Valued fields

1 Absolute values and completions

Exercise 1

1) Write $2/3$ and $-2/3$ as 5-adic expansions $\sum_{n\geqslant 0} a_n 5^n$, with $a_n \in \{0,1,2,3,4\}$.
2) Let $x = \sum_{n\geqslant 0} a_n p^n \in \mathbb{Z}_p$, with $a_n \in \{0,\dots,p-1\}$. Prove that $x \in \mathbb{Q}$ if, and only if, $(a_n)_{n\geqslant 0}$ is eventually periodic.

Exercise 2 Let x be a rational number. Determine the primes p for which $(x^n)_{n\geqslant 1}$ converges in $\mathbb{Q}_p$.

Exercise 3 Let p be a prime and a, b be nonnegative integers. Determine the closure of

$$\Sigma_{a,b} := \{a + bn : n \geqslant 0\}$$

in $\mathbb{Z}_p$. For which (a,b) is this set dense in $\mathbb{Z}_p$?

Exercise 4 Let K be a complete field endowed with a nontrivial nonarchimedean absolute value. Prove the following assertions.

1) Every open ball is also closed.
2) Every closed ball of positive radius is also open.
3) Every sphere of positive radius is both open and closed.
4) Every point of an open ball is a centre of that ball.
5) Every triangle is isosceles: among the three distances between its vertices, the largest occurs at least twice.

Is completeness needed in any of these assertions?

Exercise 5 Let p be a prime, $a \in \mathbb{Q}_p$ and $r > 0$.

1) Show that both the open ball and the closed ball with centre a and radius r are open and closed. What are their boundaries?
2) What is the connected component of a in $\mathbb{Q}_p$?

Exercise 6 Show that there exists a continuous surjection $\mathbb{Z}_p \to [0,1]$. Does there exist a continuous surjection $[0,1] \to \mathbb{Z}_p$?

Exercise 7 Let k be a field and $K = k(X)$.

1) Classify the places of K trivial on k. Can you describe the corresponding completions of K?
2) Assume $k = \mathbb{F}_q$.
   (a) Show that all places of K are trivial on k.
   (b) Prove that they satisfy a product formula.

Exercise 8 Show that 28 is a cube in $\mathbb{Z}_3$.

Exercise 9 Does there exist $z \in \mathbb{Z}[i]/(51^{100})$ such that $z^2 = 2$?

Exercise 10 This exercise concerns Hensel's lemma for systems of equations.

1) Let $f_1, \dots, f_n \in \mathbb{Z}_p[X_1, \dots, X_m]$, and let $\overline{x} \in \mathbb{F}_p^m$ satisfy $f_i(\overline{x}) = 0$ for all i. Assume that the Jacobian matrix
$$J(\overline{x}) = \left(\frac{\partial f_i}{\partial X_j}(\overline{x})\right)_{i,j}$$
has rank n.
   a) Prove that the system $f_1 = \cdots = f_n = 0$ has a solution $x \in \mathbb{Z}_p^m$ lifting $\bar{x}$.
   b) Prove uniqueness of the previous solution when $m = n$.
2) Determine all primes p for which $x^2 + 1 = 3y^2$ has a solution in $\mathbb{Z}_p^2$.
3) How many solutions in $\mathbb{Z}_5^2$ does the system
$$\begin{cases} x^2 + 1 = 3y^2, \\ x^3 + 3y^5 + y = 2 \end{cases}$$
have?

Exercise 11 Let K be a complete nonarchimedean valued field, with valuation ring A and associated valuation $v : K \to \mathbb{R} \cup \{\infty\}$. Assume that K is complete. If $P = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_0 \in K[X]$ with $a_0 a_d \neq 0$, we define $\mathcal{N}ewt_P$, the Newton polygon of P, as the unique maximal continuous function, piecewise linear with breakpoints at integral abscissas, $\mathcal{N}ewt_P : [0, d] \to \mathbb{R}$ such that

- $\mathcal{N}ewt_P(0) = v(a_0)$ and $\mathcal{N}ewt_P(d) = v(a_d)$.
- $\mathcal{N}ewt_P(j) \leqslant v(a_j)$ for all $j \in \{0, \dots, d\}$.
- The graph of $\mathcal{N}ewt_P$ is convex (i.e. has increasing slopes).

Another way to say it is that $\mathcal{N}ewt_P$ is the lower convex hull of the points $(i, v(a_i))$.

1) Draw the polygon of $X^3 - X^2 - 2X + 8 \in \mathbb{Q}_2[X]$ and of $X^3 + 2X^2 - 2X + 4 \in \mathbb{Q}_2[X]$.
2) Assume given a factorization $P(X) = \prod_{i=1}^{d}(X - x_i) \in K[X]$, with $x_i \in \overline{K}^\times$, and denote by $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_d$ the slopes of $\mathcal{N}ewt_P$ (with multiplicity). Show that, up to reordering the x_i, we have $\gamma_i = -v(x_i)$. *Hint: we can introduce the piecewise linear function on* $[0, d]$ *starting at* $v(a_0)$ *and with increasing slopes* $-v(x_i)$ *(with multiplicities) and try to compare its position to* $\mathcal{N}ewt_P$.
3) Deduce that if $P \in K[X]$ is irreducible, then its Newton polygon is a line. What are the possible reductions of P if $P \in \mathcal{O}_K[X]$ and is irreducible in $K[X]$?
4) Let $P = X^d + a_1X^{d-1} + \cdots + a_d \in K[X]$ with $a_d \neq 0$ be a separable polynomial (or assume K is of characteristic zero). Assume that the slopes of $\mathcal{N}ewt_P$ are $\lambda_1 < \cdots < \lambda_k$, with multiplicities $w_1, \ldots, w_k$. Then there exists a unique factorisation $P(X) = \prod_{i=1}^{k} g_i(X)$ with g_i monic, $\deg g_i = w_i$ and $\mathcal{N}ewt_{g_i}$ isoclinic of slope λ_i. Can you extend it to the general case?
5) Show that $X^3 - X^2 - 2X + 8$ has three distinct roots in $\mathbb{Q}_2$.
6) Deduce the following version of Hensel's lemma: if $P \in \mathcal{O}_K[X]$ is primitive and is such that
$$P \equiv \overline{h}\,\overline{g} \pmod{\mathfrak{m}_K},$$
with $\overline{h}, \overline{g}$ coprime, then there exist $h, g \in \mathcal{O}_K[X]$ such that $P = hg$, $\deg g = \deg \bar{g}$ and $h \equiv \overline{h}$, $g \equiv \overline{g} \pmod{\mathfrak{m}_K}$. *Hint: we can start by writing* $P = \prod_{i=1}^{r} P_i$ *as a product of irreducibles.*
7) Prove the Eisenstein–Dumas criterion:
Theorem. *Assume that the valuation on* K *is discrete and normalized and let* $P \in K[X]$. *Assume its Newton polygon is a line from* $(0, n)$ *to* $(d, 0)$ *with* $\gcd(n, d) = 1$. *Then* P *is irreducible.*
Deduce the classical Eisenstein criterion.

Exercise 12 Let K be a complete nonarchimedean valued field, and Ω an algebraic closure with the extended absolute value. Let $\alpha \in \Omega$ be separable over K, of degree $n > 1$, with conjugates $\alpha = \alpha_1, \ldots, \alpha_n$. Let $\beta \in \Omega$ satisfy
$$|\alpha - \beta| < \min_{2 \leqslant i \leqslant n} |\alpha - \alpha_i|.$$
1) Prove that $\sigma(\alpha) = \alpha$ for every $\sigma \in \mathrm{Aut}(\Omega / K(\beta))$.
2) Deduce that $K(\alpha) \subset K(\beta)$. This is **Krasner's lemma**.
3) What happens if $\alpha \in K$?

Exercise 13 Let K be a complete field for a nontrivial valuation v.

1) Let $f \in K[X]$ be a separable monic polynomial and write its factorization into irreducible polynomials $f = f_1 \dots f_r$. Prove that if $g \in K[X]$ is monic of degree $\deg f$ and close enough to f coefficient-wise, then g factors as $g = g_1 \dots g_r$ with $g_1, \dots, g_r$ irreducible and $\deg g_i = \deg f_i$ for all i
2) Assume that K is complete for another nontrivial valuation not equivalent to v. Prove that K is separably closed.

Exercise 14 Let K be a perfect complete field for a nontrivial nonarchimedean absolute value.

1) Let L/K be an infinite algebraic extension, endowed with the extended absolute value. For $d \geqslant 1$, set $E_d = \{x \in L : [K(x) : K] \leqslant d\}$.
   a) Use Krasner's lemma to show that E_d is closed in L.
   b) Show that E_d has empty interior. *Hint:* If a ball about a were contained in E_d, choose $b \in L$ of degree greater than $d[K(a) : K]$ and consider $a + tb$ for sufficiently small $t \in K^\times$.
   c) Apply Baire's theorem to prove that L is not complete.
2) Let $\widehat{\overline{K}}$ be the completion of an algebraic closure of K. Using Krasner's lemma, prove that $\widehat{\overline{K}}$ is algebraically closed.

2 Local fields and ramification

Exercise 15 Let K be a complete discretely valued field, with valuation ring $\mathcal{O}$, maximal ideal $\mathfrak{p} = (\pi)$ and residue field κ.

1) Prove that K is locally compact if, and only if, κ is finite. In this case, K is called a *local field*.
2) Suppose $\kappa = \mathbb{F}_q$ and $\operatorname{char} K = p > 0$.
   (i) Prove that $\mathbb{F}_q$ can be seen as a subfield of K.
   (ii) Prove $K \cong \mathbb{F}_q((t))$.
3) Suppose $\kappa = \mathbb{F}_q$ and $\operatorname{char} K = 0$.
   (i) Show that $\mathbb{Q}_p$ can be seen as a subfield of K.
   (ii) Prove that $[K : \mathbb{Q}_p] < \infty$ by constructing a finite $\mathbb{Q}_p$-basis of K.

Exercise 16 Let K be a discretely valued field, with valuation ring $\mathcal{O}$, maximal ideal $\mathfrak{p}$ and residue field κ. Set $U^{(n)} = 1 + \mathfrak{p}^n$ for $n \geqslant 1$.

1) Prove that $\mathcal{O}^\times / U^{(n)} \cong (\mathcal{O}/\mathfrak{p}^n)^\times$ and $U^{(n)}/U^{(n+1)} \cong \kappa$.

2) If K is complete, prove the topological group isomorphism

$$\mathcal{O}^\times \cong \varprojlim_n \mathcal{O}^\times / U^{(n)}.$$

Exercise 17 Let K be a complete discretely valued field, with valuation ring $\mathcal{O}$, maximal ideal $\mathfrak{p}$ and perfect residue field κ of characteristic $p > 0$.

1) Let $r \geqslant 1$. Show that $a \equiv b \pmod{\mathfrak{p}^r}$ implies $a^p \equiv b^p \pmod{\mathfrak{p}^{r+1}}$ for $a, b \in \mathcal{O}$.
2) For $\xi \in \kappa$, choose a lift $a_n \in \mathcal{O}$ of $\xi^{p^{-n}}$ for each $n \geqslant 0$. Prove that $(a_n^{p^n})_n$ converges and that its limit $\omega(\xi)$ is independent of all choices.
3) Prove that $\omega : \kappa \to \mathcal{O}$ is a multiplicative section of the reduction map $\mathcal{O} \to \kappa$. It is called the **Teichmüller lift**.
4) If $\kappa = \mathbb{F}_q$, prove that $\omega(\overline{a}) = \lim_{n\to\infty} a^{q^n}$ for every $a \in \mathcal{O}$, and that the image of ω is $\{0\} \cup \mu_{q-1}(K)$. Explain why $(a^{p^n})_n$ need not converge when κ is infinite.

Exercise 18 Let p be a prime number.

1. Compute the group of roots of unity contained in $\mathbb{Q}_p$.
2. Decide whether -2 is a square in $\mathbb{Q}_2$ and in $\mathbb{Q}_3$.
3. Let ℓ be a prime number such that $\mathbb{Q}_p$ and $\mathbb{Q}_\ell$ are isomorphic as abstract fields. Prove that $p = \ell$.

Exercise 19 Let $K/\mathbb{Q}_p$ be finite of degree d. Write $\mathcal{O}$ for its valuation ring, $\mathfrak{p} = (\pi)$ for the maximal ideal, $q = p^f$ for the residue field cardinality, and $e = v_K(p)$, with $v_K(\pi) = 1$. Put $U = \mathcal{O}^\times$ and $U^{(r)} = 1 + \mathfrak{p}^r$.

1) Using Teichmüller lifts (see Exercise 17), prove

$$K^\times \cong \pi^{\mathbb{Z}} \times \mu_{q-1}(K) \times U^{(1)}.$$

2) a) Show that $\log(1+z) = \sum_{n\geqslant 1} (-1)^{n+1} z^n/n$ converges for $z \in \mathfrak{p}$ and defines a continuous homomorphism $\log : U^{(1)} \to K$.
   b) Prove that log extends uniquely to a continuous homomorphism $\log : K^\times \to K$ with $\log p = 0$.
3) For every integer $r > e/(p-1)$, show that $\exp : z \mapsto \sum_{n\geqslant 0} z^n/n!$ and log are mutually inverse topological group isomorphisms

$$\mathfrak{p}^r \underset{\log}{\overset{\exp}{\rightleftarrows}} U^{(r)}.$$

4) (a) Using Exercise 16, show that all the quotients $U^{(1)}/U^{(r)}$ are finite p-groups.

(b) For $u \in U^{(1)}$ and $a \in \mathbb{Z}_p$, choose $(a_n)_n$ a sequence of integers converging to a and define $u^a = \lim_{n\to\infty} u^{a_n}$. Show that u^a does not depend on the choice of $(a_n)_n$, and prove that the external law $(a,u) \mapsto u^a$ defines a structure of a topological $\mathbb{Z}_p$-module on $U^{(1)}$.

5) Prove that, for some $b \geqslant 0$, there is an isomorphism:

$$K^\times \cong \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \mathbb{Z}/p^b\mathbb{Z} \times \mathbb{Z}_p^d$$

as topological groups.

6) For every integer $m \geqslant 1$, deduce

$$[K^\times : (K^\times)^m] = m[U : U^m] = \frac{m}{\|m\|}\#\mu_m(K)$$

where $\|m\| = q^{-v_K(m)}$.

Exercise 20 Let p be a prime.

1) Take $a \in \mathbb{Q}_p^\times$ and write $a = p^\alpha u$, where $u \in \mathbb{Z}_p^\times$.
   a) If p is odd, show that a is a square if, and only if, α is even and $\overline{u}$ is a square in $\mathbb{F}_p^\times$.
   b) If $p = 2$, show that a is a square if, and only if, α is even and $u \equiv 1 \pmod 8$.

2) For $a, b \in \mathbb{Q}_p^\times$, define their **Hilbert symbol** by

$$(a,b)_p = \begin{cases} 1 & \text{if } x^2 - ay^2 - bz^2 = 0 \text{ has a nonzero solution in } \mathbb{Q}_p^3, \\ -1 & \text{otherwise.} \end{cases}$$

Write $a = p^\alpha u$ and $b = p^\beta v$, with $u, v \in \mathbb{Z}_p^\times$. We aim to prove

$$(a,b)_p = \begin{cases} \left(\dfrac{\overline{(-1)^{\alpha\beta}u^\beta v^\alpha}}{p}\right) & p \neq 2, \\ (-1)^{\frac{u-1}{2}\frac{v-1}{2} + \alpha\frac{v^2-1}{8} + \beta\frac{u^2-1}{8}} & p = 2. \end{cases} \tag{1}$$

Here $(\frac{t}{p})$ is the Legendre symbol (equal to 1 or -1 according as $t \in \mathbb{F}_p^\times$ is a square or not); the exponent in the second line is read modulo 2.

   a) Show that it suffices to consider $(\alpha,\beta) \in \{(0,0),(0,1),(1,1)\}$.
   b) Using question 1), prove (1) when $(\alpha,\beta) = (0,1)$.
   c) Assume $(\alpha,\beta) = (0,0)$.
      i) For odd p, show that the equation $x^2 - \bar{u}y^2 - \bar{v}z^2 = 0$ has nonzero solutions in $\mathbb{F}_p$. Deduce formula (1) in that case.
      ii) Use question 1) to settle formula (1) in the remaining case $p = 2$.
   d) Assume $(\alpha,\beta) = (1,1)$.

i) Show that $(a,b)_p = (pu, -uv)_p$.

ii) Deduce formula (1) in this case.

3) a) Using the basic properties of Legendre's symbol, prove that the map:

$$\begin{aligned} \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times &\to \{-1,1\} \\ (a,b) &\mapsto (a,b)_p \end{aligned}$$

is bilinear.

b) Define:

$$(a,b)_\infty = \begin{cases} 1 & \text{if } x^2 - ay^2 - bz^2 = 0 \text{ has a nonzero solution in } \mathbb{R}^3, \\ -1 & \text{otherwise.} \end{cases}$$

Prove that the map:

$$\begin{aligned} \mathbb{R}^\times \times \mathbb{R}^\times &\to \{-1,1\} \\ (a,b) &\mapsto (a,b)_\infty \end{aligned}$$

is bilinear.

c) Using quadratic reciprocity, prove Hilbert's reciprocity law: for $a,b \in \mathbb{Q}^\times$, the symbol $(a,b)_p$ is equal to 1 for almost all p and $(a,b)_\infty \cdot \prod_p (a,b)_p = 1$.

Exercise 21 Let p be a prime and $a_1, \dots, a_5 \in \mathbb{Q}_p^\times$. We aim to prove that

$$a_1 x_1^2 + \cdots + a_5 x_5^2 = 0 \tag{2}$$

has a nonzero solution in $\mathbb{Q}_p^5$.

1) Explain why one may assume $v_p(a_i) \in \{0,1\}$ for every i. We make that assumption in the rest of the exercise.

2) Conclude in the case p is odd. To do so, recall that, by Exercise 20, if a and b are p-adic numbers such that $v_p(a) = v_p(b) = 0$, then equation $x^2 - ay^2 - bz^2 = 0$ has non-zero p-adic solutions.

3) Assume $p = 2$.

a) Explain why it suffices to find a primitive solution of equation 2 modulo 32.

b) Conclude using a computer or a calculator.

Exercise 22 Let K be a local field, L/K a finite extension of degree n, and π a uniformizer of $\mathcal{O}_L$.

1) When L/K is totally ramified, prove that $\mathcal{O}_L = \mathcal{O}_K[\pi]$.

2) When L/K is totally ramified, show that the minimal polynomial of π is Eisenstein.

3) Conversely, if α is a root of an Eisenstein polynomial over $\mathcal{O}_K$, prove that $K(\alpha)/K$ is totally ramified and that α is a uniformizer.

Exercise 23

1) Let K be a local field with residue field $\mathbb{F}_q$ of characteristic p. Let $n \geqslant 1$ be prime to p, let ζ be a primitive n-th root of unity, and put $L = K(\zeta)$.

   a) Prove that L/K is unramified of degree

   $$f = \min\{r \geqslant 1 : q^r \equiv 1 \pmod{n}\}.$$

   b) Show that $\mathrm{Gal}(L/K)$ is cyclic of order f, generated by $\zeta \mapsto \zeta^q$.

   c) Prove that $\mathcal{O}_L = \mathcal{O}_K[\zeta]$. *Hint:* Use Nakayama's Lemma.

2) Let $m \geqslant 1$ and let ζ be a primitive p^m-th root of unity. Put $L = \mathbb{Q}_p(\zeta)$. You may use Exercise 22 in this question.

   a) Prove that $L/\mathbb{Q}_p$ is totally ramified of degree $(p-1)p^{m-1}$ and that

   $$\mathrm{Gal}(L/\mathbb{Q}_p) \cong (\mathbb{Z}/p^m\mathbb{Z})^\times.$$

   b) Show that $\mathcal{O}_L = \mathbb{Z}_p[\zeta]$, that $1-\zeta$ is a uniformizer, and that $N_{L/\mathbb{Q}_p}(1-\zeta) = p$.

Exercise 24 Let L/K be a finite separable extension and v a discrete valuation on K. Prove that:

$$N_{L/K}(x) = \prod_{w|v} N_{L_w/K_v}(x), \qquad \mathrm{Tr}_{L/K}(x) = \sum_{w|v} \mathrm{Tr}_{L_w/K_v}(x).$$

Exercise 25

1) Factor $X^4 - 17$ into irreducible polynomials over $\mathbb{Q}_2$.

2) Describe the decomposition of (2) in the ring of integers of $K = \mathbb{Q}(\sqrt[4]{17})$, giving all ramification indices and residue degrees.

Exercise 26 Let L/K be a finite totally and tamely ramified extension of complete discretely valued fields and set $\Gamma_K = v(K^\times)$ and $\Gamma_L = v(L^\times)$. Prove that $M \mapsto v(M^\times)/\Gamma_K$ is a bijection from intermediate fields of L/K to subgroups of Γ_L/Γ_K.

Exercise 27 Let K be a finite extension of $\mathbb{Q}_p$. Let L be a finite Galois extension of K. Prove that $\mathrm{Gal}(L/K)$ is solvable.

3 Adèles and idèles

Exercise 28 Let F be a number field. Put $\mathbb{A} = \mathbb{A}_F$ and let $\mathbb{A}_f := \prod'_{v\nmid\infty} F_v$ be the ring of finite adèles.

1) Let $\widehat{\mathcal{O}}_F := \prod_{v\nmid\infty} \mathcal{O}_v$. Show that $\mathbb{A}_f \cong F \otimes_{\mathcal{O}_F} \widehat{\mathcal{O}}_F$.
2) Show that $\mathbb{A}_f = F + U$ for every open subgroup U of $\mathbb{A}_f$.
3) Let $f : \mathbb{A} \to \mathbb{C}$ be a continuous function such that $f(x+\zeta) = f(x)$ for every $x \in \mathbb{A}$ and every $\zeta \in F$, and $f(x+y) = f(x)$ for every $x \in \mathbb{A}$ and $y \in F_v$, $v \mid \infty$. Show that f is constant.
4) Deduce that $\mathbb{A}/F$ is connected. Show that

$$\mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \cong (\mathbb{R} \times \prod_p \mathbb{Z}_p)/\mathbb{Z} \cong \varprojlim_n \mathbb{R}/n\mathbb{Z}.$$

Exercise 29 Let F be a number field, $F \neq \mathbb{Q}$. Let $v_1, \dots, v_d$ be the infinite places of F. Show that for every i the image of $\mathcal{O}_F$ under the diagonal map to $\prod_{k\neq i} F_{v_k}$ is a dense subgroup.

Exercise 30 Let K be a number field, $\mathbb{A}_K$ its adèle ring and I_K its idèle group. Set:

$$I_K^1 = \{a \in I_K : \prod_v \|a_v\|_v = 1\}, \qquad C_K^1 = I_K^1/K^\times.$$

This exercise aims at proving compactness of C_K^1 and deducing class-number finiteness and Dirichlet's unit theorem, without using Minkowski's theorem on lattices.

You may use the following basic facts about Haar measure: there is a translation-invariant measure μ on $\mathbb{A}_K$, finite on compact sets, and a measurable fundamental domain D for K of finite positive measure V; moreover

$$\mu(aE) = \left(\prod_v \|a_v\|_v\right)\mu(E) \qquad (a \in I_K).$$

1) Prove that, if $E \subset \mathbb{A}_K$ is measurable and $\mu(E) > V$, then $E - E$ contains a nonzero element of K.
2) Set $B := \prod_v B_v$, where $B_v = \mathcal{O}_v$ at every finite place and $B_v = B(0, R_v)$ for some $R_v > 0$ at infinite places. We assume that the R_v have been chosen so that $\mu(B) > V$ and we put $T = B - B$. Show that for every $a \in I_K^1$ there exists $b \in K^\times$ with $b/a \in T \cap I_K^1$.
3) Show that $T \cap I_K^1$ is contained in a compact subset of I_K, and deduce that C_K^1 is compact.
4) Show that $a \mapsto \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(a_{v_{\mathfrak{p}}})}$ induces a continuous surjection $C_K^1 \to \mathrm{Cl}(K)$, where the class group has the discrete topology. Conclude that the class group is finite.
5) Let r, s be the numbers of real and complex places of K. Define

$$\lambda : \mathcal{O}_K^\times \longrightarrow H = \left\{(x_v)_{v\mid\infty} \in \mathbb{R}^{r+s} : \sum_v x_v = 0\right\}, \qquad u \longmapsto (\log\|u\|_v)_v.$$

a) Prove that $\operatorname{Ker}\lambda$ is the finite group $\mu(K)$ of roots of unity of K, and that $\lambda(\mathscr{O}_K^\times)$ is discrete in H.

b) Identify the kernel of $C_K^1 \to \mathrm{Cl}(K)$ with

$$\left(K_\infty^1 \times \prod_{v\nmid\infty} \mathscr{O}_v^\times\right)/\mathscr{O}_K^\times, \qquad \text{where } K_\infty^1 = \{x \in \prod_{v|\infty} K_v^\times : \prod_v \|x_v\|_v = 1\}.$$

Deduce that $H/\lambda(\mathscr{O}_K^\times)$ is compact.

c) Conclude that $\mathscr{O}_K^\times \cong \mu(K) \times \mathbb{Z}^{r+s-1}$.

Exercise 31

1) Show that $\mathbb{Q}^\times$ is not dense in the finite idèles $\mathbb{A}_f^\times$.

2) Show that the image of $\mathrm{SL}_2(\mathbb{Q})$ in $\mathrm{SL}_2(\mathbb{A}_f)$ is dense. *Hint:* Check that, if F is a field and $M \in \mathrm{SL}_2(F)$, one can write $M = \begin{pmatrix}1 & 0\\ w & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix}\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}$ for some $w, x, y, z \in F$.

3) Deduce that if K is an open compact subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$, then

$$\mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_f) / K \cong \mathbb{Q}^\times \backslash \mathbb{A}_f^\times / \det(K).$$