CHARACTERS OF SPEH REPRESENTATIONS AND LEWIS CAROLL IDENTITY

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1. Introduction

Let F be a non archimedian local field of characteristic zero, let A be a central division algebra over F and let $G_m = \mathbf{GL}(m, A)$. The unitary dual of G was described in [14] ([13] in the case $A = F$), and the missing part of the proof (conjectures $U0$ and $U1$ in [14]) were obtained respectively in [11] and [2]. Speh representations are building blocks for the unitary dual of G_m, in the sense that an irreducible unitary representation of G_m is always fully parabolically induced from a tensor product of Speh representations or complementary series starting from a tensor product of two copies of the same Speh representation (see section 4). Since there is a well-known formula for the character of a parabolically induced representation, in terms of the character of the induced representation, characters of irreducible unitary representations of G_m can be computed from characters of Speh representations, and a formula for them has been given by Tadic in [12] in the case $A = F$, and generalized in [15] for A arbitrary. Let us explain briefly what are these character formulas. By Langlands classification, the Grothendieck group of finite length representations of G_m has two bases - the obvious one by irreducibles and the one consisting of "standard" representations, i.e. the ones which are parabolically induced by essentially square integrable representations. Therefore, any irreducible character can be expressed as a linear combination of characters of standard representations. Tadic's result is an explicit formula for the character of Speh representations in the second basis. In general, such formulas for irreducible characters can be obtained only by combinatorially very complicated Kazhdan-Lusztig algorithms (computing relevant Kazhdan-Lusztig-(Vogan) polynomials, [8], [7]). It is remarkable that a closed formula can be given for Speh representations. For another instance of such closed formulas, for unipotent representations of complex groups, see [4].

The proof in [12] is rather indirect. It uses a character formula for Speh representations of $\mathbf{GL}(m, \mathbb{C})$ due to Zuckerman (here Speh representations are one-dimensional, i.e. in the complex case, they are just characters) and a formal argument allowing a comparison between

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$\mathbf{GL}(m, \mathbb{C})$ and $\mathbf{GL}(m, A)$ based on a formula for the end of complementary series (in the p-adic case as well as in the complex case ; for p-adic case this is the formula in Theorem 3.1). For $\mathbf{GL}(m, A)$, the irreducible constituents of the end of complementary series can be obtained by elementary arguments, but unfortunately, this was not the case for $\mathbf{GL}(m, \mathbb{C})$ (see the proof in [12], using results of S. Sahi [10]). A direct proof of the character formula was therefore desirable. It was clear from [12] that a direct argument could be given once a combinatorial identity was proved. We remarked that Tadic's formula for the characters of Speh representations is given by the determinant of a matrix, and that the combinatorial identity needed can be reinterpreted as a curious identity of determinants. We proved this identity, learning later that it was already known under the name of Lewis Caroll identity [1], [6]. We finish this introduction by remarking that in the case of $\mathbf{GL}(m, \mathbb{C})$, the argument can be reversed, and that it gives a direct proof for the composition factors of the ends of complementary series, from Zuckerman formula, much simpler than the one in [12] and [10].

Finally, let us mention that these character formulas are essential in the extension of the local Jacquet-Langlands correspondence from tempered representations to all irreducible unitary representations ([3],[15]) and also to get the global correspondence ([3]).

2. Speh representations

We follow [14] for notation. Let us denote by ν the morphism

$$\nu : G_m = \mathbf{GL}(m, A) \rightarrow \mathbb{R}_+^*, \qquad g \mapsto |RN(g)|_F$$

where $RN(g)$ is the reduced norm of the algebra $M_m(A)$ of all m by m matrices over A (with values in F) and $|.|_F$ is a normalized p-adic norm for F.

If π_i is a smooth representation of G_{n_i}, $i = 1, \ldots, r$,

$$\pi_1 \times \pi_2 \times \ldots \times \pi_r$$

is then the representation of $G_{n_1 + \cdots + n_r}$ induced parabolically from the representation $\pi_1 \otimes \pi_2 \otimes \ldots \otimes \pi_r$ of the standard Levi subgroup $G_{n_1} \times G_{n_2} \times \ldots \times G_{n_r}$ (standard parabolic subgroups are the one containing upper triangular matrices in the group).

Let $\mathcal{R}_m$ be the Grothendieck group of the category of smooth representations of G_n of finite length. Parabolic induction defines a product on

$$\mathcal{R} = \bigoplus_{m \in \mathbb{N}} \mathcal{R}_m.$$

It is understood that $G_0 = \{1\}$, a group which admits a unique irreducible representation σ_0. The image of σ_0 in $\mathcal{R}$ is the unit of $\mathcal{R}$.

Let ρ be an irreducible cuspidal representation of G_m, and let $s = s_\rho$ be the integer attached to it in [14] (this is the length of the Zelevinskii segment corresponding to the essentially square integrable (modulo center) representation obtained from ρ by the Jacquet-Langlands correspondence). Set $\nu_\rho = \nu^{s_\rho}$. It is proved in [14], generalizing results of Zelevinskii [16], that

$$\nu_\rho^{\frac{l-1}{2}} \rho \times \nu_\rho^{\frac{l-3}{2}} \rho \times \ldots \times \times \nu_\rho^{-\frac{l-3}{2}} \rho \times \nu_\rho^{-\frac{l-1}{2}} \rho$$

has a unique irreducible subrepresentation, denoted $\delta(\rho, l)$, that $\delta(\rho, l)$ is essentially square integrable (modulo center), and that all irreducible essentially square integrable (modulo center) representations are obtained in this manner. More generally, if $i \leq j$ are integers, $\delta(\rho, [i, j])$ denotes the unique irreducible subrepresentation of

$$\nu_\rho^j \rho \times \nu_\rho^{j-1} \rho \times \ldots \times \times \nu_\rho^{i+1} \rho \times \nu_\rho^i \rho$$

Let $\delta = \delta(\rho, l)$ as above. The Speh representation $u(\delta, n)$ is the unique irreducible quotient (Langlands quotient) of

$$\nu_\rho^{\frac{n-1}{2}} \delta \times \nu_\rho^{\frac{n-3}{2}} \delta \times \ldots \times \nu_\rho^{-\frac{n-3}{2}} \delta \times \nu_\rho^{-\frac{n-1}{2}} \delta.$$

If ρ is unitary, so is $\delta = \delta(\rho, l)$ and $u(\delta, n)$ ([13],[2]). The Speh representations $u(\delta(\rho, l), n)$, ρ unitary, are the building blocks in the classification of the unitary dual of the groups $\mathbf{GL}(m, A)$ (see [13],[14], and section 4).

3. End of complementary series

For $\alpha \geq 0$, and ρ an irreducible cuspidal representation of G_m as in the previous section, set

$$\pi(\rho, l, n, \alpha) = \nu_\rho^\alpha u(\delta(\rho, l), n) \times \nu_\rho^{-\alpha} u(\delta(\rho, l), n)$$

If $0 \leq \alpha < \frac{1}{2}$, $\pi(\rho, l, n, \alpha)$ is irreducible, and unitary if ρ is (complementary series attached to $u(\delta(\rho, l), n)$).

At $\alpha = \frac{1}{2}$, $\pi(\rho, l, n, \alpha)$ reduces. A formula for the composition series is given by Tadic ([15], Prop 4.1).

Theorem 3.1. *Let ρ, n, l as above. In the Grothendieck group, we have*

$$\nu_\rho^{\frac{1}{2}} u(\delta(\rho, l), n) \times \nu_\rho^{-\frac{1}{2}} u(\delta(\rho, l), n) =$$
$$u(\delta(\rho, l), n+1) \times u(\delta(\rho, l), n-1) + u(\delta(\rho, l+1), n) \times u(\delta(\rho, l-1), n).$$

When $n = 0$ or $l = 0$, we set $u(\delta(\rho, l), n) = \sigma_0$ (the unit in $\mathcal{R}$).

In particular, when ρ is unitary, all the constituents of the ends of complementary series are unitary, a result due to D. Milicic in a much more general setting ([9]).

4. Unitary dual of G_m

We are now in position to state the classification theorem of Tadic ([13],[14]). Let us denote by $\mathbf{Irr}^u$ the set of irreducible unitary smooth representations of the groups G_m, $m \in \mathbb{N}$. Conjecture $U(0)$ of Tadic, proved by Bernstein [5] in the field case, and Sécherre [11] for a division algebra asserts that if σ, π are in $\mathbf{Irr}^u$, then $\sigma \times \pi$ is again in $\mathbf{Irr}^u$. Let us denote by C^u the subset of $\mathbf{Irr}^u$ consisting of supercuspidal representations.

Theorem 4.1. *The set* $\mathbf{Irr}^u$*, endowed with the product* $\times$*, is a free commutative monoid with basis*

$$\mathcal{B} = \{u(\delta(\rho, l), n), \pi(\rho, l, n, \alpha),\ \rho \in C^u,\ n, l \in \mathbb{N},\ \alpha \in]0, \frac{1}{2}[\}.$$

Thus, if π *is an irreducible unitary smooth representation of some* G_m*, there exists* $\pi_1, \dots, \pi_k \in \mathcal{B}$*, unique up to a permutation, such that*

$$\pi = \pi_1 \times \dots \times \pi_k.$$

5. Character formula for $u(\delta(\rho, l), n)$

Let ρ, n, l as above. We will prove by induction on n a character formula for $u(\delta(\rho, l), n)$. In order to have uniform formulas, we adopt the following convention. Recall that the representations $\delta(\rho, [i, j])$ were defined above for integers i, j, $i \leq j$. If $j < i - 1$, we set (in $\mathcal{R}$) : $\delta(\rho, [i, j]) = 0$, and $\delta(\rho, [i, i-1]) = 1$.

Theorem 5.1.

$$\nu_\rho^{\frac{l+n}{2}} u(\delta(\rho, l), n) = \sum_{w \in \mathfrak{S}_n} (-1)^{l(w)} \prod_{i=1}^{n} \delta(\rho, [i, l + w(i) - 1]),$$

where $l(w)$ is the length of the permutation w in the symmetric group $\mathfrak{S}_n$ (therefore $(-1)^{l(w)}$ is the signature of w).

Proof. The identity to be proved can also be written, with $m_{i,j} = \delta(\rho, [i, l + j - 1])$,

$$\nu_\rho^{\frac{l+n}{2}} u(\delta(\rho, l), n) = \det(m_{i,j})_{1 \leq i,j \leq n}.$$

Thus $(m_{i,j})_{i,j}$ is a matrix with coefficients in $\mathcal{R}$. The formula is true for $n = 1$. Suppose it is true up to n. From Theorem 3.1, we have

$$\begin{aligned} & u(\delta(\rho, l), n+1) \times u(\delta(\rho, l), n-1) \\ & \quad = \nu_\rho^{\frac{1}{2}} u(\delta(\rho, l), n) \times \nu_\rho^{-\frac{1}{2}} u(\delta(\rho, l), n) \\ & \qquad - u(\delta(\rho, l+1), n) \times u(\delta(\rho, l-1), n). \end{aligned}$$

Using induction hypothesis, we get

$$\nu_\rho^{\frac{l+n+1}{2}} u(\delta(\rho,l),n+1) \times \nu_\rho \sum_{w\in\mathfrak{S}_{n-1}} (-1)^{l(w)} \prod_{i=1}^{n-1} \delta(\rho,[i,l+w(i)-1]) =$$

$$\nu_\rho \sum_{w\in\mathfrak{S}_n} (-1)^{l(w)} \prod_{i=1}^{n} \delta(\rho,[i,l+w(i)-1]) \times \sum_{w\in\mathfrak{S}_n} (-1)^{l(w)} \prod_{i=1}^{n} \delta(\rho,[i,l+w(i)-1])$$

$$-\sum_{w\in\mathfrak{S}_n} (-1)^{l(w)} \prod_{i=1}^{n} \delta([\rho,[i,l+w(i)]) \times \nu_\rho \sum_{w\in\mathfrak{S}_n} (-1)^{l(w)} \prod_{i=1}^{n} \delta(\rho,[i,l+w(i)-2])$$

Set $m'_{i,j} = \delta([\rho,[i+1,l+j])$ and rewrite this equality as

$$\begin{aligned} &\nu_\rho^{\frac{l+n+1}{2}} u(\delta(\rho,l),n+1) \times \det((m'_{i,j})_{1\le i,j\le n-1}) \\ &= \det((m'_{i,j})_{1\le i,j\le n}) \times \det((m'_{i,j})_{0\le i,j\le n-1}) \\ &- \det((m'_{i,j}))_{0\le i\le n-1,1\le j\le n}) \times \det((m'_{i,j}))_{1\le i\le n,0\le j\le n-1}) \end{aligned}$$

Lemma 6.1 and the fact that $\mathcal{R}$ ([14]) is a domain show that

$$\nu_\rho^{\frac{l+n+1}{2}} u(\delta(\rho,l),n+1) = \det((m'_{i,j})_{0\le i,j\le n}) = \det((m_{i,j})_{1\le i,j\le n+1}).$$

$\square$

6. Lewis Caroll identity

We give a quick proof of Lewis Caroll (*ie.* Charles Dodgson) identity. Let $\mathcal{R}$ be a ring and let V be a free $\mathcal{R}$-module, of dimension $n+1$, with basis $(e_0,\dots,e_n)$. Let γ be an endomorphism of V. Denote by $\wedge^i\gamma$ the induced endomorphism in $\bigwedge^i V$ and $M=(m_{ij})$ the matrix of γ in this basis.

Recall that if p and q are two integers such that $p+q=n+1$, we get a perfect duality

$$\bigwedge^p V \otimes \bigwedge^q V \to \bigwedge^{n+1} V$$

Thus, we have

$$(\bigwedge^p V)^* \simeq \bigwedge^q V \otimes \det{}^{-1}.$$

This shows that

$$\bigwedge^{n-1} V \otimes \det{}^{-1} \simeq (\bigwedge^2 V)^* \simeq \bigwedge^2(V^*) \simeq \bigwedge^2(\bigwedge^n V \otimes \det{}^{-1}) \simeq \bigwedge^2(\bigwedge^n V) \otimes \det{}^{-2}.$$

Finally

$$\bigwedge^{n-1} V \otimes \det = \bigwedge^{n-1} V \otimes \bigwedge^{n+1} V \simeq \bigwedge^2(\bigwedge^n V).$$

Therefore, the matrices in the natural bases deduced from $(e_0,\dots,e_n)$, respectively of

$$\wedge^{n-1}\gamma \otimes \wedge^{n+1}\gamma = \wedge^{n-1}\gamma \otimes \det(\gamma)\mathrm{Id} \quad \text{and} \quad \wedge^2(\wedge^n\gamma)$$

are equal.

The matrix of $\wedge^n\gamma$, in the basis $\tilde{e}_i = e_0 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge e_n$ is the transpose of the comatrix of M. Let us denote it by $Q = (q_{ij})_{ij}$.

The coefficient $((0,n),(0,n))$ of the matrix of $\wedge^2(\wedge^n\gamma)$ in the basis $\tilde{e}_i \wedge \tilde{e}_j$, $0 \leq i < j \leq n$, is $q_{00}q_{nn} - q_{0n}q_{n0}$.

In the basis $\tilde{e}_{ij}$, $0 \leq i < j \leq n$ of $\bigwedge^{n-1} V$, where

$$\tilde{e}_{ij} = e_0 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge \hat{e}_j \wedge \ldots \wedge e_n$$

the coefficient $(0,n),(0,n)$ of the matrix of $\wedge^{n-1}\gamma$ is the "central" minor $(n-1)\times(n-1)$ of M, *ie.* $\det((m_{ij})_{1\leq i,j\leq n-1})$.

Therefore, we obtain

$$\det((m_{ij})_{1\leq i,j\leq n-1}) \det M = q_{00}q_{nn} - q_{0n}q_{n0}$$

But coefficients of the transpose of the comatrix are also minors. Writing them as such, we get

Lemma 6.1.

$$\begin{aligned} &\det((m_{ij})_{1\leq i,j\leq n-1})\ \det((m_{ij})_{0\leq i,j\leq n}) \\ &\quad = \det\left((m_{ij})_{1\leq i,j\leq n}\right) \det((m_{ij})_{0\leq i,j\leq n-1}) \\ &\qquad - \det\left((m_{ij})_{1\leq i\leq n, 0\leq j\leq n-1}\right) \det\left((m_{ij})_{0\leq i\leq n-1, 1\leq j\leq n}\right). \end{aligned}$$

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