ENDOSCOPY FOR REAL REDUCTIVE GROUPS

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0. Introduction

The purpose of this paper is to review Diana Shelstad's results on endoscopy for real reductive groups ([Sh1], [Sh2], [Sh3], [Sh4]). This impressive body of work is a part of the general Langlands program in automorphic representation theory, where it gives the first examples of Langlands' (local) functoriality principle. It establishes a transfer of orbital integrals between a real reductive group $G(\mathbb{R})$ and one of its endoscopic groups $H(\mathbb{R})$. With hindsight, and somewhat artificially, it is easiest for us to break down the difficulties Shelstad encountered into two parts. Her first problem was to find the correct definition of the transfer factors; the second problem was to show that they indeed give the transfer of orbital integrals. In her series of papers on the subject, these two problems are intertwined, since the second one obviously gives necessary conditions for the first. Moreover, the definitions of the transfer factors grew more elaborate and conceptual as increasingly general situations were considered. Finally, Shelstad's work in the real case served as a guide to the (perhaps definitive) treament of transfer factors for all local fields in [LS], and [KS] for the twisted case. In [LS2], a somewhat indirect argument shows that, up to a global sign, the transfer factors of [LS], when specialized to the real case, are the same as those in [Sh4].

For anybody who wants to study the subject today in some depth, the natural path would be first to absorb the definitions in [LS], specialize them to the real case, and then use [Sh1], [Sh2], [Sh3] and [Sh4] to establish the transfer of orbital integrals. This requires substantial effort, and it is this effort we want to reduce as much as possible. Our point of view is to take [LS], [LS2] and [KS] as the foundations of the theory of endoscopy, and this will be our starting point. We explain in some detail how the definitions there specialize to the real case. Then we reduce the proof of the transfer of orbital integrals to a set of properties of the transfer factors (Proposition 4.4 and Lemma 4.6). Some of these properties are already established in [LS], [LS2] or [KS], and we don't repeat the proofs here. Other properties are extracted

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from Shelstad's articles. We hope that the treament given here is simpler. It follows the lines of [R], where the transfer of orbital integrals is established for real groups in the context of *twisted* endoscopy.

In the general setting of twisted endoscopy, as formulated in [KS] and specialized to real groups, one starts with a datum (G, θ, ω) where G is an algebraic reductive group defined over $\mathbb{R}$, θ is an algebraic $\mathbb{R}$-automorphism of G, and ω a quasicharacter of $G(\mathbb{R})$, the group of real points of G. Shelstad's series of papers, culminating in [Sh4], deals with the case $\theta = 1$, $\omega = 1$. In [R], we deal with the case $\omega = 1$ and θ is of finite order. In this paper, we take $\theta = 1$, but we allow ω to be non-trivial. The technical complications it induces are minor (unlike the case where θ is non-trivial). It could be argued that the results of [R] and those here could be combined to give the general case. But our motivation here is to simplify the existing litterature, so we won't go in this direction. Another difference with Shelstad's approach is that we work with the space of compactly supported smooth functions rather than the Schwartz space. For this, we use Bouaziz' results on orbital integrals of such functions ([B1]).

For a general discussion of endoscopy and the Langlands functoriality principle, we refer to [LTF], [Kn], [G].

Let us now describe in more detail the contents of the paper. The first section introduces the basic notions : reductive algebraic groups defined over $\mathbb{R}$, their L-groups, the Langlands classification of irreducible representations, stable conjugacy and finally the definition of endoscopic data. The second section describes the correspondences of points (or rather conjugacy classes) between an algebraic group G defined over $\mathbb{R}$ and one of its endoscopic groups H. The main notion here is that of Cartan subgroups of $H(\mathbb{R})$ originating in $G(\mathbb{R})$. When a Cartan subgroup $T_H(\mathbb{R})$ of $H(\mathbb{R})$ originates in $G(\mathbb{R})$, there is an isomorphism $T_H(\mathbb{R}) \simeq T_G(\mathbb{R})$ with a Cartan subgroup of $G(\mathbb{R})$, compatible with the correspondence of regular conjugacy classes. Some properties are established : if $T_H(\mathbb{R})$ originates in $G(\mathbb{R})$, all Cartan subgroups in $H(\mathbb{R})$ smaller than $T_H(\mathbb{R})$ in the Hirai order also originate in $G(\mathbb{R})$ (Lemma 2.10), and for a Cartan subgroup immediately bigger than $T_H(\mathbb{R})$ in the Hirai order, a necessary and sufficient condition is given in terms of Cayley transforms (Lemma 2.12).

In the third section, we recall properties of orbital integrals. The space of orbital integrals on $G(\mathbb{R})$ is defined. It is a locally convex topological vector space, and its dual is isomorphic to the space of (conjugation-) invariant distributions on $G(\mathbb{R})$. It can be viewed as the space of smooth compactly supported functions on the "variety" of conjugacy classes in $G(\mathbb{R})$. Cartan subgroups are transversal to regular orbits and regular elements are dense in $G(\mathbb{R})$. This explain why orbital integrals are characterized by the properties of their restrictions to Cartan subgroups ([B1]). Stable orbital integrals on $H(\mathbb{R})$ are also introduced. The main theorem of the paper is then stated (Theorem 3.6) : starting from an orbital integral on $G(\mathbb{R})$, one can define a function on a dense open set of regular elements in $H(\mathbb{R})$, by a formula involving the so-called transfer factors. This formula can be extended smoothly to all regular elements, and it gives a stable orbital integral on $H(\mathbb{R})$. In the fourth section, we reduce the proof of the main theorem to a set of properties of the transfer factors. This is purely formal, as the transfer factors are not even defined at this point. This is done in the next section, where the definitions of [LS] are specialized to the real case. In the last section, we prove the desired properties of the transfer factors. In an appendix, we recall the Langlands correspondence and the Tate-Nakayama duality for real tori, since these are ingredients of the definition of transfer factors.

1. Notation and basic definitions

1.1. **Notation for group actions.** Let A be a group, and X a set on which A acts. For all subset B in X, set:

$$Z(A,B) = \{a \in A \,|\, \forall b \in B,\ a\cdot b = b\}$$
$$N(A,B) = \{a \in A \,|\, \forall b \in B,\ a\cdot b \in B\}.$$

Different actions of a group on itself will be considered, so unless otherwise stated, the above notations will refer to the usual action by conjugation. The inner automorphism of A given by an element $a \in A$ is denoted by $\operatorname{Int} a$ and $A^a := \{b \in A \,|\, \operatorname{Int} a(b) = aba^{-1} = b\}$. If A is a topological group, the connected component of the trivial element in A is denoted by A_0. The center of A is denoted by $Z(A)$.

1.2. **Weil group.** Let $\Gamma = \{1, \sigma\} = \operatorname{Gal}(\mathbb{C}/\mathbb{R})$ be the Galois group of $\mathbb{C}$ over $\mathbb{R}$. The Weil group $W_{\mathbb{R}}$ is a non-split extension

$$1 \to \mathbb{C}^\times \to W_{\mathbb{R}} \to \Gamma \to 1.$$

More precisely, $W_{\mathbb{R}}$ is the group generated by $\mathbb{C}^*$ and an element j which projects to $\sigma \in \Gamma$ with relations

$$j^2 = -1, \quad jz = \bar{z}j.$$

1.3. **Reductive algebraic groups over $\mathbb{C}$.** For the proof of the results in this paragraph, we refer to [ABV], Chapter 2. Let G be a connected reductive algebraic group defined over $\mathbb{C}$. The group G is identified with the group of its complex points.

Following [LS], by a pair in G, we mean a couple (B,T) where B is a Borel subgroup of G and T a maximal torus in B, and by a splitting of G, we mean a triple $\mathbf{spl}_G = (B, T, \{X_\alpha\})$ where (B,T) is a pair in G and $\{X_\alpha\}$ a collection of non-zero root vectors, one for each simple root α of T in B. Two spittings $\mathbf{spl}_G = (B, T, \{X_\alpha\})$ and $\mathbf{spl}'_G = (B', T', \{X'_\alpha\})$ are conjugate under G by an element uniquely determined modulo $Z(G)$, two pairs (B,T) and (B',T') are conjugate under G by an element uniquely determined modulo T, and two Borel subgroups B and B' are conjugate under G by an element uniquely determined modulo B.

Associated to a pair (B,T), there is a based root datum

$$\Psi(G,B,T) = (X^*(T), \Delta(B,T), X_*(T), \Delta\check{}(B,T))$$

where $X^*(T)$ is the group of algebraic characters of T, $X_*(T)$ the group of 1-parameter subgroups of T, $\Delta(B,T)$ are the simple roots of the positive root sytem $R(B,T)$ of T in B, and $\Delta\check{}(B,T)$ are the simple coroots. The structure on this datum is the lattice structure of $X^*(T)$ and $X_*(T)$, together with the perfect pairing

$$X^*(T) \times X_*(T) \to \mathbb{Z},$$

the containment $\Delta(B,T) \subset X^*(T)$ and $\Delta\check{}(B,T) \subset X_*(T)$ and the bijection $\alpha \leftrightarrow \alpha\check{}$ between roots and coroots. By an automorphism of $\Psi(G,B,T)$ we mean a bijective map respecting this structure.

Since two pairs (B,T) and (B',T') are conjugate under G by an element uniquely determined modulo T, there is a canonical isomorphism between $\Psi(G,B,T)$ and $\Psi(G,B',T')$. Let us define

$$\Psi_G = (X^*, \Delta, X_*, \Delta\check{})$$

as the projective limit of the $\Psi(G,B,T)$. Thus, for each pair (B,T), there is a canonical isomorphism between $\Psi(G,B,T)$ and Ψ_G. There is an exact sequence :

$$1 \to \operatorname{Int} G \to \operatorname{Aut} G \to \operatorname{Aut} \Psi_G \to 1$$

where $\operatorname{Aut} G$ denotes the group of complex algebraic automorphisms of G.

1.4. ***L*-group data.** The dual of Ψ_G is by definition $\Psi_G\check{} = (X_*, \Delta\check{}, X^*, \Delta)$. A dual group for G is a connected reductive algebraic complex group $\hat{G}$ such that $\Psi_{\hat{G}} \simeq \Psi_G\check{}$. It is unique up to isomorphism.

Let us consider extensions (of topological groups) of the form

$$1 \to \hat{G} \to \mathcal{G} \xrightarrow{p} W_{\mathbb{R}} \to 1.$$

A splitting of this extension is a continuous homomorphism

$$s : \; W_{\mathbb{R}} \to \mathcal{G}$$

such that $p \circ s$ is the identity of $W_{\mathbb{R}}$. Each splitting defines a homomorphism η_s of $W_{\mathbb{R}}$ into the group of automorphism of $\hat{G}$. A splitting will be called admissible if for all $w \in W_{\mathbb{R}}$, $\eta_s(w)$ is complex analytic and the associated linear transformation of the Lie algebra of $\hat{G}$ is semisimple. It will be called distinguished if there is a splitting $\mathbf{spl}_{\hat{G}} = (\mathcal{B}, \mathcal{T}, \{\mathcal{X}_\alpha\})$ of $\hat{G}$ such that elements of $\mathbb{C}^\times \subset W_{\mathbb{R}}$ acts trivially on $\hat{G}$, and $\eta_s(j)$ is an automorphism of $\hat{G}$ preserving $\mathbf{spl}_{\hat{G}}$ (recall that j is an element of

$W_{\mathbb{R}}$ such that $W_{\mathbb{R}} = \mathbb{C}^\times \coprod j\mathbb{C}^\times$). Two distinguished splitting will be called equivalent if they are conjugate under $\hat{G}$.

Following Langlands we consider the category $\mathfrak{G}\hat{}$ whose objects are extensions of the above type, with $\hat{G}$ a connected reductive algebraic group defined over $\mathbb{C}$, together with an equivalence class of distinguished splittings (called special). A morphism in this category between two extensions :

$$1 \to \hat{G} \to \mathcal{G} \xrightarrow{p} W_{\mathbb{R}} \to 1.$$

$$1 \to \hat{G}' \to \mathcal{G}' \xrightarrow{p'} W_{\mathbb{R}} \to 1.$$

is an equivalence class of L-homomorphisms, *i.e.* continuous group morphisms $\xi : \mathcal{G} \to \mathcal{G}'$ such that $p' \circ \xi = p$, such that the restriction of ξ to $\hat{G}$ (with value in $\hat{G}'$) is a morphism of complex algebraic groups and such that ξ preserves admissible splittings. Two L-homomorphisms ξ_1 and ξ_2 as above are in the same class if there exists $g \in \hat{G}$ such that

$$\xi_2 = \operatorname{Int} g \circ \xi_1.$$

Suppose now that G is a connected reductive algebraic group defined over $\mathbb{R}$. The group G is identified with the group of its complex points. Let σ be the non trivial element of $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We will denote by σ_G the action of σ on G, by $G(\mathbb{R})$ the group of real points of G. The automorphism σ_G on G induces an automorphism a_σ of Ψ_G via the isomorhisms $\Psi_G \simeq \Psi(G, B, T)$, $\Psi_G \simeq \Psi(G, \sigma_G(B), \sigma_G(T))$ and $\sigma_G : \Psi(G, B, T) \simeq \Psi(G, \sigma_G(B), \sigma_G(T))$. The automorphism a_σ of Ψ_G induces an automorphism of $\Psi_G\check{}$ that we still denote by a_σ.

If we fix a splitting $\mathbf{spl}_{\hat{G}} = (\mathcal{B}, \mathcal{T}, \{\mathcal{X}_\alpha\})$ of $\hat{G}$, we can transfer the automorphism a_σ from $\Psi_{\hat{G}}$ to $\Psi(\hat{G}, \mathcal{B}, \mathcal{T})$ via the canonical isomorphism $\Psi_{\hat{G}} \simeq \Psi(\hat{G}, \mathcal{B}, \mathcal{T})$. Then there is a unique algebraic automorphism $\sigma_{\hat{G}}$ of $\hat{G}$ preserving $\mathbf{spl}_{\hat{G}}$ and inducing a_σ on $\Psi(\hat{G}, \mathcal{B}, \mathcal{T})$. (Notice that $\sigma_{\hat{G}}$ is an algebraic automorphism, while σ_G is not).

An L-group datum for G is an object

$$1 \to \hat{G} \to {}^L G \xrightarrow{p} W_{\mathbb{R}} \to 1.$$

in $\mathfrak{G}\hat{}$, such that the action of $\eta_s(j)$ on $\hat{H}$ coincide with $\sigma_{\hat{G}}$ constructed as above from a special splitting $\mathbf{spl}_{\hat{G}}$.

A realization of the L-group ${}^L G$ of G is then a semi-direct product $\hat{G} \rtimes_{\rho_G} W_{\mathbb{R}}$, where the action ρ_G of $W_{\mathbb{R}}$ on $\hat{G}$ factors through the projection on $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, with σ acting by $\sigma_{\hat{G}}$, where $\sigma_{\hat{G}}$ is constructed as above from a special splitting. Two realizations of ${}^L G$ given by two different choices of a splitting of $\hat{G}$ are isomorphic as objects in $\mathfrak{G}\hat{}$..

Remark. Suppose that $\mathcal{G}$ is a split extension of $W_{\mathbb{R}}$ by $\hat{G}$, i.e. that we have a split exact sequence

$$1 \to \hat{G} \to \mathcal{G} \to W_{\mathbb{R}} \to 1. \tag{1.1}$$

Then $\mathcal{G}$ is not always an L-group for some real form of G. Nethertheless, we can attach to $\mathcal{G}$ an L-action $\rho_{\mathcal{G}}$ of $W_{\mathbb{R}}$ on $\hat{G}$ as follows. If $c : W_{\mathbb{R}} \to G$ splits (1.1), then for $w \in W_{\mathbb{R}}$, $\operatorname{Int} c(w)$ acts by conjugacy on $\hat{G}$. Composing with an inner automorphism of $\hat{G}$, we obtain an automorphism $\rho_{\mathcal{G}}(w)$ preserving $\mathbf{spl}_{\hat{G}}$. It is straightforward to check that $\rho_{\mathcal{G}}$ doesn't depend on the choices (for instance, if we fix a splitting $\mathbf{spl}_{\hat{G}}$ of $\hat{G}$ from which we construct ${}^L G$ with L-action ρ_G, and if $\mathcal{G}$ is another realization of the L-group obtained from another choice of splitting, we get $\rho_{\mathcal{G}} = \rho_G$). This will be used in the definition of endoscopic data.

1.5. **Inner forms.** The group G is quasi-split if and only if it has an $\mathbb{R}$-splitting, i.e. one preserved by σ_G. Two connected reductive algebraic groups G and G^* defined over $\mathbb{R}$ are inner forms if there exists an isomorphism $\psi : G \to G^*$ defined over $\mathbb{C}$, and an element $u_\sigma \in G^*$ such that

$$\psi \circ \sigma(\psi)^{-1} = \psi \circ \sigma_G \circ \psi^{-1} \circ \sigma_{G^*} = \operatorname{Int} u_\sigma \tag{1.2}$$

This define an equivalence relation between real forms of connected algebraic reductive groups defined over $\mathbb{R}$. Every equivalence class contains a quasi-split form ([ABV], Proposition 2.7). It is easy to check that up to isomorphism, ${}^L G$ depends only on the inner class of real forms of G.

1.6. **Langlands classification.** Let us denote by $\mathcal{HC}_G$ the category of finite length Harish-Chandra modules of $G(\mathbb{R})$ (technically this requires the choice of a maximal compact subgroup $K(\mathbb{R})$ of $G(\mathbb{R})$, but we will ignore this and quite often we will loosely refer to this category as the category of representations of $G(\mathbb{R})$). We denote by $\Pi(G(\mathbb{R}))$ the set of irreducible representations of $G(\mathbb{R})$.

A Langlands parameter is an L-morphism $\phi : W_{\mathbb{R}} \to {}^L G$. A $\hat{G}$-conjugacy class of Langlands parameters is then a morphism in the category $\mathfrak{G}\hat{}$. Let us denote by $\Phi(G)$ the set conjugacy classes of Langlands parameters, and by $[\phi]$ the conjugacy class of ϕ. . To each $[\phi] \in \Phi(G)$ is attached a packet Π_ϕ (sometimes empty) of representations, with the following properties :

- the Π_ϕ, with $[\phi]$ running over $\Phi(G)$ form a partition of $\Pi(G(\mathbb{R}))$,
- each packet Π_ϕ is finite,
- all representations in a packet have same central and infinitesimal character,
- if a representation in some Π_ϕ is in the discrete series (resp. is tempered), then all representations in Π_ϕ are in the discrete series (resp. are tempered).

For further details about the Langlands classification, we refer to [L1], [Bo] or [ABV].

1.7. **Quasicharacter.** The action of $W_{\mathbb{R}}$ on $Z(\hat{G})$ is well defined, and so is $H^1(W_{\mathbb{R}}, Z(\hat{G}))$, where it is understood that only continuous cocycles are to be considered. There is an action of $H^1(W_{\mathbb{R}}, Z(\hat{G}))$, defined as folows. If ϕ is a Langlands parameter, and if $\mathbf{a} \in H^1(W_{\mathbb{R}}, Z(\hat{G}))$, one $\mathbf{a}[\phi]$ to be the class of

$$a\phi(w) = a(w)\phi(w), \quad (w \in W_{\mathbb{R}}),$$

where a is any choice of a cocycle representing $\mathbf{a}$ (it is easily checked that the class of $a\phi$ doesn't depend on the choice of a). Thus it also acts on the set of L-packets. To describe this, we recall a construction made in [L1], before Lemma 2.12 : to each $\mathbf{a} \in H^1(W_{\mathbb{R}}, Z(\hat{G}))$ is attached a quasicharacter $\omega_{\mathbf{a}}$ of $G(\mathbb{R})$, satisfying the following property :

$$\Pi_\phi \otimes \omega_{\mathbf{a}} := \{\pi \otimes \omega_{\mathbf{a}} \mid \pi \in \Pi_\phi\} = \Pi_{\mathbf{a}\phi}.$$

Our interest is in the study of L-packets Π_ϕ of irreducible admissible representations of $G(\mathbb{R})$ such that

$$\Pi_\phi = \Pi_\phi \otimes \omega_{\mathbf{a}} = \Pi_{\mathbf{a}\phi}. \tag{1.3}$$

Since all representations in a L-packet have same infinitesimal character, we see that this forces $\omega_{\mathbf{a}}$ to have trivial infinitesimal character. Thus we suppose that $\omega_{\mathbf{a}}$ is trivial on $G(\mathbb{R})_0$. Set $G(\mathbb{R})^+ = \ker \omega_{\mathbf{a}}$. This is an normal open subgroup of $G(\mathbb{R})$ and one can view $\omega_{\mathbf{a}}$ as a character of the finite group $G(\mathbb{R})/G(\mathbb{R})^+$.

1.8. **Endoscopic data.** Let G be a connected reductive algebraic group defined over $\mathbb{R}$. We fix a quasi-split group G^* in the inner class of real forms of G with an inner inner twist $\psi : G \to G^*$ and an element u_σ in G^* as in (1.2).

We denote by G_{sc} the universal covering of the derived group G_{der} of G^*, and if T is a maximal torus in G, T_{sc} denote the inverse image of $T \cap G_{der}$ in G_{sc} under the natural projection. We will often identify elements in G_{sc} or T_{sc} with their image in G or T without comment.

The group G^* being quasi-split, let us fix once for all a $\mathbb{R}$-splitting $\mathbf{spl}_{G^*} = (\mathbf{B}, \mathbf{T}, \{\mathbf{X}_\alpha\})$. Let us also fix a splitting $\mathbf{spl}_{\hat{G}} = (\mathcal{B}, \mathcal{T}, \{\mathcal{X}_\alpha\})$ of $\hat{G}$, from which we construct a realization of the L-group of G. Finally, let $\mathbf{a} \in H^1(W_{\mathbb{R}}, Z(\hat{G}))$ and let $\omega = \omega_{\mathbf{a}}$ bethe quasicharacter of $G(\mathbb{R})$, attached to it as in 1.7.

Definition 1.1. Following [KS], we call the quadruple $(H, \mathcal{H}, s, \xi)$ an endoscopic datum for $(G, \mathbf{a})$ if :
(i) H is a quasi-split connected reductive algebraic group defined over $\mathbb{R}$.
(ii) s is a semi-simple element in $\hat{G}$.
(iii) $\mathcal{H}$ is a split extension of $W_{\mathbb{R}}$ by $\hat{H}$ such that for a choice of a splitting $\mathbf{spl}_{\hat{H}}$ of $\hat{H}$, ρ_H and $\rho_{\mathcal{H}}$ coincide (see remark 1.4). (Both ρ_H and $\rho_{\mathcal{H}}$ depend on the choice of $\mathbf{spl}_{\hat{G}}$, but not the property $\rho_H = \rho_{\mathcal{H}}$.)

(iv) $\xi : \mathcal{H} \to {}^L G$ is an L-homomorphism.
(v) $\mathrm{Int}\, s \circ \xi = a \cdot \xi$ where a is a 1-cocycle of $W_{\mathbb{R}}$ in $Z(\hat{G})$ representing $\mathbf{a}$.
(vi) ξ maps isomorphically $\hat{H}$ into $(\hat{G}^s)_0$.

The purpose of endoscopic data for $(G, \mathbf{a})$ is to study reprsentations of $G(\mathbb{R})$ in L-packets satisfying (1.3). But $\Pi_\phi = \Pi_{a\phi}$ if and only if there exists $g \in \hat{G}$ such that

$$\mathrm{Int}\, g \circ \phi = a\phi,$$

i.e. the set $S_\phi = \{g \in \hat{G} \,|\, \mathrm{Int}\, g \circ \phi = a\phi\}$ is not empty. Assume $g \in S_\phi$ has Jordan decomposition $g = su$. Then $s \in S_\phi$ and $u \in \mathrm{Cent}_{\hat{G}}(\phi)_0$, a connected group which acts by translations on S_ϕ. Therefore, every connected component of S_ϕ contains a semisimple element.

Assume that $s \in S_\phi$ is semisimple, so that $\hat{H} := \hat{G}_0^s$ is reductive. Let $\mathcal{H}$ be the subgroup of ${}^L G$ generated by $\hat{H}$ and the image of ϕ, endowed with the induced topology, and let ξ be the inclusion of $\mathcal{H}$ in ${}^L G$. Then there is a split exact sequence

$$1 \to \hat{H} \to \mathcal{H} \to W_{\mathbb{R}} \to 1.$$

Define $\rho_{\mathcal{H}}$ as in remark 1.4, and construct an L-group ${}^L H$ from $\hat{H}$, $\rho_{\mathcal{H}}$ and a choice of a splitting $\mathbf{spl}_{\hat{H}}$. Let H be a quasi-split reductive group defined over $\mathbb{R}$ with L-group ${}^L H$. Then $(H, \mathcal{H}, s, \xi)$ is an endoscopic datum for $(G, \mathbf{a})$. If $\mathcal{H}$ is isomorphic to ${}^L H$, then $\xi : W_{\mathbb{R}} \to {}^L G$, which factorizes through $\mathcal{H}$, gives a Langlands parameter

$$\xi_H \,:\, W_{\mathbb{R}} \to \mathcal{H}.$$

In lines with Langlands functoriality principle, there should a transfer from Π_{ϕ_H} to Π_ϕ, *i.e.* a way to relate characters of representations in Π_{ϕ_H} to characters of of representations in Π_ϕ. This relationship between characters of the two groups will be obtained from a dual transfer of orbital integrals from $G(\mathbb{R})$ to $H(\mathbb{R})$. The goal of the paper is to explain how this transfer of orbital integrals is obtained. When $\mathcal{H}$ is not an L-group for H, the situation is more complicated. We need the notion of z-pair introduced below, and we get a tranfer of orbital integrals from $G(\mathbb{R})$ to some extension $H_1(\mathbb{R})$ of $H(\mathbb{R})$.

Definition 1.2. The endoscopic data $(H, s, \mathcal{H}, \xi)$ and $(H', s', \mathcal{H}', \xi')$ are isomorphic if there exists $g \in \hat{G}$, an $\mathbb{R}$-isomorphism $\alpha :\, H \to H'$ and an L-isomorphism $\beta : \mathcal{H}' \to \mathcal{H}$ such that :

- The maps induced by α and β on the based root data, $\Psi_H \xrightarrow{\alpha} \Psi_{H'}$ and $\Psi_{\hat{H}'} \xrightarrow{\beta} \Psi_{\hat{H}}$ are dual to each other.
- $\mathrm{Int}\, g \circ \xi \circ \beta = \xi'$.
- $gsg^{-1} = s'$ modulo $Z(\hat{G})Z(\xi')_0$, where $Z(\xi')$ is the centralizer in $\hat{G}$ of the image of $\mathcal{H}'$ under ξ'.

We may replace (s,ξ) by $(gsg^{-1}, \operatorname{Int} g\circ\xi)$ for any $g\in\hat{G}$ and get another equivalent endoscopic datum. Thus, up to equivalence, we can assume that $s\in\mathcal{T}$. Then $\operatorname{Int} s\cdot(\mathcal{B},\mathcal{T})=(\mathcal{B},\mathcal{T})$. Making another such replacement with a $g\in(\hat{G}^s)_0$, we may also assume that :

$$\xi(\mathcal{B}_H)=\mathcal{B}\cap(\hat{G}^s)_0 \quad\text{and}\quad \xi(\mathcal{T}_H)=\mathcal{T}. \tag{1.4}$$

For the rest of the paper, we fix G, $\hat{G}$, a splitting $\mathbf{spl}_{\hat{G}}=(\mathcal{B},\mathcal{T},\{\mathcal{X}\})$, an endoscopic datum $(H,\mathcal{H},s,\xi)$, and a splitting $\mathbf{spl}_{\hat{H}}=(\mathcal{B}_H,\mathcal{T}_H,\{\mathcal{X}_H\})$ of $\hat{H}$. We assume that $s\in\mathcal{T}$ and that $(H,\mathcal{H},s,\xi)$ satisfies (1.4).

1.9. ***z*-pair.** Since $\mathcal{H}$ is not necessarily an L-group for H (see [KS], §2.1) we need to introduce a z-extension H_1 of H. We recall the definition ([K]):

Definition 1.3. A z-extension of a connected reductive algebraic quasi-split real group H is a central extension H_1 of H:

$$1\to Z_1\to H_1\to H\to 1$$

where H_1 is a connected reductive algebraic quasi-split real group whose derived group is simply-connected and Z_1 is a central torus in H_1, isomorphic to a product of $\operatorname{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{R}^\times$.

Note that since $H^1(\Gamma,Z_1)=\{1\}$, we have also :

$$1\to Z_1(\mathbb{R})\to H_1(\mathbb{R})\to H(\mathbb{R})\to 1.$$

Dual to the exact sequence in the above definition we have

$$1\to\hat{H}\to\hat{H}_1\to\hat{Z}_1\to 1,$$

so we regard $\hat{H}$ as a subgroup of $\hat{H}_1$. This inclusion can be extended to a L-homomorphism $\xi_{H_1}:\mathcal{H}\to {}^LH_1$ (see [KS] lemma 2.2.A).

Definition 1.4. By a z-pair for $\mathcal{H}$, we mean a pair (H_1,ξ_{H_1}) where H_1 is a z-extension of H and $\xi_{H_1}:\mathcal{H}\to {}^LH_1$ a L-homomorphism that extends $\hat{H}\hookrightarrow\hat{H}_1$.

Observe that ξ_{H_1} determines a character λ_{H_1} of $Z_1(\mathbb{R})$. This character has Langlands parameter:

$$W_{\mathbb{R}}\xrightarrow{c}\mathcal{H}\xrightarrow{\xi_{H_1}}{}^LH_1\to{}^LZ_1, \tag{1.5}$$

where c is any section of $\mathcal{H}\to W_{\mathbb{R}}$ and ${}^LH_1\to{}^LZ_1$ is the natural extension of $\hat{H}_1\to\hat{Z}_1$. For a discussion of the significance of λ_{H_1} in terms of Langlands functoriality principle, see [KS], end of section 2.2. Let us just say here that if $\mathcal{H}$ is not an L-group, then there is no matching between compactly supported smooth functions on $G(\mathbb{R})$ and compactly supported smooth functions on $H(\mathbb{R})$. Instead, we will establish a matching between compactly supported smooth functions on $G(\mathbb{R})$ and smooth functions on $H_1(\mathbb{R})$, compactly supported modulo $Z_1(\mathbb{R})$ and transforming under translations by elements of $Z_1(\mathbb{R})$ according to λ_{H_1}.

2. Correspondences of points

2.1. **Correspondences of semi-simple conjugacy classes.** Recall the inner twist $\psi : G \to G^*$ and the element $u_\sigma \in G^*$ such that $\psi\sigma(\psi)^{-1} = \operatorname{Int} u_\sigma$. Then ψ induces a bijective map $\mathcal{A}_{G,G^*}$ from conjugacy classes in G to conjugacy classes in G^* which is defined over $\mathbb{R}$. Let us denote by $\mathcal{A}_{G^*,G}$ its inverse.

We recall now the points correspondences between H and G. We have fixed splitings $\mathbf{spl}_{\hat{G}^*} = (\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$ and $\mathbf{spl}_{\hat{H}} = (\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{X}_H\})$.

Theorem 2.1. *([LS] 1.3.A) There is a canonical map defined over $\mathbb{R}$:*

$$\mathcal{A}_{H/G} : \; Cl_{ss}(H) \to Cl_{ss}(G) \tag{2.1}$$

between semi-simple conjugacy classes in H and semi-simple conjugacy classes in G.

This map is obtained in the following way: suppose (B_H, T_H) is a pair in H and that (B, T) is a pair in G^*. Attached to (B_H, T_H) and $(\mathcal{B}_H, \mathcal{T}_H)$ is an isomorphism $\hat{T}_H \simeq \mathcal{T}_H$ and attached to (B, T) and $(\mathcal{B}, \mathcal{T})$ is an isomorphism $\hat{T} \simeq \mathcal{T}$. We have therefore a chain of isomorphisms:

$$\hat{T}_H \simeq \mathcal{T}_H \xrightarrow{\xi} \mathcal{T} \simeq \hat{T}$$

which yields $T_H \simeq T$. This isomorphism transports the coroots of T_H in H into a subsystem of the coroots of T in G^* and the Weyl group $W_H := W(H, T_H)$ into a subgroup of the Weyl group $W := W(G^*, T)$, and so induce a map :

$$T_H/W_H \to T/W.$$

Therefore we have:

$$\mathrm{Cl}_{ss}(H) \simeq T_H/W_H \to T/W \simeq \mathrm{Cl}_{ss}(G^*) \xrightarrow{\mathcal{A}_{G^*,G}} \mathrm{Cl}_{ss}(G)$$

yielding the map (2.1).

If T_H is defined over $\mathbb{R}$, we may choose (B, T) and B_H such that both T and $T_H \simeq T$ are defined over $\mathbb{R}$ (see [LS], 1.3.A). An $\mathbb{R}$-isomorphism $\eta : T_H \to T$ as above will be called an admissible embedding of T_H in G^*. It is uniquely determined up to $\mathcal{A}(T)$-conjugacy, that is up to composition with $\operatorname{Int} g$ where g lies in

$$\mathcal{A}(T) \; := \{g \in G^* : \; \sigma_{G^*}(g)^{-1} g \in T\}.$$

2.2. **Norms.** Let $\delta \in G(\mathbb{R})$ be a semi-simple element. Recall that δ is regular (resp. strongly regular) if $(G^\delta)_0$ (resp. G^δ) is a maximal torus of G.

Let us denote by $\mathcal{O}_\gamma$ the conjugacy class of an element $\gamma \in H$, and by $\mathcal{O}_\delta$ the conjugacy class of an element $\delta \in G$.

Definition 2.2. An element $\gamma \in H$ is G-regular (resp. strongly G-regular if $\mathcal{A}_{H/G}(\mathcal{O}_\gamma)$ is a regular (resp. stronly regular) conjugacy class in G.

Lemma 2.3. *(*[KS]*,3.3.C)*
(i) G-regular implies regular,
(ii) strongly G-regular implies strongly regular.

Definition 2.4. The stable conjugacy class of a strongly regular element $\delta \in G(\mathbb{R})$ is $\mathcal{O}_\delta \cap G(\mathbb{R})$.

We extend this definition for regular elements. Let $\delta \in G(\mathbb{R})$ be such a element. Suppose that $\delta' = g\delta g^{-1} \in G(\mathbb{R})$. Then we have $\sigma_G(g)^{-1}g \in G^\delta$. Let us denote $T_\delta = (G^\delta)_0$. Since δ is regular, T_δ is a torus.

Definition 2.5. In the above setting, we say that δ' is in the stable conjugacy class of δ if and only if $\sigma_G(g)^{-1}g \in T_\delta$.

We will see below the reason for this requirement. Note that if δ is strongly regular, then $G^\delta = T_\delta$ and the two definitions agree.

Definition 2.6. Let $\gamma \in H(\mathbb{R})$ be G-regular, and let T_H be the maximal torus of H containing γ. Fix an admissible embedding $T_H \xrightarrow{\eta} T$ of T_H in G^*. We say that γ is a norm of $\delta \in G(\mathbb{R})$ if :
(i) δ lies in the image of $\mathcal{O}_\gamma$ under $\mathcal{A}_{H/G}$.
Then, by definition, there exist $x \in G^*$ and $\delta^* \in T$ such that $\delta^* = x\psi(\delta)x^{-1}$ and $\delta^* = \eta(\gamma)$.
(ii) $\operatorname{Int} x \circ \psi : T_\delta \to T$ is defined over $\mathbb{R}$.

We will see in the proof of the next theorem that if γ is strongly G-regular, then the condition (ii) is automatically fulfilled. When it is not possible to find such an element δ, we say that γ is not a norm.

Theorem 2.7. *Let $\gamma \in H(\mathbb{R})$ be G-regular. Then γ is a norm of exactly one stable conjugacy class in $G(\mathbb{R})$ or is not a norm.*

Proof. Recall that

$$\psi = \operatorname{Int} u_\sigma \circ \sigma_{G^*} \circ \psi \circ \sigma_G = \operatorname{Int} u_\sigma \circ \sigma(\psi),$$
$$\text{thus} \quad \sigma(\operatorname{Int} x \circ \psi) = \operatorname{Int}(\sigma_{G^*}(x)u_\sigma^{-1}) \circ \psi.$$

Therefore, $\operatorname{Int} x \circ \psi : T_\delta \to T$ is defined over $\mathbb{R}$ if and only if

$$\sigma(\operatorname{Int} x \circ \psi) = \operatorname{Int}(\sigma_{G^*}(x)u_\sigma^{-1}) \circ \psi = \operatorname{Int} x \circ \psi.$$

Set $v_\sigma = xu_\sigma\sigma_{G^*}(x)^{-1}$. Thus $\mathrm{Int}\, x \circ \psi$ is defined over $\mathbb{R}$ if and only if $\mathrm{Int}\, v_\sigma^{-1}$ is trivial on T, i.e. $v_\sigma \in T$.

Furthermore, since

$$\begin{aligned} v_\sigma^{-1}\delta^* v_\sigma &= \sigma_{G^*}(x)u_\sigma^{-1}\psi(\delta)u_\sigma\sigma_{G^*}(x)^{-1} = \sigma_{G^*}(x)\sigma(\psi)(\delta)\sigma_{G^*}(x)^{-1} \\ &= \sigma_{G^*}(\delta^*) = \delta^* \end{aligned}$$

we get $v_\sigma \in (G^*)^{\delta^*}$.

From this discussion, we see that if γ is stronly G-regular, then δ^* is strongly regular, $(G^*)^{\delta^*} = T$ and $\mathrm{Int}\, x \circ \psi : T_\delta \to T$ is defined over $\mathbb{R}$.

Suppose now that δ_1 is stably conjugate to δ, i.e. there exists $g \in G$, with $\delta_1 = g\delta g^{-1} \in G$ and $\sigma_G(g)^{-1}g \in T_\delta$. Suppose also that γ is a norm for δ. Then we compute

$$\delta^* = x\psi(g)^{-1}\psi(\delta_1)(x\psi(g)^{-1})^{-1}.$$

We want to prove that γ is a norm for δ_1, so we have to check that $\mathrm{Int}\,(x\psi(g)^{-1}) \circ \psi : T_{\delta_1} \to T$ is defined over $\mathbb{R}$. From the discussion above, this is the case if and only if $\mathrm{Int}\,(x\psi(g)^{-1}u_\sigma\sigma_{G^*}(x\psi(g)^{-1})^{-1})$ is trivial on T. We compute :

$$\begin{aligned} x\psi(g)^{-1}u_\sigma\sigma_{G^*}(x\psi(g)^{-1})^{-1} &= (x\psi(g)^{-1}u_\sigma)(u_\sigma^{-1}\psi(\sigma_G(g))u_\sigma)\sigma_{G^*}(x)^{-1} \\ &= x\psi(g^{-1}\sigma_G(g))u_\sigma\sigma_{G^*}(x)^{-1} \\ &= x\psi(g^{-1}\sigma_G(g))x^{-1}v_\sigma \end{aligned}$$

Since γ is a norm for δ, $\mathrm{Int}\, v_\sigma$ is trivial on T, hence we only have to check that $\mathrm{Int}\,(x\psi(g^{-1}\sigma_G(g))x^{-1})$ is trivial on T, i.e. $\mathrm{Int}\,(g^{-1}\sigma_G(g))$ is trivial on T_δ. This is now obvious by the definition of stable conjugacy and the requirement that $g^{-1}\sigma_G(g) \in T_\delta$.

Let us now prove the other inclusion, and so suppose that γ is a norm for δ, δ_1, i.e. there exist δ^* in T and x, $x_1 \in G^*_{sc}$ such that

$$\delta^* = x\psi(\delta)x^{-1} = x_1\psi(\delta_1)x_1^{-1}$$

We are lead to :

$$\delta_1 = \psi^{-1}(x_1^{-1}x)\delta(\psi^{-1}(x_1^{-1}x))^{-1}.$$

We have then to show that $w^{-1}\sigma_G(w) \in T_\delta$, where $w = \psi^{-1}(x_1^{-1}x)$. This is equivalent to : $\mathrm{Int}\, x \circ \psi(w^{-1}\sigma_G(w)) \in T$, and:

$$\mathrm{Int}\, x \circ \psi(w^{-1}\sigma_G(w)) = \sigma_{G^*}(v_1)^{-1}\sigma_{G^*}(v),$$

where $v = v_\sigma$ as above and $v_1 = x_1u_\sigma\sigma_{G^*}(x_1)^{-1}$. This is a product of elements in T, hence $\mathrm{Int}\, x \circ \psi(w^{-1}\sigma_G(w)) \in T$. The proof of the theorem is now complete. □

Remark. Suppose that γ is a norm of δ, and that $\eta : T_H \to T$, $\delta^* = \eta(\gamma)$ and x are fixed as in the definition. We may replace x by an element $x_1 = tx$ with $t \in (G^*)^{\delta^*}$, provided that $\mathrm{Int}\, x_1 \circ \psi$ is defined over $\mathbb{R}$. As we have seen in the proof of the theorem, this is equivalent to $\mathrm{Int}\, v_\sigma^1$ trivial on T, with $v_\sigma^1 = txu_\sigma\sigma_{G^*}(x)^{-1}\sigma_{G^*}(t)^{-1}$. Since $t \in (G^*)^{\delta^*}$,

t normalizes T and since v_σ is trivial on T, we get that $t\sigma_{G^*}(t)^{-1}$ must act trivially on T, i.e. $t\sigma_{G^*}(t)^{-1} \in T$.

2.3. **Admissible embeddings in G.** Let T_H be a maximal torus of H defined over $\mathbb{R}$. If there exists a G-regular element $\gamma \in T_H(\mathbb{R})$ which is the norm of an element $\delta \in G(\mathbb{R})$, we say that T_H originates in G. Recall this means that given an admissible embedding $T_H \xrightarrow{\eta} T$ of T_H in G^* defined over $\mathbb{R}$, there exist $x \in G^*_{sc}$ and $\delta^* \in T$ such that $\delta^* = x\psi(\delta)x^{-1}$, $\delta^* = \eta(\gamma)$ and

$$\operatorname{Int} x \circ \psi : T_\delta \to T$$

is defined over $\mathbb{R}$. Let us denote T_δ by T_G.

Let $\gamma_1 \in T_H$ be another G-regular element, ${\delta_1}^* = \eta(\gamma_1) \in T$ and $\delta_1 \in T_G$ such that $\operatorname{Int} x \circ \psi(\delta_1) = {\delta_1}^*$. Since $\operatorname{Int} x \circ \psi : T_G \to T$ is defined over $\mathbb{R}$, γ_1 is a norm of δ_1. Conversely, for every regular element δ_1 in $T_G(\mathbb{R})$, there is a G-regular element γ_1 of $T_H(\mathbb{R})$ which is a norm of δ_1.

We summarize the discussion.

Proposition 2.8. *Let T_H as above be a maximal torus of H defined over $\mathbb{R}$ originating in G and let $\gamma \in T_H(\mathbb{R})$ be a G-regular element which is a norm of a regular $\delta \in G(\mathbb{R})$. Let T_G the maximal torus G containing δ, and fix T, η and x as above. Then there is an isomorphism defined over $\mathbb{R}$:*

$$\eta_x : T_H \to T_G, \qquad \eta_x = (\operatorname{Int} x \circ \psi)^{-1} \circ \eta,$$

such that $\eta_x^{-1}(\delta_1)$ is a norm of δ_1 for all regular element $\delta_1 \in T_G$. As suggested by the notation, this isomorphism does not depend on the initial choice of δ and γ but only on the admissible embedding η of T_H in G^* and the element x. We call such an isomorphism an admissible embedding of T_H in G.

Graphically,

$$T_H \xrightarrow{\ \eta\ } T \xleftarrow{\operatorname{Int} x \circ \psi} T_G.$$

Suppose that we are in the setting above, i.e. $\gamma \in T_H(\mathbb{R})$ is a norm of a regular element in $G(\mathbb{R})$. Then γ is a norm of exactly one stable conjugacy class $\mathcal{O}^{st}$ in $G(\mathbb{R})$. Let us denote by Σ_γ a system of representatives for the conjugacy classes in $G(\mathbb{R})$ contained in $\mathcal{O}^{st}$. Let us first fix an admissible embedding η of T_H in G^*. Then, for all $\delta_i \in \Sigma_\gamma$, let us choose $x_i \in G^*$ such that x_i and η define an admissible embedding $\eta_{x_i} : T_H \to T_{\delta_i}$ as above. Put $T_{G,i} = T_{\delta_i}$. Suppose that γ_1 is another G-regular element in $T_H(\mathbb{R})$. Then, we have seen that γ_1 is a norm of the $\delta_{1,i} := \eta_{x_i}(\gamma_1)$.

Lemma 2.9. *The set $\{\delta_{1,i}\}$ forms a system of representatives for the conjugacy classes in $G(\mathbb{R})$ contained in the stable conjugacy class of elements in $G(\mathbb{R})$ for which γ_1 is a norm.*

Proof. This is straightforward and more or less contained in the proof of Theorem 2.7.

If T_H originates in G, a set of isomorphisms η_{x_i} as in the lemma will be called a complete system of admissible embeddings of T_H in G.

2.4. **Admissible embeddings in $G(\mathbb{R})^+$.** Suppose that the maximal torus T_H of H originates in G. Let us fix a complete system of admissible embeddings $\eta_{x_j} : T_H \to T_j$ as in the previous paragraph. We say that T_H (we should say $T_H(\mathbb{R})$) originates in $G(\mathbb{R})^+$ if one of the Cartan subgroups $T_j(\mathbb{R})$ of $G(\mathbb{R})$ is contained in $G(\mathbb{R})^+$. We first remark that if one of the $T_j(\mathbb{R})$ is contained in $G(\mathbb{R})^+$, then all of them are. Indeed, the $T_j(\mathbb{R})$ are conjugate under G, so a well known result asserts they are conjugate under $G(\mathbb{R})$. Since $G(\mathbb{R})^+$ is normal in $G(\mathbb{R})$, this proves the assertion. The same argument also proves that the notion "originates in $G(\mathbb{R})^+$" does not depend on the choices of the η_{x_i}.

2.5. **Roots originating in H.** Suppose that we have a maximal torus T_H of H defined over $\mathbb{R}$, and an admissible embedding

$$\eta : T_H \to T.$$

We say that $\alpha \in R(G^*, T)$ originates in H when there exists a root $\alpha_H \in R(H, T_H)$ such that $\eta(\alpha_H\check{}) = \check{\alpha}$.

Another way to say this is to notice that

$$\alpha_H\check{} \in R\check{}(H, T_H) \subset X_*(T_H) \simeq X^*(\mathcal{T}_H) \simeq X^*(\mathcal{T}) \simeq X_*(T),$$

but, as it is obvious from the definition of an endoscopic datum, we get $R\check{}(H, T_H) \subset R\check{}(G, T)$. Thus $\check{\alpha}$ originates in H if it coincide with a root $\alpha_H\check{}$ through these identifications.

Let $\gamma \in T_H$ be G-regular, and $\delta^* \in G^*$ such that $\eta(\gamma) = \delta$. We have then :

$$\alpha_H(\gamma) = \alpha(\delta^*) \tag{2.2}$$

Let $\eta_x : A := T_H \to T_G$ be an admissible embedding of T_H in G. We can define roots of $R(G, T_G)$ originating in H in a similar way. It is clear from the construction that the set of coroots in $R\check{}(G, T)$ (or $R\check{}(G, T_G)$) originating in H form a subsystem of coroots.

2.6. **Cayley tranforms.** Let $\mathfrak{g}$ be a real reductive Lie algebra and $\mathfrak{g}_\mathbb{C}$ its complexification. We denote by σ the conjugation of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{g}$. Let $\mathfrak{b} \subset \mathfrak{g}$ be a Cartan subalgebra. Let $\alpha \in R(\mathfrak{g}_\mathbb{C}, \mathfrak{b}_\mathbb{C})$ be an imaginary root, choose a root vector X_α for α and fix a root vector $X_{-\alpha}$ of $-\alpha$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$, where H_α is another notation for the coroot $\alpha\check{} \in \mathfrak{b}_\mathbb{C}$. Then $\mathfrak{s}_\mathbb{C} = \mathbb{C} \cdot X_\alpha + \mathbb{C} \cdot X_{-\alpha} + \mathbb{C} \cdot H_\alpha$ is a simple complex Lie algebra invariant under conjugation, $\sigma(H_\alpha) = -H_\alpha = H_{-\alpha}$ and $\sigma(X_\alpha) = cX_{-\alpha}$ for some $c \in \mathbb{R}^*$. If $c < 0$, we can renormalize to get $\sigma(X_\alpha) = -X_{-\alpha}$ or if $c > 0$, to get $\sigma(X_\alpha) = X_{-\alpha}$. In the former case, α is compact, $\mathfrak{s} = \mathfrak{s}_\mathbb{C}^\sigma \simeq \mathfrak{su}(2)$. In the latter case, α is non-compact and

$\mathfrak{s} \simeq \mathfrak{sl}(2,\mathbb{R})$. Suppose that α is non-compact. We define a standard Cayley transform with respect to α to be an element of the adjoint group of $\mathfrak{s}_{\mathbb{C}}$ of the form $c_\alpha = \exp(-i\pi(X_\alpha + X_{-\alpha})/4)$, where X_α, $X_{-\alpha}$ are normalized as explained above (they are unique up to a scalar factor of absolute value 1, and all the standard Cayley transforms for α are conjugate in the adjoint group of $\mathfrak{s}_{\mathbb{C}}$). We have :

$$\mathfrak{b}_{\mathbb{C}} = \ker\alpha \oplus \mathbb{C}\cdot H_\alpha$$
$$\mathfrak{b} = \ker\alpha_{|\mathfrak{b}} \oplus i\mathbb{R}\cdot H_\alpha.$$

Let $\mathfrak{a}_{\mathbb{C}} := c_\alpha \cdot \mathfrak{b}_{\mathbb{C}} = \ker\alpha \oplus \mathbb{C}\cdot(X_\alpha - X_{-\alpha})$. This is a Cartan subalgebra defined over $\mathbb{R}$ and

$$\mathfrak{a} = \ker\alpha_{|\mathfrak{b}} \oplus i\mathbb{R}\cdot(X_\alpha - X_{-\alpha}).$$

The root $\beta := c_\alpha\cdot\alpha$ of $R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$ is real and $c_\alpha\cdot H_\alpha = H_\beta = i(X_\alpha - X_{-\alpha})$. Furthermore :

$$\sigma(c_\alpha) = \exp(i\pi(X_\alpha + X_{-\alpha})/4) = c_\alpha^{-1}.$$

A standard Cayley transform is a particular case of a generalized Cayley transform defined by Shelstad (see [Sh1], §2), in particular, as it is easy to check, $\sigma(c_\alpha)^{-1}c_\alpha = c_\alpha^2$ realizes the Weyl reflection s_α with respect to the root α.

It will be useful to reverse the process, and define Cayley transforms with respect to real root. If $\mathfrak{a}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ defined over $\mathbb{R}$, and β a real root, we take root vectors X_β and $X_{-\beta}$ in $\mathfrak{g}$ such that $[X_\beta, X_{-\beta}] = H_\beta$ and define $c_\beta := \exp(i\pi(X_\beta + X_{-\beta})/4)$. The root $c_\beta\cdot\beta$ of $\mathfrak{b}_{\mathbb{C}} := c_\beta\cdot\mathfrak{a}_{\mathbb{C}}$ is imaginary non-compact and we can make the choices such that $c_\beta = c_\alpha^{-1}$.

2.7. **Jump data.** Let us denote by $\mathfrak{g}_{\mathbb{C}}$ (resp. $\mathfrak{g}$) the Lie algebra of G (resp. $G(\mathbb{R})$). We say that $x \in G(\mathbb{R})$ is semi-regular when the derived algebra of $\mathfrak{g}^x$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{su}(2)$. Suppose it is $\mathfrak{sl}(2,\mathbb{R})$. Let $\mathfrak{b}$ be a fundamental Cartan subalgebra of $\mathfrak{g}^x$, and $\pm\alpha$ the roots of $\mathfrak{b}_{\mathbb{C}}$ in $\mathfrak{g}^x_{\mathbb{C}}$: they are non-compact imaginary and satisfy

$$\det(\mathrm{Id} - \mathrm{Ad}x^{-1})_{|\mathfrak{g}^\alpha_{\mathbb{C}}} = 0.$$

Let c_α be a Cayley transform with respect to α as in the previous section and let us also denote by $\mathfrak{a}$ the maximally split Cartan subalgebra of $\mathfrak{g}^x$ obtained from the Cayley transform (ie. $\mathfrak{a}_{\mathbb{C}} = c_\alpha\cdot\mathfrak{b}_{\mathbb{C}}$). Let $B(\mathbb{R})$ and $A(\mathbb{R})$ be the Cartan subgroups corresponding respectively to $\mathfrak{b}$ and $\mathfrak{a}$. We refer to these notations by saying that $(x, B(\mathbb{R}), A(\mathbb{R}), c_\alpha)$ is a jump datum for $G(\mathbb{R})$. Jump data for the group $H(\mathbb{R})$ are defined in the same way.

2.8. **Jump data and admissible embedding.** Let $(\gamma_0, T_H, T'_H, c_{\alpha_H})$ be a jump datum on H.

Lemma 2.10. *(see* [Sh1], *§2) If T_H does not originate in G, neither does T'_H.*

<u>*Proof*</u>. Suppose that T'_H does originates in G. We want to prove that T_H also originates in G.

Our hypothesis is that there exist a maximal torus T' of G^* defined over $\mathbb{R}$, with an admissible embedding $\eta' : T'_H \to T'$, elements $\gamma' \in T'_H$, $\delta^{*\prime} \in T'$ such that $\eta'(\gamma') = \delta^{*\prime}$, an element $x \in G^*_{sc}$, a maximal torus T'_G of G defined over $\mathbb{R}$ and an element $\delta' \in T'_G(\mathbb{R})$ such that $\delta^{*\prime} = x\psi(\delta')x^{-1}$ and

$$\operatorname{Int} x \circ \psi : T'_G = G_0^\delta \to T'$$

is defined over $\mathbb{R}$ (see section 4). From these data, we get an admissible embedding :

$$\eta_x : T'_H \to T'_G.$$

Let us denote by β_H the real root of $R(G, T'_H)$ obtained by taking the Cayley transform $c_{\alpha_H} \cdot \alpha_H$, and by β the corresponding root of $R(G, T'_G)$. Since η' is defined over $\mathbb{R}$, it is clear that β is real.

Set $\delta_0 = \eta_x(\gamma_0) \in T'_G(\mathbb{R})$ and $\delta_0^* = \eta(\gamma_0)$. We take a standard Cayley transform c_β with respect to β and and define $\mathfrak{t}_{G,\mathbb{C}} = c_\beta \cdot \mathfrak{t}'_{G,\mathbb{C}}$. Let T_G be the corresponding torus of G. We need also to introduce the corresponding roots and Cayley transforms in G^*. Let us denote these by β^*, c_{β^*}...

The map $\operatorname{Int} x \circ \psi$ realizes isomorphisms from G^{δ_0} onto $(G^*)^{\delta_0^*}$ and from T_G onto T. The following diagram is easily seen to commute :

$$\begin{array}{ccc} T'_G & \xrightarrow{\operatorname{Int} x\circ\psi} & T' \\ \downarrow{\scriptstyle \operatorname{Int} c_\beta} & & \downarrow{\scriptstyle Int c_{\beta^*}} \\ T_G & \xrightarrow{\operatorname{Int} x\circ\psi} & T \end{array}$$

From this, we conclude that $\operatorname{Int} x \circ \psi : T_G \to T$ is defined over $\mathbb{R}$. Indeed, since $\operatorname{Int} x\circ\psi : T'_G \to T'$ is defined over $\mathbb{R}$ and since $\sigma(c_\beta) = c_\beta^{-1}$, $\sigma(c_\beta) = c_\beta^{-1}$, this amounts to

$$\operatorname{Int} c_{\beta^*}^{-1} (\operatorname{Int} x \circ \psi)_{|T'_G} \operatorname{Int} c_\beta = \operatorname{Int} c_{\beta^*} (\operatorname{Int} x \circ \psi)_{|T'_G} \operatorname{Int} c_\beta^{-1},$$

or equivalently

$$\operatorname{Int} c_{\beta^*}^{-2} (\operatorname{Int} x \circ \psi)_{|T'_G} \operatorname{Int} c_\beta^2 = (\operatorname{Int} x \circ \psi)_{|T'_G} \tag{2.3}$$

But the action of $c_{\beta^*}^{-2}$ on T is given by the action of the Weyl group reflection $s_{\beta^*} \in W(G^*, T)$ and the action of c_β^2 on T_G is given by the action of the Weyl group reflection $s_\beta \in W(G, T_G)$. It is thus obvious that (2.3) holds.

We define η by the following commutative diagram:

$$\begin{array}{ccc} T'_H & \xrightarrow{\eta'} & T' \\ \Big\downarrow{\scriptstyle \operatorname{Int} c_{\beta_H}} & & \Big\downarrow{\scriptstyle \operatorname{Int} c_{\beta^*}} \\ T_H & \xrightarrow{\eta} & T \end{array}$$

It remains to check that η is defined over $\mathbb{R}$ to complete the proof of the lemma. The argument is similar to the one just given.

$$\begin{aligned} \sigma_{G^*}(\eta) &= \sigma_{G^*}(\operatorname{Int} c_{\beta^*} \circ \eta' \circ \operatorname{Int} c_{\beta_H}^{-1}) \\ &= \operatorname{Int}(\sigma_{G^*}(c_{\beta^*})) \circ \sigma_{G^*}(\eta') \circ (\operatorname{Int} \sigma_H(c_{\beta_H}^{-1}) \\ &= \operatorname{Int} c_{\beta^*}^{-1} \circ \eta' \circ (\operatorname{Int} c_{\beta_H}) \\ &= \operatorname{Int} c_{\beta^*} \circ s_{\beta^*} \circ \eta' \circ s_{\beta_H} \circ \operatorname{Int} c_{\beta_H}^{-1} \\ &= \operatorname{Int} c_{\beta^*} \circ \eta' \circ \operatorname{Int} c_{\beta_H}^{-1} = \eta \end{aligned}$$

□

Corollary 2.11. *If T_H does not originates in $G(\mathbb{R})^+$, neither does T'_H.*

Proof. If T_H does not originate in G, this is clear from the previous lemma. Suppose that T'_H does originate in $G(\mathbb{R})^+$ and that T_H originates in G, but not in $G(\mathbb{R})^+$. Fix an admissible embedding $\eta_x : T_H \to T_G$. This means that some connected components of $T_G(\mathbb{R})$ are not in $G(\mathbb{R})^+$. Suppose T'_H originates in $G(\mathbb{R})^+$ and fix an admissible embedding $\eta_y : T'_H \to T'_G$, with $T'_G(\mathbb{R}) \subset G(\mathbb{R})^+$. We can arrange the choices such that T'_G is obtained from T_G by a standard Cayley transform. But then it is well known that all connected components of $T_G(\mathbb{R})$ intersect $T'_G(\mathbb{R})$, and we would get $T_G(\mathbb{R}) \subset G(\mathbb{R})^+$, and obtain a contradiction. Thus T'_H does not originate in $G(\mathbb{R})^+$. □

2.9. Let $(\gamma_0, T_H, T'_H, c_{\alpha_H})$ be a jump datum on H and suppose now that T_H does originate in G. Fix a complete system of admissible embeddings $\{\eta_{x_i} : T_H \to T_{G_i}\}$. Set $\delta_{0,i} = \eta_{x_i}(\gamma_0)$ and $\alpha_i = \eta_{x_i}(\alpha_H)$, and set

$$I_{\gamma_0} := \{i \mid \alpha_i \text{ is imaginary non-compact } \}$$

Lemma 2.12. *T'_H originates in G if and only if $I_{\gamma_0} \neq \emptyset$.*

Proof. Suppose T'_H originates in G, and let $\eta'_y : T'_H \to T'_G$ be an admissible embedding. Let us denote by β_H the real root in $R(T'_H, H)$ which is obtained from α_H by the Cayley transform c_{α_H}, and denote by β the corresponding real root in $R(T'_G, G)$. By the construction made in the previous paragraph, we get an admissible embedding $\eta_y : T_H \to T_G$, and a non compact imaginary root β in $R(T_G, G)$. Since our system of admissible embedding η_{x_i} was complete, there is one i such that η_y is conjugate in $G(\mathbb{R})$ to η_{x_i}, and so α_i is imaginary non-compact. Conversely, suppose that I_{γ_0} is not empty, and for some

$i \in I_{\gamma_0}$, consider the data attached to the admissible embedding η_{x_i}. We want to construct from it an admissible embedding $T'_H \to T'_G$. It is straightforward to check that the construction in the proof of lemma 2.8 can be adapted to the present situation, by exchanging the role of T_H and T'_H, the Cayley transforms being with respect to non-compact imaginary roots rather than real roots. This will be refined below in lemma 2.14. $\square$

In the setting above, suppose I_{γ_0} is not empty.

Lemma 2.13. *If for some $i \in I_{\gamma_0}$, the reflection s_{α_i} is realized in the group $G(\mathbb{R})^{\delta_{0,i}}$, then the same is true for all j in I_{γ_0}*

Proof. Suppose $n_i \in G(\mathbb{R})^{\delta_{0,i}}$ realizes s_{α_i}. Take an arbitrary $j \in I_{\gamma_0}$, and let us take $g \in G$ such that $g \cdot T_{G_i} = T_{G_j}$ with $\sigma_G(g)^{-1}g = t \in T_{G,i}$. Then,

$$\begin{aligned}\sigma_G(gn_ig^{-1}) &= \sigma_G(g)n_i\sigma_G(g)^{-1} = gt^{-1}n_itg^{-1} = gn_in_i^{-1}t^{-1}n_itg^{-1} \\ &= g(s_{\alpha_i} \cdot t^{-1})tg^{-1}.\end{aligned}$$

Set $t = \exp X$ for some $X \in \mathfrak{t}_{G,i}$. We get

$$(s_{\alpha_i} \cdot t^{-1})t = \exp(X - s_{\alpha_i} \cdot X) = \exp(\alpha_i(X)H_{\alpha_i}).$$

Since $(\sigma \cdot \alpha_i)(t) = (-\alpha_i)(t) = \alpha_i(t^{-1})$ because α_i is imaginary, and that on the other hand

$$(\sigma \cdot \alpha_i)(t) = \overline{\alpha_i(\sigma_G(t))} = \overline{\alpha_i(t^{-1})},$$

we see that $\alpha_i(t) = e^{\alpha_i(X)} \in \mathbb{R}$. Thus $\alpha_i(X) = 0$ or $\pi \mod (2\pi)$. Let us now use the fact that α_j is imaginary non-compact to rule out the case $\alpha(X) = \pi \mod (2\pi)$. Let us choose a $\mathfrak{sl}_2$-triple $(X_{\alpha_i}, X_{-\alpha_i}, H_{\alpha_i})$ with $\sigma_G(X_{\alpha_i}) = X_{-\alpha_i}$ as we may, since α_i is imaginary non-compact. Set

$$(X_{\alpha_j}, X_{-\alpha_j}, H_{\alpha_j}) = (g \cdot X_{\alpha_i}, g \cdot X_{-\alpha_i}, g \cdot H_{\alpha_i}).$$

We have

$$\begin{aligned}&\sigma_G(X_{\alpha_j}) = \sigma_G(g)X_{-\alpha_i} = \sigma_G(g)g^{-1} \cdot X_{-\alpha_j} \\ &= gt^{-1}g^{-1} \cdot X_{-\alpha_j} = g(t^{-1} \cdot X_{-\alpha_i})g^{-1} = ((-\alpha)(t^{-1}))X_{-\alpha_j} = \alpha(t)X_{-\alpha_j}.\end{aligned}$$

Since α_j is imaginary non-compact, $\alpha_i(t)$ must be positive, and thus $\alpha(X) = 0 \mod (2\pi)$. We can now conclude that $\exp(\alpha(X)H_{\alpha_i}) = 0$ since $2\pi\mathbb{Z}H_{\alpha_i}$ is in the kernel of the exponential map, and finally, $\sigma_G(gn_ig^{-1}) = gn_ig$. Then s_{α_j} is realized by $n_j = gn_ig^{-1} \in G(\mathbb{R})^{\delta_{0,j}}$. $\square$

2.10. Suppose that $\delta_0 \in T_G(\mathbb{R})$ is semi-regular, such that the roots $\pm\alpha$ of $\mathfrak{t}_{G,\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}^{\delta_0}$ are imaginary. Let $H_\alpha \in i\mathfrak{t}_G$ be the coroot of α, and $\delta_\nu = \delta_0 \exp(i\nu H_\alpha) \in T_G(\mathbb{R})$. Then for ν sufficiently small and non-zero, δ_ν is a regular element in T_G. Let $w_\alpha \in (G^{\delta_0})_0$ be an element realizing the Weyl reflection s_α with respect to α. We have then :

$$w_\alpha\delta_0 \exp(i\nu H_\alpha)w_\alpha^{-1} = \exp\delta_0(-i\nu H_\alpha).$$

Hence, δ_ν and $\delta_{-\nu}$ are stably conjugate, and there are two possibilities: either they are in the same conjugacy class in $G(\mathbb{R})$, or they are not. Let us consider the second case.

Let ν be small enough and non-zero, so that δ_ν is regular. Let $\gamma_\nu = \eta_x^{-1}(\delta_\nu)$, and Σ_{γ_ν} be a set of representatives of conjugacy classes under $G(\mathbb{R})$ in the stable conjugacy class of elements for which γ_ν is a norm. Then we may assume that δ_ν and $\delta_{-\nu}$ are in Σ_{γ_ν}.

We can construct another admissible embedding:

$$\eta_{x\psi(w_\alpha^{-1})} : T_H \longrightarrow T_G$$
$$\gamma^\nu \mapsto \delta^{-\nu}$$

using the same admissible embedding $T_H \xrightarrow{\eta} T$ of T_H in G^* and the element $x\psi(w_\alpha)$. It is indeed easy to show that $(\mathrm{Int}\,(x\psi(w_\alpha)) \circ \psi)_{|T_G}$ is defined over $\mathbb{R}$ from the fact that $(\mathrm{Int}\, x \circ \psi)_{|T_G}$ and the reflection s_α on T_G are defined over $\mathbb{R}$. We will denote $\eta_{x\psi(w_\alpha)}$ by $\overline{\eta}_x$ for short.

In the situation of Section 2.9, suppose that for all $i \in I_{\gamma 0}$, the reflection s_{α_i} is not realized in $G(\mathbb{R})^{\delta_{0,i}}$ (see Lemma 2.13). Replacing some admissible embeddings in the complete system $\{\eta_{x_i}\}$ by conjugate ones, we can choose a subset $J_{\gamma 0}$ consisting of half the indices $i \in I_{\gamma 0}$, such that :

$$\{\eta_{x_i}\}_{i \in I_{\gamma 0}} = \{\eta_{x_j}, \bar{\eta}_{x_j}\}_{j \in J_{\gamma 0}}.$$

If the reflections s_{α_i}, $i \in I_{\gamma 0}$, are realized in $G(\mathbb{R})^{\delta_{0,i}}$, we set $J_{\gamma 0} = I_{\gamma 0}$.

Lemma 2.14. *In the setting of Section 2.9, let us suppose that $I_{\gamma 0}$ is not empty. For all $j \in J_{\gamma 0}$, construct an admissible embedding η'_{x_j} : $T'_H \longrightarrow T'_G$ as in the end of the proof of lemma 2.12. Then the set $\{\eta'_{x_j}\}_{j \in J_{\gamma 0}}$ is a complete system of admissible embeddings of T'_H in G.*

Proof. First, let us notice that that constructions of lemmas 2.8 and 2.12 are inverse of each other in an obvious sense. Notice also that if α is real or non-compact imaginary, and if w_α is an element in G^{δ_0} realizing the reflection s_α we can always define $\bar{\eta}_x$ as above, replacing x by $x\psi(w_\alpha)$. But it is clear that when w_α can be realized in $G(\mathbb{R})^{\delta_0}$, $\bar{\eta}_x$ is conjugate to η_x. This is always the case if α is real. The result follows easily. □

3. Transfer of orbital integrals

3.1. Normalization of measures. In order to define the transfer of orbital integrals we have to normalize invariant measures on the various groups in a consistent way. We chose Duflo-Vergne's normalization, defined as follows: let A be a reductive group (complex or real), and pick an A-invariant symmetric, non-degenerate bilinear form κ on $\mathfrak{a}$. Then $\mathfrak{a}$ will be endowed with the Lebesgue measure dX such that the volume of a parallelotope supported by a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{a}$ is equal to $|\det(\kappa(X_i, X_j)|^{\frac{1}{2}}$ and A will be endowed with the Haar measure tangent to dX. If M is a closed subgroup of A, such that κ is non-degenerate on $\mathfrak{m}$, we endow M with the Haar measure determined by κ as above. If $M' \subset M$ are two closed subgroups of A such that κ is non-degenerate on their respective Lie algebras, we endow M/M' with the M-invariant measure, which is the quotient of the Haar measures on M and M' defined as above. We will denote it by $d\dot{m}$.

3.2. Orbital integrals on $G(\mathbb{R})$**.** Let $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$. Its orbital integral is the function defined on $G(\mathbb{R})_{reg}$ by :

$$J_{G(\mathbb{R})}(f)(x) = |\det(\mathrm{Id} - \mathrm{Ad}x^{-1})_{\mathfrak{g}_\mathbb{C}/\mathfrak{t}_\mathbb{C}}|^{\frac{1}{2}} \int_{G(\mathbb{R})/T(\mathbb{R})} f(gxg^{-1})\, d\dot{g}$$

where $T(\mathbb{R})$ is the Cartan subgroup of $G(\mathbb{R})$ containing x and $d\dot{g}$ is the invariant measure on $G(\mathbb{R})/T(\mathbb{R})$ normalized with our conventions. Note that if x is strongly regular, then $G^x(\mathbb{R}) = T(\mathbb{R})$. These objects have been studied in [HC1] and later in [B1]. We recall their properties and for this we need some notation.

Recall that if $T(\mathbb{R})$ is a Cartan subgroup of $G(\mathbb{R})$ we have a decomposition :

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \sum_{\beta \in R(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})} \mathfrak{g}_\mathbb{C}^\beta$$

where $R(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ is the root system of $\mathfrak{t}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{g}_\mathbb{C}^\beta$ is the root space for the root β.

Let P be a system of positive imaginary roots in $R(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. We introduce Harish-Chandra normalizing factor b_P on $T(\mathbb{R})_{reg}$:

$$b_P(a) = \frac{\prod_{\alpha \in P} \det(\mathrm{Id} - \mathrm{Ad}a^{-1})_{|\mathfrak{g}_\mathbb{C}^\alpha}}{|\prod_{\alpha \in P} \det(\mathrm{Id} - \mathrm{Ad}a^{-1})_{|\mathfrak{g}_\mathbb{C}^\alpha}|}$$

Definition 3.1. Let $T(\mathbb{R})$ be a Cartan subgroup of $G(\mathbb{R})$. We will denote by $T(\mathbb{R})_{I-reg}$ (resp. $T(\mathbb{R})_{In-reg}$) the set of $a \in T(\mathbb{R})$ such that the root system of $\mathfrak{t}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}^a$ has no imaginary (resp. non-compact imaginary) roots. This implies (in both cases) that $\mathfrak{t}$ is a maximally split Cartan subalgebra of $\mathfrak{g}^a$.

With notations as above, we denote by $S(\mathfrak{t}_\mathbb{C})$ the symmetric algebra of $\mathfrak{t}_\mathbb{C}$, and we identify it with the algebra of differential operators on $T(\mathbb{R})$ which are invariant under left translations by elements of $T(\mathbb{R})$. We denote by $\partial(u)$ the differential operator corresponding to $u \in S(\mathfrak{t}_\mathbb{C})$.

Let $A(\mathbb{R})$ be a Cartan subgroup of $G(\mathbb{R})$, $y \in A(\mathbb{R})$ and ϕ a function on $A(\mathbb{R})_{reg}$. Let β be a imaginary root of $\mathfrak{a}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$, and $H_\beta \in i\mathfrak{a}$ its coroot. Then, when the limits in the following formula exist we set:

$$[\phi]_\beta^+(y) = \lim_{t \to 0^+} \phi(y \exp tiH_\beta) + \lim_{t \to 0^-} \phi(y \exp tiH_\beta).$$

Definition 3.2. Let $\mathcal{I}(G(\mathbb{R}))$ be the subspace of $\mathcal{C}^\infty(G(\mathbb{R})_{reg})$ of functions ψ which are constant on the conjugacy classes and have the properties I_1, I_2, I_3, I_4 which we now define :

I_1 : if $A(\mathbb{R})$ is a Cartan subspace of $G(\mathbb{R})$, for all compact subset K of $A(\mathbb{R})$ and for all $u \in S(\mathfrak{a}_\mathbb{C})$ we have:

$$\sup_{a\in K_{reg}} |\partial(u)\cdot \psi_{|A(\mathbb{R})}(a)| < \infty.$$

I_2 : if $A(\mathbb{R})$ is a Cartan subspace of $G(\mathbb{R})$, for all system P of positive imaginary roots in $R(\mathfrak{a}_\mathbb{C}, \mathfrak{g}_\mathbb{C})$, $b_P\psi_{|A(\mathbb{R})}$ has a smooth extension on $A(\mathbb{R})_{In-reg}$. This is equivalent to:

I_2' : $\psi_{|A(\mathbb{R})}$ has a smooth extension on $A(\mathbb{R})_{I-reg}$, and for all semi-regular element $x \in A(\mathbb{R})$ such that the roots $\pm\alpha$ of $\mathfrak{a}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}^x$ are compact imaginary, for all $u \in S(\mathfrak{a}_\mathbb{C})$,

$$[\partial(u)\cdot\psi_{|A(\mathbb{R})}]_\alpha^+(x) = 0$$

I_3 : for all jump data $(x, A(\mathbb{R}), A_1(\mathbb{R}), c_\alpha)$ and for all $u \in S(\mathfrak{a}_\mathbb{C})$,

$$[\partial(u)\cdot\psi_{|A(\mathbb{R})}]_\alpha^+(x) = \mathrm{d}(x)\, \partial(c_\alpha\cdot u)\cdot\psi_{|A_1(\mathbb{R})}(x),$$

where $\mathrm{d}(x)$ is equal to 2 if the reflection $s_\alpha \in W(G, A)$ is realized in $G(\mathbb{R})^x$ and 1 otherwise.

I_4 : if $A(\mathbb{R})$ is a Cartan subspace of $G(\mathbb{R})$, Supp $(\psi_{|A(\mathbb{R})})$ is a compact subspace of $A(\mathbb{R})$.

The space $\mathcal{I}(G(\mathbb{R}))$ is endowed with a topology of an inductive limit of Fréchet spaces, and we denote by $\mathcal{I}(G(\mathbb{R}))'$ its dual.

Theorem 3.3. *(Bouaziz* [B1]*) The map $J_{G(\mathbb{R})}$ is linear, continuous and surjective from $\mathcal{C}_c^\infty(G(\mathbb{R}))$ onto $\mathcal{I}(G(\mathbb{R}))$, and its transpose ${}^tJ_{G(\mathbb{R})}$ realizes a bijection from $\mathcal{I}(G(\mathbb{R}))'$ onto the space of invariant distributions on $G(\mathbb{R})$.*

3.3. Twisted orbital integrals on $G(\mathbb{R})$.

Recall the character ω of $G(\mathbb{R})$ with Langlands parameter $\mathbf{a} \in H^1(W_\mathbb{R}, Z(\hat{G}))$.

Definition 3.4. Let $x \in G(\mathbb{R})$ be a regular element and let $T(\mathbb{R})$ be the Cartan subgroup of $G(\mathbb{R})$ containing x. For all $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$, define $J^\omega_{G(\mathbb{R})}(f)(x)$ as follows. If $T(\mathbb{R})$ is not included in $G(\mathbb{R})^+$, put

$$J^\omega_{G(\mathbb{R})}(f)(x) = 0.$$

If $T(\mathbb{R})$ is included in $G(\mathbb{R})^+$, put

$$J^\omega_{G(\mathbb{R})}(f)(x) = |\det(\mathrm{Id} - \mathrm{Ad}x^{-1})_{\mathfrak{g}_\mathbb{C}/\mathfrak{t}_\mathbb{C}}|^{\frac{1}{2}} \int_{G(\mathbb{R})/T(\mathbb{R})} \omega(g)\, f(gxg^{-1})\, d\dot{g}.$$

Remarks. 1 - Suppose that $J^\omega_{G(\mathbb{R})}$ is not always 0. Thus there is a Cartan subgroup of $G(\mathbb{R})$ in the kernel of ω. Let us decompose $\mathfrak{g}_\mathbb{C}$ as the sum of its center and its semi-simple part $\mathfrak{g}_\mathbb{C} = \mathfrak{c}_\mathbb{C} \oplus \mathfrak{g}_\mathbb{C}^{der}$. Since ω is a character, its differential $d\omega$ is 0 on $\mathfrak{g}_\mathbb{C}^{der}$ and since $\mathfrak{c}_\mathbb{C} \subset \mathfrak{t}_\mathbb{C}$, $d\omega$ is also 0 on $\mathfrak{c}_\mathbb{C}$. Thus $d\omega$ is 0 on $\mathfrak{g}_\mathbb{C}$, and ω is 0 on $G(\mathbb{R})_0$. Thus $G(\mathbb{R})^+$ is a normal open subgroup of $G(\mathbb{R})$. This condition was already noticed in Section 1.7 from other considerations.

2 - Let us fix a system of representatives $\{g_1, \dots, g_r\}$ for the cosets of $G(\mathbb{R})^+$ in $G(\mathbb{R})$. If $T(\mathbb{R})$ is included in $G(\mathbb{R})^+$ and $x \in T(\mathbb{R})$ is regular,

$$J^\omega_{G(\mathbb{R})}(f)(x) = \sum_{i=1}^r \omega(g_i)\, J_{G(\mathbb{R})^+}(f^{g_i^{-1}})(x)$$

where $f^{g_i^{-1}}$ is the function $x \mapsto f(g_i x g_i^{-1})$.

3.4. Stable orbital integrals on $H_1(\mathbb{R})$. Recall the z-pair (H_1, ξ_{H_1}) from Section 1.9, the resulting central extension of $H(\mathbb{R})$:

$$1 \to Z_1(\mathbb{R}) \to H_1(\mathbb{R}) \to H(\mathbb{R}) \to 1$$

and the character λ_{H_1} of $Z_1(\mathbb{R})$ (see Eq. 1.5).

Let $\mathcal{C}^\infty_{c,Z_1}(H_1(\mathbb{R}), \lambda_{H_1})$ be the space of smooth functions f^{H_1} on $H_1(\mathbb{R})$ with compact support modulo $Z_1(\mathbb{R})$ and such that:

$$f^{H_1}(zh) = \lambda_{H_1}(z)^{-1} f^{H_1}(h) \quad (h \in H_1(\mathbb{R})_{reg},\, z \in Z_1(\mathbb{R}))$$

The orbital integral of such a function is given by :

$$J_{H_1(\mathbb{R})}(f^{H_1})(\gamma_1) = |\det(\mathrm{Id} - \mathrm{Ad}\gamma_1^{-1})_{\mathfrak{h}_\mathbb{R}/\mathfrak{t}_\mathbb{R}}|^{\frac{1}{2}} \int_{H_1(\mathbb{R})/T_1(\mathbb{R})} f(h\gamma_1 h^{-1})\, d\dot{g}$$

where $\gamma_1 \in H_1(\mathbb{R})$ is regular and $d\dot{h}$ is the invariant measure on $H_1(\mathbb{R})/T_1(\mathbb{R})$ normalized with our conventions. This is a well-defined converging integral since $\mathrm{Supp}\, f \cap \mathcal{O}_{\gamma_1}$ is compact.

The stable orbital integral of the function $f^{H_1} \in \mathcal{C}^\infty_{c,Z_1}(H_1(\mathbb{R}), \lambda_{H_1})$ is defined by :

$$J^{st}_{H_1}(f^{H_1})(\gamma_1) = \sum_{\gamma_i} J_{H_1(\mathbb{R})}(f^{H_1})(\gamma_i)$$

where the sum is taken over a system of representatives of conjugacy class in $H_1(\mathbb{R})$ in the stable conjugacy class of γ_1.

In [B1], A. Bouaziz gave a characterization of stable orbital integrals of compactly supported functions on a real algebraic reductive connected group. We rephrase his results for functions in $\mathcal{C}^\infty_{c,Z_1}(H_1(\mathbb{R}), \lambda_{H_1})$, indicating briefly how the proof can be adapted.

The map $\gamma_1 \mapsto J^{st}_{H_1(\mathbb{R})}(f^{H_1})(\gamma_1)$ is smooth on $H_1(\mathbb{R})_{reg}$, stably invariant (i.e. constant on stable conjugacy classes) and satisfies for all $h \in H_1(\mathbb{R})_{reg}$, $z \in Z_1(\mathbb{R})$

$$J^{st}_{H_1(\mathbb{R})}(f^{H_1})(zh) = \lambda_{H_1}(z)^{-1} J^{st}_{H_1}(f^{H_1})(h). \tag{3.1}$$

Let $\mathcal{I}^{st}(H_1(\mathbb{R}),\lambda_{H_1})$ be the subspace of $\mathcal{C}^\infty(H_1(\mathbb{R})_{reg})$ of functions ψ which are constant on the stable conjugacy classes satisfing the properties $I_1^{st}=I_1$, I_2^{st}, I_3^{st}, I_4^{st}, I_5^{st}.

I_2^{st}: if $A(\mathbb{R})$ is a maximal torus of H_1 defined over $\mathbb{R}$, for all system P of positive imaginary roots in $R(\mathfrak{a}_\mathbb{C},\mathfrak{h}_{1,\mathbb{C}})$, $b_P\psi_{|A(\mathbb{R})}$ has a smooth extension on $A(\mathbb{R})_{st-In-reg}$, where $A(\mathbb{R})_{st-In-reg}$ is the set of $a\in A(\mathbb{R})$ such that all elements in $A(\mathbb{R})$ stably conjugate to a are in $A(\mathbb{R})_{In-reg}$. Note that this rather subtle definition is not really necessary here since for H_1 quasi-split, we have $A(\mathbb{R})_{st-In-reg}=A(\mathbb{R})_{I-reg}$ ([Sh1], prop. 4.11). So in fact I_2^{st} reduces to :

$I_2^{st\prime}$: $\psi_{|A(\mathbb{R})}$ has a smooth extension on $A(\mathbb{R})_{I-reg}$.

I_3^{st}: for all jump data $(x,A(\mathbb{R}),A_1(\mathbb{R}),c_\alpha)$, and for all $u\in S(\mathfrak{a}_\mathbb{C})$,

$$[\partial(u)\cdot\psi_{|A(\mathbb{R})}]_\alpha^+(x)=2\partial(c_\alpha\cdot u)\cdot\psi_{|A_1(\mathbb{R})}(x)$$

Note that the right-hand side is well defined because $x\in A(\mathbb{R})_{1,I-reg}$.

I_4^{st}: if A is a maximal torus of H_1 defined over $\mathbb{R}$, $\mathrm{Supp}\,(\psi_{|A(\mathbb{R})})$ is a compact subspace of $A(\mathbb{R})$ modulo $Z_1(\mathbb{R})$.

I_5^{st}: for all $h\in H_1(\mathbb{R})$, $z\in Z_1(\mathbb{R})$:

$$\psi(zh)=\lambda_{H_1}(z)^{-1}\psi(h) \tag{3.2}$$

The space $\mathcal{I}^{st}(H_1(\mathbb{R}),\lambda_{H_1})$ is endowed with a topology of an inductive limit of Fréchet spaces and we denote by $\mathcal{I}^{st}(H_1(\mathbb{R}),\lambda_{H_1})'$ its dual. For all functions $f^{H_1}\in\mathcal{C}^\infty_{c,Z_1}(H_1(\mathbb{R}),\lambda_{H_1})$, $J^{st}_{H_1(\mathbb{R})}(f^{H_1})\in\mathcal{I}^{st}(H_1(\mathbb{R}),\lambda_{H_1})$ (see [Sh1] and [B1] §6 for the case $H_1=H$, i.e. orbital integrals of smooth functions with compact support, and see below for an argument of how this can be adapted to the general case). The last property is a easy consequence of (3.1).

We can now define stable distributions on $H_1(\mathbb{R})$ as the closure of the subspace of $\mathcal{C}^\infty_{c,Z_1}(H_1(\mathbb{R}),\lambda_{H_1})'$ generated by distributions of the form

$$f\mapsto J^{st}_{H_1}(f)(x)$$

for regular x in $H_1(\mathbb{R})$. We will denote this space by $\mathrm{Dist}(H_1(\mathbb{R}),\lambda_{H_1})^{st}$.

Theorem 3.5. *(see* [B1]*, théorème 6.1) The map :*

$$J^{st}_{H_1};\mathcal{C}^\infty_{c,Z_1}(H_1(\mathbb{R}),\lambda_{H_1})\to\mathcal{I}^{st}(H_1(\mathbb{R}),\lambda_{H_1})$$

is linear, continuous, surjective and its transpose ${}^tJ^{st}_{H_1}$ *realizes an isomorphism between* $\mathcal{I}^{st}(H_1(\mathbb{R}),\lambda_{H_1})'$ *and the space of stable invariant distributions* $\mathrm{Dist}(H_1(\mathbb{R}),\lambda_{H_1})^{st}$.

Proof. Suppose that the extension

$$1\to Z_1(\mathbb{R})\to H_1(\mathbb{R})\to H(\mathbb{R})\to 1 \tag{3.3}$$

is split, and let $c:\ H(\mathbb{R})\to H_1(\mathbb{R})$ be a section. This section provides an isomorphism :

$$\mathrm{Restr}:\mathcal{C}^\infty_{c,Z_1}(H_1(\mathbb{R}),\lambda_{H_1})\to\mathcal{C}^\infty_c(H(\mathbb{R}))$$

by restricting a function to $c(H(\mathbb{R})) \simeq H(\mathbb{R})$.

Thus $\mathcal{C}^\infty_{c,Z_1}(H_1(\mathbb{R}),\lambda_{H_1})' \simeq \mathrm{Distr}(H(\mathbb{R}))$ and we have the following commuting diagram :

$$\begin{array}{ccc} \mathcal{C}^\infty_{c,Z_1}(H_1(\mathbb{R}),\lambda_{H_1}) & \xrightarrow{\text{Restr}} & \mathcal{C}^\infty_c(H(\mathbb{R})) \\ \Big\downarrow J^{st}_{H_1(\mathbb{R})} & & \Big\downarrow J^{st}_{H(\mathbb{R})} \\ \mathcal{I}^{st}(H_1(\mathbb{R}),\lambda_{H_1}) & \xrightarrow{\text{Restr}} & \mathcal{I}^{st}(H(\mathbb{R})) \end{array}$$

The results follow easily from Bouaziz' results in this case.

When (3.3) is not split, we use the fact that $H_1(\mathbb{R}) \simeq H_{1,der}(\mathbb{R}) \times Z(H_1)(\mathbb{R})/F$ where F is a finite subgroup and $Z(H_1)(\mathbb{R})$ is the center of $H_1(\mathbb{R})$. The theorem is established for functions in $\mathcal{C}^\infty_{c,Z_1}(H_{1,der}(\mathbb{R}) \times Z(H_1)(\mathbb{R}),\lambda_{H_1})$. We deduce the statement for $\mathcal{I}^{st}(H_1(\mathbb{R}),\lambda_{H_1})$ from the following commutative diagram :

$$\begin{array}{ccc} \mathcal{C}^\infty_{c,Z_1}(H_{1,der}(\mathbb{R}) \times Z(H_1)(\mathbb{R}),\lambda_{H_1}) & \xrightarrow{\text{M}} & \mathcal{C}^\infty_c(H_1(\mathbb{R}),\lambda_{H_1}) \\ \Big\downarrow J^{st}_{H_{1,der}(\mathbb{R})\times Z(H_1)(\mathbb{R})} & & \Big\downarrow J^{st}_{H_1(\mathbb{R})} \\ \mathcal{I}^{st}(H_{1,der}(\mathbb{R}) \times Z(H_1)(\mathbb{R}),\lambda_{H_1}) & \xrightarrow{\text{M}} & \mathcal{I}^{st}(H_1(\mathbb{R}),\lambda_{H_1}) \end{array}$$

where $M(\phi)(\gamma) = \sum_{z\in F} \phi(\gamma z)$ for any function ϕ on $H_{1,der}(\mathbb{R})\times Z(H_1)(\mathbb{R})$. □

Let $T_{H_1}(\mathbb{R})$ be a Cartan subgroup of $H_1(\mathbb{R})$ with projection $T_H(\mathbb{R})$ on $H(\mathbb{R})$. There is an exact sequence:

$$\{0\} \to \mathfrak{z}_1 \to \mathfrak{h}_1 \to \mathfrak{h} \to \{0\} \tag{3.4}$$

of Lie algebras, with $\mathfrak{z}_1$ central in $\mathfrak{h}_1$. Since such a sequence always splits, we may, by fixing a section of 3.4 identify $\mathfrak{h}$ with a subalgebra of $\mathfrak{h}_1$, and :

$$\mathfrak{h}_1 = \mathfrak{z}_1 \oplus \mathfrak{h} \tag{3.5}$$

This decomposition (3.5) induces:

$$\mathfrak{t}_{H_1} = \mathfrak{z}_1 \oplus \mathfrak{t}_H \tag{3.6}$$

Furthermore, the decompositions:

$$\mathfrak{h}_{1,\mathbb{C}} = \mathfrak{t}_{H_1,\mathbb{C}} \oplus \sum_{\alpha\in R(T_{H_1},H_1)} \mathfrak{h}^\alpha_{1,\mathbb{C}}$$

$$\mathfrak{h}_{\mathbb{C}} = \mathfrak{t}_{H,\mathbb{C}} \oplus \sum_{\alpha\in R(T_H,H)} \mathfrak{h}^\alpha_{\mathbb{C}}$$

provide identification between $R(T_{H_1},H_1) \simeq R(T_H,H)$ and $\mathfrak{h}^\alpha_{1,\mathbb{C}} \simeq \mathfrak{h}^\alpha_{\mathbb{C}}$.

Let $\gamma_1 \in T_{H_1}(\mathbb{R})$ and γ its projection on $T_H(\mathbb{R})$. Let $\alpha \in R(T_{H_1},H_1) \simeq R(T_H,H)$; then $\alpha(\gamma_1) = \alpha(\gamma)$. Thus, if P is a system of positive imaginary roots in $R(T_{H_1},H_1) \simeq R(T_H,H)$, we get $b_P(\gamma_1) = b_P(\gamma)$.

We end this section by the following remark concerning differential operators from $S(\mathfrak{t}_{H_1,\mathbb{C}})$. It is clear that for all smooth function ψ on $H_1(\mathbb{R})_{reg}$ satisfying (3.2) and for all $u \in S(\mathfrak{z}_{1,\mathbb{C}})$,

$$\partial(u) \cdot \psi = \mathrm{d}\lambda_{H_1}^{-1}(u)\psi$$

i.e. ψ is a eigenfunction for all $u \in S(\mathfrak{z}_{1,\mathbb{C}})$. Thus, to check properties $I_1^{st}, I_2^{st}, I_3^{st}$ we have only to consider differential operators coming from $S(\mathfrak{t}_{H,\mathbb{C}})$ (since (3.6) yields $S(\mathfrak{t}_{H_1,\mathbb{C}}) = S(\mathfrak{t}_{H,\mathbb{C}}) \otimes S(\mathfrak{z}_{1,\mathbb{C}})$).

3.5. Transfer of orbital integrals. If γ_1 is an element in $H_1(\mathbb{R})$, γ will denote its prjection on $H(\mathbb{R})$. Notion and terminology relative to $H(\mathbb{R})$ will be transfered to $H_1(\mathbb{R})$ using the projection. (For instance, the notion of G-regular element.) Langlands and Shelstad have defined absolute transfer factors $\Delta(\gamma_1, \delta)$ where $\gamma_1 \in H_1(\mathbb{R})$ is G-regular and $\Delta(\gamma_1, \delta) = 0$ if γ is not a norm of $\delta \in G(\mathbb{R})$. This transfer factor is a product of four terms Δ_I, Δ_{II}, Δ_{III_1} and Δ_{III_2} (we omit their term Δ_{IV} since it is already included in our definition of orbital integrals). We will recall the properties of these transfer factors when we need them.

We say that the function $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$ and the function $f^{H_1} \in \mathcal{C}_c^\infty(H(\mathbb{R}), \lambda_{H_1})$ have matching orbital integrals if

$$J_{H_1(\mathbb{R})}^{st}(f^{H_1})(\gamma_1) = \sum_{\delta \in \Sigma_\gamma} \Delta(\gamma_1, \delta) J_G^\omega(f)(\delta) \tag{3.7}$$

for every G-regular $\gamma_1 \in H_1(\mathbb{R})$. The sum (which might be empty, in which case the right-hand side is 0) is taken over a set of representative of conjugacy classes under $G(\mathbb{R})$ of elements $\delta \in G(\mathbb{R})$ for which γ is a norm and such that the unique Cartan subgroup in $G(\mathbb{R})$ containing δ is contained in $G(\mathbb{R})^+$.

In this case, we can rewrite (3.7) it as

$$J_{H_1(\mathbb{R})}^{st}(f^{H_1})(\gamma_1) = \sum_{\delta \in \Sigma_\gamma} \Delta(\gamma_1, \delta) \sum_{i=1}^{r} \omega(g_i) J_{G(\mathbb{R})^+}(f^{g_i^{-1}})(\delta). \tag{3.8}$$

The principal result of the whole theory is that for every function $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$, there is a function $f^{H_1} \in \mathcal{C}_c^\infty(H_1(\mathbb{R}), \lambda_{H_1})$ having matching orbital integrals with f. Using the terminology of the previous paragraphs, we can rephrase it in the following form :

Theorem 3.6. *For all $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$, there is a function $f^{H_1} \in \mathcal{C}_c^\infty(H_1(\mathbb{R}), \lambda_{H_1})$ having matching orbital integrals with f, ie. there exists an element* $\mathrm{Trans}\,(f) \in \mathcal{I}^{st}(H_1(\mathbb{R}), \lambda_{H_1})$ *such that :*

$$\mathrm{Trans}\,(f)(\gamma_1) = \sum_{\delta \in \Sigma_\gamma} \Delta(\gamma_1, \delta) \sum_{i=1}^{r} \omega(g_i) \psi^i(\delta) \tag{3.9}$$

for all G-regular element γ_1 of $H_1(\mathbb{R})$ originating in $G(\mathbb{R})^+$, with $\psi^i = J_{G(\mathbb{R})^+}(f^{g_i^{-1}})$, and

$$\text{Trans}\,(f)(\gamma_1) = 0 \tag{3.10}$$

If γ_1 does not originate in $G(\mathbb{R})^+$. Futhermore, Trans (f) *is defined on regular, non-G-regular elements of $H_1(\mathbb{R})$ by smooth extension.*

The next section will be devoted to the proof of theorem 3.6.

4. Proof of theorem 3.6

4.1. We sketch briefly the proof of the theorem before going into details. Let us first remark that the right-hand side of (3.9) is well defined, i.e. does not depend on the choices of representatives in Σ_γ. This is a consequence of the following lemma :

Lemma 4.1. *(*[LS]*, Lemma 4.1.C).*

$$\Delta(\gamma_1, \delta) = \Delta(\gamma_1, \delta')$$

when δ and δ' are conjugate in G.

The fact that Trans (ψ) is constant on stable conjugacy classes is proved in the same lemma of [LS] :

Lemma 4.2. *$\Delta(\gamma_1, \delta)$ is unchanged when γ_1 is replaced with a stably conjugate element in $H_1(\mathbb{R})$.*

We will show how Trans (f) is defined on regular element of $H_1(\mathbb{R})$ (but not G-regular) by a smooth extension of (3.9). This is lemma 4.3 of [Sh2].

To prove the theorem, we have to establish that properties $I_1^{st}, \dots, I_5^{st}$ hold for Trans (f). As notations suggest, properties $I_1^{st}, \dots, I_4^{st}$ for Trans (ψ) are consequences of properties $I_1, \dots, I_4$ for the ψ^i. Some are immediate $I_1^{st}, I_2^{st}, I_4^{st}$, the other one I_3^{st} requiring extra work. The last property I_5^{st}, is established by the following lemma:

Lemma 4.3. *(*[KS]*, lemma 5.1.C)*
$\Delta(z\gamma_1, \delta) = \lambda_{H_1}(z)^{-1}\Delta(\gamma_1, \delta)$ where $\gamma_1 \in H_1(\mathbb{R})$ is a regular element and $z \in Z_1(\mathbb{R})$.

4.2. Transfer of differential operators. If $T_H(\mathbb{R})$ originates in $G(\mathbb{R})^+$, then there exist a Cartan subgroup $T_G(\mathbb{R})$ in $G(\mathbb{R})$, and an isomorphism :

$$\eta_x : T_H \to T_G$$

defined over $\mathbb{R}$ such that $\eta_x^{-1}(\delta)$ is a norm of δ for all regular $\delta \in T_G(\mathbb{R})$. Recall that this map depends on the choice of a admissible embedding

$T_H \xrightarrow{\eta} T$ of T_H in G^* and an element $x \in G^*$. Differentiating and complexifying, we get isomorphisms over $\mathbb{R}$:

$$\mathfrak{t}_{H,\mathbb{C}} \xrightarrow{\eta} \mathfrak{t}_{\mathbb{C}} \xleftarrow{(\mathrm{Int}\, x \circ \psi)} \mathfrak{t}_{G,\mathbb{C}} \tag{4.1}$$

Let us denote by η_x again the isomorphism between the extreme terms of (4.1) and also for the induced isomorphism between $S(\mathfrak{t}_{G,\mathbb{C}})$ and $S(\mathfrak{t}_{H,\mathbb{C}})$. This will enable us to transfer differential operators.

Let ϕ be a smooth function defined on an open subset $\mathcal{V}$ of $T_G(\mathbb{R})$ and let $\tilde{\phi}$ be its pull-back on by η_x Take $u \in S(\mathfrak{t}_{H,\mathbb{C}})$ and $u' = \eta_x(u) \in S(\mathfrak{t}_{G,\mathbb{C}})$. It is clear that we have for all $\delta \in \mathcal{V}$:

$$\partial(u') \cdot \phi(\delta) = \partial(u) \cdot \tilde{\phi}(\eta_x^{-1}(\delta)). \tag{4.2}$$

4.3. Local behaviour of transfer factors. We continue in the setting of the previous paragraphs, ie. $T_{H_1}(\mathbb{R})$ is a Cartan subgroup of $H_1(\mathbb{R})$ with projection $T_H(\mathbb{R})$ on $H(\mathbb{R})$ originating in $G(\mathbb{R})^+$. Let γ_{01} be an element in $T_{H_1}(\mathbb{R})$. We need to study $\varphi_{|T_{H_1}(\mathbb{R})}$ in a neighborhood of γ_{01}. Let γ_0 be the projection of γ_1 in $T_H(\mathbb{R})$. Fix an admissible embedding η_x of T_H in G and let $\delta_0 = \eta_x(\gamma_0) \in T_G(\mathbb{R})$.

Because of the property I_5^{st} of φ, it is sufficient to study its restriction in a neighborhood $\mathcal{U}_{\gamma_{01}}$ of γ_{01} in

$$\gamma_{01} \exp \mathfrak{t}_{H,\mathbb{R}} \subset T_{H_1}(\mathbb{R}).$$

It is clear that the projection map induces a local topological isomorphim

$$\gamma_{01} \exp \mathfrak{t}_H \to \gamma_0 \exp \mathfrak{t}_H. \tag{4.3}$$

Suppose that $\mathcal{U}_{\gamma_{01}}$ is sufficiently small, so that the above local isomorphism restrict to a topological isomorphism from $\mathcal{U}_{\gamma_{01}}$ onto its image that we call $\mathcal{U}_{\gamma_0}$. Let us still denote by η_x the embedding of $\mathcal{U}_{\gamma_{01}}$ in $T_G(\mathbb{R})$ obtained from the composition (4.3) and $\eta_x : T_H \to T_G$ and call $\mathcal{V}_{\delta_0}$ its image. Then, for any regular element $\delta \in \mathcal{V}_{\delta_0}$, we define

$$\Delta_{\eta_x} : \mathcal{V}_{\delta_0} \to \mathbb{C}$$
$$\delta \mapsto \Delta(\eta_x^{-1}(\delta), \delta)$$

We will need the following properties of this function :

Proposition 4.4. *In the setting above :*

(i) Δ_{η_x} is smooth on $(\mathcal{V}_{\delta_0})_{reg}$. For all compact subset K of $\mathcal{V}_{\delta_0}$, Δ_{η_x} is bounded on K_{reg}. There exists $\lambda \in \mathfrak{t}_{G,\mathbb{C}}^*$ and $c \in \mathbb{C}$ such that

$$\Delta_{\eta_x}(\delta_0 \exp X) = c\, e^{\lambda(X)}$$

for all $X \in \mathfrak{t}_G$ such that $\delta_0 \exp X \in (\mathcal{V}_{\delta_0})_{reg}$. Furthermore, Δ_{η_x} has a smooth continuation on $(\mathcal{V}_{\delta_0})_{I-reg}$. Let τ_λ be the algebra automorphism *of $S(\mathfrak{t}_{G,\mathbb{C}})$ mapping $X \in \mathfrak{t}_{G,\mathbb{C}}$ to $X+\lambda(X)$. We have the following identity of differential operators on $T_G(\mathbb{R})$: for all $u \in S(\mathfrak{t}_{G,\mathbb{C}})$*

$$\partial(u) \circ \Delta_{\eta_x} = \Delta_{\eta_x} \circ \partial(\tau_\lambda(u)).$$

Suppose δ_0 is semi-regular element, such that the roots $\pm\alpha$ of $\mathfrak{t}_{G,\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}^{\delta^0}$ are imaginary. Let $H_\alpha \in i\mathfrak{t}_G$ be the coroot of α, and $\delta_\nu = \delta_0 \exp(i\nu H_\alpha)$. For ν small enough and non-zero, δ_ν is regular. Set $\gamma_{1,\nu} = \eta_x^{-1}(\delta_\nu) \in H_1(\mathbb{R})$.

(ii) if α is compact and $\gamma_0 \in T_H(\mathbb{R})_{reg}$, then

$$\lim_{\nu\to 0^+} \Delta(\gamma_{1,\nu},\delta_\nu) = -\lim_{\nu\to 0^-} \Delta(\gamma_{1,\nu},\delta_\nu),$$

(iii) if α is compact and γ_0 is semi-regular in $H(\mathbb{R})$, then

$$\lim_{\nu\to 0^+} \Delta(\gamma_{1,\nu},\delta_\nu) = \lim_{\nu\to 0^-} \Delta(\gamma_{1,\nu},\delta_\nu),$$

(iv) if α is non-compact and $\gamma_0 \in T_H(\mathbb{R})_{reg}$, then the reflection s_α with respect to the root α is not realized in $G(\mathbb{R})^{\delta_0}$, i.e. δ_ν and $\delta_{-\nu}$ are not conjugate in $G(\mathbb{R})$. Furthermore

$$\lim_{\nu\to 0^+} \Delta(\gamma_{1,\nu},\delta_\nu) = -\lim_{\nu\to 0^-} \Delta(\gamma_{1,\nu},\delta_\nu),$$

$$\Delta(\gamma_{1,\nu},\delta_\nu) = -\Delta(\gamma_{1,\nu},\delta_{-\nu}),$$

(v) if α is non-compact imaginary and γ_0 is semi-regular, then

$$\lim_{\nu\to 0^+} \Delta(\gamma_{1,\nu},\delta_\nu) = \lim_{\nu\to 0^-} \Delta(\gamma_{1,\nu},\delta_\nu),$$

$$\Delta(\gamma_{1,\nu},\delta_\nu) = \Delta(\gamma_{1,\nu},\delta_{-\nu}),$$

The proof relies on the fine analysis of the local properties of transfer factors. It is postponed to Section 6.

4.4. Smooth extension to $H_1(\mathbb{R})_{reg}$. In the setting of the previous sections, suppose that $\gamma_0 \in H(\mathbb{R})$ is regular but not G-regular. If $T_H(\mathbb{R})$ does not originate in $G(\mathbb{R})^+$ then by definition, the restriction of φ to G-regular elements in $T_{H_1}(\mathbb{R})$ is zero, and there is a smooth extention of $\varphi_{|T_{H_1}(\mathbb{R})}$ around γ_{10}. Since γ_{10} is regular, there is a neighborhood $\mathcal{U}$ of γ_{10} in $T_{H_1}(\mathbb{R})$ such that $H_1(\mathbb{R})\cdot\mathcal{U}$ is a neighborhood of γ_{10} in $H_1(\mathbb{R})$. Thus, by invariance, φ is zero in a neighborhood of γ_{10}.

Suppose that $T_H(\mathbb{R})$ originates in $G(\mathbb{R})^+$. Fix a complete system of admissible embeddings $\eta_{x_j} : T_H \to T_j$. Put $\delta_{0,j} = \eta_{x_j}(\gamma_0)$.

Suppose that one of the $\delta_{0,j}$ is in $T_j(\mathbb{R})_{I-reg}$. Then it will be the case for all of them. To see this, take two of them, say $\delta_{0,j}$ and $\delta_{0,k}$. Then, they are conjugate by an element $g \in G$, such that $\operatorname{Int} g$ sends $G^{\delta_{0,j}}$ isomorphically onto $G^{\delta_{0,k}}$ and $\mathfrak{t}_{j,\mathbb{C}}$ isomorphically onto $\mathfrak{t}_{k,\mathbb{C}}$, this latter being defined over $\mathbb{R}$. Thus $\operatorname{Int} g$ sends the roots of $\mathfrak{t}_{j,\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}^{\delta_{0,j}}$ bijectively onto the roots of $\mathfrak{t}_{k,\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}^{\delta_{0,k}}$, respecting their types (real, complex or imaginary). It follows from proposition 4.4 (i) and property I_2 of the ψ^i

for all $i = 1, \dots, r$ that there is a smooth extension of $\varphi_{|T_{H_1}(\mathbb{R})}$ around γ_{10}.

Now suppose that $\delta_{0,j} \notin T_j(\mathbb{R})_{I-reg}$. We want to study the behaviour of $\delta_j \mapsto (\Delta_{\eta_{x_j}} \psi^i_{|T_j(\mathbb{R})})(\delta_j)$ around $\delta_{0,j}$. We drop locally the indices 'i' and 'j', since we are working with only one of them at a time. Assume that δ_0 is semi-regular. Then the roots $\pm\alpha$ of $\mathfrak{t}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}^{\delta_0}$ are imaginary. Suppose they are of compact type. Then for all $u \in S(\mathfrak{t}_{H,\mathbb{C}})$, setting $u' = \eta_x(u)$,

$$\begin{aligned}
&[\partial(u') \cdot (\Delta_{\eta_x} \psi_{|T(\mathbb{R})})]^-_\alpha(\delta_0) \\
&= \lim_{\nu\to 0^+} \partial(u') \cdot (\Delta_{\eta_x} \psi_{|T(\mathbb{R})})(\delta_\nu) - \lim_{\nu\to 0^-} \partial(u') \cdot (\Delta_{\eta_x} \psi_{|T(\mathbb{R})})(\delta_\nu) \\
&= \lim_{\nu\to 0^+} \Delta_{\eta_x}(\delta_\nu)\partial(\tau_\lambda(u)) \cdot \psi_{|T(\mathbb{R})}(\delta_\nu) - \lim_{\nu\to 0^-} \Delta_{\eta_x}(\delta_\nu)\partial(\tau_\lambda(u')) \cdot \psi_{|T(\mathbb{R})}(\delta_\nu) \\
&= \left(\lim_{\nu\to 0^+} \Delta_{\eta_x}(\delta_\nu)\right) [\partial(\tau_\lambda(u')) \cdot \psi_{|T(\mathbb{R})}]^+_\alpha = 0
\end{aligned}$$

We have used Proposition 4.4, (i), (ii) and property I_2 of the ψ. Thus, the contribution of η_x to the right-hand side of (3.9) is smooth around δ_0.

Let us see what happens in the other case, i.e. when the roots $\pm\alpha$ are non-compact. Lemma 4.4 (iv) asserts that in that case δ_ν and $\delta_{-\nu}$ are not conjugate in $G(\mathbb{R})$, i.e. the reflection s_α with respect to the root α is not realized in $G(\mathbb{R})^{\delta_0}$.

Then, we are in the situation considered at the end of paragraph 2.10. To check that φ has a smooth extension around γ_0, we have to look at the contributions the right-hand side of (3.9) of η_x and $\overline{\eta}_x$. We need the following lemma :

Lemma 4.5. *Let $u' = \eta_x(u)$ and $\overline{u}' = \overline{\eta}_x(u)$. Then $\overline{u}' = (u')^{s_\alpha}$*

The proof is obvious. Let us resume our computation :

$$\begin{aligned}
&\lim_{\nu\to 0^+} \partial(u') \cdot (\Delta_{\eta_x}\psi_{|T(\mathbb{R})})(\delta_\nu) - \lim_{\nu\to 0^-} \partial(u') \cdot (\Delta_{\eta_x} \psi_{|T(\mathbb{R})})(\delta_\nu) \\
&+ \lim_{\nu\to 0^+} \partial(\overline{u}') \cdot (\Delta_{\bar{\eta}_x}\psi_{|T(\mathbb{R})})(\delta_{-\nu}) - \lim_{\nu\to 0^-} \partial(\overline{u}') \cdot (\Delta_{\bar{\eta}_x}\psi_{|T(\mathbb{R})})(\delta_{-\nu}) = \\
&\lim_{\nu\to 0^+} \Delta_{\eta_x}(\delta_\nu)\, \partial(\tau_\lambda(u')) \cdot \psi_{|T(\mathbb{R})}(\delta_\nu) - \lim_{\nu\to 0^-} \Delta_{\eta_x}(\delta_\nu)\, \partial(\tau_\lambda(u')) \cdot \psi_{|T(\mathbb{R})}(\delta_\nu) \\
&+ \lim_{\nu\to 0^-} \Delta_{\bar{\eta}_x}(\delta_\nu)\partial(\tau_\lambda(\overline{u}')) \cdot \psi_{|T(\mathbb{R})}(\delta_\nu) - \lim_{\nu\to 0^+} \Delta_{\bar{\eta}_x}(\delta_\nu)\partial(\tau_\lambda(\overline{u}')) \cdot \psi_{|T(\mathbb{R})}(\delta_\nu).
\end{aligned}$$

We have $\lim_{\nu\to 0^+} \Delta_{\bar{\eta}_x}(\delta_\nu) = \lim_{\nu\to 0^+}(\Delta_{\eta_x})(\delta_\nu)$ (Proposition 4.4, (iv)). Hence, if $\partial(\tau_\lambda(u')) = \partial(\tau_\lambda(\overline{u}')) = \partial(\tau_\lambda(u'))^{s_\alpha}$, the whole expression cancels. To complete the proof, it remains to check what happens when $\partial(\tau_\lambda(u')) = -\partial(\tau_\lambda(\overline{u}')) = -\partial(\tau_\lambda(u'))^{s_\alpha}$, the general case being deduced by linearity. Under this latter assumption we use Proposition 4.4(iv) to obtain :

$$[\partial(u') \cdot (\Delta_{\eta_x} \psi_{|T_G(\mathbb{R})})]^-_\alpha(\delta_0) = (\lim_{\nu\to 0^+} \Delta_{\eta_x}(\delta_\nu))[\partial(\tau_\lambda(u')) \cdot \psi_{|T_G(\mathbb{R})}]^+_\alpha(\delta_0)$$

and a well-know principle of Harish-Chandra ([HC1]), asserts that this is zero. For the same reason,

$$[\partial(\overline{u}')\cdot(\Delta_{\bar{\eta}_x}\,\psi_{|T_G(\mathbb{R})})]^-_\alpha(\delta_0)=0.$$

To conclude, we have proved that if $\delta_{0,j}\notin T_j(\mathbb{R})_{I-reg}$ is semi-regular for one j, then the same is true for all of them, and that the various contributions to the right-hand side of (3.9), when suitably grouped, extends to smooth functions around the $\delta_{0,j}$'s. Another principle of Harish-Chandra ([HC2]) asserts that these results still hold if the $\delta_{0,j}$'s are not semi-regular. Thus, Trans$\,(\psi)=\varphi$ has a smooth extension to $H_1(\mathbb{R})_{reg}$.

4.5. Properties $I_1^{st}, I_2^{st}, I_4^{st}$. Let T_{H_1} be a maximal torus of H_1 defined over $\mathbb{R}$. If $T_{H_1}(\mathbb{R})$ does not originate in $G(\mathbb{R})^+$, then the support of $\varphi_{|T_{H_1}(\mathbb{R})}$ is empty. If $T_{H_1}(\mathbb{R})$ originates in $G(\mathbb{R})^+$, we choose a complete system of admissible embeddings of T_H. For each Cartan subgroup $T_G(\mathbb{R})$ in $G(\mathbb{R})$ (and contained in $G(\mathbb{R})^+$) in this complete system, the restriction of the ψ^i to $T_G(\mathbb{R})$ has compact support (I_4), for all $i=1,\dots,r$. It follows easily from the definition that the support of $\varphi_{|T_H}$ is compact.

The two other properties are local, so it is sufficient to check them in the setting of Proposition 4.4. Then, they are immediate consequences of I_1, I_2 for the ψ^i and Proposition 4.4 (i).

4.6. Jump relations. In this section, we will prove the jump relations for $\varphi=$ Trans$\,(f)$. Let $(\gamma_{10},T_{H_1}(\mathbb{R}),T'_{H_1}(\mathbb{R}),c_{\alpha_H})$ be a jump datum on $H_1(\mathbb{R})$.

Corollary 2.11 asserts that if T_H does not originate in $G(\mathbb{R})^+$, nor does T'_H. In that case, $\varphi_{|T_{H_1}}\equiv 0$ and $\varphi_{|T'_{H_1}}\equiv 0$, and I_3^{st} is satisfied at γ_{10}.

We suppose now that T_{H_1} does originate in $G(\mathbb{R})^+$. We fix a complete system of admissible embeddings $\{\eta_{x_j}\}:T_H\to T_j$ as before. We suppose first that $\delta_{0,j}=\eta_{x_j}^{-1}(\gamma_0)$ is semi-regular (recall that if one is, all of them are) and we denote by $\pm\alpha_j$ the roots of $(\mathfrak{t}_{i,\mathbb{C}})$ in $\mathfrak{g}_{\mathbb{C}}^{\delta_{j,0}}$. It means that α_H and α_j have corresponding coroots through η_{x_j}. Recall that $\delta_{j,\nu}=\delta_{j,0}\exp(i\nu H_{\alpha_j})$. Let $\gamma_{j,\nu}=\eta_{x_i}^{-1}(\delta_{j,\nu})=\gamma_{01}\exp(i\nu H_{\alpha_H})$.

Lemma 2.12 asserts that if for all j, $\delta_{j,0}\in T_{j,In-reg}$, then T'_{H_1} does not originate in G. It is then easily check that the two side of the jump relation are 0, and so that I_3^{st} is satisfied.

We are now dealing with the case where both T_{H_1} and T'_{H_1} originate in $G(\mathbb{R})^+$. Let $u\in S(\mathfrak{t}_{H,\mathbb{C}})$ and $v_j=\eta_{x_j}(u)$. We have then :

$$[\partial(u)\cdot\varphi_{|T_{H_1}(\mathbb{R})}]^+_{\alpha_H}(\gamma_{10})=\sum_i\omega(g_i)\sum_j[\partial(v_j)\cdot(\Delta_{\eta_{x_j}}\psi^i_{|T_j(\mathbb{R})})]^+_{\alpha_j}(\delta_{j,0})$$

As in the previous paragraph, thanks to Proposition 4.4 (ii), if $\delta_{j,0}\in T_j(\mathbb{R})_{In-reg}$ the contribution of this term to the right-hand side is 0, so we are left only with the ones such that the root α_j is imaginary and non-compact in $\Delta(\mathfrak{t}_{j,\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^{\delta_{j,0}})$, *ie.* with the set I_{γ_0}.

A general principle of Harish-Chandra asserts that if $\partial(u)^{s_{\alpha_H}} = -\partial(u)$, then the jump relations are satisfied, the jump being 0. In the following computations we assume that $\partial(u)^{s_{\alpha_H}} = \partial(u)$, the general case being deduced by linearity. In particular, in the second case considered above, we have $v_j = \overline{v}_j$. Assuming that J_{γ_0} is 'half of' I_{γ_0}, *ie.* that the reflections with respect to the relevant roots are not realized in the real groups, we compute

$$[\partial(u)\cdot\varphi_{|T_{H_1}(\mathbb{R})}]^+_{\alpha_H}(\gamma_{01}) = \sum_i \omega(g_i)\sum_j [\partial(v_j)\cdot(\Delta_{\eta_{x_j}}\psi^i_{|T_j(\mathbb{R})})]^+_{\alpha_j}(\delta_{j,0}) \tag{4.4}$$

$$= \sum_i \omega(g_i)\sum_{j\in I_{\gamma_0}} [\partial(v_j)\cdot(\Delta_{\eta_{x_j}}\psi^i_{|T_j(\mathbb{R})})]^+_{\alpha_j}(\delta_{j,0}) \tag{4.5}$$

$$\begin{aligned} = \sum_i \omega(g_i)\sum_{j\in J_{\gamma_0}} &(\lim_{\nu\to 0}\Delta_{\eta_{x_j}}(\delta_{j,\nu}))\, \partial(c_{\alpha_j}\cdot\tau_\lambda(v_j))\cdot\psi^i_{|T'_j(\mathbb{R})}(\delta_{j,0}) \\ &+(\lim_{\nu\to 0}\Delta_{\overline{\eta}_{x_j}}(\delta_{j,\nu}))\, \partial(c_{\alpha_j}\cdot\tau_\lambda(\overline{v}_j))\cdot\psi^i_{|T'_j(\mathbb{R})}(\delta_{j,0}) \end{aligned} \tag{4.6}$$

$$= \sum_i \omega(g_i)\sum_{j\in J_{\gamma_0}} 2(\lim_{\nu\to 0}\Delta_{\eta_{x_j}}(\delta_{j,\nu}))\, \partial(c_{\alpha_j}\cdot\tau_\lambda(v_j))\cdot\psi^i_{|T'_j(\mathbb{R})}(\delta_{j,0}) \tag{4.7}$$

$$= \sum_i \omega(g_i)\sum_{j\in J_{\gamma_0}} 2\partial(c_{\alpha_j}\cdot v_j)\cdot(\Delta_{\eta'_{x_j}}\psi^i_{|T'_j(\mathbb{R})})(\delta_{j,0}) \tag{4.8}$$

$$= 2\,\partial(c_{\alpha_H}\cdot u)\cdot\varphi_{|T'_{H_1}(\mathbb{R})}(\gamma_{01}) \tag{4.9}$$

Let us make some comments on these computations. The first line (4.4) is obtained by using the local expression of φ around γ_{01}. The contribution of the indices not in I_{γ_0} being 0, we get (4.5). To get (4.6) and then (4.7), we use Proposition 4.4 (i) and (iv) and the jump relations I_3 for the ψ^i's. Lemma 2.14 and the following Lemma yield (4.8). A similar computation proves the jump relations when $J_{\gamma_0} = I_{\gamma_0}$.

Lemma 4.6. *We have*

$$\lim_{\nu\to 0}\Delta_{\eta_{x_j}}(\delta_{j,\nu}) = \Delta_{\eta'_{x_j}}(\delta_{j,0})$$

and $(\eta'_{x_j})^{-1}(c_{\alpha_j}\cdot v_j) = c_{\alpha_H}\cdot u$.

The proof is also postponed to Section 6.

So far, we have supposed that the $\delta_{j,0}$'s are semi-regular. A well-known result of Harish-Chandra enable us to relax this asumption. Thus the φ satifies the jump relations, and complete the proof of theorem 3.6. □

5. Transfer factors

We will give in this section some details on the definition of transfer factors in [LS], in order to be able to establish some of their properties that we need to complete the proof of the transfer. This also makes this paper more self-contained. On the other hand, we will need also to use some other properties of transfer factors that are established in [LS], [LS2] or [KS]. For these, we simply recall the results and we refer the reader to these papers for the proofs. We also take advantage of the fact that our base field is $\mathbb{R}$ to give somehow simplier or more explicit expressions for the various factors. In particular, the general definitions of [LS], [LS2] or [KS] use the notions of a-data and χ-data which are redundant in our case. Indeed let (B, T) be a pair for G^*, with T defined over $\mathbb{R}$. Let us now explain how we fix the a-data and χ-data on $R(T, G^*)$ or $R(\mathbf{T}, G^*)$. If $\alpha \in R(T, G^*)$ is imaginary positive (*ie.* if $\alpha \in R(T, B)$ and $\sigma_T(\alpha) = -\alpha$), set $a_\alpha = i$, $a_{-\alpha} = -i$, and for all $z \in \mathbb{C}^*$, $\chi_\alpha(z) = z/|z|$, $\chi_{-\alpha}(z) = \bar{z}/|z|$. If α is not imaginary, set $a_\alpha = 1.$ and $\chi_\alpha(z) = 1$. Take $h \in G^*$ such that $(B, T) = \operatorname{Int} h \cdot (\mathbf{B}, \mathbf{T})$ and transport χ and a-data to $R(\mathbf{T}, G^*)$.

It will be convenient to use the following notation : if $z = \exp w \in \mathbb{C}^*$, and if $\alpha \in R(T, G^*)$, we sometimes write $z^{\check{\alpha}}$ for $\exp(wH_\alpha)$.

5.1. An overview of the transfer factors.

In this section, we briefly discuss the role of the various factors in the tranfer of orbital integrals, as a motivation for the definitions to come, but also to shed some light on the path Shelstad took to find them.

Recall from the statement of theorem 3.6 that we want to match certain orbital integrals on $G(\mathbb{R})$ (or rather some linear combination of orbital integrals in a stable orbit) to stable orbital integrals on $H_1(\mathbb{R})$, that orbital integrals are determined by their restrictions to Cartan subgroups, and that we have a correspondence between Cartan subgroups of $G(\mathbb{R})$ and of $H_1(\mathbb{R})$. From definition 3.2, we see that restriction of orbital integrals to Cartan subgroups become more regular when multiplied by a certain factor (denoted b_P there). It is therefore natural to multiply orbital integrals on $G(\mathbb{R})$ and $H_1(\mathbb{R})$ by the corresponding factors, to make them as smooth as possible. The quotient of this factor for $G(\mathbb{R})$ and the one for $H_1(\mathbb{R})$ is essentially what is called Δ_{II} below, and we see it appears as a requirement from invariant harmonic analysis.

The basic idea of endoscopy is that we should put some weights on different orbital integrals on $G(\mathbb{R})$ belonging to a single stable orbit, to form a linear combination which should match some stable orbital integrals on $H_1(\mathbb{R})$. Conjugacy classes in a stable conjugacy class are parametrized by some first cohomology group and the weights will be given by a pairing (Tate-Nakayama) between this cohomology group and a group obtained from the endoscopic data. The corresponding transfer factor is denoted Δ_{III_1} below.

As mentionned in section 1.8, the transpose of the transfer of orbital integrals, which is a map from stable invariant distributions on $H_1(\mathbb{R})$ to invariant distributions on $G(\mathbb{R})$, should induce a correspondence of characters of representations of $H_1(\mathbb{R})$ in some L-packet to characters of representations of $G(\mathbb{R})$ in the corresponding L-packet for G. Consideration of infinitesimal characters of representations of these packets shows that one need to incorporate a correction character (in Shelstad's terminology) to match orbital integrals. This character (of a Cartan subgroup) is the Δ_{III_2} factor.

So far, all the factors are there to deal with the behaviour of orbital integrals on a given Cartan subgroup. The most subtle factor is the remaining one, Δ_I, which deals with the compatibility between pairs of adjacent Cartan subgroups, *i.e.* jump relations. This factor is just a sign, again given by a Tate-Nakayama pairing as for Δ_{III}, but it accomplishes a miracle : not only jump relations will be satisfied, but also the global transfer factor Δ will become canonical, independent of all the choices made to define each factor.

5.2. Some general constructions. G^* is quasi-split and $\mathbf{spl}_{G^*} = \{\mathbf{B}, \mathbf{T}, \{\mathbf{X}_\alpha\}\}$ is a splitting over $\mathbb{R}$. Then Γ acts on the Weyl group $W(G^*, \mathbf{T})$ by automorphisms and one can form the semi-direct product

$$W(G^*, \mathbf{T}) \rtimes \Gamma.$$

For all $\theta \in W(G^*, \mathbf{T}) \rtimes \Gamma$ one can lift θ to an element $n(\theta)$ in $G^* \rtimes \Gamma$ as follows :

• For all simple root α in $R(\mathbf{T}, G^*)$ notice that its coroot H_α and the element X_α which are part of the splitting $\mathbf{spl}_{G^*}$ determines an $\mathfrak{sl}_2$-triple $\{X_\alpha, X_{-\alpha}, H_\alpha\}$ and an embedding of $\mathfrak{sl}_2$ in the Lie algebra of G^*. This embedding exponentiates to a homomorphism $\phi_\alpha : SL_2 \to G^*$. Set

$$n(s_\alpha) = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is clear that $\operatorname{Int} n(s_\alpha)$ realizes the reflection s_α on $\mathbf{T}$. Another expression for $n(s_\alpha)$ is

$$n(s_\alpha) = \exp X_\alpha \exp X_{-\alpha} \exp X_\alpha.$$

• If $w = s_{\alpha_1} \dots s_{\alpha_r}$ is a reduced decomposition in $W(G^*, \mathbf{T})$, set

$$n(w) = n(s_{\alpha_1}) \dots n(s_{\alpha_r}).$$

A well-known result of Chevalley-Steinberg implies that $n(w)$ is independent of the choosen reduced decomposition of w.

• If $\theta = w \times \epsilon$, set $n(\theta) = n(w) \times \epsilon$.

There is an exact sequence

$$1 \to \mathbf{T} \to N(G^*, \mathbf{T}) \rtimes \Gamma \to W(G^*, \mathbf{T}) \rtimes \Gamma \to 1$$

and $n : W(G^*, \mathbf{T}) \rtimes \Gamma \to\to N(G^*, \mathbf{T}) \rtimes \Gamma$ is a section. This gives a 2-cocycle t from $W(G^*, \mathbf{T}) \ltimes \Gamma$ with values in $\mathbf{T}$, defined by

$$n(\theta_1)n(\theta_2) = t(\theta_1, \theta_2)n(\theta_1\theta_2), \quad \theta_1, \theta_2 \in W(G^*, \mathbf{T}) \rtimes \Gamma$$

Lemma 5.1. [LS] *Lemma 2.1.A. The cocycle t is given by*

$$t(\theta_1, \theta_2) = \prod_{\alpha>0,\, \theta_2\alpha<0,\, \theta_1\theta_2\alpha>0} \exp(i\pi H_\alpha), \quad \theta_1, \theta_2 \in W(G^*, \mathbf{T}) \ltimes \Gamma$$

Notice that since $\exp(2i\pi H_\alpha) = 1$, $t(\theta_1, \theta_2)^{-1} = t(\theta_1, \theta_2)$.

5.3. Definition of an invariant. Let (B, T) be a pair for G^*, with T defined over $\mathbb{R}$. Choose $h \in G^*$ such that $(B, T) = \operatorname{Int} h \cdot (\mathbf{B}, \mathbf{T})$. Denote by σ_T both the action of Γ on T and its transport to $\mathbf{T}$ by $\operatorname{Int} h$. Let $w(\sigma_T) \in W(G^*, \mathbf{T})$ be the class of $h^{-1}\sigma_{G^*}(h)$. Then on $\mathbf{T}$, $\sigma_T = w(\sigma_T) \times \sigma$. By Lemma 5.1 $n(\sigma_T) := n(w(\sigma_T) \times \sigma) = n(w(\sigma_T)) \times \sigma$ satisfies

$$n(\sigma_T)\sigma(n(\sigma_T)) = \prod_{\alpha>0, \sigma_T\alpha<0} \exp(i\pi H_\alpha) = \partial x_\sigma$$

where

$$x_\sigma = \prod_{\alpha>0, \sigma_T\alpha<0} \exp\left(i\frac{\pi}{2}H_\alpha\right),$$

the product being over positive roots α in $R(\mathbf{T}, G^*)$ such that $\sigma_T\alpha = -\alpha$. Then

$$m_\sigma = x_\sigma n(\sigma_T)$$

is a 1-cocycle of Γ in $N(G^*, \mathbf{T})$. Because x_σ lies in T, the image of m_σ in $W(G^*, T)$ coincide with $w(\sigma_T)$, therefore

$$hm_\sigma\sigma_{G^*}(h)^{-1} = h(m_\sigma\sigma_{G^*}(h)^{-1}h)h^{-1}$$

lies in T and is evidently a 1-cocycle. Let us denote by $\lambda(T)$ its class in $H^1(\Gamma, T)$. It is readily verified that $\lambda(T)$ doesn't depend on the choice of h, which is defined modulo left translation by an element in T. Thus $\lambda(T)$ depends on the choice of B and the choice of $\mathbf{spl}_{G^*}$ only.

Lemma 5.2. *The cohomology class $\lambda(T)$ in $H^1(\Gamma, T)$ does not depend on the choice of B. If $\mathbf{spl}_{G^*}$ is replaced by $\mathbf{spl}_{G^*} = g \cdot \mathbf{spl}_{G^*}$ with $g \in G^*$ such that $g\sigma_{G^*}(g)^{-1} \in Z(G^*)$, then $\lambda(T)$ is replaced by $\lambda'(T) = \lambda(T)\mathbf{g}$ where $\mathbf{g}$ is the class of $g\sigma_{G^*}(g)^{-1}$ in $H^1(\Gamma, T)$. Suppose that $g \in \mathcal{A}(T)$ and $T' = g \cdot T$. Let $\mathbf{g}$ be the class of the cocycle $\sigma_{G^*}(g)^{-1}g$ in $H^1(\Gamma, T)$. Then*

$$\lambda(T') = \mathbf{g} \operatorname{Int} g \cdot (\lambda(T)) \in H^1(\Gamma, T').$$

Proof. See [LS] (2.3).

5.4. Embeddings of L-groups. We continue with a pair (B,T) in G^*, with T defined over $\mathbb{R}$.

If $w \in W_{\mathbb{R}}$, let us write $w = w_0 \times \epsilon_w$, with $\epsilon_w = 1$ or σ and $w_0 \in \mathbb{C}^*$. Define

$$r(w) = \prod_{\alpha \in R(\mathbf{T},\mathbf{B}), \sigma_T\alpha<0} (w_0/|w_0|)^{\alpha^\vee}.$$

Lemma 5.3. *The coboundary of r is the inflation of t to $W_{\mathbb{R}}$.*

Proof. $\partial r(w,y) = r(w)w \cdot r(y)r(wy)^{-1}$

$$= \prod_{\alpha \in R(\mathbf{T},\mathbf{B}), \sigma_T\alpha<0} (w_0/|w_0|)^{\alpha^\vee}\epsilon_w \cdot \left((y_0/|y_0|)^{\alpha^\vee}\right) ((wy)_0/|(wy)_0|)^{-\alpha^\vee}$$

This equals 1 if $w = w_0 \times 1$ or $y = y_0 \times 1$ and $(-1)^{-\alpha^\vee}$ if $w = w_0 \times \sigma$ and $y = y_0 \times \sigma$. □

We want to construct a L- homomorphism $\xi_T : {}^LT \to {}^LG$ such that ξ_T maps $\hat{T}$ isomorphically onto $\mathcal{T}$, this isomorphism being the one attached to the pairs (B,T) and $(\mathcal{B},\mathcal{T})$.

To specify ξ_T, we have only to give a homomorphim

$$w \mapsto \xi_T(w) = \xi_T^0(w) \times w,$$

where $\xi_T^0(w) \in N(\hat{G},\mathcal{T})$, and where, in addition, if $w = w_0 \times \sigma$, then $\operatorname{Int}\xi_T(w)$ acts on $\mathcal{T}$ as the transport by ξ_T of the action σ_T on T and if $w = w_0 \times 1$, then $\xi_T(w)$ acts trivially on $\mathcal{T}$. As above, we write $\sigma_T = w(\sigma_T) \times \sigma$ in $W(\hat{G},\mathcal{T}) \ltimes \Gamma$.

If $w = w_0 \times \epsilon_w \in W_{\mathbb{R}}$, let us define $n(w)$ to be $n(w(\sigma_T)) \times w$ if $\epsilon_w = \sigma$ and $1 \times w$ if $\epsilon = 1$.

From Lemma 5.1, if $w_i = w_{i0} \times \epsilon_i \in \Gamma$, $i = 1,2$,

$$n(w_1)n(w_2)n(w_1w_2)^{-1} = t(\epsilon_1,\epsilon_2).$$

Since t splits over $W_{\mathbb{R}}$ (Lemma 5.3)

$$w \mapsto r(w)n(w)$$

defines an admissible embedding ξ_T of ${}^LT$ in ${}^LG$, depending on B and $\mathbf{spl}_{\hat{G}}$.

Lemma 5.4. *The $\hat{G}$-conjugacy class of ξ_T is independent of the choices of B and $\mathbf{spl}_{\hat{G}}$. Suppose that $g \in \mathcal{A}(T)$ and $T' = g \cdot T$. It gives an L-isomorphism ${}^LT \to {}^LT'$. The $\hat{G}$-conjugacy classes of ξ_T and $\xi_{T'}$ are canonically identified via this isomorphism.*

Proof. See [LS] (2.4)

5.5. Suppose that $\gamma_1 \in H_1(\mathbb{R})_{reg}$ projects on a G-regular element $\gamma \in H(\mathbb{R})$, which is a norm of $\delta \in G(\mathbb{R})^+$ (otherwise we set $\Delta(\gamma_1, \delta) = 0$). Let us denote by T_{H_1} (resp. T_H) the maximal torus of H_1 where γ_1 (resp. γ) lies. Suppose that (B_H, T_H) is a pair in H and that (B, T) is a pair in G^* from which we get an admissible embedding $\eta : T_H \to T$ of T_H in G^*. Set $\delta^* = \eta(\gamma)$. Let $x \in G_{sc}$ giving an admissible embedding $\eta_x : T_H \to T_G$ of T_H in G and let us assume that $T_G(\mathbb{R}) \subset G(\mathbb{R})^+$. The transfer factor $\Delta(\gamma_1, \delta)$ of [LS] is a product of four terms :

$$\Delta(\gamma_1, \delta) = \Delta_I(\gamma_1, \delta)\Delta_{II}(\gamma_1, \delta)\Delta_{III_1}(\gamma_1, \delta)\Delta_{III_2}(\gamma_1, \delta).$$

The last term Δ_{IV} of [LS] is already included in our definition of orbital integrals, so it does not appear here. This absolute transfer factor is not canonical. What is canonical is the relative transfer factor $\Delta(\gamma_1, \delta; \gamma_1', \delta_1')$ defined for two pairs of elements (γ_1, δ) and (γ_1', δ_1') as above. For all terms exept Δ_{III_1}, the corresponding factor of $\Delta(\gamma_1, \delta; \gamma_1', \delta_1')$ is just the quotient of the corresponding factors of $\Delta(\gamma_1, \delta)$ and $\Delta(\gamma_1', \delta')$ respectively. Only Δ_{III_1} is a genuine relative factor. The absolute and non canonical Δ is obtained by fixing a pair (γ_1', δ_1') satisfying the conditions of the begining of the paragraph, and a non zero value $\Delta(\gamma_1', \delta_1')$. Then for all pair (γ_1, δ),

$$\Delta(\gamma_1, \delta_1) = \Delta(\gamma_1, \delta; \gamma_1', \delta_1')\Delta(\gamma_1', \delta_1').$$

5.6. Δ_I**.** Let us denote by σ_{T_H} the action of σ on $\mathcal{T}_H$, given by the transport of the action of σ on $\hat{T}_H$ through the isomorphism $\hat{T}_H \simeq \mathcal{T}_H$ given by the pairs (B_H, T_H) and $(\mathcal{B}_H, \mathcal{T}_H)$. There is another action of σ on $\mathcal{T}_H$, namely $\rho_{\mathcal{H}}(1 \times \sigma) = \rho_H(1 \times \sigma) = \sigma_{\hat{H}}$. These two actions differs by the action of an element of the Weyl group of $\mathcal{T}_H$ in $\hat{H}$, thus the action σ_{T_H} is induced by the adjoint action of an element $h \in \mathcal{H}$ normalizing $\mathcal{T}_H$. Let us write

$$\xi(h) = g \times w \in {}^L G = \hat{G} \rtimes_{\rho_G} W_{\mathbb{R}}$$

Since $\xi(h)$ and w normalize $\mathcal{T}$, g also normalize $\mathcal{T}$. From the property $\operatorname{Int} s \circ \xi = a \otimes \xi$ we get

$$\xi(h)s\xi(h^{-1}) = sa(w).$$

Writting $s_H = \xi^{-1}(s)$, we get :

$$\sigma_{T_H}(s_H) = hs_Hh^{-1} = s_H\xi^{-1}(a(w)). \qquad (5.1)$$

Since s is central in $\xi(\hat{H})$, $s \in \mathcal{T}$, and the preimage of s in $\hat{T}_H$ through the isomorphisms $\hat{T}_H \simeq \mathcal{T}_H \overset{\xi}{\simeq} \mathcal{T}$ is independent of the choice of B_H. Thus its image s_T in $\hat{T}$ depends only on the admissible embedding $T_H \to T$. Equation (5.1) is an equality in $\mathcal{T}_H$. We can push it to an equality in $\hat{T}$ and get

$$s_T^{-1}\sigma_T(s_T) = a(w) \in Z(\hat{G}).$$

Notice that the embedding of $Z(\hat{G})$ in $\hat{T}$ is canonical. Let $\hat{T}_{ad} := T/Z(\hat{G})$. It is easy to check that $\hat{T}_{ad} = \widehat{T_{sc}}$. Furthermore, from the previous equation, we see that the image of s_T in $\hat{T}_{ad}$ is Γ-invariant, and thus define a class $\mathbf{s}_T$ in

$$\pi_0 = \pi_0(\hat{T}_{ad}^{\Gamma}) = \hat{T}_{ad}^{\Gamma}/(\hat{T}_{ad}^{\Gamma})_0.$$

Recall the Tate-Nakayama pairing (see Appendix 6.4)

$$\langle .,. \rangle : H^1(\Gamma, T_{sc}) \times \pi_0(\hat{T}_{ad}^{\Gamma}) \to \{\pm 1\}.$$

Then $\Delta_I(\gamma,\delta) = \langle \lambda(T_{sc}), \mathbf{s}_T \rangle$, where $\lambda(T_{sc})$ was defined in Section 5.3.

5.7. Δ_{II}**.** Let us denote by $R_I(T,G^*)$ the set of imaginary roots in $R(T,G^*)$, and by $R_I(T,B)$ the set of positive imaginary roots. We use similar notation $R_I(T_H,H)$ and $R_I(T_H,B_H)$ for the roots of T_H in H. Set

$$\begin{aligned}\Delta_{II}(\gamma_1,\delta) &= \prod_{\alpha\in R_I(T,B)} \frac{\alpha(\delta^*)-1}{|\alpha(\delta^*)-1|} \times \prod_{\alpha_H\in R_I(T_H,B_H)} \frac{|\alpha_H(\gamma)-1|}{\alpha_H(\gamma)-1} \\ &= \prod_{\alpha\in R_I(T,B),\ \text{not from } H} \frac{\alpha(\delta^*)-1}{|\alpha(\delta^*)-1|}.\end{aligned}$$

The equality results from the fact that $\alpha_H(\gamma_1) = \alpha(\delta^*)$ if $\eta(\alpha_H) = \alpha$.

5.8. $\Delta_{III_1} = \Delta_1$**.** This is the only relative term in the transfer factors. Let γ_1, γ, δ^*, δ, x, T_H, B_H, T, B, T_G, η, η_x as above and suppose we have also data γ_1', γ', $\delta^{*\prime}$, δ', x', T_H', B_H', T', B', T_G', η', η_x' satisfying the same properties. The relative transfer factor $\Delta_1(\gamma_1',\delta';\gamma_1,\delta)$ is defined as follows. Set $v_\sigma = x u_\sigma \sigma_{G^*}(x)^{-1}$ and $v_\sigma' = x' u_\sigma \sigma_{G^*}(x')^{-1}$. Then v_σ (resp. v_σ') is a cochain of Γ in T_{sc} (resp. T_{sc}'), which is well-defined up to a coboundary since x and (resp. x') is defined up to left multiplication by an element in T (resp. T'). Further, $\partial v_\sigma = \partial v_\sigma' = \partial u_\sigma$ takes value in Z_{sc}, the center of G_{sc}^*. Let U be the torus

$$U = T_{sc} \times T_{sc}'/\{(z^{-1},z),\ z\in Z_{sc}\}.$$

Then $(v_\sigma^{-1}, v_\sigma')$ defines an element of $H^1(\Gamma, U)$ which is independent of the choices of u_σ, x and x'. Let us write this class as

$$\mathbf{inv}(\gamma_1,\delta;\gamma_1',\delta').$$

Recall that $\hat{T}_{ad}$ is $\hat{T}/Z(\hat{G})$ and that $\hat{T}_{ad} = \widehat{T_{sc}}$. Dually, $T_{ad} = T/Z(G)$ has dual group $\hat{T}_{sc}$. Then the center $\hat{Z}_{sc}$ of the simply-connected covering of the derived group of $\hat{G}$ (a finite group isomorphic to Z_{sc}) is canonically embedded in $\hat{T}_{sc}$ and $\hat{T}_{sc}'$. Set

$$\hat{U} = \hat{T}_{sc} \times \hat{T}_{sc}'/\{(z,z),\ z\in \hat{Z}_{sc}\}.$$

One checks readily that $\hat{U}$ is indeed the dual of U.

To the endoscopic datum s, we attach $\mathbf{s}_U \in \pi_0(\hat{U}^\Gamma)$. Suppose that $\tilde{s}$ lies in the preimage of s under $\mathcal{T}_{sc} \to \mathcal{T}$. From the isomorphisms $\hat{T} \simeq \hat{T}$ and and $\hat{T} \simeq \hat{T}'$, we obtain $\hat{\mathcal{T}}_{sc} \simeq \hat{T}_{sc}$ and $\hat{\mathcal{T}}_{sc} \simeq \hat{T}'_{sc}$. We denote by $\tilde{s}_T$ and $\tilde{s}_{T'}$ the images of $\tilde{s}$ under these isomorphisms. Then s_U is the image of $(\tilde{s}_T, \tilde{s}_{T'})$ in $\hat{U}$. It is independent of the choice of $\tilde{s}$ and it is Γ-invariant. We denote by $\mathbf{s}_U$ the image of s_U in $\pi_0(\hat{U}^\Gamma)$. We can now define Δ_1 :

$$\Delta_1(\gamma_1, \delta; \gamma_1', \delta') = \langle \mathrm{inv}(\gamma_1, \delta; \gamma_1', \delta'), \mathbf{s}_U \rangle.$$

5.9. $\Delta_{III_2} = \Delta_2$. To get a better idea of this factor, it is useful to start with the case where $\mathcal{H} \simeq {}^L H$, which makes the definition much simplier. In this case, we take obviously $(H_1, \xi_{H_1}) = (H, \mathrm{Id})$. Consider data γ, δ^*, δ, x, T_H, B_H, T, B, T_G, η, η_x as above. Then, we have the following diagram

$$\begin{array}{ccc} {}^L T_H & \xrightarrow{\xi_{T_H}} & {}^L H \\ \eta \downarrow & & \downarrow \xi \\ {}^L T & \xrightarrow{\xi_T} & {}^L G \end{array}$$

Notice that we have denoted again by η the L-isomorphim induced from the $\mathbb{R}$-isomorphism $\eta : T_H \simeq T$. The L-embeddings ξ_T and ξ_{T_H} are constructed in Section 5.4

This diagram is not commutative. Identifying t in ${}^L T$ with its preimage in ${}^L T_H$, we can write

$$\xi \circ \xi_{T_H}(t \times w) = a_T(w) \xi_T(t \times w)$$

for all $t \times w$ in ${}^L T$. Then a_T is a cocycle of $W_\mathbb{R}$ in $\mathcal{T}$ (for the of $W_\mathbb{R}$ action through σ_T), and it defines a class $\mathbf{a}_T$ in $H^1(W_\mathbb{R}, \mathcal{T})$. Let us transport this through the isomorphism $\mathcal{T} \simeq \hat{T}$ to a class in $H^1(W_\mathbb{R}, \hat{T})$ that we still denote by $\mathbf{a}_T$. Then

$$\Delta_2(\gamma, \delta) = \langle \mathbf{a}_T, \delta^* \rangle$$

where $\langle .\,, . \rangle$ is the Langlands pairing between T and $H^1(W_\mathbb{R}, \hat{T})$ (see Appendix 6.4).

Let us now consider the general case. We cannot compare directely the L-embeddings $\xi_T : {}^L T \to {}^L G$ and $\xi_{T_H} : {}^L T_H \to {}^L H$ as above. We should rather compare ξ_T and $\xi_{T_{H_1}} : {}^L T_{H_1} \to {}^L H_1$, but this will be indirect. Let us first remark that $\mathbf{spl}_{\hat{H}} = \{\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{X}\}\}$ determines a splitting $\mathbf{spl}_{\hat{H}_1} = \{\mathcal{B}_{H_1}, \mathcal{T}_{H_1}, \{\mathcal{X}\}\}$ of $\hat{H}_1$, and that any pair (B_H, T_H) in H determines a pair (B_{H_1}, T_{H_1}) in H_1. Thus the isomorphism $\hat{T}_H \simeq \mathcal{T}_H$ given by these choices of pairs extends uniquely to an isomorphism $\hat{T}_{H_1} \simeq \mathcal{T}_{H_1}$. The L-embedding $\xi_{T_{H_1}} : {}^L T_{H_1} \to {}^L H_1$ is the extension of this isomorphism which was constructed in Section 5.4.

Let us consider the subgroup $\mathcal{U}$ of $\mathcal{H}$ consisting of elements u normalizing $\mathcal{T}_H$ and acting by conjugacy on $\mathcal{T}_H$ as σ_{T_H} if u projects on $\sigma \in \Gamma$, and trivially if u projects trivially Γ. It is easy to see that $\mathcal{U}$ projects surjectively onto $W_{\mathbb{R}}$ and that the kernel of this projection is $\mathcal{T}_H$, *ie.* we have an exact sequence

$$1 \to \mathcal{T}_H \to \mathcal{U} \to W_{\mathbb{R}} \to 1 \tag{5.2}$$

The restriction of $\xi_{H_1} : \mathcal{H} \to {}^L H_1$ to $\mathcal{U}$ is easily seen to have its image included in the image of $\xi_{T_{H_1}}$, thus there exists an unique L-homomorphism $\alpha_0 : \mathcal{U} \to {}^L T_{H_1}$ such that $\xi_{H_1} = \xi_{T_{H_1}} \circ \alpha_0$ on $\mathcal{U}$. Let us also set α to be the composition of α_0 and $t \mapsto t^{-1}$ on ${}^L T_{H_1}$.

For $u \in \mathcal{U}$, write $\xi(u) \in {}^L G$ as $\xi(u) = g \times w \in \hat{G} \times W_{\mathbb{R}}$. Since T and T_H are isomorphic other $\mathbb{R}$, $\xi(u)$ acts on $\mathcal{T}$ by conjugacy as σ_T if $w = w_0 \times \sigma$ and trivially if $w = w_0 \times 1$. Thus it is also clear that $\xi(\mathcal{U})$ is included in the image of ξ_T and therefore there exists an unique L-embedding $\beta : \mathcal{U} \to {}^L T$ such that $\xi = \xi_T \circ \beta$.

So far, we have defined

$$(\alpha, \beta) : \mathcal{U} \to {}^L T_{H_1} \times {}^L T \simeq {}^L(T_{H_1} \times T). \tag{5.3}$$

Let us consider the fiber product $T_1 := T_{H_1} \times_{T_H} T$. This is an algebraic torus defined over $\mathbb{R}$. Since $T_H \simeq T$ over $\mathbb{R}$, $T_1 \simeq T_{H_1}$ and

$$\hat{T}_1 = \hat{T}_{H_1} \times \hat{T} / \{(t^{-1}, \eta^{-1}(t)), t \in T_H\}.$$

The natural projection $\hat{T}_{H_1} \times \hat{T} \to \hat{T}_1$ extends to an L-homomorphism

$$^L(T_{H_1} \times T_H) \to {}^L T_1. \tag{5.4}$$

The composition of (5.3) and (5.4) gives a L-homomorphism

$$\mathcal{U} \to {}^L T_1$$

which has $\mathcal{T}_H$ in its kernel. Since $\mathcal{U}/\mathcal{T}_H \simeq W_{\mathbb{R}}$ (Eq (5.2)), we get a L-homomorphism

$$W_{\mathbb{R}} \to {}^L T_1$$

or equivalently, a 1-cocycle of $W_{\mathbb{R}}$ in $\hat{T}_1$. As we have observed, $T_1 \simeq T_{H_1}$, so we can transport this cocycle to a cocycle with value in $\hat{T}_{H_1}$. Let $\mathbf{a}_T$ be its class in $H^1(W_{\mathbb{R}}, \hat{T}_{H_1})$. We can now give the definition of Δ_2 :

$$\Delta_2(\gamma_1, \delta) = \langle \mathbf{a}_T, \gamma_1 \rangle$$

where the pairing is the Langlands pairing for tori.

6. Proof of Proposition 4.4 and Lemma 4.6

6.1. Proof of (i)**.** The setting is that of Proposition 4.4. Let us fix $\delta \in (\mathcal{V}_{\delta_0})_{reg}$, $\gamma_1 = \eta_x^{-1}(\delta)$ and $X \in \mathfrak{t}_G$ sufficiently close from 0. The rest of the notation is recalled in the following diagram :

$$\begin{array}{ccccc} \gamma_1 \exp X_H \in T_{H_1}(\mathbb{R}) & & & & \\ \downarrow & & & & \\ \gamma \exp X_H \in T_H(\mathbb{R}) & \xrightarrow{\eta} & \delta^* \exp X^* \in T(\mathbb{R}) & \xleftarrow{\mathrm{Int}\, x \circ \psi} & \delta \exp X \in T_G(\mathbb{R}) \end{array}$$

The transfer factor we want to examine is :

$$\begin{aligned} \Delta(\exp X) &:= \Delta(\gamma_1 \exp X_H, \delta \exp X) \\ &= \Delta(\gamma_1 \exp X_H, \delta \exp X; \gamma_1, \delta)\Delta(\gamma_1, \delta) \end{aligned}$$

where $\Delta(.,.;.,.)$ is the canonically defined relative transfer factor of [LS].

As in before, we write $\Delta(X)$ as a product of four terms (recall that the last term of [LS] has been included in our definition of orbital integrals).

$$\Delta(X) = \Delta_I(X)\Delta_{II}(X)\Delta_1(X)\Delta_2(X)$$

The term Δ_1 is a quotient ([LS], section 3.2). Because the two elements $\gamma \exp X_H$ and γ lie in the same maximal torus T_H and the numerator and denominator of Δ_I each depend on the torus rather than the individual elements, we find that $\Delta_I(X) = 1$.

Let us now examine $\Delta_{II}(X)$. It is again a quotient

$$\begin{aligned} \Delta_{II}(X) &= \frac{\Delta_{II}(\gamma_1 \exp X_H, \delta \exp X)}{\Delta_{II}(\gamma_1, \delta)} \\ &= \prod \frac{\alpha(\delta^* \exp X^*) - 1}{|\alpha(\delta^* \exp X^*) - 1|} \frac{|\alpha(\delta^*) - 1|}{\alpha(\delta^*) - 1} \end{aligned} \tag{6.1}$$

where the product is over the imaginary roots in $R(T, B)$ not from H.

Let us write $\alpha(\delta^*) = z^2$ and $\alpha(\exp X^*) = w^2$ with z and w on the unit circle. Then

$$\frac{\alpha(\delta^* \exp X^*) - 1}{|\alpha(\delta^* \exp X^*) - 1|} \frac{|\alpha(\delta^*) - 1|}{\alpha(\delta^*) - 1} = \frac{zw(zw - \bar{z}\bar{w})}{|zw(zw - \bar{z}\bar{w})|} \frac{|z(z - \bar{z})|}{z(z - \bar{z})} = \frac{w}{|w|}$$

because X is close from 0. Thus

$$\Delta_{II}(X) = \prod_{\alpha \in R_I(B,T),\ \text{not from } H} \exp \alpha(X^*/2). \tag{6.2}$$

We now take a closer look on $\Delta_1(X) = \Delta_1(\gamma_1 \exp X_H, \delta \exp X; \gamma_1, \delta)$. Here, the general construction of 5.8 simplifies because $T' = T$, $v'_\sigma = v_\sigma$ and thus the class $\mathbf{inv}(\gamma_1 \exp X_H, \delta \exp X; \gamma_1, \delta)$ in $H^1(\Gamma, U)$ is the class of the cocycle $(v_\sigma^{-1}, v_\sigma)$. Since in this case $\mathbf{s}_U$ is the class of (s_T, s_T) in $\pi_0(\hat{U}^\Gamma)$, we see that

$$\Delta_1(\gamma_1 \exp X_H, \delta \exp X; \gamma_1, \delta) = \langle \mathrm{inv}(\gamma_1 \exp X_H, \delta \exp X; \gamma_1, \delta), \mathbf{s}_U \rangle = 1.$$

Finally, $\Delta_2(X)$ is obtained by evaluating the character given by the class $\mathbf{a}_T$ on $\exp X_H$.

The obstruction to extend Δ_{η_x} to a smooth function on $(\mathcal{V}_{\delta_0})$ comes from the Δ_{II} term. More precisely, with notation of section 5.7, there might be an obstruction at the point $\delta_0 \exp X$ if $\alpha(\delta_0^* \exp X^*) = 1$ with α imaginary, not from H. Thus Δ_{η_x} extends to a smooth function on $(\mathcal{V}_{\delta_0})_{I-reg}$. Since Δ_{II} is locally bounded, Δ is also locally bounded. We see also that Δ has the form $\Delta(X) = c \exp \lambda(X)$ for some constant c and some $\lambda \in \mathfrak{t}_{\mathbb{C}}$, from direct consideration of the explicit form of the non trivial factors Δ_{II} and Δ_2. This finishes the proof of (i).

6.2. The notation is now as in the rest of Proposition 4.4. If γ_0 is semi-regular, then α is not from H and Δ_{η_x} has a smooth continuation at δ_0. This can be rewritten

$$\lim_{\nu \to 0^+} \Delta(\gamma_1^\nu, \delta^\nu) = \lim_{\nu \to 0^-} \Delta(\gamma_1^\nu, \delta^\nu)$$

proving the corresponding assertions in (iii) and (v). If γ_0 is regular, we are in the opposite situation, *ie.* α is from H. The obstruction to the smoothness of Δ_{η_x} at δ_0 comes from the factor

$$\frac{\alpha(\delta^{\nu,*}) - 1}{\alpha(\delta^{\nu,*}) - 1}$$

and it is easy to see that

$$\lim_{\nu \to 0^+} \frac{\alpha(\delta^{\nu,*}) - 1}{\alpha(\delta^{\nu,*}) - 1} = - \lim_{\nu \to 0^-} \frac{\alpha(\delta^{\nu,*}) - 1}{\alpha(\delta^{\nu,*}) - 1},$$

proving the assertions in (ii) and (iv).

6.3. For the remaining assertions of Proposition 4.4, we need to compare

$$\Delta(\gamma_{1,\nu}, \delta_\nu) \quad \text{and} \quad \Delta(\gamma_{1,\nu}, \delta_{-\nu}).$$

By [KS], Theorem 5.1.D, the quotient of these two terms is given by the quantity

$$\langle \mathbf{inv}(\delta^\nu, \delta^{-\nu}), \mathbf{s}_T \rangle,$$

where $\mathbf{inv}(\delta^\nu, \delta^{-\nu}) \in H^1(\Gamma, T)$ is defined as follows. Let w_α be an element in $G_{sc}^{\delta_0}$ realizing the Weyl group reflection s_α. We have

$$w_\alpha \delta^\nu w_\alpha^{-1} = \delta^{-\nu}.$$

Thus $w_\alpha^{-1} \sigma_G(w_\alpha)$ is a 1-cocycle of Γ in $T_{G,sc}$. Its class in $H^1(\Gamma, T_{G,sc})$ and identifying $H^1(\Gamma, T_{sc}) \simeq H^1(\Gamma, T_{G,sc})$ we get $\mathbf{inv}(\delta^\nu, \delta^{-\nu})$. The element $\mathbf{s}_T \in \pi_0(\hat{T}^\Gamma)$ was defined in Section 5.6. Let us identify the cohomology class of $w_\alpha^{-1} \sigma_G(w_\alpha)$ in $H^1(\Gamma, T_{G,sc})$ using the isomorphism

$$H^1(\Gamma, T_{G,sc}) \simeq X_*(T_{G,sc})^{-\sigma_*} / (1 - \sigma_*) X_*(T_{G,sc})$$

of Proposition B.2. For this, choose a morphism

$$\phi_\alpha \ : SL(2, \mathbb{C}) \to G_{sc}^*$$

defined over $\mathbb{R}$ and sending the imaginary non compact root of the circle torus in $SL(2,\mathbb{R})$ to α. An easy computation in $SL(2,\mathbb{C})$ shows that we can choose w_α such that $w_\alpha^{-1}\sigma_G(w_\alpha)$ is the image of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and thus one has $w_\alpha^{-1}\sigma_G(w_\alpha) = \exp(i\pi H_\alpha)$ which get identify with the class of $\alpha\check{}$ in $X_*(T_{G,sc})^{-\sigma_*}/(1-\sigma_*)X_*(T_{G,sc})$. Using the explicit form of the Tate-Nakayama pairing in Proposition B.2, we see that $\langle \mathbf{inv}(\delta^\nu,\delta^{-\nu}), \mathbf{s}_T\rangle = 1$ if and only if $\tilde{\alpha}\check{}(s) = 1$, where $\tilde{\alpha}$ is the transport of α in $R(\mathbf{T}, G^*)$, *ie.* if and only if $\alpha \in R(T_{G,sc}, G^*_{sc}) \simeq R(T_G, G^*)$ comes from H. The values of the Tate-Nakayama pairing are in $\{\pm 1\}$. Thus $\Delta(\gamma_{1,\nu},\delta_\nu) = \Delta(\gamma_{1,\nu},\delta_{-\nu})$ when γ_0 is singular in $H(\mathbb{R})$, and $\Delta(\gamma_{1,\nu},\delta_\nu) = -\Delta(\gamma_{1,\nu},\delta_{-\nu})$ when γ_0 is regular in $H(\mathbb{R})$. In this case, by Lemma 4.1 δ_ν and $\delta_{-\nu}$ can not be conjugate in $G(\mathbb{R})$. This finishes the proof of Proposition 4.4

6.4. Proof of Lemma 4.6. The second assertion of the lemma follows from an easy computation. We concentrate now on the first. The statement and the proof of this result is the main technical improvment from Shelstad treatment in her series of papers. The reason is that at the time, she was missing the conceptual definition of the Δ_I factor, in contrast with the other factors which are already implicit in her work. The virtue of the definition of Δ_I is that it makes Lemma 4.6 true, and it makes the global transfer factor Δ independent of all the choices. Concerning the proof of the Lemma, the approach is also sensibly different. It becomes now a simple consequence of the descent technique in [LS2]. Indeed, the main theorem there (Theorem 1.6.A) allows us to reduce the proof of Lemma 4.6 to groups of rank one, namely the centralizer of the semi-regular elements $\delta_{0,j}$ in the lemma. Explicit computations for $SL(2,\mathbb{R})$ are in [LS]. This finishes the proof.

Appendix A. Langlands correspondence for real tori

A.1. Characters. Let $\mathbb{T}$ be a torus defined over $\mathbb{R}$, with character module $X^*(\mathbb{T})$ and set $X_*(\mathbb{T}) = \mathrm{Hom}(X^*(\mathbb{T}),\mathbb{Z})$ for the lattice of one-parameter subgroups. Identify $\mathbb{T}$ with the group of its complex points. We have identifications $\mathfrak{t}_\mathbb{C} = X_*(\mathbb{T}) \otimes_\mathbb{Z} \mathbb{C}$ and $\mathfrak{t}^*_\mathbb{C} = X^*(\mathbb{T}) \otimes_\mathbb{Z} \mathbb{C}$ such that the canonical pairing between $X^*(\mathbb{T})$ and $X_*(\mathbb{T})$ gives the pairing between $\mathfrak{t}_\mathbb{C}$ and $\mathfrak{t}^*_\mathbb{C}$. Recall that $\ker[\exp : \mathfrak{t}_\mathbb{C} \to \mathbb{T}] = 2i\pi X_*(\mathbb{T})$. Let us denote by σ the Galois involution of $\mathbb{T}$, by $d\sigma$ its differential, which is a Galois involution of $\mathfrak{t}_\mathbb{C} = \mathrm{Lie}(\mathbb{T}) = X_*(\mathbb{T}) \otimes_\mathbb{Z} \mathbb{C}$ and respectively by σ^* and σ_* the induced actions on $X^*(\mathbb{T})$ and $X_*(\mathbb{T})$. With these notations we have

$$d\sigma(\lambda\check{} \otimes z) = \sigma_*(\lambda\check{}) \otimes \overline{z}. \tag{A.1}$$

Set $T := T(\mathbb{R})$, the group of real points of $\mathbb{T}$. An element $h = \exp X$, $X \in \mathfrak{t}_{\mathbb{C}}$ belongs to T if and only if

$$d\sigma(X) - X \in 2i\pi X_*(\mathbb{T}).$$

We write $X = X_I + X_R$ where

$$X_I = \frac{1}{2}(X - d\sigma(X)) \text{ and } X_R = \frac{1}{2}(X + d\sigma(X)).$$

Then $X_R \in \mathfrak{t}$ and since $d\sigma(X) - X = 2X_I$, we find that $X_I \in i\pi X_*(\mathbb{T})$. Furthermore, writing $X_I = \check{\lambda} \otimes i\pi$, we see that

$$d\sigma(X_I) = -X_I = \sigma_*(\check{\lambda}) \otimes -i\pi,$$

and thus $\sigma_*(\check{\lambda}) = \check{\lambda}$.

Thus we decompose h as a product h_1h_2 where $h_1 = \exp(X_R) \in T_0 = \exp \mathfrak{t}$ and

$$h_2 = \exp X_I \in F = \{\exp i\pi\check{\lambda} \mid \check{\lambda} \in X_*(\mathbb{T}),\, \sigma_*(\check{\lambda}) = \check{\lambda}\}. \tag{A.2}$$

Therefore $T = T_0F$, with

$$T_0 \cap F = \{\exp i\pi\check{\lambda} \mid \check{\lambda} = \check{\mu} + \sigma_*(\check{\mu}),\, \check{\mu} \in X_*(\mathbb{T})\}\} \tag{A.3}$$

Let us prove this assertion. Suppose that $\exp i\pi\check{\lambda} = \exp X$ with $X = \check{\nu} \otimes z \in \mathfrak{t}$, i.e. $d\sigma(X) = \sigma_*(\check{\nu}) \otimes \overline{z} = \check{\nu} \otimes z$. For some $\check{\mu} \in X_*(\mathbb{T})$, we have

$$\check{\lambda} \otimes i\pi - \check{\nu} \otimes z = \check{\mu} \otimes 2i\pi. \tag{A.4}$$

Applying σ_* and complex conjugacy to (A.4) successively on the first and second factor of $X_*(\mathbb{T}) \otimes \mathbb{C}$ we get:

$$\check{\lambda} \otimes i\pi - \sigma_*(\check{\nu}) \otimes z = \sigma_*(\check{\mu}) \otimes 2i\pi \tag{A.5}$$

$$\check{\lambda} \otimes -i\pi - \check{\nu} \otimes \overline{z} = \check{\mu} \otimes -2i\pi. \tag{A.6}$$

The sum of (A.5) and (A.6) gives:

$$-(\sigma_*(\check{\nu}) \otimes z + \check{\nu} \otimes \overline{z}) = (\sigma_*(\check{\mu}) - \check{\mu}) \otimes 2i\pi \tag{A.7}$$

But since $\sigma_*(\check{\nu}) \otimes \overline{z} = \check{\nu} \otimes z$, we have $\sigma_*(\check{\nu}) \otimes z = \check{\nu} \otimes \overline{z}$, and by (A.5) and (A.7) we get

$$\check{\lambda} \otimes i\pi + \frac{1}{2}(\sigma_*(\check{\mu}) - \check{\mu}) \otimes 2i\pi = \sigma_*(\check{\mu}) \otimes 2i\pi. \tag{A.8}$$

We conclude that $\check{\lambda} = \sigma_*(\check{\mu}) + \check{\mu})$. The converse is obvious. □

Given a pair of elements (μ, λ) in $\mathfrak{t}_{\mathbb{C}}^*$ we set, for all $X \in \mathfrak{t}_{\mathbb{C}}$ such that $\exp X \in T$,

$$\xi_{(\mu,\lambda)}(\exp X) = e^{\mu(X_R)+2\lambda(X_I)}.$$

Then $\xi_{(\mu,\lambda)}$ is a well-defined character on T if and only if $\xi_{(\mu,\lambda)}(\exp X) = 1$ when $X \in 2i\pi X_*(\mathbb{T})$. This condition is satisfied if and only if $\mu(X_R) + 2\lambda(X_I) \in 2i\pi\mathbb{Z}$. Set $X = 2i\pi\check{\nu}$, with $\check{\nu} \in X_*(\mathbb{T})$. From the definitions, we get that $X_R = i\pi(\check{\nu} - \sigma_*(\check{\nu}))$ whereas $X_I = i\pi(\check{\nu} + \sigma_*(\check{\nu}))$. So a necessary and sufficient condition for (μ, λ) to define a character of T is that for all $\check{\nu} \in X_*(\mathbb{T})$,

$$\text{(A.9)} \qquad \mu(i\pi(\check{\nu} - \sigma_*(\check{\nu}))) + 2\lambda(i\pi(\check{\nu} + \sigma_*(\check{\nu}))) \in 2i\pi\mathbb{Z}.$$

An easy computation shows that this amounts to

$$\text{(A.10)} \qquad \frac{1}{2}(\mu - \sigma^*(\mu)) + (\lambda + \sigma^*(\lambda)) \in X^*(\mathbb{T}).$$

Furthermore, $\xi_{(\mu,\lambda)} = \xi_{(\mu',\lambda')}$ if and only if

$$\text{(A.11)} \qquad \mu = \mu' \quad \text{and } \lambda - \lambda' \in X^*(\mathbb{T}) + \{\nu - \sigma^*(\nu) | \, \nu \in \mathfrak{t}^*_{\mathbb{C}}\}$$

Let us summarize what we have obtained :

Proposition A.1. *The characters of the T are all of the form $\xi_{(\mu,\lambda)}$ with μ and λ satisfying*

$$\frac{1}{2}(\mu - \sigma^*(\mu)) + (\lambda + \sigma^*(\lambda)) \in X^*(\mathbb{T}),$$

and $\xi_{(\mu,\lambda)} = \xi_{(\mu',\lambda')}$ if and only if

$$\mu = \mu' \quad \textit{and } \lambda - \lambda' \in X^*(\mathbb{T}) + \{\nu - \sigma^*(\nu) | \, \nu \in \mathfrak{t}^*_{\mathbb{C}}\}$$

A.2. Langlands parameters. We recall some definitions. The Weil group $W_{\mathbb{R}}$ of $\mathbb{R}$ is an extension of $\Gamma := \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ by $\mathbb{C}^*$, i.e. we have an exact sequence:

$$\text{(A.12)} \qquad 1 \to \mathbb{C}^* \to W_{\mathbb{R}} \to \Gamma \to 1.$$

As a set $W_{\mathbb{R}} \simeq \mathbb{C}^* \times \Gamma$ and multiplication is defined by
(A.13)

$$(1 \times \sigma)(z \times 1) = (\overline{z} \times \sigma); \; (z \times 1)(1 \times \sigma) = (z, \sigma); \; (1 \times \sigma)^2 = (-1 \times 1).$$

Let us denote by $\hat{\mathbb{T}}$ the dual torus of $\mathbb{T}$, i.e. $\hat{\mathbb{T}}$ is a complex algebraic torus with identifications $X^*(\hat{\mathbb{T}}) = X_*(\mathbb{T})$ and $X_*(\hat{\mathbb{T}}) = X^*(\mathbb{T})$. It particular, we have $\mathrm{Lie}(\hat{\mathbb{T}}) = \mathfrak{t}^*_{\mathbb{C}}$ and $\mathrm{Lie}(\hat{\mathbb{T}})^* = \mathfrak{t}_{\mathbb{C}}$. Let $\sigma_{\hat{\mathbb{T}}}$ denote the *algebraic* action on $\hat{\mathbb{T}}$ inducing respectively σ^* and σ_* on $X^*(\mathbb{T})$ and $X_*(\mathbb{T})$. The L-group of $\mathbb{T}$, ${}^LT$ is an the semi-direct product of $\hat{\mathbb{T}}$ and $W_{\mathbb{R}}$, the action ρ_T of $W_{\mathbb{R}}$ being given by

$$\text{(A.14)} \qquad \rho_T(z \times 1) = 1\,; \;\; \rho_T(z \times \sigma) = \sigma_{\hat{\mathbb{T}}}$$

Definition A.2. An homomorphism $\phi : W_R \to {}^LT$ is called an L-homomorphism if it satisfies:

(i) ϕ is continous

(ii) Let π be the projection from ${}^LT$ on $W_{\mathbb{R}}$. Then $\pi \circ \phi = \mathrm{Id}_{W_{\mathbb{R}}}$.

The group $\hat{\mathbb{T}}$ acts on L-homomorphisms by conjugation on the image. A Langlands parameter is a conjugacy class of L-homomorphisms.

To specify an L-homomorphism, we need to specify two things: the restiction ϕ_0 of ϕ to $\mathbb{C}^*$, and the element $\phi(1 \times \sigma)$. These are subjects to various constraints coming from the defining relations in $W_{\mathbb{R}}$. First of all, let us write

$$\phi(1 \times \sigma) = \exp 2i\pi\lambda \times (1 \times \sigma)$$

for some λ in $\mathfrak{t}^*_{\mathbb{C}}$ (λ is defined up to an element of $X^*(\mathbb{T})$). The restriction ϕ_0 is given by a pair of (μ, ν) of elements of $\mathfrak{t}^*_{\mathbb{C}}$, with $\mu - \nu \in X^*(\mathbb{T})$, such that

$$\phi_0(\exp z) = \exp(z\mu + \overline{z}\nu). \tag{A.15}$$

The first two relations in $W_{\mathbb{R}}$ force

$$\phi(1 \times \sigma)\phi_0(z)\phi(1 \times \sigma)^{-1} = \phi(\overline{z}). \tag{A.16}$$

Using (A.15), we find that $\nu = \sigma^*(\mu)$. The third relation in $W_{\mathbb{R}}$ gives:

$$\phi(1 \times \sigma)^2 = \phi_0(-1). \tag{A.17}$$

Using $\phi(1 \times \sigma) = \exp 2i\pi\lambda \times (1 \times \sigma)$, we find that the left hand side of (A.17) is $\exp(2i\pi(\lambda + \sigma^*(\lambda)))$. The right hand side may be computed by setting $z = i\pi$ in (A.15). What we obtain is:

$$\exp(2i\pi(\lambda + \sigma^*(\lambda))) = \exp(i\pi(\mu - \sigma^*(\mu))). \tag{A.18}$$

Thus we find that

$$(\lambda + \sigma^*(\lambda)) - \frac{1}{2}(\mu - \sigma^*(\mu)) \in X^*(\mathbb{T}). \tag{A.19}$$

We will denote by $\phi_{\mu,\lambda}$ the L-homomorphism given by such a pair (μ, λ).

Lemma A.3. *(i) Suppose we have pairs (μ, λ) and (μ', λ') satisfying the congruence (A.19). Then $\phi_{\mu,\lambda} = \phi_{\mu',\lambda'}$ if and only if $\mu = \mu'$ and $\lambda - \lambda' \in X^*(\mathbb{T})$.*

(ii) The L-homomorphism $\phi_{\mu,\lambda}$ and $\phi_{\mu',\lambda'}$ define the same Langlands parameter if and only if $\mu = \mu'$ and $\lambda - \lambda' \in X^*(\mathbb{T}) + \{\nu - \sigma^*(\nu) | \nu \in \mathfrak{t}^*_{\mathbb{C}}\}$

the proof is straightforward. Notice that (A.19) is the same as (A.10) and that (ii) is the same as A.11 and thus that we have a bijection between Langlands parameters and characters of T: this is the Langlands correspondance for (real) tori.

Theorem A.4. *(Langlands correspondence for real tori). The characters of T are in one-to-one correspondence with the $\hat{\mathbb{T}}$-conjugacy classes of L-morphisms $\phi : W_{\mathbb{R}} \to {}^LT$, the correspondence is given by*

$$\xi_{(\lambda,\mu)} \longleftrightarrow \phi_{\lambda,\mu}$$

where λ, μ are elements in $\mathfrak{t}_{\mathbb{C}}$ satisfying (A.10).

Let us remark the following equivalent definition of the set of Langlands parameters.

Proposition A.5. *The set of Langlands parameters is in bijection with $H^1(W_{\mathbb{R}}, \hat{\mathbb{T}})$.*

This is immediate from the definitions.

Appendix B. Tate-Nakayama duality

We state some results as one can find them in chapter 9 of [ABV]. Let $\mathbb{T}$ be an algebraic torus over $\mathbb{C}$. There is a natural inclusion-reversing bijection between subgroups of $\mathbb{T}$ and sublattices of $X^*(\mathbb{T})$. To an algebraic subgroup $\mathbb{S}$ corresponds the sublattice L of characters which are trivial on $\mathbb{S}$. Dually, to a sublattice L of $X^*(\mathbb{T})$ corresponds the algebraic subgroup of elements annihilating the characters in L.

Suppose that $\mathbb{S}_1 \subset \mathbb{S}_2$ correspond to $L_2 \subset L_1$. Then the restriction of characters defines a natural isomorphism

(B.1) $$\mathrm{Hom}_{alg}(\mathbb{S}_1/\mathbb{S}_2, \mathbb{C}^*) \simeq L_2/L_1.$$

Suppose that $\mathbb{S}$ correspond to a sublattice L. Then $\mathbb{S}_0$ the identity connected component of $\mathbb{S}$ corresponds to the lattice

(B.2) $$L^0 = \{\tau \in X^*(\mathbb{T}) \text{ such that } n\tau \in L \text{ for some positive integer } n\}.$$

Suppose that θ is an algebraic automorphism of $\mathbb{T}$ and θ^* is the transpose automorphism of $X^*(\mathbb{T})$. Then $\mathbb{T}^\theta$ correspond to

(B.3) $$(1-\theta^*)X^*(\mathbb{T}) = \{\tau - \theta^*(\tau),\, \tau \in X^*(\mathbb{T})\}$$

Dually the sublattice of fixed points $X^*(\mathbb{T})^{\theta^*}$ corresponds to the (connected) subgroup

(B.4) $$(1-\theta)\mathbb{T} = \{t\theta(t)^{-1},\, t \in \mathbb{T}\} = (\mathbb{T}^{-\theta})_0$$

Suppose $\mathbb{T}$ is defined over $\mathbb{R}$, that is that we are given an Galois action σ on $\mathbb{T}$. The notations are as in the previous section, and $\sigma_{\hat{\mathbb{T}}}$ will denote the *algebraic* action on $\hat{\mathbb{T}}$ inducing respectively σ^* and σ_* on $X^*(\mathbb{T})$ and $X_*(\mathbb{T})$. We consider the following component group:

(B.5) $$\pi_0(\hat{\mathbb{T}}^{\sigma_{\hat{\mathbb{T}}}}) = \hat{\mathbb{T}}^{\sigma_{\hat{\mathbb{T}}}}/(\hat{\mathbb{T}}^{\sigma_{\hat{\mathbb{T}}}})_0 = \hat{\mathbb{T}}^{\sigma_{\hat{\mathbb{T}}}}/(1+\sigma_{\hat{\mathbb{T}}})\hat{\mathbb{T}}.$$

From (B.3) and (B.4) we find that

(B.6) $$\mathrm{Hom}_{alg}(\pi_0(\hat{\mathbb{T}}^{\sigma_{\hat{\mathbb{T}}}}), \mathbb{C}^*) \simeq X_*(\mathbb{T})^{-\sigma_*}/(1-\sigma_*)X_*(\mathbb{T}).$$

Let us write $\mathbb{T}^{fin}$ for the subgroup of $\mathbb{T}$ consisting of elements of finite order. Define

(B.7) $$\mathfrak{t}_{\mathbb{Q}} := X_*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{Q} \subset \mathfrak{t}_{\mathbb{C}}.$$

Then we have:

Lemma B.1. *(*[ABV]*, lemma 9.9)*

(i) The normalized exponential mapping e*, given by* $e(\tau) = \exp(2i\pi\tau)$ *defines an isomorphism*

$$\mathfrak{t}_{\mathbb{C}}/X_*(\mathbb{T}) \simeq \mathbb{T}.$$

(ii) The preimage of $\mathbb{T}^{fin}$ *under this isomorphism is precisely* $\mathfrak{t}_{\mathbb{Q}}$*, so we have a natural isomorphism:*

$$\mathfrak{t}_{\mathbb{Q}}/X_*(\mathbb{T}) \simeq \mathbb{T}^{fin}.$$

Consider now the following subgroups:

$$\mathbb{T}^{-\sigma_T,fin} = \{t \in \mathbb{T} \text{ such that } t\sigma_T(t) \text{ has finite order }\}$$
$$\mathbb{T}^{-\sigma_T} = \{t \in \mathbb{T} \text{ such that } t\sigma_T(t) = 1\}$$
$$\mathbb{T}_0^{-\sigma_T} = \{s\sigma_T(s)^{-1},\, s \in \mathbb{T}\}$$

Proposition B.2. *The mapping $\tau \mapsto e(\tau/2)$ maps the -1 eigenspace $\mathfrak{t}_{\mathbb{Q}}^{-\sigma_*}$ into $\mathbb{T}^{-\sigma_T,fin}$. The preimages of the subgroups $\mathbb{T}_0^{-\sigma_T} \subset \mathbb{T}^{-\sigma_T}$ are the lattices*

$$(1-\sigma_*)X_*(\mathbb{T}) \subset X_*(\mathbb{T})^{-\sigma_*}.$$

There is a natural isomorphism:

$$X_*(\mathbb{T})^{-\sigma_*}/(1-\sigma_*)X_*(\mathbb{T}) \simeq \mathbb{T}^{-\sigma_T}/\mathbb{T}_0^{-\sigma_T}$$

Notice that $\mathbb{T}^{-\sigma_T}/\mathbb{T}_0^{-\sigma_T}$ is nothing but $H^1(\Gamma,\mathbb{T})$. From (B.6) and (B.2) we get a perfect pairing

$$\pi_0(\hat{\mathbb{T}}^{\hat{\sigma}_{\mathbb{T}}}) \times H^1(\Gamma,\mathbb{T}) \to \mathbb{C}^* \qquad \text{(B.8)}$$

which is the Tata-Nakayama pairing.

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