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On a conjecture of Kato and Kuzumaki

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1. Introduction

In 1986, Kato and Kuzumaki stated a set of conjectures which aimed at giving a diophantine characterization of cohomological dimension of fields in terms of Milnor K-theory and projective hypersurfaces of small degree ([2]). To understand the motivations for their conjectures, we need to introduce the classical notion of C_i-fields.

C_i-fields

Definition (Artin, Lang). Let $i \geq 0$ be an integer. A field L is C_i if, for any positive integers n and d such that $n \geq d^i$, every hypersurface in $\mathbb{P}^n_L$ of degree d has a rational point.

Here are some of the main properties of C_i-fields:

- A field has the C_0-property if, and only if, it is algebraically closed.
- Finite fields are C_1 (Chevalley-Warning).
- If L is a C_i-field, then the field of rational functions $L(t)$ and the field of Laurent series $L((t))$ are C_{i+1} (Tsen-Lang-Nagata, Greenberg).

Motivation for the conjectures

The properties of the cohomological dimension of fields are very similar to the properties of C_i-fields. For instance, finite fields have cohomological dimension 1, and if L is a field of cohomological dimension i, then the fields $L(t)$ and $L((t))$ have cohomological dimension $i+1$.

These similarities suggest that there could be a link between C_i-fields and fields of cohomological dimension i. It turns out that a C_1-field has cohomological dimension ≤ 1 and a C_2-field has cohomological dimension ≤ 2. However, in general, it is not known if a C_i-field has cohomological dimension at most i. As for the converse, it is known to be false for any $i \geq 1$ (for instance, a p-adic field has cohomological dimension 2 but is not C_2 according to work of Terjanian).

Kato and Kuzumaki's idea to avoid these problems and to characterize cohomological dimension of fields in diophantine terms consists in introducing variants of the C_i-properties involving Milnor K-theory.

2. Kato and Kuzumaki's conjectures

Definition of Milnor K-theory

Let L be any field and let q be a non-negative integer. The q**-th Milnor** K**-theory** group of L is by definition the group $K_0^M(L) = \mathbb{Z}$ if $q = 0$ and:

$$K_q^M(L) := \frac{(L^\times)^{\otimes q}}{\langle x_1 \otimes ... \otimes x_q | \exists i, j, i \neq j, x_i + x_j = 1 \rangle}$$

if $q > 0$.

Norm map in Milnor K-theory

According to work of Kato, when L' is a finite extension of a field L, one can construct **norm homomorphisms** $N_{L'/L} : K_q^M(L') \to K_q^M(L)$ for each q satisfying the following properties:

- for $q = 0$, the map $N_{L'/L} : K_0^M(L') \to K_0^M(L)$ is given by multiplication by $[L' : L]$;
- for $q = 1$, the map $N_{L'/L} : K_1^M(L') \to K_1^M(L)$ coincides with the usual norm $L'^\times \to L^\times$;
- there is a compatibility condition between the norm maps for different values of q.

C_i^q-fields

Let L be a field. For each L-scheme of finite type Z, we denote by $N_q(Z/L)$ the subgroup of $K_q^M(L)$ generated by the images of the maps $N_{L'/L} : K_q^M(L') \to K_q^M(L)$ when L' describes the finite extensions of L such that $Z(L') \neq \emptyset$.

Definition. For $i \geq 0$, we say that L satisfies the C_i^q**-property** if, for every finite extension M of L and for every hypersurface Z in $\mathbb{P}^n_M$ of degree d with $d^i \leq n$, we have $N_q(Z/M) = K_q^M(M)$.

In particular:

- A field L is C_i^0 if, for each finite extension M of L, every hypersurface Z in $\mathbb{P}^n_M$ of degree d with $d^i \leq n$ has index 1. In particular, a C_i-field has the C_i^0-property.
- A field L is C_0^q if, for each tower of finite extensions $L''/L'/L$, the norm $N_{L''/L'} : K_q^M(L'') \to K_q^M(L')$ is surjective.

Statement of the conjecture

Conjecture (Kato, Kuzumaki, 1986, [2]). For $i \geq 0$ and $q \geq 0$, a perfect field is C_i^q if, and only if, it is of cohomological dimension at most $i+q$.

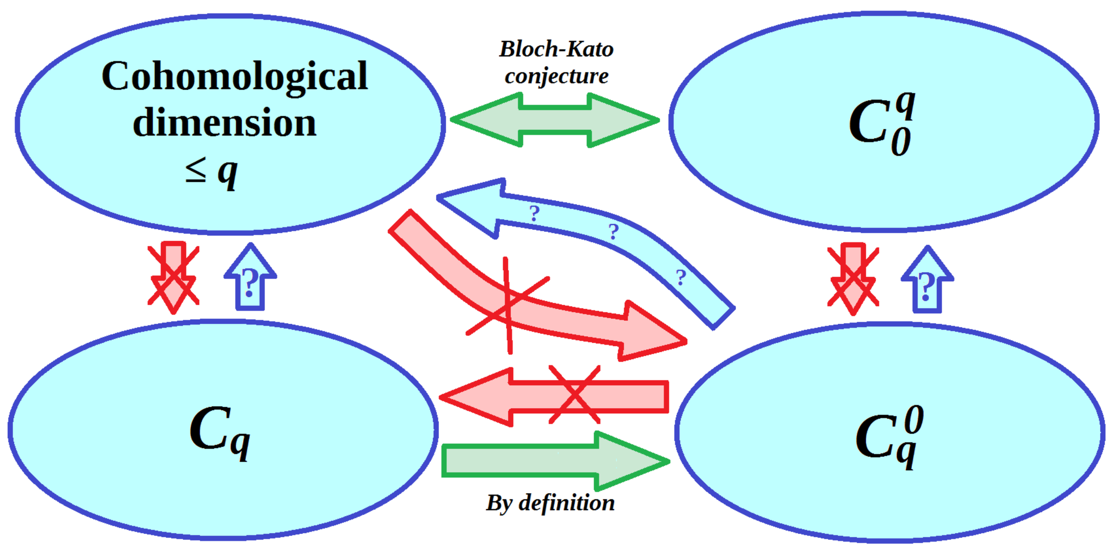
Known results and counter-examples

By using the Bloch-Kato conjecture, one can show that a field of characteristic zero is C_0^q if, and only if, it is of cohomological dimension at most q.

However, Kato and Kuzumaki's conjectures are nowadays known to be wrong in general. For example, Merkurjev constructed in 1992 a field of characteristic 0 and of cohomological dimension 2 which did not satisfy property C_2^0. Similarly, Colliot-Thélène and Madore produced in 2004 a field of characteristic 0 and of cohomological dimension 1 which did not satisfy property C_1^0.

These counter-examples were all constructed by a method using transfinite induction due to Merkurjev and Suslin. **The conjecture of Kato and Kuzumaki is therefore still completely open for fields that usually appear in number theory or in algebraic geometry.**

Summary



3. Wittenberg's work ([3])

In 2015, Wittenberg made an important step forward concerning Kato and Kuzumaki's conjectures ([3]). By introducing a stronger version of the C_1^1-property which behaves much better with respect to dévissage, he proved the C_1^1-property for several fields of cohomological dimension 2:

Theorem (Wittenberg, 2015, [3]). The field $\mathbb{C}((t_1))((t_2))$, p-adic fields and totally imaginary number fields satisfy property C_1^1.

However, Wittenberg's article leaves open the question of the C_1^1-property for other usual fields of cohomological dimension 2: the field of rational functions $\mathbb{C}(x,y)$, the field of Laurent series in two variables $\mathbb{C}((x,y))$, and the fields $\mathbb{C}(x)((y))$ and $\mathbb{C}((x))(y)$. It is worth noting that, apart from $\mathbb{C}((x,y))$, the previous fields are known not to not to satisfy the strong variant of the C_1^1-property introduced by Wittenberg.

4. Main results ([1])

Number fields

We first consider the case of number fields. The main result is a local-global principle in the context of the conjecture of Kato and Kuzumaki for varieties containing a geometrically integral closed subscheme. Such a result was previously only known for smooth, projective, geometrically irreducible varieties thanks to work by Kato and Saito or for proper varieties of Euler-Poincaré characteristic equal to 1 according to Wittenberg's work ([3]):

Theorem A. Let K be a number field and let Ω_K be the set of places of K. Let Z be a K-variety containing a geometrically integral closed subscheme. For each $v \in \Omega_K$, let K_v be the completion of K with respect to v and Z_v be the K_v-scheme $Z \times_K K_v$. Then:

$$\mathrm{Ker}\left(K^\times/N_1(Z/K) \to \prod_{v \in \Omega_K} K_v^\times/N_1(Z_v/K_v)\right) = 0.$$

By combining Theorem **A** with a result of Kollár stating that over a field of characteristic 0 any hypersurface in $\mathbb{P}^n$ of degree d with $d \leq n$ contains a geometrically integral closed subscheme and with the C_1^1-property for p-adic fields, **we recover the** C_1^1**-property for totally imaginary number fields**.

This proof has the advantage of being very explicit. For instance, it allows one to see that, if $K = \mathbb{Q}(i)$ and Z is the conic of equation $x^2 + 3y^2 + 5z^2 = 0$ in $\mathbb{P}^2$, then:

$$K^\times = \langle N_{K(\sqrt{a})/K}(K(\sqrt{a})^\times) | a \in \{3, 5, 15, 17\} \rangle$$

and each of the extensions $K(\sqrt{a})$ appearing in the previous formula satisfies $Z(K(\sqrt{a})) \neq \emptyset$.

Global fields of positive characteristic

By proving a similar statement to Theorem **A** for global fields of positive caracteristic, we get the following result:

Theorem B. Let K be the function field of a curve over a finite field of characteristic $p > 0$ and let Z be a hypersurface of degree d in $\mathbb{P}^n_K$ such that $d \leq n$. Then the exponent of the group $K^\times/N_1(Z/K)$ is a power of p.

Function fields of complex varieties

Thanks to a surprisingly simple argument and some computations in Milnor K-theory, we prove:

Theorem C. Let k be an algebraically closed field of characteristic 0. Then the function field of an n-dimensional integral k-variety satisfies the C_i^q-property for all $i \geq 0$ and $q \geq 0$ such that $i + q = n$.

The field $\mathbb{C}(x_1, ..., x_m)((t))$

In contrast with theorem **C**, the proof of the following result is particularly difficult and technical:

Theorem D. Let k be an algebraically closed field of characteristic zero. Let L be the function field of an n-dimensional integral k-variety. Then the complete field $L((t))$ satisfies the C_i^q-property for all $i \geq 0$ and $q \geq 0$ such that $i+q = n+1$.

5. Summary in dimension 2

Summary for usual fields of cohomological dimension 2:

Field	C_2	C_2^0	C_1^1	C_0^2
$\mathbb{Q}_p$	✗	?	✓	✓
$\mathbb{Q}(i)$	✗	?	✓	✓
$\mathbb{C}((t_1))((t_2))$	✓	✓	✓	✓
$\mathbb{C}(x)((t))$	✓	✓	✓	✓
$\mathbb{C}((t))(x)$	✓	✓	?	✓
$\mathbb{C}((x,y))$	?	?	?	✓
$\mathbb{C}(x,y)$	✓	✓	✓	✓

References

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[3] Olivier Wittenberg. Sur une conjecture de Kato et Kuzumaki concernant les hypersurfaces de Fano. *Duke Math. J.*, 164(11):2185–2211, 2015.

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