DIRAC OPERATORS IN REPRESENTATION THEORY A SELECTED SURVEY

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1. Introduction

Dirac operators were introduced into representation theory of real reductive groups by Parthasarathy [24] with the aim of constructing discrete series representations (see also Atiyah-Schmid [1]). In [25], the same author used the Dirac operator to establish a very useful criterion for unitarizability of representations, the so-called Parthasarathy Dirac-inequality.

In the 1990's, D. Vogan revisited the theory with the aim of sharpening Parthasarathy-Dirac inequality [28]. He introduced the notion of Dirac cohomology for Harish-Chandra modules and stated a conjecture for the infinitesimal character of a module having non vanishing Dirac cohomology. This conjecture was proved in [8]. At the same time, Kostant introduced a Dirac operator in a much more general setting [15], [17], [16], often named "cubic Dirac operator", because its definition involve a cubic term in a Cliford algebra.

Since then, a vast literature on the subject has been published. See [9] for a nice introduction, and [21] for a different point of view.

We will first focus on the role of Dirac operators in the representation theory of $(\mathfrak{g}, K)$-modules. Then we will introduce Kostant cubic Dirac operator in the general setting and state the main results about it (computation of its square, Huang-Pandzic theorem). We make further study in the case of cubic dirac operators for Levi subalgebra, which allows us to relate Dirac and Lie algebra cohomology for finite dimensional modules and unitary $(\mathfrak{g}, K)$-modules in the hermitian symmetric case. We conclude by a few remarks on the construction of discrete series, thereby completing a circle of ideas.

2. Preliminary material

We recall some well known facts about quadratic spaces and Clifford algebras, mainly to introduce notation. A *quadratic space* (V, B) is a finite dimensional vector space over a field $\mathbb{K}$ of characteristic $\neq 2$ endowed with a non-degenerate symmetric bilinear form. In these notes, we will be interested only in the case $\mathbb{K} = \mathbb{C}$. The Clifford algebra $\mathrm{Cl}(V; B)$ of (V, B) is the quotient of the tensor algebra $T(V)$ by the two-sided ideal generated by elements of the form

$$v \otimes w + w \otimes v + 2B(v, w)\, 1, \qquad (v, w \in V).$$

This is the convention of [8] and [9], which differs by a sign with the convention in [15] and [11]. Of course, over $\mathbb{C}$ there is no substantial difference between the two conventions.

Notice that all the terms in the above expression are of even degrees (2 or 0). Thus, the graded algebra structure on $T(V)$ induces a filtered algebra structure on $\mathrm{Cl}(V; B)$, but also a structure of $\mathbb{Z}_2$-graded algebra (*i.e.* a super algebra structure). Simply put, the $\mathbb{Z}_2$-grading and the filtration are defined by the condition that the generators $v \in V$

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of $\mathrm{Cl}(V;B)$ are odd, of filtration degree 1. We denote by $\mathbf{Gr}(\mathrm{Cl}(V;B))$ the graded super algebra obtained from the filtration on $\mathrm{Cl}(V;B)$. We get a decomposition

$$\mathrm{Cl}(V;B) = \mathrm{Cl}^{\bar{0}}(V;B) \oplus \mathrm{Cl}^{\bar{1}}(V;B).$$

The (super)algebra $\mathrm{Cl}(V;B)$ is very close to the exterior algebra $\bigwedge V$. We view the latter as a graded supercommutative superalgebra. In fact, $\mathrm{Cl}(V;B)$ and $\bigwedge V$ are isomorphic as super vector spaces (Chevalley isomorphism). More precisely there is a representation

$$f_{\mathrm{Cl}} : \mathrm{Cl}(V;B) \longrightarrow \mathrm{End}(\bigwedge V)$$

of $\mathrm{Cl}(V;B)$ in the exterior algebra $\bigwedge V$, which allows one to define the *symbol map*

$$\sigma : \mathrm{Cl}(V;B) \longrightarrow \bigwedge V, \qquad x \mapsto f_{\mathrm{Cl}}(x) \cdot 1$$

where $1 \in \mathbb{K}$ is regarded as an element of degree 0 in $\bigwedge V$. The Chevalley isomorphism is the map induced from $\mathbf{Gr}(\mathrm{Cl}(V;B))$ on $\bigwedge V$ by the symbol map. Its inverse is the quantization map given by graded symmetrization, that is, for $v_1, \ldots, v_k \in V$,

$$q(v_1 \wedge \ldots \wedge v_k) = \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \mathrm{sgn}(s)\, v_{s(1)} v_{s(2)} \cdots v_{s(k)}.$$

Let us denote by $\mathbf{O}(V,B)$ the group of linear isometries of (V,B). Its Lie algebra $\mathfrak{o}(V;B)$ relates to the "quadratic" elements of the Clifford algebra, more precisely, the elements $q(x)$, $x \in \bigwedge^2 V$ span a Lie subalgebra of $\mathrm{Cl}(V,B)$. Let us denote by $\{.,.\}$ the induced bracket on $\bigwedge^2 V$, *i.e.*

$$[q(x), q(y)]_{\mathrm{Cl}} = q(\{x,y\}), \qquad (x, y \in \bigwedge^2 V).$$

Here $[.\,,.]_{\mathrm{Cl}}$ denotes the supercommutator bracket on $\mathrm{Cl}(V;B)$. Then the transformation

$$x \mapsto A_x, \quad A_x(v) = [q(x), v]_{\mathrm{Cl}}, \quad (v \in V)$$

defines an element A_x of $\mathfrak{o}(V;B)$, and the map

$$\bigwedge^2 V \to \mathfrak{o}(V;B), \quad x \mapsto A_x \tag{2.1}$$

is a Lie algebra isomorphism. A formula for its inverse λ is given as follows. Let $(e_i)_i$ be a basis of V with dual basis (with respect to B) $(e^i)_i$, then if $A \in \mathfrak{o}(V;B)$,

$$\lambda(A) = \frac{1}{4} \sum_i A(e_i) \wedge e^i \in \bigwedge^2 V.$$

We assume from now on that $\mathbb{K} = \mathbb{C}$. The special orthogonal group $\mathbf{SO}(V;B)$ admits a central extension $\mathbf{Spin}(V,B)$ by $\mathbb{Z}_2$, which may be realized in the group $\mathrm{Cl}(V;B)^{\times}$ of invertible elements of the Clifford algebra.

Let us now recall some facts about Clifford modules.

Theorem 2.1. (*i*) *Suppose that $n = \dim_{\mathbb{C}}(V)$ is even. Then there are :*
- *two isomorphism classes of irreducible $\mathbb{Z}_2$-graded $\mathrm{Cl}(V,B)$-modules,*
- *one isomorphism class of irreducible ungraded $\mathrm{Cl}(V,B)$-modules,*
- *two isomorphism classes of irreducible $\mathrm{Cl}^{\bar{0}}(V,B)$-modules.*

(*ii*) *Suppose that $n = \dim_{\mathbb{C}}(V)$ is odd. Then there are*
- *one isomorphism class of irreducible $\mathbb{Z}_2$-graded $\mathrm{Cl}(V,B)$-modules,*
- *two isomorphism classes of irreducible ungraded $\mathrm{Cl}(V,B)$-modules,*
- *one isomorphism class of irreducible $\mathrm{Cl}^{\bar{0}}(V,B)$-modules,*

Let us denote by S the choice of an irreducible $\mathbb{Z}_2$-graded $\mathrm{Cl}(V,B)$-modules.

3. Dirac operator and $(\mathfrak{g}, K)$-modules

3.1. **Notation and structural facts.** Let G be a connected real reductive Lie group with Cartan involution θ such that $K = G^\theta$ is a maximal compact subgroup of G. Let us denote by $\mathfrak{g}_0$ the Lie algebra of G, $\mathfrak{g}$ its complexification, with Cartan involutions also denoted by θ.

We fix an invariant nondegenerate symmetric bilinear form B on $\mathfrak{g}_0$, extending the Killing form on the semisimple part of $\mathfrak{g}_0$. Let

$$\mathfrak{g}_0 = \mathfrak{k}_0 \overset{\theta}{\oplus} \mathfrak{p}_0, \qquad \mathfrak{g} = \mathfrak{k} \overset{\theta}{\oplus} \mathfrak{p}$$

be the Cartan decomposition of $\mathfrak{g}_0$ and $\mathfrak{g}$ respectively. We assume furthermore that in extending the Killing form, we made sure that $B_{|\mathfrak{p}_0}$ remains definite positive and $B_{|\mathfrak{k}_0}$ definite negative.

Let $\mathrm{Cl}(\mathfrak{p}) = \mathrm{Cl}(\mathfrak{p}; B)$ be the Clifford algebra of $\mathfrak{p}$ with respect to B. We denote by $\widetilde{K}$ the spin double cover of K, i.e., the pull-back of the covering map $\mathbf{Spin}(\mathfrak{p}_0) \to \mathbf{SO}(\mathfrak{p}_0)$ by the adjoint action map $\mathrm{Ad}_{|\mathfrak{p}_0} : K \to \mathbf{SO}(\mathfrak{p}_0)$. The compact groups $\mathbf{Spin}(\mathfrak{p}_0)$ and $\mathbf{SO}(\mathfrak{p}_0)$ embed in their complexification $\mathbf{Spin}(\mathfrak{p})$ and $\mathbf{SO}(\mathfrak{p})$, so we get the following diagram :

$$\begin{tikzcd}
\widetilde{K} \arrow[r] \arrow[d] & \mathbf{Spin}(\mathfrak{p}_0) \arrow[r, hook] \arrow[d] & \mathbf{Spin}(\mathfrak{p}) \arrow[r, hook] \arrow[d] & \mathrm{Cl}^{\bar{0}}(\mathfrak{p})^\times \\
K \arrow[r, "\mathrm{Ad}_{|\mathfrak{p}_0}"] & \mathbf{SO}(\mathfrak{p}_0) \arrow[r, hook] & \mathbf{SO}(\mathfrak{p}) &
\end{tikzcd}$$

The complexification of the differential at the identity of the Lie group morphism $\mathrm{Ad}_{|\mathfrak{p}_0}$: $K \to \mathbf{SO}(\mathfrak{p}_0)$, is the Lie algebra morphism

$$\mathrm{ad}_{|\mathfrak{p}} : \mathfrak{k} \to \mathfrak{so}(\mathfrak{p}), \quad X \mapsto \mathrm{ad}(X)_{|\mathfrak{p}}$$

Let us denote by α the composition of this map with the identification $\mathfrak{so}(\mathfrak{p}) \simeq \bigwedge^2 \mathfrak{p}$ of (2.1) and the inclusion $q : \bigwedge^2 \mathfrak{p} \hookrightarrow \mathrm{Cl}(\mathfrak{p})$:

$$\alpha : \mathfrak{k} \to \mathrm{Cl}(\mathfrak{p}). \tag{3.1}$$

There is an explicit expression for $\alpha : \mathfrak{k} \to \mathrm{Cl}(\mathfrak{p})$: if $(Y_i)_i$ is a basis of $\mathfrak{p}$ with dual basis $(Z_i)_i$, then for any $X \in \mathfrak{k}$,

$$\alpha(X) = \frac{1}{4} \sum_{i,j} B([Z_i, Z_j], X)\; Y_i Y_j. \tag{3.2}$$

A key role will be played in what follows by the associative $\mathbb{Z}_2$-graded superalgebra $\mathcal{A} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$. The $\mathbb{Z}_2$-grading comes from the $\mathbb{Z}_2$-grading on $\mathrm{Cl}(\mathfrak{p})$, *i.e.* elements in $U(\mathfrak{g}) \otimes 1$ are even. We will often use the super Lie algebra structure on $\mathcal{A}$ given by the (super)commutator bracket.

The group K acts on $U(\mathfrak{g})$ through $K \subset G$ by the adjoint action, and on $\mathrm{Cl}(\mathfrak{p})$ through the map $\widetilde{K} \to \mathrm{Cl}^{\bar{0}}(\mathfrak{p})^\times$ in the first row of the diagram above and conjugation in $\mathrm{Cl}(\mathfrak{p})$ (this action of $\widetilde{K}$ on $\mathrm{Cl}(\mathfrak{p})$ factors through K) . Thus we get a linear action of K on $\mathcal{A} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$. Differentiating this action at the identity, and taking the complexification, we get a Lie algebra representation of $\mathfrak{k}$ in $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$. This represention can be described as follows. The map (3.1) is used to define a map

$$\Delta : \mathfrak{k} \longrightarrow \mathcal{A} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}), \quad \Delta(X) = X \otimes 1 + 1 \otimes \alpha(X)$$

which is a morphism of Lie algebra (it takes values in the even part of the super Lie algebra $\mathcal{A}$). Thus it extends to an algebra morphism

$$\Delta : U(\mathfrak{k}) \longrightarrow \mathcal{A} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}). \tag{3.3}$$

The action of an element $X \in \mathfrak{k}$ on $\mathcal{A}$ is then given by the adjoint action of $\Delta(X)$, *i.e.* $a \in \mathcal{A} \mapsto [\Delta(X), a]$. We denote by $\mathcal{A}^K$ (resp. $\mathcal{A}^{\mathfrak{k}}$) the subalgebra of K-invariants (resp. $\mathfrak{k}$-invariants) in $\mathcal{A}$. Since K is assumed to be connected, $\mathcal{A}^K = \mathcal{A}^{\mathfrak{k}}$.

We can now introduce the Dirac operator D :

Definition 3.1. if $(Y_i)_i$ is a basis of $\mathfrak{p}$ and $(Z_i)_i$ is the dual basis with respect to B, then

$$D = \sum_i Y_i \otimes Z_i \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$$

is independent of the choice of basis $(Y_i)_i$ and K-invariant for the adjoint action on both factors. Then D is an element of $\mathcal{A}^K$.

3.2. **An equivalence of category.** We are interested in the representation theory of G, and the category of representations we consider is $\mathcal{M}(\mathfrak{g}, K)$, the category of Harish-Chandra modules (*i.e.* $(\mathfrak{g}, K)$-modules for the pair Harish-Chandra pair $(\mathfrak{g}, K)$ (*cf.* [13])). The main idea of the subject is to study $X \in \mathcal{M}(\mathfrak{g}, K)$ by considering the natural action of the remarkable operator D on $X \otimes S$, where S is a module of spinors for $C(\mathfrak{p})$.

Let us make first some remarks about the correspondence $X \mapsto X \otimes S$. Recall that a generalized Harish-Chandra pair ([13], Chapter I) $(\mathscr{A}, \mathsf{K})$ is :
- a complex associative unital algebra $\mathscr{A}$,
- a compact Lie group K, its Lie algebra $\mathfrak{k}_0$ with complexification $\mathfrak{k}$.
- a locally K-finite action of K on $\mathscr{A}$ by algebra automorphisms.
- an algebra homomorphism $\iota : U(\mathfrak{k}) \to \mathscr{A}$ satisfying

$$k \cdot (\iota(u)) = \iota(k \cdot u), \quad (u \in U(\mathfrak{k})), \, (k \in \mathsf{K}).$$

On the left hand side, the dot represents the given K-action on $\mathscr{A}$, while on the right hand side, it represents the usual adjoint action.

Of course, to say that $(\mathfrak{g}, K)$ is an Harish-Chandra pair means that $(U(\mathfrak{g}), K)$ is a generalized Harish-Chandra pair with the obvious adjoint action of K on $U(\mathfrak{g})$ and inclusion ι of $U(\mathfrak{k})$ in $U(\mathfrak{g})$. Now, if we come back to the notation of the previous section, we can easily check that $(\mathcal{A} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}), \widetilde{K})$ is also a generalized Harish-Chandra pair, the action of $\widetilde{K}$ on $\mathcal{A}$ being the one described in the previous section, and ι is the map denoted by Δ in (3.3).

In the same way, Harish-Chandra modules (or $(\mathscr{A}, \mathsf{K})$-modules) for a generalized Harish-Chandra pair $(\mathscr{A}, \mathsf{K})$ are left modules M for $\mathscr{A}$ which are also spaces for a locally finite representation of K, subject to the compatibility conditions

$$k(am) = (k \cdot a)(km), \quad (k \in \mathsf{K}), \, (a \in \mathscr{A}), \, (m \in M),$$

$$(\iota(u))m = um, \quad (u \in U(\mathfrak{k})), \, (m \in M).$$

We denote by $\mathcal{M}(\mathscr{A}, \mathsf{K})$ the category of Harish-Chandra modules for the generalized Harish-Chandra pair $(\mathscr{A}, \mathsf{K})$. For the pair $(U(\mathfrak{g}), \mathsf{K})$, this is just the category $\mathcal{M}(\mathfrak{g}, K)$ of Harish-Chandra modules in the usual sense.

Now, if X is a $(\mathfrak{g}, K)$-module, then $X \otimes S$ is a $(\mathcal{A}, \widetilde{K})$-module : $\mathcal{A} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$ acts on $X \otimes S$ in the obvious way, while $\widetilde{K}$ acts on X (through K) and on S (through $\mathbf{Spin}(\mathfrak{p}_0) \subset \mathrm{Cl}(\mathfrak{p})$), and thus on the tensor product $X \otimes S$. This defines a functor

$$X \mapsto X \otimes S, \qquad \mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(\mathcal{A}, \widetilde{K}).$$

In fact, using that $\mathrm{Cl}(\mathfrak{p})$ (resp. $\mathrm{Cl}^{\bar{0}}(\mathfrak{p})$) is a simple algebra with unique irreducible module S when $\dim\mathfrak{p}$ is even (resp. odd), we deduce immediately

Proposition 3.2. *The functor $X\mapsto X\otimes S$ from $\mathcal{M}(\mathfrak{g},K)$ to $\mathcal{M}(\mathcal{A},\widetilde{K})$ is an equivalence of categories.*

Its inverse is the functor $M\mapsto \mathrm{Hom}_{\mathrm{Cl}(\mathfrak{p})}(S,M)$ (resp. $M\mapsto \mathrm{Hom}_{\mathrm{Cl}^{\bar{0}}(\mathfrak{p})}(S,M)$) if $\dim\mathfrak{p}$ is even (resp. odd).

Thus we don't loose any information by passing from X to $X\otimes S$ and back. As we said above, the main idea is to use the action of the Dirac operator on $X\otimes S$ to obtain information about X. Let us now put this principle into perspective by discussing a theorem of Harish-Chandra.

3.3. **On a theorem of Harish-Chandra.** A well known theorem due to Harish-Chandra [6], asserts that an irredible $(\mathfrak{g},K)$-module is characterized by the action of $U(\mathfrak{g})^K$ on any non-trivial K-isotypic component. A simplified algebraic proof of this result was given by Lepowsky-McCollum [20]. The original idea of Harish-Chandra was to use this fact to study and classify irreducible $(\mathfrak{g},K)$-modules. For instance, a proof of the subquotient theorem of Harish-Chandra, which says that any irreducible repredentation of G is equivalent to a subquotient of a principal series representation,is based on this fact ([31], Thm. 3.5.6). The action of $U(\mathfrak{g})^K$ on any the trivial K-isotypic component of a spherical representation is also well understood. The problem with continuing this approach is that the associative algebra $U(\mathfrak{g})^K$ is highly non commutative and very little is known about its structure and representation theory (although see [18] for some recent progress). In his thesis [29], Vogan made some use of this idea in his classification of irreducible $(\mathfrak{g},K)$-modules via their lowest K-type. In [23], we proved an analogue of Harish-Chandra result, but for irreducible modules in the category $\mathcal{M}(\mathcal{A},\widetilde{K})$. Our goal was to take advantage of the fact that the algebra $\mathcal{A}^K=(U(\mathfrak{g})\otimes\mathrm{Cl}(\mathfrak{p}))^K$ is slightly better understood than $U(\mathfrak{g})^K$. At least, we can give a interesting non trivial element of it, namely the Dirac operator D. We will se below that in fact $\mathcal{A}^K$ has the structure of a differential superalgebra, the differential being given by the adjoint action of D. For the moment, let us give more details about what we have just said about Harish-Chandra's theorem and its generalization in [23].

Let us consider a generalized Harish-Chandra pair $(\mathscr{A},\mathsf{K})$ and the category $\mathcal{M}(\mathscr{A},\mathsf{K})$ of Harish-Chandra modules for this pair. The key fact is that $\mathcal{M}(\mathscr{A},\mathsf{K})$ is equivalent to the category of non-degenerate modules over an algebra with idempotents (or algebra with an approximate identity), namely the Hecke algebra $R(\mathscr{A},\mathsf{K})$ constructed in [13], §I.5. As a vector space, $R(\mathscr{A},\mathsf{K})$ is isomorphic to

$$\mathcal{A}\otimes_{U(\mathfrak{k})}R(\mathsf{K}),$$

where $R(K)$ is the convolution algebra of K-finite distributions on K (see [13], Definition I. 115). The algebra product is a little bit subtle, and we refer to [13] for details.

The relevant result is then Proposition 3.8 below, due to J. Bernstein, whose main application is in the theory of reductive p-adic groups ([27], Section I.3). Bernstein told us that the idea of his proof came from the treatment of Harish-Chandra's result given by Godement in [5]. We start by recalling a few basic facts about idempotented algebras. The reference for these results is [27], Section I.3.

Definition 3.3. Let A be a ring (possibly without unit). We say that A is an idempotented ring if for any finite subset $\{a_1,\dots,a_n\}$ of A, there exists an idempotent e in A ($e^2=e$) such that $a_i=ea_ie$ for all i.

Definition 3.4. A module M for the idempotented ring A is non-degenerate if for any $m \in M$, there exists an idempotent e in A such that $e \cdot m = m$.

For an A-module M, we denote by M_A the non-degenerate part of M, i.e., the largest non-degenerate submodule of M.

Let us remark that a ring with unit is an idempotented ring, and that non-degenerate modules are in this case simply the unital modules, i.e., the modules on which the unit of the ring acts as the identity.

We denote by $\mathcal{M}(A)$ the category of non-degenerate left modules for the idempotented ring A. When A is a ring with unit, $\mathcal{M}(A)$ is the category of left unital A-modules.

Let A be an idempotented $\mathbb{C}$-algebra, and let e be an idempotent element of A. Let M be a non-degenerate A-module. Then M decomposes as

$$M = e \cdot M \oplus (1-e) \cdot M \tag{3.4}$$

Notice that eAe is an algebra with unit e, and that $e \cdot M$ is a unital eAe-module.

Let us define the functor :

$$j_e : \mathcal{M}(A) \to \mathcal{M}(eAe), \quad M \mapsto e \cdot M.$$

The functor j_e is exact.

Let us denote by:

— $\mathcal{M}(A, e)$ the full subcategory of $\mathcal{M}(A)$ of modules M such that $M = Ae \cdot M$.

— $\mathbf{Irr}(A)$ the set of isomorphism classes of simple non-degenerate A-modules,

— $\mathbf{Irr}(eAe)$ the set of isomorphism classes of simple unital eAe-modules,

— $\mathbf{Irr}(A, e)$ the subset of $\mathbf{Irr}(A)$ of modules M satisfying $e \cdot M \neq 0$.

Modules in $\mathcal{M}(A, e)$ are thus the modules M in $\mathcal{M}(A)$ generated by $e \cdot M$, and $\mathbf{Irr}(A, e)$ is the set of isomorphism classes of irreducible objects in $\mathcal{M}(A, e)$.

Lemma 3.5. *Consider the induction functor i:*

$$i : \mathcal{M}(eAe) \to \mathcal{M}(A), \qquad Z \mapsto A \otimes_{eAe} Z.$$

Then $j_e \circ i$ is naturally isomorphic to the identity functor of $\mathcal{M}(eAe)$, i.e.,

$$j_e \circ i(Z) \cong Z, \qquad Z \in \mathcal{M}(eAe), \tag{3.5}$$

these isomorphisms being natural in Z.

One deduces from this that $A \cdot (e \cdot i(Z)) = i(Z)$, thus the functor i takes values in $\mathcal{M}(A, e)$.

Of course, it is possible that j_e annihilates some modules in $\mathcal{M}(A)$, and therefore one cannot hope to obtain all non-degenerate A-modules from modules in $\mathcal{M}(eAe)$ by induction. Nevertheless, we get all irreducible modules in $\mathcal{M}(A, e)$.

Definition 3.6. Let M be a non-degenerate A-module and let $e \in \mathrm{Idem}(A)$. Let us define the non-degenerate A-module :

$$M_e := M/F(eA, M), \quad \text{where } F(eA, M) = \{m \in M \mid eA \cdot m = 0\}.$$

Lemma 3.7. *Let M be a non-degenerate A-module. Then $(M_e)_e = M_e$.*

Proposition 3.8. *The map $M \mapsto e \cdot M$ gives a bijection from $\mathbf{Irr}(A, e)$ onto $\mathbf{Irr}(eAe)$, with inverse given by $W \mapsto (A \otimes_{eAe} W)_e$.*

We now describe the consequences for the category $\mathcal{M}(\mathcal{A}, \widetilde{K})$ of Harish-Chandra modules for a generalized pair $(\mathcal{A}, \widetilde{K})$. As we said above, this category is naturally equivalent to the category of non-degenerate $R(\mathcal{A}, \widetilde{K})$-modules ([13] Chapter I).

Let (γ, F_γ) be an irreducible finite-dimensional representation of $\widetilde{K}$. Let us denote by $\Theta_{\tilde{\gamma}}$ the character of the contragredient representation $(\tilde{\gamma}, F_\gamma^*)$, and let

$$\chi_\gamma = \frac{\dim(V_\gamma)}{\mathrm{vol}\,(K)} \Theta_{\tilde{\gamma}}\, dk$$

be the idempotent element of $R(\widetilde{K})$ giving the projection operators on $\widetilde{K}$-isotypic components of type γ. Then $1 \otimes \chi_\gamma$ defines an idempotent of $R(\mathcal{A}, \widetilde{K})$.

Theorem 3.9. *The algebra*

$$(1 \otimes \chi_\gamma) \cdot R(\mathcal{A}, \widetilde{K}) \cdot (1 \otimes \chi_\gamma)$$

is isomorphic to

$$\mathcal{A}^K \otimes_{U(\mathfrak{k})^K} \mathrm{End}(F_\gamma).$$

Then for any $(\mathcal{A}, \widetilde{K})$-module V, $(1 \otimes \chi_\gamma) \cdot V$ is the $\widetilde{K}$-isotypic component $V(\gamma)$ of V. Thus Theorem 3.9 and the computation in its proof give

Theorem 3.10. *Let us fix an irreducible finite-dimensional representation (γ, F_γ) of $\widetilde{K}$. Then the map $V \mapsto V(\gamma)$ from the set of equivalence classes of irreducible $(\mathcal{A}, \widetilde{K})$-modules V with non-zero $\widetilde{K}$-isotypic component $V(\gamma)$ to the set of equivalence classes of simple unital $\mathcal{A}^K \otimes_{U(\mathfrak{k})^K} \mathrm{End}(V_\gamma)$-modules is a bijection, with inverse given by*

$$W \mapsto \left(R(\mathcal{A}, \widetilde{K}) \otimes_{\left[\mathcal{A}^K \otimes_{U(\mathfrak{k})^K} \mathrm{End}(F_\gamma)\right]} W \right)_{1 \otimes \chi_\gamma}.$$

To resume, the idea we want to use is the following : to study an $(\mathfrak{g}, K)$-module X, one would like to study the action of $U(\mathfrak{g})^K$ on a (non-zero) K-isotypic component of X, but since a little is known about $U(\mathfrak{g})^K$, we will instead study the action of $(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^K$ a (non-zero) K-isotypic component of $X \otimes S$. The structure of $(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^K$ is better (but not completely) understood than the he structure of $U(\mathfrak{g})^K$. We will now see what can be said from that fact that it contains the Dirac operator D.

3.4. **The square of the Dirac operator.** The most important property of D is the formula

$$D^2 = -\mathrm{Cas}_\mathfrak{g} \otimes 1 + \Delta(\mathrm{Cas}_\mathfrak{k}) + (\|\rho_\mathfrak{k}\|^2 - \|\rho_\mathfrak{g}\|^2) 1 \otimes 1 \tag{3.6}$$

due to Parthasarathy [24] (see also [9]). Here $\mathrm{Cas}_\mathfrak{g}$ (respectively $\mathrm{Cas}_\mathfrak{k}$) denotes the Casimir element of $U(\mathfrak{g})$. The constant $(\|\rho_\mathfrak{k}\|^2 - \|\rho_\mathfrak{g}\|^2)$ is explained below.

We will discuss the proof later, in the more general setting of Kostant cubic Dirac operator.

This has several important consequences. To state them, we need more notation. Let us fix a maximal torus T in K, its Lie algebra $\mathfrak{t}_0$, with complexification $\mathfrak{t}$. Let $\mathfrak{a}$ denotes the centralizer of $\mathfrak{t}$ in $\mathfrak{p}$. Then

$$\mathfrak{h} := \mathfrak{t} \oplus \mathfrak{a}$$

is a fundamental Cartan subalgebra of $\mathfrak{g}$, and the above decomposition also gives an imbedding $\mathfrak{t}^* \to \mathfrak{g}^*$. Let $R = R(\mathfrak{g}, \mathfrak{h})$ denotes the root system of $\mathfrak{h}$ in $\mathfrak{g}$, $W = W(\mathfrak{g}, \mathfrak{h})$ its Weyl group. Let us also choose a positive root system R^+ in R. As usual, ρ denotes the half-sum of positive roots, an element in $\mathfrak{h}^*$. Similarly, we introduce the root system $R_\mathfrak{k} = R(\mathfrak{k}, \mathfrak{t})$, its Weyl group $W_\mathfrak{k}$, a positive root system $R_\mathfrak{k}^+$, compatible with R^+, and half-sum of positive roots $\rho_\mathfrak{k}$.

The bilinear form B on $\mathfrak{g}$ restricts to a non degenerate symmetric bilinear form on $\mathfrak{h}$, which is definite positive on the real form $i\mathfrak{t}_0 \oplus \mathfrak{a}$. We denote by $\langle\, .\, , .\rangle$ the induced form on $i\mathfrak{t}_0^* \oplus \mathfrak{a}$ and in the same way its extension to $\mathfrak{h}^*$. The norm appearing in (3.6) is defined for any $\lambda \in \mathfrak{h}^*$ by $\|\lambda\|^2 = \langle \lambda, \lambda \rangle$.

Recall the Harish-Chandra algebra isomorphism

$$\gamma_{\mathfrak{g}} : \mathfrak{Z}(\mathfrak{g}) \simeq S(\mathfrak{h})^W \tag{3.7}$$

between the center $\mathfrak{Z}(\mathfrak{g})$ of the envelopping algebra $U(\mathfrak{g})$ and the W-invariants in the symmetric algebra $S(\mathfrak{h})$ on $\mathfrak{h}$. Accordingly, a character χ of $\mathfrak{Z}(\mathfrak{h})$ is given by an element of $\mathfrak{h}^*$ (or rather its Weyl group orbit). If $\lambda \in \mathfrak{h}^*$, we denote by χ_λ the corresponding character of $\mathfrak{Z}(\mathfrak{g})$. We say that a module X in $\mathcal{M}(\mathfrak{g}, K)$ has infinitesimal character λ if any $z \in \mathfrak{Z}(\mathfrak{g})$ acts on X by the scalar operator $\chi_\lambda(z)\mathrm{Id}_X$.

Assume that X has infinitesimal character $\Lambda \in \mathfrak{h}^*$. Let (γ, F_γ) be an irreducible representation of $\widetilde{K}$ with highest weight $\tau = \tau_\gamma \in \mathfrak{t}^*$. We denote the corresponding $\widetilde{K}$-isotypic component of $X \otimes S$ by $(X \otimes S)(\gamma)$. Then D^2 acts on $(X \otimes S)(\gamma)$ by the scalar

$$-\|\Lambda\|^2 + \|\tau + \rho_{\mathfrak{k}}\|^2. \tag{3.8}$$

In particular, we see that the kernel of D^2 on $X \otimes S$ is a direct sum of full $\widetilde{K}$-isotypic components of $X \otimes S$: these are exactly those $(X \otimes S)(\gamma)$ for which

$$\|\tau + \rho_{\mathfrak{k}}\|^2 = \|\Lambda\|^2. \tag{3.9}$$

Another useful consequence of (3.6) is

$$D^2 \text{ is in the center of the algebra } \mathcal{A}^K. \tag{3.10}$$

4. Dirac operator and unitarizable of $(\mathfrak{g}, K)$-modules

We continue with the notation of the previous section. Assume now that the $(\mathfrak{g}, K)$-module X is endowed with a definite positive invariant Hermitian product $\langle . , . \rangle_X$. Invariance means that elements in $\mathfrak{g}_0$ act as skew-symmetric operators on X, *i.e.*

$$\langle X.v, w \rangle_X = -\langle v, X \cdot w \rangle_X, \qquad (v, w \in X),\ (X \in \mathfrak{g}_0).$$

For $X \in \mathfrak{g}$, this means that the adjoint of X is $-\bar{X}$, where the bar denotes complex conjugation with respect to the real form $\mathfrak{g}_0$ of $\mathfrak{g}$. We say that the $(\mathfrak{g}, K)$-module is unitarizable.

There is a well-known a definite positive Hermitian product $\langle . , . \rangle_S$ on S so that the elements of $\mathfrak{p}_0 \subset \mathcal{C}(\mathfrak{p})$ act as skew-symmetric operators on S (see [31] or [9] 2.3.9). If X is a unitarizable $(\mathfrak{g}, K)$-module, $X \otimes S$ is then equiped with the definite positive Hermitian product tensor product of $\langle . , . \rangle_X$ and $\langle . , . \rangle_S$, denoted by $\langle . , . \rangle_{X \otimes S}$.

It is clear that D is symmetric with respect to $\langle . , . \rangle_{X \otimes S}$. In particular D^2 is a positive symmetric operator on $X \otimes S$. From (3.8) we get

Proposition 4.1 (Parthasarathy-Dirac inequality)**.** *Assume that the unitarizable* $(\mathfrak{g}, K)$*-module* X *has infinitesimal character* $\Lambda \in \mathfrak{h}^*$*. Let* (γ, F_γ) *be an irreducible representation of* $\widetilde{K}$ *with highest weight* $\tau = \tau_\gamma \in \mathfrak{t}^*$ *such that* $(X \otimes S)(\tau) \neq 0$*. Then*

$$\|\tau + \rho_{\mathfrak{k}}\|^2 \geq \|\Lambda\|^2.$$

Remark 4.2. If the $(\mathfrak{g}, K)$-module X is unitarizable and has an infinitesimal character, it is then also easily seen that D acts semisimply on $X \otimes S$. In particular

$$\ker D^2 = \ker D. \tag{4.1}$$

Similarly, if X is finite-dimensional, then there is a natural inner product on $X \otimes S$ such that D is skew-symmetric with respect to this inner product. So D acts semisimply on $X \otimes S$ and (4.1) holds also for finite-dimensional X (but now $D^2 \leq 0$).

Parthasarathy's Dirac inequality gives a powerful criterion of unitarizability for irreducible $(\mathfrak{g}, K)$-modules. This can be slightly refined as follows ([22]).

Proposition 4.3. *Assume that G is semisimple with no compact simple factor. Let X be a $(\mathfrak{g}, K)$-module, let $\langle\, .\, , .\rangle_X$ be any definite positive Hermitian product on X and let $\langle\, .\, , .\rangle_{X\otimes S}$ be the Hermitian form on $X \otimes S$ constructed as above. Then X is unitary with respect to $\langle\, .\, , .\rangle_X$ (i.e. invariant) if and only if D is a symmetric operator on $X \otimes S$ with respect to $\langle\, .\, , .\rangle_{X\otimes S}$.*

Example 4.4. Spherical principal series of $\mathbf{SL}(2, \mathbb{R})$

Spherical principal series of $\mathbf{SL}(2, \mathbb{R}) \times \mathbf{SL}(2, \mathbb{R})$

5. Dirac cohomology

Let us now review Vogan's definition of Dirac cohomology.

Definition 5.1. Let $X \in \mathcal{M}(\mathfrak{g}, K)$. The Dirac operator D acts on $X \otimes S$. Vogan's Dirac cohomology of X is the quotient

$$H^D_{\mathrm{V}}(X) = \ker D/(\ker D \cap \operatorname{Im} D).$$

Since $D \in \mathcal{A}^K$, $\widetilde{K}$ acts on $\ker D$, $\operatorname{Im} D$ and $H^D_{\mathrm{V}}(X)$.

This is particularly helpful if X is unitary, in which case there is a natural inner product on $X \otimes S$ such that D is symmetric with respect to this inner product. If furthermore X is admissible, then it follows that D acts semisimply on $X \otimes S$ and so

$$\ker D^2 = \ker D = H^D_{\mathrm{V}}(X). \tag{5.1}$$

In this case, the Dirac cohomology of X is a sum the full isotypic components $X \otimes S(\gamma)$ such that (3.9) holds.

For general X, (4.1) does not hold, but note that D is always a differential on $\ker D^2$, and $H^D_{\mathrm{V}}(X)$ is the usual cohomology of this differential.

Let us state the main result of [8], which gives a strong condition on the infinitesimal character of a $(\mathfrak{g}, K)$-module X with non zero Dirac cohomology.

Proposition 5.2. *Let $X \in \mathcal{M}(\mathfrak{g}, K)$ be a Harish-Chandra module with infinitesimal character $\Lambda \in \mathfrak{h}^*$. Assume that (γ, F_γ) is an irreducible representation of $\widetilde{K}$ with highest weight $\tau = \tau_\gamma \in \mathfrak{t}^*$ such that $(X \otimes S)(\gamma)$ contibutes to $H^D_{\mathrm{V}}(X)$. Then*

$$\Lambda = \tau + \rho_{\mathfrak{k}} \quad \textit{up to conjugacy by the Weyl group } W. \tag{5.2}$$

Thus for unitary X, (3.9) is equivalent to the stronger condition (5.2), provided that γ appears in $X \otimes S$.

6. Why is Dirac cohomology an interesting invariant ?

Many interesting modules have non-vanishing Dirac cohomology :

- Finite dimensional representations (Kostant). Dirac cohomology is given by to extremal weights.
- Discrete series, and more generally Vogan-Zuckerman $A_{\mathfrak{q}}(\lambda)$-modules ([7]).
- Highest weight modules [11], [10], [12]
- Unipotent representations [2], [3]

Dirac cohomology is related to other kinds of cohomological invariants

- $\mathfrak{n}$-cohomology for highest weight-modules [11], [12]
-$(\mathfrak{g}, K)$-cohomology for $A_{\mathfrak{q}}(\lambda)$-modules [7],

7. Dirac cohomology and $(\mathfrak{g}, K)$-cohomology

An important problem in the theory of automorphic forms is to compute cohomology of locally symmetric spaces. Matsushima's formula ([?]) relates this problem to computation of $(\mathfrak{g}, K)$-cohomology of irreducible unitary Harish-Chandra modules for the corresponding semisimplegroup G. Vogan and Zuckerman [30] have classified all irreducible unitary Harish-Chandra modules X such that $H^*(\mathfrak{g}, K, X \otimes F^*) \neq 0$ where F is a finite-dimensional representation of G. They are precisely the cohomologically induced modules $A_{\mathfrak{q}}(\lambda)$ with the same infinitesimal character as F. Moreover, they have explicitely computed the cohomology.

Let $X \in \mathcal{M}(\mathfrak{g}, K)$ be an irreducible unitary Harish-Chandra module and suppose that X has the same infinitesimal character as an irreducible finite dimensional representation F (this is an obvious necessary condition for $H^*(\mathfrak{g}, K, X \otimes F^*)$ to be non zero).

A calculation ([31], Chapter 9) shows that if $\dim \mathfrak{p}$ is even :

$$H^*(\mathfrak{g}, K; X \otimes F^*) = \mathrm{Hom}_{\widetilde{K}}(H_D(F), H_D(X)),$$

and if $\dim \mathfrak{p}$ is odd :

$$H^*(\mathfrak{g}, K; X \otimes F^*) = \mathrm{Hom}_{\widetilde{K}}(H_D(F), H_D(X)) \oplus \mathrm{Hom}_{\widetilde{K}}(H_D(F), H_D(X)).$$

8. Dirac cohomology of some $(\mathfrak{g}, K)$-modules

In [7] the Dirac cohomology of finite dimensional representations is computed, generalizing the result in [15] which assumed that G and K have equal rank.

Let us recall the setting : G is a connected real reductive groupe with maximal compact subgroup K, $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ is a Cartan fondamental subalgebra of $\mathfrak{g}_0$, $l_0 = \dim_{\mathbb{R}} \mathfrak{a}_0$. We have root system $R = R(\mathfrak{g}, \mathfrak{h})$, $R_{\mathfrak{k}} = R(\mathfrak{k}, \mathfrak{t})$, and the non reduced root system $R(\mathfrak{g}, \mathfrak{t})$. Let $W = W(\mathfrak{g}, \mathfrak{h})$, $W_{\mathfrak{k}} = W(\mathfrak{k}, \mathfrak{t})$ and $W(\mathfrak{g}, \mathfrak{t})$ be the corresponding Weyl groups. Note that $W(\mathfrak{g}, \mathfrak{t})$ can be identified with θ-stable elements in W. We choose compatible positive root systems $R^+ \subset R$, $R^+_{\mathfrak{k}} \subset R_{\mathfrak{k}}$ and $R(\mathfrak{g}, \mathfrak{t})^+ \subset R(\mathfrak{g}, \mathfrak{t})$. Denote by $C_{\mathfrak{g}}(\mathfrak{h})$, $C_{\mathfrak{k}}(\mathfrak{t})$, and $C_{\mathfrak{g}}(\mathfrak{t})$ the corresponding closed Weyl chambers. Note that $C_{\mathfrak{g}}(\mathfrak{t})$ is contained in $C_{\mathfrak{g}}(\mathfrak{h})$ and in $C_{\mathfrak{k}}(\mathfrak{t})$. Let

$$W(\mathfrak{g}, \mathfrak{t})^1 = \{w \in W(\mathfrak{g}, \mathfrak{t}) \mid w(C_{\mathfrak{g}}(\mathfrak{t})) \subset C_{\mathfrak{k}}(\mathfrak{t})\}$$

Let us denote by E_μ the irreducible finite dimensional representation of $\mathfrak{k}$ with highest weight μ.

Theorem 8.1. *Let V_λ be an irreducible finite dimensional $(\mathfrak{g}, K)$-module with highest weight λ. If $\lambda \neq \theta(\lambda)$, then $H_D(V_\lambda) = 0$. If $\lambda = \theta(\lambda)$, then as a $\mathfrak{k}$-module, the Dirac cohomology of V_λ is*

$$H_D(V_\lambda) = \bigoplus_{w \in W(\mathfrak{g}, \mathfrak{t})^1} 2^{[l_0/2]}\, E_{w\cdot(\lambda+\rho)-\rho_{\mathfrak{k}}}.$$

In [7] the Dirac cohomology of $A_{\mathfrak{q}}(\lambda)$-modules is computed. The notation is the same as above, but we need more. Fix some element $H \in i\mathfrak{t}_0$ and let $\mathfrak{l}$ be the 0-eigenspace of $\mathrm{ad}(H)$ in $\mathfrak{g}$ and $\mathfrak{U}$ the sum of the eigenspaces of $\mathrm{ad}(H)$ in $\mathfrak{g}$ for the positive eigenvalues. Then

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$$

is a paraolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ and nilpotent radical $\mathfrak{u}$. Note that $\mathfrak{l}$, $\mathfrak{u}$ and $\mathfrak{q}$ are stable under θ, and that $\mathfrak{l}$ is stable under complex conjugation, so that $\mathfrak{l}_0$ is a real form of $\mathfrak{l}$. Let us denote by L the connected subgroup of G with Lie algebra $\mathfrak{l}_0$. Assume that our choice of positive root system R^+ is compatible with $\mathfrak{q}$ in the sense that

$$R(\mathfrak{u}) = \{\alpha \in R \,|\, \mathfrak{g}_\alpha \in \mathfrak{u}\}$$

is contained in R^+. We note also that $\mathfrak{h} \subset \mathfrak{l}$. We take $R_{\mathfrak{l}}^+ = R(\mathfrak{l}, \mathfrak{h}) \cap R^+$ as a choice of positive root system in $R_{\mathfrak{l}} = R(\mathfrak{l}, \mathfrak{h})$. Likewise $R(\mathfrak{l}, \mathfrak{t}) \subset R(\mathfrak{g}, \mathfrak{t})$ and $R^+(\mathfrak{l}, \mathfrak{t}) = R(\mathfrak{l}, \mathfrak{t}) \cap R^+(\mathfrak{g}, \mathfrak{t})$.

Let $\lambda \in \mathfrak{l}^*$ be the complexified differential of a unitary character σ of L satisfying

$$\langle \alpha, \lambda_{\mathfrak{t}} \rangle \geq 0$$

(In particular, σ is in the good range). We can them form the Vogan-Zuckerman module $A_{\mathfrak{q}}(\lambda)$ ([13]). It is the unique irreducible unitary $(\mathfrak{g}, K)$-module satisfying

(i) the restriction of $A_{\mathfrak{q}}(\lambda)$ to K contains the irreducible representation with highest weight $\mu(\lambda, \mathfrak{q}) = \lambda_{\mathfrak{t}} + 2\rho(\mathfrak{u} \cap \mathfrak{p})$ where $\rho(\mathfrak{u} \cap \mathfrak{p})$ is the half-sum of all $\mathfrak{t}$-weights in $\mathfrak{u} \cap \mathfrak{p}$ counted with multiplicity.

(ii) $A_{\mathfrak{q}}(\lambda)$ has infinitesimal character $\lambda + \rho$.

(iii) if an irreducible representation with highest weight μ of K occurs in $A_{\mathfrak{q}}(\lambda)$, then μ is of th form

$$\mu = \mu(\lambda, \mathfrak{q}) + \sum_{\alpha \in R(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t})} n_\alpha \, \alpha,$$

where the n_α's are non-negative integers. In particular the lowest K-type of $A_{\mathfrak{q}}(\lambda)$ is the one with highest weight $\mu(\lambda, \mathfrak{q})$ which occurs with multiplicity one.

The set $W(\mathfrak{l}, \mathfrak{t})^1$ is defined with respect to $\mathfrak{l}$ as it was defined above with respect to $\mathfrak{g}$. Notice that $W(\mathfrak{l}, \mathfrak{t})^1 = (\mathfrak{l}, \mathfrak{t}) \cap (\mathfrak{g}, \mathfrak{t})^1$.

Theorem 8.2. *If $\lambda \neq \theta(\lambda)$, then $H_D(A_{\mathfrak{q}}(\lambda)) = 0$. If $\lambda = \theta(\lambda)$ then as a $\mathfrak{k}$-module, the Dirac cohomology of $A_{\mathfrak{q}}(\lambda)$ is*

$$H_D(A_{\mathfrak{q}}(\lambda)) = \bigoplus_{w \in W(\mathfrak{l}, \mathfrak{t})^1} 2^{[l_0/2]} \, E_{w\cdot(\lambda+\rho)-\rho_c}.$$

[10] : Dirac cohomology of Wallach representations is computed. Wallach representations are special families of irreducible unitary highest weight modules. So their Dirac cohomology coincide, up to a twist, with their $\mathfrak{n} = \mathfrak{p}^+$-chomology, which is computed by Enright, but the computation there is more explicit.

9. Kostant's cubic Dirac operator

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In [15], B. Kostant introduces cubic Dirac operators, which are a generalization of Dirac operators considered above. The setting is the following. Let $\mathfrak{g}$ be a complex reductive Lie algebra, and fix an invariant nondegenerate symmetric bilinear form B on $\mathfrak{g}$ extending the Killing form on the semisimple part of $\mathfrak{g}$. Suppose that $\mathfrak{r}$ is a reductive subalgebra of $\mathfrak{g}$ such that the restriction of B to $\mathfrak{r}$ is non-degenerate. Let $\mathfrak{s}$ be the orthogonal complement of $\mathfrak{r}$ with respect to B. Then the restriction of B to $\mathfrak{s}$ is non degenerate and we have a decomposition

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}.$$

Let $\mathrm{Cl}(\mathfrak{s})$ be the Clifford algebra of $\mathfrak{s}$.

As in Section 2, the exterior algebra $\bigwedge \mathfrak{s}$ is identified with the Clifford algebra $\mathrm{Cl}(\mathfrak{s})$ as a vector space via Chevalley isomorphism. Thus, for elements u, v in $\bigwedge \mathfrak{s}$, we distinguish between their Clifford product uv and their exterior product $u \wedge v$. Exterior product by u in $\bigwedge \mathfrak{s}$ is denoted by $e(u) \in \mathrm{End}(\bigwedge \mathfrak{s})$. The bilinear form B on $\mathfrak{s}$ extends to a non degenerate bilinear form on $\bigwedge \mathfrak{s}$, still denoted by B, and this gives an identification of $\bigwedge \mathfrak{s}$ with its dual $(\bigwedge \mathfrak{s})^*$. For $X \in \mathfrak{s}$, we denote by $\iota(X)$ the transpose of $e(X)$ with respect to B. For any $v \in \bigwedge \mathfrak{s}$ and any $X \in \mathfrak{s}$, the Clifford product Xv is then given by

$$Xv = (e(X) + \iota(X))(v), \tag{9.1}$$

and the Clifford relations extend to

$$Xv - (-1)^k vX = 2\iota(X)w \tag{9.2}$$

for $v \in \bigwedge^k \mathfrak{s}$ and any $X \in \mathfrak{s}$.

The restriction of the fundamental 3-form of $\mathfrak{g}$ gives an element $\nu \in \bigwedge^3 \mathfrak{s}$ characterized by the identity

$$B(\nu, X \wedge Y \wedge Z) = \frac{1}{2} B(X, [Y, Z]), \qquad (X, Y, Z \in \mathfrak{s}) \tag{9.3}$$

Let $(X_i)_i$ be an orthonormal basis of $\mathfrak{s}$. Then Kostant's cubic Dirac operator $D(\mathfrak{g}, \mathfrak{r})$ is the element

$$D(\mathfrak{g}, \mathfrak{r}) = \sum_i X_i \otimes X_i + 1 \otimes \nu \tag{9.4}$$

of $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})$. The terminology is explained by the fact that the element ν is of degree 3.

The element ν is expressed in terms of the basis $(X_i)_i$ as

$$\nu = \frac{1}{2} \sum_{i<j<k} B([X_i, X_j], X_k)\; X_i X_j X_k \tag{9.5}$$

and since the x_i are orthogonal, we have also

$$\nu = \frac{1}{2} \sum_{i<j<k} B([X_i, X_j], X_k)\; X_i \wedge X_j \wedge X_k \tag{9.6}$$

One can now easily see that D in independent of the choice of the basis $(X_i)_i$.

Let us recall briefly some construction of Section 3, which carry over immediately to the more general setting of this section. Firstly, there is the map $\alpha : \mathfrak{r} \to \mathfrak{so}(\mathfrak{s}) \to \mathrm{Cl}(\mathfrak{s})$ which is the composition of the action map $\mathfrak{r} \to \mathfrak{so}(\mathfrak{s})$ followed by the standard inclusion of $\mathfrak{so}(\mathfrak{s})$ into $\mathrm{Cl}(\mathfrak{s})$ using the identification $\mathfrak{so}(\mathfrak{s}) \cong \bigwedge^2 \mathfrak{s}$.

From this, we construct the diagonal embedding of $U(\mathfrak{r})$ in $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})$. We embed $\mathfrak{r}$ in $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})$ via

$$X \longmapsto X \otimes 1 + 1 \otimes \alpha(X), \qquad X \in \mathfrak{r},$$

This embedding is then extended naturally to a morphism of (super)algebras :

$$\Delta : U(\mathfrak{r}) \to U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}). \tag{9.7}$$

The $\mathbb{Z}$-graduation on the tensor algebra of $\mathfrak{s}$ induces a $\mathbb{Z}_2$-graduation on the Clifford algebra. This gives $\mathrm{Cl}(\mathfrak{s})$ the structure of a super-algebra. The trivial graduation on $U(\mathfrak{g})$ gives a $\mathbb{Z}_2$-graduation on $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})$. With this in place, we can state :

Lemma 9.1. *The cubic Dirac operator $D(\mathfrak{g}, \mathfrak{r})$ is $\mathfrak{r}$-invariant, i.e. it (super)commutes with the image of $U(\mathfrak{r})$ by Δ. We write $D(\mathfrak{g}, \mathfrak{r}) \in (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}))^{\mathfrak{r}}$.*

We now state the two main general results about cubic Dirac operators. The first one is the computation of their square in [15], Theorem 2.16 :

Theorem 9.2. *In the setting as above,*

$$D(\mathfrak{g},\mathfrak{r})^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Delta(\Omega_{\mathfrak{r}}) + (||\rho_{\mathfrak{r}}||^2 - ||\rho||^2) 1 \otimes 1,$$

where $\Omega_{\mathfrak{g}}$ (resp. $\Omega_{\mathfrak{r}}$) denotes the Casimir element in $\mathfrak{Z}(\mathfrak{g})$ (resp. $\mathfrak{Z}(\mathfrak{r})$).

We give some ingredients of a simple proof due to N. Prudhon [26]. It is based a relation between Dirac operators in a situation where we have two subalgebras $\mathfrak{r}$ and $\mathfrak{l}$ of $\mathfrak{g}$ with

$$\mathfrak{g} \supset \mathfrak{r} \supset \mathfrak{l}$$

such that the restriction of the invariant form B on both $\mathfrak{r}$ and $\mathfrak{l}$ is non degenerate. Let us write the orthogonal decompositions

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}, \quad \mathfrak{g} = \mathfrak{r} \oplus \mathfrak{m}, \quad \mathfrak{m} = \mathfrak{s} \oplus \mathfrak{m}_{\mathfrak{r}}, \quad \mathfrak{r} = \mathfrak{l} \oplus \mathfrak{m}_{\mathfrak{r}}.$$

Via the natural inclusion, all these operators can be seen as living in $U(\mathfrak{g}) \otimes \mathcal{C}(\mathfrak{m})$.

In that case, denoting by $\bar{\otimes}$ the graded tensor product, we have an isomorphism

$$\mathrm{Cl}(\mathfrak{m}) = \mathrm{Cl}(\mathfrak{s}) \bar{\otimes} \mathrm{Cl}(\mathfrak{m}_{\mathfrak{r}})$$

which allows us to identify $\mathrm{Cl}(s)$ and $\mathrm{Cl}(\mathfrak{m}_{\mathfrak{r}})$ as subalgebras of $\mathrm{Cl}(\mathfrak{m})$. Furthermore, we may consider the following Dirac operators

$$D(\mathfrak{g},\mathfrak{l}) \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{m}), \quad D(\mathfrak{g},\mathfrak{r}) \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}), \quad D(\mathfrak{r},\mathfrak{l}) \in U(\mathfrak{r}) \otimes \mathrm{Cl}(\mathfrak{m}_{\mathfrak{r}}).$$

The diagonal embedding Δ of $U(\mathfrak{r})$ in $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})$ in (9.7) gives a diagonal embedding still denoted by Δ

$$\Delta : U(\mathfrak{r}) \otimes \mathrm{Cl}(\mathfrak{m}_{\mathfrak{r}}) \to U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}) \bar{\otimes} \mathrm{Cl}(\mathfrak{m}_{\mathfrak{r}}) \simeq U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{m}). \tag{9.8}$$

Proposition 9.3. *In the setting above,*
(i) $D(\mathfrak{g},\mathfrak{l}) = D(\mathfrak{g},\mathfrak{r}) + \Delta(D(\mathfrak{r},\mathfrak{l})$
(ii) The components $D(\mathfrak{g},\mathfrak{r})$ et $\Delta(D(\mathfrak{r},\mathfrak{l})$ (super)commute.

Now Theorem 3.6 in general will be a consequence the proposition with $\mathfrak{l} = 0$ and the computation of the square of $D(\mathfrak{g}, 0)$ and $D(\mathfrak{r}, 0)$.

The second result was proved by Huang and Pandzic in the setting of section 3 in [8], and Kostant noticed that the statement and the proof are still valid in the general case [17]. To state it, we need to introduce some material. Recall that the Clifford algebra $\mathrm{Cl}(\mathfrak{s})$ is $\mathbb{Z}_2$-graded, $\mathrm{Cl}(\mathfrak{s}) = \mathrm{Cl}(\mathfrak{s})^{\bar{0}} \oplus \mathrm{Cl}(\mathfrak{s})^{\bar{1}}$. Thus the algebra $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})$ is a $\mathbb{Z}_2$-graded algebra. The diagonal embedding $\Delta : U(\mathfrak{r}) \to U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})$ gives the right-hand side a structure of $\mathfrak{r}$-module through the adjoint action, and this action respects the $\mathbb{Z}_2$-grading. Thus the algebra of $\mathfrak{r}$-invariants $(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}))^{\mathfrak{r}}$ is still endowed with a $\mathbb{Z}_2$-grading. It is easy to see that $D(\mathfrak{g},\mathfrak{r})$ is $\mathfrak{r}$-invariant.

Let us introduce the operator

$$d = \mathrm{ad} D : a \mapsto [D, a]$$

on the superalgebra $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})$, where the bracket is the superbracket. Then, using the super-Jacobi identity, we get $d^2 = (\mathrm{ad}D)^2 = \mathrm{ad}(D^2)$. The result in theorem 9.2 shows that D^2 supercommutes with all elements in $(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}))^{\mathfrak{r}}$. Furthermore, D being $\mathfrak{r}$-invariant, $\mathrm{ad}D$ is $\mathfrak{r}$-equivariant and thus induces

$$d : (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}))^{\mathfrak{r}} \to (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}))^{\mathfrak{r}}$$

with $d^2 = 0$.

Define the cohomology of d to be $\ker d/\operatorname{Im} d$ on $(U(\mathfrak{g})\otimes C(\mathfrak{s}))^{\mathfrak{r}}$. The theorem of Huang and Pandzic computes this cohomology. Let us remark first that $\Delta(\mathfrak{Z}(\mathfrak{r}))$ is in the kernel of D.

Theorem 9.4. *(Huang-Pandzic [8]) On $(U(\mathfrak{g})\otimes \mathrm{Cl}(\mathfrak{s}))^{\mathfrak{r}}$, we have*

$$\ker d = \Delta(\mathfrak{Z}(\mathfrak{r})) \oplus \operatorname{Im} d.$$

The proof reduces to the exactness of the Koszul complex $S(\mathfrak{s})\otimes \bigwedge^{\bullet}\mathfrak{s}$ for $\mathfrak{s}$. It uses the filtration on $U(\mathfrak{g})$ and the corresponding graduation $\mathbf{Gr}(U(\mathfrak{g}))\simeq S(\mathfrak{g})$, which gives

$$\mathbf{Gr}(U(\mathfrak{g})\otimes \mathrm{Cl}(\mathfrak{s})) \simeq S(\mathfrak{g})\otimes \bigwedge \mathfrak{s} \simeq S(\mathfrak{r})\otimes S(\mathfrak{s})\otimes \bigwedge \mathfrak{s}.$$

Corollary 9.5. *Let us apply this to an element of the form $z\otimes 1$, where $z\in\mathfrak{Z}(\mathfrak{g})$, which is obviously $\mathfrak{r}$-invariant and in the kernel of d. We get that $z\otimes 1$ can be written as*

$$z\otimes 1 = \Delta(z_1) + Da + aD$$

for some $a\in (U(\mathfrak{g})\otimes C(\mathfrak{s}))^{\mathfrak{r}}$ (in the odd part of the superalgebra), and some $z_1\in\mathfrak{Z}(\mathfrak{r})$.

Let us now identify z_1 explicitely. It is always possible to find in $\mathfrak{g}$ a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{h}$ such that

$$\mathfrak{b} = \mathfrak{b}\cap\mathfrak{r}\oplus\mathfrak{b}\cap\mathfrak{s},\quad \mathfrak{h} = \mathfrak{h}\cap\mathfrak{r}\oplus\mathfrak{h}\cap\mathfrak{s}$$

Let us write these decompositions simply as

$$\mathfrak{b} = \mathfrak{b}_{\mathfrak{r}}\oplus\mathfrak{b}_{\mathfrak{s}},\quad \mathfrak{h} = \mathfrak{h}_{\mathfrak{r}}\oplus\mathfrak{h}_{\mathfrak{s}}$$

Let us denote by $W_{\mathfrak{g}}$ the Weyl group of the root system $R(\mathfrak{h},\mathfrak{g})$ and $W_{\mathfrak{r}}$ the Weyl group of the root system $R(\mathfrak{h}_{\mathfrak{r}},\mathfrak{r})$. Seeing the symmetric algebra $S(\mathfrak{h})$ (resp. $S(\mathfrak{h}_{\mathfrak{r}})$) as the algebra of polynomial functions on $\mathfrak{h}^*$ (resp. on $\mathfrak{h}_{\mathfrak{r}}^*$), we notice that restriction of functions from $\mathfrak{h}^*$ to $\mathfrak{h}_{\mathfrak{r}}$ induces a morphism

$$\mathrm{res}: S(\mathfrak{h})^{W_{\mathfrak{g}}} \to S(\mathfrak{h}_{\mathfrak{r}})^{W_{\mathfrak{r}}}$$

Let us denote by $\gamma_{\mathfrak{g}}$ and $\gamma_{\mathfrak{r}}$ respectively the Harish-Chandra isomorphisms

$$\mathfrak{Z}(\mathfrak{g}) \xrightarrow{\gamma_{\mathfrak{g}}} S(\mathfrak{h})^{W_{\mathfrak{g}}},\quad \mathfrak{Z}(\mathfrak{r}) \xrightarrow{\gamma_{\mathfrak{r}}} S(\mathfrak{h}_{\mathfrak{r}})^{W_{\mathfrak{r}}}.$$

Proposition 9.6. *There is a unique algebra morphism $\eta_{\mathfrak{r}}:\mathfrak{Z}(\mathfrak{g})\to\mathfrak{Z}(\mathfrak{r})$ such that*

$$\begin{tikzcd}
\mathfrak{Z}(\mathfrak{g}) \arrow[r, "\eta_{\mathfrak{r}}"] \arrow[d, "\gamma_{\mathfrak{g}}"'] & \mathfrak{Z}(\mathfrak{r}) \arrow[d, "\gamma_{\mathfrak{r}}"'] \\
S(\mathfrak{h})^{W_{\mathfrak{g}}} \arrow[r, "\mathrm{res}"] & S(\mathfrak{h}_{\mathfrak{r}})^{W_{\mathfrak{r}}}
\end{tikzcd}$$

commutes.

Proposition 9.7. *In the setting of the corollary above,*

$$z\otimes 1 = \Delta(\eta_{\mathfrak{r}}(z)) + Da + aD$$

for some $a\in (U(\mathfrak{g})\otimes \mathrm{Cl}(\mathfrak{s}))^{\mathfrak{r}}$ (in the odd part of the superalgebra).

Let us now define, following Vogan, the Dirac cohomology of a $\mathfrak{g}$-module V. Notice that $U(\mathfrak{g})\otimes\mathrm{Cl}(\mathfrak{s})$ acts on $V\otimes S$.

Definition 9.8. The Dirac cohomology $H_D(\mathfrak{g},\mathfrak{r};V)$ is the quotient of the kernel of D acting on $V\otimes S$ by $\ker D\cap\operatorname{Im} D$. Since D is $\mathfrak{r}$-invariant, $\ker D$, $\operatorname{Im} D$ and $H_D(\mathfrak{g},\mathfrak{r};V)$ are naturally $\mathfrak{r}$-modules.

Remark 9.9. Notice that D^2 doesn't act by 0 on $V \otimes S$. So D is not a differential on $V \otimes S$, and $\operatorname{Im} D$ is not included in $\ker D$.

The center of the envelopping algebra $\mathfrak{Z}(\mathfrak{g})$ acts naturally on $H_D(\mathfrak{g}, \mathfrak{r}; V)$, since it commutes with D in $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})$. The following result is an analog of the Casselman-Osborne Lemma.

Proposition 9.10. *The action of an element $z \otimes 1$ in $\mathfrak{Z}(\mathfrak{g}) \otimes 1$ on $H_D(\mathfrak{g}, \mathfrak{r}; V)$ coincide with the action of $\eta_{\mathfrak{r}}(z) \in U(\mathfrak{r})$ (ie. with the action of $\Delta(\eta_{\mathfrak{r}}(z))$).*

This is a consequence of proposition 9.7.

We have already seen one of the main instances of these general definitions, namely the case $\mathfrak{r} = \mathfrak{k}$ and $\mathfrak{s} = \mathfrak{p}$ of section 3. Notice that if X, Y, Z are in $\mathfrak{p}$, $[X, Y]$ is in $\mathfrak{k}$, so $B([X, Y], Z) = 0$. Thus the second sum in (9.4) (the cubic part) vanishes. In the next section, we consider another important particular choice of $/frr$ and $/frs$.

10. Cubic Dirac operators for Levi subalgebras

The second kind of cubic Dirac operator we consider arises in the following setting. Suppose that $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a parabolic subalgebra of $\mathfrak{g}$, that $\mathfrak{q}^- = \mathfrak{l} \oplus \mathfrak{u}^-$ is the opposite parabolic subalgebra, and set $\mathfrak{s} = \mathfrak{u} \oplus \mathfrak{u}^-$. Then

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{s}. \tag{10.1}$$

Furthermore, the restrictions of the Killing form B to $\mathfrak{l}$ and $\mathfrak{s}$ are non-degenerate, and the above decomposition is orthogonal. Thus we can form the cubic Dirac operator $D(\mathfrak{g}, \mathfrak{l})$ as above. It will be convenient to use a different kind of basis of $\mathfrak{s}$ to express this operator. Indeed, since $\mathfrak{u}$ and $\mathfrak{u}^-$ are isotropic subspaces in perfect duality under B, we can identify $\mathfrak{u}^*$ with $\mathfrak{u}^-$; this identification is $\mathfrak{l}$-invariant. Let $u_1, \dots, u_n$ be a basis of $\mathfrak{u}$, and let $u_1^-, \dots, u_n^-$ be the dual basis of $\mathfrak{u}^-$. Consider the following dual bases of $\mathfrak{s}$:

$$b_1 = u_1, \dots, b_n = u_n,\ b_{n+1} = u_1^-, \dots, b_{2n} = u_n^-;$$
$$d_1 = u_1^-, \dots, d_n = u_n^-,\ d_{n+1} = u_1, \dots, d_{2n} = u_n.$$

We can now write Kostant's cubic Dirac operator $D = D(\mathfrak{g}, \mathfrak{l})$ as

$$D = \sum_{i=1}^{2n} d_i \otimes b_i + 1 \otimes \nu.$$

It is clear that the first sum in the expression for D breaks up into two pieces:

$$\sum_{i=1}^{n} u_i^- \otimes u_i + \sum_{i=1}^{n} u_i \otimes u_i^- = A + A^-,$$

where A respectively A^- denote the two summands. Moreover, since for any i, j, k,

$$B([u_i, u_j], u_k) = B([u_i^-, u_j^-], u_k^-) = 0,$$

the cubic element v also splits into two parts, as

$$\nu = -\frac{1}{2} \sum_{i<j} \sum_{k} B([u_i^-, u_j^-], u_k)\, u_i \wedge u_j \wedge u_k^- - \frac{1}{2} \sum_{i<j} \sum_{k} B([u_i, u_j], u_k^-)\, u_i^- \wedge u_j^- \wedge u_k = a + a^-,$$

where a respectively a^- denote the two summands. One can pass from exterior multiplication to Clifford multiplication and get

$$a = -\frac{1}{4}\sum_{i,j}[u_i^-, u_j^-]u_i u_j = -\frac{1}{4}\sum_{i,j} u_i u_j [u_i^-, u_j^-].$$

Similarly,

$$a^- = -\frac{1}{4}\sum_{i,j}[u_i, u_j]u_i^- u_j^- = -\frac{1}{4}\sum_{i,j} u_i^- u_j^- [u_i, u_j].$$

In performing the calculation, one has to deal with expressions like $\sum_{i,j} B([u_i^-, u_j^-], u_i)u_j$. This expression is however zero, since it is an $\mathfrak{l}$-invariant element of $\mathfrak{u}$, and there are no nonzero $\mathfrak{l}$-invariants in $\mathfrak{u}$. For future reference, let us write the decomposition of D we obtained as

$$D = A + A^- + 1 \otimes \nu = A + A^- + 1 \otimes a + 1 \otimes a^- = C + C^-, \tag{10.2}$$

where C, C^- are the following elements of $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})$:

$$C = A + 1 \otimes a \qquad \text{and} \qquad C^- = A^- + 1 \otimes a^-.$$

We are interested in the action of these elements on the $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})$ - module $V \otimes S$, where V is a $\mathfrak{g}$-module, and S is the spin module for the Clifford algebra $\mathrm{Cl}(\mathfrak{s})$. As mentioned above, we use the identification of S with $\bigwedge^{\cdot} \mathfrak{u}$, given explicitly in [15] and [17]. Namely, one can construct S as the left ideal in $C(\mathfrak{s})$ generated by the element $u_\top^- = u_1^- \dots u_n^-$. One then has $S = (\bigwedge^{\cdot} \mathfrak{u})u_\top^-$, which is isomorphic to $\bigwedge^{\cdot} \mathfrak{u}$ as a vector space, and the action of $\mathrm{Cl}(\mathfrak{s})$ is given by left Clifford multiplication. Explicitly, $u \in \mathfrak{u}$ and $u^- \in \mathfrak{u}^-$ act on $Y = Y_1 \wedge \dots \wedge Y_p \in \bigwedge^p \mathfrak{u}$ by

$$u \cdot Y = u \wedge Y;$$

$$u^- \cdot Y = 2\sum_{i=1}^{p}(-1)^{i+1}B(u^-, Y_i)Y_1 \wedge \dots \hat{Y_i} \dots \wedge Y_p.$$

Namely, since $\mathfrak{u}$ and $\mathfrak{u}^-$ are isotropic, the Clifford and wedge products coincide on each of them; in particular, $u^- u_\top^- = 0$.

It is quite clear that the action of C^- on $V \otimes S = V \otimes \bigwedge \mathfrak{u}$ is 2 times the action of the $\mathfrak{u}$-homology differential ∂, which is given by

$$\partial(v \otimes Y_1 \wedge \dots \wedge Y_p) = \sum_{i=1}^{p}(-1)^i Y_i \cdot v \otimes Y_1 \wedge \dots \hat{Y_i} \wedge \dots \wedge Y_p +$$
$$\sum_{1 \le i < j \le p}(-1)^{i+j} v \otimes [Y_i, Y_j] \wedge Y_1 \wedge \dots \wedge \hat{Y_i} \wedge \dots \wedge \hat{Y_j} \wedge \dots Y_p$$

on $V \otimes \bigwedge^p \mathfrak{u}$. Namely, acting on a typical basis element $v \otimes u_{i_1} \wedge \dots \wedge u_{i_k}$, $u_i \otimes u_i^-$ will produce zero if i is different from all i_j, and it will produce $2(-1)^{j+1} u_{i_j} v \otimes u_{i_1} \wedge \dots \hat{u}_{i_j} \dots \wedge u_{i_k}$ if $i = i_j$. Moreover, $[u_i, u_j]u_i^- u_j^-$ will act as zero unless both i and j appear among $i_1, \dots, i_k$, and if i and j do appear, then u_i and u_j get contracted while the commutator gets inserted, exactly as in the formula for ∂.

To understand the action of C, we first make the following identifications:

$$V \otimes \textstyle\bigwedge^p \mathfrak{u} \cong \mathrm{Hom}((\bigwedge^p \mathfrak{u})^*, V) \cong \mathrm{Hom}(\bigwedge^p(\mathfrak{u}^*), V) \cong \mathrm{Hom}(\bigwedge^p \mathfrak{u}^-, V).$$

The last space is the space of p-cochains for the $\mathfrak{u}^-$-cohomology differential d, given by the usual formula

$$(d\omega)(X_0\wedge\ldots\wedge X_p)=\sum_{i=0}^{p}(-1)^i\; X_i\cdot\omega(X_0\wedge\ldots\wedge\hat{X}_i\wedge\ldots\wedge X_p)+ \sum_{0\leq i<j\leq p}(-1)^{i+j}\;\omega([X_i,X_j]\wedge X_0\wedge\ldots\wedge\hat{X}_i\wedge\ldots\wedge\hat{X}_j\wedge\ldots\wedge X_p)$$

The following lemma will help us understand the action of C. The proof is a straightforward calculation, starting from the fact that the identification $\bigwedge^p(\mathfrak{u}^*)=(\bigwedge^p\mathfrak{u})^*$ is given via $(f_1\wedge\cdots\wedge f_p)(X_1\wedge\cdots\wedge X_p)=\det f_i(X_j)$.

Lemma 10.1. *Through the above identifications, the differential* $d:V\otimes\bigwedge^p\mathfrak{u}\to V\otimes\bigwedge^{p+1}\mathfrak{u}$ *is given by*

$$d(v\otimes Y_1\wedge\ldots\wedge Y_p)=\sum_{i=1}^{n}u_i^-\cdot v\otimes u_i\wedge Y_1\wedge\ldots\wedge Y_p + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{p}v\otimes u_i\wedge Y_1\wedge\ldots\wedge[u_i^-,Y_j]_{\mathfrak{u}}\wedge\ldots\wedge Y_p,$$

where $[u_i^-,Y_j]_{\mathfrak{u}}$ *denotes the projection of* $[u_i^-,Y_j]$ *on* $\mathfrak{u}$.

It is now clear that the action of A gives the first (single) sum in the expression for d. On the other hand, the element $1\otimes a=-\frac{1}{4}\otimes\sum_{i,j}u_iu_j[u_i^-,u_j^-]$ transforms $v\otimes Y_1\wedge\cdots\wedge Y_p\in V\otimes\bigwedge^p\mathfrak{u}$ into

$$-\frac{1}{4}v\otimes\sum_{i,j}u_iu_j\;2\sum_{k=1}^{p}(-1)^{k+1}B([u_i^-,u_j^-],Y_k)\,Y_1\wedge\ldots\hat{Y}_k\cdots\wedge Y_p$$
$$=\frac{1}{2}v\otimes\sum_{i,j,k}(-1)^{k+1}B([u_i^-,Y_k],u_j^-)\,u_i\wedge u_j\wedge Y_1\wedge\ldots\hat{Y}_k\cdots\wedge Y_p.$$

Now we sum $\sum_j B([u_i^-,Y_k],u_j^-)\,u_j=[u_i^-,Y_k]_{\mathfrak{u}}$, and after commuting $[u_i^-,Y_k]_{\mathfrak{u}}$ into its proper place, we get the second (double) sum in the expression for d.

So we see that $D=C+C^-$, where C (resp. C^-) acts on $V\otimes S=V\otimes\bigwedge\mathfrak{u}$ as the $\mathfrak{u}^-$-cohomology differential d (resp. up to a factor -2, as the $\mathfrak{u}$-homology differential ∂). To compare the $\mathfrak{l}$-actions under this identification, note that the natural action of $\mathfrak{l}$ on $V\otimes S$ is the tensor product of the restriction of the $\mathfrak{g}$-action on V and the spin action on S. On the other hand, the usual $\mathfrak{l}$ action on $\mathfrak{u}^-$-cohomology and $\mathfrak{u}$-homology is given by the adjoint action on $\bigwedge^{\cdot}\mathfrak{u}^-$ and $\bigwedge^{\cdot}\mathfrak{u}$. Thus, our identification of $V\otimes\bigwedge^{\cdot}\mathfrak{u}$ with $V\otimes S$ is not an $\mathfrak{l}$-isomorphism. However, as was proved in [16], Proposition 3.6, the two actions differ only by a twist with the one dimensional $\mathfrak{l}$-module $Z_{-\rho(\mathfrak{u})}$ of weight $-\rho(\mathfrak{u})$.

Remark 10.2. The fact that C and C^- act on $V\otimes S$ as differentials is by no means an accident, as they in fact square to zero in $U(\mathfrak{g})\otimes\mathrm{Cl}(\mathfrak{s})$. A simple direct way of seeing this uses Kostant's formula [15], Theorem 2.16 for the square of D:

$$D^2=-\Omega_{\mathfrak{g}}\otimes 1+\Delta(\Omega_{\mathfrak{l}})+(\|\rho_{\mathfrak{l}}\|-\|\rho\|)1\otimes 1.$$

It follows from this expression for D^2 that D^2 commutes with all $\mathfrak{l}$-invariant elements of $U(\mathfrak{g})\otimes\mathrm{Cl}(\mathfrak{s})$. In particular, D^2 commutes with $E=-\frac{1}{2}\sum_i 1\otimes u_i^*u_i$. An easy direct calculation shows that

$$[E,C]=C;\qquad\qquad [E,C^-]=-C^-.$$

Thus
$$0 = [E, D^2] = [E, C^2 + CC^- + C^-C + (C^-)^2] = 2C^2 - 2(C^-)^2,$$
hence $C^2 = (C^-)^2$. Combined with $[E, C^2] = 2C^2$ and $[E, (C^-)^2] = -2(C^-)^2$ this implies $C^2 = (C^-)^2 = 0$.

To finish this section, let us note that D is independent not only of the choice of basis (u_i) but also of the choice of $\mathfrak{u} \subset \mathfrak{s}$. On the other hand, E, C and C^- do depend on the choice of $\mathfrak{u}$.

The results of this section suggest there may be a relation between Dirac cohomology for the pair $(\mathfrak{g}, \mathfrak{l})$, $\mathfrak{u}$-homology and $\mathfrak{u}^-$-cohomology. We get results in some particular cases studied in the next sections.

11. Hodge decomposition for $\mathfrak{p}^-$-cohomology (Hermitian symmetric case)

In this section, we study the following special case : suppose we are in the setting of Section 3, but also that the pair $(\mathfrak{g}, \mathfrak{k})$ is hermitian symmetric. Then $\mathfrak{k}$ is also a Levi subalgebra of $\mathfrak{g}$, $\mathfrak{p}$ decomposes as $\mathfrak{p}^+ \oplus \mathfrak{p}^-$ and we are also in the setting of the previous section with $\mathfrak{l} = \mathfrak{k}$, $\mathfrak{u} = \mathfrak{p}^+$ and $\mathfrak{u}^- = \mathfrak{p}^-$.

Suppose furthermore that V is a unitary $(\mathfrak{g}, K)$-module. In this case, d and δ are minus adjoints of each other with respect to a positive definite form on $V \otimes S$, and we can use a variant of standard Hodge decomposition to conclude that the Dirac cohomology, $\mathfrak{p}^-$-cohomology and $\mathfrak{p}^+$-homology are all isomorphic to the space of "harmonics".

More precisely, suppose that V posseses a positive definite hermitian form invariant with respect to the real form $\mathfrak{g}_0$. We denote by $^-$ the conjugation with respect to $\mathfrak{g}_0$.

Note that our parabolic $\mathfrak{k} \oplus \mathfrak{p}^+ = \mathfrak{l} \oplus \mathfrak{u}$ is θ-stable, and that $\mathfrak{p}^- = \mathfrak{u}^- = \bar{\mathfrak{u}}$ is indeed the complex conjugate of $\mathfrak{u}$. Furthermore, we can choose the basis u_i so that after suitable normalization, we can take $u_i^* = \bar{u}_i$.

We consider the positive definite form on $S \cong \bigwedge^{\cdot} \mathfrak{u}$, given by $\langle X, Y \rangle_{pos} := 2B(X, \bar{Y})$ on $\mathfrak{u}$, and extended to all of S in the usual way, using the determinant. Notice that we have $\langle u_i, u_j \rangle_{pos} = 2\delta_{ij}$. This form is $\mathfrak{k}$-invariant.

Lemma 11.1. *With respect to the form $\langle X, Y \rangle_{pos}$ on S, the adjoint of the operator $u_i \in C(\mathfrak{s})$ is $-u_i^*$.*

The adjoint of the operator u_i on V is $-\bar{u}_i$. So the adjoint of $u_i \otimes u_i^*$ on $V \otimes S$ is $u_i^* \otimes u_i$. Here we consider the tensor product hermitian form on $V \otimes S$; this form will again be denoted by $\langle ., . \rangle_{pos}$.

Corollary 11.2. *With respect to the form $\langle ., . \rangle_{pos}$ on $V \otimes S$, the adjoint of C is C^-. Therefore D is self-adjoint on $V \otimes S$.*

Of course, we knew already from Section 3 that D is self-adjoint with respect to the positive definite form $\langle ., . \rangle_{pos}$ on $V \otimes S$, and that in particular in this case the operators D and D^2 have the same kernel on $V \otimes S$, which is also the Dirac cohomology of V. The new ingredient is the assertion about C and C^-. Recall that these induce, up to a scalar factor, the differentials d and δ of $\mathfrak{u}^-$-cohomology and $\mathfrak{u}$-homology.

It is now easy to obtain a variant of the usual Hodge decomposition. The following arguments are well known; see e.g. [31], Scholium 9.4.4. We first have

Lemma 11.3. *(a)* $\ker D = \ker d \cap \ker \delta$;

(b) $\operatorname{Im} \delta$ *is orthogonal to* $\ker d$ *and* $\operatorname{Im} d$ *is orthogonal to* $\ker \delta$.

Proof. (a) Since $D = d + \delta$, it is clear that $\ker d \cap \ker \delta$ is contained in $\ker D$. On the other hand, if $Dx = 0$, then $dx = -\delta x$, hence $\delta dx = -\delta^2 x = 0$. So $\langle dx, dx\rangle_{pos} = \langle -\delta dx, x\rangle_{pos} = 0$, hence $dx = 0$. Now $Dx = 0$ implies that also $\delta x = 0$.

(b) is obvious since d and δ are minus adjoint to each other.

Combining Corollary 4.6, Lemma 4.7 and the fact $\ker D = \ker D^2$, we get

Theorem 11.4. *Let $(\mathfrak{g}, \mathfrak{k})$ be a hermitian symmetric pair and set $\mathfrak{l} = \mathfrak{k}$ and $\mathfrak{u} = \mathfrak{p}^+$. Let V be an irreducible unitary $(\mathfrak{g}, K)$-module. Then:*

(a) $V \otimes S = \ker D \oplus \operatorname{Im} d \oplus \operatorname{Im} \delta$;

(b) $\ker d = \ker D \oplus \operatorname{Im} d$;

(c) $\ker \delta = \ker D \oplus \operatorname{Im} \delta$.

In particular, The Dirac cohomology of V is equal to $\mathfrak{p}^-$-cohomology and to $\mathfrak{p}^+$-homology, up to modular twists:

$$\ker D \cong H^{\cdot}(\mathfrak{p}^-, V) \otimes Z_{\rho(\mathfrak{p}^-)} \cong H_{\cdot}(\mathfrak{p}^+, V) \otimes Z_{\rho(\mathfrak{p}^-)}.$$

More precisely, (up to modular twists) the Dirac cohomology $\ker D$ is the space of harmonic representatives for both $\mathfrak{p}^-$-cohomology and $\mathfrak{p}^+$-homology.

12. Dirac cohomology of finite dimensional modules

Let $\mathfrak{g}$ be a complex finite dimensional semi-simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ a Borel subalgebra containing $\mathfrak{h}$, and $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$ the opposite Borel subalgebra. Set $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{n}^-$. Thus $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$. This is a special case of the situation considered in (10.1). Recall that the spin module S for the Clifford algebra $\mathrm{Cl}(\mathfrak{s})$ has been identified as a vector space with $\bigwedge \mathfrak{n}$.

Let $\mathfrak{u}$ be a compact form of $\mathfrak{g}$, and let U be the corresponding compact adjoint group. Let us denote by $X \mapsto \bar{X}$ the complex conjugation in $\mathfrak{g}$ with respect to the real form $\mathfrak{u}$. Then

$$\langle X, Y\rangle = -2B(X, \bar{Y}), \quad (X, Y \in \mathfrak{g})$$

defines a definite positive U-invariant hermitian form on $\mathfrak{g}$. This hermitian form restricts to $\mathfrak{n}$, and can be extended to $\bigwedge \mathfrak{n}$ in the usual way, using the determinant.

Lemma 12.1. *The adjoint of the operator $X \in \mathfrak{s} \subset \mathrm{Cl}(\mathfrak{s})$ acting on $S \simeq \bigwedge \mathfrak{n}$ with respect to the hermitian form defined above is $\bar{X}$.*

This is easily checked by an explicit calculation.

Assume that $\mathfrak{h}$ is the complexification of a Cartan subalgebra of $\mathfrak{u}$, *ie.*

$$\mathfrak{h} = \mathfrak{h} \cap \mathfrak{u} \oplus \mathfrak{h} \cap i\mathfrak{u}.$$

In this case, all the roots in $R(\mathfrak{h}, \mathfrak{g})$ are imaginary, or equivalently

$$\mathfrak{n}^- = \bar{\mathfrak{n}}.$$

We have then

Lemma 12.2. *The cubic part $v = a + a^-$ of the Dirac operator $D(\mathfrak{g}, \mathfrak{h})$ is anti-self adjoint with respect to the form $\langle ., .\rangle$ on S. More precisely a is adjoint to $-a^-$.*

This follows from the previous lemma and the formulas for a and a^-.

Let V be a finite dimensional $\mathfrak{g}$-module. Then V can be endowed with a definite positive U-invariant hermitian form. Let us call such a form admissible. For such a form, consider the tensor product form on $V \otimes S$ obtained by tensoring with the form on S introduced above, that we still denote by $\langle ., .\rangle$.

Lemma 12.3. *The adjoint of $X \in \mathfrak{g}$ acting on V with respect to an admissible form is $-\bar{X}$. The adjoint of the operator A acting on $V \otimes S$ is $-A^-$. Thus the Dirac operator*

$$D = D(\mathfrak{g}, \mathfrak{h}) = A + A^- + 1 \otimes a + 1 \otimes a^-$$

is anti-self-adjoint with respect to $\langle ., . \rangle$.

Corollary 12.4. *Let V be a finite dimensional representation of $\mathfrak{g}$. The operator $D = D(\mathfrak{g}, \mathfrak{h})$ acting on $V \otimes S$ is semi-simple. In particular* $\ker D = \ker D^2$ *and* $\operatorname{Im} D \cap \ker D = 0$. *Thus*

$$H_D(\mathfrak{g}, \mathfrak{h}; V) = \ker D = \ker D^2.$$

The following result is an analog of Kostant theorem [14]. It computes Dirac cohomology of an irreducible finite dimensional representation.

Theorem 12.5. *Let V be the irreducible finite dimensional representation of V with highest weight μ. Then, as a $\mathfrak{h}$-module*

$$H_D(\mathfrak{g}, \mathfrak{h}; V) = \bigoplus_{w \in W} \mathbb{C}_{w \cdot (\mu + \rho)}$$

As for Kostant's theorem [14], the proof is in two steps : first the weights $\mathbb{C}_{w \cdot (\mu+\rho)}$ occur in $V \otimes S$ with multiplicity one and the corresponding weight spaces are in the kernel of D^2 (this can be checked directly from the formula for D^2). Secondly, no other weights can occur in the Dirac cohomology because of proposition 9.10.

This result can be sharpened. As a representation of $\mathfrak{h}$, the spinor module S can be splitted in

$$S = S^+ \oplus S^-$$

where, in the identification $S \simeq \bigwedge \mathfrak{n}$, S^+ (resp. S^-) corresponds to the even (resp. odd) part in $\bigwedge \mathfrak{n}$.

Since D is an odd element in $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})$ its action on $V \otimes S$, for any $\mathfrak{g}$-module V exchanges $V \otimes S^+$ and $V \otimes S^-$:

$$D : V \otimes S^+ \leftrightarrow V \otimes S^-$$

The **index of the Dirac operator** acting on $V \otimes S$ is the virtual representation

$$V \otimes S^+ - V \otimes S^-$$

of $\mathfrak{h}$.

Theorem 12.6. *Let V be the irreducible finite dimensional representation of V with highest weight μ. Then, as virtual representations of $\mathfrak{h}$*

$$V \otimes S^+ - V \otimes S^- = \sum_{w \in W} (-1)^{l(w)} \mathbb{C}_{w \cdot (\mu+\rho)}.$$

This is easily obtained from the previous theorem and some knowledge of the weight structure of $S = S^+ \oplus S^-$.

Corollary 12.7. *(Weyl character formula) The character of the finite dimensional representation of V with highest weight μ is given by*

$$\mathrm{ch}(V) = \frac{\sum_{w \in W} (-1)^{l(w)} \mathbb{C}_{w \cdot (\mu+\rho)}}{\mathrm{ch}(S^+ - S^-)}.$$

The character $\mathrm{ch}(S^+ - S^-) = \sum_{w \in W} (-1)^{l(w)} \mathbb{C}_{w \cdot \rho}$ *is the usual Weyl denominator.*

All these results are proved in much more generality in [17]. Namely, Kostant considers there the general situation

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$$

of the begining of section 9. We have restricted here our attention to the case $\mathfrak{r} = \mathfrak{h}$.

13. Realization of finite dimensional modules

Let us now consider the problem of realizing irreducible finite dimensional representations of $\mathfrak{g}$. As above, with assume that $\mathfrak{u}$ is a compact real form of $\mathfrak{g}$ and that U is a connected, simply-connected compact semi-simple group with Lie algebra $\mathfrak{u}$. Let T be a maximal torus of U and let $\mathfrak{h}$ be the complexification of $\mathfrak{t} = \mathrm{Lie}(T)$. Let us start with the one dimensional representation $\mathbb{C}_\lambda$ of T, $\lambda \in i\mathfrak{t}^*$. Let $\mathcal{L}_\lambda$ (resp. $\mathcal{L}_\lambda^{\pm}$) be the vector bundle on U/T corresponding to the representation $\mathbb{C}_\lambda \otimes S$ of T (resp. $\mathbb{C}_\lambda \otimes S^{\pm}$) and $\mathcal{C}_c^\infty(U/T, \mathcal{L}_\lambda)$ (resp. $\mathcal{C}_c^\infty(U/T, \mathcal{L}_\lambda^{\pm})$) the space of smooth compactly supported sections of these vector bundles . Writing

$$\mathcal{C}_c^\infty(U/T, \mathcal{L}_\lambda) \simeq \mathcal{C}^\infty(U) \otimes_T (\mathbb{C}_\lambda \otimes S)$$

one see that $\mathcal{C}_c^\infty(U/T, \mathcal{L}_\lambda)$ becomes a $\mathfrak{g}$-module, $X \in \mathfrak{g}$ acting by (right) differentiation on the first factor. Thus $\mathcal{C}_c^\infty(U/T, \mathcal{L}_\lambda)$ is a $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})$-module. In particular, the Dirac operator $D = D(\mathfrak{g}, \mathfrak{h})$ acts on this space. Extend this to

$$\text{Ł}^2(U/T, \mathcal{L}_\lambda) \simeq L^2(U) \otimes_T (\mathbb{C}_\lambda \otimes S)$$

Let us rewrite

$$L^2(U/T, \mathcal{L}_\lambda) \simeq L^2(U) \otimes_T (\mathbb{C}_\lambda \otimes S) \simeq \mathrm{Hom}_T(\mathbb{C}_{-\lambda}, L^2(U) \otimes S).$$

Now, the action of D on the right-hand side of the above equality is only on the range $L^2(U) \otimes S$. By Peter-Weyl theorem, one has

$$L^2(U) = \bigoplus_{\nu \in \mathcal{P}^+} V_\nu \otimes V_\nu^*$$

Again, D is anti-self-adjoint. It follows that

$$\ker D = \bigoplus_{\nu \in \mathcal{P}^+} V_\nu \otimes \ker\{D \text{ on } \mathrm{Hom}_T(\mathbb{C}_{-\lambda}, V_\nu^* \otimes S).\}$$

The contragredient representation V_ν^* has lowest weight $-\nu$. Thus theorem 12.5 implies that $\ker D \neq 0$ if and only if $-\nu - \rho$ is conjugate to $-\lambda$, *ie.* $\nu + \rho$ is conjugate to λ. In fact, we obtain :

Theorem 13.1. *([19]) One has* $\ker D = V_{w\cdot\lambda+\rho}$ *if there exists* $w \in W$ *such that* $w \cdot \lambda - \rho$ *is dominant, and* $\ker D = 0$ *otherwise.*

There is a more precise version of this result involving the index of the Dirac operator, as in theorem 12.6 above. In this case, the action of D on $\mathcal{C}_c^\infty(U/T, \mathcal{L}_\lambda)$ exchanges the two factors $\mathcal{C}_c^\infty(U/T, \mathcal{L}_\lambda^{\pm})$ (and thus, with obvious notation $L^2(U/T, \mathcal{L}_\lambda^{\pm})$.

Theorem 13.2. *([19]) One has* $\mathrm{Index}(D) = (-1)^{l(w)} V_{w\cdot\lambda-\rho}$ *if there exists* $w \in W$ *such that* $w \cdot \lambda + \rho$ *is dominant, and* $\mathrm{Index}(D) = 0$ *otherwise.*

This result provides an explicit realization of irreducible finite dimensional representation of $\mathfrak{g}$. It is essentially equivalent to the Borel-Weil-Bott theorem, Dirac operators and Dirac cohomology playing the role of $\mathfrak{n}$-cohomology. In fact, Dirac cohomology and $\mathfrak{n}$-cohomology (or $\mathfrak{n}$-homology) coincide up to a shift for finite dimensional modules. Indeed we have seen that the differential for the Lie algebra cohomology with respect to

$\mathfrak{n}^-$ is given by the action of C on $V \otimes S$, and differential for the Lie algebra homology with respect to $\mathfrak{n}$ is given by the action of C^-. Furthermore, $V \otimes S$ admits a hermitian product and C, C^- are adjoint to each other for the Hilbert space structure. Thus Dirac cohomology, as we have already noticed, is simply the kernel of D, and also the kernel of D^2. Since

$$CC^- + C^-C = D^2$$

A Hogde theorem type of argument shows that $\ker C = \operatorname{Im} C \oplus \ker D^2$ and $\ker C^- = \operatorname{Im} C^- \oplus \ker D^2$.

Isimeqn some cases, similar arguments work for unitary Harish-Chandra modules. One needs self-adjointness of the Dirac oparator in order to apply the same kind of techniques, and this limits the range of applicability ([11]).

14. Realization of discrete series

The setting is the one of section 3.1. Assume that G is semi-simple and that G and K have equal rank. We know from Harish-Chandra that the latter condition is necessary for G to have discrete series.

The realization of discrete series in [24] and [1] is somehow similar to the one above for finite dimensional representations. One start with G/K rather than U/T and with an irreducible finite dimensional representation of K. Tensoring with the spin module S, we form the corresponding vector bundle over G/K. In the argument, we replace the use of Peter-Weyl theorem by the L^2-index theorem. (We have reverted the historical order : the construction of finite dimensional representation is technically much simplier, but requires the cubic Dirac operator which was introduced by Kostant much later.)

15. Main properties of Dirac cohomology and alternate definitions

Suppose that we are again in the most general setting of Section 9 but with $\mathfrak{g}$ semisimple and $\mathfrak{r}$ of equal rank. Then the spin mudule splits

$$S = S^+ \oplus S^-$$

and accordingly, we have odd and even part in the Dirac cohomology of a $\mathfrak{g}$-module V

$$H_D(V) = H_D^+(V) \oplus = H_D^-(V)$$

Proposition 15.1. *Suppose that*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

is an exact sequence of $\mathfrak{g}$-modules having infinitesimal character. Then there is a natural six-terms exact sequence

$$\begin{tikzcd} H_D^+(U) \arrow[r] & H_D^+(V) \arrow[r] & H_D^+(W) \arrow[d] \\ H_D^-(W) \arrow[u] & H_D^-(V) \arrow[l] & H_D^-(U) \arrow[l] \end{tikzcd}$$

The fact that this holds only for modules with infinitesimal character have led Pandzic and Somberg (preprint) to propose an alternate definition of Dirac cohomology where this property holds without restriction. By an alternate definition of Dirac cohomology, we mean a new cohomology $H_D^{new}(V)$ wich coincide with the previous one in the most interesting cases (for instance unitary $(\mathfrak{g}, K)$-modules, or finite dimensional representations, *ie.* when the Dirac operator acts semisimply), and for which the Hunag-Pandzic theorem still holds. The goal being that this new theory has better fnctorial properties.

Another altenate definition is given in [23], the goal there being that Dirac cohomology becomes a functor with a left (or right) adjoint, which will then deserves the name of Dirac induction. It is close to the induction functor defined in Lemma 3.5.

16. Proofs

16.1. **Proof of proposition 3.2.** Here the $\mathfrak{g}$-action on $\mathrm{Hom}_{C(\mathfrak{p})}(S,M)$ is on M only, while the K action descends from the $\widetilde{K}$-action given by

$$(k\cdot f)(s)=k(f(k^{-1}\cdot s)),\qquad k\in\widetilde{K},\ f\in\mathrm{Hom}_{C(\mathfrak{p})}(S,M),\ s\in S.$$

We claim that this is the inverse of $X\mapsto X\otimes S$. In fact, since S is the only simple $C(\mathfrak{p})$-module and since the category of $C(\mathfrak{p})$-modules is semisimple, for every $C(\mathfrak{p})$-module M there is an isomorphism of $C(\mathfrak{p})$-modules

$$M\cong\mathrm{Hom}_{C(\mathfrak{p})}(S,M)\otimes S,$$

given from right to left by the evaluation map. This isomorphism is easily seen to respect the $(\mathcal{A},\widetilde{K})$-action.

Likewise, since $\mathrm{Hom}_{C(\mathfrak{p})}(S,S)\cong\mathbb{C}$ by Schur's Lemma, we have a $(\mathfrak{g},K)$-isomorphism

$$\mathrm{Hom}_{C(\mathfrak{p})}(S,X\otimes S)\cong\mathrm{Hom}_{C(\mathfrak{p})}(S,S)\otimes X\cong X$$

for any $(\mathfrak{g},K)$-module X.

16.2. **Proof of proposition 3.9.** We begin by proving a generalization of the well-known Schur orthogonality relations. Let K be a compact Lie group. Let us fix a Haar measure μ_K on K. If f is integrable on K, we will simply write $\int_K f(k)\,dk$ for $\int_K f(k)\,d\mu_K(k)$. Let (π_1,V_1), (π_2,V_2) be two irreducible finite dimensional representations of K. Let $\langle .\,,.\rangle_1$ and $\langle .\,,.\rangle_2$ be some invariant Hermitian products respectively on V_1 and V_2.

Proposition 16.1. *Let f be a smooth function on K. Then*

$$\int_K f(k)\ \langle\pi(k)\cdot v_1,v_1^*\rangle\ \langle\pi(k)^{-1}\cdot v_2,v_2^*\rangle\ dk=\frac{\int_K f(k)\,dk}{\dim(V)}\langle v_1,v_2^*\rangle\langle v_2,v_1^*\rangle,\tag{16.1}$$

for all v_1,v_2 in V and for all v_1^*,v_2^* in V^*.

To prove this, one uses the fact that $V\otimes V^*$ is an irreducible representation of $K\times K$, with contragredient $V^*\otimes V$ and the following lemma:

Lemma 16.2. *Let G be a compact Lie group, and W be an irreducible finite dimensional representation of G. If $B:W\times W^*\to\mathbb{C}$ is a G-invariant bilinear form, then there exists a constant $c\in\mathbb{C}$ such that*

$$B(w,w^*)=c\,\langle w,w^*\rangle,\qquad (w\in W),\ (w^*\in W^*).$$

In particular, if B is nonzero, then B is non-degenerate.

The proof is an easy consequence of Schur lemma (see [27], Prop II.1.9).

To prove Proposition 16.1, we apply Lemma 16.2 for $G=K\times K$ and $W=V\otimes V^*$. The left-hand side of (16.1) defines a $K\times K$-invariant bilinear form B on $(V\otimes V^*)\times(V^*\otimes V)$. Thus, Lemma 16.2 gives a constant $c\in\mathbb{C}$ such that for all $v_1,v_2\in V$ and $v_1^*,v_2^*\in V^*$,

$$\int_K f(k)\ \langle\pi(k)\cdot v_1,v_1^*\rangle\langle\pi(k)^{-1}\cdot v_2,v_2^*\rangle\ dk=c\ \langle v_1,v_2^*\rangle\langle v_2,v_1^*\rangle.$$

It remains to evaluate the value of c. To do this, we fix a basis $\{z_i\}_i$ of V, with dual basis $\{z_i^*\}_i$. Then we write the above formula for $v_1^*=z_i^*$, and $v_2=z_i$, and add up the results over i. The details are left to the reader. $\square$

Proof. For $a \in \mathcal{A}$, $T \in R(K)$, we have

$$(a \otimes T) \cdot (1 \otimes \chi_\gamma) = a \otimes T * \chi_\gamma.$$

But $T * \chi_\gamma = \chi_\gamma * T$ is the projection of T on $R(K)_\gamma$ (see [13], Proposition I.24, I.30 and Equation (1.37)). Furthermore ([13], Proposition I.39)

$$R(K)_\gamma = R(K)_{l,\gamma} = R(K)_{r,\tilde{\gamma}} \cong F_\gamma \otimes_{\mathbb{C}} F_\gamma^* \cong \mathrm{End}(F_\gamma),$$

the isomorphism between $F_\gamma \otimes_{\mathbb{C}} F_\gamma^*$ and $R(K)_\gamma$ being given by

$$v \otimes v^* \mapsto \langle v, \tilde{\gamma}(.) \cdot v^* \rangle \, \mu_K = \langle \gamma(.)^{-1} \cdot v, v^* \rangle \, \mu_K.$$

Thus we see that

$$R(\mathcal{A}, K) \cdot (1 \otimes \chi_\gamma) \cong \mathcal{A} \otimes_{U(\mathfrak{k})} (F_\gamma \otimes_{\mathbb{C}} F_\gamma^*)$$

To compute $(1 \otimes \chi_\gamma) \cdot (a \otimes T) \cdot (1 \otimes \chi_\gamma)$, we may now assume that $T = T * \chi_\gamma = \chi_\gamma * T$ is of the form

$$T = \langle v, \tilde{\gamma}(.) \cdot v^* \rangle = \langle \gamma(.)^{-1} \cdot v, v^* \rangle \, \mu_K,$$

and we need to evaluate :

$$(1 \otimes \chi_\gamma) \cdot (a \otimes T).$$

According to [13], Proposition I.104, one may compute this product by introducing a basis of the (finite-dimensional) space generated by $a \in \mathcal{A}$ as a representation of K. Let $\{a_j\}_j$ be such a basis, with dual basis $\{a_j^*\}_j$. Then

$$(1 \otimes \chi_\gamma) \cdot (a \otimes T) = \sum_j a_j \otimes (\langle \mathrm{Ad}(.)a, a_j^* \rangle \chi_\gamma) * T.$$

Let us give another expression for the element

$$(\langle \mathrm{Ad}(.)a, a_i^* \rangle \chi_\gamma) * T$$

of $R(K)$. As a matter of notation, recall that $\langle \mathrm{Ad}(.)a, a_j^* \rangle \chi_\gamma$ is the result of the multiplication of the distribution χ_γ in $R(K)$ by the smooth function $\langle \mathrm{Ad}(.)a, a_i^* \rangle$ on K, an element in $R(K)$. For a test function $\phi \in \mathcal{C}^\infty(K)$,

$$
\begin{aligned}
&\langle\ (\langle \mathrm{Ad}(.)a, a_j^*\rangle \chi_\gamma) * T, \phi\rangle = \int_{K\times K} \phi(xy)\, \langle \mathrm{Ad}(x)a, a_j^*\rangle\ d\chi_\gamma(x) dT(y) \\
&= \int_{K\times K} \phi(xy)\, \langle \mathrm{Ad}(x)a, a_j^*\rangle \frac{\dim(V_\gamma)}{\mathrm{vol}\,(K)} \Theta_{\tilde\gamma}(x) \langle \tilde\gamma(y)\cdot v^*, v\rangle\, dxdy \\
&= \int_K \int_K \phi(y) \langle \mathrm{Ad}(x)a, a_j^*\rangle \frac{\dim(V_\gamma)}{\mathrm{vol}\,(K)} \left(\sum_i \langle \tilde\gamma(x)\cdot v_i^*, v_i\rangle \right) \langle \tilde\gamma(x^{-1}y)\cdot v^*, v\rangle\, dxdy \\
&= \int_K \phi(y) \sum_i \frac{\dim(V_\gamma)}{\mathrm{vol}\,(K)} \left(\int_K \langle \mathrm{Ad}(x)a, a_j^*\rangle \langle \tilde\gamma(x) v_i^*, v_i\rangle \langle \tilde\gamma(x)^{-1}\cdot(\tilde\gamma(y)\cdot v^*), v\rangle\, dx \right) dy \\
&= \int_K \phi(y) \sum_i \left(\int_K \langle \mathrm{Ad}(x)a, a_j^*\rangle\, dx \right) \frac{1}{\mathrm{vol}\,(K)} \langle v_i^*, v\rangle \langle \tilde\gamma(y)\cdot v^*, v_i\rangle dy \\
&= \left(\int_K \langle \mathrm{Ad}(x)a, a_j^*\rangle\, dx \right) \frac{1}{\mathrm{vol}\,(K)} \int_K \phi(y) \langle\ \sum_i \langle \tilde\gamma(y)\cdot v^*, v_i\rangle v_i^*, v\rangle\, dy \\
&= \left(\int_K \langle \mathrm{Ad}(x)a, a_j^*\rangle\, dx \right) \frac{1}{\mathrm{vol}\,(K)} \int_K \phi(y) \langle \tilde\gamma(y)\cdot v^*, v\rangle dy \\
&= \left(\int_K \langle \mathrm{Ad}(x)a, a_j^*\rangle\, dx \right) \frac{1}{\mathrm{vol}\,(K)}\ \langle T, \phi\rangle
\end{aligned}
$$

In the third line, we have written $\Theta_{\tilde\gamma}$ as a trace, choosing a basis $\{v_i\}_i$ of F_γ with dual basis $\{v_i^*\}_i$, and we also made a change of variable $y \mapsto x^{-1}y$. In the fourth line, we arrange the terms so that an expression like the left-hand side of (16.1) becomes apparent. Then, we simplify the expression using (16.1). The rest of the computation is clear.

Thus we obtain,

$$
(\langle \mathrm{Ad}(.)a, a_i^*\rangle \chi_\gamma)\, T = \left(\int_K \langle \mathrm{Ad}(x)a, a_i^*\rangle\, dx \right) \frac{1}{\mathrm{vol}\,(K)}\ T
$$

and

$$
\begin{aligned}
(1\otimes \chi_\gamma)\cdot (a\otimes T) &= \sum_i a_i \otimes \left(\int_K \langle \mathrm{Ad}(x)a, a_i^*\rangle\, dx \right) \frac{1}{\mathrm{vol}\,(K)}\ T \\
&= \frac{1}{\mathrm{vol}\,(K)} \left(\int_K \sum_i \langle \mathrm{Ad}(x)a, a_i^*\rangle a_i dx \right) \otimes T \\
= \frac{1}{\mathrm{vol}\,(K)} \left(\int_K \mathrm{Ad}(x)a\, dx \right) \otimes T
\end{aligned}
$$

But $a \mapsto \dfrac{1}{\mathrm{vol}\,(K)} \left(\int_K \mathrm{Ad}(x)a\, dx \right)$ is the projection operator from $\mathcal{A}$ to $\mathcal{A}^K$. The assertion in the theorem is now clear. □

16.3. **Proof of Proposition 9.3.** Let $(X_i)_{i=1,\dots,s}$ be an orthonormal basis of $\mathfrak{s}$, and $(X_j)_{j=s+1,\dots,s+r}$ be an orthonormal basis of $\mathfrak{m}_\mathfrak{r}$, so that $(X_k)_{k=1,\dots s+r}$ is an orthonormal basis of $\mathfrak{m}$. The expression for $D(\mathfrak{g},\mathfrak{l})$ is

$$
D(\mathfrak{g},\mathfrak{l}) = \sum_{i=1,\dots,s+r} X_i \otimes X_i + 1 \otimes \nu_\mathfrak{m}
$$

where

$$\nu_{\mathfrak{m}} = \frac{1}{2} \sum_{1\le i<j<k\le s+r} B([X_i, X_j], X_k)\; X_i X_j X_k.$$

We can write this last term as the sum of the following terms :

$$\nu_{\mathfrak{s}} = \frac{1}{2} \sum_{1\le i<j<k\le s} B([X_i, X_j], X_k)\; X_i X_j X_k. \tag{16.2}$$

$$\frac{1}{2} \sum_{1\le i\le s<j<k\le s+r} B([X_i, X_j], X_k)\; X_i X_j X_k, \tag{16.3}$$

$$\frac{1}{2} \sum_{1\le i<j\le s<k\le s+r} B([X_i, X_j], X_k)\; X_i X_j X_k, \tag{16.4}$$

$$\nu_{\mathfrak{m}_{\mathfrak{r}}} = \frac{1}{2} \sum_{s+1\le i<j\le s<k\le s+r} B([X_i, X_j], X_k)\; X_i X_j X_k, \tag{16.5}$$

Notice that (16.3) is zero because $\mathfrak{r}$ is a subalgebra, so the terms appearing there are

$$B([X_i, X_j], X_k) = B(X_i, [X_j, X_k]) = 0$$

since $[X_j, X_k] \in \mathfrak{r}$ and $X_i \in \mathfrak{s}$ is orthogonal to $\mathfrak{r}$.

Thus, we may write $D(\mathfrak{g}, \mathfrak{l})$ as the sum of

$$D(\mathfrak{g}, \mathfrak{r}) = \sum_{i=1,\ldots,s} X_i \otimes X_i + 1 \otimes \nu_{\mathfrak{s}}$$

and

$$\sum_{i=s+1,\ldots,s+r} X_i \otimes X_i + 1 \otimes \nu_{\mathfrak{m}_{\mathfrak{r}}} + 1 \otimes \frac{1}{2} \sum_{1\le i<j\le s<k\le s+r} B([X_i, X_j], X_k)\; X_i X_j X_k.$$

Let us show now that this last term equals $\Delta(D(\mathfrak{r}, \mathfrak{l}))$, $\Delta)$ being the diagonal embbeding (9.8). Indeed, in $U(\mathfrak{g}) \otimes (\mathcal{C}(\mathfrak{s}) \bar{\otimes} \mathcal{C}(\mathfrak{m}_{\mathfrak{r}}) \simeq U(\mathfrak{g}) \otimes \mathcal{C}(\mathfrak{m})$ one has

$$\begin{aligned}
\Delta(D(\mathfrak{r}, \mathfrak{l})) &= \sum_{i=s+1,\ldots,s+r} X_i \otimes (1 \bar{\otimes} X_i) + 1 \otimes (\alpha(X_i) \bar{\otimes} X_i) + 1 \otimes (1 \bar{\otimes} \nu_{\mathfrak{m}_{\mathfrak{r}}}) \\
&= \sum_{i=s+1,\ldots,s+r} X_i \otimes (1 \bar{\otimes} X_i) + \sum_{i=s+1,\ldots,s+r} 1 \otimes \left(\frac{1}{4} \sum_{1\le j,k\le s} B([X_j, X_k], X_i)\; X_j X_k \right) \bar{\otimes} X_i \\
&\quad + 1 \otimes (1 \bar{\otimes} \nu_{\mathfrak{m}_{\mathfrak{r}}}) \\
&= \sum_{i=s+1,\ldots,s+r} X_i \otimes X_i + 1 \otimes \nu_{\mathfrak{m}_{\mathfrak{r}}} + 1 \otimes \frac{1}{2} \sum_{1\le i<j\le s<k\le s+r} B([X_i, X_j], X_k)\; X_i X_j X_k.
\end{aligned}$$

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