On composition of torsors

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Abstract

Let K be a field, let X be a connected smooth K-scheme and let G, H be two smooth connected K-group schemes. Given $Y \to X$ a G-torsor and $Z \to Y$ an H-torsor, we study whether one can find an extension E of G by H so that the composite $Z \to X$ is an E-torsor. We give both positive and negative results, depending on the nature of the groups G and H.

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1. Introduction

Consider a field K, a smooth connected K-scheme X and two smooth connected K-group schemes G and H. In the present article, we are interested in the following question about compositions of torsors:

Question 1.1. Let $Y \to X$ be a G-torsor, and let $Z \to Y$ be H-torsor. Can one find an extension of K-group schemes $(1 \to H \to E \to G \to 1)$, together with an E-torsor structure on the composite $Z \to X$, such that the following holds.

1. The action of E on Z extends that of H.

2. The G-torsors $Z/H \to X$ and $Y \to X$ are isomorphic.

Of course, one does not expect to get a positive answer to this question in full generality. The goal of the article is to give both positive and negative results, depending on the nature of the groups G and H.

Particular cases of Question 1.1 have been considered in [HS05], [BD13], [BDLM20] and [ILA21], as well as [Bri12], [Bri13] and [Bri20]. In [HS05], [BD13] and [ILA21], compositions of torsors are used to study obstructions to the local-global principle and to weak approximation over various arithmetically interesting fields, while in [BDLM20] they are used to study invariants of reductive groups. In [Bri12], [Bri13] and [Bri20], vector bundles (usual and projective) over abelian varieties, which are essentially compositions of torsors, are studied as interesting geometrical objects in their own right.

In the present article, we study compositions of torsors in a systematic way, at least in the case where K has zero characteristic. The main positive result in this direction goes as follows.

Theorem 1.2. *Let K be a field of characteristic 0. Let X be a connected smooth K-scheme. Let G, H be smooth connected K-group schemes. Let $Y \to X$ be a G-torsor and let $Z \to Y$ be an H-torsor. Assume one of the following:*

- *H is an abelian variety.*
- *H is a semi-abelian variety and G is linear.*

Then there exists a canonical extension of K-group schemes

$$1 \to H \to E \to G \to 1,$$

together with a canonical structure of an E-torsor on the composite $Z \to X$.

In order to prove this result, we first give in Section 2 an abstract statement (Theorem 2.1) for torsors and groups satisfying certain technical conditions. In Section 3, we prove that these conditions are met in the cases given in Theorem 1.2. In Section 4, we present a weaker version of Theorem 1.2 that works over arbitrary fields (cf. Theorem 4.1).

Theorem 1.2 covers a certain number of the previously known results in the literature: [BD13, Lem. 2.13] deals with the case where $H = \mathbb{G}_{\mathrm{m}}$ and G is linear, [BDLM20, Thm. A.1.5] deals with the case where $X = \mathrm{Spec}(K)$, H is a special torus and G is reductive, while [ILA21, Thm. A.1] deals with the case where H is a torus and $\mathrm{Pic}(\bar{G}) = 0$. In [Bri12], [Bri13] and [Bri20], Brion studies homogeneous bundles over abelian varieties, getting results that are related to our main theorem in the case where $X = \mathrm{Spec}(K)$ and G is an abelian variety, although they do not deal directly with compositions of torsors with such G (see however [Bri12, Cor. 3.2] and compare with our Theorem 2.1 and Proposition 2.4).

Theorem 1.2 does not cover all cases dealt with by Harari and Skorobogatov in [HS05, Prop. 1.4], since they consider H to be of multiplicative type, and hence it may be non-connected. However, using a variant of the abstract Theorem 2.1 (cf. Theorem 2.3), we recover their result. Since they also provide an abstract result in their article (cf. [HS05, Thm. 1.2]), we compare this result with ours at the end of Section 2.

Finally, in Section 5, we present a certain number of counterexamples to Question 1.1. Table 1 summarizes both the positive and negative results we obtain in characteristic zero.

H \ G	t.	u.	s.s.	a.v.
t.	✓	✓	✓	✗
u.	✗	✗	✗	✗
s.s.	✗	✗	✗	✗
a.v.	✓	✓	✓	✓

t. :	torus
u. :	unipotent
s.s. :	semisimple
a.v. :	abelian variety
✓ :	positive answer
✗ :	negative answer

Table 1: Answer to Question 1.1 for several types of groups G and H over a field of characteristic zero.

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2. Abstract results

In this section, unless otherwise stated, K is an arbitrary field. For a K-scheme W, we denote by X_W, Y_W, Z_W, H_W, G_W the W-schemes obtained by base change from X, Y, Z, H, G respectively. We start by proving the following abstract theorem, which will be the key tool to settle Theorem 1.2.

Theorem 2.1. *Let K be a field. Let X be a smooth K-scheme. Let G, H be smooth connected K-group schemes with H abelian. Let $Y \to X$ be a G-torsor and let $Z \to Y$ be an H-torsor. Finally, let $\mathcal{M}$ be the sheaf over the smooth site over K associated to the presheaf given by $(W \mapsto H(Y_W)/H(W))$. Assume the following:*

(i) The class of $Z_\Omega \to Y_\Omega$ in $H^1(Y_\Omega, H_\Omega)$ is $G(\Omega)$-invariant for every algebraically closed field Ω/K.

(ii) The sheaf $\mathcal{M}$ is torsion-free and representable by a discrete K-group scheme M whose connected components are étale over K.

(iii) Every M-torsor over G is representable.

Then there exists a canonical extension of K-group schemes

$$1 \to H \to E \to G \to 1,$$

together with a canonical structure of an E-torsor on the composite $Z \to X$.

The following Lemma, which we use in the proof of Theorem 2.1 below, is standard. Nonetheless, we provide a sketch of proof.

Lemma 2.2. *Let K be a field and let G be a smooth connected K-group scheme. Denote by $K(G)/K$ the function field of G/K, and by Ω an algebraic closure of $K(G)$. Let $\mathcal{E}$ be a group functor over the fppf site of K, equipped with a K-homomorphism $\pi : \mathcal{E} \to G$. Assume the following:*

1. *The functor $\mathcal{E}$, on K-algebras, commutes with filtered direct limits.*
2. *The arrow $\pi(\Omega) : \mathcal{E}(\Omega) \to G(\Omega)$ is surjective.*

Then, there exists a (finite) fppf cover $(V_i)_{i\in I}$ of G such that the arrow $\pi(V_i) : \mathcal{E}(V_i) \to G(V_i)$ is surjective, for all $i \in I$.

Proof. Denote by $g \in G(\Omega)$ the generic point of G. Let $e \in \mathcal{E}(\Omega)$ be a lift of g, which exists by condition 2. Write Ω as the direct limit (union) $\varinjlim A_j$, of its flat $K[G]$-subalgebras A_j. By condition 1, e belongs to $\mathcal{E}(A_j)$ for some j. In geometric terms, there exists a non-empty open subscheme $U \subset G$, and a flat morphism $V = \mathrm{Spec}(A_j) \to U$, which lifts via π to an element of $\mathcal{E}(V)$. To conclude, we then use the fact that translates $\gamma \cdot V$, for $\gamma \in G(\Omega)$, form a flat cover of G, and that these γ's lift to $\mathcal{E}(\Omega)$. ☺

Proof of Theorem 2.1. Consider the group $\mathrm{Aut}^H_{X_W}(Z_W)$ of X_W-automorphisms φ of Z_W that are compatible with the action of H in the sense that the following diagram commutes:

$$\begin{array}{ccc} H \times Z_W & \xrightarrow{a_W} & Z_W \\ \downarrow{\scriptstyle \mathrm{id}\times\varphi} & & \downarrow{\scriptstyle \varphi} \\ H \times Z_W & \xrightarrow{a_W} & Z_W, \end{array}$$

where a denotes the morphism defining the action of H on Z and a_W the corresponding morphism after base change. The functor $W \mapsto \mathrm{Aut}^H_{X_W}(Z_W)$ defines a group presheaf over the smooth site over K. Denote by $\underline{\mathrm{Aut}}^H_X(Z)$ the corresponding sheaf and consider the subsheaf $\underline{\mathrm{Aut}}^H_Y(Z)$ defined by taking the subgroup $\mathrm{Aut}^H_{Y_W}(Z_W)$ of $\mathrm{Aut}^H_{X_W}(Z_W)$ for each W. Since every element in $\mathrm{Aut}^H_{X_W}(Z_W)$ induces an X_W-automorphism of Y_W, we have an exact sequence of sheaves

$$1 \to \underline{\mathrm{Aut}}^H_Y(Z) \to \underline{\mathrm{Aut}}^H_X(Z) \xrightarrow{\pi} \underline{\mathrm{Aut}}_X(Y),$$

where $\underline{\mathrm{Aut}}_X(Y)$ denotes the sheaf of X-automorphisms of Y.

Note now that G is naturally a closed subgroup of $\underline{\mathrm{Aut}}_X(Y)$. Taking the pullback via π, we get an exact sequence of sheaves

$$1 \to \mathcal{A} \to \mathcal{E}' \xrightarrow{\pi} G,$$

where $\mathcal{A} = \underline{\mathrm{Aut}}^H_Y(Z)$. Now, the functor $V/Y \mapsto \mathrm{Aut}^H_V(Z \times_Y V)$ over the smooth site of Y is represented, as a Y-scheme, by H_Y (cf. for instance [Gir71, III.§1.5]). In other words, $\mathcal{A}(W) = \mathrm{Aut}^H_{Y_W}(Z_W) = H(Y_W)$ and hence $\mathcal{M} = \mathcal{A}/H$. Thus, by (ii), we get an exact sequence of group sheaves over K

$$1 \to H \to \mathcal{A} \to M \to 1,$$

where M is a discrete K-group scheme whose connected components are étale over K; in other words, M is a disjoint union of (possibly infinitely many) spectra of finite separable field extensions of K. By Galois descent, it follows that $\mathcal{A}$ is represented by a K-scheme A. Moreover, since H is connected, it corresponds to the neutral connected component of A. In particular, H is characteristic in A.

On the other hand, since Z is of finite type over K, the functor $\underline{\mathrm{Aut}}_K(Z)$ commutes with direct limits. Hence, so does the closed subfunctor $\mathcal{E}' \subset \underline{\mathrm{Aut}}_K(Z)$. Then, by (i) and Lemma 2.2, we get that the arrow $\pi : \mathcal{E}' \to G$ is surjective in the fppf site. In particular, $\mathcal{E}'$ is an A-torsor in this site. Recalling now that both G and A are smooth, we get that $\mathcal{E}'$ is in fact an A-torsor in the smooth site and hence we get an exact sequence of smooth sheaves

$$1 \to A \to \mathcal{E}' \xrightarrow{\pi} G \to 1. \tag{$\mathcal{S}$}$$

Since A is normal in $\mathcal{E}'$ and H is characteristic in A, we obtain that H is normal in $\mathcal{E}'$. We may quotient then by H in order to get an exact sequence

$$1 \to M \to \mathcal{F} \xrightarrow{\bar{\pi}} G \to 1. \tag{$\bar{\mathcal{S}}$}$$

As before, $\mathcal{F}$ corresponds to an M-torsor over the scheme G, which is representable by (iii). Thus we have an exact sequence of K-group *schemes*

$$1 \to M \to F \to G \to 1.$$

Since M is discrete and torsion-free by (ii), and since G is connected, we see that the neutral connected component $F^0 \subset F$, is mapped isomorphically to G by $\bar{\pi}$. This provides a splitting of extension ($\bar{\mathcal{S}}$). As a consequence, extension ($\mathcal{S}$) comes from an extension of group sheaves

$$1 \to H \to \mathcal{E} \to G \to 1.$$

Using [Mil80, III, Thm. 4.3.(a),(c)] and Chevalley's Theorem (cf. [Con02, Thm. 1.1] or [BLR90, §9.2, Thm. 1]), we see that $\mathcal{E}$ is in fact representable by a K-group scheme E. As a subgroup of $\underline{\mathrm{Aut}}_X^H(Z)$, it acts on Z, and it is immediate to check then that $Z \to X$ is an E-torsor, which enjoys the required properties. To conclude, note that the construction above is canonical. ☺

In the previous theorem, the assumptions that H is abelian and that G and H are both connected can be removed when M is the trivial group. In that way, one gets the following result, which implies [HS05, Prop. 1.4].

Theorem 2.3. *Let K be a field. Let X be a smooth K-scheme. Let G, H be smooth K-group schemes. Let $Y \to X$ be a G-torsor and let $Z \to Y$ be an H-torsor. Assume the following:*

(i) The class of $Z_\Omega \to Y_\Omega$ in $H^1(Y_\Omega, H_\Omega)$ is $G(\Omega)$-invariant for every algebraically closed field Ω/K.

(ii) The sheaf of sets $\mathcal{M}$ over the smooth site over K associated to the presheaf given by $(W \mapsto H(Y_W)/H(W))$ is trivial.

Then there exists a canonical extension of K-group schemes

$$1 \to H \to E \to G \to 1,$$

together with a canonical structure of an E-torsor on the composite $Z \to X$.

Proof. The proof starts exactly as the one above, except for the following modification. Instead of considering the groups $\mathrm{Aut}_{X_W}^H(Z_W)$ and $\mathrm{Aut}_{Y_W}^H(Z_W)$ for smooth $W \to K$ and the corresponding sheaves $\underline{\mathrm{Aut}}_X^H(Z)$ and $\underline{\mathrm{Aut}}_Y^H(Z)$, we consider the groups $\mathrm{Aut}_{X_W}^{H'}(Z_W)$ and $\mathrm{Aut}_{Y_W}^{H'}(Z_W)$ and the corresponding sheaves $\underline{\mathrm{Aut}}_X^{H'}(Z)$ and $\underline{\mathrm{Aut}}_Y^{H'}(Z)$, where H' is the Y-group scheme obtained by twisting H_Y by the torsor $Z \to Y$ (H' is actually H_Y when H is abelian). This group scheme acts naturally on Z on the left compatibly with the right action of H (cf. [Gir71, III.§1.5]). In particular, we still have the equality $\mathrm{Aut}_{Y_W}^{H'}(Z_W) = H(Y_W)$ by *loc. cit.* and an exact sequence analog to sequence ($\mathcal{S}$):

$$1 \to \mathcal{A}' \to \mathcal{E} \xrightarrow{\pi} G \to 1,$$

with $\mathcal{A}' = \underline{\mathrm{Aut}}_Y^{H'}(Z)$. The assumption on the sheaf M tells us then that $\mathcal{A}'$ is actually H, and hence the exact sequence becomes

$$1 \to H \to \mathcal{E} \xrightarrow{\pi} G \to 1.$$

Thus $\mathcal{E}$ is an H-torsor, which is then representable by the same argument used in the previous proof. We get then an extension of K-groups

$$1 \to H \to E \to G \to 1.$$

And again, since E is by definition a subgroup of $\underline{\mathrm{Aut}}_X^H(Z)$, it is immediate to check that E acts on Z and that $Z \to X$ is an E-torsor. The fact that the construction is canonical is once again easy to see.

☺

The $G(\Omega)$-invariance of the H_Ω-torsor $Z_\Omega \to Y_\Omega$ is, in a wide variety of cases, a strictly necessary hypothesis in order to get a positive answer to Question 1.1. More precisely:

Proposition 2.4. *Let K be an algebraically closed field of characteristic 0. Let*

$$1 \to H \to E \to G \to 1,$$

be an extension of smooth K-group schemes with G connected. Assume that the unipotent radical of H is trivial. Let X be a smooth K-scheme, let $Z \to X$ be an E-torsor and let $Y := Z/H$, so that $Z \to Y$ is an H-torsor and $Y \to X$ is a G-torsor. Then the class of $Z \to Y$ in $H^1(Y, H)$ is $G(K)$-invariant.

Proof. Define C to be the centralizer of H in E. We claim that C surjects onto G via the projection. Since C is the kernel of the natural arrow $E(K) \to \mathrm{Aut}(H)$ given by conjugation, the claim amounts to proving that the induced morphism $G(K) \to \mathrm{Out}(H)$ is trivial, where $\mathrm{Out}(H) := \mathrm{Aut}(H)/\mathrm{Inn}(H)$.

By Lemma 3.3, which we prove in the following section, we know that $G(K)$ is generated by its infinitely divisible elements, while $\mathrm{Out}(H)$ has no such elements. Indeed, this group is finite for reductive groups (cf. [Dem65, Thm. 5.2.3]), while it is a subgroup of $\mathrm{GL}_n(\mathbb{Z})$ for abelian varieties (as follows from [Mil86, Thm. 10.15]). In the general case, our hypothesis on H and Chevalley's Theorem (cf. [Con02, Thm. 1.1]) ensure that H is an extension

$$1 \to L \to H \to A \to 1,$$

of an abelian variety A and a reductive linear group L. Since L is a characteristic subgroup of H and scheme morphisms from an abelian variety to a linear group are constant, one easily sees that $\mathrm{Aut}(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(L) \times \mathrm{Aut}(A)$. We deduce the same property for $\mathrm{Out}(H)$, which implies the claim.

Consider now $g \in G(K)$ and let us prove that the torsor $g^*Z \to Y$, defined as the left vertical arrow of the fiber product

$$\begin{array}{ccc} g^*Z & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\;g\;} & Y, \end{array}$$

is isomorphic to the torsor $Z \to Y$. Let $c \in C(K) \subset E(K)$ be a preimage of g. Then we have a commutative square

$$\begin{array}{ccc} Z & \xrightarrow{\;c\;} & Z \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\;g\;} & Y. \end{array}$$

Then, by the universal property of the fiber product, we get a Y-morphism $Z \to g^*Z$, which we claim it is H-equivariant. This is a straightforward computation that uses the fact that $c \in C(K)$ commutes with H. This proves that the class of $Z \to Y$ is g-invariant and hence $G(K)$-invariant.

Remark 2.5. A result similar to Theorem 2.3 can be found in [HS05, Thm. 1.2]. However, the assumptions are slightly different:

Harari and Skorobogatov assume that every morphism $Z_{\bar{K}} \to H_{\bar{K}}$ is trivial. This is easily seen to imply the triviality of $\mathcal{M}$ and hence our assumption (ii).

On the other hand, they assume that every automorphism of $Y_{\bar{K}}$ given by an element $g \in G(\bar{K})$ can be lifted to an automorphism of $Z_{\bar{K}}$. Our assumption (i) implies this, of course, but it is not clear whether they are equivalent assumptions, even though ours seems to be always necessary, as it can be seen from Proposition 2.4.

In any case, assumption (ii) of Theorem 2.3 is met for instance when H is affine, G is anti-affine and X is connected and proper. Indeed, in this case $\mathcal{O}(Y) = K$ and hence $H(Y \times W) = H(W)$ for geometrically integral W by [Bri21, Lem. 5.2]. This implies the triviality of $\mathcal{M}$. These are milder hypotheses than those considered by Harari and Skorobogatov in [HS05, Prop. 1.4], who deal for instance with the case of H of multiplicative type and Y proper.

3. Proof of Theorem 1.2

It will suffice to prove that the assumptions (i)–(iii) of Theorem 2.1 are met under each of the hypotheses of Theorem 1.2. We fix then a field K **of characteristic** 0 and keep the other notations as above: X is a connected smooth K-scheme; G, H are smooth connected K-group schemes; $Y \to X$ is a G-torsor and $Z \to Y$ is an H-torsor; $\mathcal{M}$ is the sheaf over the smooth site over K associated to the presheaf given by $(W \mapsto H(Y_W)/H(W))$.

3.1 Proof of (ii) and (iii)

We prove (ii). All the properties mentioned in the statement (representability, discreteness, no torsion, étale connected components) are étale-local over K, so we may and will assume that K is algebraically closed. Then Lemma A.1, which we prove in the appendix, tells us that, for any integral K-scheme W, we have that $H(Y_W)/H(W) = H(Y)/H(K)$ is a finitely generated free abelian group. This implies that $\mathcal{M}$ is torsion-free and representable by a discrete K-group scheme M whose connected components are étale over K.

We prove (iii). By [SGA7, Exp. 8, Prop. 5.1], we know that $H^1(G, \mathbb{Z}) = 0$. Since M is locally isomorphic to a subgroup of $\mathbb{Z}^n$, we see that every M-torsor over G is locally trivial, hence representable.

3.2 Proof of (i) when H is an abelian variety

We prove the following result, which clearly implies assumption (i) in Theorem 2.1:

Proposition 3.1. *Let K be an algebraically closed field of characteristic* 0*. Let G be a smooth connected algebraic group and let H be an abelian variety. Let X be a smooth K-scheme and let $Y \to X$ be a G-torsor. Then the action of $G(K)$ on $H^1(Y, H)$ is trivial.*

We first need to prove some lemmas on the structure of the groups involved.

Lemma 3.2. *The group $H^1(Y, H)$ is torsion of cofinite type, i.e. its m-torsion subgroup is finite for every $m \in \mathbb{N}$.*

Proof. The group $H^1(Y,H)$ is torsion according to [Ray70, Cor. XIII.2.4, Prop. XIII.2.6]. Moreover, by [SGA4, Th. 5.2 of Exp. XVI], for each integer $n > 0$, the group $H^1(Y, H[n])$ is finite, and hence so is its quotient $H^1(Y,H)[n]$. ☺

Lemma 3.3. *The group $G(K)$ is spanned by its divisible subgroups.*

Proof. Write $G = G_{\mathrm{aff}}G_{\mathrm{ant}}$ where G_{aff} is the largest connected affine subgroup of G and G_{ant} is the largest anti-affine subgroup of G (cf. [BSU13, Thm. 1.2.4]). Every element g of $G_{\mathrm{aff}}(K)$ can be written as:

$$g = su_1...u_r$$

where s is a semisimple element of $G_{\mathrm{aff}}(K)$ and each u_i is contained in a subgroup of G_{aff} isomorphic to $\mathbb{G}_{\mathrm{a}}$. Hence $G_{\mathrm{aff}}(K)$ is spanned by its divisible subgroups. Moreover, the anti-affine group G_{ant} is connected commutative (cf. [BSU13, Thm. 1.2.1]), and hence $G_{\mathrm{ant}}(K)$ is a divisible group. We deduce that $G(K)$ is spanned by its divisible subgroups. ☺

Lemma 3.4. *Let Γ be a profinite group. Then, Γ has no non-trivial infinitely divisible elements.*

Proof. The statement is clear if Γ is finite. It thus also holds for inverse limits of finite groups. ☺

Proof of Proposition 3.1. By Lemma 3.2, we have an isomorphism of abstract groups:

$$H^1(Y,H) \cong \bigoplus_p (F_p \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{r_p})$$

where p runs through the set of all prime numbers, F_p is a finite abelian p-group and $r_p \geq 0$. We deduce that:

$$\mathrm{Aut}\left(H^1(Y,H)\right) \cong \prod_p \mathrm{Aut}_{\mathbb{Z}_p}(F_p \oplus \mathbb{Z}_p^{r_p}),$$

which is a profinite group. Lemmas 3.4 and 3.3 then imply that every morphism from $G(K)$ to $\mathrm{Aut}\left(H^1(Y,H)\right)$ is trivial. ☺

3.3 Proof of (i) when H is semi-abelian and G is linear

As before, we prove the following result, which clearly implies assumption (i) in Theorem 2.1:

Proposition 3.5. *Let K be an algebraically closed field of characteristic 0. Let G be a smooth connected linear group and let H be a semi-abelian variety. Let X be a smooth K-scheme and let $Y \to X$ be a G-torsor. Then the action of $G(K)$ on $H^1(Y,H)$ is trivial.*

Again, we start with a lemma on the structure of the groups involved.

Lemma 3.6. *There is an exact sequence of abelian groups with $G(K)$-action*

$$0 \to N \to H^1(Y,H) \to M \to 0,$$

such that:

- *M is a torsion group of cofinite type.*
- *The torsion subgroup $N_{\mathrm{tors}} \subseteq N$ is of cofinite type.*
- *The action of $G(K)$ on both M and N is trivial.*

Proof. We have an exact sequence:

$$0 \to T \to H \to A \to 0$$

where T is a torus and A is an abelian variety. It induces a cohomology exact sequence:

$$H^1(Y,T) \xrightarrow{f} H^1(Y,H) \xrightarrow{g} H^1(Y,A),$$

whose arrows are clearly $G(K)$-equivariant since the action is on Y. Put $M := \mathrm{im}(g)$ and $N := \mathrm{im}(f)$. By Lemma 3.2, $H^1(Y,A)$ is torsion of cofinite type, and hence so is M. Moreover, by Proposition 3.1, the group $G(K)$ acts trivially on $H^1(Y,A)$, and hence on M. On the other hand, note that $H^1(Y,T) \cong \mathrm{Pic}(Y)^{\dim(T)}$. Since G is linear, a result of Sumihiro (cf. [Bri18, Thm. 5.2.1]) tells us that the action of $G(K)$ on $H^1(Y,T)$ is trivial. ☺

Proof of Proposition 3.5. By Lemma 3.6, the group $G(K)$ acts trivially on M and N. Hence the action of $G(K)$ on $H^1(Y,H)$ corresponds to a morphism from $G(K)$ to $\mathrm{Hom}(M,N)$. The abelian group M is torsion, and hence $\mathrm{Hom}(M,N) = \mathrm{Hom}(M,N_{\mathrm{tors}})$. Moreover, M is of cofinite type, and so is the group N_{tors}. We can therefore write:

$$M \cong \bigoplus_p \left(F_p \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{r_p}\right),$$
$$N_{\mathrm{tors}} \cong \bigoplus_p \left(F'_p \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{r'_p}\right),$$

where p runs through the set of all prime numbers, F_p and F'_p are finite abelian p-groups and $r_p, r'_p \geq 0$. It follows that $\mathrm{Hom}(M, N_{\mathrm{tors}})$ is a profinite group. Hence, it has no non-trivial infinitely divisible elements by Lemma 3.4. Thus, every morphism from $G(K)$ to $\mathrm{Hom}(M,N_{\mathrm{tors}})$ is trivial by Lemma 3.3. We deduce that the action of $G(K)$ on $H^1(Y,H)$ is trivial. ☺

We finish this section with an example that shows that one really needs to assume K to be algebraically closed in Proposition 3.5 (and hence in assumption (i) of Theorems 2.1 and 2.3).

Example 3.7. Let K be a field and let L/K be a separable quadratic field extension such that the norm $N_{L/K} : L^\times \to K^\times$ is not surjective. Consider the extension of algebraic K-groups

$$1 \to R^1_{L/K}(\mathbb{G}_{\mathrm{m}}) \to R_{L/K}(\mathbb{G}_{\mathrm{m}}) \xrightarrow{N_{L/K}} \mathbb{G}_{\mathrm{m}} \to 1,$$

where $R_{L/K}$ denotes Weil scalar restriction and $N_{L/K}$ is the norm of L/K. Set $G = Y := \mathbb{G}_{\mathrm{m}}$, $H := R^1_{L/K}(\mathbb{G}_{\mathrm{m}})$ and $X := \mathrm{Spec}(K)$. The extension above provides a class

$$x_0 := [R_{L/K}(\mathbb{G}_{\mathrm{m}}) \to \mathbb{G}_{\mathrm{m}}] \in H^1(Y,H).$$

This class is not invariant under the action of $G(K)$. Indeed, the action of $G(K)$ is described as follows:

$$\lambda \cdot x = x + p^*\delta(\lambda),$$

where $\lambda \in G(K)$, $x \in H^1(Y,H)$, $p : Y \to K$ is the structure morphism and $\delta : G(K) \to H^1(K,H)$ is the connecting map in Galois cohomology. In particular, since the arrow

$$p^* : H^1(K,H) \to H^1(Y,H),$$

is injective, we have $\lambda \cdot x = x$ if and only if $\lambda \in N_{L/K}(L^\times)$, which does not hold in general.

4. Positive characteristic

In this section, we present a weaker version of Theorem 1.2 that works in positive characteristic.

Theorem 4.1. *Let K be a field. Let X be a connected smooth K-scheme. Let G, H be smooth connected K-group schemes. Let $Y \to X$ be a G-torsor and let $Z \to Y$ be an H-torsor. Assume one of the following:*

- *H, G are abelian varieties and X is proper.*
- *H is a torus and G is linear.*

Then there exists a canonical extension of K-group schemes

$$1 \to H \to E \to G \to 1,$$

together with a canonical structure of an E-torsor on the composite $Z \to X$.

Proof. The proof of this result is given once again by Theorem 2.1, which is valid over an arbitrary field. We need to prove then that assumptions (i)–(iii) of Theorem 2.1 are met. The proof of (ii) and (iii) is exactly the same we gave in Section 3.1, since we did not use any hypothesis on the characteristic of the base field and Lemma A.1 holds over separably closed fields.

Thus, we are only left with (i). In the second case, this is a direct consequence of Sumihiro's result we used before (cf. [Bri18, Thm. 5.2.1]). In the first case, (i) is implied by Proposition 4.2 here below. ☺

Proposition 4.2. *Let K be an algebraically closed field. Let X be a smooth projective K-scheme. Let G and H be abelian varietes and let $Y \to X$ be a G-torsor. Then the action of $G(K)$ on $H^1(Y,H)$ is trivial.*

Proof. Without loss of generality, we can assume that X (and hence Y) is connected. Note that that $G(K)$ is divisible and that $H^1(Y,H)$ is torsion by [Ray70, Cor. XIII.2.4, Prop. XIII.2.6]. Moreover, for each integer n coprime to p, the group $H^1(Y,H)[n]$ is a quotient of the finite group $H^1(Y,H[n])$, and hence is finite. By proceeding as in Proposition 3.1, we see that $G(K)$ acts trivially on the q-primary part of $H^1(Y,H)$ for every prime $q \neq p$. It is enough then to prove that $G(K)$ acts trivially on $H^1(Y,H)[p^n]$ for every $n \in \mathbb{N}$. We proceed by induction on n.

For $n = 1$, we know that $H^1(Y,H)[p]$ is a quotient of $H^1_{\text{fppf}}(Y,H[p])$ and $H[p] \cong (\mathbb{Z}/p\mathbb{Z})^a \times (\mu_p)^b \times (\alpha_p)^c$ for some integers a, b, c (cf. [Sha86]). It is therefore enough to prove that the action of $G(K)$ on $H^1_{\text{fppf}}(Y,\mathbb{Z}/p\mathbb{Z})$ and $H^1_{\text{fppf}}(Y,\mu_p)$ and $H^1_{\text{fppf}}(Y,\alpha_p)$ is trivial.

Since Y is proper over K, [Mil80, Cor. VI.2.8] ensures the finiteness of $H^1_{\text{fppf}}(Y,\mathbb{Z}/p\mathbb{Z})$, which implies the triviality of the action in this case.

Now by Kummer theory, we have an exact sequence:

$$0 \to K[Y]^\times/(K[Y]^\times)^p \to H^1_{\text{fppf}}(Y,\mu_p) \to \text{Pic}(Y_{\bar{K}})[p] \to 0.$$

Using the properness of Y once more, we have $K[Y] = K$, and hence the quotient $K[Y]^\times/(K[Y]^\times)^p$ is trivial. Moreover, the group $\text{Pic}(Y)[p]$ is always finite. Hence $H^1_{\text{fppf}}(Y,\mu_p)$ is finite, which implies the triviality of the action in this case.

We deal now with $H^1_{\text{fppf}}(Y,\alpha_p)$. Let A be a K-algebra. Then $H^0(Y_A,\mathcal{O}_{Y_A}) = A$ and $H^1(Y_A,\mathcal{O}_{Y_A}) = H^1_{\text{fppf}}(Y,\mathcal{O}_Y) \otimes_K A$, so that, after taking cohomology of the extension of fppf sheaves (over Y_A)

$$0 \to \alpha_p \to \mathbb{G}_a \xrightarrow{\text{Frob}} \mathbb{G}_a \to 0,$$

we get an exact sequence

$$0 \to A/A^p \to H^1_{\text{fppf}}(Y_A,\alpha_p) \to H^1(Y,\mathcal{O}_Y) \otimes_K A.$$

Taking $A = K$, we get an inclusion of finite-dimensional K-vector spaces

$$H^1_{\text{fppf}}(Y,\alpha_p) \subset H^1(Y,\mathcal{O}_Y).$$

It then suffices to show that $G(K)$ acts trivially on $H^1_{\text{fppf}}(Y,\mathcal{O}_Y)$. To do so, observe that the G-action on Y, induces an action of the abstract group $G(A)$ on the A-scheme Y_A, and hence an A-linear action of $G(A)$ on $H^1_{\text{fppf}}(Y,\mathcal{O}_Y) \otimes_K A$. Being functorial in A, it arises from a morphism of algebraic K-groups $\rho : G \to \text{GL}(H^1_{\text{fppf}}(Y,\mathcal{O}_Y))$, which is trivial because G is an abelian variety. This concludes the proof for $n = 1$.

Consider now the following exact sequence

$$0 \to H^1(Y,H)[p] \to H^1(Y,H)[p^{n+1}] \to H^1(Y,H)[p^n],$$

and let I be the image of the rightmost arrow. By the inductive assumption, the group $G(K)$ acts trivially on $H^1(Y,H)[p]$ and on I. Hence the action of $G(K)$ on $H^1(Y,H)[p^{n+1}]$ corresponds to a morphism $G(K) \to \text{Hom}(I, H^1(Y,H)[p])$. But this morphism is trivial since $G(K)$ is divisible and $\text{Hom}(I, H^1(Y,H)[p])$ is p-torsion. We deduce that $G(K)$ acts trivially on $H^1(Y,H)[p^{n+1}]$, as wished. ☺

5. Counterexamples

In this section, we provide examples of towers of torsors that do not admit a torsor structure under an extension of the two involved groups. We treat every negative case considered in Table 1.

5.1 Examples where H is a torus

As it is suggested by the proofs in Section 3.1, when H is a torus, assumptions (ii) and (iii) of Theorem 2.1 are satisfied in all generality. According to Table 1, it is then assumption (i), on the $G(\Omega)$-invariance of the H-torsor $Z \to Y$ that must fail in order to get counterexamples.

Example 5.1. Assume that K is algebraically closed, $X = \mathrm{Spec}(K)$, $G = Y = E$ is an elliptic curve, and $H = \mathbb{G}_\mathrm{m}$. Then the group $E(K)$ acts on $H^1(Y, H) = \mathrm{Pic}(E)$ via the following formula:

$$Q \cdot [D] = [D] + \deg(D) \cdot ([Q] - [O]), \quad Q \in E(K),\ [D] \in \mathrm{Pic}(E).$$

This action is not trivial, and hence one can find a class in $\mathrm{Pic}(E)$ that is not $G(K)$-invariant. By Proposition 2.4, this class represents an H-torsor $Z \to Y$ such that the composition $Z \to K$ is not a torsor under an extension E of G by H.

5.2 Examples where H is unipotent

We continue with the case in which H is unipotent. The following example covers the cases in which G is either a torus, a unipotent group or a semisimple group.

Example 5.2. Let $H = \mathbb{G}_\mathrm{a}$, X an elliptic curve over an algebraically closed field K and Y the trivial G-torsor with G either $\mathbb{G}_\mathrm{a}$, $\mathbb{G}_\mathrm{m}$ or SL_n (with $n \geq 2$). On the one hand, by Künneth's formula we have $H^1(Y, \mathbb{G}_a) = H^1(X, \mathbb{G}_\mathrm{a}) \otimes_K \mathcal{O}(G) = \mathcal{O}(G)$, which is an infinite-dimensional K-vector space.

On the other hand, every extension E of G by $\mathbb{G}_\mathrm{a}$ is split by the basic theory of linear groups. Even more, if $G = \mathbb{G}_\mathrm{a}$ or $G = \mathrm{SL}_n$, then the extension is simply the direct product, while if $G = \mathbb{G}_\mathrm{m}$, then it corresponds to the semi-direct product $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$ with $\mathbb{G}_\mathrm{m}$ acting on $\mathbb{G}_\mathrm{a}$ by multiplication by k-th powers for some $k \in \mathbb{Z}$. In particular, these extensions are parametrized by $\mathbb{Z}$. In any case, we have a split exact sequence:

$$1 \to H^1(X, \mathbb{G}_\mathrm{a}) \to H^1(X, E) \to H^1(X, G) \to 1,$$

and we know that, if $Z \to X$ was actually an E-torsor, then its image in $H^1(X, G)$ would be the trivial torsor since $Y \to X$ is trivial. We deduce then that the moduli space of torsors lifting $Y \to X$ to an E-torsor for some extension E of G by $\mathbb{G}_\mathrm{a}$ is either a one-dimensional vector space or a countable union of one-dimensional spaces. In either case, since $H^1(Y, \mathbb{G}_\mathrm{a})$ is infinite-dimensional, there exists a $\mathbb{G}_a$-torsor $Z \to Y$ such that the composite $Z \to X$ is not a torsor under an extension of G by $\mathbb{G}_\mathrm{a}$.

This example leads to a more general construction for towers of $\mathbb{G}_\mathrm{a}$-torsors over curves of genus ≥ 2.

Example 5.3. Let X/K be a smooth projective curve, of genus $g \geq 2$. Then, the K-vector space $H^1(X, \mathcal{O}_X)$ is g-dimensional. Let $Y \to X$ be a non-trivial $\mathbb{G}_\mathrm{a}$-torsor, whose class in $H^1(X, \mathcal{O}_X)$ we denote by y. Using the correspondence between $\mathbb{G}_\mathrm{a}$-torsors and extensions of vector bundles of $\mathcal{O}_X$ by itself, Y is determined by an extension

$$\mathcal{E}: \qquad 0 \to \mathcal{O}_X \xrightarrow{s} E \xrightarrow{\pi} \mathcal{O}_X \to 0.$$

We then have

$$Y = \operatorname{Spec}\left(\varinjlim_n \operatorname{Sym}^n E\right),$$

so that

$$H^1(Y, \mathcal{O}_Y) = H^1(X, \varinjlim_n \operatorname{Sym}^n E) = \varinjlim_n H^1(X, \operatorname{Sym}^n E).$$

Now, to compute this direct limit, one can use the symmetric powers of $\mathcal{E}$:

$$\operatorname{Sym}^n \mathcal{E}: \qquad 0 \to \operatorname{Sym}^{n-1} E \xrightarrow{\times s} \operatorname{Sym}^n E \xrightarrow{\pi^n} \mathcal{O}_X \to 0,$$

where

$$\pi^n(e_1 \otimes \ldots \otimes e_n) := \pi(e_1) \ldots \pi(e_n).$$

Denote by $y^n \in H^1(X, \operatorname{Sym}^n E)$ the class of $\operatorname{Sym}^n \mathcal{E}$. We have a commutative diagram of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Sym}^n(E) & \xrightarrow{\times s} & \operatorname{Sym}^{n+1}(E) & \xrightarrow{\pi^{n+1}} & \mathcal{O}_E & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi^n} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \times(n+1)} & & \\ 0 & \longrightarrow & \mathcal{O}_E & \xrightarrow{s} & E & \xrightarrow{\pi} & \mathcal{O}_E & \longrightarrow & 0, \end{array}$$

where g is given by the formula

$$g(e_0 \ldots e_n) = \sum_0^n \pi(e_0) \ldots \widehat{\pi(e_i)} \ldots \pi(e_n) e_i,$$

where $\widehat{}$ denotes an omitted variable. Since E is non-split, and since $(n+1) \in K^\times$, we get that (the class of) $\operatorname{Sym}^n \mathcal{E}$ does not belong to the image of $H^1(X, \operatorname{Sym}^{n-1} E) \to H^1(X, \operatorname{Sym}^n E)$; in particular, it is non-split.

Now, the cohomology exact sequence associated to $\operatorname{Sym}^n \mathcal{E}$ gives:

$$K \to H^1(X, \operatorname{Sym}^{n-1} E) \xrightarrow{s_n} H^1(X, \operatorname{Sym}^n E) \to H^1(X, \mathcal{O}_X) \to 0,$$

where the image of the leftmost arrow is precisely the subspace generated by the class of $\operatorname{Sym}^n \mathcal{E}$. This tells us that, if we set $V_n := H^1(X, \operatorname{Sym}^n E)/\langle y^n \rangle$, we have an exact sequence

$$0 \to V_n \to V_{n+1} \to H^1(X, \mathcal{O}_X)/\langle y \rangle \to 0.$$

Since $\dim(H^1(X, \mathcal{O}_X)) = g \geq 2$, we see that the direct limit of the V_n's has infinite dimension, so that the same holds for $H^1(Y, \mathcal{O}_Y) = \varinjlim_n H^1(X, \operatorname{Sym}^n E)$.

Assume now, that for every $\mathbb{G}_a$-torsor $Z \to Y$, we can find an extension of X-group schemes

$$1 \to \mathbb{G}_a \to \Gamma \to \mathbb{G}_a \to 1,$$

such that $Z \to X$ can be equipped with the structure of a Γ-torsor. Since K has characteristic zero, Γ is the affine space of a vector bundle F over X, fitting into an extension of vector bundles over X

$$\mathcal{F}: \qquad 0 \to \mathcal{O}_X \xrightarrow{s} F \xrightarrow{\pi} \mathcal{O}_X \to 0.$$

Using the same computation as above, we get that $H^1(X, F)$ is $(2g-1)$-dimensional. Thus, the moduli space of torsors under extensions of $\mathbb{G}_a$ by itself is $(3g-1)$-dimensional. This contradicts the fact that $H^1(Y, \mathcal{O}_Y)$ is infinite-dimensional.

Remark 5.4. Note that in Examples 5.2 and 5.3 the base scheme X is always proper. On the opposite side, when the base is affine, we get a particular case where the answer to Question 1.1 is positive with H unipotent as follows:

Assume that H is a *split* unipotent group, that G is linear and that X is *affine*. Let Y be a G-torsor over X and let Z be an H-torsor avec Y. Then Y is affine, and hence $H^1(Y, \mathbb{G}_{\mathrm{a}}) = 0$. We deduce that $H^1(Y, H)$ is trivial, so that $Z = Y \times H$. In particular, Z is a $(G \times H)$-torsor over X.

We also provide an example in which the base scheme X is not proper over K.

Example 5.5. Let $G = A$ be an abelian variety, $H = \mathbb{G}_{\mathrm{a}}$ and let X be the spectrum of a function field L over K. Then $\mathrm{Ext}(A, \mathbb{G}_{\mathrm{a}}) \simeq H^1(A, \mathcal{O}_A)$ by [Ser75, VII.17, Thm. 7], which is a K-vector space for group extensions over K and an L-vector space of the same dimension if we do the corresponding base change (and the restriction arrow is the obvious injection). This tells us immediately that there are extensions of A by $\mathbb{G}_{\mathrm{a}}$ over L that do not come from extensions over K. In particular, these extensions are towers of torsors over L that cannot have a torsor structure under an extension defined over K (if an extension were a torsor under another extension, they would have the same underlying variety and hence define the same element in $H^1(A, \mathcal{O}_A)$). Obviously, these extensions can be built over a suitable (smooth affine) K-scheme, if one wants X to be more than just a single point.

5.3 Examples where H is semisimple

We finish this section with examples in which H is semisimple. This completes the study of all cases in Table 1.

Example 5.6. Let $H = \mathrm{PGL}_n$ with $n \geq 2$ and let G be either $\mathbb{G}_{\mathrm{a}}^m$, $\mathbb{G}_{\mathrm{m}}^m$, PGL_m, or an abelian variety A. Consider an H-torsor $Z \to G$, and the trivial G-torsor $G \to K$ below it. If Question 1.1 had a positive answer for this tower, then Z would be an E-torsor for some extension E of G by H. However, in all four cases for G (assuming $n \gg m$ if $G = \mathbb{G}_{\mathrm{a}}^m$ and assuming K algebraically closed if $G = A$) we have that the only possible extension is the direct product $E = G \times H$. Indeed, the first three cases are elementary results from the theory of linear groups, and the case $G = A$ comes from [BSU13, Prop. 3.1.1]. This implies in particular that the class in $H^1(G, H)$ of $Z \to G$ must come from $H^1(K, H)$. Thus, any class in $H^1(G, H)$ which does not come from $H^1(K, H)$ gives a negative answer to Question 1.1.

In the case $G = \mathbb{G}_{\mathrm{a}}^m$, examples of non-constant Azumaya algebras over $\mathbb{G}_{\mathrm{a}}^2$ are known to exist as soon as $\mathrm{Br}(K) \neq 0$ (cf. [OS71, Prop. 2]). These correspond to elements in $H^1(G, H)$ that do not come from $H^1(K, H)$.

In the case $G = \mathbb{G}_{\mathrm{m}}^m$, a simple computation using residue maps with respect to the irreducible divisors in $\mathbb{P}^m \smallsetminus \mathbb{G}_{\mathrm{m}}^m$ (cf. for instance [CTS21, Thm. 3.7.2]) tells us that $\mathrm{Br}(G)/\mathrm{Br}(K) \neq 0$. Taking an Azumaya algebra representing an element in $\mathrm{Br}(G) \smallsetminus \mathrm{Br}(K)$ gives us in turn a class in $H^1(G, H)$ which does not come from $H^1(K, H)$ (cf. [CTS21, Thm. 3.3.2] for the equivalence between Brauer classes and Azumaya algebras).

In the case $G = \mathrm{PGL}_m$, the subgroup of algebraic classes in $\mathrm{Br}(G)/\mathrm{Br}(K)$ is isomorphic to $H^1(K, \mathbb{Z}/m\mathbb{Z})$ (cf. [San81, Lem. 6.9(iii)]), so that one can find non-constant classes as well by choosing a suitable ground field K.

Finally, in the case $G = A$, it is also well-known that $\mathrm{Br}(A)/\mathrm{Br}(K)$ is non-trivial in general, even over an algebraically closed field (cf. [Ber72, p. 182]). We then conclude as before.

Remark 5.7. Given that all the examples above use the adjoint group $H = \mathrm{PGL}_n$, one could wonder whether Question 1.1 has a positive answer when H is semi-simple and simply connected. This question remains open.

A. An elementary proof of Rosenlicht's Lemma

We prove the following lemma, due to Rosenlicht in the case of an algebraically closed field (cf. [Ros61]).

Lemma A.1. *Let H be a semi-abelian variety over a field K. Let V and W be geometrically integral K-varieties. Then, the following holds.*

1. *The abelian group $H(W)/H(K)$ is finitely generated and free.*
2. *If K is separably closed, the sequence*

$$0 \to H(K) \xrightarrow{h \mapsto (h,-h)} H(V) \times H(W) \xrightarrow{\pi_V^* + \pi_W^*} H(V \times_K W) \to 0,$$

is exact, where

$$\begin{aligned} \pi_V^* : H(V) &\to H(V \times_K W) \\ h &\mapsto h \circ \pi_V . \end{aligned}$$

Proof. In both statements, W and V can be replaced by a non-empty open subvariety. In particular, by generic smoothness, we can thus assume that V and W are smooth over K.

Let us prove the first assertion. Denoting by $\bar{K}$ a separable closure of K, the natural arrow

$$H(W)/H(K) \to H(\bar{W})/H(\bar{K})$$

is injective, so that we may assume $K = \bar{K}$. By definition, there is an exact sequence

$$1 \to T \to H \xrightarrow{\pi} A \to 1,$$

with T a torus and A an abelian variety. Since K is separably closed and T is smooth, the sequence

$$1 \to T(W)/T(K) \to H(W)/H(K) \xrightarrow{\pi} A(W)/A(K),$$

is also exact. It will suffice then to treat the cases $H = A$, or $H = \mathbb{G}_\mathrm{m}$.

Up to shrinking W, we can assume there is a smooth K-morphism $W \to U$ of relative dimension one and with geometrically connected fibers, where $U = D(f)$ is a principal open subset of some affine space. Denote by $K' = K(U)$ the field of functions of U, and set $W' := W \times_U K'$. Then W' is a smooth K'-curve, and there is an exact sequence

$$0 \to H(U)/H(K) \to H(W)/H(K) \to H(W')/H(K').$$

Thus, the problem is further reduced to two particular cases: $W = D(f)$ is a principal open subset of some affine space, or W is a smooth curve over K. The first case is trivial for abelian varieties (a morphism from a rational variety to an abelian variety is constant). For $\mathbb{G}_{\mathrm{m}}$, it is dealt with by a straightforward direct computation. It remains to treat the case of a smooth affine curve W/K. The case $H = \mathbb{G}_{\mathrm{m}}$ is once again a straightforward computation, using the smooth proper curve C compactifying W and the fact that $H(C) = H(K)$. For H an abelian variety, using [Mil86, Thm. 6.1], the statement is equivalent to $\mathrm{Hom}_{\mathrm{gp}}(\mathrm{Jac(C)}, H)$ being a free abelian group of finite rank, which holds by [Mil86, Thm. 10.15].

In order to establish the second assertion, we only need to check the surjectivity of $\pi_V^* + \pi_W^*$. Pick rational points $v_0 \in V(K)$ and $w_0 \in W(K)$, which exist since K is separably closed and V, W are smooth over K. For $f \in H(V \times_K W)$, set

$$\tilde{f}(v, w) := f(v, w) - f(v_0, w) - f(v, w_0) + f(v_0, w_0).$$

Then, the composite

$$\pi \circ \tilde{f} : V \times_K W \to A$$

vanishes on $\{v_0\} \times_K W$ and on $V \times_K \{w_0\}$. Using [Mil86, Thm. 3.4], we get $\pi \circ \tilde{f} = 0$. In other words, $\tilde{f}$ takes values in T. Thus, in order to conclude, it suffices to prove the exactness when $H = \mathbb{G}_{\mathrm{m}}$. Fix $f \in H(V \times_K W) = \mathcal{O}^\times_{V \times_K W}$. Replacing f by $(v, w) \mapsto f(v, w) f(v_0, w)^{-1}$, we may assume that $f = 1$ on $\{v_0\} \times_K W$. To conclude, we have to prove that f factors through the projection $\pi_V : V \times W \to V$, i.e. that f does not depend on W. Using that the smooth K-variety W is covered by smooth K-curves, we easily reduce to the case where W is a curve.

If W an open subset of $\mathbb{G}_{\mathrm{m}}$, this is once again a straightforward computation. In general, for a given $v_1 \in V(K)$, set

$$\begin{aligned} g_1 : W &\to \mathbb{G}_{\mathrm{m}}, \\ w &\mapsto f(v_1, w). \end{aligned}$$

We have to show that g_1 is constant. Assume it is not. Then, it is finite of degree $d \geq 1$. Up to shrinking W, we may assume that g_1 is a composite arrow $W \to W' \to U \subset \mathbb{G}_{\mathrm{m}}$ with $W \to W'$ purely inseparable and $W' \to U$ finite and étale.

Assume first that $W = W'$. Denote by $\tilde{W} \to U$ the Galois closure of g_1. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{O}^\times_{V \times W} & \xrightarrow{\rho_1} & \mathcal{O}^\times_W \\ \Big\downarrow N & & \Big\downarrow N \\ \mathcal{O}^\times_{V \times U} & \xrightarrow{\rho_1} & \mathcal{O}^\times_U, \end{array}$$

where N is the (multiplicative) norm with respect to the finite étale morphism g_1 and ρ_1 denotes the restrictions to the fiber above v_1 (in particular, it maps f to g_1). We claim that $N(f) \in \mathcal{O}^\times_{V \times_K U}$ is trivial on $\{v_0\} \times U$. Indeed, we have $N(f) = f_1 f_2 \ldots f_d$, where $f = f_1, f_2, \ldots, f_d$ are the images of f with respect to the different embeddings of $\mathcal{O}_{V \times_K W}$ in $\mathcal{O}_{V \times_K \tilde{W}}$. These are trivial on $\{v_0\} \times \tilde{W}$, whence the claim. Since we know the conclusion of the Lemma for $W = U$, we get that $N(f) \in \mathcal{O}^\times_{V \times U}$ does not depend on U. Using commutativity of the diagram above, we compute:

$$\rho_1(N(f)) = N(g_1) = g_1^d.$$

Thus, g_1^d is constant. Hence so is g_1, which finishes the proof when $W = W'$. In general, for a finite purely inseparable morphism of degree p^r, the norm is given by $N(x) = x^{p^r}$, so that a straightforward variant of the proof above applies. ☺

Remark A.2. The second statement, over a non-separably closed K, is false in general. Indeed, surjectivity fails when $V = W$ is a non-trivial H-torsor. This is essentially the only counterexample, as surjectivity holds whenever $H^1(K, H) = 0$ (e.g. for $H = \mathbb{G}_{\mathrm{m}}$).

Remark A.3. When H is a torus, the proof of point 2) of Lemma A.1 that we provide above uses affine geometry, combined with a norm argument. In this sense, it is an "inner" proof. This is a more elementary approach than the use of a normal compactification of W in Rosenlicht's original proof.

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