On a conjecture of Kato and Kuzumaki

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Abstract. In 1986, Kato and Kuzumaki stated several conjectures in order to give a diophantine characterization of cohomological dimension of fields in terms of projective hypersurfaces of small degree and Milnor K-theory. We establish these conjectures for finite extensions of $\mathbb{C}(x_1,...,x_n)$ and $\mathbb{C}(x_1,...,x_n)((t))$, and we prove new local-global principles over number fields and global fields of positive characteristic in the context of Kato and Kuzumaki's conjectures.

Introduction

In 1986, in the article [KK86], Kato and Kuzumaki stated a set of conjectures which aimed at giving a diophantine characterization of cohomological dimension of fields. For this purpose, they introduced variants of the C_i-properties of fields involving Milnor K-theory and projective hypersurfaces of small degree, and they hoped that these variants would characterize fields of small cohomological dimension.

More precisely, fix a field L and two non-negative integers q and i. Let $K_q^M(L)$ be the q-th Milnor K-group of L. For each finite extension L' of L, one can define a norm morphism $N_{L'/L} : K_q^M(L') \to K_q^M(L)$ (see section 1.7 of [Kat80]). Thus, if Z is a scheme of finite type over L, one can introduce the subgroup $N_q(Z/L)$ of $K_q^M(L)$ generated by the images of the norm morphisms $N_{L'/L}$ when L' describes the finite extensions of L such that $Z(L') \neq \emptyset$. One then says that the field L is C_i^q if, for each $n \geq 1$, for each finite extension L' of L and for each hypersurface Z in $\mathbb{P}^n_{L'}$ of degree d with $d^i \leq n$, one has $N_q(Z/L') = K_q^M(L')$. For example, the field L is C_i^0 if, for each finite extension L' of L, every hypersurface Z in $\mathbb{P}^n_{L'}$ of degree d with $d^i \leq n$ has a 0-cycle of degree 1. The field L is C_0^q if, for each tower of finite extensions $L''/L'/L$, the norm morphism $N_{L''/L'} : K_q^M(L'') \to K_q^M(L')$ is surjective.

Kato and Kuzumaki conjectured that, for $i \geq 0$ and $q \geq 0$, a perfect field is C_i^q if, and only if, it is of cohomological dimension at most $i+q$. This conjecture generalizes a question raised by Serre in [Ser65] asking whether the cohomological dimension of a C_i-field is at most i. In 2005, in an unpublished paper [Acq05], Acquista proved Kato and Kuzumaki's conjecture for $i = 0$: in other words, a perfect field is C_0^q if, and only if, it is of cohomological dimension at most q. As it was already pointed out at the end of Kato and Kuzumaki's original paper [KK86], such a result also follows from the Bloch-Kato conjecture, which has been established by Rost and Voevodsky. However, it turns out that the conjectures of Kato and Kuzumaki are wrong in general. For example, Merkurjev constructed in [Mer91] a field of characteristic 0 and of cohomological dimension 2 which did not satisfy property C_2^0. Similarly, Colliot-Thélène and Madore produced in [CTM04] a field of characteristic 0 and of cohomological dimension 1 which did not satisfy property C_1^0. These counter-examples were all constructed by a method using transfinite induction due to Merkurjev and Suslin. The conjecture of Kato and Kuzumaki is therefore still completely open for fields that usually appear in number theory or in algebraic geometry.

Very recently, in [Wit15], Wittenberg made an important step forward : he proved that p-adic fields, the field $\mathbb{C}((t_1))((t_2))$ and totally imaginary number fields all satisfy property C_1^1. His method consists in introducing and proving a property which is stronger than property C_1^1 : more precisely, he says that a field L is strongly C_1^1 if, for each finite extension L' of L, each proper scheme Z over L' and each coherent sheaf E on Z, the Euler-Poincaré characteristic $\chi(Z,E)$ kills the abelian group $K_q^M(L')/N_q(Z/L')$. It turns out that this notion behaves much better with respect to dévissage than the C_1^1-property of Kato and Kuzumaki : this allows Wittenberg to use methods that had been previously developped in [ELW15].

Wittenberg's article leaves open the question of the C_1^1-property for the following fields : the field of rational functions $\mathbb{C}(x,y)$, the field of Laurent series in two variables $\mathbb{C}((x,y))$, and the fields $\mathbb{C}(x)((y))$ and $\mathbb{C}((x))(y)$. That the property is satisfied by $\mathbb{C}(x,y)$ and $\mathbb{C}(x)((y))$ is a particular case of the general theorems that are established in the present paper (see theorems **C** and **D** below).

The article is divided into three parts that can be read almost independently and that deal with Kato and Kuzumaki's conjectures for different fields. In the first section, we focus on the cases of number fields and of function fields of curves over finite fields. In the case of number fields, we establish a local-global principle in the context of the conjecture of Kato and Kuzumaki for varieties containing a geometrically integral closed subscheme. Such a result was previously only known for smooth, projective, geometrically irreducible varieties (see theorem 4 of [KS83]) or for proper varieties of Euler-Poincaré characteristic equal to 1 (proposition 6.2 of [Wit15]) :

Theorem A. (Theorem 1.4, number field case)
Let K be a number field and let Ω_K be the set of places of K. Let Z be a K-variety containing a geometrically integral closed subscheme. For each $v \in \Omega_K$, let K_v be the completion of K with respect to v and Z_v be the K_v-scheme $Z \times_K K_v$. Then :

$$Ker\left(K^\times/N_1(Z/K) \to \prod_{v\in\Omega_K} K_v^\times/N_1(Z_v/K_v)\right) = 0.$$

This theorem, which is established by using hilbertianity properties of number fields as well as results due to Demarche and Wei ([DW14]) concerning the local-global principle for torsors under normic tori, then allows us to deduce a simplified and more effective proof of Wittenberg's result concerning the C_1^1-property for totally imaginary number fields (see corollary 1.9 and paragraph 1.2.2). The explicitness of our proof allows us to give new and more precise results in some situations (see proposition 1.14).

In the case of global fields of positive characteristic, we prove a local-global principle similar to the one in theorem **A** but which involves a variant of the group $N_1(Z/K)$:

Theorem B. (Theorem 1.4, function field case)
Let K be the function field of a curve over a finite field of characteristic $p > 0$ and let Ω_K be the set of places of K. Let Z be a proper K-scheme containing a geometrically irreducible closed subscheme. For $v \in \Omega_K$, let K_v be the completion of K with respect to v and Z_v be the K_v-scheme $Z \times_K K_v$. Let $N_1^s(Z/K)$ be the subgroup of $K^\times$ spanned by the images of the norm homomorphisms $N_{L_s/K} : L_s^\times \to K^\times$ where L describes finite extensions of K such that $Z(L) \neq \emptyset$ and L_s stands for the separable closure of K in L. Then :

$$Ker\left(K^\times/N_1^s(Z/K) \to \prod_{v\in\Omega_K} K_v^\times/N_1^s(Z_v/K_v)\right) = 0.$$

This theorem then allows us to prove that global fields of positive characteristic have the C_1^1-property "away from p" (corollary 1.18).

In the second part, by means of a surprisingly simple argument, we prove Kato and Kuzumaki's conjectures for function fields over $\mathbb{C}$ of arbitrary dimension :

Theorem C. (Theorem 2.2)
Let k be an algebraically closed field of characteristic 0. Then the function field of an n-dimensional integral k-variety satisfies the C_i^q-property for all $i \geq 0$ and $q \geq 0$ such that $i + q = n$.

In particular, this shows that the field $\mathbb{C}(x, y)$ satisfies the C_1^1-property, and hence answers question (3) in paragraph 7.3 of [Wit15] positively.

In the third and last part, we prove Kato and Kuzumaki's properties for complete discrete valuation fields whose residue field is the function field of a variety over an algebraically closed field of characteristic zero :

Theorem D. (Theorem 3.9)
Let k be an algebraically closed field of characteristic zero. Let K be the function field of an n-dimensional integral k-variety. Then the complete field $K((t))$ satisfies the C_i^q-property for all $i \geq 0$ and $q \geq 0$ such that $i + q = n + 1$.

This theorem, whose proof relies on subtle refinements of Artin's approximation theorem, implies in particular that $\mathbb{C}(x)((t))$ is C_1^1.

Remark 0.1. The C_1^1-property for the fields $\mathbb{C}(x, y)$ and $\mathbb{C}(x)((t))$, which is a special case of theorems **C** and **D**, cannot be obtained by the methods developped in [Wit15] because those fields are not strongly C_1^1 (see remark 7.6 of [Wit15]).

Preliminaries

Let L be any field and let q be a non-negative integer. The q-th Milnor K-group of L is by definition the group $K_0^M(K) = \mathbb{Z}$ if $q = 0$ and :

$$K_q^M(L) := \underbrace{L^\times \otimes_{\mathbb{Z}} ... \otimes_{\mathbb{Z}} L^\times}_{q \text{ times}} / \langle x_1 \otimes ... \otimes x_q | \exists i, j, i \neq j, x_i + x_j = 1\rangle$$

if $q>0$. For $x_1,...,x_q \in L^\times$, the symbol $\{x_1,...,x_q\}$ denotes the class of $x_1 \otimes ... \otimes x_q$ in $K_q^M(L)$. More generally, for r and s non-negative integers such that $r+s=q$, there is a natural pairing :

$$K_r^M(L) \times K_s^M(L) \to K_q^M(L)$$

which we will denote $\{\cdot,\cdot\}$.

When L' is a finite extension of L, one can construct a norm homomorphism $N_{L'/L}: K_q^M(L') \to K_q^M(L)$ (see section 1.7 of [Kat80]) satisfying the following properties :

- for $q=0$, the map $N_{L'/L}: K_0^M(L') \to K_0^M(L)$ is given by multiplication by $[L':L]$;
- for $q=1$, the map $N_{L'/L}: K_1^M(L') \to K_1^M(L)$ coincides with the usual norm $L'^\times \to L^\times$;
- if r and s are non-negative integers such that $r+s=q$, we have $N_{L'/L}(\{x,y\}) = \{x, N_{L'/L}(y)\}$ for $x \in K_r^M(L)$ and $y \in K_s^M(L')$;
- if L'' is a finite extension of L', we have $N_{L''/L} = N_{L'/L} \circ N_{L''/L'}$.

For each L-scheme of finite type, we denote by $N_q(Z/L)$ the subgroup of $K_q^M(L)$ generated by the images of the maps $N_{L'/L}: K_q^M(L') \to K_q^M(L)$ when L' describes the finite extensions of L such that $Z(L') \neq \emptyset$. In particular, $N_0(Z/L)$ is the subgroup of $\mathbb{Z}$ generated by the index of Z (ie the gcd of the degrees $[L':L]$ when L' describes the finite extensions of L such that $Z(L') \neq \emptyset$). For $i \geq 0$, we say that L satisfies the C_i^q-property if, for every finite extension L' of L and for every hypersurface Z in $\mathbb{P}^n_{L'}$ of degree d with $d^i \leq n$, we have $N_q(Z/L') = K_q^M(L')$. In particular, L is C_i^0 if, for each finite extension L' of L, every hypersurface Z in $\mathbb{P}^n_{L'}$ of degree d with $d^i \leq n$ has index 1.

The field L is C_0^q if, for each tower of finite extensions $L''/L'/L$, the norm $N_{L''/L'}: K_q^M(L'') \to K_q^M(L')$ is surjective. As it was already pointed out by Kato and Kuzumaki at the end of [KK86], by using the Bloch-Kato conjecture which identifies the groups $K_q^M(L)/n$ and $H^q(L, \mu_n^{\otimes q})$ for n prime to the characteristic of L and which has been proved by Rost and Voevodsky, one can show that a field of characteristic zero is C_0^q if, and only if, it is of cohomological dimension at most q.

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1. Global fields

1.1 Proof of theorems A and B

This section is devoted to number fields and function fields of curves over finite fields. The main goal consists in establishing theorems **A** and **B**. Whenever K is a global field, Ω_K stands for the set of places of K, and for $v \in \Omega_K$, we denote by K_v the completion of K with respect to v and by $\mathcal{O}_v$ the ring of integers in K_v.

We start with a preliminary lemma concerning Hilbertian fields. For a definition of Hilbertian fields, the reader may refer to section 12.1 of [FJ08].

Lemma 1.1. *Let K be a Hilbertian field and fix an algebraic closure $\overline{K}$ of K. Let F be a finite Galois extension of K and let Y be a geometrically integral K-variety. Then there exists a finite extension F_0 of K such that $Y(F_0) \neq \emptyset$ and $F_0 \cap F = K$.*

Proof. Of course, we can assume that dim $Y > 0$. By applying Bertini's theorem to an open dense quasi-projective subset of Y, one shows that Y contains a quasi-projective geometrically integral curve C over K. Since Y is geometrically reduced, one can find a curve C' in $\mathbb{P}^2_K$ birationally equivalent to C. Let $g \in K[X,Y,Z]$ be a homogeneous polynomial which is irreducible over $\overline{K}$ and such that C' is the curve defined by the equation $g = 0$. Let U' be a non-empty subset of C' which is isomorphic to an open subset of C. We now distinguish two cases :

- if K has characteristic $p > 0$, we know that $g \in K[X,Y,Z] \setminus K[X^p,Y^p,Z^p]$, and we can therefore assume without loss of generality that $g \in K[X,Y,Z] \setminus K[X^p,Y,Z]$. Hence we may consider an integer $m \geq 1$ and a polynomial $h \in K[Y,Z] \setminus \{0\}$ such that p does not divide m and the coefficient of X^m in g is h. We also consider the set :

$$H := \{(y,z) \in F^2 | g(X,y,z) \in F[X] \text{ is irreducible}, h(y,z) \neq 0\}.$$

- if K has characteristic 0, we can assume without loss of generality that $g \notin K[Y,Z]$ and we consider the set :

$$H := \{(y,z) \in F^2 | g(X,y,z) \in F[X] \text{ is irreducible}\}.$$

  To unify notations with the positive characteristic case, we also set $h(Y,Z) := 1 \in K[Y,Z]$.

As g is irreducible over F and separable in the variable X, the set H is by definition a separable Hilbert subset of F^2. According to corollary 12.2.3 of [FJ08], H contains a separable Hilbert subset H' of K^2. Since K is a Hilbertian field, the second paragraph of section 12.1 of [FJ08] implies that H' is Zariski dense in K^2. In particular, the set H' is infinite, and there exists an infinite number of pairs $(y,z) \in K^2$ such that $g(X,y,z)$ is irreducible over F and $h(y,z) \neq 0$. Each of these pairs corresponds to a point $w \in (C')^{(1)}$ such that $K(w) \cap F = K$ and the extension $K(w)/K$ is separable. Since $C' \setminus U'$ is finite, we conclude that there exists $w \in (U')^{(1)}$ such that $K(w)$ is a finite separable extension of K satisfying $K(w) \cap F = K$. By setting $F_0 = K(w)$, we get $Y(F_0) \neq \emptyset$ and $F_0 \cap F = K$. □

Corollary 1.2. *Let K be a Hilbertian field and fix an algebraic closure $\overline{K}$ of K. Let F be a finite Galois extension of K and let Y be a geometrically irreducible K-variety. Then there exists a finite extension F_0 of K such that $Y(F_0) \neq \emptyset$ and $F_0 \cap F = K$.*

Proof. If K has characteristic 0, the corollary immediately follows from lemma 1.1. Assume that K has positive characteristic. Let K' be a purely inseparable finite extension of K such that $(Y_{K'})^{\mathrm{red}}$ is geometrically integral. By lemma 1.1, there exists a finite extension F_1 of K' such that $Y(F_1) \neq \emptyset$ and $F_1 \cap (K' \cdot F) = K'$, where $K' \cdot F$ denotes the subfield of $\overline{K}$ generated by K' and F. Then we also have $F_1 \cap F = K$. □

We now introduce a variant of the group $N_1(Z/K)$ which will allow us to treat in a unified way number fields and function fields of curves over finite fields :

Definition 1.3. *Let K be a field and let Z be a K-scheme of finite type. We denote by $N_1^s(Z/K)$ the subgroup of $K^\times$ spanned by the images of the norm morphisms $N_{L_s/K} : L_s^\times \to K^\times$ where L describes finite extensions of K such that $Z(L) \neq \emptyset$ and L_s stands for the separable closure of K in L.*

Note that, if K is a field of characteristic 0 and Z is a K-scheme of finite type, then $N_1^s(Z/K) = N_1(Z/K)$. We are now ready to prove the main theorem of this section :

Theorem 1.4. *Let K be a number field or the function field of a curve over a finite field. Let Z be a K-variety containing a geometrically irreducible closed subscheme. For $v \in \Omega_K$, we denote by Z_v the K_v-scheme $Z \times_K K_v$. Then :*

$$Ker\left(K^\times/N_1^s(Z/K) \to \prod_{v\in\Omega_K} K_v^\times/N_1^s(Z_v/K_v)\right) = 0.$$

Notation 1.5. Whenever M denotes a Galois module over K, we define the first Tate-Shafarevich group of M by :

$$Ш^1(K,M) := \text{Ker}\left(H^1(K,M) \to \prod_{v\in\Omega_K} H^1(K_v,M)\right).$$

Proof. In the sequel, we fix an algebraic closure $\overline{K}$ of K : all finite extensions of K will therefore be considered as subfields of $\overline{K}$.

Now fix $x \in K^\times$ whose class modulo $N_1^s(Z/K)$ lies in :

$$\text{Ker}\left(K^\times/N_1^s(Z/K) \to \prod_{v\in\Omega_K} K_v^\times/N_1^s(Z_v/K_v)\right).$$

We want to prove that $x \in N_1^s(Z/K)$. To do so, we consider a finite normal extension L of K such that $Z(L) \neq \emptyset$. Let L_s be the separable closure of K in L : it is a finite Galois extension of K. Let $S \subseteq \Omega_K$ be the set of places v of K satisfying one of the following properties :
(i) v is finite and the extension L_s/K is ramified at v ;
(ii) v is finite and x is not a unit in $\mathcal{O}_v$;
(iii) v is infinite.
Of course, S is a finite subset of Ω_K.

Now fix $v \in \Omega_K$. Two main cases arise :
- Assume in the first place that $v \in \Omega_K \setminus S$. In this case, v is a finite place, and as the extension L_{sv}/K_v is unramified, we know that $N_{L_{sv}/K_v}(L_{sv}^\times)$ contains $\mathcal{O}_v^\times$. Since $x \in \mathcal{O}_v^\times$, we conclude that $x \in N_{L_{sv}/K_v}(L_{sv}^\times)$.
- Assume now that $v \in S$ and fix an algebraic closure $\overline{K_v}$ of K_v. By assumption, $x \in N_1^s(Z_v/K_v)$. Let then $M_1^{(v)}, ..., M_{n_v}^{(v)}$ be finite extensions of K_v contained in

$\overline{K_v}$ such that, if $M_{i,s}^{(v)}$ denotes the separable closure of K_v in $M_i^{(v)}$, we have $x \in \left\langle N_{M_{i,s}^{(v)}/K_v}({M_{i,s}^{(v)}}^\times)|1 \leq i \leq n_v \right\rangle \subseteq K_v^\times$ and $Z(M_i^{(v)}) \neq \emptyset$ for each i. According to Greenberg's approximation theorem if v is finite (theorem 1 of [Gre66]) and Tarski-Seidenberg principle if v is real (corollary 4.1.6 de [Pir11]), we have $Z(M_i^{(v)} \cap \overline{K}) \neq \emptyset$. We can therefore consider a finite extension $L_i^{(v)}$ of K contained in $M_i^{(v)} \cap \overline{K}$ such that $Z(L_i^{(v)}) \neq \emptyset$. Let $L_{i,s}^{(v)}$ be the separable closure of K in $L_i^{(v)}$. The valuation on $M_i^{(v)}$ induces by restriction a place w of $L_i^{(v)}$ which divides v and such that the completion of $L_{i,s}^{(v)}$ with respect to w is a subextension of $M_{i,s}^{(v)}/K_v$. Hence :

$$N_{M_{i,s}^{(v)}/K_v}({M_{i,s}^{(v)}}^\times) \subseteq N_{L_{i,s}^{(v)} \otimes_K K_v/K_v}((L_{i,s}^{(v)} \otimes_K K_v)^\times) \subseteq K_v^\times.$$

Since $x \in \left\langle N_{M_{i,s}^{(v)}/K_v}({M_{i,s}^{(v)}}^\times)|1 \leq i \leq n_v \right\rangle \subseteq K_v^\times$, we deduce that :

$$x \in \left\langle N_{L_{i,s}^{(v)} \otimes_K K_v/K_v}((L_{i,s}^{(v)} \otimes_K K_v)^\times)|1 \leq i \leq n_v \right\rangle \subseteq K_v^\times.$$

To summmarize, we have just proved that, if T is the normic torus $R^1_{E/K}(\mathbb{G}_m)$ with $E = L_s \times \prod_{v \in S} \prod_{i=1}^{n_v} L_{i,s}^{(v)}$ and if $[x]$ stands for the image of x in

$$H^1(K,T) = K^\times / N_{E/K}(E^\times),$$

then :

$$[x] \in Ш^1(K,T). \tag{1}$$

Let now F be the smallest finite Galois extension of K containing L_s and all the $L_{i,s}^{(v)}$. Since Z contains a geometrically irreducible closed subscheme, corollary 1.2 shows that Z has a point in a finite extension F_0 of K such that $F_0 \cap F = K$. Denote by $F_{0,s}$ the separable closure of K in F_0.

According to theorem 1 of [DW14], since $F_{0,s} \cap F = K$ and the extension F/K is Galois, we have :

$$Ш^1(K,Q) = 0$$

where Q denotes the normic torus $R^1_{E'/K}(\mathbb{G}_m)$, with $E' = L_s \times F_{0,s} \times \prod_{v \in S} \prod_{i=1}^{n_v} L_{i,s}^{(v)}$. Noting that the torus T naturally embeds in Q and using (1), we conclude that the class of x in $H^1(K,Q)$ is trivial. Since $Z(L) \neq \emptyset$, $Z(F_0) \neq \emptyset$ and $Z(L_i^{(v)}) \neq \emptyset$ for each v and each i, this shows that $x \in N_1^s(Z/K)$ as desired. □

Remark 1.6. Let K be a number field and keep the notations and the assumptions of theorem 1.4. The proof implies that, if $L_1, ..., L_r$ are finite extensions of K such that :

$$\langle N_{L_i \otimes_K K_v/K_v}((L_i \otimes_K K_v)^\times)|1 \leq i \leq r \rangle = K_v^\times$$

for each $v \in \Omega_K$, then there exists a finite extension L_{r+1} of K such that :

$$\langle N_{L_i/K}(L_i^\times)|1 \leq i \leq r+1 \rangle = K^\times.$$

Moreover, if L is a finite Galois extension of K containing all the L_i, the field L_{r+1} can be chosen to be any finite extension of K which is linearly disjoint from L.

1.2 Number fields

In this paragraph, we focus on the case when K is a number field. We give a new proof of the C_1^1-property for totally imaginary number fields, and we see how this proof allows one to study some concrete examples.

1.2.1 Property C_1^1 for totally imaginary number fields

In theorem 1.4, the assumption that Z contains a geometrically integral closed subscheme cannot be removed. Indeed, one can for example choose $K = \mathbb{Q}$, take for Z a variety over $L = \mathbb{Q}(\sqrt{13}, \sqrt{17})$ having a rational point in L and see Z as a K-variety. In this case, theorem 1.4 fails since the affine $\mathbb{Q}$-variety defined by the equation :

$$N_{L/\mathbb{Q}}(x + y\sqrt{13} + z\sqrt{17} + t\sqrt{221}) = -1$$

does not satisfy the local-global principle.

Nevertheless, the assumption that Z contains a geometrically integral closed subscheme can be slightly weakened :

Corollary 1.7. *Let K be a number field and let Z be K-variety. For $v \in \Omega_K$, we denote by Z_v the K_v-scheme $Z \times_K K_v$. Assume that there exist finite extensions $K_1, ..., K_r$ of K such that Z_{K_i} contains a geometrically integral closed subscheme for each i and the gcd of the degrees $[K_i : K]$ is 1. Then :*

$$Ker\left(K^\times/N_1(Z/K) \to \prod_{v\in\Omega_K} K_v^\times/N_1(Z_v/K_v)\right) = 0.$$

Remark 1.8. This corollary was previously only known for smooth, projective, geometrically irreducible K-varieties (theorem 4 of [KS83]) and for proper varieties with Euler-Poincaré characteristic equal to 1 (proposition 6.2 of [Wit15]). It generalizes those results according to proposition 3.3 of [Wit15].

Proof. According to theorem 1.4, for each i, we have :

$$\mathrm{Ker}\left(K_i^\times/N_1(Z_{K_i}/K_i) \to \prod_{w\in\Omega_{K_i}} K_{i,w}^\times/N_1(Z_{K_{i,w}}/K_{i,w})\right) = 0.$$

Therefore a restriction-corestriction argument shows that the group

$$\mathrm{Ker}\left(K^\times/N_1(Z/K) \to \prod_{v\in\Omega_K} K_v^\times/N_1(Z_v/K_v)\right)$$

is of $[K_i : K]$-torsion for each i, hence trivial. □

Wittenberg has recently proved property C_1^1 for totally imaginary number fields (theorem 6.1 of [Wit15]). Theorem 1.4 allows us to obtain this result by a different method. The passage from local results to global results is simpler and more explicit than in section 6 of [Wit15].

Corollary 1.9. *Let K be a number field and let Z be a hypersurface of degree d in $\mathbb{P}^n_K$ such that $d \leq n$ and $N_1(Z_v/K_v) = K_v^\times$ for each real place v of K. Then $N_1(Z/K) = K^\times$.*

Proof. By exercise I.7.2(c) of [Har77], we know that the Euler-Poincaré characteristic $\chi(Z, \mathcal{O}_Z) := \sum_{i \geq 0} \dim_K H^i_{\mathrm{Zar}}(Z, \mathcal{O}_Z)$ is equal to 1. Hence, proposition 3.3 of [Wit15] establishes the existence of finite extensions $K_1, ..., K_r$ of K satisfying the assumptions of corollary 1.7. We deduce that :

$$\mathrm{Ker}\left(K^\times/N_1(Z/K) \to \prod_{v \in \Omega_K} K_v^\times/N_1(Z_v/K_v)\right) = 0.$$

But by corollary 5.5 of [Wit15], we have $N_1(Z_v/K_v) = K_v^\times$ for each finite place v of K. By assumption, we also know that $N_1(Z_v/K_v) = K_v^\times$ for each infinite place v of K. We conclude that $N_1(Z/K) = K^\times$. □

Remark 1.10. Instead of using proposition 3.3 of [Wit15] and corollary 1.7 to prove that :

$$\mathrm{Ker}\left(K^\times/N_1(Z/K) \to \prod_{v \in \Omega_K} K_v^\times/N_1(Z_v/K_v)\right) = 0,$$

we could have combined theorem 2 of [Kol07] (which asserts that a projective hypersurface in $\mathbb{P}^n_K$ of degree d with $d \leq n$ always contains a geometrically integral closed subscheme) with theorem 1.4. The proof of proposition 3.3 of [Wit15] is nevertheless more elementary than the one of theorem 2 of [Kol07].

1.2.2 Some concrete examples over number fields

It is interesting to notice that the proof we have given of the C_1^1-property for totally imaginary number fields is quite explicit : by this, we mean that in many numerical examples, it allows us to establish more precise results than just the C_1^1-property. To see this, we first establish the following lemma :

Lemma 1.11. *Let $n \geq 1$ be an integer. Let M be a field of characteristic prime to n. Fix an algebraic closure $\overline{M}$ of M. Assume that M contains all n-th roots of unity and that $M^\times/M^{\times n}$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2$. Let $a_0, ..., a_n$ be $n+1$ elements of $M^\times$. For $0 \leq i, j \leq n$ with $i \neq j$, set $M_{ij} = M\left(\sqrt[n]{a_i a_j^{-1}}\right)$. Then :*

$$M^\times = \left\langle N_{M_{ij}/M}(M_{ij}^\times) | 1 \leq i, j \leq n, i \neq j \right\rangle.$$

Proof. Write $n = p_1^{r_1} ... p_s^{r_s}$ with $p_1, ..., p_s$ pairwise distinct prime numbers and $r_1, ..., r_s$ positive integers. Since $\left\langle N_{M_{ij}/M}(M_{ij}^\times) | 1 \leq i, j \leq n, i \neq j \right\rangle$ contains $M^{\times n}$ and :

$$M^\times/M^{\times n} \cong \prod_{t=1}^{s} M^\times/M^{\times p_t^{r_t}},$$

it is enough to show that for each $t \in \{1, ..., s\}$, the group $M^\times$ is spanned by the subgroups $M^{\times p_t^{r_t}}$ and $N_{M_{ij}/M}(M_{ij}^\times)$ for $1 \leq i, j \leq n, i \neq j$.

We henceforth fix $t \in \{1,...,s\}$. If there exist integers i and j with $0 \leq i,j \leq n$ and $i \neq j$ such that $a_i a_j^{-1} \in M^{\times p_t^{r_t}}$, there is nothing to prove. We can therefore assume that $a_i a_j^{-1} \notin M^{\times p_t^{r_t}}$ for all $0 \leq i,j \leq n$ with $i \neq j$.

For $0 \leq i,j \leq n$ with $i \neq j$, let e_{ij} be the largest divisor of $p_t^{r_t}$ such that there exists $y_{ij} \in M^\times$ satisfying $y_{ij}^{e_{ij}} = a_i a_j^{-1}$. The following properties are satisfied :

(i) for $0 \leq i,j \leq n$ with $i \neq j$, the integer $p_t^{r_t}$ does not divide e_{ij}, because $a_i a_j^{-1} \notin M^{\times p_t^{r_t}}$,

(ii) for $0 \leq i,j \leq n$ with $i \neq j$, the order of y_{ij} in $M^\times/M^{\times p_t^{r_t}}$ is $p_t^{r_t}$ because $M^\times/M^{\times p_t^{r_t}}$ is isomorphic to $(\mathbb{Z}/p_t^{r_t}\mathbb{Z})^2$,

(iii) for $1 \leq i,j \leq n$ with $i \neq j$, one has $y_{ij}^{e_{ij}} = y_{i0}^{e_{i0}} \cdot y_{j0}^{-e_{j0}}$,

and we want to prove that the group $M^\times/M^{\times p_t^{r_t}}$ is spanned by the $N_{M_{ij}/M}(M_{ij}^\times)$ for $0 \leq i,j \leq n, i \neq j$. Since the group $N_{M_{ij}/M}(M_{ij}^\times)$ contains y_{ij} for each i and j and $M^\times/M^{\times p_t^{r_t}}$ is isomorphic to $(\mathbb{Z}/p_t^{r_t}\mathbb{Z})^2$, it is enough to prove the following abstract sublemma provided that one chooses $\Lambda = M^\times/M^{\times p_t^{r_t}}$, $x_{ij} = y_{ij}$ for $1 \leq i,j \leq p_t^{r_t}$ with $i \neq j$ and $x_{ii} = y_{i0}$ for $i \in \{1,...,p_t^{r_t}\}$. □

Sublemma 1.12. *Let p be a prime number and $r \geq 1$ an integer. Set $n = p^r$ and let $\Lambda = (\mathbb{Z}/n\mathbb{Z})^2$. For each $i \in \{1,...,n\}$ and each $j \in \{1,...,n\}$, let x_{ij} be an element of Λ and let e_{ij} be a positive integer. Assume that :*
(i) for $1 \leq i,j \leq n$, the integer p^r does not divide e_{ij}.
(ii) for each i and each j, the order of x_{ij} in Λ is n.
(iii) for each i and each j such that $i \neq j$, one has $e_{ij}x_{ij} = e_{ii}x_{ii} - e_{jj}x_{jj}$.
Then Λ is spanned by all the x_{ij}.

Proof. Consider an automorphism ϕ of Λ such that $\phi(x_{1,1}) = (1,0)$. By assumptions (i) and (ii), we have $e_{ij}x_{ij} \neq 0$ for all i and j. Hence $\phi(e_{11}x_{11}),...,\phi(e_{nn}x_{nn})$ are pairwise distinct and non-zero. According to the pigeonhole principle, we deduce that we are in one of the following situations :

- **Case 1 :** there exists $i_0 \in \{1,...,n\}$ such that $\phi(e_{i_0i_0}x_{i_0i_0}) \in \{0\} \times (\mathbb{Z}/n\mathbb{Z} \setminus \{0\})$. We then conclude that x_{11} and $x_{i_0i_0}$ span Λ.
- **Case 2 :** there exist $i_0 \in \{1,...,n\}$ and $j_0 \in \{1,...,n\}$ such that $\phi(e_{i_0i_0}x_{i_0i_0}) - \phi(e_{j_0j_0}x_{j_0j_0}) \in \{0\} \times (\mathbb{Z}/n\mathbb{Z} \setminus \{0\})$. We conclude that x_{11} and $x_{i_0j_0}$ span the group Λ.

□

In the sequel, we will also need the following easy lemma :

Lemma 1.13. *Let n be a positive integer and let $q(n)$ be the number of prime divisors of n. Let X be a generating set of $\Delta := \mathbb{Z}/n\mathbb{Z}$. Then X contains a subset X' which has at most $q(n)$ elements and which spans Δ.*

Proof. We proceed by induction on $q(n)$.

- If $q(n) = 1$, then $n = p^a$ for some prime number p and some integer a. The set X contains an element x which is not divisible by p, and one can simply choose $X' = \{x\}$.
- Now let q be a positive integer and assume that the lemma is known when $q(n) \leq q$. Take $n \geq 1$ such that $q(n) = q+1$ and write $n = p_1^{a_1}...p_{q+1}^{a_{q+1}}$ for $p_1,...,p_{q+1}$ pairwise distinct prime numbers and $a_1,...,a_{q+1}$ positive integers. The set X contains an element x which is not divisible by p_{q+1}. The quotient group $\Delta/\langle x \rangle$ is spanned by the image

$\overline{X}$ of X in $\Delta / \langle x \rangle$. Since $\Delta / \langle x \rangle$ is a cyclic group whose order m satisfies $q(m) \leq q$, the induction hypothesis shows that one can find a subset $\overline{X_0}$ of $\overline{X}$ which has at most q elements and which spans $\Delta / \langle x \rangle$. By choosing any lifting $X_0 \subseteq X$ of $\overline{X_0} \subseteq \overline{X}$ having at most q elements, one sees that $\{x\} \cup X_0$ is a subset of X which has at most $q(n)$ elements and which spans Δ.

□

Lemma 1.11 applies to p-adic fields containing n^{th} roots of unity and such that p does not divide n. From this, our proof of Kato and Kuzumaki's conjecture yields the following proposition :

Proposition 1.14. *Let $n \geq 1$ be an integer. Let K be a totally imaginary number field containing n^{th} roots of unity. Let $f \in K[X_0, ..., X_n]$ be a homogeneous polynomial of degree n of the form :*

$$f = a_0 X_0^n + ... + a_n X_n^n + g(X_0, ..., X_n)$$

where each monomial appearing in g contains at least three different variables. Set :

$$N = \frac{n(n+1)}{2} + 1 + [K : \mathbb{Q}] q(n)(q(n)+1),$$

where $q(n)$ denotes the number of prime divisors of n. Then there exist N finite extensions $K_1, ..., K_N$ of K such that :
(i) the equation $f = 0$ has non-trivial solutions in K_i for each i,
(ii) $K^\times$ is spanned by the subgroups $N_{K_i/K}(K_i^\times)$ for $1 \leq i \leq N$.

Proof. For $0 \leq i < j \leq n$, consider the field $K_{ij} = K\left(\sqrt[n]{a_i a_j^{-1}}\right)$. Fix v a place of K not dividing n and denote by $k(v)$ the residue field of K_v. Since K contains n^{th} roots of unity, n divides the order of $k(v)^\times$. Hence proposition II.5.7 of [Neu99] implies that $K_v^\times / K_v^{\times n} \cong (\mathbb{Z}/n\mathbb{Z})^2$. Lemma 1.11 then shows that $K_v^\times$ is spanned by the subgroups $N_{K_{ij} \otimes_K K_v / K_v}((K_{ij} \otimes_K K_v)^\times)$.

Fix now v a place of K dividing n. Since the maximal unramified extension of K_v is a C_1-field (theorem 12 of [Lan52]), there exists a finite unramified extension $L_{v,0}$ of K_v such that the equation $f = 0$ has a non-trivial solution in $L_{v,0}$. As $L_{v,0}/K_v$ is unramified, the group $N_{L_{v,0}/K_v}(L_{v,0}{}^\times)$ contains $\mathcal{O}_v^\times$. Moreover, by corollary 5.5 of [Wit15], the group $K_v^\times$ is spanned by the images of the norm morphisms N_{M/K_v} when M describes finite extensions of K_v such that the equation $f = 0$ has non-trivial solutions in M : hence, by applying lemma 1.13 to the group $\Delta = (K_v^\times / K_v^{\times n}) \big/ (\mathcal{O}_v^\times / \mathcal{O}_v^{\times n})$ (which is isomorphic to $\mathbb{Z}/n\mathbb{Z}$), one can find $q(n)$ finite extensions $L_{v,1}, ..., L_{v,q(n)}$ of K_v such that the equation $f = 0$ has non-trivial solutions in $L_{v,i}$ for each i and the subgroup of $K_v^\times$ spanned by the subgroups $N_{L_{v,i}/K_v}(L_{v,i}^\times)$ contains a uniformizer. Hence the group $K_v^\times$ is spanned by the subgroups $N_{L_{v,i} \otimes_K K_v / K_v}(L_{v,i}^\times)$ for $0 \leq i \leq q(n)$. By Greenberg's approximation theorem, we deduce that there exist finite extensions $M_{v,0}, M_{v,1}, ..., M_{v,q(n)}$ of K such that the equation $f = 0$ has non-trivial solutions in $M_{v,i}$ for $0 \leq i \leq q(n)$ and $K_v^\times$ is spanned by the subgroups $N_{M_{v,i} \otimes_K K_v / K_v}((M_{v,i} \otimes_K K_v)^\times)$.

Let M be a Galois extension of K containing all the K_{ij} and all the $M_{v,i}$. Let L be a finite field extension of K which is linearly disjoint from M and such that the equation

$f = 0$ has a non-trivial zero in L. Such an extension L exists by theorem 2 of [Kol07] and by lemma 1.1. Then, by remark 1.6, the group $K^\times$ is spanned by the subgroups $N_{K_{ij}/K}(K_{ij}^\times)$ (for $0 \leq i < j \leq n$), $N_{M_{v,i}/K}(M_{v,i}^\times)$ (for $0 \leq i \leq q(n)$) and $N_{L/K}(L^\times)$. The corollary follows since the number of finite extensions of K that enter the game here is at most N. $\square$

Here is a concrete example :

Example 1.15. Consider the case where $K = \mathbb{Q}(i)$ and :

$$f = X_0^2 + 2X_1^2 + aX_2^2 \in K[X_0, X_1, X_2]$$

for some $a \in \mathbb{Z}$ such that a is congruent to 1, 3, 9, 11, 17, 19, 25 or 27 modulo 32. Let v_2 be the unique place of K above 2 and note that we have :

$$1 + i = N_{K_{v_2}(\sqrt{2})/K_{v_2}} \left(1 + \frac{1-i}{2}\sqrt{2}\right),$$

hence :

$$1 + i \in N_{K_{v_2}(\sqrt{2})/K_{v_2}} \left(K_{v_2}(\sqrt{2})^\times\right). \tag{2}$$

Moreover, one easily checks that the assumptions on a imply that the extension $K_{v_2}(\sqrt{a})/K_{v_2}$ is unramified. Hence

$$\mathcal{O}_{v_2}^\times \subseteq N_{K_{v_2}(\sqrt{a})/K_{v_2}} \left(K_{v_2}(\sqrt{a})^\times\right). \tag{3}$$

From the inclusions (2) and (3), we get that the group $K_{v_2}^\times$ is spanned by the subgroups $N_{K_{v_2}(\sqrt{2})/K_{v_2}} \left(K_{v_2}(\sqrt{2})^\times\right)$ and $N_{K_{v_2}(\sqrt{a})/K_{v_2}} \left(K_{v_2}(\sqrt{a})^\times\right)$. By using lemma 1.11, we deduce that for each place v of K, the group $K_v^\times$ is spanned by the subgroups $N_{K_v(\sqrt{b})/K_v} \left(K_v(\sqrt{b})^\times\right)$ for $b \in \{2, a, 2a\}$. One can then easily check that the extensions $K(\sqrt{2}, \sqrt{a})$ and $K(\sqrt{a+2})$ are always linearly disjoint over K. Therefore $K^\times$ is spanned by the subgroups $N_{K(\sqrt{b})/K} \left(K(\sqrt{b})^\times\right)$ for $b \in \{2, a, 2a, a+2\}$. Of course, for such b, the equation $f = 0$ has non-trivial solutions in $K(\sqrt{b})$.

1.3 Global fields of positive characteristic

In this paragraph, we focus on the case when K is a global field of positive characteristic. We prove of the C_1^1-property « away from p », and, as in the case of number fields, we see how the proof allows one to study some concrete examples.

We start by introducing a variant of the group $N_1(Z/K)$ which will allow us to study the C_1^1-property « away from p » for global fields of positive characteristic :

Definition 1.16. *Let K be a field of characteristic $p > 0$. Let Z be a K-scheme of finite type. We denote by $N_1^p(Z/K)$ the set of $x \in K^\times$ such that there exists an integer $r \geq 1$ satisfying $x^{p^r} \in N_1(Z/K)$.*

The following proposition is a consequence of theorem 1.4 :

Proposition 1.17. *Let K be the function field of a curve over a finite field of characteristic $p>0$. Let Z be a K-variety containing a geometrically irreducible closed subscheme. For $v \in \Omega_K$, we denote by Z_v the K_v-scheme $Z \times_K K_v$. Then the abelian group :*

$$Ker\left(K^\times/N_1(Z/K) \to \prod_{v\in\Omega_K} K_v^\times/N_1^p(Z_v/K_v)\right)$$

is a p-primary group.

Proof. Consider an element $x \in K^\times$ whose class modulo $N_1(Z/K)$ lies in :

$$\mathrm{Ker}\left(K^\times/N_1(Z/K) \to \prod_{v\in\Omega_K} K_v^\times/N_1^p(Z_v/K_v)\right).$$

By assumption, for each $v \in \Omega_K$, there exists $r_v \geq 0$ such that $x^{p^{r_v}} \in N_1(Z_v/K_v) \subseteq N_1^s(Z_v/K_v)$. Furthermore, there exists an integer $m \geq 0$ such that $x^m \in N_1^s(Z_v/K_v)$ for each $v \in \Omega_K$. We conclude that there exists $r \geq 0$ such that $x^{p^r} \in N_1^s(Z_v/K_v)$ for each $v \in \Omega_K$. According to theorem 1.4, this shows that $x^{p^r} \in N_1^s(Z/K)$. We can therefore consider finite extensions $K_1, ..., K_n$ of K such that, if $K_{i,s}$ denotes the separable closure of K in K_i, we have $x^{p^r} \in \left\langle N_{K_{i,s}/K}(K_{i,s}^\times) | 1 \leq i \leq n\right\rangle$ and $Z(K_i) \neq \emptyset$ for each i. Since all the degrees $[K_i : K_{i,s}]$ are powers of p, this implies that there exists an integer $r' \geq 0$ such that $(x^{p^r})^{p^{r'}} \in \langle N_{K_i/K}(K_i^\times) | 1 \leq i \leq n\rangle$. We conclude that $x^{p^{r+r'}} \in N_1(Z/K)$, which finishes the proof of the corollary. □

We are now ready to prove the C_1^1-property « away from p » for global fields of characteristic p.

Theorem 1.18. *Let K be the function field of a curve over a finite field of characteristic $p>0$ and let Z be a hypersurface of degree d in $\mathbb{P}^n_K$ such that $d \leq n$. Then the exponent of the group $K^\times/N_1(Z/K)$ is a power of p.*

For the proof, it is useful to recall from [Wit15] that a field L is said to be strongly C_1^1 away from p if, for each finite extension L' of L, each proper scheme Z over L' and each coherent sheaf E on Z, the Euler-Poincaré characteristic $\chi(Z,E)$ kills every element of $K_q^M(L')/N_q(Z/L')$ whose order is not divisible by p.

Proof. If A is a torsion abelian group, we denote by $A\{p'\}$ the subgroup of A constituted by elements of A whose order is not divisible by p. For each proper K-scheme Z, we define :

$$H_1(Z/K) = K^\times/N_1(Z/K)$$

and we denote by n_Z the exponent of the abelian group $H_1(Z/K)\{p'\}$ if Z is non-empty or 0 otherwise. We say that Z satisfies property P if Z is normal. We are now going to check the three assumptions that appear in proposition 2.1 of [Wit15].

(1) This is obvious, because a morphism of proper K-schemes $Y \to Z$ induces a surjective morphism $H_1(Y/K) \to H_1(Z/K)$.

(2) Let Z be a proper normal K-scheme. Let K' be the algebraic closure of K in $K(Z)$. Then Z is naturally endowed with a structure of proper geometrically irreducible K'-scheme. According to theorem 1.17 :

$$\operatorname{Ker}\left(H_1(Z/K') \to \prod_{v\in\Omega_K} H_1(Z_v/K'_v)\right)\{p'\} = 0.$$

Moreover, since K'_v is strongly C_1^1 away from p for each $v \in \Omega_K$ according to corollary 4.7 of [Wit15], the group $H_1(Z_v/K'_v)\{p'\}$ is killed by $\chi_{K'}(Z, \mathcal{O}_Z)$. We deduce that the group $H_1(Z/K')\{p'\}$ is also killed by $\chi_{K'}(Z, \mathcal{O}_Z)$. But $\chi_K(Z, \mathcal{O}_Z) = [K' : K]\chi_{K'}(Z, \mathcal{O}_Z)$. Hence a restriction-corectriction argument shows that $\chi_K(Z, \mathcal{O}_Z)$ kills $H_1(Z/K)\{p'\}$. The integer n_Z has therefore to divide $\chi_K(Z, \mathcal{O}_Z)$.

(3) It suffices to choose the normalization morphism.

We can therefore apply proposition 2.1 of [Wit15] and deduce that the field K is strongly C_1^1 away from p. The corollary then follows from the fact an $(n-1)$-dimensional projective hypersurface of degree d with $d \leq n$ has Euler-Poincaré characteristic 1. □

Remark 1.19. While corollary 1.9 was already proved in [Wit15], corollary 1.18 is new.

In the same way as in the case of number fields, one can get more precise results. For example, one can prove the following proposition similarly to proposition 1.14 :

Proposition 1.20. *Let $n \geq 1$ be an integer. Let K the function field of a curve over a finite field and assume that K contains n^{th} roots of unity. Let $f \in K[X_0, ..., X_n]$ be a homogeneous polynomial of degree n of the form :*

$$f = a_0 X_0^n + ... + a_n X_n^n + g(X_0, ..., X_n)$$

where each monomial appearing in g contains at least three different variables. Assume that the projective hypersurface defined by $f = 0$ is geometrically irreducible. Set :

$$N = \frac{n(n+1)}{2} + 1.$$

Then there exist N finite extensions $K_1, ..., K_N$ of K such that :
(i) the equation $f = 0$ has non-trivial solutions in K_i for each i,
(ii) $K^\times$ is spanned by the subgroups $N_{K_i/K}(K^\times)$ for $1 \leq i \leq N$.

2. Function fields of varieties over an algebraically closed field

In this section, we are going to establish Kato and Kuzumaki's conjectures for function fields of varieties over an algebraically closed field of characteristic 0. We have already recalled that the Bloch-Kato conjecture implies that a field of characteristic 0 is C_0^q if, and only if, it is of cohomological dimension at most q. The proposition that follows is a particular case of this result. Anyway, we give an elementary proof, because its ideas will be useful in the sequel in order to establish theorems 2.2 and 3.9 :

Proposition 2.1. *Let k be an algebraically closed field of characteristic 0. Then the field $K = k(t_1, ..., t_q)$ satisfies property C_0^q.*

Proof. We proceed by induction on q. The result is obvious for $q=0$. Assume now that we have proved the proposition for some $q \geq 0$ and consider the field $K=k(t_1,...,t_{q+1})$. Let L_1 be a finite extension of K and L_2 be a finite extension of L_1. Let $u_1,...,u_{q+1}$ be elements of $L_1^\times$. We are going to prove that $\{u_1,...,u_{q+1}\} \in N_{L_2/L_1}(K_{q+1}^M(L_2))$.

To do so, we construct a family $(w_1,...,w_s)$ of elements in $L_1^\times$ in the following way :
- if $u_1,...,u_{q+1}$ are not algebraically independent over k, we consider a transcendence basis $(v_1,...,v_r)$ of the extension $L_1/k(u_1,...,u_{q+1})$ and we set $(w_1,...,w_s)=(u_1,...,u_{q+1},v_1,...,v_{r-1})$.
- if $u_1,...,u_{q+1}$ are algebraically independent over k, we set $(w_1,...,w_s)=(u_1,...,u_q)$.

Let M_1 (resp. M_2) be the algebraic closure of $k(w_1,...,w_s)$ in L_1 (resp. L_2). Let C_1 (resp. C_2) be a geometrically integral curve over M_1 (resp. M_2) such that $M_1(C_1)=L_1$ (resp. $M_2(C_2)=L_2$). Since $\overline{M_1}(C_1)$ is a C_1 field and $\overline{M_2}(C_2)/\overline{M_1}(C_1)$ is a finite extension, propositions 10 and 11 of section X.7 of [Ser79] imply that $u_{q+1} \in N_{\overline{M_2}(C_2)/\overline{M_1}(C_1)}(\overline{M_2}(C_2)^\times)$. And so there exist a finite extension F of M_2 and $y \in F(C_2)^\times$ such that $u_{q+1} = N_{F(C_2)/F(C_1)}(y)$. Moreover, by the inductive assumption, M_1 satisfies property C_0^q, and hence there exists $x \in K_q^M(F)$ such that $\{u_1,...,u_q\} = N_{F/M_1}(x)$. We deduce that :

$$\begin{aligned} N_{L_2/L_1}(N_{F(C_2)/L_2}(\{x,y\})) &= N_{F(C_2)/M_1(C_1)}(\{x,y\}) \\ &= N_{F(C_1)/M_1(C_1)}(N_{F(C_2)/F(C_1)}(\{x,y\})) \\ &= N_{F(C_1)/M_1(C_1)}(\{x,u_{q+1}\}) \\ &= \{u_1,...,u_{q+1}\}. \end{aligned}$$

We have therefore proved that $\{u_1,...,u_{q+1}\} \in N_{L_2/L_1}(K_{q+1}^M(L_2))$. As a consequence, the field K satisfies the C_0^q-property. □

We are now ready to establish Kato and Kuzumaki's conjectures for the function field of a variety over an algebraically closed field of characteristic 0 :

Theorem 2.2. *Let k be an algebraically closed field of characteristic 0. Then the function field of a q-dimensional integral k-variety satisfies the C_i^j-property for all $i \geq 0$ and $j \geq 0$ such that $i+j=q$.*

Proof. Let K be the function field of a q-dimensional integral k-variety. Let $i \geq 0$ and $j \geq 0$ be integers such that $i+j=q$. If $j=0$, there is nothing to prove because the field K is C_q. If $i=0$, the result follows from the previous proposition. Hence we can now assume that $i \neq 0$ and $j \neq 0$.
Fix a finite extension L of K. Let Z be a hypersurface of degree d in $\mathbb{P}_L^n$ with $d^i \leq n$ and let $u_1,...,u_j$ be elements of $L^\times$. We will show that the symbol $\{u_1,...,u_j\}$ is in $N_j(Z/K)$. Let $(v_1,...,v_r)$ be a transcendence basis of the extension $L/k(u_1,...,u_j)$ (with $r \geq 0$). Let M be the algebraic closure of $k(u_1,...,u_j,v_{q-j+1},...,v_r)$ in L (so that the transcendence degree of M/k is j) and let X be a geometrically integral M-variety of dimension i such that $M(X)=L$. Since the field $\overline{M}(X)$ is C_i, the variety Z has points in $\overline{M}(X)$. Therefore, there exists a finite extension F of M such that $Z(F(X)) \neq \emptyset$. Moreover, since the norm $N_{F/M}: K_j^M(F) \to K_j^M(M)$ is surjective according to proposition 2.1 and $\{u_1,...,u_j\} \in K_j^M(M)$, we get $\{u_1,...,u_j\} \in N_{F/M}(K_j^M(F))$. As a consequence, $\{u_1,...,u_j\} \in N_{F(X)/M(X)}(K_j^M(F(X)))$, and K has the C_i^j-property. □

Remark 2.3. In the previous theorem, we have in fact proved that, if L is a finite extension of $k(t_1, ..., t_q)$ and Z is a hypersurface of degree d in $\mathbb{P}^n_L$ with $d^i \leq n$, then for each j-symbol $x \in K_j^M(L)$, there exists a finite extension M of L such that $Z(M) \neq \emptyset$ and $x \in N_{M/L}(K_j^M(M))$. In particular, if $i = q-1$ and $j = 1$, for each element x in $L^\times$, there exists a finite extension M of L such that $Z(M) \neq \emptyset$ and $x \in N_{M/L}(M^\times)$.

3. Local fields with a function field as residue field

3.1 Problem and strategy

The goal of this section is to prove the conjectures of Kato and Kuzumaki for complete discrete valuation fields whose residue field is the function field of a variety over an algebraically closed field of characteristic 0. In particular, this applies to the field $\mathbb{C}(x)((t))$, for which properties C_0^2 and C_2^0 are already known and for which we are going to establish property C_1^1.

The main difficulty we face in order to establish the C_1^1-property for the field $K = \mathbb{C}(x)((t))$ lies in proving that, if Z is a hypersurface in $\mathbb{P}^n_K$ of degree $d \leq n$, then $t \in N_1(Z/K)$. To do so, we are going to show that if we adjoin all the roots of t to K, then the field K_∞ we obtain is C_1 : this will imply that $Z(K_\infty) \neq \emptyset$. In order to establish the C_1-property for K_∞, we will have to establish a modular criterion allowing us to determine whether an affine variety over K_∞ has a rational point (corollary 3.8). For this purpose we will heavily use the constructions of the article [Gre66] by Greenberg.

3.2 Greenberg's approximation theorem revisited

We start by recalling theorem 1 of [Gre66] :

Theorem 3.1. *(Theorem 1 of [Gre66])*
Let R be a henselian discrete valuation ring with field of fractions K. Let t be a uniformizer of R. Let $\underline{F} = (F_1, ..., F_r)$ be a system of r polynomials in n variables with coefficients in R. We assume that K has characteristic 0. Then there exist integers $N \geq 1$, $c \geq 1$ and $s \geq 0$ (depending exclusively on the ideal $\underline{F}R[\underline{X}]$ of $R[\underline{X}]$ generated by $F_1, ..., F_r$) such that, for each $\nu \geq N$ and each $\underline{x} \in R^n$ satisfying :

$$\underline{F}(\underline{x}) \equiv 0 \mod t^\nu,$$

there exists $\underline{y} \in R^n$ such that :

$$\underline{y} \equiv \underline{x} \mod t^{[\nu/c]-s} \quad \text{and} \quad \underline{F}(\underline{y}) = 0.$$

In particular, if the system $\underline{F} = 0$ has solutions modulo t^m for each $m \geq 1$, then it has a solution in R.

From now on, fix a henselian discrete valuation ring R with field of fractions K. Assume that K has characteristic 0 and fix an algebraic closure $\overline{K}$ of K. Let t be a uniformizer of R and choose a compatible system $\{t^{1/q}\}_{q\geq 1}$ of roots of t in $\overline{K}$: by this, we mean that the elements $t^{1/q}$ of $\overline{K}$ satisfy the relation $(t^{1/(qq')})^{q'} = t^{1/q}$ for each $q, q' \geq 1$. For $q \geq 1$, we set $K_q = K(t^{1/q})$ and $R_q = \mathcal{O}_{K_q}$. We also set $K_\infty = \bigcup_{q\geq 1} K_q$

and $R_\infty = \bigcup_{q\geq 1} R_q$. We want to establish a similar result to theorem 3.1 for the field K_∞. In that respect, we start by proving a simple lemma in commutative algebra.

Definition 3.2. *We say that an ideal I of $R[\underline{X}]$ is t-saturated if, for each $f \in R[\underline{X}]$ such that $tf \in I$, we have $f \in I$.*

Remark 3.3. Of course, the previous definition is independent of the choice of the uniformizer t. But since in the sequel we will have to replace R by R_q, it will be useful to systematically track a uniformizer of the ring we will be working on.

Lemma 3.4. *Let I be an ideal of $R[\underline{X}]$.*
(i) If I is t-saturated, then $IR_q[\underline{X}]$ is $t^{1/q}$-saturated for each $q \geq 1$.
(ii) If I is radical and t-saturated, then $IR_q[\underline{X}]$ is radical for each $q \geq 1$.

Proof. (i) Assume that I is t-saturated. Fix an integer $q \geq 1$ and let $f \in R_q[\underline{X}]$ such that $t^{1/q}f \in IR_q[\underline{X}]$. Write $t^{1/q}f = \sum_{i=1}^r f_i g_i$, with $f_i \in I$ et $g_i \in R_q[\underline{X}]$. For each $i \in \{1, ..., n\}$, let h_i be a polynomial in $R[\underline{X}]$ such that $t^{1/q}$ divides $g_i - h_i$ in R_q (ie the valuation of $g_i - h_i$ is strictly positive) : this can be achieved because R and R_q have the same residue field. We can now write :

$$t^{1/q}f = \sum_{i=1}^r f_i(g_i - h_i) + \sum_{i=1}^r f_i h_i.$$

Thus, t divides $\sum_{i=1}^r f_i h_i$ in $R[\underline{X}]$. Since I is t-saturated and $\sum_{i=1}^r f_i h_i \in I$, we deduce that $\frac{\sum_{i=1}^r f_i h_i}{t} \in I$. The equality :

$$f = \sum_{i=1}^r f_i \frac{g_i - h_i}{t^{1/q}} + t^{(q-1)/q} \cdot \frac{\sum_{i=1}^r f_i h_i}{t}$$

then implies that $f \in IR_q[\underline{X}]$ and hence the ideal $IR_q[\underline{X}]$ is $t^{1/q}$-saturated.

(ii) Assume that I is radical and t-saturated. Fix $q \geq 1$ and let f be a polynomial in $R_q[\underline{X}]$ such that $f^n \in IR_q[\underline{X}]$ for some $n > 0$. Since I is radical, one immediately checks that $IK[\underline{X}]$ is also radical. This implies that $IK_q[\underline{X}]$ is also radical, because the extension K_q/K is separable. Hence $f \in IK_q[\underline{X}]$. This means that there exists $r \geq 1$ such that $t^{r/q}f \in IR_q[\underline{X}]$. Since $IR_q[\underline{X}]$ is $t^{1/q}$-saturated (by part (i)), we deduce that $f \in IR_q[\underline{X}]$. Hence the ideal $IR_q[\underline{X}]$ is indeed radical.

□

In order to prove a similar result to theorem 3.1 for K_∞, we need to work simultaneously with all the fields K_q, which all satisfy theorem 3.1. More precisely, if we fix a system of polynomial equations over R, we can see it as a system of equations with coefficients in R_q for each $q \geq 1$: theorem 3.1 then gives us integers N_q, c_q and s_q, and our goal in the sequel is to control these integers when q varies. It is therefore quite natural to introduce the following technical definition :

Definition 3.5. *Let $\underline{F} = (F_1, ..., F_r)$ be a system of r polynomials in n variables with coefficients in R.*
(i) Fix $q \in \mathbb{N}$. We say that a triplet $(N, c, s) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}_0$ is associated to $(R, t, q, \underline{F})$ if it satisfies the following property : for each $\nu \geq N$ and each $\underline{x} \in R_q^n$ such that

$$\underline{F}(\underline{x}) \equiv 0 \mod t^{\nu/q},$$

there exists $\underline{y} \in R_q^n$ *such that*

$$\underline{y} \equiv \underline{x} \mod t^{([\nu/c]-s)/q} \quad et \quad \underline{F}(\underline{y}) = 0.$$

(ii) We say that a 4-tuple $(q_0, N, c, s) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}_0$ *is* $(R, t, \underline{F})$*-admissible if, for each* $q \geq 1$*, the triplet* (qN, c, qs) *is associated to* $(R, t, qq_0, \underline{F})$*.*

We are now ready to state the following theorem :

Theorem 3.6. *Let* R *be a henselian discrete valuation ring with field of fractions* K*. Assume that* K *has characteristic 0 and fix a uniformizer* t *of* R*. For* $q \geq 1$*, we set* $K_q = K(t^{1/q})$ *and* $R_q = \mathcal{O}_{K_q}$*. Let* $\underline{F} = (F_1, ..., F_r)$ *be a system of* r *polynomials in* n *variables with coefficients in* R*. Then there exists a 4-tuple* (q_0, N, c, s) *which is* $(R, t, \underline{F})$*-admissible.*

In order to establish this theorem, we are going to use considerably the constructions developped in the proof of theorem 3.1 (see [Gre66]).

Proof. Denote by V the affine K-variety defined by $\underline{F} = 0$ and let m be its dimension. We are going to prove by induction on m that there exists a $(R, t, \underline{F})$-admissible 4-tuple of integers.

- If $m = -1$ (ie $V = \emptyset$), then there exists $u \in (R \cap \underline{F}R[\underline{X}]) \setminus \{0\}$. The 4-tuple $(1, \mathrm{val}_R(u) + 1, 1, 0)$ is then $(R, t, \underline{F})$-admissible, since for these values of q_0, N, c, s, the assumption appearing in the definition 3.5(i) fails.
- Assume now that $m \geq 0$.
  - Assume in the first place that $\underline{F}R[\underline{X}]$ is radical and t-saturated, and that V_{K_∞} is irreducible. In this case, lemma 3.4 shows that the ideal $\underline{F}R_q[\underline{X}]$ of $R_q[\underline{X}]$ is radical for each $q \geq 1$. Let J be the jacobian matrix of $\underline{F}$ and let $\underline{D}$ be the system of minors of size $n - m$ in J. By the inductive assumption, there exists a 4-tuple (q_0', N', c', s') which is $(R, t, \underline{F}, \underline{D})$-admissible. For $I \subseteq \{1, ..., r\}$ with $|I| = n - m$, denote by $\underline{F}_I$ the system constituted by the polynomials F_i for $i \in I$. Let V_I be the K-variety defined by the system $\underline{F}_I = 0$. Let V_I^+ be the union of the irreducible components of V_I which are m-dimensional and different from V. Let $\underline{G}_I$ be a system of generators of the ideal of V_I^+ in $R[\underline{X}]$. By the inductive assumption, there exists a 4-tuple $(q_{0,I}, N_I, c_I, s_I)$ which is $(R, t, \underline{G}_I, \underline{F})$-admissible. Set :

$$q_0 = q_0' \prod_{\substack{|I|=n-m \\ I \subseteq \{1,...,n\}}} q_{0,I}.$$

According to the proof of theorem 1 of [Gre66], for each $q \geq 1$, the triplet $(N^{(q)}, c^{(q)}, s^{(q)})$ defined by :

$$N^{(q)} = 2 + 2qq_0 \max\left\{\frac{N'}{q_0'}, \max\left\{\frac{N_I}{q_{0,I}} | I \subseteq \{1, ..., n\}, |I| = n - m\right\}\right\}$$
$$c^{(q)} = 2 \max\left\{c', \max\left\{c_I | I \subseteq \{1, ..., n\}, |I| = n - m\right\}\right\}$$
$$s^{(q)} = 1 + qq_0 \max\left\{\frac{s'}{q_0'}, \max\left\{\frac{s_I}{q_{0,I}} | I \subseteq \{1, ..., n\}, |I| = n - m\right\}\right\}$$

is associated to $(R, t, qq_0, \underline{F})$. We deduce that the 4-tuple (q_0, N, c, s) defined by :

$$\begin{aligned} N &= 2 + 2q_0 \max\left\{\frac{N'}{q_0'}, \max\left\{\frac{N_I}{q_{0,I}} | I \subseteq \{1, ..., n\}, |I| = n - m\right\}\right\} \\ c &= 2\max\left\{c', \max\left\{c_I | I \subseteq \{1, ..., n\}, |I| = n - m\right\}\right\} \\ s &= 1 + q_0 \max\left\{\frac{s'}{q_0'}, \max\left\{\frac{s_I}{q_{0,I}} | I \subseteq \{1, ..., n\}, |I| = n - m\right\}\right\} \end{aligned}$$

is $(R, t, \underline{F})$-admissible.

- We do not make any assumptions anymore. Let $q_0' \geq 1$ be an integer such that the irreducible components $W_1, ..., W_u$ of $V_{K_{q_0'}}$ remain irreducible over K_∞. For each $j \in \{1, ..., u\}$, let I_j' be the prime ideal of $K_{q_0'}[\underline{X}]$ defining W_j. Consider the ideal :

$$I_j := I_j' \cap R_{q_0'}[\underline{X}].$$

Let $\underline{G}_j$ be a system of generators of I_j. The ideal I_j is radical and $t^{1/q_0'}$-saturated. Moreover, the $K_{q_0'}$-variety defined by I_j is W_j : it is a variety of dimension at most m and $(W_j)_{K_\infty}$ is irreducible. We deduce that there exists a 4-tuple $(q_{0,j}, N_j, c_j, s_j)$ which is $(R_{q_0'}, t^{1/q_0'}, \underline{G}_j)$-admissible. Note now that there exists an integer $w \in \mathbb{N}^*$ such that $(I_1'...I_u')^w \subseteq \underline{F}K_{q_0'}[\underline{X}]$. Hence there exists $v \in \mathbb{N}^*$ such that :

$$t^{uvw/q_0'}(I_1...I_u)^w \subseteq \underline{F}R_{q_0'}[\underline{X}]. \tag{4}$$

Set :

$$q_0 = q_0' \prod_j q_{0,j},$$

and consider an integer $q \geq 1$. Denote by val : $R_{qq_0} \to \mathbb{Z} \cup \{\infty\}$ the valuation on R_{qq_0}, and introduce the integers :

$$\begin{aligned} N^{(q)} &= uw\frac{qq_0}{q_0'}\left(\max\left\{\frac{N_j}{q_{0,j}}\right\} + v\right) \\ c^{(q)} &= uw\max\{c_j\} \\ s^{(q)} &= 1 + \frac{qq_0}{q_0'}\left(v + \max\left\{\frac{s_j}{q_{0,j}}\right\}\right). \end{aligned}$$

Fix $\nu \geq N^{(q)}$ and $\underline{x} \in R_{qq_0}^n$ such that $\underline{F}(\underline{x}) \equiv 0 \mod t^{\nu/(qq_0)}$. If we are given polynomials $g_1 \in I_1R_{qq_0}[\underline{X}], ..., g_u \in I_uR_{qq_0}[\underline{X}]$, the inclusion (4) implies that :

$$\nu \leq \mathrm{val}\left[t^{uvw/q_0'}\left(\prod_{j=1}^u g_j(\underline{x})\right)^w\right] = \frac{qq_0}{q_0'}uvw + w\sum_j \mathrm{val}(g_j(\underline{x})).$$

Hence there exists an integer $j_0 \in \{1, ..., u\}$ such that :

$$\mathrm{val}(g_{j_0}(\underline{x})) \geq \frac{\nu}{uw} - \frac{qq_0}{q_0'}v.$$

Since this is true whatever the chosen polynomials $g_1 \in I_1R_{qq_0}[\underline{X}], ..., g_u \in I_uR_{qq_0}[\underline{X}]$ are, we conclude that there exists $j_1 \in \{1, ..., u\}$ such that :

$$\forall g \in I_{j_1}R_{qq_0}[\underline{X}], \mathrm{val}(g(\underline{x})) \geq \frac{\nu}{uw} - \frac{qq_0}{q_0'}v. \tag{5}$$

As $\nu \geq N^{(q)}$, we also have :

$$\frac{\nu}{uw} - \frac{qq_0}{q_0'}v \geq \frac{qq_0}{q_0'q_{0,j_0}}N_{j_0}. \tag{6}$$

Since the 4-tuple $(q_{0,j_0}, N_{j_0}, c_{j_0}, s_{j_0})$ is $(R_{q_0'}, t^{1/q_0'}, \underline{G}_{j_0})$-admissible, the triplet $(\frac{qq_0}{q_0'q_{0,j}}N_{j_0}, c_{j_0}, \frac{qq_0}{q_0'q_{0,j_0}}s_{j_0})$ is associated to $\left(R_{q_0'}, t^{1/q_0'}, \frac{qq_0}{q_0'}, \underline{G}_{j_0}\right)$. We then deduce from (5) and (6) that there exists $\underline{y} \in R^n_{qq_0}$ such that :

$$\underline{y} \equiv \underline{x} \mod t^{\mu/(qq_0)} \quad \text{et} \quad \underline{G}_{j_0}(\underline{y}) = 0,$$

where $\mu = [\frac{\nu}{c_{j_0}uw}] - \frac{qq_0}{q_0'c_{j_0}}v - 1 - \frac{qq_0}{q_0'q_{0,j_0}}s_{j_0}$. This implies that :

$$\underline{y} \equiv \underline{x} \mod t^{([\nu/c^{(q)}]-s^{(q)})/(qq_0)} \quad \text{and} \quad \underline{F}(\underline{y}) = 0,$$

and hence the triplet $(N^{(q)}, c^{(q)}, s^{(q)})$ is associated to $(R, t, qq_0, \underline{F})$. Therefore the 4-tuple (q_0, N, c, s) defined by :

$$N = uw\frac{q_0}{q_0'}\left(\max\left\{\frac{N_j}{q_{0,j}}\right\} + v\right)$$
$$c = uw\max\{c_j\}$$
$$s = 1 + \frac{q_0}{q_0'}\left(v + \max\left\{\frac{s_j}{q_{0,j}}\right\}\right)$$

is $(R, t, \underline{F})$-admissible.

□

Corollary 3.7. *Let R be a henselian discrete valuation ring with field of fractions K. Assume that K has characteristic 0 and fix a uniformizer t of R. For $q \geq 1$, we set $K_q = K(t^{1/q})$ and $R_q = \mathcal{O}_{K_q}$. We also set $K_\infty = \bigcup_{q\geq 1} K_q$ and $R_\infty = \bigcup_{q \geq 1} R_q$. Let $\underline{F} = (F_1, ..., F_r)$ be a system of r polynomials in n variables with coefficients in R_∞. There exists $M \in \mathbb{Q}_{>0}$, $\gamma \in \mathbb{N}$ and $\sigma \in \mathbb{Q}_{>0}$ satisfying the following property : for each rational number $\mu \geq M$ and each $\underline{x} \in R^n_\infty$ such that*

$$\underline{F}(\underline{x}) \equiv 0 \mod t^{\mu},$$

there exists $\underline{y} \in R^n_\infty$ such that

$$\underline{y} \equiv \underline{x} \mod t^{\mu/\gamma - \sigma} \quad \textit{and} \quad \underline{F}(\underline{y}) = 0.$$

Proof. By replacing R by R_q for some sufficiently large q, we can assume that the system $\underline{F}$ has coefficients in R. According to theorem 3.6, there exists a $(R, t, \underline{F})$-admissible 4-tuple (q_0, N, c, s). Set $M = N/q_0$, $\gamma = c$ and $\sigma = \frac{s+1}{q_0}$. Consider $\mu \in \mathbb{Q}$ such that $\mu \geq M$ and write $\mu = a/b$ with $a, b \in \mathbb{N}$. Assume that there exists $\underline{x} \in R^n_\infty$ such that $\underline{F}(\underline{x}) \equiv 0 \mod t^{\mu}$. Let $q_1 \geq 1$ be such that $\underline{x} \in R^n_{q_1}$. We know that, for each $q \geq 1$, the triplet (qN, c, qs) is associated to $(R, t, qq_0, \underline{F})$. In particular, the triplet (bq_1N, c, bq_1s) is associated to $(R, t, bq_1q_0, \underline{F})$. Since $\underline{F}(\underline{x}) \equiv 0 \mod t^{\mu}$ and $\mu \geq N/q_0$, we deduce that there exists $\underline{y} \in R^n_{bq_0q_1}$ such that $\underline{F}(\underline{y}) = 0$ and :

$$\underline{y} \equiv \underline{x} \mod t^{\lambda}$$

with $\lambda = \frac{1}{bq_1q_0}\left(\left[\frac{aq_1q_0}{c}\right] - bq_1s\right)$. This finishes the proof because $\lambda \geq \frac{\mu}{c} - \sigma$. □

Corollary 3.8. *Under the assumptions of corollary 3.7, if the congruence* $\underline{F}(\underline{x}) \equiv 0 \mod t^\nu$ *has solutions in* R_∞ *for each integer* $\nu \geq 1$*, then the equation* $\underline{F}(\underline{x}) = 0$ *has solutions in* R_∞*.*

3.3 Statement for the field $\mathbb{C}(x_1, ..., x_m)((t))$

We are finally ready to establish Kato and Kuzumaki's conjectures for complete discrete valuation fields whose residue field is the function field of a variety over an algebraically closed field of characteristic 0 :

Theorem 3.9. *Let* k *be an algebraically closed field of characteristic zero and fix* $m \geq 1$*. Let* Y *be an* m*-dimensional integral* k*-variety and set* $K = k(Y)((t))$*. Then the complete field* K *satisfies the* C_i^j*-property for all* $i \geq 0$ *and* $j \geq 0$ *such that* $i + j = m + 1$*.*

Proof. Since the field K is C_{m+1} and has cohomological dimension $m+1$, we can assume that $j \neq 0$ and $i \neq 0$. In the sequel, we fix an algebraic closure $\overline{K}$ of K. All fields will be understood as subfields of $\overline{K}$.

Let Z be a hypersurface of $\mathbb{P}^n_K$ of degree d, with $d^i \leq n$. We want to prove that $N_j(Z/K) = K_j^M(K)$.

- Fix first a j-tuple $(u_1, ..., u_j) \in k(Y)^{\times j}$. We are going to prove that $\{u_1, ..., u_j\} \in N_j(Z/K)$. For this purpose, let $(v_1, ..., v_r)$ be a transcendence basis of the extension $k(Y)/k(u_1, ..., u_j)$ and denote by M the algebraic closure of the field $k(u_1, ..., u_j, v_{m-j+1}, ..., v_r)$ in K : it is a field of transcendence degree j over k. Let Y' be a geometrically integral M-variety of dimension $i-1$ such that $k(Y) = M(Y')$. The field $\overline{M}(Y')$ is C_{i-1}, and therefore the field :
$$K_M := \bigcup_{F/M \text{ finite}} F(Y')((t))$$
is C_i. We deduce that $Z(K_M) \neq \emptyset$, and hence there exists a finite extension F of M such that $Z(F(Y')((t))) \neq \emptyset$. Since M is C_0^j, we have $\{u_1, ..., u_j\} \in N_{F/M}(K_j^M(F))$, and hence $\{u_1, ..., u_j\} \in N_j(Z/K)$ as desired.
- Fix now a $(j-1)$-tuple $(u_1, ..., u_{j-1}) \in k(Y)^{\times j-1}$. We are going to prove that $\{u_1, ..., u_{j-1}, t\} \in N_j(Z/K)$. For this purpose, consider a homogeneous polynomial $f \in k(Y)[[t]][X_0, ..., X_n]$ defining Z. Let $(v_1, ..., v_r)$ be a transcendence basis of the extension $k(Y)/k(u_1, ..., u_{j-1})$ and denote by M the algebraic closure of $k(u_1, ..., u_{j-1}, v_{m-j+2}, ..., v_r)$ in K : it is a field of transcendence degree $j-1$ over k. Let Y' be a geometrically integral M-variety of dimension i such that $k(Y) = M(Y')$. We set :
$$K_M := \bigcup_{F/M \text{ finite}} F(Y')((t)),$$
$$R_M := \bigcup_{F/M \text{ finite}} F(Y')[[t]].$$
The ring R_M is a henselian discrete valuation ring with fraction field K_M, uniformizer t and residue field $\overline{M}(Y')$. We also set :
$$K_\infty := \bigcup_{q \geq 1} K_M(t^{1/q}),$$
$$R_\infty := \bigcup_{q \geq 1} R_M[t^{1/q}].$$

Let $\mathfrak{m}_\infty$ be the maximal ideal of R_∞ and fix an integer $\nu \geq 1$. Let $f_\nu \in M((t))(Y')[X_0, ..., X_n]$ and $g_\nu \in R_\infty[X_0, ..., X_n]$ be homogeneous polynomials of degree d such that :

$$f = f_\nu + t^\nu g_\nu.$$

Since $\overline{M((t))}(Y')$ is C_i and is contained in K_∞, there exists $(x_0, ..., x_n) \in R_\infty^{n+1} \setminus \mathfrak{m}_\infty^{n+1}$ such that $f_\nu(x_0, ..., x_n) = 0$. We therefore have :

$$f(x_0, ..., x_n) \equiv 0 \mod t^\nu.$$

Since this is satisfied for each $\nu \geq 1$, we deduce from corollary 3.8 that $Z(K_\infty) \neq \emptyset$. We can then consider a finite extension F/M and an integer $q \geq 1$ such that $Z(F(Y')((t^{1/q}))) \neq \emptyset$. As M has the C_0^{j-1}-property, there exists $x \in K_{j-1}^M(F)$ such that $N_{F/K}(x) = \{u_1, ..., u_{j-1}\}$. Hence :

$$\begin{aligned} N_{F(Y')((t^{1/q}))/K}(\{x, t^{1/q}\}) &= N_{F(Y')((t))/M(Y')((t))}\left(N_{F(Y')((t^{1/q}))/F(Y')((t))}(\{x, t^{1/q}\})\right) \\ &= N_{F(Y')((t))/M(Y')((t))}(\{x, \pm t\}) \\ &= \{u_1, ..., u_{j-1}, \pm t\}. \end{aligned}$$

We conclude that $\{u_1, ..., u_{j-1}, t\} \in N_j(Z/K)$.
Since the group $K_j^M(K)/d$ is spanned by symbols of the form $\{u_1, ..., u_j\}$ and $\{u_1, ..., u_{j-1}, t\}$ with $(u_1, ..., u_j) \in k(Y)^{\times j}$, we get $N_j(Z/K) = K_j^M(K)$. □

Remark 3.10. Let k be an algebraically closed field of characteristic 0 and let Y be an integral k-variety of dimension m. The previous proof shows in fact that, if M is an extension of transcendence degree $j-1$ over k contained in $k(Y)$ and if Y' is an integral M-variety such that $M(Y') = k(Y)$, then the field :

$$K_\infty = \bigcup_{q \geq 1} \bigcup_{F/M \text{ finite}} F(Y')((t))(t^{1/q})$$

is C_{m+1-j}. In particular, the field $\bigcup_{q\geq 1} \mathbb{C}(x)((t))(t^{1/q})$ is C_1.

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