Endoscopy for real groups

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"Endoscopy for real groups"

"Twisted endoscopy for real groups"

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I. Introduction.

Endodscopy is a major theme in the framework of Langlands functoriality conjectures. This is a vast and technical subject, with both local and global aspect. My goal today is very limited : i want to show how the theory emerged in the real case in the work of Diana Shelstad in the 80's, from natural consideration in harmonic analysis of real reductive groups. I particular, i would like to present the transfer factors of Langlands-Shelstad (Kottwitz Shelstad for the twisted case), as they appeared in Shelstad's papers. I hope this *a posteriori* survey of Shelstad's work will help to understand these rather subtle and difficult objects (the transfer factors).

II. Conjucacy classes / Stable conjugacy classes

$\mathbb{G}$: connected algebraic reductive group / $\mathbb{R}$.

$G = \mathbb{G}(\mathbb{R})$: real points

$\mathfrak{g} = \mathbf{Lie}(G), \quad \mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

$\Gamma = \mathbf{Gal}(\mathbb{C}/\mathbb{R}) =< \sigma >$

Let $\gamma \in G$ regular (semisimple), $T_\gamma = \mathbb{T}_\gamma(\mathbb{R})$: unique Cartan subgroup of G containing γ.

Def : $\gamma, \gamma' \in G_{reg}$ are stably conjugate if there exists $g \in \mathbb{G} = \mathbb{G}(\mathbb{C})$ s.t.

$\gamma' = g\gamma g^{-1}$ and $\sigma(g)g^{-1} \in \mathbb{T}_\gamma$

Rmk : If γ, γ' are conjugate in G, they are stably conjugate. So stable conjugacy classes are unions of usual conjugacy classes.

T : Cartan subgroup of G.

$T = T_I T_R = T_I \exp \mathfrak{a}, \quad \mathfrak{a} = \mathrm{Lie}(T_R)$: compact/ split decomposition of T.

$\mathbb{M} = Z(\mathbb{G}, \mathfrak{a})$, $M = \mathbb{M}(\mathbb{R}) = Z(G, \mathfrak{a})$.

R : root system of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{g}$

W: Weyl group

$\alpha \in R$ is real if $\sigma(\alpha) = \alpha$, imaginary if $\sigma(\alpha) = -\alpha$, complex if $\sigma(\alpha) \neq \pm\alpha$.

$R_I \subset R$: system of imaginary roots $\simeq R(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$.

W_I : Weyl group of R_I,
$W(M, T) = N_M(T)/T$: real Weyl group

$W(M, T) \subset W_I \subset W^{\Gamma}$.

So the action of W_I on $\mathbb{T}$ preserves T.

Prop: $\gamma \in G_{reg}$, $T = T_\gamma$.
If γ is strongly regular,

$$\{w \cdot \gamma\}_{\bar{w} \in W_I / W(M,T)}$$

is a system of representatives of conjugacy classes in the stable conjugacy class of γ. In general, we get all classes, but not uniquely.

III. Orbital integrals

G acts on itself by conjugation. One would like to define a functional space of test functions on "the quotient variety".

Problem : the quotient is not a variety....

$$\mathcal{C}_c^\infty(G) \overset{dual}{\leftrightarrow} \mathbf{Dist}(G)$$

test functions / distributions

$$\mathbf{Dist}(G)^G \hookrightarrow \mathbf{Dist}(G)$$

Problem Find functional space $I(G)$ with surjective map

$$\mathcal{C}_c^\infty(G) \longrightarrow I(G)$$

and $I(G)' \simeq \mathbf{Dist}(G)^G$

Answer : $I(G)$: space of orbital integrals :

$f \in \mathcal{C}_c^\infty(G)$, $\gamma \in G_{reg}$

$$J_G(f)(\gamma) = |D_G(\gamma)|^{1/2} \int_{G/T_\gamma} f(g\gamma g^{-1})\, d\dot{g}$$

$d\dot{g}$: invariant measure, normalized in a coherent way for all Cartan subgroups

$J_G(f) \in \mathcal{C}^\infty(G_{reg})$ has certain properties (Harish-Chandra) :

- all derivatives locally bounded

- compact support on Cartan subgroups

- smooth extention on some parts of T_{sing}

- when no smooth extention on a wall, jump is explicitly given.

Thm (A. Bouaziz) These properties characterize orbital integrals.

$I(G)$: LF space

Stable orbital integrals

$J_G^{st}(f)(\gamma) = \sum_{\bar{w} \in W_I / W(M,T)} J_G(f)(w \cdot \gamma)$

Image of J_G^{st} is characterized by same kind of properties.

$I^{st}(G)$: space of stable orbital integrals.

$\gamma \in G_{reg}$, set

$I(\gamma, .) \in \mathbf{Dist}(G)^G, \quad I(\gamma, f) = J_G(f)(\gamma)$.

$I^{st}(\gamma, .) \in \mathbf{Dist}(G)^G, \quad I(\gamma, f) = J_G^{st}(f)(\gamma)$.

Bouaziz thm implies that

$< I(\gamma, .), \gamma \in G_{reg} >$ is weakly dense in $\mathbf{Dist}(G)^G$

Def: $\mathbf{Dist}(G)^{st} = \overline{< I^{st}(\gamma, .), \gamma \in G_{reg} >}$

So $\mathbf{Dist}(G)^{st} \hookrightarrow \mathbf{Dist}(G)^G \hookrightarrow \mathbf{Dist}(G)$ and $\mathcal{C}_c^\infty(G) \to I(G) \to I^{st}(G)$

Prop If $\Theta \in \mathbf{Dist}(G)^G$ is given by a locally L^1 function F_Θ (for instance a character), then $\Theta \in \mathbf{Dist}(G)^{st}$ iff F_Θ is constant on stable conjugacy classes.

IV. Characters

So far, i've talked about invariant distribution on G defined geometrically (orbital integrals). They appear in the geometric side of the trace formula. Let us consider now distributions appearing in the spectral side, i.e. characters of representations.

$\mathcal{M}(G)$: category of Harish-Chandra modules

$\Pi(G)$: (classes of) irreducible objects

If $(\pi, V) \in \mathcal{M}(G)$, $\Theta_\pi \in \mathbf{Distr}(G)^G$ is the distribution character of π. It is given by a locally L^1 function F_{Θ_π}, analytic on G_{reg}.

$\Pi_{temp}(G)$: (classes of) irreducible tempered representations.

Thm (Arthur) $< \Theta_\pi, \pi \in \Pi_{temp}(G) >$ is weakly dense in $\mathbf{Distr}(G)^G$.

also follows easily from Bouaziz thm.

V. L-packets

Grouping orbital integrals in a given stable conjugacy class, we obtain stable distributions. So there must be a way to group together distribution characters Θ_π to get stable distributions.

Thm (Langlands) There is a partition of $\Pi(G)$ in "L-packets"

$$\Pi(G) = \coprod_\phi \Pi_\phi$$

satisfying many properties, among them :

- Π_ϕ are finite

- if one π in Π_ϕ is tempered, all of them are

- if $\pi_1, \pi_2 \in \Pi_\phi$ have same central and infinitesimal characters

....

- For $G = GL(n, \mathbb{R})$, all L-packets are singletons.

Example Discrete series for $SL(2, \mathbb{R})$: π_n, $n \in \mathbb{Z}^*$. Then $\{\pi_n, \pi_{-n}\}$ is a L-packet.

Also $\{\pi_0^-, \pi_0^+\}$: limits of discrete series

Thm (Shelstad) If $\Pi_\phi \subset \Pi_{temp}(G)$, then

$\Theta_\phi = \sum_{\pi \in \Pi_\phi} \Theta_\pi$

is a stable distribution.

Rmk It works only with tempered packets. For non-tempered, one has to consider instead more complicated Arthur packets to get stable distributions (see Adams-Barbasch-Vogan).

Parameters:

$W_{\mathbb{R}}$ Weil group of $\mathbb{C}/\mathbb{R}$.

$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_{\mathbb{R}} \longrightarrow \Gamma \longrightarrow 1$

non split exact sequence.

${}^L G = \hat{G} \ltimes W_{\mathbb{R}}$: Langlands L-group

$\Phi(G) = \{\phi : W_{\mathbb{R}} \longrightarrow {}^L G$ s.t..... $\}/ \sim \hat{G}$

VI. Endoscopic group

Take $\mathbb{H}$, $\mathbb{G}$ as before, and suppose there is map

$\xi : {}^L H \longrightarrow {}^L G$ (satisfying some properties...)

so that one can defined geometric correspondences between stable conjugacy classes in H and G (H and G "share" some Cartan subgroups).

>From a Langlands parameter $\phi_H : W_\mathbb{R} \longrightarrow {}^L H$ one get a Langlands parameter
$\xi \circ \phi_H := \phi_G : W_\mathbb{R} \longrightarrow {}^L G$.

We want a "character identity" :

$$\Theta_{\phi_H} \longleftrightarrow \sum_{\pi \in \Pi_{\phi_G}} a_\pi \, \Theta_\pi \quad (*)$$

If we can find enough different H to invert $(*)$, we would write each Θ_π as a linear combination of stable distributions : useful for applications of the trace formula.

So, we want

$$Trans : \mathrm{Distr}(H)^{st} \to \mathrm{Distr}(G)^{G}$$

We can get such a map as the transpose of a map

$$trans : I(G) \to I^{st}(H)$$

Basic idea : $W_I/W(M,T)$ which parametrizes conjugacy classes inside a stable conjugacy class, can be viewed as a subset of a subgroup of $H^1(\Gamma, \mathbb{T})$. One can use Tate-Nakayama's pairing

$$H^1(\Gamma, \mathbb{T}) \times \pi_0(\hat{T}^{\Gamma}) \to \mathbb{Z}$$

to put weights on each conjugacy class inside the stable one.

Shelstad found out what H to consider, giving geometric correspondence between stable conjugacy classes in H and G

$\gamma_H \longleftrightarrow \gamma_G$

and defined a map

$$trans : I(G) \to I^{st}(H)$$

defined on γ_H in H_{reg} such that $\gamma_G \in G_{reg}$ by :

$$trans(\psi)(\gamma_H) = \sum_{\delta \in \Sigma_G} \Delta(\delta, \gamma_H)\psi(\delta) \quad (**)$$

Σ_G : system of representatives of conjugacy classes in the stable conjugacy class of γ_G.

$(**)$ should have smooth extension to regular γ_H such that γ_G is not regular in G.

$\Delta(\delta, \gamma_H)$: transfer factors

$\Delta = \Delta_I \Delta_{II} \Delta_{III_1} \Delta_{III_2}$

Rmk : In LS or KS, there is also a Δ_{IV}. We have included it in our definition of orbital integrals $|D_G(\delta)|^{1/2}/D_H(\gamma_H)|^{1/2}$

Δ_{III_1} : correspond to the basic idea of endoscopic groups : it is a sign $\{\pm 1\}$ put on each conjugacy classes in the stable conjugacy class of γ_G (depends on the choice of the base point γ_G).

Δ_{II} : $\delta \mapsto \psi$ can be made more regular by multiplying by a factor

$$b_{R_I^+}(\delta) = \prod_{\alpha \in R_I^+} (\alpha(\delta) - 1)/|(\alpha(\delta) - 1|$$

Choose positive system of imaginary roots R_I^+, $R_{I,H}^+$, then

$\Delta_{II}(\delta, \gamma_H) = b_{R_I^+}(\delta)/b_{R_{I,H}^+}(\gamma_H)$.

$\boldsymbol{\Delta_{III_2}}$: $\gamma_H \in \boldsymbol{T_H}$, $\boldsymbol{\delta} \in \boldsymbol{T}$, $\mathbb{T}_H \simeq \boldsymbol{T}$ over $\mathbb{R}$.
${}^L\boldsymbol{T_H} \simeq {}^L\boldsymbol{T}$ and some choices of positive root systems give

$$ {}^L\boldsymbol{T_H} \to {}^L\boldsymbol{H}, \qquad {}^L\boldsymbol{T} \to {}^L\boldsymbol{G} $$

$$\begin{array}{ccc} {}^L\boldsymbol{T_H} & \longrightarrow & {}^L\boldsymbol{H} \\ \downarrow & & \downarrow \boldsymbol{\xi} \\ {}^L\boldsymbol{T} & \longrightarrow & {}^L\boldsymbol{G} \end{array}$$

But the diagram is not commutative : defect given by $\boldsymbol{a} \in \boldsymbol{H^1}(\boldsymbol{W}_{\mathbb{R}}, \mathbb{T})$

By Langlands correspondance for tori, this gives a character $\boldsymbol{\chi_a}$ of $\boldsymbol{T}$,

$\boldsymbol{\Delta_{III_1}}(\boldsymbol{\delta}, \gamma_H) = \boldsymbol{\chi_a}(\boldsymbol{\delta})$

"correction character" in Shelstad terminology.

Δ_I : maybe the most subtle : it is a sign $\{\pm 1\}$ depending on the pair (T_H, T), given also by Tate-Nakayama duality, which does two things (miracle!)

- Δ becomes independent of all choices

- jump formula are satisfied for $trans(\psi)$.

Thm (Shelstad) For tempered L-packets, we get what we want :

$Trans : \mathrm{Distr}(H)^{st} \to \mathrm{Distr}(G)^G$

$\Theta_{\phi_H} \mapsto \sum_{\pi \in \Pi_{\phi_G}} (\pm 1)\Theta_\pi$

± 1 interpreted as a pairing between Π_{ϕ_G} viewed as a subset of the character group of S_{ϕ_G} and

$S_{\phi_G} =$
$Z(\phi_G(W_R), \hat{G})/Z(\phi_G(W_R), \hat{G})_0 Z(\hat{G})^\Gamma$

There are enough endoscopic groups to invert the system.