Duality and local-global principle over two-dimensional henselian local rings

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According to Sansuc, over number fields, Brauer-Manin obstruction is the only obstruction to the local-global principle for torsors under linear connected algebraic groups. In a recent article ([1]), Colliot-Thélène, Parimala and Suresh introduce a new kind of obstruction to the local-global principle over function fields of regular integral schemes of any dimension, and they ask whether it is the only obstruction to the local-global principle for torsors under linear connected algebraic groups over the Laurent series field $\mathbb{C}((x, y))$. In this talk, I will explain why this question has a positive answer.

1 Fields of interest

In this report, we are interested in finite extensions of the Laurent series field in two variables $\mathbb{C}((x, y))$. More generally, we adopt the following notations :

k : algebraically closed field of characteristic 0.
R : integral, local, normal, henselian, 2-dimensional k-algebra with residue field k.
$\mathcal{X} := \operatorname{Spec} R$.
$X := \mathcal{X} \setminus \{s\}$ where s is the closed point of $\mathcal{X}$.
$X^{(1)}$: set of codimension 1 points in X.
K : the fraction field of R.

We are interested in the field K.

2 Brauer-Hasse-Noether exact sequence

In this paragraph, we want to understand the Brauer group of K. To do so, consider a desingularization $f : \tilde{\mathcal{X}} \to \mathcal{X}$ of $\mathcal{X}$ such that :

- f is projective and $\tilde{\mathcal{X}}$ is an integral, regular, 2-dimensional scheme ;
- $f : f^{-1}(X) \to X$ is an isomorphism ;
- the special fiber $Y := f^{-1}(s)$ is a strict normal crossing divisor of $\tilde{\mathcal{X}}$.

Such a desingularization always exists.

Now observe that Y is a projective k-curve whose irreducible components are smooth. Let $g_1, ..., g_n$ be the genera of the irreducible components of Y. Let also Γ be the graph attached to Y : by definition, this is the graph whose vertices are the irreducible components of Y and whose edges connect two vertices if, and only if, the corresponding irreducible components intersect. Denote by c the first Betti number of Γ.

Theorem 2.1. *There is an exact sequence :*

$$0 \to (\mathbb{Q}/\mathbb{Z})^{c+2\sum g_i} \to Br\,K \to \bigoplus_{v \in X^{(1)}} Br\,K_v \to \mathbb{Q}/\mathbb{Z} \to 0$$

where K_v is the completion of K at v for $v \in X^{(1)}$ and the middle map is the restriction map.

The proof uses the Gersten conjecture for the regular scheme $\tilde{\mathcal{X}}$ (such a result is due to Panin) and requires to carry out a geometrical and combinatorial study of the desingularization $\tilde{\mathcal{X}}$.

3 Duality theorems

3.1 Duality in étale cohomology

Let $j : U \hookrightarrow X$ be an open immersion, with U non-empty. Let F be a finite étale group scheme over U. By using the Brauer-Hasse-Noether exact sequence of the previous paragraph, one can define a natural pairing :

$$AV : H^r(U, F) \times H^{3-r}(X, j_! F') \to H^3(X, j_! \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z},$$

where $F' = \underline{Hom}(F, \mathbb{G}_m)$ is the Cartier dual of F.

Theorem 3.1. *The pairing AV is a perfect pairing of finite groups for each integer $r \in \{0, 1, 2, 3\}$.*

There are two proofs for this theorem : one can proceed "by hand" by making quite subtle dévissages to reduce to the case when F is constant and then use the Brauer-Hasse-Noether exact sequence of the previous paragraph, or one can use Gabber's general results on the existence of dualizing complexes ([3]).

3.2 Duality in Galois cohomology

For each Galois module M over K, we define its Tate-Shafarevich groups by :

$$Ш^r(K, M) := \text{Ker}\left(H^r(K, M) \to \prod_{v \in X^{(1)}} H^r(K_v, M) \right).$$

By using extensively theorem 3.1, one can prove the following duality theorem :

Theorem 3.2. *Let T be a K-torus. Let $\hat{T}$ be its module of characters. Then there is a natural pairing :*

$$PT : Ш^1(K, T) \times Ш^2(K, \hat{T}) \to \mathbb{Q}/\mathbb{Z}$$

which is non-degenerate on the left, and whose right-kernel is the maximal divisible subgroup of $Ш^2(K, \hat{T})$.

4 Obstructions to the local-global principle

Recall the Brauer-Hasse-Noether exact sequence :

$$\operatorname{Br} K \to \bigoplus_{v \in X^{(1)}} \operatorname{Br} K_v \xrightarrow{\theta} \mathbb{Q}/\mathbb{Z} \to 0.$$

When Z is a smooth K-variety, one can introduce the set of adelic points $Z(\mathbb{A}_K)$ of Z and then define a pairing :

$$BM : Z(\mathbb{A}_K) \times \operatorname{Br} Z \to \mathbb{Q}/\mathbb{Z}, ((p_v)_{v \in X^{(1)}}, \alpha) \mapsto \theta((p_v^* \alpha)_v).$$

By using theorem theorem 3.2 and by comparing the pairings PT and BM, it is possible to describe the obstructions to local-global principle for torsors under linear connected algebraic groups over K :

Theorem 4.1. *Let G be a linear connected algebraic group over K. Let Z be a K-torsor under G. If the orthogonal of $\operatorname{Br} Z$ in $Z(\mathbb{A}_K)$ for the pairing BM is non-empty, then Z has a rational point.*

References

[1] J.-L. Colliot-Thélène, R. Parimala, and V. Suresh : *Lois de réciprocité supérieures et points rationnels*, Transactions of the American Mathematical Society, **368**, 4219–4255 (2016).

[2] D. Izquierdo : *Dualité et principe local-global pour les anneaux locaux henséliens de dimension 2*, with an appendix by Joël Riou, to appear in Algebraic Geometry.

[3] J. Riou : *Exposé XVII. Dualité*, in Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents, Astérisque, **363-364**, 351–453 (2014).