fibration on the moduli space of stable Higgs bundles).

Not much is known about the singular fibers of Lagrangian fibrations. One can try to study their cohomology by applying the decomposition theorem; this produces certain perverse sheaves on the base of the Lagrangian fibration. In some cases, such as Ngô's support theorem [7], these perverse sheaves are controlled by what happens on the smooth locus, but in general, their behavior is a mystery. Two beautiful conjectures by Shen, Yin and Maulik [12, 6] try to shed some light on the perverse sheaves in the decomposition theorem, by relating them to the sheaves of holomorphic forms on the holomorphic symplectic manifold. They also predict some surprising symmetries, similar in spirit to the symmetries one finds among the Hodge numbers of a compact hyperkähler manifold.

Let M be a holomorphic symplectic manifold of dimension $2n$ that is Kähler but not necessarily compact, and let $\pi\colon M \to B$ be a Lagrangian fibration. In the talk, I explained how one can use Saito's theory of Hodge modules [8] and the BGG correspondence [1, 2] between graded modules over symmetric and wedge algebras to establish a relationship between the following two seemingly unrelated objects: the derived direct image $\mathbf{R}\pi_* \Omega_M^{n+i}$ of the sheaf of $(n+i)$-forms on M, and the i-th perverse sheaf P_i in the decomposition theorem for π. It turns out that we need to take the associated graded with respect to a certain filtration on both sides: in the case of $\mathbf{R}\pi_* \Omega_M^{n+i}$, this is the *perverse filtration* coming from the decomposition theorem; in the case of P_i, it is the *Hodge filtration* of P_i, viewed as a Hodge module. A special case of the main theorem is Matsushita's theorem [5], which says that

$$\mathbf{R}\pi_* \mathscr{O}_M \cong \bigoplus_{i=0}^{N} \Omega_B^i[-i].$$

Another general principle, which is already evident from the proof of Matsushita's theorem, is that the holomorphic symplectic structure and the Kähler structure together give rise to interesting symmetries.

I mentioned several applications of the main result. One is a relative Hard Lefschetz theorem for the action of the holomorphic symplectic form (which complements the usual relative Hard Lefschetz theorem for the action of the Kähler form); another one is a proof for the symmetry conjecture by Shen and Yin [12]. The main result also leads to a different proof for the "numerical perverse = Hodge" symmetry for compact hyperkähler manifolds [13] that does not rely on the existence of a hyperkähler metric. Perhaps the most useful feature of the present work is that no restrictions on the singular fibers are needed: the main theorem applies for example to the entire Hitchin fibration on the moduli space of semistable Higgs bundles (provided that the rank and the degree are coprime).

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Hyperbolicity and fundamental groups of quasi-projective varieties

Ya Deng
(joint work with Benoit Cadorel, Katsutoshi Yamanoi)

The concept of pseudo Picard hyperbolicity and pseudo Brody hyperbolicity has been introduced for complex algebraic varieties. A complex quasi-projective normal variety X is said to be pseudo Picard hyperbolic if there exists a proper Zariski closed subset $Z \subsetneqq X$ such that any holomorphic map $f : \mathbb{D}^* \to X$ from the punctured disk $\mathbb{D}^*$ with an essential singularity at the origin is contained in Z. Similarly, X is called pseudo Brody hyperbolic if there exists a proper Zariski closed subset $Z \subsetneqq X$ such that any non-constant holomorphic map $f : \mathbb{C} \to X$ is contained in Z. It is worth noting that pseudo Picard hyperbolicity implies pseudo Brody hyperbolicity, which is a weaker form of hyperbolicity.

Another concept that has been studied extensively is the notion of log general type. A variety X is said to be *strongly of log general type* if there exists a proper Zariski closed subset $Z \subsetneqq X$ such that any closed positive-dimensional subvariety V of X that is not of log general type is contained in Z.

In a recent paper by Cadorel, Yamanoi, and the reporter [2], the strong version of the Green-Griffiths-Lang conjecture has been studied for varieties that admit a big and reductive representation of their (topological) fundamental group $\pi_1(X)$. This conjecture states that the four hyperbolicity properties, namely, pseudo Picard hyperbolicity, pseudo Brody hyperbolicity, log general type, and strongly of log general type are equivalent for a given variety X. We were able to prove this conjecture for the aforementioned class of varieties.

Theorem 1 ([2, Theorem 0.4])**.** Let X be a complex smooth quasi-projective variety and $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a big and reductive representation. Then for any automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, the strong Green-Griffiths-Lang conjecture holds for the conjugate variety $X^\sigma := X \times_\sigma \mathbb{C}$, i.e. the following properties are equivalent:

(1) X^σ is of log general type.
(2) X^σ is strongly of log general type.
(3) X^σ is pseudo Picard hyperbolic.
(4) X^σ is pseudo Brody hyperbolic.

Recall that a representation $\varrho : \pi_1(X) \to G(\mathbb{C})$ is said to be *big*, or *generically large* in [10], if for any closed subvariety $Z \subset X$ containing a *very general* point of X, $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is infinite, where Z^{norm} denotes the normalization of Z. It is worth noting that a stronger notion of largeness exists, where ϱ is called *large* if $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is infinite for any closed subvariety Z of X.

We introduce four special subsets of X that measure the non-hyperbolicity locus from different perspectives.

Definition 2 (Special subsets)**.** Let X be a smooth quasi-projective variety.

(1) $\mathrm{Sp}_{\mathrm{sab}}(X) := \overline{\bigcup_f f(A_0)}^{\mathrm{Zar}}$, where f ranges over all non-constant rational maps $f : A \dashrightarrow X$ from all semi-abelian varieties A to X such that f is regular on a Zariski open subset $A_0 \subset A$ whose complement $A \backslash A_0$ has codimension at least two;
(2) $\mathrm{Sp}_{\mathrm{h}}(X) := \overline{\bigcup_f f(\mathbb{C})}^{\mathrm{Zar}}$, where f ranges over all non-constant holomorphic maps from $\mathbb{C}$ to X;
(3) $\mathrm{Sp}(X) := \overline{\bigcup_V V}^{\mathrm{Zar}}$, where V ranges over all positive-dimensional closed subvarieties of X which are not of log general type;
(4) $\mathrm{Sp}_{\mathrm{p}}(X) := \overline{\bigcup_f f(\mathbb{D}^*)}^{\mathrm{Zar}}$, where f ranges over all holomorphic maps from the punctured disk $\mathbb{D}^*$ to X with essential singularity at the origin.

Another strong version of the Green-Griffiths-Lang conjecture asserts that the four special subsets defined in Definition 2 should coincide. We establish this conjecture under the assumption that $\pi_1(X)$ admits a large and reductive representation, as stated in the following theorem.

Theorem 3 ([2, Theorem 0.6])**.** Let X be a smooth quasi-projective variety and $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a large and reductive representation. Then for any automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$,

(a) the four special subsets defined in Definition 2 are the same, i.e.,

$$\mathrm{Sp}(X^\sigma) = \mathrm{Sp}_{\mathrm{sab}}(X^\sigma) = \mathrm{Sp}_{\mathrm{h}}(X^\sigma) = \mathrm{Sp}_{\mathrm{p}}(X^\sigma).$$

(b) These special subsets are conjugate under automorphism σ, i.e.,

$$\mathrm{Sp}_\bullet(X^\sigma) = \mathrm{Sp}_\bullet(X)^\sigma,$$

where $\mathrm{Sp}_\bullet$ denotes any of Sp, $\mathrm{Sp}_{\mathrm{sab}}$, Sp_{h} or Sp_{p}.

(c) $\mathrm{Sp}(X^\sigma)$ is a proper Zariski closed subset of X^σ if and only if X is of log general type.

In [2], we also prove the following result:

Theorem 4 ([2, Theorem 0.1]). Let X be a complex quasi-projective normal variety and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a big representation such that the Zariski closure of $\varrho(\pi_1(X))$ is a semisimple algebraic group. Then, for any automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, the variety X^σ is strongly of log general type and pseudo Picard hyperbolic.

We remark that the condition in Theorem 4 is sharp. Theorem 4 are new even in the case where X is projective. When the variety X in Theorem 4 is projective, Campana-Claudon-Eyssidieux [5, Theorem 1] proved that X is of general type and Yamanoi [12, Proposition 2.1] proved that X does not admit Zariski dense entire curves $f : \mathbb{C} \to X$.

It is noteworthy that the condition of bigness for the representations ϱ in Theorem 4 is not particularly restrictive, as demonstrated by the following result:

Corollary 5 ([2, Corollary 0.2]). *Let X be a complex quasi-projective normal variety and let G be a semisimple algebraic group over $\mathbb{C}$. If $\varrho : \pi_1(X) \to G(\mathbb{C})$ is a Zariski dense representation, then there exist a finite étale cover $\nu : \widehat{X} \to X$, a birational and proper morphism $\mu : \widehat{X}' \to \widehat{X}$, a dominant morphism $f : \widehat{X}' \to Y$ with connected general fibers, and a big and Zariski dense representation $\tau : \pi_1(Y) \to G(\mathbb{C})$ such that*

(a) $f^*\tau = (\nu \circ \mu)^*\varrho$.

(b) *the variety Y is pseudo Picard hyperbolic and strongly of log general type.*

In particular, X is neither weakly special nor Brody special.

Note that by Campana [4], a quasi-projective variety X is *weakly special* if for any finite étale cover $\widehat{X} \to X$ and any birational modification $\widehat{X}' \to \widehat{X}$, there exists no dominant morphism $\widehat{X}' \to Y$ with Y a positive-dimensional quasi-projective normal variety of log general type. By [8] a quasi-projective variety is *Brody special* if it contains a Zariski dense entire curve.

Corollary 5 generalizes the previous work by Mok [11], Corlette-Simpson [6], and Campana-Claudon-Eyssidieux [5], in which they proved similar factorisation results.

On the other hand, Campana's abelianity conjecture [4, 11.2] predicts that a smooth quasi-projective variety X that is *special* or *Brody special* has a virtually abelian fundamental group. When a special variety X is projective, it is known that all linear quotients of $\pi_1(X)$ are virtually abelian (cf. [3, Theorem 7.8]). The same conclusion is valid for any Brody special smooth projective variety X (cf. [12, Theorem 1.1]). While it is natural to expect similar results for smooth quasi-projective varieties, we construct in [2] an example of a quasi-projective surface that is special and Brody special, whose fundamental group is linear and nilpotent but not virtually abelian. This provides a counterexample to Campana's conjecture in the general case. In the same work, we prove the following theorem:

Theorem 6 ([2, Theorem 0.8])**.** Let X be a special or Brody special smooth quasi-projective variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a linear representation. Then $\varrho(\pi_1(X))$ is *virtually nilpotent.*

To prove the above theorems, in [2] we develop new features in non-abelian Hodge theories in both archimedean and non-archimedean settings, geometric group theory, and Nevanlinna theory. Along the way, two difficult theorems are established, which are of significant interest in their own right. One such technique is a reduction theorem for Zariski dense representations $\varrho : \pi_1(X) \to G(K)$, where G is a reductive algebraic group defined over a non-Archimedean local field K.

Theorem 7 ([2, Theorem 0.11])**.** Let X be a complex quasi-projective manifold, and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a reductive representation where K is a non-archimedean local field. Then there exists a quasi-projective normal variety S_ϱ and a dominant morphism $s_\varrho : X \to S_\varrho$ with connected general fibers, such that for any connected Zariski closed subset T of X, the following properties are equivalent:

(a) the image $\rho(\mathrm{Im}[\pi_1(T) \to \pi_1(X)])$ is a bounded subgroup of $G(K)$.
(b) For every irreducible component T_o of T, the image $\rho(\mathrm{Im}[\pi_1(T_o^{\mathrm{norm}}) \to \pi_1(X)])$ is a bounded subgroup of $G(K)$.
(c) The image $s_\varrho(T)$ is a point.

When X is projective, this theorem was proved in [9, 7]. One of the building blocks of the proof of Theorem 7 is based on previous results by Brotbek, Daskalopoulos, Mese, and the reporter [1] on the existence of harmonic mappings to Bruhat-Tits buildings (an extension of Gromov-Schoen's theorem to quasi-projective cases) and the construction of logarithmic symmetric differential forms via these harmonic mappings.

Another significant building block is the following theorem.

Theorem 8 ([2, Theorem 0.13])**.** Let X be a quasi-projective variety. Assume that there is a morphism $a : X \to A$ such that $\dim X = \dim a(X)$ where A is a semi-abelian variety (e.g., when X has maximal quasi-Albanese dimension). Then the following properties are equivalent:

(a) X is of log general type.
(b) X is strongly of log general type.
(c) X is pseudo Picard hyperbolic.
(d) X is pseudo Brody hyperbolic.

The proof of Theorem 8 is heavily based on Nevanlinna theory.

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The Riemann-Schottky problem via singularities of theta divisors

Ruijie Yang
(joint work with Christian Schnell)

This talk is about the classical Riemann-Schottky problem: determine which complex principally polarized abelian varieties (p.p.a.v.) arise as Jacobians of complex curves. This problem has a long history, going back to the work of Riemann, and there are many results. For a recent summary, see Grushevsky's survey [8]. More precisely, in this talk we would like to approach this problem using singularities of theta divisors, which can be traced back to the work of Andreotti-Mayer [1]. There is a precise question posed by Casalaina-Martin in 2008 [4, Question 4.7].

Question 1. Let (A, Θ) be a principally polarized abelian variety. If (A, Θ) is indecomposable as p.p.a.v., it is true that

$$\dim \mathrm{Sing}_m(\Theta) \leq \dim A - 2m + 1, \quad \forall m \geq 2? \tag{1}$$

Here $\mathrm{Sing}_m(\Theta) := \{x \in \Theta \mid \mathrm{mult}_x(\Theta) \geq m\}$. Moreover, if the equality is achieved in (1) by any $m \geq 2$, is it true that A is either the Jacobian of a hyperelliptic curve or the intermediate Jacobian of a cubic threefold?

If this question is true, then it implies a conjecture of Debarre, proposed by Grushevsky [8, Conjecture 5.5] and a conjecture of Grushevsky [8, Conjecture 5.12]. Here are some evidences.

(1) If $A = \mathrm{Jac}(C)$ for a smooth projective curve C, then it is true by the Riemann Singularity Theorem and Marten's work on Brill-Noether varieties.

(2) If A is the Prym variety associated to an étale double cover, then it is true by the work of Casalaina-Martin [5]. As a corollary, the conjecture holds when $\dim A \leq 5$ (under this assumption, every p.p.a.v. is a Prym variety).
(3) If the theta divisor Θ has only isolated singularities, Mustata and Popa [13] show that the dimension bound (1) in the question holds, using their theory of Hodge ideals.

In this talk, I present the following result, based on the joint work with Christian Schnell, as a new result in higher dimension to the Riemann-Schottky problem from the point view of Question 1.

Theorem 2. *Let (A, Θ) be an indecomposable principally polarized abelian variety. Assume the "center of minimal exponent" of (A, Θ) is a one-dimensional scheme Y, then Y must be a smooth hyperelliptic curve and*

$$\dim \operatorname{Sing}_m(\Theta) \leq \dim A - 2m + 1, \quad \forall m \geq 2.$$

Moreover, if the equality is achieved by any $m \geq 2$, then either $A = \operatorname{Jac}(Y)$ or $\dim A = 2m$, $g(Y) = 2m - 1$, the minimal exponent of Θ is $(2m-1)/m$ and Θ has a unique singular point of multiplicity m.

To explain the notion of "center of minimal exponent" for a pair (X, D), recall that if f is a holomorphic function on $\mathbb{C}^n$, the minimal exponent $\tilde{\alpha}_f$ [15] is the smallest root of $\tilde{b}_f(-s)$, where $\tilde{b}_f(s) = b_f(s)/(s+1)$ is the reduced Bernstein-Sato polynomial of f introduced by Saito [16]. This can be generalized to any nonzero divisor D on a complex manifold by setting

$$\tilde{\alpha}_D := \min_{x \in D} \tilde{\alpha}_{f_x},$$

where f_x is a local defining equation of D and $f_x(x) = 0$. By the work of Lichtin [11] and Kollár [10, Theorem 10.6], it is known that $\operatorname{lct}(D) = \min\{\tilde{\alpha}_D, 1\}$, where $\operatorname{lct}(D)$ is the log canonical threshold of D. The numerical invariant $\operatorname{lct}(D)$ can be sheafified using the multiplier ideal sheaves $\{\mathcal{J}(X, \alpha D)\}_{\alpha \in \mathbb{Q}}$ as the minimum jumping number. The minimal exponent is sheafified in our work as the first jumping number of what we called "higher multiplier ideals" $\{\mathcal{J}_k(X, \alpha D)\}_{k \in \mathbb{N}, \alpha \in \mathbb{Q}}$ and the "center of minimal exponent" of (X, D) is constructed using these ideal sheaves, which generalizes the notion of minimal log canonical center of (X, D).

Before discussing the higher multiplier ideals, let me explain briefly why these invariants of singularities are related to Question 1. If (A, Θ) is a principally polarized abelian variety, Kollár [9, Theorem 17.13] showed that $\operatorname{lct}(\Theta) = 1$, i.e. (A, Θ) is log canonical, which implies that

$$\dim \operatorname{Sing}_m(\Theta) \leq \dim A - m, \quad \forall m \geq 2.$$

Later, Ein and Lazarsfeld [7] proved that if one assumes (A, Θ) is indecomposable, then Θ is normal and has rational singularities, moreover $\dim \operatorname{Sing}_m(\Theta) \leq \dim A - m - 1$. By a result of Saito [15], the rationality of Θ is equivalent to

$$\tilde{\alpha}_\Theta > 1.$$

These results show that the dimension bound on the multiplicity locus $\mathrm{Sing}_m(\Theta)$ is intimately related to singularities of the pair (A, Θ) from the point view of birational geometry. Since $\mathrm{lct}(\Theta) = 1$, to get the optimal bound predicted in Question 1, we use the minimal exponent as the new invariant and compute several examples including the p.p.a.v. in the boundary case of Question 1.

Theorem 3. *Let (A, Θ) be a principally polarized abelian variety.*

- *If $A = \mathrm{Jac}(C)$ for a smooth projective curve C, then $\tilde{\alpha}_\Theta \leq 2$. If C is hyperelliptic, then $\tilde{\alpha}_\Theta = 3/2$.*
- *If A is the intermediate Jacobian of a cubic threefold, then $\tilde{\alpha}_\Theta = \frac{5}{3}$.*

In particular, we see that all boundary cases in Question 1 satisfy $1 < \tilde{\alpha}_\Theta < 2$. The proof of Theorem 3 goes in the following way: first there is a lower bound of the minimal exponent $\tilde{\alpha}_D$ using log resolutions of (X, D) by Mustata-Popa [14] and Dirks-Mustata [6] and a new upper bound obtained in our work

$$\tilde{\alpha}_D \leq \min_{m \geq 2} \frac{\mathrm{codim}_X(\mathrm{Sing}_m(D))}{m}, \tag{2}$$

where $\mathrm{Sing}_m(D)$ is the locus of singular points of multiplicity $\geq m$. Then we find an explicit log resolution of (A, Θ) where A is the Jacobian of a hyperelliptic curve [18]. For the case of cubic threefold, Mumford [12, p. 348] proved that Θ has only an isolated singularity whose projectivized tangent cone is the original cubic threefold, therefore a log resolution can be achieved by blowing up the cone point. The reason $\tilde{\alpha}_\Theta \in (1, 2)$ is relevant is because by way of contradiction if we assume $\dim \mathrm{Sing}_m(\Theta) \geq \dim A - 2m + 1$ for some $m \geq 2$, then $\tilde{\alpha}_\Theta \leq \frac{2m-1}{m} < 2$ by (2).

Finally, let us explain the notion of "center of minimal exponent". Using the theory of complex Hodge modules in the MHM project by Sabbah and Schnell and the language of D-algebras developed by Beilinson and Bernstein [2], we study the notion of "twisted complex Hodge modules" and construct ideal sheaves, called "higher multiplier ideals", which sheafify the minimal exponent.

Theorem 4. *Let X be a complex manifold and let D be an effective divisor, then for any $k \in \mathbb{N}$ and $\alpha \in \mathbb{Q}$, there exists a coherent ideal sheaf $\mathcal{J}_k(X, \alpha D) \subseteq \mathcal{O}_X$ such that*

- *If $k = 0$, we have $\mathcal{J}_0(X, \alpha D) = \mathcal{J}(X, \alpha D)$, the usual multiplier ideal sheaf.*
- *Fix $k \geq 0$, then $\{\mathcal{J}_k(\alpha D)\}_{\alpha \in \mathbb{Q}}$ is discrete, right continuous and form a decreasing sequence: $\mathcal{J}_k(X, \alpha D) \subseteq \mathcal{J}_k(X, \beta D)$ if $\alpha \geq \beta$.*
- *The minimal exponent is the first jumping number of all higher multiplier ideals:*

$$\tilde{\alpha}_D = \min\{k + \alpha, k \in \mathbb{N}, \alpha \in [0, 1) \mid \mathcal{J}_k(X, \alpha D) \subsetneq \mathcal{O}_X\}.$$

When D is a global hypersurface, these ideals recover the "microlocal multiplier ideals" studied by M. Saito [17]. The first result is a reinterpretation of a result of Budur and Saito [3]. Assume $\tilde{\alpha}_D = k + \alpha$ with $\alpha \in [0, 1)$, the "center of minimal exponent" is defined as a subscheme of the zero locus of $\mathcal{J}_k(X, \alpha D)$ using a further weight filtration on $\mathcal{J}_k(X, (\alpha - \epsilon)D)/\mathcal{J}_k(X, \alpha D)$ from the theory of complex Hodge modules.

To finish the proof of Theorem 2, we further derive global results of the higher multiplier ideals $\mathcal{J}_k(X, \alpha D)$ including Nadel-type vanishing on abelian varieties and use the geometry of the linear series $|2\Theta|$.

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L^2-extension of Hodge objects

Junchao Shentu
(joint work with C. Zhao)

The purpose of this talk is to explain the works [18, 19, 20] on the transcendental aspects of a intersection complex and the lowest Hodge piece of a pure Hodge module.

Transcendental aspect of the intersection complex. The intersection cohomology was defined by Goresky and MacPherson in [4] as a cohomology theory on singular spaces that satisfies Poincaré duality. Later it turns out that the intersection cohomology of a complex projecitve variety satisfies the hard Lefschetz Theorem and admits a natural pure Hodge structure ([1, 13, 14]). On the transcendental aspect, Cheeger [3] discovered a cohomology theory, called L^2-cohomology, which satisfies Poincaré duality for essentially the same class of spaces as that Goresky and MacPherson had considered. Based on Cheeger's calculations on conical singularities [3], Cheeger-Goresky-MacPherson formulated the following conjecture:

Conjecture 1. *Let $X \subset \mathbb{CP}^N$ be a projective variety. Then there is a natural isomorphism*

$$H^k_{(2),\max}(X_{\mathrm{reg}}, g_{\mathrm{FS}}) \simeq IH^k(X), \quad \forall k$$

where g_{FS} is the metric on X_{reg} induced from the Fubini-Study metric on $\mathbb{CP}^N$.

Cheeger [3] confirmed this conjecture in the case when X has only conical singularities. Later, Hsiang-Pati [5] and Nagase [11] proved the conjecture for X being a normal surface. When X has only isolated singularities, Saper [15, 16] constructed a family of complete Kähler metrics, whose L^2-cohomology is isomorphic to the intersection cohomology. Based on Saper's work, Ohsawa [12] successfully verified Conjecture 1 for this case.

Conjecture 1 has been generalized to spaces with coefficients in a polarized variation of Hodge structure and has been demonstrated by many authors; see for example [21, 22, 2, 7, 9, 17]. These works suggest the following problem.

Let X be a compact Kähler space and $X^o \subset X_{\mathrm{reg}}$ a dense Zariski open subset. Let $\mathbb{V} = (\mathcal{V}, \nabla, F^\bullet, Q)$ be a polarized variation of Hodge structure on X^o, with h_Q its Hodge metric. The L^2-de Rham complex of sheaves $\mathcal{D}^\bullet_{X,\mathbb{V};ds^2,h_Q}$ is defined by assigning to every open subset $U \subset X$ the complex of C^∞ $\mathbb{V}$-valued forms α on $U \cap X^o$ such that α and $d\alpha$ are locally square integrable at every point of U. Let $IC_X(\mathbb{V})$ be the intermediate extension of $\mathbb{V}$.

Problem 2. *Find a complete Kähler metric ds^2 on X^o (independent of the choice of $\mathbb{V}$) such that*

(1) $\mathcal{D}^\bullet_{X,\mathbb{V};ds^2,h_Q}$ is a complex of fine sheaves, and
(2) there is a quasi-isomorphism

$$\mathcal{D}^\bullet_{X,\mathbb{V};ds^2,h_Q} \simeq IC_X(\mathbb{V}).$$

In particular, there is a natural isomorphism

$$H^k_{(2)}(X^o, \mathbb{V}; ds^2, h_Q) \simeq IH^k(X, \mathbb{V})$$

for every k.

In [20], the authors give a solution to Problem 2. Let $\mathbb{V}$ be a semisimple complex local system on X^o endowed with the Corlette-Jost-Zuo metric h ([6],[10]).

Theorem 3 (C. Zhao and –, [20]). *There exists a complete Kähler metric ds^2_{Dist} on X^o such that*

(1) $\mathcal{D}^\bullet_{X,\mathbb{V};ds^2_{\mathrm{Dist}},h}$ is a complex of fine sheaves, and
(2) there is a canonical quasi-isomorphism

$$\mathcal{D}^\bullet_{X,\mathbb{V};ds^2_{\mathrm{Dist}},h} \simeq IC_X(\mathbb{V}).$$

In particular, when $\mathbb{V} = \mathbb{C}_X$ is the trivial local system endowed with the trivial metric, one has an isomorphism

$$H^k_{(2)}(X^o, ds^2_{\mathrm{Dist}}) \simeq IH^k(X), \quad \forall k.$$

The metric ds^2_{Dist} is independent of the choice of $(\mathbb{V}, h)$. However, it is not canonical and depends on a certain desingularization of (X, X^o). There are some unsolved problems concerning the transcendental aspects of the intersection complex.

- Is there a canonical complete Kähler metric ds^2 on X^o so that Theorem 3 holds for ds^2?
- Does the quasi-isomorphism in Theorem 3 preserve the L^2-Hodge filtration on the left handside and Saito's Hodge filtration on the right handside?
- Is there an extremal Kähler metric in the quasi-isometric class of ds^2_{Dist} in Theorem 3?

Transcendental aspect of the lowest Hodge piece of a pure Hodge module. Let X be a complex projective variety and $X^o \subset X_{\mathrm{reg}}$ a Zariski open subset. Let $\mathbb{V} = (\mathcal{V}, \nabla, F^\bullet, Q)$ be a polarized variation of Hodge structure on X^o. Kollár [8] introduced a coherent sheaf $S_X(\mathbb{V})$ associated with $\mathbb{V}$ and made a conjecture on various properties on the higher direct images of $S_X(\mathbb{V})$ [8, §5], including the torsion freeness, a Kollár type vanishing theorem, and a decomposition theorem. Saito settled Kollár's conjecture Hodge theoretically, depending on the observation that $S_X(\mathbb{V})$ is isomorphic to the lowest Hodge piece of the intermediate extension $IC_X(\mathbb{V})$ as a Hodge module.

In [18] we give a transcendental proof to Kollár's conjecture, based on the L^2-Dolbeault resolution of $S_X(\mathbb{V})$. Let us consider a Hermitian vector bundle (E, h) on X^o. Define the subsheaf $S_X(E, h) \subset j_*(K_{X^o} \otimes E)$ consisting of the holomorphic forms which are locally square integrable at every point of X where $j : X^o \to X$ denotes the open immersion. An interesting example is the following.

Theorem 4 (C. Zhao and –,[18]). *Let $\mathbb{V} = (\mathcal{V}, \nabla, F^\bullet, Q)$ be a polarized variation of Hodge structure on X^o. Let $S(\mathbb{V}) := F^{\max\{p|F^p \neq 0\}}$ and let h_Q be the Hodge metric associated with the polarization Q. Then there is an isomorphism*

$$S_X(S(\mathbb{V}), h_Q) \simeq S_X(\mathbb{V}).$$

This example, applied to the following “meta” Kollár package, implies Kollár's conjecture.

Theorem 5 (C. Zhao and –,[19]). *Let $f : X \to Y$ be a proper locally Kähler morphism from a complex space X to an irreducible complex space Y. Assume that every irreducible component of X is mapped onto Y. Let $X^o \subset X_{\mathrm{reg}}$ be a dense Zariski open subset and (E, h) a Hermitian vector bundle on X^o. Let F be a Nakano semi-positive vector bundle on X. Assume that (E, h) is Nakano semipositive and $S_X(E, h)$ is coherent. Then the following statements hold.*

- (torsion freeness): *$R^q f_*(S_X(E, h) \otimes F)$ is torsion free for every $q \geq 0$ and vanishes if $q > \dim X - \dim Y$.*
- (injectivity theorem): *If L is a semi-positive holomorphic line bundle so that $L^{\otimes l}$ admits a nonzero holomorphic global section s for some $l > 0$, then the canonical morphism*

$$R^q f_*(\times s) : R^q f_*(S_X(E, h) \otimes F \otimes L^{\otimes k}) \to R^q f_*(S_X(E, h) \otimes F \otimes L^{\otimes k+l})$$

  is injective for every $q \geq 0$ and every $k \geq 1$.
- (vanishing theorem): *If Y is a projective algebraic variety and L is an ample line bundle on Y, then*

$$H^q(Y, R^p f_*(S_X(E, h) \otimes F) \otimes L) = 0, \quad \forall q > 0, \quad \forall p \geq 0.$$

- (decomposition theorem): *Assume moreover that X is a compact Kähler space. Then $Rf_*(S_X(E, h) \otimes F)$ splits in $D(Y)$, i.e.,*

$$Rf_*(S_X(E, h) \otimes F) \simeq \bigoplus_q R^q f_*(S_X(E, h) \otimes F)[-q] \in D(Y).$$

  As a consequence, the Leray spectral sequence

$$E_2^{pq} : H^p(Y, R^q f_*(S_X(E, h) \otimes F)) \Rightarrow H^{p+q}(X, S_X(E, h) \otimes F)$$

  degenerates at the E_2 page.

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The Kobayashi Conjecture for Compact Hyperkähler Manifolds

Christian Lehn
(joint work with Ljudmila Kamenova)

This is a report about a joint work with Ljudmila Kamenova [2], where we prove Kobayashi's conjecture for compact hyperkähler varieties under certain natural assumptions. In particular, the result applies to any currently known compact hyperkähler manifold.

1. Kobayashi's conjecture

We start with a reminder on Kobayashi hyperbolicity. Let X be a complex variety. The Kobayashi pseudometric d_X is defined as the largest pseudometric on X such that $f^*d_X \leq d_P$ for all holomorphic maps $f : \Delta \to X$ where d_P is the Poincaré metric on the unit disk Δ. We call X *Kobayashi hyperbolic* if d_X is a metric. We will be concerned with the opposite case: Kobayashi's conjecture [4, Problem F.2, p. 405] predicts that for varieties with trivial canonical bundle, this pseudometric vanishes identically.

2. Assumptions and Main Result

Our assumptions concern the second Betti number of the hyperkähler variety and the existence of Lagrangian fibrations. More precisely, we prove:

Theorem 1. *Let X be a compact hyperkähler manifold with $b_2(X) \geq 7$. Suppose that every compact hyperkähler manifold deformation equivalent to X satisfies the rational SYZ conjecture. Then the Kobayashi pseudometric d_X vanishes identically.*

Let us recall the rational SYZ conjecture: it states that given a compact hyperkähler manifold X with a nontrivial line bundle L such that $q_X(L) = 0$ where q_X denotes the Beauville–Bogomolov–Fujiki form, some multiple of L gives a rational Lagrangian fibration $f : X \dashrightarrow B$.

In [2], we actually prove a somewhat stronger result than Theorem 1. Our result also holds for singular varieties, more precisely, for primitive symplectic varieties in the sense of [1]. As in the known examples of compact hyperkähler manifolds, we always have second Betti number at least 7 and also the SYZ conjecture is known to hold in those cases, our result in particular applies to all currently known examples of compact hyperkähler manifolds. Thereby, it completes the results by Kamenova–Lu–Verbitsky [3].

3. Previous Results and Outline of the Argument

Verbitsky had shown that any irreducible symplectic manifold with second Betti number at least 5 is non-hyperbolic (building on Campana's result that any twistor family of irreducible symplectic manifolds contains at least one non-hyperbolic member).

The vanishing of the Kobayashi pseudometric has previously been shown by Kamenova–Lu–Verbitsky [3] assuming SYZ and that the second Betti number be at least 13. Their idea was to deform to a variety with two transverse Lagrangian fibrations. Such a variety is in particular chain connected by abelian varieties and their degenerations and the restriction of d_X to those vanishes. Then, they use the ergodicity of the monodromy action on the marked moduli space and they prove the upper semi-continuity of the Kobayashi diameter to transport the result to any other manifold of the same deformation type.

Our key discovery is that for the pseudometric to vanish it is already enough to have *one* Lagrangian fibration instead of two. Our proof uses an inductive argument (passing through singular symplectic varieties, even if one starts with a smooth one) involving birational geometry and then a theorem of Campana on almost holomorphic maps associated to covering families of analytic cycles.

4. Open Questions

Unlike for smooth hyperkähler varieties, there are examples known with second Betti number smaller than 7, in fact, even with $b_2 = 3$. While the SYZ conjecture is considered a major open problem, removing the hypotheses on the second Betti number might be an interesting problem to think about. One possibility would be to pass to a terminalization and then to a finite quasi-étale cover (as holomorphic maps are distance decreasing for the Kobayashi metrics). While this helps in examples, getting rid of the b_2-assumption in general would certainly require a new idea.

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