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Singular Monge-Ampère equations

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Summary

The objective of this thesis is to present the developments in pluripotential theory made with my collaborators and how these were used to study the existence and uniqueness of singular canonical metrics on (singular) Kähler varieties.

Finding canonical (Kähler-Einstein, constant scalar curvature, extremal) metrics on compact Kähler manifolds is one of the central questions in differential geometry. Given a Kähler metric ω on a compact Kähler manifold X, one looks for a Kähler potential φ such that $\omega_\varphi := \omega + dd^c\varphi$ is "canonical".

In this thesis we will focus on two type of canonical metrics, *Kähler-Einstein* metrics and *constant scalar curvature Kähler* (cscK) metrics. In both cases the problem of looking for such special metrics can be reformulated in terms of complex Monge-Ampère equations. This explains also why the study of complex Monge-Ampère operators has played an important role in Kähler geometry in the last 50 years.

In parallel to the study of the existence and regularity of solutions of complex Monge-Ampère equations (that represents the core of the proof of the Calabi conjecture given by Yau in the end of the 70's), pluripotential theory was introduced at the beginning of the 80's.

Pluripotential theory is the several complex variables analogue of classical potential theory in the complex plane (or on Riemann surfaces). While the latter is a linear theory associated to the Laplacian of a Kähler metric, the former is highly non-linear and associated to the complex Monge-Ampère operator $\varphi \mapsto (\omega + dd^c\varphi)^n$.

Since the beginning of the theory, pluripotential tools were found to be very flexible and suitable to study degenerate complex Monge-Ampère equations, that in turn were related to the study of singular (canonical) metrics.
Indeed, given (Y, ω_Y) a singular variety Y endowed with a Kähler metric ω_Y and $\pi : X \to Y$ a resolution of singularities, the Kähler-Einstein equation on Y writes as a degenerate complex Monge-Ampère equation on the smooth manifold X, where the reference form $\theta := \pi^\star\omega_Y$ is degenerate, i.e. it is not necessarily Kähler but merely semi-positive. Pluripotential theory in the compact setting has its roots in the early 2000s [GZ05, GZ07, BEGZ10].
Motivated by the study of singular Kähler-Einstein metrics, with Tamas Darvas and Chinh Lu [DDNL18c, DDNL19, DDNL18a, DDNL18c, DDNL21] we made a systematic study of pluripotential theory and we developed it in the very general setting where the reference form θ represents a big cohomology class.

The main class of objects studied by pluripotential theory is the class of quasi-plurisubharmonic functions (qpsh for short). A classical way to construct qpsh functions is to use an envelope construction. The study of envelops first in domains of $\mathbb{C}^n$ and later on compact Kähler manifold attracted a lot of attention from people working in pluripotential theory ([Zah77], [Sic81], [BT76]) and then from the Kähler community ([Che00, Dar17, Ber15, RWN14, CTW19, DDNL18c], to cite only a few).

Important applications of envelopes are for example the trascendental holomorphic Morse inequalities on projective manifolds [WN19a] and the regularity of geodesics in the space of Kähler potentials

$$\mathscr{H}_\omega := \{\varphi \in C^\infty(X, \mathbb{R}),\ \omega_\varphi > 0\}.$$

The work with Stefano Trapani [DNT20] fits in these circle of ideas. More precisely, we get a better understanding of the support of the Monge-Ampére measures of envelopes.

The study of the geometry of the space of Kähler metrics is closely connected with the uniqueness and existence of canonical Kähler metrics, since these metrics are critical points of suitable functionals defined on $\mathscr{H}_\omega$.

Mabuchi introduced a Riemannian structure on the space of Kähler potentials $\mathscr{H}_\omega$. As shown by Chen [Che00], $\mathscr{H}_\omega$ endowed with the Mabuchi d_2 distance is a metric space. Darvas [Dar17] showed that its metric completion coincides with a finite energy class $\mathscr{E}^2(X, \omega)$ of quasi-plurisubharmonic functions previously introduced by Guedj and Zeriahi [GZ07] (for completely different reasons!). Other Finsler geometries d_p, $p \geq 1$, on $\mathscr{H}_\omega$ were studied by Darvas [Dar15] who showed that the metric completion of $\mathscr{H}_\omega$ w.r.t. d_p is yet another Monge-Ampère energy class, denoted by $\mathscr{E}^p(X, \omega)$. This led to several spectacular results related to a longstanding conjecture that relates the existence of cscK metrics with the properness of a functional, called K-energy (see [DR17], [BDL20], [CC20a, CC20b, CC18]). More precisely, while in [DR17] the authors proved the existence of a Kähler-Einstein metric on a Fano manifold if and only if the Ding functional is proper (w.r.t. d_1); in [CC20a, CC20b, CC18]) the authors, using a combination of novel PDE techniques, proved the existence of a cscK metric if and only if the K-energy is proper (w.r.t. d_1). The key step of the last result is to establish uniform estimates. In a joint work with Alix Deruelle [DDN21] we give a simpler proof of such estimates also using pluripotential theory. This paper represents the first step of a possible generalization of the existence of cscK metrics on a singular Kähler variety.

Moreover, employing the same technique as in [DR17] and extending the L^1-Finsler structure of [Dar15] to big and semipositive classes in a work joint with Vincent Guedj [DNG18], we establish analogous results for singular normal Kähler varieties. Motivated by the same geometric applications, we also studied the L^p ($p \geq 1$) Finsler geometry in big and semipositive cohomology classes via an approximation method.

This study was then generalized to big and nef classes in the work with Chinh Lu [DNL20].

The Ding functional and the K-energy play a central role in the variational approach to look for canonical metrics. Indeed, their critical points are Kähler-Einstein and cscK

metrics, respectively. The leading term of the Ding functional is the Monge-Ampère energy while the leading term of the K-energy is an entropy term.
In the work joint with Chinh Lu and Vincent Guedj [DNGC21] we compare the two notions of probability measures with finite energy and with finite entropy.

Going back to earlier developments, Donaldson conjectured that a constant scalar curvature metric exists in a Kähler class if and only if the K-energy has certain growth along the geodesic rays of this space [Don99]. This is closely related to the notion of K-stability and is the focus of intense research to this day. Motivated by this picture, there is special interest in regularity of geodesic segments and rays, as well as their geometric significance (see [Che00, Tos18, CTW19, RWN14] to name only very few works in a fast expanding literature).

Given their importance in the above mentioned applications, with Tamas Darvas and Chinh Lu we were interested to see how one can construct weak geodesic rays inside $(\mathcal{E}^1(X,\theta), d_1)$. This is done in [DDNL18c, DDNL18a, DDNL21]. In particular, in [DDNL21] we characterize the limit of weak geodesic rays through the study of the space of singularity types of quasi-plurisubharmonic functions.

A better understanding of singularities of quasi-plurisubharmonic functions and their masses allowed us to work in the big setting and to define relative Monge-Ampère energy classes $\mathcal{E}^p(X,\theta,\phi)$ w.r.t. a model potential ϕ [DDNL18b, DDNL19, Min19b, Tru20]. In this classes we looked for (weak) solutions of complex Monge-Ampère equations with prescribed singularities. The resolution of such equations gives the existence of singular Kähler-Einstein metrics with prescribed singularities.

We describe now the organization of the thesis. It is divided into two chapters:

1. The first chapter is devoted to introduce some new notions in pluripotential theory together with some of the developments made with my collaborators. These results are fundamental and they are used in a deep way in all of the proofs of the results presented in Chapter 2.

   We start with the notion of envelopes and their regularity. We then continue with a particular example of envelopes, the geodesic segments and rays, and we present a construction of the latter in the big setting.

   We then present some new results concerning Monge-Ampère energy classes that give a positive answer to some open questions in pluripotential theory. This is the place where we define as well the "relative" version of them.

   We continue the discussion with the theory of generalized capacities that will be essential for later purposes (more precisely for the resolution of complex Monge-Ampère equations with prescribed singularities).

   We conclude the Chapter introducing the space of singularity types.

2. In Chapter 2 we present the geometric results we obtained on the existence (and uniqueness) of singular canonical metrics. All the latter results make use of the pluripotentials tools introduced in Chapter 1.

In particular we present the results of existence (and uniqueness) for singular Kähler-Einstein metrics (on a singular Kähler variety) together with its family version, singular Kähler-Einstein metrics with prescribed singularities (on a smooth Calabi-Yau or general type manifold). We end the Chapter discussing the case of cscK metrics.

List of presented works

The following publications are avaible on my webpage https://sites.google.com/site/edinezza/

1. Uniform estimates for cscK metrics, with Alix Deruelle (2021), *Annales de la Faculté des Sciences de Toulouse*.
2. Finite entropy vs finite energy, with V. Guedj, C. Lu (2021), *Commentari Mathematici Helvetici*, arXiv:2006.07061.
3. Families of singular Kähler-Einstein metrics, with H. Guenancia, V. Guedj (2020), accepted in *JEMS*.
4. Monge-Ampère measures on contact sets, avec S. Trapani (2020), to appear in *Math. Res. Letters*.
5. The metric geometry of singularity types, with T. Darvas, C. Lu, *Journal für die reine und angewandte Mathematik* (2020), arXiv:1909.00839.
6. L^p metric geometry of big and nef cohomology classes, with C. Lu (2019), *Acta Mathematica Vietnamica*, arXiv:1808.06308.
7. Log-concavity of volume and complex Monge-Ampère equations with prescribed singularity, with T. Darvas, C. Lu (2018), *Mathematische Annalen*, 1-38, (2019).
8. L^1 metric geometry of big cohomology classes, with T. Darvas, C. Lu (2018), *Ann. Inst. Fourier*, arXiv:1802.00087.
9. Monotonicity of non-pluripolar products and complex Monge-Ampére equations with prescribed singularity, with T. Darvas, C. Lu (2018), *Analysis & PDE*, 11-8 (2018), 2049-2087.
10. Geometry and topology of the space of Kähler metrics on singular varieties, with. V. Guedj (2017), *Compositio Mathematica*.
11. On the singularity type of full mass currents in big cohomology classes, with T. Darvas, C. Lu, *Compositio Math.*, Vol. 154 (2018), 380-409.

List of non-presented works

1. Regularity of push-forward of Monge-Ampère measures, with C. Favre (2017), *Ann. Inst. Fourier*, arXiv:1712.09884.

2. The Monge-Ampère class $\mathscr{E}$, contribution in the book “Complex and Symplectic Geometry", Springer INdAM Series 21, Springer, 2017, 51-59.

3. Divisorial Zariski Decomposition and some properties of full mass currents, with E. Floris. S. Trapani, *Annali della Scuola Normale Superiore*, issue 4, Vol. XVII (2017).

4. Uniqueness and short time regularity of the weak Kähler-Ricci flow, with C. H. Lu, *Advances in Mathematics*, Vol. 305C (2016), 953-993.

5. Finite Pluricomplex energy measures, *Potential Analysis*, (2015), DOI: 10.1007/s11118-015-9503-4.

6. Generalized Monge-Ampère Capacities, with C. H. Lu, *International Mathematics Research Notices*, (2014), DOI: 10.1093/imrn/rnu166.

7. Complex Monge-Ampère equations on quasiprojective varieties, with C. H. Lu, *Journal für die reine und angewandte Mathematik*, (2014), DOI: 10.1515/crelle-2014-0090.

8. Stability of Monge-Ampère energy classes, *J. Geom. Anal.*, Vol. 25 (2015), no. 4, 2565-2589.

9. Hitchhiker’s guide to the fractional Sobolev spaces, with G. Palatucci, E. Valdinoci, *Bulletin des Sciences Mathématiques*, Vol. 136 (2012), no. 5, 521-573.

The papers 5,6,7 and 8 are part of my Ph.D thesis.

Contents

Chapter 1

Pluripotential Theory

We recall some notions and facts also in order to fix notations.
Let (X, ω) be a compact Kähler manifold of dimension n and fix θ a smooth closed real $(1,1)$-form. A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is called quasi-plurisubharmonic if locally $u = \rho + \varphi$, where ρ is smooth and φ is a plurisubharmonic function. We say that u is θ-plurisubharmonic (θ-psh for short) if it is quasi-plurisubharmonic and $\theta_u := \theta + dd^c u \geq 0$ in the weak sense of currents on X. We let $\mathrm{PSH}(X, \theta)$ denote the space of all θ-psh functions on X. The cohomology class $\{\theta\} \in H^{1,1}(X, \mathbb{R})$ is *big* if there exists $\psi \in \mathrm{PSH}(X, \theta)$ such that $\theta + dd^c\psi \geq \varepsilon\omega$ for some $\varepsilon > 0$.

A potential $u \in \mathrm{PSH}(X, \theta)$ has *analytic singularities* if it can be written locally as $u(z) = c \log \sum_{j=1}^{k} |f_j(z)|^2 + h(z)$, where $c > 0$, the $f_j's$ are holomorphic functions and h is smooth. By the fundamental approximation theorem of Demailly [Dem92], if $\{\theta\}$ is big there are plenty of θ-psh functions with analytic singularities. Following [BEGZ10] the ample locus of $\{\theta\}$ (denoted by $\mathrm{Amp}\,(\theta)$) is defined to be the set of all $x \in X$ such that there exists a θ-psh function on X with analytic singularities, smooth in a neighborhood of x. It follows from [Bou04] that there exists a θ-psh function ψ with analytic singularities such that $\mathrm{Amp}\,(\theta)$ is the open set on which ψ is smooth and $\psi = -\infty$ on $X \setminus \mathrm{Amp}\,(\theta)$.

Given $u, v \in \mathrm{PSH}(X, \theta)$, we say that

- u is more singular than v, i.e., $u \preceq v$, if there exists $C \in \mathbb{R}$ such that $u \leq v + C$;
- u has the same singularity as v, i.e., $u \simeq v$, if $u \preceq v$ and $v \preceq u$.

The classes $[u]$ of this latter equivalence relation are called *singularity types.*

When θ is non-Kähler, elements of $\mathrm{PSH}(X, \theta)$ can be quite singular, and we distinguish the potential with the smallest singularity type in the following manner:

$$V_\theta := \sup\{u \in \mathrm{PSH}(X, \theta) \text{ such that } u \leq 0\}.$$

A function $u \in \mathrm{PSH}(X, \theta)$ is said to have minimal singularities if it has the same singularity type as V_θ, i.e., $[u] = [V_\theta]$. By the analysis above it follows that V_θ is locally bounded in the Zariski open set $\mathrm{Amp}\,(\theta)$.

Given $\theta^1, ..., \theta^n$ closed smooth $(1,1)$-forms representing big cohomology classes and $u_j \in \mathrm{PSH}(X, \theta^j)$, $j = 1, ...n$, following the construction of Bedford-Taylor [BT82, BT87] in the local setting, it has been shown in [BEGZ10] that the sequence of positive measures

$$\mathbb{1}_{\bigcap_j \{u_j > V_{\theta^j} - k\}} \theta^1_{\max(u_1, V_{\theta^1} - k)} \wedge \ldots \wedge \theta^n_{\max(u_n, V_{\theta^n} - k)} \tag{1.0.1}$$

has total mass (uniformly) bounded from above and is non-decreasing in $k \in \mathbb{R}$, hence converges weakly as $k \to +\infty$ to the so called *non-pluripolar product*

$$\theta^1_{u_1} \wedge \ldots \wedge \theta^n_{u_n}.$$

The resulting positive measure does not charge pluripolar sets. In the particular case when $u_1 = u_2 = \ldots = u_n = u$ and $\theta^1 = ... = \theta^n = \theta$ we will call θ^n_u the non-pluripolar measure of u_i, which generalizes the usual notion of volume form in case θ_u is a smooth Kähler form. As a consequence of Bedford-Taylor theory it can be seen that the measures in (1.0.1) all have total mass less than $\int_X \theta^n_{V_\theta} := \mathrm{vol}(\theta)$, in particular, after letting $k \to \infty$ we notice that $\int_X \theta^n_u \leq \int_X \theta^n_{V_\theta}$. In fact it was recently proved in [WN19b, Theorem 1.2] that for any $u, v \in \mathrm{PSH}(X, \theta)$ the following monotonocity property holds for the total masses:

$$u \preceq v \Longrightarrow \int_X \theta^n_u \leq \int_X \theta^n_v.$$

This result, together with the generalization [DDNL18b, Theorem 1.1], opened the door to the development of relative finite energy pluripotential theory, that is going to be treated in Section 1.3.

It is important to mention that the generalization of Witt-Nÿstrom's result we gave in the joint paper with Tamas Darvas and Chinh Lu (see Section 1.3), and all the contributions we made in pluripotential theory in the big setting deeply relies on the following lower-semicontinuity property of non-pluripolar products:

Theorem 1.1 ([DDNL18b, DDNL19]) *Let $\theta^j, j \in \{1, \ldots, n\}$ be smooth closed real $(1,1)$-forms on X whose cohomology classes are big. Suppose that for all $j \in \{1, \ldots, n\}$ we have $u_j, u_j^k \in \mathrm{PSH}(X, \theta^j)$ such that $u_j^k \to u_j$ in capacity as $k \to \infty$. Then for all positive bounded quasi-continuous functions χ we have*

$$\liminf_{k \to +\infty} \int_X \chi \theta^1_{u_1^k} \wedge \ldots \wedge \theta^n_{u_n^k} \geq \int_X \chi \theta^1_{u_1} \wedge \ldots \wedge \theta^n_{u_n}.$$

If additionally,

$$\int_X \theta^1_{u_1} \wedge \ldots \wedge \theta^n_{u_n} \geq \limsup_{k \to \infty} \int_X \theta^1_{u_1^k} \wedge \ldots \wedge \theta^n_{u_n^k}, \tag{1.0.2}$$

then $\theta^1_{u_1^k} \wedge \ldots \wedge \theta^n_{u_n^k} \to \theta^1_{u_1} \wedge \ldots \wedge \theta^n_{u_n}$ in the weak sense of measures on X.

We recall that a sequence $\{u^k\}_k$ converges in capacity to u if for any $\delta > 0$ we have

$$\lim_{k \to +\infty} \mathrm{Cap}_\omega\{|u^k - u| \geq \delta\} = 0,$$

where Cap_ω is the Monge-Ampère capacity associated to ω (see [GZ17, Definition 4.23]).

1.1 Envelopes

Starting from the works of Zaharjuta [Zah77] and Siciak [Sic81], which years later have been taken over by Bedford and Taylor [BT76, BT82, BT87], *envelopes of plurisubharmonic functions* started to be of interest and to play an important role in the development of the pluripotential theory on domains of $\mathbb{C}^n$.

When, relying on the Bedford and Taylor theory in the local case, the foundations of a pluripotential theory on compact Kähler manifolds has been developed [GZ05, GZ07], *envelopes of quasi-plurisubharmonic functions* started to be intensively studied.

The two basic (and related) questions are about the regularity of envelopes and the behavior of their Monge-Ampère measures.

In the following we are going to work with some well known envelope constructions:

$$P_\theta(f),\ P_\theta(f_1,\ldots,f_k),\ P_\theta[\varphi](f),\ P_\theta[\varphi].$$

Given $f, f_1, \ldots f_k$ functions on X bounded from above, we consider the "rooftop envelopes"

$$P_\theta(f) := (\sup\{v \in \mathrm{PSH}(X,\theta),\ v \le f)\})^*$$

and

$$P_\theta(f_1,\ldots,f_k) := P_\theta(\min(f_1,\ldots,f_k)) = (\sup\{v \in \mathrm{PSH}(X,\theta),\ v \le \min(f_1,\ldots f_k)\})^*.$$

Then, given a θ-psh function φ, the above procedure allows us to introduce

$$P_\theta[\varphi](f) := \Big(\lim_{C\to+\infty} P_\theta(\varphi + C, f)\Big)^*.$$

Note that by definition we have $P_\theta[\varphi](f) = P_\theta[\varphi](P_\theta(f))$. When $f = 0$, we will simply write $P_\theta[\varphi] := P_\theta[\varphi](0)$ and refer to this potential as the *envelope of the singularity type* $[\varphi]$. Also, in the following when no confusion can arise we drop the dependence of the smooth form and we simply write $P(f), P[\varphi]$.
We emphasize that the functions $P_\theta(f)$, $P_\theta(f_1,\ldots,f_k)$ and $P_\theta[\varphi](f)$ are either θ-psh or identically equal to $-\infty$. Observe that if $-C \le f \le C$, then $V_\theta - C \le P_\theta(f) \le V_\theta + C$; hence $P_\theta(f)$ is a well defined θ-psh function.

The study of such envelopes has lead to several works. In particular, in a joint paper with Stefano Trapani [DNT20] we study the support of the Monge-Ampère measure of such envelopes and we prove a very general result:

Theorem 1.2 ([DNT20]) *Let θ be smooth closed real $(1,1)$-form on X such that the cohomology class $\{\theta\}$ is pseudoeffective. Let φ be a θ-plurisubharmonic function and $f \in C^{1,\bar{1}}(X)$. Assume $\varphi \le f$. Then the non-pluripolar product θ^n_φ satisfies the equality*

$$\mathbf{1}_{\{\varphi=f\}}\theta^n_\varphi = \mathbf{1}_{\{\varphi=f\}}\theta^n_f. \tag{1.1.3}$$

Here we denote by $C^{1,\bar{1}}(X)$ the space of continuous function with bounded distributional laplacian w.r.t. ω. Elliptic regularity and Sobolev's embedding theorem imply that $C^{1,\bar{1}}(X) \subset W^{2,p}$ for all $p \geq 1$, and $\bigcap_{p\geq 1} W^{2,p} \subset C^{1,\alpha}$ for any $\alpha \in (0,1)$. Here $W^{2,p}$ denotes the Sobolev space of functions with all derivatives up to second order in L^p.

As an almost immediate consequence we get:

Corollary 1.3 ([DNT20]) *Let $\varphi \in \mathrm{PSH}(X,\theta)$ and $f \in C^{1,\bar{1}}(X)$ be such that $\varphi \leq f$. We have:*

i) $\theta^n_{P_\theta(f)} = \mathbf{1}_{\{P_\theta(f)=f\}}\theta^n_f$.

ii) $\theta^n_{P[\varphi](f)} = \mathbf{1}_{\{P[\varphi](f)=f\}}\theta^n_f$.

The result in *i*) was already known in the case of a smooth barrier function f and $\theta \in c_1(L)$ where L is a big line bundle over X [Ber09] or when $\{\theta\}$ is big and nef (not necessarily representing the first Chern class of a line bundle) [Ber19]. It is also worth to mention that at the best of our knowledge, the equality in *ii*) is new even in the case of a Kähler class.

The fact that the Monge-Ampère measure of an envelope is supported in the contact set $\{P_\theta(h) = h\}$ is used in a key way in several works dealing with problems of an algebraic geometric flavor. I can mention for example the work of Witt-Nyström [WN19a] where he solves a conjecture of Boucksom-Demailly-Păun-Peternell on the duality between the pseudoeffective and the movable cone on a projective manifold.

It is important to observe that one cannot expect Theorem 1.2 to hold when the barrier function f is singular. The following counterexample shows indeed that (1.1.3) does not hold when f is merely continuous.
Let $\mathbb{B} \subset X$ be a small open ball and let ρ be a smooth potential such that $\omega = dd^c\rho$ in a neighborhood of $\overline{\mathbb{B}}$. Using the Poisson formula we solve the Dirichlet problem

$$(dd^c(\rho + v))^n = 0 \quad \text{in } \mathbb{B}, \qquad v|_{\partial\mathbb{B}} = 0.$$

Since the boundary data is continuous, we have existence of a continuous solution $v \geq 0$ which is ω-psh in $\mathbb{B}$. We then define

$$f := \begin{cases} v & \text{in } \mathbb{B} \\ 0 & \text{in } X \setminus \mathbb{B}. \end{cases}$$

By construction $f \geq 0$ is a continuous function and, since $\max(v, 0) = v$, f is also ω-psh. On the other hand we observe that

$$\int_{X\setminus\mathbb{B}} \omega_f^n = \int_X \omega_f^n = \int_X \omega^n > \int_{X\setminus\mathbb{B}} \omega^n.$$

Since $\{f = 0\} \subseteq X \setminus \mathbb{B}$, we then deduce that the two measures $\mathbf{1}_{\{f=0\}}\omega^n$ and $\mathbf{1}_{\{f=0\}}\omega_f^n$ can not coincide.

Regularity questions about envelopes for functions f that are less regular have been also addressed in the literature: Guedj, Lu and Zeriahi [GLZ19] proved that if f is a continuous function, $P_\theta(f)$ is also continuous. They also proved that its Monge-Ampère measure (w.r.t. any big class $\{\theta\}$) is supported on the contact set $\{P_\theta(f) = f\}$.

It is important to mention that the equality

$$\theta^n_{P_\theta(f)} = \mathbf{1}_{\{P_\theta(f)=f\}}\theta^n_f$$

can be proved as a consequence of the $C^{1,\bar{1}}$-regularity of the envelope $P_\theta(f)$ on $\mathrm{Amp}\,(\theta)$, that is now unavailable in its full generality. There are several recent works which prove the regularity $C^{1,\bar{1}}$ in less degenerate cases but the problem in the case of a big class is still open.
Assume that $\{\theta\}$ is big and nef, Berman [Ber19], using PDE methods, proved that the envelope $P_\theta(h)$ is in $C^{1,\bar{1}}$ on $\mathrm{Amp}\,(\theta)$. The optimal regularity $C^{1,1}$ in the Kähler case was then proved independently by [Tos18] and [CZ19], while the big and nef case was settled in [CTW19].

As the list of papers suggests the community of Kähler geometers is particularly interested in the subject. But all known techniques seems not to work in the big case.

Question 1: Let $\{\theta\}$ be a big class and let f be a $C^{1,\bar{1}}$ function. Is $P_\theta(f)$ a $C^{1,\bar{1}}$ function in $\mathrm{Amp}\,(\theta)$?

1.2 Weak geodesics segments & rays

Geodesic segments connecting Kähler potentials were first introduced by Mabuchi [Mab87]. Semmes [Sem92] and Donaldson [Don99] independently realized that the geodesic equation can be reformulated as a degenerate homogeneous complex Monge-Ampère equation. The best regularity of a geodesic segment connecting two Kähler potentials is known to be $C^{1,1}$ (see [Che00], [Bło12], [CTW19]).

In the context of a big cohomology class, the regularity of geodesics is very delicate. To avoid this issue in a work in collaboration with Tamas Darvas and Chinh Lu [DDNL18c] we follow an idea of Berndtsson [Ber15] and we adapt the definition of (sub)geodesics to the context of big cohomology classes as well.

Fix $0 < \ell \leq \infty$. For a curve $(0, \ell) \ni t \mapsto u_t \in \mathrm{PSH}(X, \theta)$ we define its complexification as a function in $X \times D_\ell$,

$$X \times D_\ell \ni (x, z) \mapsto U(x, z) := u_{\log|z|}(x),$$

where $D_\ell := \{z \in \mathbb{C} \mid 1 < |z| < e^\ell\}$, and π is the projection on X.

We say that $t \mapsto u_t$ is a subgeodesic segment (resp. ray) if $U(x, z) \in \mathrm{PSH}(X \times D_\ell, \pi^*\theta)$ with $\ell < \infty$ (resp. $U(x, z) \in \mathrm{PSH}(X \times D_\infty, \pi^*\theta)$).

For $\varphi, \psi \in \mathrm{PSH}(X, \theta)$, we let $\mathscr{S}_{(0,\ell)}(\varphi, \psi)$ denote the set of all subgeodesic segments $(0, \ell) \ni t \mapsto u_t \in \mathrm{PSH}(X, \theta)$ that satisfy $\limsup_{t\to 0} u_t \leq \varphi$ and $\limsup_{t\to\ell} u_t \leq \psi$.

Now, for $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, the *weak (Mabuchi) geodesic segment* connecting φ and ψ is defined as the upper envelope of all subgeodesic segments in $\mathscr{S}_{(0,\ell)}(\varphi,\psi)$, i.e.

$$\varphi_t := \sup_{\mathscr{S}_{(0,\ell)}(\varphi,\psi)} u_t. \tag{1.2.4}$$

For general $\varphi, \psi \in \mathrm{PSH}(X,\theta)$ it is possible that φ_t is identically equal to $-\infty$ for any $t \in (0,\ell)$, meaning that geodesic segments connecting two general θ-psh functions may not exist. But when $\varphi, \psi \in \mathscr{E}^p(X,\theta)$ it was shown that $P(\varphi,\psi) \in \mathscr{E}^p(X,\theta)$ as well (this is a very nice result we proved in [DDNL18c] from which it follows the convexity of the energy class $\mathscr{E}$). Since $P(\varphi,\psi) \leq \varphi_t$, we obtain that $\varphi_t \in \mathscr{E}^p(X,\theta)$ for any $t \in [0,\ell]$. By $\mathbb{R}$-invariance each subgeodesic segment is in particular t-convex, hence we get that

$$\varphi_t \leq \left(1 - \frac{t}{\ell}\right)\varphi + \frac{t}{\ell}\psi, \ \forall t \in [0,\ell]. \tag{1.2.5}$$

Consequently the upper semicontinuous regularization (with respect to both variables x, z) of $t \to \varphi_t$ is again in $\mathscr{S}_{(0,\ell)}(\varphi,\psi)$, hence so is $t \to \varphi_t$.

In particular, if φ and ψ have minimal singularity type, the function $h := |\varphi - \psi|$ is bounded and $t \to u_t := \max\left(\varphi - \|h\|_{L^\infty}\frac{t}{\ell}, \psi - \|h\|_{L^\infty}\frac{\ell-t}{\ell}\right)$ is a subgeodesic. Therefore $\varphi_t \geq u_t$ for any $t \in (0,\ell)$ and hence $\varphi_t \in \mathrm{PSH}(X,\theta)$ has minimal singularity type for any $t \in (0,\ell)$. Moreover, by this last fact and (1.2.5) it follows that $\lim_{t\to 1}\varphi_t = \varphi$ and $\lim_{t\to\ell}\varphi_t = \psi$. Consequently, in the particular case when φ, ψ have minimal singularity type, it is natural to extend the curves $(0,\ell) \ni t \to \varphi_t \in \mathrm{PSH}(X,\theta)$ at the endpoints by $\varphi_0 := \varphi$ and $\varphi_1 := \psi$. As we will see, a similar pattern will arise when $\varphi, \psi \in \mathscr{E}^1(X,\omega)$. Collecting and expanding the above thought we proved:

Proposition 1.4 ([DDNL18c]) *Let $t \mapsto \varphi_t$ be the weak Mabuchi geodesic joining $\varphi_0, \varphi_\ell \in \mathrm{PSH}(X,\theta)$ with minimal singularity type, constructed as above. Then for $C := \sup_X |\varphi_\ell - \varphi_0|/\ell > 0$ we have that*

$$|\varphi_t - \varphi_{t'}| \leq C|t - t'|, \quad t, t' \in [0,\ell].$$

Additionally, for the complexification $\Phi(x,z) := \varphi_{\log|z|}(x)$ we have

$$(\pi^*\theta + dd^c\Phi)^{n+1} = 0 \text{ in } \mathrm{Amp}\,(\theta) \times D_\ell,$$

where equality is understood in the weak sense of measures.

Finally we say that a curve $[0,+\infty) \ni t \to \varphi_t \in \mathrm{PSH}(X,\theta)$ is a *weak geodesic ray*, with minimal singularity type, if for any fixed $\ell > 0$ $[0,\ell] \ni t \to \varphi_t \in \mathrm{PSH}(X,\theta)$ is a weak geodesic segment joining φ_0 and φ_ℓ, potentials with minimal singularity.

We shall say that weak geodesic rays are not easy to construct. As opposed to this, test curves can be easily constructed and they can also be conveniently "maximized". Suppose $\phi \in \mathrm{PSH}(X,\theta)$ has minimal singularity. Roughly speaking, we say that $\mathbb{R} \ni \tau \mapsto$

$\psi_\tau \in \text{PSH}(X, \theta)$ it is a test curve, if it is τ-concave, $\psi_\tau = \phi \in \text{PSH}(X, \theta)$ for all $\tau \leq -C_\psi$, and $\psi_\tau = -\infty$ for all $\tau \geq C_\psi$, for some constant $C_\psi > 0$. Additionally a test curve is maximal, if it satisfies:

$$P[\psi_\tau](\phi) = \psi_\tau, \quad \tau \in \mathbb{R}.$$

Test curves were introduced by Ross and Witt Nyström [RWN14] in order to deal with geodesic rays in the Kähler case. To this end, we emphasize that the construction of Ross and Witt Nyström not only generalizes to the big case, but in [DDNL18a] we show that their very flexible method gives all possible weak geodesic rays (with minimal singularity) in a unique manner. More precisely we point out a duality between rays and maximal test curves, via the partial Legendre transform:

Theorem 1.5 ([DDNL18a]) *The correspondence $\psi \mapsto \check{\psi}$ gives a bijective map between maximal τ-usc test curves $\tau \mapsto \psi_\tau$ and weak geodesic rays with minimal singularity type $t \mapsto u_t$. The inverse of this map is $u \mapsto \hat{u}$.*

Here $\check{\psi}$ and $\hat{u}$ represent the partial (inverse) Legendre transforms of ψ and u respectively, defined by:

$$\check{\psi}_t := \sup_{\tau \in \mathbb{R}} (u_\tau + t\tau), \quad \hat{u}_\tau := \inf_{t \geq 0} (u_t - t\tau).$$

As a corollary we recover the main analytic result of [RWN14] in the big context:

Corollary 1.6 ([DDNL18a]) *Let $\tau \mapsto \psi_\tau$ be a test curve such that $\psi_{-\infty} = \phi$. Define*

$$w_t = \sup_{\tau \in \mathbb{R}} (P[\psi_\tau](\phi) + t\tau), \quad t \geq 0.$$

Then the curve $t \mapsto w_t$ is a weak geodesic ray, with minimal singularity, emanating from ϕ.

1.3 Energy Classes

Relative finite energy class $\mathcal{E}(X, \theta, \phi)$. As we briefly mentioned the mass is monotone w.r.t. the singularity type [DDNL18b], meaning that given $u_i, v_i \in \text{PSH}(X, \theta)$, $i = 1, \cdots, n$

$$u_i \preceq v_i \Longrightarrow \int_X \theta_{u_1} \wedge \cdots \wedge \theta_{u_n} \leq \int_X \theta_{v_1} \wedge \cdots \wedge \theta_{v_n},$$

and

$$u_i \simeq v_i \Longrightarrow \int_X \theta_{u_1} \wedge \cdots \wedge \theta_{u_n} = \int_X \theta_{v_1} \wedge \cdots \wedge \theta_{v_n}.$$

It is worth noticing that the reverse implication in the latter statement is not true, meaning that there are examples of θ-psh functions not having the same singularity type but having the same mass. One can then wonder if, given $u \in \text{PSH}(X, \theta)$, there exists a *least singular* potential that is less singular than u but has the same full mass as u. As we will see this is indeed the case.

The ceiling operator and model potentials. In joint works with Tamas Darvas and Chinh Lu [DDNL21], we introduce the *ceiling* operator $\mathscr{C} : \mathrm{PSH}(X,\theta) \mapsto \mathrm{PSH}(X,\theta)$ defined by

$$\mathscr{C}(u) := \mathrm{usc}(\sup \mathscr{F}_u),$$

where

$$\mathscr{F}_u := \left\{ v \in \mathrm{PSH}(X,\theta) \;\middle|\; [u] \leq [v],\; v \leq 0,\; \int_X \theta_v^k \wedge \theta_{V_\theta}^{n-k} = \int_X \theta_u^k \wedge \theta_{V_\theta}^{n-k},\; k \in \{0, ..., n\} \right\}.$$

As it turns out, there is no reason to take the upper semi-continuous regularization in the definition above, as $\mathscr{C}(u)$ is a candidate in its defining family $\mathscr{F}_u$. Indeed we show that for any $u \in \mathrm{PSH}(X,\theta)$ and $u \leq 0$,

$$\mathscr{C}(u) = \lim_{\varepsilon \mapsto 0^+} P[(1-\varepsilon)u + \varepsilon V_\theta] \in \mathscr{F}_u. \tag{1.3.6}$$

In particular, if $\phi, \psi \in \mathrm{PSH}(X,\theta)$ with $[\phi] \leq [\psi]$ then $\mathscr{C}(\phi) \leq \mathscr{C}(\psi)$, i.e., $\mathscr{C}$ is monotone increasing. In the important particular case of non-vanishing mass, i.e. $\int_X \theta_u^n > 0$, it can be checked that the equality of the full masses $\int_X \theta_u^n = \int_X \theta_v^n$ implies equality at all the intermediate levels, $\int_X \theta_v^k \wedge \theta_{V_\theta}^{n-k} = \int_X \theta_u^k \wedge \theta_{V_\theta}^{n-k}$, for any k. As a consequence $\mathscr{C}(u)$ can be expressed as

$$\mathscr{C}(u) = \sup \left\{ v \in \mathrm{PSH}(X,\theta) \;\middle|\; [u] \leq [v], v \leq 0 \text{ and } \int_X \theta_u^n = \int_X \theta_v^n \right\}$$

We then say that a potential $\phi \in \mathrm{PSH}(X,\theta)$ is a *model* potential if $\phi = \mathscr{C}(\phi)$, i.e., if ϕ is a fixed point of $\mathscr{C}$. Similarly, the corresponding singularity types $[\phi]$ are called model type singularities.

Examples of model potentials are functions with analytic singularities. As a more specific example we have that the potential V_θ is a model potential.

In the case of non-vanishing mass we can show that $P[u] = \mathscr{C}(u)$. We conjecture that his is the case in general as well, i.e. that for any $u \in \mathrm{PSH}(X,\theta)$ we have that $P[u] = \mathscr{C}(u)$.

Although we treated both the non-vanishing mass case and the zero mass case up to a good level of generality, in what follows we decide, for simplicity, to present the theory in the non-vanishing mass case. So, we will always assume that all θ-psh functions we work with have strictly positive mass.

Fixing a model potential $\phi \in \mathrm{PSH}(X,\theta)$, it is natural to consider the set of ϕ-relative *full mass potentials*:

$$\mathscr{E}(X,\theta,\phi) := \left\{ u \in \mathrm{PSH}(X,\theta), [u] \leq [\phi] \text{ such that } \int_X \theta_u^n = \int_X \theta_\phi^n \right\}.$$

Observe that when $\phi = V_\theta$, the relative class $\mathscr{E}(X,\theta,V_\theta)$ is nothing else than the full Monge-Ampère energy class $\mathscr{E}(X,\theta)$, previously introduced by [GZ07].

Using envelopes we conveniently characterized membership in $\mathscr{E}(X,\theta,\phi)$:

Theorem 1.7 ([DDNL18c]) *Suppose $\phi \in \mathrm{PSH}(X,\theta)$, $\phi = P[\phi]$ and $\int_X \theta_\phi^n > 0$. Then $u \in \mathscr{E}(X,\theta,\phi)$ if and only if $u \in \mathrm{PSH}(X,\theta)$, $[u] \le [\phi]$ and $P_\theta[u] = \phi$. Also, if $u \in \mathscr{E}(X,\theta,\phi)$, then*

$$\nu(u,x) = \nu(\phi,x), \quad \text{for any } x \in X.$$

Here $\nu(u,x)$ denotes the Lelong number of u at the point x.
The last statement in the particular case of $\phi = V_\theta$ and $u \in \mathscr{E}(X,\theta)$ positively answers to an open question in pluripotential theory asked in [BEGZ10].

The finite energy classes $\mathscr{E}^p$. Fix $p \ge 1$. We start recalling the definition of the Monge-Ampère energy I_p and of the class $\mathscr{E}^p(X,\theta)$.
Given any θ-psh function u with minimal singularities (that w.l.o.g. we can suppose such that $u \le V_\theta$), we define the Monge-Ampère energy as

$$I_p(u) := \frac{1}{n+1}\sum_{k=0}^{n}\int_X -(V_\theta - u)^p \theta_u^k \wedge \theta_{V_\theta}^{n-k}.$$

We then define the Monge-Ampère energy for arbitrary $u \in \mathrm{PSH}(X,\theta)$ as

$$I_p(u) := \inf\{I_p(v) \,\Big|\, v \in \mathrm{PSH}(X,\theta),\ v \text{ has minimal singularities, and } u \le v\}.$$

In the case $p = 1$ we simply use the notation I instead of I_1.

We let $\mathscr{E}^p(X,\theta)$ denote the set of all $u \in \mathrm{PSH}(X,\theta)$ such that $I_p(u)$ is finite.
In the Kähler case, these classes were introduced in [GZ07] in relation to the existence of weak solutions for Monge-Ampère equations with degenerate data, but it then played a crucial role in the study of the space of Kähler potentials

$$\mathscr{H}_\omega := \{\varphi \in \mathscr{C}^\infty(X,\mathbb{R}) \,\Big|\, \omega_\varphi := \omega + dd^c\varphi > 0\}.$$

Darvas [Dar15] introduced indeed a family of distances in $\mathscr{H}_\omega$:

Définition 1.1 *Let $\varphi_0, \varphi_1 \in \mathscr{H}_\omega$. For $p \ge 1$, we set*

$$d_{p,\omega}(\varphi_0,\varphi_1) := \inf\{\ell_p(\psi) \,|\, \psi \text{ is a smooth path joining } \varphi_0 \text{ to } \varphi_1\},$$

where $\ell_p(\psi) := \int_0^1 \left(\frac{1}{V}\int_X |\dot\psi_t|^p \omega_{\psi_t}^n\right)^{1/p} dt$ and $V := \mathrm{vol}(\omega) = \int_X \omega^n$.

It was then proved in [Dar17] (generalizing Chen's original arguments [Che00]) that d_p defines a distance on $\mathscr{H}_\omega$, and for all $\varphi_0, \varphi_1 \in \mathscr{H}_\omega$,

$$d_{p,\omega}(\varphi_0,\varphi_1) = \left(\frac{1}{V}\int_X |\dot\varphi_t|^p \omega_{\varphi_t}^n\right)^{1/p}, \quad \forall t \in [0,1], \tag{1.3.7}$$

where $t \mapsto \varphi_t$ is the Mabuchi geodesic (defined in Section 1.2).

By [Dem92, BK07], potentials in $\mathscr{E}^p(X,\omega)$ can be approximated from above by smooth Kähler potentials. As shown in [Dar17] the metric d_p can be extended for potentials in $\varphi_0, \varphi_1 \in \mathscr{E}^p(X,\omega)$: if φ_i^k are smooth strictly ω-psh functions decreasing to φ_i, $i = 0,1$ then the limit

$$d_{p,\omega}(\varphi_0,\varphi_1) := \lim_{k\to+\infty} d_{p,\omega}(\varphi_0^k,\varphi_1^k)$$

exists and it is independent of the approximants. By [Dar15], d_p defines a metric on $\mathscr{E}^p(X,\omega)$ and $(\mathscr{E}^p(X,\omega), d_{p,\omega})$ is a complete geodesic metric space.

In the work [DNL20] joint with Chinh Lu we deal with the case of $\{\theta\}$ being a big and nef class. In this context we define a distance d_p on $\mathscr{E}^p(X,\theta)$ and prove that the space $(\mathscr{E}^p(X,\theta), d_p)$ is a complete geodesic metric space.
Typically there are no smooth potentials in $\mathrm{PSH}(X,\theta)$ but the following class contains plenty of potentials sufficiently regular for our purposes:

$$\mathscr{H}_\Delta := \{\varphi \in \mathrm{PSH}(X,\theta) \,\big|\, \varphi = P_\theta(f),\ f \in \mathscr{C}(X,\mathbb{R}),\ dd^c f \le C(f)\omega\}.$$

Here $C(f)$ denotes a positive constant which depends only on f. Note that any $u = P_\theta(f) \in \mathscr{H}_\Delta$ has minimal singularities because, for some constant $C > 0$, $V_\theta - C$ is a candidate defining $P_\theta(f)$.

Theorem 1.8 ([DNL20]) *Assume that $\varphi_0 := P_\theta(f_0), \varphi_1 := P_\theta(f_1) \in \mathscr{H}_\Delta$. Let $d_{p,\varepsilon}$ be the Mabuchi distance w.r.t. $\omega_\varepsilon := \theta + \varepsilon\omega$, where ω is a Kähler form and $\varepsilon > 0$. We define*

$$d_p(\varphi_0,\varphi_1) := \lim_{\varepsilon\to 0} d_{p,\varepsilon}(\varphi_{0,\varepsilon},\varphi_{1,\varepsilon}),$$

where $\varphi_{0,\varepsilon} := P_{\omega_\varepsilon}(f_0)$ and $\varphi_{1,\varepsilon} := P_{\omega_\varepsilon}(f_1)$. Then, the limit exists and is independent of the choice of ω. Moreover d_p, defined as above, is a distance on $\mathscr{H}_\Delta$.

Observe that by nefness of θ, $\omega_\varepsilon := \theta + \varepsilon\omega$ represents a Kähler cohomology class for any $\varepsilon > 0$. Note that ω_ε is not necessarily a Kähler form but there exists a smooth potential $h_\varepsilon \in \mathscr{C}^\infty(X,\ \mathbb{R})$ such that $\omega'_\varepsilon := \omega_\varepsilon + dd^c h_\varepsilon$ is a Kähler form. Observe that, a priori, by Darvas [Dar15], it is the Mabuchi distance $d_{p,\omega'_\varepsilon}$ to be well defined on $\mathscr{E}^p(X,\omega'_\varepsilon)$. What is hidden behind the statement of the above Theorem is that we can actually show that the Mabuchi distance behaves well when we change the Kähler representative in $\{\omega_\varepsilon\}$.

Given $\varphi_0, \varphi_1 \in \mathscr{E}^p(X,\theta)$, we then define

$$d_p(\varphi_0,\varphi_1) := \lim_{j\to+\infty} d_p(P_\theta(f_{0,j}), P_\theta(f_{1,j})),$$

where $f_{i,j}$ is a sequence of smooth functions decreasing to φ_i, $i = 0,1$.

Theorem 1.9 ([DNL20]) *The space $(\mathscr{E}^p(X,\theta), d_p)$ is a complete geodesic metric space which is the completion of $(\mathscr{H}_\Delta, d_p)$.*

The above result extends results in another works of mine [DNG18] to the context of big and nef cohomology classes. Also, the case when $p = 1$ was established in a work in collaboration with Tamas Darvas and Chinh Lu [DDNL18a] in the more general case of big cohomology classes using the Monge-Ampère energy I. It is worth to underline that the distance d_1 defined by approximation in Theorem 1.8 coincides with the one defined [DDNL18a] and that I am going to describe in the next lines.

The starting point of the definition of d_1 in the setting of a big class is that, given $u, v \in \mathcal{E}^1(X, \theta)$, it has been shown in [DDNL18c] that $P(u, v)$ belongs to $\mathcal{E}^1(X, \theta)$. Consequently, we can define $d_1(u, v)$ as the following finite quantity:

$$d_1(u, v) = I(u) + I(v) - 2I(P(u, v)). \tag{1.3.8}$$

Thus defined, d_1 is symmetric, and non-degeneracy is a simple consequence of the domination principle. The main difficulty is to show that the triangle inequality also holds. We accomplish this, and we are also able to show that the resulting metric space $(\mathcal{E}^1(X, \theta), d_1)$ is complete, with metric geodesics running between any two points. The construction of these geodesic segments has been recalled in Section 1.2. We record all of this in a theorem:

Theorem 1.10 ([DDNL18a]) *$(\mathcal{E}^1(X, \theta), d_1)$ is a complete geodesic metric space.*

As we pointed out, in the Kähler case the d_1 metric is introduced quite differently. In the Kähler case, formula (1.3.8) is a result of a theorem ([Dar15, Corollary 4.14]), but in the big case we take it as our definition for d_1!

Though there is no apparent connection with the infinite dimensional L^1 Finsler geometry in the case when the reference form is big. By the double estimate below, we will still refer to d_1 as the L^1 metric of $\mathcal{E}^1(X, \theta)$. Indeed, by this double inequality, it seems that one should think of d_1 as a kind of L^1 metric with "moving measures":

$$d_1(u, v) \leq \int_X |u - v|\theta_u^n + \int_X |u - v|\theta_v^n \leq 3 \cdot 2^n(n + 1)d_1(u, v), \qquad u, v \in \mathcal{E}^1(X, \theta).$$

Energy & Entropy.

The class $\mathcal{E}^1$ is the most studied in the literature since it appears in a natural way in the variational approach to look for Kähler-Einstein metrics. More precisely, the Monge-Ampère energy I is the leading term of the Ding functional. Maximizing such a functional is equivalent to the search for Kähler-Einstein metrics on Fano manifolds. In parallel to this, the notion of probability measures with finite entropy [BBE+19] has played an important role in recent developments in Kähler geometry: indeed, a constant scalar curvature Kähler metric minimizes another functional, the so-called K-energy, whose leading term is an entropy. The purpose of the work joint with Chinh Lu and Vincent Guedj [DNGC21] is to compare these two notions.

We consider $\mu = f\omega^n$, $0 \leq f$, $\mu(X) = \int_X \omega^n$, a probability measure with finite entropy

$$0 \leq \mathrm{Ent}_{\omega^n}(\mu) := \int_X f \log f \; \omega^n < +\infty.$$

Since μ is absolutely continuous with respect to the volume form ω^n, it is in particular "non-pluripolar" hence it follows from [GZ07] that there exists a unique full mass potential $\varphi \in \mathscr{E}(X, \omega)$ such that $\sup_X \varphi = 0$ and

$$(\omega + dd^c\varphi)^n = \mu.$$

It has been observed in [BBE+19] that

$$\mathrm{Ent}(X, \omega) \subset \mathscr{E}^1(X, \omega),$$

and the injection $\mathrm{Ent}(X, \omega) \hookrightarrow \mathscr{E}^1(X, \omega)$ is compact, where $\mathrm{Ent}(X, \omega)$ is the set of ω-psh functions whose Monge-Ampère measure has finite entropy. However all computable examples suggest that $\mathrm{Ent}(X, \omega)$ is actually contained in a higher energy class $\mathscr{E}^p(X, \omega)$ for some $p > 1$ depending on the dimension. We confirm this experimental observation by showing the following:

Theorem 1.11 ([DNGC21]) *Let $\mu = (\omega + dd^c\varphi)^n = f\omega^n$ be a probability measure with finite entropy $\mathrm{Ent}_{\omega^n}(\mu) = \int_X f \log f\omega^n < +\infty$. Then*

$$\varphi \in \mathscr{E}^{\frac{n}{n-1}}(X, \omega).$$

Moreover the inclusion $\mathrm{Ent}(X, \omega) \hookrightarrow \mathscr{E}^p(X, \omega)$ is compact for any $p < \frac{n}{n-1}$.

This exponent is sharp when $n \geq 2$. If $n = 1$ then φ is continuous, hence it belongs to $\mathscr{E}^p(X, \omega)$ for all $p > 0$.

The case of Riemann surfaces deserves a special treatment: finite entropy potentials turn out to be bounded (and even continuous), but this is no longer the case in higher dimension. The proof of Theorem 1.11 relies on a Moser-Trudinger inequality which provides a strong integrability property of finite energy potentials. This is the content of our second main result:

Theorem 1.12 ([DNGC21]) *Fix $p > 0$. There exist positive constants $c, C > 0$ depending on X, ω, n, p such that, for all $\varphi \in \mathscr{E}^p(X, \omega)$ with $\sup_X \varphi = -1$,*

$$\int_X \exp\left(c|I_p(\varphi)|^{-1/n}|\varphi|^{1+\frac{p}{n}}\right)\omega^n \leq C.$$

Theorem 1.12 is an interesting variant of Trudinger's inequality on compact Kähler manifolds. The case $p = 1$ settles a conjecture of Aubin (called Hypothèse fondamentale [Aub84]) which is motivated by the search for Kähler-Einstein metrics on Fano manifolds. The conjecture was previously proved by Berman-Berndtsson [BB11] under the assumption that the cohomology class of ω is the first Chern class of an ample holomorphic line bundle.

The relative finite energy class $\mathscr{E}^1$. In [DDNL18b], we develop the variational approach in order to study degenerate complex Monge-Ampère equations with prescribed singularities (see Section 2.3). In order to do so we need to understand the relative version of the Monge-Ampère energy, and its bounded locus $\mathscr{E}^1(X,\theta,\phi)$.

For $u \in \mathscr{E}(X,\theta,\phi)$ with relatively minimal singularities, we define the Monge-Ampère energy of u relative to ϕ as

$$\mathrm{I}_\phi(u) := \frac{1}{n+1}\sum_{k=0}^{n}\int_X (u-\phi)\theta_u^k \wedge \theta_\phi^{n-k}.$$

In the next theorem we collect basic properties of the Monge-Ampère energy:

Theorem 1.13 ([DDNL18b]) *Suppose $u, v \in \mathscr{E}(X,\theta,\phi)$ have relatively minimal singularities. The following hold:*
(i) $\mathrm{I}_\phi(u) - \mathrm{I}_\phi(v) = \frac{1}{n+1}\sum_{k=0}^{n}\int_X (u-v)\theta_u^k \wedge \theta_v^{n-k}$.
(ii) If $u \le \phi$ then, $\int_X (u-\phi)\theta_u^n \le I_\phi(u) \le \frac{1}{n+1}\int_X (u-\phi)\theta_u^n$.
(iii) I_ϕ *is non-decreasing and concave along affine curves. Additionally, the following estimates hold: $\int_X (u-v)\theta_u^n \le I_\phi(u) - I_\phi(v) \le \int_X (u-v)\theta_v^n$.*

We can thus define the Monge-Ampère energy for arbitrary $u \in \mathrm{PSH}(X,\theta,\phi)$ using a familiar formula:

$$I_\phi(u) := \inf\{I_\phi(v) \mid v \in \mathscr{E}(X,\theta,\phi),\ v \text{ has relatively minimal singularities, and } u \le v\}.$$

We then show that if $u \in \mathrm{PSH}(X,\theta,\phi)$ then $\mathrm{I}_\phi(u) = \lim_{t\to\infty} \mathrm{I}_\phi(\max(u,\phi-t))$. We let $\mathscr{E}^1(X,\theta,\phi)$ denote the set of all $u \in \mathrm{PSH}(X,\theta,\phi)$ such that $\mathrm{I}_\phi(u)$ is finite.

It is important to stress that the definition of I_ϕ is given and its properties are proved in the case of a model potential ϕ with small unbounded locus (i.e. ϕ locally bounded outside a closed complete pluripolar set $A \subset X$). The latter assumption is heavily used in order to justify integration by parts.
In later works Mingchen Xia [Min19a] and Chinh Lu [Chi21] proved a very general integration by parts formula. This result allows us to work with the relative Monge-Ampère I_ϕ without assuming that the model potential ϕ has small unbounded locus.

In analogy with what we saw in the paragraph 1.3, one could wonder whether it is possible to define relative Monge-Ampère classes $\mathscr{E}^p(X,\theta,\phi)$ and if the geometry of these classes in interesting. A positive answer is given by Mingchen Xia [Min19b]. The case $p = 1$ was independently treated in [Tru20].

1.4 The theory of Capacities

We recall the circle of ideas related to the ϕ-relative Monge-Ampère capacity. This notion has its roots in [DNL17, DNL15], and it was treated in detail in a couple of works in collaboration with Tamas Darvas and Chinh Lu [DDNL18b, DDNL19].
We start by introducing the main concepts. For this we fix $\chi \in \mathrm{PSH}(X,\theta)$.

Définition 1.2 *Let E be a Borel subset of X. We define the χ-relative capacity of E as*

$$\mathrm{Cap}_\chi(E) := \sup\left\{\int_E \theta_u^n \;\Big|\; u \in \mathrm{PSH}(X,\theta),\ \chi - 1 \le u \le \chi\right\}. \tag{1.4.9}$$

It can be proved that Cap_χ is inner regular, i.e.,

$$\mathrm{Cap}_\chi(E) = \sup\{\mathrm{Cap}_\chi(K) \;\big|\; K \subset E\ ;\ K \text{ is compact}\}.$$

Moreover it is elementary to see that Cap_χ is continuous along increasing sequences, i.e., if $\{E_j\}_j$ increases to E then

$$\mathrm{Cap}_\chi(\cup E_j) = \lim_j \mathrm{Cap}_\chi(E_j).$$

In particular, if ψ is a quasi-psh function then the function $t \mapsto \mathrm{Cap}_\chi(\psi < \chi - t)$ is right-continuous on $\mathbb{R}$.
Such capacity can be compared with the one defined in [GZ05] and [BEGZ10]. We indeed show that if $\chi = (1-\varepsilon)w + \varepsilon V_\theta$, where $w \in \mathrm{PSH}(X,\theta)$, $w \le 0$ and $\varepsilon \in (0,1)$, then for any Borel subset $E \subset X$ one has

$$\mathrm{Cap}_\theta(E) := \mathrm{Cap}_{V_\theta}(E) \le \varepsilon^{-n}\mathrm{Cap}_\chi(E).$$

The *relative χ-extremal function* of E is defined as

$$h_{E,\chi} := \sup\{u \in \mathrm{PSH}(X,\theta) \;\big|\; u \le \chi - 1 \text{ on } E \text{ and } u \le \chi \text{ on } X\}.$$

The *global χ-extremal function* of E is defined as

$$V_{E,\chi} := \sup\{u \in \mathrm{PSH}(X,\theta,\chi) \;\big|\; u \le \chi \text{ on } E\}.$$

We set $M_\chi(E) := \sup_X V^*_{E,\chi}$, where $V^*_{E,\chi}$ denotes the upper semicontinuous regularization of $V_{E,\chi}$. The Alexander-Taylor capacity is then defined as $T_\chi(E) := \exp(-M_\chi(E))$.

We obtain that sets with zero capacity are small, i.e. given a Borel set $B \subset X$, then $\mathrm{Cap}_\chi(B) = 0$ if and only if B is pluripolar.
In similar spirit, we mention that $M_\chi(B) = +\infty$ implies that $\mathrm{Cap}_\chi(B) = 0$.
In order to use Cap_χ in an effective manner, additional assumptions need to be made on the potential χ. We assume that $\chi := \phi$, where ϕ is a model potential and has non-collapsing mass:

$$\mathscr{C}(\phi) = P[\phi] = \phi \quad\text{and}\quad \int_X \theta_\phi^n > 0.$$

For elementary reasons $h^*_{E,\phi}$ is a θ-psh function on X which has the same singularity type as ϕ, in fact $\phi - 1 \le h^*_{E,\phi} \le \phi$. A similar conclusion holds for $V^*_{E,\phi}$ if E is non-pluripolar, more precisely:

$$\phi \le V^*_{E,\phi} \le \phi + M_\phi(E).$$

Indeed, the first estimate is trivial, while for the second one we notice that every candidate potential of $V^*_{E,\phi} - M_\phi(E)$ is non-positive and more singular than ϕ. Hence the supremum of all these potentials has to be less than $P[\phi] = \phi$.

The Monge-Ampère measures of the relative extremal function $h^*_{E,\phi}$ and the global extremal function $V^*_{E,\phi}$ are quite well understood. We indeed prove that given E a (non-pluripolar) Borel set then

(i) $\theta^n_{h^*_{E,\phi}}$ vanishes in the open set $\{h^*_{E,\phi} < 0\} \setminus \bar{E}$. Moreover if $E = K$ is a compact subset of X then

$$\mathrm{Cap}_\phi(K) = \int_K \theta^n_{h^*_{K,\phi}} = \int_X (\phi - h^*_{K,\phi})\theta^n_{h^*_{K,\phi}}.$$

(ii) $\theta^n_{V^*_{E,\phi}}$ vanishes in $X \setminus \overline{E}$.

Lastly we point out that the Alexander-Taylor and Monge-Ampère capacities are related by the following estimates:

$$1 \leq \left(\frac{\int_X \theta^n_\phi}{\mathrm{Cap}_\phi(K)} \right)^{1/n} \leq \max(1, M_\phi(K)),$$

for any compact subset $K \subset X$ with $\mathrm{Cap}_\phi(K) > 0$.

1.5 The space of Singularity Types

In [DDNL21] we study the space of singularities classes $[w], w \in \mathrm{PSH}(X, \theta)$ that we denote by $\mathscr{S}(X, \theta)$. This latter space plays an important role in transcendental algebraic geometry, as its elements represent the building blocks of multiplier ideal sheaves, log-canonical thresholds, etc., bridging the gap between the algebraic and the analytic viewpoint on the subject.

The space $\mathrm{PSH}(X, \theta)$ has a natural complete metric space structure given by the L^1 metric. However the L^1 metric does not naturally descend to $\mathscr{S}(X, \theta)$ making the study of variation of singularity type quite awkward and cumbersome.

On the other hand, "approximating" an arbitrary singularity type $[u]$ with one that is much nicer goes back to the beginnings of the subject. Perhaps the most popular of these approximation procedures is the one that uses Bergman kernels, as first advocated in this context by Demailly [Dem92]. Here, using Ohsawa-Takegoshi type theorems one obtains a (mostly decreasing) sequence $[u_j]$ that in favorable circumstances approaches $[u]$ in the sense that multiplier ideal sheaves, log-canonical thresholds, vanishing theorems, intersection numbers etc. can be recovered in the limit (see for example [Bou02a, Bou02b, Bou04]). Still, no metric topology seems to be known that could quantify the effectiveness or failure of the "convergence" $[u_j] \mapsto [u]$. In the work with Tamas Darvas and Chinh Lu we propose an alternative remedy to this.

We introduce a natural (pseudo)metric $d_{\mathscr{S}}$ on $\mathscr{S}(X,\theta)$ and point out that it fits well with some already existing approaches in the literature. The precise definition of $d_{\mathscr{S}}$ uses the language of geodesic rays from our previous works and that we already described in Section 1.2.

By $\mathscr{R}(X,\theta)$ we denote the space of finite energy geodesic rays emanating from V_θ:

$$\mathscr{R}(X,\theta) := \{[0,\infty) \ni t \mapsto u_t \in \mathscr{E}^1(X,\theta) \text{ s.t. } u_0 = V_\theta \text{ and } t \mapsto u_t \text{ is a } d_1 \text{ geodesic ray}\}.$$

As shorthand convention we will use the notation $\{u_t\}_t \in \mathscr{R}(X,\theta)$ when referring to rays. We then introduce the chordal L^1 geometry on $\mathscr{R}(X,\theta)$:

$$d_1^c(\{u_t\}_t, \{v_t\}_t) := \lim_{t\to\infty} \frac{d_1(u_t, v_t)}{t}. \tag{1.5.10}$$

Note that (basically) from [BDL20] it follows that $t \mapsto d_1(u_t, v_t)$ is convex, hence the map $t \mapsto d_1(u_t, v_t)/t$ is increasing, implying that the limit in (1.5.10) is well defined. We also show that:

Theorem 1.14 ([DDNL21]) *The space $(\mathscr{R}(X,\theta), d_1^c)$ is a complete metric space.*

For $\{u_t\}_t \in \mathscr{R}(X,\theta)$ it is natural to introduce the radial Monge–Ampère energy $I\{\cdot\} : \mathscr{R}(X,\theta) \to \mathbb{R}$ by the formula $I\{u_t\} = \lim_t \frac{I(u_t)}{t} = I(u_1)$, where in the last equality we have used the linearity of I along geodesic rays (previously proved in [DDNL18c]).

We then observe that $\mathscr{S}(X,\theta)$ embeds naturally in $\mathscr{R}(X,\theta)$, endowing the former space with a natural pseudo-metric structure.

Given $\psi \in \mathrm{PSH}(X,\theta)$ with $\psi \leq 0$, we consider a geodesic ray $\{r[\psi]_t\}_t \in \mathscr{R}(X,\theta)$ whose potentials have minimal singularities. The specific construction is as follows. Let $[0,l] \ni t \mapsto r(\psi)_t^l \in \mathscr{E}^1(X,\omega)$ be the geodesic segment with minimal singularity type joining $r(\psi)_0^l = V_\theta$ and $r(\psi)_l^l = \max(\psi, V_\theta - l)$. It can be shown that for any fixed $t > 0$ the family $\{r(\psi)_t^l\}_{l\geq 0}$ is increasing as $l \to \infty$, and its limit equals the geodesic ray with minimal singularity type $t \mapsto r[\psi]_t$. Along the way we also obtain the lower bound $\max(\psi, V_\theta - t) \leq r[\psi]_t$ for all $t \in [0,\infty)$.

Since $\psi \leq \psi'$ implies that $r[\psi]_t \leq r[\psi']_t$ and $r[\psi]_t = r[\psi + C]_t$, $C \in \mathbb{R}$, we obtain that the construction of the ray only depends on the singularity type, giving us a map:

$$r[\cdot] : \mathscr{S}(X,\theta) \to \mathscr{R}(X,\theta). \tag{1.5.11}$$

The basic idea will be to pull back the metric geometry of $\mathscr{R}(X,\theta)$ to $\mathscr{S}(X,\theta)$ via this map.

Via our embedding in (1.5.11), we can also introduce the Monge-Ampère energy of singularity types

$$I_{\mathscr{S}}[\psi] := I\{r[\psi]_t\}.$$

Theorem 1.15 ([DDNL21]) *For $\psi \in \mathrm{PSH}(X,\theta)$ we have*

$$I_{\mathscr{S}}[\psi] = -\int_X \theta_{V_\theta}^n + \frac{1}{n+1} \sum_{j=0}^{n} \int_X \theta_{V_\theta}^j \wedge \theta_\psi^{n-j}. \tag{1.5.12}$$

Finally, we consider the L^1 (pseudo)metric structure of $\mathscr{S}(X,\theta)$, by pulling back the chordal metric structure from $\mathscr{R}(X,\theta)$:

$$d_{\mathscr{S}}([\psi],[\chi]) := d_1^c(\{r[\psi]_t\}_t, \{r[\chi]_t\}_t).$$

In case $[u] \leq [v]$, using (1.5.12), the expression for $d_{\mathscr{S}}([u],[v])$ is especially simple:

Proposition 1.16 ([DDNL21]) *If $[u],[v] \in \mathscr{S}(X,\theta)$ is such that $[u] \leq [v]$ then*

$$d_{\mathscr{S}}([u],[v]) = \frac{1}{n+1}\sum_{j=0}^{n}\bigg(\int_X \theta_{V_\theta}^j \wedge \theta_v^{n-j} - \int_X \theta_{V_\theta}^j \wedge \theta_u^{n-j}\bigg).$$

When $[u] \not\leq [v]$, then a similar simple expression for $d_{\mathscr{S}}([u],[v])$ may not be available, however one can find a useful expression that totally governs the behavior of $d_{\mathscr{S}}([u],[v])$:

Proposition 1.17 ([DDNL21]) *There exists an absolute constant $C > 1$ only dependent on n such that:*

$$d_{\mathscr{S}}([u],[v]) \leq \sum_{j=0}^{n}\bigg(2\int_X \theta_{V_\theta}^j \wedge \theta_{\max(u,v)}^{n-j} - \int_X \theta_{V_\theta}^j \wedge \theta_v^{n-j} - \int_X \theta_{V_\theta}^j \wedge \theta_u^{n-j}\bigg) \leq C d_{\mathscr{S}}([u],[v]).$$

As we will see in the Theorem below, $d_{\mathscr{S}}([u],[v]) = 0$ when the singularities of u and v are essentially indistinguishable (the Lelong numbers, multiplier ideal sheaves, mixed masses of $[u]$ and $[v]$ are the same). More precisely, $d_{\mathscr{S}}([u],[v]) = 0$ if and only if u and v belong to the same relative full mass class. In particular, $u \in \mathscr{E}(X,\theta)$ if and only if $d_{\mathscr{S}}([u],[V_\theta]) = 0$. Consequently, the degeneracy of $d_{\mathscr{S}}$ is quite natural:

Theorem 1.18 ([DDNL21]) *$(\mathscr{S}(X,\theta), d_{\mathscr{S}})$ is a pseudo-metric space. More precisely, the following are equivalent:*
(i) $d_{\mathscr{S}}([\psi],[\chi]) = 0$.
(ii) $r[\psi] = r[\chi]$.
(iii) $\mathscr{C}(\psi) = \mathscr{C}(\chi)$.

Given the $d_{\mathscr{S}}$-continuity of $[u] \to \int_X \theta_u^n$ it is quite natural to introduce the following subspaces for any $\delta \geq 0$:

$$\mathscr{S}_\delta(X,\theta) := \{[u] \in \mathscr{S}(X,\theta) : \int_X \theta_u^n \geq \delta\}.$$

These spaces are $d_{\mathscr{S}}$-closed, and they are also complete:

Theorem 1.19 ([DDNL21]) *For any $\delta > 0$ the space $(\mathscr{S}_\delta(X,\theta), d_{\mathscr{S}})$ is complete.*

Unfortunately the space $(\mathscr{S}(X,\theta), d_{\mathscr{S}})$ is not complete. This is quite natural however, as issues may arise if the non-pluripolar mass vanishes in the $d_{\mathscr{S}}$-limit. In the paper (see [DDNL21, Section 4.2] we give an explicit example of this phenomenon.

Chapter 2

Canonical metrics

2.1 Kähler-Einstein metrics

Given a compact Kähler manifold (X, ω) of complex dimension n, the geometric problem of looking for a Kähler-Einstein (KE) metric $\omega_\varphi := \omega + dd^c\varphi \in \{\omega\}$ is equivalent to solving a complex Monge-Ampère equation

$$(\omega + dd^c\varphi)^n = e^{-\lambda\varphi} f \omega^n$$

where $0 < f \in C^\infty(X, \mathbb{R})$ is a data of the problem and $\lambda \in \mathbb{R}$.

A Kähler manifold with a KE metric with $\lambda > 0$ is called Fano; if $\lambda = 0$ then we refer to the manifold as Calabi-Yau; if $\lambda < 0$ the manifold is said to be of general type.

The positive curvature case ($\lambda > 0$) is the most complicated: Kähler-Einstein metrics do not always exist. Recently, X.X. Chen, S. Donaldson and S. Sun [CDS15a, CDS15b, CDS15c] proved the *Yau-Tian-Donaldson conjecture* for Kähler-Einstein metrics: a Fano manifold X admits a Kähler-Einstein metric if and only if it is K-stable (where the K-stability is a property of the manifold of an algebraic nature).

The existence and uniqueness of the solution $\varphi \in C^\infty(X)$ in the case $\lambda < 0$ were independently proved by Aubin [Aub78] and Yau [Yau78] while the case $\lambda = 0$ was settled by Yau in [Yau78]. Yau's proof of the Calabi conjecture relies on the continuity method and the final goal is to establish *uniform estimates* for the solution φ, i.e. estimates of the type

$$\|\varphi\|_{C^k} \leq C_k \qquad \forall k \in \mathbb{N},$$

where C is a positive constant under control, that does not depend on φ.

The most difficult step is the first one: C^0-estimate. Once the C^0 and the C^2-estimates are in hand, all the higher order estimates can be obtained from them using Evans-Krylov theory, Schauder's estimates and a bootstrap argument. Yau, in his proof notably provided the crucial C^0 a priori estimate making a clever use of Moser's iteration techniques.

In [Koł98], Kołodziej generalized the C^0 a priori estimate. The strength of the latter is that it can be applied to a larger family of Monge-Ampère equations:

$$(\omega + dd^c\varphi)^n = f\omega^n,$$

where we merely ask $0 \leq f \in L^p$ for some $p > 1$. The beautiful proof of this result uses techniques from pluripotential theory.

More precisely, in the case of Kołodziej's approach, the key pluripotential tool is the notion of *Monge-Ampère capacity.* Given a Borel set $E \subset X$, the capacity of E is defined as

$$\mathrm{Cap}_\omega(E) := \sup\left\{ \int_E \omega_v^n \;\Big|\; v \in \mathrm{PSH}(X,\omega),\ -1 \leq v \leq 0 \right\}.$$

Observe that the measure ω_v^n is well-defined since v is a bounded ω-psh function.

Kołodziej's idea is to show that the Monge-Ampère capacity of the sub-level sets $\{\varphi < -t\}$ vanishes for $t > 0$ large enough ($t \geq T_\infty$). This gives the L^∞-bound since *qpsh*-functions on a compact manifold are bounded from above.

Kołodziej's work made clear that pluripotential methods were powerful enough to hope to treat *degenerate* complex Monge-Ampère equations. His uniform estimate can indeed be applied to complex Monge-Ampère equations of the type

$$(\omega + dd^c\varphi)^n = f dV$$

where dV is a smooth volume form, $0 \leq f \in L^p(dV)$ for some $p > 1$.

This opened the way to the works of Berman, Boucksom, Eyssidieux, Guedj and Zeriahi ([GZ05], [GZ07], [BEGZ10], [BBGZ13], etc...) as we are going to discuss in the next Sections.

2.2 Singular KE

The motivation to look for singular Kähler-Einstein metrics comes from the Minimal Model Program (MMP) in birational geometry.
The MMP is part of the birational classification of projective manifolds and the problem is the following: we fix a smooth manifold and we consider its birational equivalence class. The goal is to find the "simplest manifold" (the *Minimal Model*) in this class. The case of surfaces has been studied by the Italian school (guided by Castelnuovo, Enriques and Severi) in the XX century. In higher dimension the situation is more complicated since there exist manifolds which do not have a smooth minimal model. For this reason if we want to have a chance to classify (projective) manifolds we need to work with singular varieties.

One can still make sense of the Kähler-Einstein equation on a singular variety. Such equation reduces to a degenerate complex Monge-Ampère equation. Indeed, given (Y, ω_Y) a singular variety Y endowed with a Kähler metric ω_Y and $\pi : X \to Y$ a resolution of singularities, the KE equation on Y writes as a complex Monge-Ampère equation on the smooth manifold X of type:

$$(\theta + dd^c\varphi)^n = e^{-\lambda\varphi}\mu, \qquad \lambda \in \mathbb{R}. \tag{2.2.1}$$

Here the word "degenerate" stands for the fact that

- the reference form $\theta := \pi^\star \omega_Y$ is not necessarily Kähler but merely semi-positive. Indeed the $(1,1)$-form θ is strictly positive where π is biholomorphic but it vanishes along the exceptional divisor E.

- the measure μ is not necessarily a smooth volume form (it can have divisorial singularities for example). It is just a positive non-pluripolar measure with $\mu(X) = \int_X (\pi^\star \omega_Y)^n > 0$.

We first look for a *weak* solution of (2.2.1) and then we study the regularity of it. When $\lambda \leq 0$, the existence and uniqueness (up to constant when $\lambda = 0$) of a weak solution $\varphi \in \mathscr{E}(X, \theta)$ is guaranteed by [BEGZ10].

The regularity problem (still in the case $\lambda \leq 0$) was worked out by Eyssidieux, Guedj and Zeriahi in [EGZ09]: they proved that when $\mu = f\,dV$ and f has very specific divisorial singularities, i.e. $f = \frac{|\sigma_1|^{2k} + \cdots |\sigma_p|^{2k}}{|\tau_1|^{2\ell} + \cdots |\tau_q|^{2\ell}}$, with $k, \ell \in \mathbb{R}^+$ with σ_i, τ_i holomorphic sections of some line bundle, then the solution φ is smooth outside an analytic set (that is $\mathrm{Amp}\,(\theta)^c \cup \cap_i \{\sigma_i = 0\} \cup \cap_i \{\tau_i = 0\}$).

As consequence they showed that general type varieties, as well as a very wide class of Calabi-Yau varieties (precisely, polarized $\mathbb{Q}$-Calabi-Yau varieties), admit a unique *singular Kähler-Einstein* metric. This notion can be understood as smooth and Kähler-Einstein in the usual sense on the ample locus of the canonical bundle and in a generalized sense (of currents) on the exceptional locus.

Moreover, in the same article it is shown that the local potential of this singular Kähler-Einstein metric is bounded on all X.

In the last years there have been a lot of attempts to establish the global continuity of the potential. This was established in a very recent preprint by Guedj, Guenancia and Zeriahi [GGZ21].

A more general case (when the reference form represents a big class) was treated in [QT21] where the author proves that singular Kähler-Einstein metrics on log canonical varieties of general type have continuous potentials on the ample locus outside of the non-klt part.

Nevertheless, the problem of understanding the asymptotic behavior of a singular KE metric near the boundary of the ample locus of $\theta = \pi^* \omega_Y$ (or equivalently near the singular points of Y) is still widely open.

The strategy to prove the existence and regularity of a solution to (2.2.1) is to establish a uniform a priori estimate. The first crucial step is the C^0-estimates: in the case $\lambda < 0$ this is a simple consequence of the maximum principle; on the other hand, in the case $\lambda = 0$ this estimate is quite delicate.

In a joint paper with Vincent Guedj and Henri Guenancia we revisit the proof by Yau [Yau78], as well as its recent generalizations [Koł98, EGZ09], and establish the following:

Theorem 2.1 ([DNGG21]) *Let $\{\theta\}$ be a big cohomology class of volume $V > 0$. Let ν and $\mu = f\,\nu$ be probability measures, with $0 \leq f \in L^p(\nu)$ for some $p > 1$. Assume the assumptions (H1)-(H2) or (H1)-(H2') are satisfied:*

(H1) $\exists \alpha > 0, A_\alpha > 0$ *such that* $\forall \psi \in PSH(X,\theta)$, $\int_X e^{-\alpha(\psi - \sup_X \psi)} d\nu \leq A_\alpha$;

(H2) there exists $C > 0$ *such that* $(\int_X |f|^p \, d\nu)^{1/p} \leq C$*;*

(H2') there exists $C, \varepsilon > 0$ *such that* $\int_X |f| |\log f|^{n+\varepsilon} \, d\nu \leq C$.

Let φ *be the unique* θ*-psh function with minimal singularities such that*

$$V^{-1}(\theta + dd^c\varphi)^n = \mu,$$

and $\sup_X \varphi = 0$*. Then* $-M \leq \varphi - V_\theta \leq 0$ *where*

$$M = 1 + C^{1/n} A_\alpha^{1/nq} \, e^{\alpha/nq} b_n \left[5 + e\alpha^{-1} C (q!)^{1/q} A_\alpha^{1/q}\right],$$

where b_n *is a uniform constant such that* $\exp(-1/x) \leq b_n^n x^{2n}$ *for all* $x > 0$.

The importance of such a result is that gives a very explicit expression of the lower bound for $\varphi - V_\theta$. As we are going to see in Section 2.4, the above Theorem will be fundamental in order to develop the first steps of pluripotential theory in family.

The above Theorem has to be thought as a (relative) C^0-estimates for φ. But, what about higher regularity?

Question 2:

- Let $\mu = f\, dV$, $f > 0$, a smooth volume form with $\mu(X) = \mathrm{vol}(\{\theta\})$, and let φ be the unique θ-psh function such that $\theta_\varphi^n = \mu$ normalized with $\sup_X \varphi = 0$. Is φ smooth in $\mathrm{Amp}\,(\theta)$?

- Or more generally, assume that the density f is smooth outside some analytic set D. Is φ is smooth in $\mathrm{Amp}\,(\theta) \cap D^c$?

When f is smooth and $\beta := \{\theta\}$ is a Kähler class, a positive answer is given by Yau's theorem. Building on the latter result, in [BEGZ10] they prove that this is still the case when f is smooth and α is big and nef. Their strategy relies on the approximation of α by Kähler classes β_ε.

The general case of a big cohomology class is widely open and as I am going to describe in the next sections, achievements in this direction will lead the way to answer to other regularity issues related to geometric problems, such as the regularity of singular Kähler-Einstein metrics, and the existence and regularity of singular constant scalar curvature metrics.

The case $\lambda > 0$ is completely different since solutions of the Monge-Ampère equation $\theta_\varphi^n = e^{-\lambda\varphi}\mu$ do not always exist. As consequence singular Kähler-Einstein metrics of positive curvature are more difficult to construct. It is already so in the smooth case [CDS15c]. Their first properties have been obtained in [BEGZ10, BBE$^+$19].

Pushing further these works, in a joint paper with Vincent Guedj we provide a necessary and sufficient analytic condition for their existence, generalizing a result of Tian [Tia97] and Phong-Song-Sturm-Weinkove [PSSW08]. More precisely, we work in the following singular context: we consider (Y, D) a log Fano pair, that is a klt pair (Y, D) such that Y is projective and $-(K_Y + D)$ is ample and we fix a reference smooth strictly psh metric ϕ_Y on $-(K_Y + D)$, with curvature ω_Y. There is then a canonical measure attached to ϕ_Y, the so-called adapted measure μ_{ϕ_Y}. The volume of (Y, D) is

$$V := c_1(Y, D)^n = \int_Y \omega_Y^n.$$

A *Kähler-Einstein metric* T for the log Fano pair (Y, D) is a finite energy current $T \in c_1(Y, D)$ such that $T^n = V \cdot \mu_T$, where μ_T is the adapted measure associated to the potential of the $(1, 1)$-current T.
If we choose a log resolution $\pi : X \to Y$, the equation becomes

$$(\theta + dd^c\varphi)^n = e^{-\varphi}\widetilde{\mu}_0$$

where $\theta = \pi^*\omega_Y$ is semipositive and big and $\widetilde{\mu}_0 = \prod_i |f_i|^{2a_i} dV$.

Following and idea of Darvas-Rubinstein [DR17], we then prove the following:

Theorem 2.2 ([DNG18]) *Let (Y, D) be a log Fano pair. It admits a unique Kähler-Einstein metric iff there exists $\varepsilon, M > 0$ such that for all $\varphi \in \mathscr{H}_\theta := \pi^*\mathscr{H}_{\omega_Y}$,*

$$\mathscr{F}(\varphi) \leq -\varepsilon d_1(0, \varphi) + M.$$

Here the distance d_1 is the L^1-distance w.r.t. the semi-positive form θ (see Section 1.3) and the functional $\mathscr{F}$, known as the *Ding functional*, is defined as

$$\mathscr{F}(\varphi) := I(\varphi) + \log\left[\int_{\tilde{X}} e^{-\varphi} d\widetilde{\mu}_0\right],$$

where I is the Monge-Ampère energy w.r.t. θ. Let me observe that the statement of the above Theorem is independent on the resolution π.
It is worth to mention that such result relies on a deep study and understand of the geodesics in the space $\mathscr{H}_\theta := \pi^*\mathscr{H}_{\omega_Y} = \{\varphi \in \mathrm{PSH}(X, \theta),\ \theta_\varphi := \theta + dd^c\varphi > 0 \text{ in } \mathrm{Amp}\,(\theta)\}$ and on a consistent definition of the distances d_p w.r.t. the degenerate form θ. A large part of our work is indeed dedicated to this. This ideas were then taken into account, simplified and generalized in the subsequent work with Chinh Lu [DNL20] (see Section 1.3).

2.3 Singular KE with prescribed singularities

In a series of works [DDNL18b, DDNL19] with Chinh Lu and Tamas Darvas, we studied solutions to complex Monge-Ampère equations with prescribed singularity. One starts with a potential $u \in \mathrm{PSH}(X,\theta)$ and a density $0 \leq f \in L^p(X)$, $p > 1$, and looks for a solution $u \in \mathrm{PSH}(X,\theta)$ such that $\theta_u^n = f\omega^n$ and $[u] = [\phi]$. The compatibility condition $\int_X \theta_\phi^n = \int_X f\omega^n > 0$ is necessary for the probability of this equation. Beyond this normalization condition, as it turns out, the necessary and sufficient condition for the well posedness is that ϕ is a model potential (i.e. $\phi = \mathscr{C}(\phi) = P[\phi]$). The result we achieved states as

Theorem 2.3 ([DDNL19]) *Let $\lambda \geq 0$. Assume ϕ is a model potential and that μ is a non-pluripolar positive measure on X such that $\mu(X) = \int_X \theta_\phi^n > 0$. Then there exists a unique (up to constant when $\lambda = 0$) $u \in \mathscr{E}(X,\theta,\phi)$ such that $\theta_u^n = e^{\lambda u}\mu$.*

In addition to this, in the particular case when $\mu = f\omega^n$ with $f \in L^p(X,\omega^n)$, $p > 1$ we have that

$$\phi - C\Big(\lambda, p, \omega, \int_X \theta_\phi^n, \|f\|_{L^p}\Big) \leq u \leq \phi \leq 0.$$

When θ is Kähler and $\phi = 0$, the first part of the Theorem is due to [GZ07] while the second part reduces to Kołodziej's L^∞-estimate [Koł98] in the context of the Calabi-Yau theorem [Yau78]. In the general case $\{\theta\}$ a big class and $\phi = V_\theta$, then the above result was proved in [BBGZ13] and [EGZ09].

This result is a significant generalization of Kołodziej's L^∞ estimate [Koł98] to our relative context and it uses new pluripotential tools: the generalized Monge-Ampère capacity Cap_χ in the big setting (see Section 1.4). In [DDNL18b] and [DDNL19] we use two different approaches to prove such a result. In particular, in [DDNL18b] we develop the variational approach using the relative Monge-Ampère energy I_ϕ (see paragraph 1.3) following the ideas in [BBE+19].
On the other hand, in [DDNL19] we exploit the theory of capacities. One of the building blocks for the arguments is the following:

Theorem 2.4 ([DDNL19]) *Fix $a \in [0,1), A > 0$, $\chi \in \mathrm{PSH}(X,\theta)$ and $0 \leq f \in L^p(X,\omega^n)$ for some $p > 1$. Assume that $u \in \mathrm{PSH}(X,\theta)$, normalized by $\sup_X u = 0$, satisfies*

$$\theta_u^n \leq f\omega^n + a\theta_\chi^n.$$

Assume also that

$$\int_E f\omega^n \leq A[\mathrm{Cap}_\chi(E)]^2, \tag{2.3.2}$$

for every Borel subset $E \subset X$. If $P[u]$ is less singular than ψ then

$$\chi - C\Big(\|f\|_{L^p}, p, (1-a)^{-1}, A\Big) \leq u.$$

The ideas behind the proof of the above result are very similar to the ones in [Koł98], with the "only" difference that before applying them we had to develop the whole theory of generalized capacities (see Section 1.4) in the setting of big cohomology classes.
We give a sketch below of the arguments we used in order to get the reader more familiar with these generalized capacities. For $t > 0$ we set $g(t) := [\mathrm{Cap}_\chi(u < \chi - t)]^{1/n}$. It can be

proved that $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a decreasing right-continuous function and that $g(+\infty) = 0$. Let $s \in [0,1]$ and suppose $v \in \mathrm{PSH}(X, \theta)$ satisfies $\chi - 1 \le v \le \chi$. Since $P[u]$ is less singular then χ, the comparison principle gives

$$\begin{aligned} s^n \int_{\{u<\chi-t-s\}} \theta_v^n &\le s^n \int_{\{u<(1-s)\chi+sv-t\}} \theta_v^n \le \int_{\{u<(1-s)\chi+sv-t\}} \theta_{(1-s)\chi+sv}^n \\ &\le \int_{\{u<(1-s)\chi+sv-t\}} \theta_u^n \le \int_{\{u<\chi-t\}} \theta_u^n, \end{aligned}$$

hence taking supremum over all candidates v we arrive at

$$s^n \mathrm{Cap}_\chi(u<\chi-t-s) \le \int_{\{u<\chi-t\}} \theta_u^n. \tag{2.3.3}$$

Also, for each $t > 0$, since $P[u]$ is less singular than χ, it follows once again from the comparison principle and our assumptions that

$$\int_{\{u<\chi-t\}} \theta_u^n \le \int_{\{u<\chi-t\}} f\omega^n + a\int_{\{u<\chi-t\}} \theta_\chi^n \le \int_{\{u<\chi-t\}} f\omega^n + a\int_{\{u<\chi-t\}} \theta_u^n.$$

Since $a \in [0,1)$ we thus get

$$\int_{\{u<\chi-t\}} \theta_u^n \le \frac{1}{1-a}\int_{\{u<\chi-t\}} f\omega^n.$$

Combining this with (2.3.3) we then get

$$s^n \mathrm{Cap}_\chi(u<\chi-t-s) \le \frac{1}{1-a}\int_{\{u<\chi-t\}} f\omega^n. \tag{2.3.4}$$

Therefore, combining (2.3.2) with (2.3.4) we obtain

$$s^n \mathrm{Cap}_\chi(u<\chi-t-s) \le \frac{A}{1-a}[\mathrm{Cap}_\chi(u<\chi-t)]^2, \tag{2.3.5}$$

which implies

$$sg(t+s) \le Bg^2(t),\ \ \forall t>0, \forall s \in [0,1],$$

where $B = (A/(1-a))^{1/n}$.
By an application of Hölder's inequality, there is a constant $t_0 > 0$ depending only on $a, p, \|f\|_p$ such that

$$\int_{\{u<\chi-t_0\}} f\omega^n \le \int_X \frac{|\chi-u|}{t_0} f\omega^n \le \frac{\|f\|_p}{t_0}\left(\int_X |u-\max(u,\chi)|^q\omega^n\right)^{1/q} \le \frac{1-a}{(2B)^n}, \tag{2.3.6}$$

where $q > 1$ is the conjugate exponent of p.
It is a technical but crucial point to observe that in the last line above both u and $\max(u,\chi)$ satisfy $\sup_X u = 0, \sup_X \max(u,\chi) = 0$, hence the constant t_0 can be chosen to be only dependent on $p, \|f\|_p, (1-a)^{-1}, A$ (but not on u and χ).

It then follows from (2.3.4) and (2.3.6) that $g(t_0+1) \le (2B)^{-1}$. Thus $g(t_0+3)=0$. We can then conclude that $u \ge \chi - t_0 - 3$ almost everywhere on X, hence everywhere as desired.

We emphasize that when $\mu = f\omega^n$ with $0 \le f \in L^p$ for some $p>1$, Theorem 2.3 gives the *relative* L^∞-estimate $|u-\phi| \le C$. The following question on higher regularity is then natural:

Question 3: Let $\phi \in \mathrm{PSH}(X,\theta) \cap C^\infty(\Omega)$, where Ω is a (dense) open subset of X. Assume μ is a smooth volume form such that $\mu(X) = \int_X \theta_\phi^n > 0$. Let $\varphi \in \mathrm{PSH}(X,\theta)$ be the unique solution of

$$\begin{cases} \theta_\varphi^n = \mu, \\ [\varphi] = [\phi], \\ \sup_X \varphi = 0. \end{cases}$$

Is φ smooth in $\mathrm{Amp}\,(\theta) \cap \Omega$?

Question 3 can be viewed yet as an another generalization of Question 2, even if much harder. In this setting indeed even the "baby" case when $\theta = \omega$ is a Kähler form and f is smooth is completely open.

Another question one could wonder about is *stability* of such solutions. More precisely, one might ask what happens if one considers a family of such equations, where the prescribed singularity type $[\phi_j]$ converges to some fixed singularity type $[\phi]$. In such a case we show that the solutions ψ_j converge to ψ in capacity as expected, further evidencing the practicality of the $d_{\mathscr{S}}$-topology:

Theorem 2.5 ([DDNL21]) *Given $\delta > 0$ and $p > 1$ suppose that:*
∘ $[\phi_j], [\phi] \in \mathscr{S}_\delta(X,\omega)$, $j \ge 0$ *satisfy* $[\phi_j] = [P[\phi_j]]$, $[\phi] = [P[\phi]]$ *and* $d_{\mathscr{S}}([\phi_j],[\phi]) \to 0$.
∘ $f_j, f \ge 0$ *are such that* $\|f\|_{L^p}, \|f_j\|_{L^p}$, $p>1$, *are uniformly bounded and* $f_j \to_{L^1} f$.
∘ $\psi_j, \psi \in \mathrm{PSH}(X,\theta)$, $j \ge 0$ *satisfy* $\sup_X \psi_j = 0$, $\sup_X \psi = 0$ *and*

$$\begin{cases} \theta_{\psi_j}^n = f_j\omega^n \\ [\psi_j] = [\phi_j] \end{cases}, \quad \begin{cases} \theta_\psi^n = f\omega^n \\ [\psi] = [\phi]. \end{cases}$$

Then ψ_j converges to ψ in capacity, in particular $\|\psi_j - \psi\|_{L^1} \to 0$.

Solutions of complex Monge-Ampère equations are linked to existence of special Kähler metrics. In particular, we can think of the solution to $\theta_u^n = f\omega^n$ as a potential with prescribed singularity type and prescribed Ricci curvature in the philosophy of the Calabi-Yau theorem. As an immediate application of the resolution of the Monge-Ampère equation $\theta_u^n = e^{\lambda u} f\omega^n$ with prescribed singularities $[u] = [\phi]$, we obtain existence of singular *Kähler-Einstein* (KE) metrics with prescribed singularity type on Kähler manifolds of general type.

Corollary 2.6 ([DDNL18b]) *Let X be a smooth projective manifold with canonical ample ($K_X > 0$) and let h be a smooth Hermitian metric on K_X with $\theta := \Theta(h) > 0$. Suppose also that $\phi \in \mathrm{PSH}(X, \theta)$ is a model potential, has small unbounded locus and $\int_X \theta_\phi^n > 0$. Then there exists a unique singular KE metric $he^{-\phi_{KE}}$ on K_X ($\theta^n_{\phi_{KE}} = e^{\phi_{KE}+f_\theta}\theta^n$, where f_θ is the Ricci potential of θ satisfying $\mathrm{Ric}\ \theta = \theta + dd^c f_\theta$), with $\phi_{KE} \in \mathrm{PSH}(X, \theta)$ having the same singularity type as ϕ.*

An analogous result also holds on Calabi-Yau manifolds. For the sake of completeness, we should mention that the existence of singular Kähler-Einstein metrics with prescribed singularities on a Fano manifold is studied in [Tru21].

As another application of the resolution of complex Monge-Ampère equations with prescribed singularities we confirm the log-concavity conjecture of Boucksom-Eyssidieux-Guedj-Zeriahi [BEGZ10, Conjecture 1.23], informally referred to as the "log-concavity conjecture" of total masses:

Theorem 2.7 ([DDNL19]) *Let $\theta^1, \dots, \theta^n$ be smooth closed real $(1,1)$-forms and $u_j \in \mathrm{PSH}(X, \theta^j)$. Then*

$$\int_X \theta^1_{u_1} \wedge \ldots \wedge \theta^n_{u_n} \geq \left(\int_X (\theta^1_{u_1})^n\right)^{\frac{1}{n}} \cdots \left(\int_X (\theta^n_{u_n})^n\right)^{\frac{1}{n}}. \tag{2.3.7}$$

In particular, $\theta_u \mapsto (\int_X \theta_u^n)^{\frac{1}{n}}$ is concave, and so is the map $\theta_u \mapsto \log(\int_X \theta_u^n)$.

If equality holds in (2.3.7), it does not necessarily mean that the singularity types of the u_j are the same up to scaling (as one would perhaps expect). Still, it remains an interesting question to characterize the conditions under which equality is attained.

The proof of the above result is a clever combination of all the ingredients we have in hands and here below we present a sketch. Without loss of generality we can assume that the classes of $\{\theta^j\}$ are big and their masses are non-zero. Otherwise the right-hand side of the inequality to be proved is zero. In fact, after re-scaling, we can assume that $\int_X \omega^n = \int_X (\theta^j_{u_j})^n = 1,\ j \in \{1, \dots, n\}$.
We know from [DDNL18b] (see Section 1.3) that $P_{\theta^j}[u_j]$ is a model potential and that

$$\int_X \theta^1_{P_{\theta^1}[u_1]} \wedge \ldots \wedge \theta^n_{P_{\theta^n}[u_n]} = \int_X \theta^1_{u_1} \wedge \ldots \wedge \theta^n_{u_n}.$$

For each j, Theorem 2.3 insures existence of $\varphi_j \in \mathcal{E}(X, \theta^j, P_\theta[u_j])$ such that $\left(\theta^j_{\varphi_j}\right)^n = \omega^n$, therefore

$$\int_X \theta^1_{\varphi_1} \wedge \ldots \wedge \theta^n_{\varphi_n} = \int_X \theta^1_{P_{\theta^1}[u_1]} \wedge \ldots \wedge \theta^n_{P_{\theta^n}[u_n]}.$$

Finally, an application of the mixed inequalities for the pluripolar products [BEGZ10] gives $\theta^1_{\varphi_1} \wedge \ldots \wedge \theta^n_{\varphi_n} \geq \omega^n$. The result follows after we combine the above identities and we integrate.

2.4 Singular KE in family

The solution of the (singular) Calabi Conjecture [Yau78, EGZ09] provides a very powerful existence theorem for Kähler-Einstein metrics with negative or zero Ricci curvature. It is important to study the ways in which these canonical metrics behave when they are moving in families. In the joint work with Vincent Guedj and Henri Guenancia [DNGG21] we consider the case when both the complex structure and the Kähler class vary and we try and understand how the corresponding metrics can degenerate.

In all what follows, given a positive real number r, we denote by $\mathbb{D}_r := \{z \in \mathbb{C}; |z| < r\}$ the open disk of radius r in the complex plane. If $r = 1$, we simply write $\mathbb{D}$ for $\mathbb{D}_1$.

Let $\mathscr{X}$ be an irreducible and reduced complex Kähler space. We let $\pi : \mathscr{X} \to \mathbb{D}$ denote a proper, surjective holomorphic map such that each fiber $X_t = \pi^{-1}(t)$ is a n-dimensional, reduced, irreducible, compact Kähler space, for any $t \in \mathbb{D}$.

We pick a covering $\{U_\alpha\}_\alpha$ of $\mathscr{X}$ by open sets admitting an embedding $j_\alpha : U_\alpha \hookrightarrow \mathbb{C}^N$ for some $N \geq n+1$. Moreover, we fix a Kähler form ω on $\mathscr{X}$. Up to refining the covering, the datum of ω is equivalent to the datum of Kähler metrics on open neighborhoods of $j_\alpha(U_\alpha) \subset \mathbb{C}^N$ that agree on each intersection $U_\alpha^{\mathrm{reg}} \cap U_\beta^{\mathrm{reg}}$. Equivalently, ω is a genuine Kähler metric on $\mathscr{X}_{\mathrm{reg}}$ such that $(j_\alpha)_*(\omega|_{U_\alpha^{\mathrm{reg}}})$ is the restriction of a Kähler metric defined on an open neighborhood of $j_\alpha(U_\alpha) \subset \mathbb{C}^N$.

One important property that Kähler metrics satisfy is that their pull back under a modification is a smooth form (i.e. locally the restriction of a smooth form under a local embedding in $\mathbb{C}^N$); in particular, it is dominated by a Kähler form.

For each $t \in \mathbb{D}$, we set

$$\omega_t := \omega_{|X_t}.$$

We fix a smooth, closed differential $(1,1)$-form Θ on $\mathscr{X}$ and set $\theta_t = \Theta_{|X_t}$. Up to shrinking $\mathbb{D}$, one will always assume that there exists a constant $C_\Theta > 0$ such that

$$-C_\Theta \omega \leq \Theta \leq C_\Theta \omega. \tag{2.4.8}$$

In particular, one has the inclusion $\mathrm{PSH}(X_t, \theta_t) \subseteq \mathrm{PSH}(X_t, C_\Theta \omega_t)$. We assume that the cohomology classes $\{\theta_t\} \in H^{1,1}(X_t, \mathbb{R})$ are psef, i.e. the sets $\mathrm{PSH}(X_t, \theta_t)$ are non-empty for all t.

The problem we focus on is to get a better understanding of the deformation of a family of metrics $\theta_t + dd^c \varphi_t$. The idea is to apply the previous uniform estimates Theorem 2.1 when the complex structure of the underlying manifold is moving, but in order to do so we need insure that the assumptions (H1) and (H2) are satisfied (uniformly w.r.t. the fiber).

One of the main results of the paper allows us to check hypothesis (H1), as soon as the mean value of sup-normalized θ_t-psh functions is uniformly controlled.

Theorem 2.8 ([DNGG21]) *In the above setting, there exist $\alpha > 0$ and constants $A_\alpha, C > 0$ such that for all $t \in \overline{\mathbb{D}}_{1/2}$ and for all $\varphi_t \in \mathrm{PSH}(X_t, \theta_t)$ with $\sup_{X_t} \varphi_t = 0$,*

$$\int_{X_t} e^{-\alpha \varphi_t} \omega_t^n \leq C \exp\left\{-A_\alpha \int_{X_t} \varphi_t \omega_t^n\right\}. \tag{2.4.9}$$

The above result has to be thought as a family version of Skoda's integrability theorem. The proof basically follows the ideas in Zeriahi [Zer01]. It is worth mentioning that, in parallel to the very general case (whose proof is very tricky), we provide a very explicit result in the projective case:

Proposition 2.9 ([DNGG21]) *Let $V \subseteq \mathbb{P}^N$ be a projective variety of complex dimension n and degree d. Let $\omega = \omega_{\mathrm{FS}}|_V$ and $\varphi \in \mathrm{PSH}(V, \omega)$ be such that $\sup_V \varphi = 0$. Then*

$$\int_V e^{-\frac{1}{nd}\varphi}\omega^n \leq (4n)^n \cdot d \cdot \exp\left\{-\frac{1}{nd}\int_V \varphi\omega^n\right\}.$$

Going back to assumption (H1) in Theorem 2.1: it is classical that one can compare the supremum and the mean value of θ-psh functions on a fixed compact Kähler variety. We conjecture that the following results holds:

Conjecture 2.1 *There exists a constant $C > 0$ such that: the inequality*

$$\sup_{X_t} \varphi_t - C \leq \frac{1}{V}\int_{X_t} \varphi_t \, {\omega_t}^n \leq \sup_{X_t} \varphi_t$$

holds for all $t \in \overline{\mathbb{D}}_{1/2}$ and for every function $\varphi_t \in \mathrm{PSH}(X_t, \theta_t)$.

In the paper, we propose a large class of families for which the conjecture holds.

Assumption 2.2 *We consider the following settings:*

1. *The map π is projective,*
2. *The map π is locally trivial,*
3. *The fibers X_t are smooth for $t \neq 0$,*
4. *The fibers X_t have isolated singularities for every $t \in \mathbb{D}$.*

Proposition 2.10 ([DNGG21]) *Assume 2.2 is satisfied. Then Conjecture 2.1 holds. That is, there exists a constant $C > 0$ such that: the inequality*

$$\sup_{X_t} \varphi_t - C \leq \frac{1}{V}\int_{X_t} \varphi_t \, {\omega_t}^n \leq \sup_{X_t} \varphi_t$$

holds for all $t \in \overline{\mathbb{D}}_{1/2}$ and for every function $\varphi_t \in \mathrm{PSH}(X_t, \theta_t)$.

By combining the above two results we get that the assumption (H1) is satisfied (uniformly on t). The assumptions (H2)/(H2') have to be checked for each specific case.

Families of manifolds of general type

Let $\mathscr{X}$ be an irreducible and reduced complex space endowed with a Kähler form ω and a proper, holomorphic map $\pi : \mathscr{X} \to \mathbb{D}$. We assume that for each $t \in \mathbb{D}$, the (schematic) fiber X_t is a n-dimensional Kähler manifold X_t of general type, i.e. such that its canonical bundle K_{X_t} is big. In particular, $\mathscr{X}$ is automatically non-singular and the map π is smooth. We fix Θ a closed differential $(1,1)$-form on $\mathscr{X}$ which represents $c_1(K_{\mathscr{X}/\mathbb{D}})$ and set $\theta_t = \Theta_{|X_t}$.

It follows from [BEGZ10], a generalization of the Aubin-Yau theorem [Aub78, Yau78], that there exists a unique Kähler-Einstein current on X_t. This is a positive closed current T_t in $c_1(K_{X_t})$ which is a smooth Kähler form in the ample locus $\mathrm{Amp}\,(K_{X_t})$, where it satisfies the Kähler-Einstein equation

$$\mathrm{Ric}(T_t) = -T_t.$$

It can be written $T_t = \theta_t + dd^c\varphi_t$, where φ_t is the unique θ_t-psh function with minimal singularities that satisfies the complex Monge-Ampère equation

$$(\theta_t + dd^c\varphi_t)^n = e^{\varphi_t + h_t}\omega_t^n \quad \text{on } \mathrm{Amp}\,(K_{X_t}),$$

where h_t is such that $\mathrm{Ric}(\omega_t) - dd^c h_t = -\theta_t$ and $\int_{X_t} e^{h_t}\omega_t^n = \mathrm{vol}(K_{X_t})$. For $x \in \mathscr{X}$, set

$$\phi(x) := \varphi_{\pi(x)}(x) \tag{2.4.10}$$

and consider

$$V_\Theta = \sup\{u \in \mathrm{PSH}(\mathscr{X}, \Theta);\ u \le 0\}. \tag{2.4.11}$$

We prove that conditions (H1) and (H2) are satisfied in this setting. It then follows from Theorem 2.1 and the plurisubharmonic variation of the T_t's ([CGP17]) that $\phi - V_\Theta$ is uniformly bounded on compact subsets of $\mathscr{X}$:

Theorem 2.11 ([DNGG21]) *Let $\pi : \mathscr{X} \to \mathbb{D}$ be a smooth Kähler family of manifolds of general type, let $\Theta \in c_1(K_{\mathscr{X}/\mathbb{D}})$ be a smooth representative and let ϕ be the Kähler-Einstein potential as in* (2.4.10). *Given any compact subset $\mathscr{K} \Subset \mathscr{X}$, there exists a constant $M_{\mathscr{K}}$ such that the following inequality*

$$-M_{\mathscr{K}} \le \phi - V_\Theta \le M_{\mathscr{K}}$$

holds on $\mathscr{K}$, where V_Θ is defined by (2.4.11).

The same results can be proved if the family $\pi : \mathscr{X} \to \mathbb{D}$ is replaced by a smooth family $\pi : (\mathscr{X}, B) \to \mathbb{D}$ of pairs (X_t, B_t) of log general type, i.e. such that (X_t, B_t) is klt and $K_{X_t} + B_t$ is big for all $t \in \mathbb{D}$.

Families of $\mathbb{Q}$-Calabi-Yau varieties

A $\mathbb{Q}$-Calabi-Yau variety is a compact, normal Kähler space X with canonical singularities such that the $\mathbb{Q}$- line bundle K_X is torsion. Up to taking a finite, quasi-étale cover one can assume that $K_X \sim_{\mathbb{Z}} \mathscr{O}_X$. Given any Kähler class α on X, it follows from [EGZ09] and [Pău08] that there exists a unique singular Ricci flat Kähler metric $\omega_{\mathrm{KE}} \in \alpha$, i.e. a closed, positive current $\omega_{\mathrm{KE}} \in \alpha$ with globally bounded potentials inducing a smooth, Ricci-flat Kähler metric on X_{reg}.

Now, we can consider families of such varieties and ask how the bound on the potentials vary. This is the content of the following result

Theorem 2.12 ([DNGG21]) *Let $\mathscr{X}$ be a normal, $\mathbb{Q}$-Gorenstein Kähler space and let $\pi : \mathscr{X} \to \mathbb{D}$ be a proper, surjective, holomorphic map. Let α be a relative Kähler cohomology class on $\mathscr{X}$ represented by a relative Kähler form ω. Assume additionally that*

- *The relative canonical bundle $K_{\mathscr{X}/\mathbb{D}}$ is trivial.*
- *The central fiber X_0 has canonical singularities.*
- *Assumption 2.2 is satisfied.*

Up to shrinking $\mathbb{D}$, each fiber X_t is a $\mathbb{Q}$-Calabi-Yau variety. Let $\omega_{\mathrm{KE},t} = \omega_t + dd^c\varphi_t$ be the singular Ricci-flat Kähler metric in α_t, normalized by $\int_{X_t} \varphi_t\omega_t^n = 0$. Then, given any compact subset $K \Subset \mathbb{D}$, there exists $C = C(K) > 0$ such that

$$\mathrm{osc}_{X_t}\varphi_t \leq C$$

for any $t \in K$, where $\mathrm{osc}_{X_t}(\varphi_t) = \sup_{X_t}\varphi_t - \inf_{X_t}\varphi_t$.

In the case of a projective smoothing (i.e. when $\mathscr{X}$ admits a π-ample line bundle and X_t is smooth for $t \neq 0$), the result above has been obtained previously by Rong-Zhang [RZ11] by using Moser iteration process.

2.5 CscK metrics

Up to now we focused on Monge-Ampère equations related to the construction of (singular) Kähler-Einstein metrics. But, these metrics do not always exist and the obstructions to their existence are very well-known.

The next in line version of special metrics involves the notion of scalar curvature. Given a Kähler form ω on a compact Kähler manifold X we define its *scalar curvature* as

$$S(\omega) := \mathrm{tr}_\omega(\mathrm{Ric}(\omega)) = n\frac{\mathrm{Ric}(\omega) \wedge \omega^{n-1}}{\omega^n}.$$

We then look for Kähler metrics with *constant scalar curvature* (cscK for short). In particular, we look for a Kähler metric $\omega_\varphi \in \{\omega\}$ such that $S(\omega_\varphi) = \bar{S}$, $\bar{S} \in \mathbb{R}$.

Integrating both sides with respect to ω_φ^n, we find that $\bar{S}$ is a cohomological constant equal to $nV^{-1}c_1(X)\cdot\{\omega\}^{n-1}$, where $V=\int_X \omega^n$.

In the 90's Tian made an influential conjecture stating that the existence of a cscK metric is equivalent to the properness of an energy functional, called (Mabuchi) *K-energy*. There were several attempts by many authors in this direction. The conjecture was first proved in the (Fano) KE case by Darvas and Rubinstein [DR17]; the fact that the existence of a cscK metric implies the properness of the K-energy is due to [BDL20], while the reverse implication was proved more recently by Chen and Cheng [CC20a, CC20b, CC18].

The K-energy is a functional on the space of Kähler potentials $\mathscr{H}_\omega$ defined as

$$K(\varphi):=\mathrm{Ent}_{\omega^n}(\omega_\varphi^n)+J_{-\mathrm{Ric}(\omega)}(\varphi)$$

where we recall that the entropy of the measure ω_φ^n is defined as

$$\mathrm{Ent}_{\omega^n}(\omega_\varphi^n):=\int_X \log\frac{\omega_\varphi^n}{\omega^n}\,\omega_\varphi^n\geq 0$$

while

$$J_{-\mathrm{Ric}(\omega)}(\varphi):=\frac{1}{n!}\sum_{k=0}^{n-1}\int_X \varphi\,(-\mathrm{Ric}(\omega))\wedge\omega^k\wedge\omega_\varphi^{n-1-k}-\frac{1}{(n+1)!}\sum_{k=0}^{n}\int_X C_\omega\,\varphi\,\omega^k\wedge\omega_\varphi^{n-k},$$

where C_ω is a cohomological constant.

Tian introduced the notion of "J-properness" on the space $\mathscr{H}$. We say that K is J-proper if there exist $A',B'>0$ such that for any $\varphi\in\mathscr{H}$ we have

$$K(\varphi)\geq A'\,J(\varphi)-B'. \tag{2.5.12}$$

The precise formulation of Tian's conjecture is that existence of cscK metrics in $\mathscr{H}$ is equivalent to J-properness of the K-energy.

As is going to be clear in the following, the resolution of this conjecture is very much related to the developments in pluripotential theory in the last years.

First of all, we need to point out that thanks to [BEGZ10], the K-energy can be extended as a functional on the all energy class $\mathscr{E}^1(X,\omega)$,

$$K:\mathscr{E}^1(X,\omega)\to(-\infty,+\infty].$$

The relevance of this fact is that $\mathscr{E}^1$ turned out to be the metric completion of $\mathscr{H}$ when endowed with L^1-type Mabuchi metric d_1 [Dar17].
One can also show that d_1 metric growth is comparable to J. As a consequence, condition (2.5.12) is equivalent to asking that there exist $A,B>0$ such that for any $\varphi\in\mathscr{H}$ we have

$$K(\varphi)\geq Ad_1(0,\varphi)-B.$$

Going back to the cscK equation: it is easy to see that it can be re-written as a system of coupled equations. Indeed, if we set $\omega_\varphi^n=e^F\omega^n$, then tracing the pointwise equality

$$\mathrm{Ric}(\omega_\varphi)=\mathrm{Ric}(\omega)-dd^c\log\frac{\omega_\varphi^n}{\omega^n}$$

with respect to ω_φ leads to

$$\bar{S} = S(\omega_\varphi) = \mathrm{tr}_{\omega_\varphi}(\mathrm{Ric}(\omega)) - \Delta_{\omega_\varphi} F.$$

It then follows that the cscK equation can be re-written as a system of coupled equations:

$$\omega_\varphi^n = e^F \omega^n, \qquad \Delta_{\omega_\varphi} F = -\bar{S} + \mathrm{tr}_{\omega_\varphi}(\mathrm{Ric}(\omega)), \tag{2.5.13}$$

where F and φ are the unknown functions.

The main difference between the KE equation and the cscK equation is that while the first one involves second order derivatives, in the latter one has to deal with derivatives up to the fourth order. This of course increases the difficulty. Nevertheless, the “innocent” observation that allows to re-write the cscK equation as in (2.5.13) reduces in a significant way the technical difficulties.

Moreover, the “artificial” Monge-Ampère equation $\omega_\varphi^n = e^F \omega^n$ helps us in believing that the strategy used in the KE case can be adapted and can lead to meaningful results. This is indeed the case as showed by Chen and Cheng [CC20a, CC20b, CC18]: they deform the system into a one parameter family of coupled equations and they establish uniform estimates. In this concern the key result that Chen and Cheng [CC20a] are able to obtain states as follows:

Key Result (Chen-Cheng [CC20a]): Assume ω_φ is a cscK metric for some smooth function φ on X normalized such that $\sup_X \varphi = 0$. Then all the derivatives of φ can be estimated in terms of $\mathrm{Ent}(\varphi)$, i.e. for each $k \geq 0$, there exists a positive constant $C_k = C(k, \mathrm{Ent}(\varphi))$ such that

$$\|\varphi\|_{C^k} \leq C_k.$$

It is worth underlining that, once the C^0 and C^2 estimates are in hand, higher order estimates follow from standard regularity results for complex Monge-Ampère equations.

At this point it could be yet unclear how the above result implies the implication of the conjecture “properness $\Longrightarrow \exists$ cscK”. The reasoning goes as follows: in the framework of the continuity method (specific to this setting) it suffices to prove uniform estimates for cscK potentials. Indeed, such estimates generalize easily to potentials which are solutions of the intermediate equations we have to deal with in the continuity method. We then assume φ be such that ω_φ is a cscK metric.
Since K is proper, there exists $B > 0$ such that $K(\varphi) \geq A d_1(0, \varphi) - B$. On the other side, since ω_φ is a cscK metric we do know (by convexity of the K-energy along geodesics in $\mathscr{H}_\omega$ [BB17]) that φ is a minimizer of the K-energy, hence $K(\varphi) \leq C$ for some $C > 0$. Thus $d_1(0, \varphi) \leq C_1$. From the latter inequality we obtain a control on the $\sup_X \varphi$ since by [Dar17] we know that $d_1(0, \varphi) \geq \sup_X \varphi - C_2$. Therefore

$$J_{-\mathrm{Ric}(\omega)}(\varphi) \leq \left(\sup_X \varphi\right) C_0 \int_X \omega^n \leq C_0(C_1 + C_2).$$

where C_0 depends on an upper bound of $\mathrm{Ric}(\omega)$.
The upper bound of $K(\varphi)$ and of $J_{-\mathrm{Ric}(\omega)}(\varphi)$ finally imply a (uniform) upper bound of the entropy term $\mathrm{Ent}(\omega_\varphi)$. The Theorem by Chen and Cheng insures that we have uniform estimates along the continuity method.

2.6 Singular cscK metrics

The next step will be to answer to the (natural) urge of studying the singular case:

Problem. Study the existence, uniqueness and regularity of singular cscK on varieties.

One can expect an analogue of the smooth case to hold, i.e. the existence of a singular cscK is equivalent to the properness of the K-energy on the singular space. But this is just a hope at this point and it has to be exploited in a rigorous way.

In the singular setting, one can still make sense of (2.5.13). The latter reduces to a system of equations (on a smooth resolution) which is *degenerate*:

$$\theta_\varphi^n = e^F \mu, \qquad (dd^c F + \chi) \wedge \theta_\varphi^{n-1} = -\frac{\bar{S}}{n} e^F \mu \tag{2.6.14}$$

where χ is a smooth $(1,1)$-form.
Here the reference form θ is not necessarily Kähler but merely semi-positive and big, and μ is not necessarily a smooth volume form (it can have divisorial singularities for example). Observe that, when $\mu = \omega^n$ and $\theta = \omega$, then (2.6.14) is exactly (2.5.13). Once again, the LHS of (2.6.14) is understood as the non-pluripolar Monge-Ampère measure of φ.
We are then tempted to apply pluripotential methods and get a weak solution for (2.6.14) but this equation has not been treated in this sense yet.
The regularity can be treated by looking to a perturbed system (for which we know the existence of a smooth solution) and then passing to the limit. The first attempt in this direction is to perturb the equation by adding some extra positivity (for example, adding $\varepsilon\omega$ where ω is a Kähler form and $\varepsilon > 0$). More precisely, we study

$$\omega_{\varphi_\varepsilon}^n = e^{F_\varepsilon} \mu, \qquad (dd^c F_\varepsilon + \chi) \wedge \omega_{\varphi_\varepsilon}^{n-1} = -\frac{\bar{S}_\varepsilon}{n} e^{F_\varepsilon} \mu \tag{2.6.15}$$

where $\omega_{\varphi_\varepsilon} = \omega_\varepsilon + dd^c \varphi_\varepsilon$. Here $\omega_\varepsilon = \theta + \varepsilon\omega$ is the perturbed metric.
We assume that for any $\varepsilon > 0$ there exists a smooth solution φ_ε of (2.6.15) and we need to establish uniform estimates that do not depend on ε. If we do so, we can then pass to the limit as $\varepsilon \to 0$, ensuring uniform estimates for φ and F on each compact subset $K \subset \mathrm{Amp}\,(\theta)$.
Establishing uniform estimates is a key step. As I emphasized in the story of the Kähler-Einstein case, one of the key steps is the C^0-estimate. The original proof of the C^0-estimate in [CC20a] uses the Alexandroff maximum principle (for the real Monge-Ampère operator): these kind of techniques unfortunately are not very flexible and not adapted in the singular case.

The desire for a pluripotential version of the proof of the C^0-estimates is then justified. This is the reason that brought to a joint work with Alix Deruelle [DDN21] where we present an alternative proof of the C^0-estimates.
The equation we look at is

$$\omega_\varphi^n = e^F \omega^n, \qquad \Delta_{\omega_\varphi} F = -\bar{S} + \mathrm{tr}_{\omega_\varphi}(\mathrm{Ric}(\omega)),$$

where ω is a Kähler form. Let ψ be the unique smooth solution of

$$\omega_\psi^n = b^{-1} e^F \sqrt{F^2+1}\, \omega^n, \qquad \sup_X \psi = 0,$$

where $b = \int_X e^F \sqrt{F^2+1}\, \omega^n$ in order to have $\int_X \omega_\psi^n = \int_X \omega^n = 1$. The existence of a smooth solution ψ is guaranteed by Yau's theorem. Moreover, a simple observation ensures that if $\mathrm{Ent}(\varphi)$ is uniformly bounded, so is b. The key result to which we provide an alternative proof making use of pluripotential theory, states as following:

Theorem 2.13 ([CC20a],[DDN21]) *Given $\varepsilon \in (0,1)$, there exists $C = C(\varepsilon, \omega, b)$ such that*

$$F + \varepsilon\psi - A\varphi \le C,$$

where $A > 0$ is a uniform constant depending only on the lower bound of the Ricci curvature.

A sketch of the proof is provided with the aim of both identifying a delicate point and showing how dealing with singular settings can be much easier once we master pluripotential theory.

Let $H := F + \varepsilon\psi - A\varphi$, A_0 be such that $\mathrm{Ric}(\omega) \ge -A_0\, \omega$ and $A = A_0 + 1$.

- By the maximum principle, applied to H, we can then infer that at a maximum point x_0 we have
$$F(x_0) \le C_0, \quad C_0 = C_0(\varepsilon, A_0, \omega).$$

- Claim: There exists $C_1 > 0$ depending on ε, A and b such that
$$\varepsilon\psi - A\varphi \le C_1.$$
The proof of the claim is based on the observation that for any $a, \delta \in (0,1)$ we have
$$\omega_\varphi^n \le a\omega_{\delta\psi}^n + e^{\frac{b}{a\delta^n}} \omega^n.$$
This (at first sight harmless) inequality allows us to invoke a deep result in [DDNL19]. We can then infer that $\varphi \ge \delta\psi - C_3$, where $C_3 = C_3(a, b, \delta)$.

  The naive idea behind this result is that if the Monge-Ampère measure of our solution φ is *dominated* by the "right quantity", then we can provide a *sub-solution*, hence a lower bound.

Starting from the information that $F + \varepsilon\psi - A\varphi \leq C$, $\varepsilon \in (0,1)$ and $C = C(\varepsilon, \omega, b)$ one can conclude that the functions ψ, φ, F are uniformly bounded by a constant that only depends on ω and b. This makes use once again of pluripotential theory and in particular of a powerful integrability result which is known as a uniform version of Skoda's integrability theorem.

At this stage the arguments work for a reference form that is Kähler: the first item involves indeed the maximum principle. But, nevertheless, they opened the way to the feasibility of establishing the "pluripotential C^0-estimate" when the reference form is only semi-positive and big.

Establishing uniform estimates does not yet give existence of singular cscK metrics. Our starting point was indeed to assume the existence of φ_ε smooth solutions of (2.6.15). By the deep result of Chen and Cheng we do know that this is equivalent to the properness of the K-energy w.r.t. the perturbed metric ω_ε. We then need to exploit the K-energy condition.

One has to understand the relation between the properness of the K-energy w.r.t. θ and the properness of the K-energy w.r.t. ω_ε. The latter not being clear because $\mathrm{PSH}(X, \omega_\varepsilon)$ is bigger than $\mathrm{PSH}(X, \theta)$, hence we have more functions for which we should test the properness of the functional. More precisely, the properness of the K-energy w.r.t. θ reads as

$$K(\varphi) \geq A d_1(0, \varphi) - B, \qquad A, B > 0$$

for any $\varphi \in \mathscr{E}^1(X, \theta)$. Checking this property w.r.t. ω_ε amounts to proving the above inequality for any $\varphi_\varepsilon \in \mathscr{E}^1(X, \omega_\varepsilon)$. An accurate study of the relation between the energy classes $\mathscr{E}^1(X, \theta)$ and $\mathscr{E}^1(X, \omega_\varepsilon)$ is then also necessary.

The last (but not least) question to treat concerns the uniqueness. In the smooth case this is a consequence of the convexity of the K-energy along geodesics in the space of Kähler potentials $\mathscr{H}_\omega$. This is a very delicate problem because geodesics in the space of Kähler potentials are not smooth.

The convexity of the K-energy along weak geodesics was conjectured by Chen and proved by Berman and Berndtsson in a paper published in [BB17]. Their proof is based on the plurisubharmonic variation of Bergman kernels.

In the singular context of a merely semi-positive and big form, the study of the convexity properties of the K-energy then require the preliminary study of the regularity of geodesics in the space of singular Kähler varieties and of how Bergman kernels behave when we work with a big line bundle.

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