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Hypercontractivity for Markov semi-groups ☆

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A B S T R A C T

We investigate in a systematic way hypercontractivity property in Orlicz spaces for Markov semi-groups related to homogeneous and non homogeneous diffusions in $\mathbb{R}^n$. We provide an explicit construction of a family of Orlicz functions for which we prove that the associated hypercontractivity property is equivalent to a suitable functional inequality.

1. Introduction

The first aim of this paper is to give a unified setting for strong contractivity properties of Markov semi-group to be satisfied with respect to suitable family of Luxembourg norms in Orlicz spaces.

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Initiated by Nelson in the late sixties [16–18] in quantum field theory, the notion of hypercontractivity of the Ornstein-Ulhenbeck process was put in light by Gross' seminal work [10]. One of the main observation of Gross is that hypercontractivity is equivalent to the so called log-Sobolev inequality. See also [8,23] for earlier papers on related topic.

More precisely, let γ_n be the standard Gaussian measure on $\mathbb{R}^n$. Then the Ornstein-Uhlenbeck semi-group $(P_t)_{t\geq 0}$, whose infinitesimal generator is $L := \Delta - x \cdot \nabla$ (with the dot sign standing for the Euclidean scalar product), is reversible with respect to γ_n and satisfies the following remarkable *hypercontractivity property*: for any $f \colon \mathbb{R}^n \to \mathbb{R}$ smooth enough it holds

$$\|P_t f\|_{q(t)} \leq \|P_s f\|_{q(s)}, \qquad s \leq t$$

where $q(t) = 1 + (q(0)-1)e^{2t}$, $q(0) \geq 1$, and $\|g\|_p^p := \int |g|^p d\gamma_n$, $p \geq 1$. Such a contraction property is equivalent [10] to the following log-Sobolev inequality: for any $f \colon \mathbb{R}^n \to \mathbb{R}$ smooth enough, it holds

$$\mathrm{Ent}_{\gamma_n}(f^2) := \int f^2 \log f^2 d\gamma_n - \int f^2 d\gamma_n \log \int f^2 d\gamma_n \leq 2 \int |\nabla f|^2 d\gamma_n.$$

Using Gross' paper and Γ^2-calculus of Bakry-Emery [5,2], it can be immediately proved that any semi-group associated to a diffusion of the form $L := \Delta - \nabla V \cdot \nabla$, with V satisfying $\mathrm{Hess}(V) \geq \rho > 0$, as a matrix, enjoys the hypercontractivity property as above with reference measure having density e^{-V} with respect to the Lebesgue measure and $q(t) = 1 + (q(0)-1)e^{(4/\rho)t}$, $q(0) \geq 1$.

From the seventies, both the hypercontractivity property and the log-Sobolev inequality found a huge amount of applications in various fields, including Analysis (isoperimetry, concentration of measure phenomenon, convex geometry), Statistical mechanics, Information Theory and others. Giving an exhaustive presentation of the literature is out of reach. We refer to the textbooks [1,14,11,6,15,19] for an introduction and references.

Now, let $\Phi \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous convex function satisfying $\Phi(x) = 0$ iff $x = 0$. Later on we may call such a function a *Young function.*[1] Then, given $f \colon \mathbb{R}^n \to \mathbb{R}$ such that $\int \Phi(\alpha f) d\gamma_n < +\infty$ for some $\alpha > 0$, one can define the so-called Luxembourg norm associated to Φ and γ_n as

$$\|f\|_\Phi = \inf\left\{\lambda > 0 : \int \Phi\left(\frac{|f|}{\lambda}\right) d\gamma_n \leq 1\right\}.$$

The power function $\Phi(x) = |x|^p$, $p \geq 1$, trivially corresponds to the usual $\mathbb{L}_p$-norm introduced above $\|f\|_\Phi = \|f\|_p$. The space of all functions with finite Luxembourg norm

[1] Note however that, usually, one does not require in the definition of a Young function neither the regularity assumption, nor the condition $\phi(x) = 0$ iff $x = 0$.

will be denoted by $\mathbb{L}_\Phi(\gamma_n)$ (or simply $\mathbb{L}_\Phi$ when there is no confusion, note however that norms are always computed with an underlying measure).

With this definition at hand, for the family of Young functions $\Phi_t(x) = |x|^{q(t)}$, $x \in \mathbb{R}$, $t \geq 0$ with $q(t) = 1+(q(0)-1)e^{2t}$, $q(0) \geq 1$, the hypercontractivity above can be restated as follows

$$\|P_t f\|_{\Phi_t} \leq \|P_s f\|_{\Phi_s}, \qquad s \leq t.$$

In other words, the Ornstein-Uhlenbeck semi-group is a contraction along the family of Orlicz spaces $(\mathbb{L}_{\Phi_t})_{t\geq 0}$.

Following Gross' ideas, in [3] the authors proved that some contraction property along a different type of family of Orlicz spaces could hold. Consider the following infinitesimal generator[2] in dimension n, $L := \Delta - \nabla V \cdot \nabla$, with $V(x) = \sum_{i=1}^n |x_i|^\alpha$, $\alpha \in [1,2]$, $x \in \mathbb{R}^n$. Denote by $(P_t)_{t\geq 0}$ the associated semi-group and by $\mu(dx) = Z^{-1}e^{-V(x)}dx$, $x \in \mathbb{R}^n$, the associated reversible probability measure, $Z := \int e^{-V(x)}dx$ being the normalization constant. Finally define $\Phi_t(x) = |x|^p e^{q(t)F(x)}$ with $F(x) := \log(1+x)^{2(\alpha-1)/\alpha} - \log(2)^{2(\alpha-1)/\alpha}$, $q(t) = Ct$ for some constant $C > 0$, and $p > 1$. By construction $\mathbb{L}_{\Phi_t} \subset \mathbb{L}_{\Phi_s} \subset \mathbb{L}_p$ for any $s \leq t$ and $\mathbb{L}_{p+\varepsilon} \not\subset \mathbb{L}_{\Phi_t}$ for any $\varepsilon > 0$, $t \geq 0$ and $\alpha \in [1,2)$ (since $e^{q(t)F(x)} \ll |x|^\varepsilon$ near infinity, for any $\alpha \in [1,2)$).

In [3] it is proved that $(P_t)_{t\geq 0}$ is a contraction along the family of Orlicz spaces $(\mathbb{L}_{\Phi_t})_{t\geq 0}$: namely that, for any $s \leq t$, it holds $\|P_t f\|_{\Phi_t} \leq \|P_s f\|_{\Phi_s}$. Moreover, such a contraction property is equivalent to the following, known as F-Sobolev inequality ([20,24]): for any $f \colon \mathbb{R}^n \to \mathbb{R}$ it holds

$$\int f^2 F\left(\frac{f^2}{\int f^2 d\mu}\right) d\mu \leq C' \int |\nabla f|^2 d\mu$$

where C' is a constant that depends on C and α. Note that $\alpha = 2$ corresponds to the Gaussian case depicted above. Such inequalities and contraction properties were used to establish dimension free isoperimetric inequalities and concentration properties for μ [3,4]. We refer the reader to [25] for explicit criterion for a F-Sobolev inequality to hold, and to [24] for associated contraction property of the semi-group.

Motivated by the previous two fundamental examples, the aim of this paper is to investigate on contraction properties $\|P_t f\|_{\Phi_t} \leq \|P_s f\|_{\Phi_s}$, $s \leq t$, along abstract general family of Orlicz spaces $(\mathbb{L}_{\Phi_t})_{t\geq 0}$, together with possible connection with functional inequalities of F-Sobolev type.

The second objective of the paper is to explore a more general setting which would include inhomogeneous diffusion operators associated to one parameter families of probability measures that we now introduce. Consider $L_t := \Delta - \nabla V_t \cdot \nabla$, $t \geq 0$, on $\mathbb{R}^n$, with V_t

[2] More precisely one should consider a regularized version of $|x|^\alpha$ in a neighborhood of the origin. For the sake of simplicity we may avoid such technical considerations in this introduction, that are irrelevant for our purpose, and we refer to [3] for details.

smooth enough and such that $\int e^{-V_t} = 1$ so that $\mu_t(dx) = e^{-V_t(x)}dx$ is a probability measure on $\mathbb{R}^n$ for all $t \geq 0$. The associated semi-group will be denoted by $(P_s^{(t)})_{s\geq 0}$ (we refer to e.g. [2,11] for its construction and related technicalities) which is reversible in $\mathbb{L}_2(\mu_t)$. One wishes to obtain contraction bounds of the type $\|P_t^{(t)} f\|_{\Phi_t} \leq m(t,s)\|P_s^{(s)} f\|_{\Phi_s}$, $s \leq t$ for some function m possibly equal to 1.

Thus in the more general setting we not only change with time the Orlicz functions, but also the underlying probability measures. Here we are interested in a class of flows through Orlicz spaces and how it relates to an action of contractions.

Besides interesting generalizations, we hope that our results can be used in the future to study linear and nonlinear parabolic time dependent problems. Note that in case of a time dependent parabolic problem of the form

$$\begin{aligned} \partial_t u &= Lu + \beta_t \cdot \nabla u \equiv L_t u \\ u_{|t=0} &= f \end{aligned}$$

under suitable conditions on the coefficient β_t, one can hope to approximate the solution on small intervals $s \in [t_n, t_{n+1}]$ by $P_{s-t_n}^{(t_n)} u_{t_n}$. Then one needs to setup a suitable framework to control convergence of such approximation when $\sup_n |t_{n+1} - t_n| \to 0$. While we mention here as an example a linear problem, we remark that nonlinear semigroups with hypercontractivity properties has been studied in [9] and one could possibly extend the above given idea to the nonlinear time dependent parabolic problems.

Moreover, as suggested to us by a referee, since Gross's theorem is established for symmetric Markov processes associated with Dirichlet forms, it is reasonable to conjecture that most of the result of this paper can be extended to such an abstract framework and leave this to future investigation.

After Section 2, that collects some technical facts about Orlicz functions/norms, we deal in Section 3 with the homogeneous setting.

Our first main theorem is Theorem 3.9 that asserts that, for any properly chosen family of Orlicz spaces (see Section 3.2), we have

$$\|P_t f\|_{\Phi_t} \leq \|P_s f\|_{\Phi_s}$$

if and only if some inequality of log-Sobolev type holds. Theorem 3.9 encompasses the above two known fundamental examples and can therefore be seen as a generalization of Gross' theorem.

Section 4 is devoted to the time-inhomogeneous setting. Our second main result is Theorem 4.1 which constitutes some analog of Gross' theorem for inhomogeneous Markov semi-groups.

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2. Technical preparations

In this section we collect some useful technical facts on various aspects of Young Functions and Luxembourg norms.

2.1. Youngs functions

An even continuous convex function $\Phi\colon \mathbb{R} \to \mathbb{R}_+$ satisfying $\Phi(x) = 0$ iff $x = 0$ is called a *Young function*. If in addition $\lim_{x\to 0} \Phi(x)/x = 0$, $\lim_{x\to\infty} \Phi(x)/x = +\infty$, and $\Phi(\mathbb{R}) \subset \mathbb{R}^+$, Φ is called a *nice Young function*, or *N-function* [21].

Classical examples include $\Phi(x) = |x|^p$, $p \geq 1$ which is a nice Young function only for $p > 1$; $\Phi(x) = e^{|x|} - |x| - 1$, $\Phi(x) = e^{|x|^\delta} - 1$, $\delta > 1$ are nice Young functions.

We say that Φ satisfies the Δ_2-condition if for some $K > 0$ and all $x \geq 0$, it holds $\Phi(2x) \leq K\Phi(x)$. A useful consequence of the Δ_2-condition is the fact that $x\Phi'(x)$ compares to Φ. More precisely,

$$\Phi(x) \leq x\Phi'(x) \leq (K-1)\Phi(x). \tag{2.1}$$

The first inequality follows from the convexity property of Φ and $\Phi(0) = 0$, while the second is a consequence of the Δ_2-condition and $\Phi(2x) - \Phi(x) = \int_x^{2x} \Phi'(t)dt \geq \Phi'(x)x$.

Given a Young function Φ and a probability measure μ, for any $f\colon \mathbb{R}^n \to \mathbb{R}$ we set

$$\|f\|_\Phi := \inf\left\{\lambda > 0 : \int \Phi\left(\frac{|f|}{\lambda}\right) d\mu \leq 1\right\} \in [0,\infty]$$

with the convention that $\inf\emptyset = +\infty$. When useful we may write $\|f\|_{\Phi,\mu}$ to emphasize the underlying measure.

2.2. Derivative of Luxembourg norm

Here we give an explicit expression of the derivative with respect to time of the following function $t \mapsto \|P_t f\|_{\Phi_t}$ which will constitute the starting point of our investigations.

In the sequel we will use the following notations. Given a family of twice differentiable Young functions $(\Phi_t)_{t\geq 0} = (\Phi(t,x))_{t\geq 0}$, we denote by $\dot{\Phi}_t$ the derivative with respect to t, and by Φ_t' and Φ_t'' the first and second order derivative with respect to the second variable x.

Consider the inhomogeneous diffusion generator $L_t := \Delta - \nabla V_t \cdot \nabla$, $t \geq 0$, on $\mathbb{R}^n$, with V_t sufficiently smooth and such that $\int e^{-V_t} dx = 1$ so that $\mu_t(dx) = e^{-V_t(x)}dx$ is a probability measure on $\mathbb{R}^n$ for all $t \geq 0$. Denote by $(P_s^{(t)})_{s\geq 0}$ the associated semigroup. By construction L_t is symmetric in $\mathbb{L}_2(\mu_t)$ and the following integration by parts formula holds for any differentiable function $\Psi\colon \mathbb{R} \to \mathbb{R}$, any $f, g\colon \mathbb{R}^n \to \mathbb{R}$, such that $\Psi(f), \nabla\Psi(f) \in \mathbb{L}_2(\mu_t)$ and g is in the domain of L_t.

$$\int \Psi(f) L_t g d\mu_t = -\int \Psi'(f) \nabla f \cdot \nabla g d\mu_t. \tag{2.2}$$

As we explain in Appendix A, formally we have

$$\partial_t P_t^{(t)} f = L_t f + \mathcal{V}_t f$$

We prove the following differential property.

Lemma 2.1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth bounded function not equal to zero a.e. and $(\Phi_t)_{t\geq 0}$ be a family of $\mathcal{C}^2$ Young functions. Let $N(t) := \|P_t^{(t)} f\|_{\Phi_t}$ and $g := \frac{P_t^{(t)} f}{N(t)}$, $t \geq 0$. Suppose $\nabla V_t \cdot \nabla \dot{V}_t - \Delta \dot{V}_t = -L_t \dot{V}_t$ is μ_t-integrable. Then, it holds*

$$\begin{aligned} N'(t) \int g \Phi_t'(g) d\mu_t = N(t) &\left(\int \dot{\Phi}_t(g) d\mu_t - \int \Phi_t''(g) |\nabla g|^2 d\mu_t - \int \Phi_t(g) \dot{V}_t d\mu_t \right) \\ &+ \int \int_0^t P_{t-s}^{(t)} f \nabla P_s^{(t)} (\Phi_t'(g)) \cdot \nabla \dot{V}_t ds d\mu_t \\ &- \int \left[\nabla V_t \cdot \nabla \dot{V}_t - \Delta \dot{V}_t \right] \int_0^t P_{t-s}^{(t)} f P_s^{(t)} (\Phi_t'(g)) ds d\mu_t \end{aligned}$$

In particular, when V_t does not depend on t (homogeneous case), the latter reduces to

$$N'(t) \int g \Phi_t'(g) d\mu = N(t) \left(\int \dot{\Phi}_t(g) d\mu - \int \phi_t''(g) |\nabla g|^2 d\mu \right).$$

Remark 2.2. We assumed $\mathcal{C}^2$ for Young functions for simplicity. Most of the results in this paper can easily be understood for any Young function using the notion of second order derivative in the sense of Aleksandrov.

Proof. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth bounded function not equal to zero a.e. From the definition of the Luxembourg norm, we observe that for any $t \geq 0$, $\int \Phi_t \left(\frac{P_t^{(t)} f}{N(t)} \right) d\mu_t = 1$. Therefore, taking the derivative, we get

$$\int \dot{\Phi}_t(g) d\mu_t + \int \Phi_t'(g) \frac{d}{dt} \left(\frac{P_t^{(t)} f}{N(t)} \right) d\mu_t - \int \Phi_t(g) \dot{V}_t d\mu_t = 0$$

where as already mentioned the dot stands for the derivative with respect to the variable t. Observe that,

$$\frac{d}{dt} \left(\frac{P_t^{(t)} f}{N(t)} \right) = \frac{\dot{P}_t^{(t)} f}{N(t)} + \frac{L_t P_t^{(t)} f}{N(t)} - \frac{P_t^{(t)} f N'(t)}{N(t)^2} = \frac{\dot{P}_t^{(t)} f}{N(t)} + L_t g - \frac{N'(t)}{N(t)} g$$

where we set

$$\dot{P}_t^{(t)} f := \lim_{\varepsilon \to 0} \frac{P_t^{(t+\varepsilon)} f - P_t^{(t)} f}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \frac{d}{ds} \left(P_s^{(t+\varepsilon)} (P_{t-s}^{(t)} f) \right) ds$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t P_s^{(t+\varepsilon)} \left([L_{t+\varepsilon} - L_t] P_{t-s}^{(t)} f \right) ds$$

$$= \int_0^t P_s^{(t)} \left(-\nabla \dot{V}_t \cdot \nabla P_{t-s}^{(t)} f \right) ds.$$

Therefore, using (2.2), we get

$$\int \Phi_t'(g) \frac{d}{dt} \left(\frac{P_t^{(t)} f}{N(t)} \right) d\mu_t = -\frac{1}{N(t)} \int \int_0^t \Phi_t'(g) P_s^{(t)} \left(\nabla \dot{V}_t \cdot \nabla P_{t-s}^{(t)} f \right) ds d\mu_t$$
$$- \int \Phi_t''(g) |\nabla g|^2 d\mu_t - \frac{N'(t)}{N(t)} \int g \Phi_t'(g) d\mu_t.$$

The previous computations lead to

$$\frac{N'(t)}{N(t)} \int g \Phi_t'(g) d\mu_t = \int \dot{\Phi}_t(g) d\mu_t - \int \Phi_t''(g) |\nabla g|^2 d\mu_t - \int \Phi_t(g) \dot{V}_t d\mu_t$$
$$- \frac{1}{N(t)} \int \int_0^t \Phi_t'(g) P_s^{(t)} \left(\nabla \dot{V}_t \cdot \nabla P_{t-s}^{(t)} f \right) ds d\mu_t$$

and we are left with the study of the last term on the right hand side of the latter. By reversibility of the semi-group, we have

$$\int \Phi_t'(g) P_s^{(t)} \left(\nabla \dot{V}_t \cdot \nabla P_{t-s}^{(t)} f \right) d\mu_t = \int P_s^{(t)} (\Phi_t'(g)) \nabla \dot{V}_t \cdot \nabla P_{t-s}^{(t)} f d\mu_t$$
$$= \int P_{t-s}^{(t)} f \nabla_t^* \left(P_s^{(t)} (\Phi_t'(g)) \nabla \dot{V}_t \right) d\mu_t$$

where ∇_t^* is the adjoint of ∇ in $\mathbb{L}_2(\mu_t)$, namely such that $\int u \nabla v d\mu_t = \int v \nabla_t^* u d\mu_t$. One can see that $\nabla_t^* = -\mathrm{div} + \nabla V_t$ where ∇V_t acts multiplicatively. Therefore,

$$\int \Phi_t'(g) P_s^{(t)} \left(\nabla \dot{V}_t \cdot \nabla P_{t-s}^{(t)} f \right) d\mu_t = - \int P_{t-s}^{(t)} f \nabla P_s^{(t)} (\Phi_t'(g)) \cdot \nabla \dot{V}_t d\mu_t$$
$$+ \int P_{t-s}^{(t)} f P_s^{(t)} (\Phi_t'(g)) \left[\nabla V_t \cdot \nabla \dot{V}_t - \Delta \dot{V}_t \right] d\mu_t$$

From this the desired result follows. $\square$

2.3. Expansion of the square

Here we may recall the *expansion of the square* method or, as it is called in [13], U-bounds. That is the bounds obtained by using Leibnitz rule together with integration by parts as follows. Let $U \colon \mathbb{R}^n \to \mathbb{R}$ be such that $\int e^{-U} dx < \infty$. Then, for any differentiable function $f \colon \mathbb{R}^n \to \mathbb{R}$, one has

$$\int f^2(|\nabla U|^2 - 2\Delta U)e^{-U} dx \leq 4 \int |\nabla f|^2 e^{-U} dx. \tag{2.3}$$

In fact, expanding the square, one has

$$0 \leq \int |\nabla(fe^{-U/2})|^2 dx = \int |\nabla f|^2 e^{-U} dx - \int f\nabla f \cdot \nabla U e^{-U} dx + \frac{1}{4}\int f^2 |\nabla U|^2 e^{-U} dx.$$

The expected inequality (2.3) then follows by applying an integration by parts on the cross term.

The *expansion of the square* method revealed to be very powerful. It can be used for instance to prove Hardy's inequality with optimal constant on $\mathbb{R}^d$, $d \geq 3$, or Poincaré inequality for the Gaussian measure. We refer the interested reader to [13,7] for more results and references.

3. Hypercontractivity for homogeneous Markov semi-groups

In this section our aim is to introduce the notion of *standard Orlicz family* that will play a key role for proving the equivalence between some functional inequality and a hypercontractivity property along the corresponding family of Orlicz spaces. We need first to analyze how to get a hypercontractivity property along a general family of Orlicz spaces.

All along the section we set $L = \Delta - \nabla V \cdot \nabla$, with V smooth enough and such that $\mu(dx) = e^{-V} dx$ is a probability measure on $\mathbb{R}^n$. We denote by $(P_t)_{t \geq 0}$ the associated semi-group which is reversible with respect to μ. Orlicz spaces and their corresponding Luxembourg norms are understood with respect to μ.

3.1. Hypercontractivity along Orlicz spaces

Using Lemma 2.1 we first prove that hypercontractivity is a direct and immediate consequence of some family of functional inequalities. Our second result shows how that family can, under some assumptions, be reduced to one single functional inequality of log-Sobolev-type.

Proposition 3.1. *Let $(\Phi_t)_{t \geq 0}$ be a family of $\mathcal{C}^2$ Young functions. Assume that for any $t \geq 0$, any sufficiently smooth function f, we have*

$$\|f\|_{\Phi_t}^2 \int \dot{\Phi}_t \left(\frac{f}{\|f\|_{\Phi_t}} \right) d\mu \leq \int \Phi_t'' \left(\frac{f}{\|f\|_{\Phi_t}} \right) |\nabla f|^2 d\mu. \tag{3.1}$$

Then, for any $t \geq s$,

$$\|P_t f\|_{\Phi_t} \leq \|P_s f\|_{\Phi_s}.$$

Proof. We need to prove is that $N : t \mapsto \|P_t f\|_{\Phi_t}$ is non-increasing. Lemma 2.1 asserts that

$$\frac{N'(t)}{N(t)} \int g \Phi_t'(g)\, d\mu = \int \dot{\Phi}_t(g)\, d\mu - \int \Phi_t''(g) |\nabla g|^2 d\mu$$

where $g := \frac{P_t f}{N(t)}$. Since for any $t \geq 0$, Φ_t is a Young function, it satisfies $x\Phi_t'(x) \geq 0$ for any $x \in \mathbb{R}$. It follows by (3.1) that $N'(t) \leq 0$ which is the expected result. $\square$

Using an isometry between $\mathbb{L}_{\Phi_t}$ and $\mathbb{L}_{\Phi_s}$, we may translate (3.1) for Φ_s into a similar inequality for Φ_t, therefore reducing the family of inequalities (3.1) possibly to a single one.

Proposition 3.2. *Let $(\Phi_t)_{t \geq 0}$ be a family of $\mathcal{C}^2$ Young functions. Assume that for some $t, s \geq 0$ there exist two positive constants $C(t,s)$ and $\widetilde{C}(t,s)$ such that*
(i)

$$\dot{\Phi}_t(\Phi_t^{-1}) \leq C(t,s) \dot{\Phi}_s(\Phi_s^{-1}),$$

(ii)

$$\frac{\Phi_t''}{{\Phi_t'}^2} \circ \Phi_t^{-1} \geq \widetilde{C}(t,s) \frac{\Phi_s''}{{\Phi_s'}^2} \circ \Phi_s^{-1}.$$

Assume furthermore that for some constant $c > 0$ and for any f (smooth enough), it holds

$$\|f\|_{\Phi_s}^2 \int \dot{\Phi}_s \left(\frac{f}{\|f\|_{\Phi_s}} \right) d\mu \leq c \int \Phi_s'' \left(\frac{f}{\|f\|_{\Phi_s}} \right) |\nabla f|^2 d\mu. \tag{3.2}$$

Then, it holds

$$\|f\|_{\Phi_t}^2 \int \dot{\Phi}_t \left(\frac{f}{\|f\|_{\Phi_t}} \right) d\mu \leq c \frac{C(t,s)}{\widetilde{C}(t,s)} \int \Phi_t'' \left(\frac{f}{\|f\|_{\Phi_t}} \right) |\nabla f|^2 d\mu.$$

for any f for which the right hand side is well defined.

Proof. Let

$$\begin{array}{rcl} I_{s,t} : \mathbb{L}_{\Phi_t} & \to & \mathbb{L}_{\Phi_s} \\ f & \mapsto & \|f\|_{\Phi_t} \Phi_s^{-1} \circ \Phi_t \left(\frac{f}{\|f\|_{\Phi_t}} \right). \end{array}$$

For any $f \in \mathbb{L}_{\Phi_t}$, by the very definition of the Luxembourg norm, it holds $\|I_{s,t}(f)\|_{\Phi_s} = \|f\|_{\Phi_t}$. Therefore, $I_{s,t}(f)$ is an isometry between the two Orlicz spaces $\mathbb{L}_{\Phi_t}$ and $\mathbb{L}_{\Phi_s}$. Applying (3.2) to $I_{s,t}(f)$ leads to

$$\|f\|_{\Phi_t}^2 \int \dot{\Phi}_s \left(\Phi_s^{-1} \circ \Phi_t \Big(\frac{f}{\|f\|_{\Phi_t}} \Big) \right) d\mu \leq$$
$$c \int \Phi_s'' \circ \Phi_s^{-1} \circ \Phi_t \left(\frac{f}{\|f\|_{\Phi_t}} \right) \frac{\Phi_t' \left(\frac{f}{\|f\|_{\Phi_t}} \right)^2 |\nabla f|^2}{\Phi_s' \circ \Phi_s^{-1} \circ \Phi_t \left(\frac{f}{\|f\|_{\Phi_t}} \right)^2} d\mu.$$

The result follows by (i) and (ii). $\square$

The simplest example is given by the $\mathbb{L}_p$ scale $\Phi_t(x) = |x|^{q(t)}$ for some function q that we assume to be increasing. Then, it holds

$$\dot{\Phi}_t(\Phi_t^{-1}) = \frac{q'(t)}{q(t)} x \log x \qquad \text{and} \qquad \frac{\Phi_t''}{{\Phi_t'}^2} \circ \Phi_t^{-1} = \frac{q(t)-1}{q(t)} \frac{1}{x}.$$

Therefore assumptions (i) and (ii) hold with $C(t,s) = \frac{q'(t)q(s)}{q(t)q'(s)}$ and $\widetilde{C}(t,s) = \frac{(q(t)-1)q(s)}{q(t)(q(s)-1)}$. In particular, the choice $q(t) = 1 + e^{(4/\rho)t}$, $\rho > 0$, guarantees that $C(t,s) = \widetilde{C}(t,s)$ for all s, t. Hence, the family of inequalities (3.2) are all equivalent to (3.2) with $s = 0$, which reads

$$\mathrm{Ent}_\mu(f^2) \leq \rho \int |\nabla f|^2 d\mu$$

since $\Phi_0 = |x|^2$ (and therefore $\dot{\Phi}_0(x) = (2/\rho) x^2 \log x^2$ and $\Phi_0''(x) = 2$). This is the log-Sobolev inequality and therefore Proposition 3.2 is just one direction in Gross' theorem [10].

In the above example, both $\dot{\Phi}_t(\Phi_t^{-1})$ and $\frac{\Phi_t''}{{\Phi_t'}^2} \circ \Phi_t^{-1}$ are of the form $a(t)b(x)$. Based on this simple observation, we may construct a generic family of Orlicz functions that, by construction, will automatically satisfies assumptions (i) and (ii) of the latter. This is the object of the next section.

3.2. The standard Orlicz family

We define a large class of family of N-functions that we will call the *standard Orlicz family*.

Definition 3.3 *(standard Orlicz family).* Let $F:(0,\infty)\to\mathbb{R}$ be a $\mathcal{C}^2$ increasing function with $F(1)=0$. Assume that $(0,\infty)\ni x\mapsto xF(x)$ is convex and that $1/xF(x)$ is not integrable at $x=0$, $x=1$ and $x=+\infty$. Let $\mathcal{F}_1:(0,1)\to\mathbb{R}$ and $\mathcal{F}_2:(1,+\infty)\to\mathbb{R}$ be two primitives of $x\mapsto 1/(xF(x))$.

Let Φ_0 be a nice Young function and x_o the unique positive point such that $\Phi_0(x_o)=1$. We assume that $-\left(\frac{\Phi_0}{\Phi_0'}\right)' F(\Phi_0)-\Phi_0F'(\Phi_0)$ is non-increasing on $\mathbb{R}^+$ and that Φ_0 is of class $\mathcal{C}^2$ on $(0,\infty)$.

Given an increasing function $\lambda\colon[0,\infty)\to[0,\infty)$, with $\lambda(0)=0$, we define the family of functions $(\Phi_t)_{t\geq 0}$ by

$$\Phi_t(x)=\begin{cases}0 & \text{for } x=0\\ \mathcal{F}_1^{-1}\left(\mathcal{F}_1(\Phi_0(x))+\lambda(t)\right) & \text{for } x\in(0,x_o)\\ 1 & \text{for } x=x_o\\ \mathcal{F}_2^{-1}\left(\mathcal{F}_2(\Phi_0(x))+\lambda(t)\right) & \text{for } x\in(x_o,+\infty).\end{cases}\qquad \forall t>0.$$

We shall call the family $(\Phi_t)_{t\geq 0}$ the *standard Orlicz family built from F, Φ_0 and λ.*

Remark 3.4. The Lemma below will prove that all Φ_t are indeed Young functions and in fact nice Young functions. This justifies the terminology "Orlicz family". Also, it is not difficult to check that the definition above does not depend on the choice of the primitives: any two different primitives lead to the same final function Φ_t.

Example 3.5. As an example consider $F(x)=\log(x)$ and any nice Young function Φ_0. Then, $\mathcal{F}_1(x)=\log(\log(1/x))$, $x\in(0,1)$ and $\mathcal{F}_2(x)=\log(\log(x))$, $x>1$ so that $\mathcal{F}_1^{-1}(x)=e^{-e^x}$ and $\mathcal{F}_2^{-1}(x)=e^{e^x}$, $x\in\mathbb{R}$. Hence, $\Phi_t(x)=\Phi_0^{e^{\lambda(t)}}$. This corresponds to an $\mathbb{L}_p$ scale when $\Phi_0(x)=|x|^q$ for some $q>1$. More specifically, if $q(t)=1+e^{(4/\rho)t}$ and $\lambda(t)=\log(q(t)/2)$, with $\Phi_0(x)=x^2$, we have $\Phi_t(x)=|x|^{q(t)}$ and we are back to Gross' setting.

The more general choices $F(x)=\log(1+x)^\beta-\log(2)^\beta$, $\beta\in(0,1)$, can also be considered, but lead to non explicit $\mathcal{F}_1$ and $\mathcal{F}_2$. Although one can easily give an asymptotic of the corresponding $\Phi_t(x)$, when x tends to 0 or $+\infty$. For instance, Φ_t is equivalent to $\Phi_0 e^{a_\beta\lambda(\log\phi_0)^\beta}$ when x tends to infinity, where a_β is a numerical constant that depends only on β. This amounts to the family of Young functions $x^2e^{ctF(x)}$ considered in [3, Section 7].

In the next lemma we collect some property of the standard Orlicz families.

Lemma 3.6. *Let F, Φ_0 and λ satisfying the assumptions of Definition 3.3 and let $(\Phi_t)_{t\geq 0}$ be the standard Orlicz family built from F, Φ_0 and λ. Then,*
(i) $\mathcal{F}_1$ and $\mathcal{F}_2$ are $\mathcal{C}^2$ functions respectively on $(0,1)$ and $(1,+\infty)$. $\mathcal{F}_1$ is decreasing with $\lim_{x\to 0}\mathcal{F}_1(x)=-\lim_{x\to 1}\mathcal{F}_1(x)=+\infty$. While $\mathcal{F}_2$ is increasing with $\lim_{x\to 1}\mathcal{F}_2(x)=$

$-\lim_{x\to+\infty} \mathcal{F}_2(x) = -\infty$. *In particular* Φ_t *is well defined and continuous. Moreover, for* $t \geq s$, $\Phi_t \leq \Phi_s$ *on* $(0, x_0)$ *and* $\Phi_t \geq \Phi_s$ *on* $(x_0, +\infty)$.
(ii) For any $t \geq 0$, Φ_t *is a nice Young function of class* $\mathcal{C}^2$ *on* $(0, \infty)$ *(with* $\Phi_t'(x_o) = \Phi_0'(x_o)$ *and* $\Phi_t''(x_o) = \Phi_0''(x_o)$*).*
(iii) For any $t \geq 0$, $\dot{\Phi}_t \circ \Phi_t^{-1} = \lambda'(t)xF(x)$.
(iv) For any $t \geq s \geq 0$. $\frac{\Phi_t''}{{\Phi_t'}^2} \circ \Phi_t^{-1} \geq \frac{\Phi_s''}{{\Phi_s'}^2} \circ \Phi_s^{-1}$.
(v) Assume that λ *tends to infinity at infinity. Then for any* $f \in \mathbb{L}_\infty$, $\lim_{t\to+\infty} \|f\|_{\Phi_t} = \frac{1}{x_0}\|f\|_\infty$.

Proof. Points (i) and (iii) are simple consequences of the definitions of the object involved.

It is not difficult but tedious to prove that for all $t \geq 0$, Φ_t is $\mathcal{C}^2$ (we omit the proof). Using that $\Phi_t \leq \Phi_0$ on $(0, x_o)$ and since Φ_0 is a nice Young function we deduce that $\lim_{x\to 0} \frac{\Phi_t(x)}{x} = 0$. Similarly, $\lim_{x\to\infty} \frac{\Phi_t(x)}{x} = +\infty$. In order to prove that Φ_t is a nice Young function it therefore remains to show that Φ_t is convex. For $x \neq x_0$, a simple differentiation gives

$$\begin{aligned} \frac{\Phi_t''}{{\Phi_t'}^2} &= \mathcal{F}'(\Phi_t)\left(-\frac{\mathcal{F}''(\Phi_t)}{\mathcal{F}'(\Phi_t)^2} + \frac{\mathcal{F}''(\Phi_0)}{\mathcal{F}'(\Phi_0)^2} + \frac{\Phi_0''}{\mathcal{F}'(\Phi_0){\Phi_0'}^2}\right) \\ &= \mathcal{F}'(\Phi_t)\left(\left(\frac{1}{\mathcal{F}'}\right)'(\Phi_t) - \left(\frac{1}{\mathcal{F}'}\right)'(\Phi_0) + \frac{\Phi_0''}{\mathcal{F}'(\Phi_0){\Phi_0'}^2}\right) \end{aligned} \tag{3.3}$$

where $\mathcal{F} = \mathcal{F}_1$ when $x \in (0, x_0)$ and $\mathcal{F} = \mathcal{F}_2$ when $x > x_0$. A Taylor expansion of $\left(\frac{1}{\mathcal{F}'}\right)'$ at the first order insures that

$$\frac{\Phi_t''}{{\Phi_t'}^2} = \mathcal{F}'(\Phi_t)\left((\Phi_t - \Phi_0)\left(\frac{1}{\mathcal{F}'}\right)''(\theta) + \frac{\Phi_0''}{\mathcal{F}'(\Phi_0){\Phi_0'}^2}\right)$$

for some $\theta \in (\Phi_t, \Phi_0)$ when $x \in (0, x_0)$ and $\theta \in (\Phi_0, \Phi_t)$ when $x > x_0$. Since $x \mapsto xF(x)$ is convex, $\frac{1}{\mathcal{F}'}$ is convex. It follows that $\left(\frac{1}{\mathcal{F}'}\right)''(\theta) \geq 0$ and thus that $(\Phi_t - \Phi_0)\left(\frac{1}{\mathcal{F}'}\right)''(\theta) + \frac{\Phi_0''}{\mathcal{F}'(\Phi_0){\Phi_0'}^2}$ has the same sign as $\mathcal{F}'(\Phi_t)$. This proves that Φ_t is convex.

Next, we deal with Point (iv). From (3.3) we have with the same notation as before

$$\frac{\Phi_t''}{{\Phi_t'}^2} \circ \Phi_t^{-1}(x) = -\frac{\mathcal{F}''(x)}{\mathcal{F}'(x)} + \mathcal{F}'(x)\left(\frac{\mathcal{F}''(\Phi_0)}{\mathcal{F}'(\Phi_0)^2} + \frac{\Phi_0''}{\mathcal{F}'(\Phi_0){\Phi_0'}^2}\right) \circ \Phi_t^{-1}(x).$$

Note that by hypothesis,

$$\frac{\mathcal{F}''(\Phi_0)}{\mathcal{F}'(\Phi_0)^2} + \frac{\Phi_0''}{\mathcal{F}'(\Phi_0){\Phi_0'}^2} = -\left(\frac{\Phi_0}{\Phi_0'}\right)' F(\Phi_0) - \Phi_0 F'(\Phi_0)$$

is non-increasing. Thus, by Point (ii) and using the sign of $\mathcal{F}'(x)$ on each domain $(0, x_0)$ and $(x_0, +\infty)$, we have

$$\frac{\Phi_t''}{\Phi_t'^2}\circ\Phi_t^{-1}(x) \geq -\frac{\mathcal{F}''(x)}{\mathcal{F}'(x)} + \mathcal{F}'(x)\left(\frac{\mathcal{F}''(\Phi_0)}{\mathcal{F}'(\Phi_0)^2} + \frac{\Phi_0''}{\mathcal{F}'(\Phi_0)\Phi_0'^2}\right)\circ\Phi_s^{-1}(x)$$
$$= \frac{\Phi_s''}{\Phi_s'^2}\circ\Phi_s^{-1}(x)$$

which is the expected result.

Finally we will prove Point (v). Let $f \in \mathbb{L}_\infty$. Then, $\int \Phi_t\Big(\frac{x_0|f|}{\|f\|_\infty}\Big)d\mu \leq \Phi_t(x_0) = 1$. Hence, by definition of the norm, $\|f\|_{\Phi_t} \leq \frac{1}{x_0}\|f\|_\infty$. In order to prove the bound from below, fix $\varepsilon > 0$ small enough. Then note that for any $x > x_0$, $\lim_{t\to+\infty}\Phi_t(x) = +\infty$. Thus

$$\int \Phi_t\left(\frac{|f|x_0}{\|f\|_\infty(1-\varepsilon)}\right)d\mu \geq \int\limits_{\{|f|\geq\|f\|_\infty(1-\frac{\varepsilon}{2})\}} \Phi_t\left(\frac{|f|x_0}{\|f\|_\infty(1-\varepsilon)}\right)d\mu$$
$$\geq \Phi_t\left(x_0\Big(1+\frac{\varepsilon}{2(1-\varepsilon)}\Big)\right)\mu\left(\{|f|\geq\|f\|_\infty(1-\frac{\varepsilon}{2})\}\right) \geq 1$$

provided t is large enough. It follows that $\frac{1}{x_0}\|f\|_\infty(1-\varepsilon) \leq \|f\|_{\Phi_t}$ for t large enough. This leads to the expected result and achieves the proof of the lemma. □

Remark 3.7. When $\lambda(t) = \alpha t$ for some $\alpha > 0$, the standard Orlicz family enjoy a shift type property. Indeed, in that case $\Phi_t = \mathcal{F}_i^{-1}(\mathcal{F}_i(\Phi_s) + \lambda(t-s))$, $i = 1, 2$. Therefore, the standard Orlicz families $(\Phi_t)_{t\geq 0}$ built from Φ_0, F and λ and $(\Psi_t)_{t\geq 0}$ built from Φ_s, F and λ, satisfy $\Psi_t = \Phi_{t+s}$ for any $t \geq 0$.

Remark 3.8. When $\Phi_0(x) = x^2$,

$$-\left(\frac{\Phi_0}{\Phi_0'}\right)' F(\Phi_0) - \Phi_0 F'(\Phi_0) = -\frac{1}{2}F(x^2) - x^2F'(x^2).$$

Thus, this function is non-increasing if and only if $\frac{3}{2}F'(x) + xF''(x) \geq 0$ if and only if $x \mapsto xF(x^2)$ is convex. Thus, in that case, one can only assume that $x \mapsto xF(x^2)$ is convex (which implies that $x \mapsto xF(x)$ is convex).

3.3. Gross-Orlicz' theorem

Thanks to the above definition of the standard Orlicz family, we can state one of our main results which generalizes Gross's theorem.

Theorem 3.9 *(Gross-Orlicz). Let $(\Phi_t)_{t\geq 0}$ be a standard Orlicz family built from F, Φ_0 and λ satisfying the hypotheses of Lemma 3.6. Let $c > 0$. Then the following are equivalent*
(i)

$$\|f\|^2_{\Phi_0} \int \Phi_0\Big(\frac{f}{\|f\|_{\Phi_0}}\Big) F\left(\Phi_0\Big(\frac{f}{\|f\|_{\Phi_0}}\Big)\right) d\mu \le c \int \Phi_0''\left(\frac{f}{\|f\|_{\Phi_0}}\right) |\nabla f|^2 d\mu; \tag{3.4}$$

for any function f for which the right hand side is well defined;
(ii) $\forall t \ge s \ge 0$, it holds

$$\|P_t f\|_{\Phi_t} \le \|P_s f\|_{\Phi_s},$$

for any function f for which the right hand side is well defined.

Moreover $(i) \Rightarrow (ii)$ with any (increasing) λ such that $\frac{\Phi_t''}{\Phi_t'^2} \circ \Phi_t^{-1} \ge c\lambda'(t) \frac{\Phi_0''}{\Phi_0'^2} \circ \Phi_0^{-1}$ for any $t \ge 0$ (in particular, any λ satisfying $\lambda'(t) \le 1/c$ would do); and $(ii) \Rightarrow (i)$ with $c = 1/\lambda'(0)$.

Proof. We first prove that (i) implies (ii). Item (iv) of Lemma 3.6 guarantees that $\frac{\Phi_t''}{\Phi_t'^2} \circ \Phi_t^{-1} \ge c\lambda'(t) \frac{\Phi_0''}{\Phi_0'^2} \circ \Phi_0^{-1}$ with $\lambda(t) = t/c$. Hence, the set of functions λ, increasing, satisfying $\frac{\Phi_t''}{\Phi_t'^2} \circ \Phi_t^{-1} \ge c\lambda'(t) \frac{\Phi_0''}{\Phi_0'^2} \circ \Phi_0^{-1}$ for any $t \ge 0$ is non empty and we may fix one of them.

Consider the standard Orlicz family $(\Phi_t)_{t\ge 0}$ built from F, Φ_0 and λ.

Note that by definition of the standard Orlicz family, $\dot\Phi_0 = \lambda'(0)\Phi_0 F(\Phi_0)$. Thus Inequality (3.4) reads as

$$\|f\|^2_{\Phi_0} \int \dot\Phi_0\left(\frac{f}{\|f\|_{\Phi_0}}\right) d\mu \le \lambda'(0) c \int \Phi_0''\left(\frac{f}{\|f\|_{\Phi_0}}\right) |\nabla f|^2 d\mu.$$

From the properties proved in Lemma 3.6 we can apply Proposition 3.2 with $C(t,0) = \frac{\lambda'(t)}{\lambda'(0)}$ and $\widetilde{C}(t,0) = c\lambda'(t)$. We get that, for any $t \ge 0$, any smooth function f satisfies

$$\begin{aligned}\|f\|^2_{\Phi_t} \int \dot\Phi_t\left(\frac{f}{\|f\|_{\Phi_t}}\right) d\mu &\le \lambda'(0) c \frac{C(t,0)}{\widetilde{C}(t,0)} \int \Phi_t''\left(\frac{f}{\|f\|_{\Phi_t}}\right) |\nabla f|^2 d\mu \\ &= \int \Phi_t''\left(\frac{f}{\|f\|_{\Phi_t}}\right) |\nabla f|^2 d\mu.\end{aligned}$$

The result of Point (ii) follows by Proposition 3.1.

Now we prove that (ii) implies (i). Let $N(t) = \|P_t f\|_{\Phi_t}$. By Lemma 2.1 at $t = 0$, we infer that

$$\begin{aligned}\frac{N'(0)}{N(0)} \int \frac{f}{\|f\|_{\Phi_0}} \Phi_0'\left(\frac{f}{\|f\|_{\Phi_0}}\right) d\mu &= \int \dot\Phi_0\left(\frac{f}{\|f\|_{\Phi_0}}\right) d\mu \\ &\quad - \int \Phi_0''\left(\frac{f}{\|f\|_{\Phi_0}}\right) |\nabla \frac{f}{\|f\|_{\Phi_0}}|^2 d\mu.\end{aligned}$$

The hypercontractivity property of point (ii) insures that $N'(0) \le 0$. Thus, since $x\Phi_0'(x) \ge 0$, we get that

$$\|f\|_{\Phi_0}^2 \int \dot{\Phi}_0\left(\frac{f}{\|f\|_{\Phi_0}}\right) d\mu \le \int \Phi_0''\left(\frac{f}{\|f\|_{\Phi_0}}\right) |\nabla f|^2 d\mu.$$

The result follows by the formula $\dot{\Phi}_0 = \lambda'(0)\Phi_0 F(\Phi_0)$ proved in Lemma 3.6. □

Remark 3.10. When $\Phi_0(x) = x^2$, Inequality (3.4) reads as

$$\int f^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu \le 2c \int |\nabla f|^2 d\mu.$$

This is the usual F-Sobolev inequality introduced by Rosen [20] (see also Wang [24]), and corresponds to the log-Sobolev inequality when $F(x) = \log x$.

When $F = \log$, one can consider $\lambda(t) = \log(1 + e^{(4/\rho)t}) - \log 2$, with $\rho = 2c$. Then, as already mentioned in Example 3.5, the standard Orlicz family built from $\Phi_0(x) = x^2$, F and λ, is $\Phi_t(x) = |x|^{q(t)}$ with $q(t) = 1 + e^{(4/\rho)t}$. In that case Theorem 3.9 is nothing but Gross' equivalence between the log-Sobolev inequality and the hypercontractivity in $\mathbb{L}_p$ scale recalled in the introduction.

Theorem 3.9 has to be compared to [3, Theorem 6]. When $F = \log$, [3, Theorem 6] asserts that $\|P_t f\|_{\widetilde{q}(t)} \le \|f\|_2$ with $\widetilde{q}(t) = 2e^{\rho t}$ which is off by a factor of 2 in the exponential (though capturing the exponential character of the $\mathbb{L}_p$ scale).

Furthermore, for $F(x) := \log(1+x)^\beta - \log(2)^\beta$, $\beta \in (0,1)$, [3, Theorem 6 and Corollary 34] does not give an hypercontractivity property, but only hyper-boundedness (see section 3.5 below for more on hyper-boundedness). One of the main differences comes from the fact that in [3] the authors deals with an explicit family of Young functions which imposes in some situation stronger assumptions. This happens for the second assumption of [3, Theorem 6] which reads in our setting as $\Phi_t(x)F(x^2) \le \ell(t)\Phi_t(F(\Phi_t(x))) + m$. We do not need such an assumption here.

To conclude with the comparison between the two theorems, we observe that the first assumption of [3, Theorem 6] is implied by $x \mapsto xF(x^2)$ convex, see Remark 3.8 above and [3, Proposition 7].

Notice that, for the following smooth version of $|x|^\alpha$, $\alpha \in (1,2)$,

$$u_\alpha(x) = \begin{cases} |x|^\alpha & \text{for } |x| > 1 \\ \frac{\alpha(\alpha-2)}{8}x^4 + \frac{\alpha(4-\alpha)}{4}x^2 + (1 - \frac{3}{4}\alpha + \frac{1}{8}\alpha^2) & \text{for } |x| \le 1 \end{cases}$$

it has been proved in [3, Proposition 33] that the probability measure $d\mu_\alpha^n(x) = \prod_{i=1}^n Z_\alpha^{-1} e^{-u_\alpha(x_i)} dx_i$ on $\mathbb{R}^n$ satisfies (3.4) with $F(x) = \log(1+x)^\beta - \log(2)^\beta$ with $\beta = 2(1 - \frac{1}{\alpha})$, $\Phi_0(x) = x^2$ and some constant $c = c(\alpha) > 0$ (that does not depend on n). This in turn leads to the hypercontractivity property for the standard Orlicz family built from F, Φ_0 and any λ satisfying $\lambda'(t) \le 1/c$.

3.4. Perturbation of Orlicz families and hypercontractivity

In this section we show how to translate the hypercontractivity property from one family of Young functions to another.

Proposition 3.11. *Let $(\Psi_t)_{t\geq 0}$ and $(\Phi_t)_{t\geq 0}$ be two families of Young functions and $(P_t)_{t\geq 0}$ be a linear semi-group acting on a set of functions $\mathcal{A}$ onto itself. Assume that for some $t \geq 0$,*
(i) any $f \in \mathcal{A}$ satisfies $\|P_t f\|_{\Psi_t} \leq \|f\|_{\Psi_0}$,
(ii) the function $\Psi_t^{-1} \circ \Phi_t$ is convex, satisfies $\Psi_t^{-1} \circ \Phi_t \leq \Psi_0^{-1} \circ \Phi_0$ and $\Psi_t^{-1} \circ \Phi_t(\mathcal{A}) \subset \mathcal{A}$,
(iii) for any function $f \in \mathcal{A}$, any convex function F, $F(P_t f) \leq P_t(F(f))$.
Then, any $f \in \mathcal{A}$ satisfies

$$\|P_t f\|_{\Phi_t} \leq \|f\|_{\Phi_0}.$$

Proof. By definition of the norm and Jensen's inequality given in (iii), together with (ii), we have

$$\begin{aligned} 1 &= \int \Phi_t \left(\frac{P_t f}{\|P_t f\|_{\Phi_t}} \right) d\mu = \int \Psi_t \circ \Psi_t^{-1} \circ \Phi_t \left(P_t \frac{f}{\|P_t f\|_{\Phi_t}} \right) d\mu \\ &\leq \int \Psi_t \left(P_t \Psi_t^{-1} \circ \Phi_t \left(\frac{f}{\|P_t f\|_{\Phi_t}} \right) \right) d\mu. \end{aligned}$$

This implies by definition of the norm and the hypercontractivity for the family Ψ_t (given in (i)) that

$$1 \leq \|P_t \Psi_t^{-1} \circ \Phi_t \Big(\frac{f}{\|P_t f\|_{\Phi_t}} \Big) \|_{\Psi_t} \leq \|\Psi_t^{-1} \circ \Phi_t \Big(\frac{f}{\|P_t f\|_{\Phi_t}} \Big) \|_{\Psi_0}.$$

It follows that $1 \leq \int \Psi_0 \circ \Psi_t^{-1} \circ \Phi_t \Big(\frac{f}{\|P_t f\|_{\Phi_t}} \Big) d\mu$. Hence by point (ii), $1 \leq \int \Phi_0 \Big(\frac{f}{\|P_t f\|_{\Phi_t}} \Big) d\mu$. In turn, $\|f\|_{\Phi_0} \geq \|P_t f\|_{\Phi_t}$. This ends the proof. □

Example 3.12. Assume that $\Phi_t = \Psi_t \circ F$ for a fixed Young function F. Then hypotheses (ii) and (iii) are automatically satisfied. For instance, it is known that the linear semi-group with diffusion generator $L = \Delta - \nabla U \nabla$ with $\mathrm{Hess}(U) \geq \rho > 0$ satisfies log-Sobolev inequality with constant $2/\rho$ and in turn is hypercontractive in the sense that

$$\|P_t f\|_{q(t)} \leq \|f\|_2, \qquad \text{with } q(t) := 1 + e^{(4/\rho)t}.$$

Now let $\Psi_t(x) = |x|^{q(t)}$. Hence, for any Young function F, the previous proposition asserts that

$$\|P_t f\|_{F^{q(t)}} \leq \|f\|_{F^2}.$$

Similarily from [3] we learn that the semi-group associated to $L = \Delta - \nabla U \nabla$ with $U(x) = |x|^\alpha$ (more precisely a smoothed version of $|x|^\alpha$), $1 \le \alpha \le 2$, is hypercontractive in the Orlicz' spaces family $\mathbb{L}_{\Phi_t}$ with $\Phi_t = x^2 e^{ct \log(1+|x|^2)^{2(1-\frac{1}{\alpha})}}$ for some constant c. It follows that for any Young function F, the semi-group $(P_t)_{t\ge 0}$ is also hypercontractive in the Orlicz' spaces family $\mathbb{L}_{\Psi_t}$ with $\Psi_t = F(x)^2 e^{ct \log(1+|F(x)|^2)^{2(1-\frac{1}{\alpha})}}$.

3.5. Hypercontractivity versus hyper-boundedness

In this section, we deal with perturbation arguments that allows one to get some hyper-boundedness property starting from hypercontractivity.

Theorem 3.13. *Let $(\Phi_t)_{t\ge 0}$ and $(\Psi_t)_{t\ge 0}$ be two standard Orlicz families built respectively from F and $\widetilde{F}$, Φ_0 and λ, both satisfying the hypotheses of Definition 3.3.*

Assume that for any $\varepsilon > 0$ there exists $D(\varepsilon) \ge 0$ such that all $x \ge 0$ satisfy

$$\widetilde{F}(x) \le \varepsilon F(x) + D(\varepsilon).$$

Suppose also that for any f and any $t \ge 0$, it holds

$$\|P_t f\|_{\Phi_t} \le \|f\|_{\Phi_0}.$$

Then, for any $s_2 \ge s_1 \ge 0$, any $t \ge 0$, any $\mathcal{C}^1$ increasing function $q : \mathbb{R}^+ \to \mathbb{R}^+$ with $q(0) = s_1$ and $q(t) = s_2$, it holds

$$\|P_t f\|_{\Psi_{s_2}} \le \|f\|_{\Psi_{s_1}} \exp\left\{ \int_0^t q'(u)\lambda'(q(u)) D\left(\frac{\lambda'(0)}{q'(u)\lambda'(q(u))}\right) du \right\} \qquad \forall f \in \mathbb{L}_{\Psi_{s_1}}.$$

Proof. Our aim is to use the hypercontractivity property in Orlicz spaces $\mathbb{L}_{\Phi_t}$ together with Theorem 3.9 to get a functional inequality involving the Young functions Φ_t, and then use the assumption on F and $\widetilde{F}$ to get a similar inequality for $\widetilde{F}$.

Fix $s_2 \ge s_1 \ge 0$, $t \ge 0$ and a $\mathcal{C}^1$ increasing function $q : \mathbb{R}^+ \to \mathbb{R}^+$ with $q(0) = s_1$ and $q(t) = s_2$. Fix $f \in \mathbb{L}_{\Psi_{s_1}}$ and let $N(u) := \|P_u f\|_{\Psi_{q(u)}}$ and $g := P_u f / N(u)$. Applying Lemma 2.1 to $\widetilde{\Psi}(t,x) := \Phi_{q(t)}(x)$, and observing that $\frac{\partial}{\partial t}\widetilde{\Psi}(t,x) := q'(t)\dot{\Psi}_{q(t)}(x)$, we get

$$\frac{N'(u)}{N(u)} \int g \Psi'_{q(u)}(g)\, d\mu = q'(u) \int \dot{\Psi}_{q(u)}(g)\, d\mu - \int \Psi''_{q(u)}(g)\, |\nabla g|^2\, d\mu.$$

Since $\Psi_{q(u)}$ is a nice Young function, $x\Psi'_{q(u)}(x) \ge \Psi_{q(u)}(x)$ for any $x \ge 0$, any u. Therefore $\int g \Psi'_{q(u)}(g)\, d\mu \ge 1$ and in turn, when $N'(u) \ge 0$, we have

$$\frac{N'(u)}{N(u)} \le q'(u) \int \dot{\Psi}_{q(u)}(g)\, d\mu - \int \Psi''_{q(u)}(g)\, |\nabla g|^2\, d\mu.$$

Now, thanks to Theorem 3.9, the hypercontractivity assumption guarantees that

$$\|f\|_{\Phi_0}^2 \int \Phi_0\left(\frac{f}{\|f\|_{\Phi_0}}\right) F\left(\Phi_0\left(\frac{f}{\|f\|_{\Phi_0}}\right)\right) d\mu \leq \frac{1}{\lambda'(0)} \int \Phi_0''\left(\frac{f}{\|f\|_{\Phi_0}}\right) |\nabla f|^2 d\mu.$$

By our assumption, item (*iii*) of Lemma 3.6 (recall that $\Psi_0 = \Phi_0$), we have

$$\dot{\Psi}_0 = \lambda'(0)\Phi_0 \widetilde{F}(\Phi_0) \leq \lambda'(0)\varepsilon\Phi_0 F(\Phi_0) + \lambda'(0)D(\varepsilon)\Phi_0.$$

Therefore, for any f with $\|f\|_{\Phi_0} = 1$,

$$\begin{aligned}\int \dot{\Psi}_0(f)\, d\mu &\leq \lambda'(0)\varepsilon \int \Phi_0(f)\, F(\Phi_0(f))\, d\mu + \lambda'(0)D(\varepsilon) \\ &\leq \varepsilon \int \Phi_0''(f)\, |\nabla f|^2\, d\mu + \lambda'(0)D(\varepsilon) = \varepsilon \int \Psi_0''(f)\, |\nabla f|^2\, d\mu + \lambda'(0)D(\varepsilon).\end{aligned}$$

Recall the isometry

$$\begin{array}{rcl} I_{s,t} : \mathbb{L}_{\Psi_t} & \to & \mathbb{L}_{\Psi_s} \\ f & \mapsto & \|f\|_{\Psi_t} \Psi_s^{-1} \circ \Psi_t\left(\frac{f}{\|f\|_{\Psi_t}}\right) \end{array}$$

from Proposition 3.2 that we may use with $s = 0$ and $t = q(u)$. The previous inequality applied to $I_{0,q(u)}(f)$ ensures that for any f with $\|f\|_{\Psi_{q(u)}} = \|I_{0,q(u)}(f)\|_{\Psi_0} = \|I_{0,q(u)}(f)\|_{\Phi_0} = 1$, it holds

$$\int \dot{\Psi}_0 \circ \Psi_0^{-1} \circ \Psi_{q(u)}(f) d\mu \leq \varepsilon \int \frac{\Psi_0''}{{\Psi_0'}^2} \circ \Psi_0^{-1} \circ \Psi_{q(u)}(f)\, |\nabla f|^2\, {\Psi_{q(u)}'}^2(f) d\mu + \lambda'(0)D(\varepsilon).$$

It follows from items (*iii*) and (*iv*) of Lemma 3.6 that

$$\frac{\lambda'(0)}{\lambda'(q(u))} \int \dot{\Psi}_{q(u)}(f) d\mu \leq \varepsilon \int \Psi_{q(u)}''(f)\, |\nabla f|^2\, d\mu + \lambda'(0)D(\varepsilon).$$

This leads to

$$\frac{N'(u)}{N(u)} \leq \frac{q'(u)\lambda'(q(u))}{\lambda'(0)} \left(\varepsilon \int \Psi_{q(u)}''(f)\, |\nabla f|^2\, d\mu + \lambda'(0)D(\varepsilon)\right) - \int \Psi_{q(u)}''(f)\, |\nabla f|^2\, d\mu$$

for any u such that $N'(u) \geq 0$. The latter being valid for any $\varepsilon > 0$, choose ε such that $q'(u)\lambda'(q(u))\varepsilon = \lambda'(0)$. Therefore, provided that $N'(u) \geq 0$ it holds

$$\frac{N'(u)}{N(u)} \leq q'(u)\lambda'(q(u)) D\left(\frac{\lambda'(0)}{q'(u)\lambda'(q(u))}\right).$$

This bound trivially holds when $N'(u) < 0$. Hence

$$\log \|P_t f\|_{\Psi_{s_2}} - \log \|P_0 f\|_{\Psi_{s_1}} = \int_0^t \frac{d}{du} \log \|P_u f\|_{\Psi_{q(u)}} du = \int_0^t \frac{N'(u)}{N(u)} du$$
$$\leq \int_0^t q'(u)\lambda'(q(u)) D\left(\frac{\lambda'(0)}{q'(u)\lambda'(q(u))}\right) du.$$

The result follows. $\square$

As an example of application, consider, for $\beta \in (0,1]$, $F_\beta(x) = (\log(1+x))^\beta - (\log 2)^\beta$. It is not difficult to check that for any $\varepsilon > 0$ and any $\beta' < \beta$, it holds

$$F_{\beta'}(x) \leq \varepsilon F_\beta(x) + D(\varepsilon)$$

with

$$D(\varepsilon) := -(\log 2)^{\beta'} + \varepsilon(\log 2)^\beta + \left(\frac{\beta'}{\beta}\right)^{\frac{\beta'}{\beta-\beta'}} \frac{\beta-\beta'}{\beta} \left(\frac{1}{\varepsilon}\right)^{\frac{\beta'}{\beta-\beta'}}.$$

Now let

$$u_\alpha(x) = \begin{cases} |x|^\alpha & \text{for } |x| > 1 \\ \frac{\alpha(\alpha-2)}{8}x^4 + \frac{\alpha(4-\alpha)}{4}x^2 + (1 - \frac{3}{4}\alpha + \frac{1}{8}\alpha^2) & \text{for } |x| \leq 1 \end{cases}$$

be a smooth version of $|x|^\alpha$, $\alpha \in (1,2)$. Define the probability measure $d\mu_\alpha^n(x) = \prod_{i=1}^n Z_\alpha^{-1} e^{-u_\alpha(x_i)} dx_i$ on $\mathbb{R}^n$. As already mentioned in Remark 3.10, it follows from [3, Proposition 33] that Inequality (3.4) holds for F_β, $\Phi_0(x) = x^2$ and some $c = c(\alpha) > 0$ and therefore that the semi-group $(P_t)_{t\geq 0}$ associated to μ_α^n is hypercontractive along the standard Orlicz family $(\Phi_t)_{t\geq 0}$ built from F_β, Φ_0 and any λ satisfying $\lambda'(t) \leq 1/c$. Fix for simplicity $\lambda(t) = t/c$.

Consider the standard Orlicz families $(\Psi_t)_{t\geq 0}$ built from $F_{\beta'}$, $\Phi_0(x) = x^2$ and λ.

The previous theorem shows that (for $s_1 = 0$ and $s_2 = s$)

$$\|P_t f\|_{\Psi_s} \leq m(t)\|f\|_2$$

where

$$m(t) = \inf_q \exp\left\{\frac{1}{c}\int_0^t q'(u) D\left(\frac{1}{q'(u)}\right) du\right\}$$

where the infimum is running over all increasing $q\colon [0,t] \to \mathbb{R}_+$ with $q(0) = 0$ and $q(t) = s$. We stress that the Luxembourg norm is computed here with reference measure μ_α^n.

One has

$$q'(u)D\left(\frac{1}{q'(u)}\right) = -q'(u)(\log 2)^{\beta'} + (\log 2)^{\beta} + C_{\beta,\beta'}(q'(u))^{\frac{\beta}{\beta-\beta'}}$$

where we set, for simplicity, $C_{\beta,\beta'} := \left(\frac{\beta'}{\beta}\right)^{\frac{\beta'}{\beta-\beta'}} \frac{\beta-\beta'}{\beta}$. Therefore,

$$\inf_q \left\{ \int_0^t q'(u)D\left(\frac{1}{q'(u)}\right) du \right\} = -s(\log 2)^{\beta'} + t(\log 2)^{\beta} + C_{\beta,\beta'} \inf_q \left\{ \int_0^t (q'(u))^{\frac{\beta}{\beta-\beta'}} du \right\}.$$

Since $\beta/(\beta-\beta') \geq 1$, by Holder's inequality (and equality cases in Holder's inequality), it is easy to see that

$$\inf_q \left\{ \int_0^t (q'(u))^{\frac{\beta}{\beta-\beta'}} du \right\} = s^{\frac{\beta}{\beta-\beta'}} t^{-\frac{\beta'}{\beta-\beta'}}.$$

As a conclusion

$$m(t) = \exp\left\{ \frac{1}{c} \left(-s(\log 2)^{\beta'} + t(\log 2)^{\beta} + C_{\beta,\beta'} s^{\frac{\beta}{\beta-\beta'}} t^{-\frac{\beta'}{\beta-\beta'}} \right) \right\}.$$

Note in particular that the factor in the exponential explodes for a fixed t, when s goes to infinity. This must be since the semi-group associated to μ_α^n can not be ultracontractive.

4. Contraction property for inhomogeneous Markov semi-groups

In this section we deal with the time-dependent diffusion operators $L_t := \Delta - \nabla V_t \cdot \nabla$, $t \geq 0$, on $\mathbb{R}^n$, with V_t smooth enough and such that $\int e^{-V_t} = 1$. Recall that the associated semi-group $(P_s^{(t)})_{s\geq 0}$ is reversible with respect to the probability measure $\mu_t(dx) := e^{-V_t(x)}dx$. All along this section Luxembourg norms are understood with respect to μ_t. We may omit such a dependence when not needed and write otherwise $\|\cdot\|_{\Phi_t,\mu_t}$.

In order to obtain contraction bounds for $P_t^{(t)} f$, one could try to use the following natural simple strategy. For t fixed, one may assume that $\mathrm{Hess}(V_t) \geq \rho_t$ for some $\rho_t > 0$ so that Gross' theorem applies and leads to

$$\|P_s^{(t)} f\|_{q_t(s),\mu_t} \leq \|f\|_{2,\mu_t} \qquad s,t \geq 0,$$

with, say, $q_t(s) \leq 1 + e^{(4/\rho_t)s}$ where we set $\|g\|_{q,\mu_t} := (\int |g|^q d\mu_t)^{\frac{1}{q}}$ for the $\mathbb{L}_q$ norm of g with respect to μ_t (we choose $q_t(0) = 2$ for simplicity). Applying the latter at time $s = t$ leads to

$$\|P_t^{(t)} f\|_{q_t(t),\mu_t} \leq \|f\|_{2,\mu_t}.$$

Now observe that the latter might be very weak if ρ_t is small. Moreover and more essentially one would like to deal with a norm on the right hand side independent of t (say related to μ_0). Before achieving this program, by means of Lemma 2.1, let us end this introduction with an easy (and very specific) example of inhomogeneous Markov semi-group whose hypercontractivity property can be derived from the results of the previous section.

Consider for instance $V_t = U_1$ for $t \in [0, T]$ and $V_t = U_2$ for $t > T$ where U_1 and U_2 are associated to hypercontractivity properties in Orlicz spaces $\mathbb{L}_{\Phi_t^{(1)}}$ and $\mathbb{L}_{\Phi_t^{(2)}}$ respectively. Then, we can argue that $\|P_t^{(t)} f\|_{\Phi_t^{(1)}} \leq \|P_s^{(s)} f\|_{\Phi_s^{(1)}}$, for any $s \leq t \leq T$, and then $\|P_t^{(t)} f\|_{\Phi_t^{(2)}} \leq \|P_s^{(s)} f\|_{\Phi_s^{(2)}}$ for any $T < s \leq t$. Therefore, if the two families of Orlicz spaces coincide at time T, *i.e.* $\Phi_T^{(1)} = \Phi_T^{(2)}$, and potentially modulo some extra assumptions on the Young functions $\Phi_t^{(1)}$, $\Phi_t^{(2)}$, if we set $\Phi_t = \Phi_t^{(1)}$ for $t \in [0, T]$ and $\Phi_t := \Phi_t^{(2)}$ for $t \geq T$, one has $\|P_t^{(t)} f\|_{\Phi_t} \leq \|P_s^{(s)} f\|_{\Phi_s}$ for all $s \leq t$. As already mentioned this is however very specific and somehow artificially inhomogeneous. We would like to deal with examples of potentials V_t that evolve all along the time t.

In the next section we will deal under the restricted hypothesis of the log-Sobolev inequality (4.1) related to the Orlicz family $\Phi_t(x) = |x|^{q(t)}$ ($\mathbb{L}_p$ scale). This makes the presentation more precise and easier by reducing some technicalities. However, it already encompasses many of the difficulties. The last section (that comes after) will finally deal with a more general setting.

We stress that the results below are a first step in the understanding of contraction properties for inhomogeneous Markov semi-groups. Many remain to be discovered and we believe that our investigations open new lines of research with possible application, as mentioned in the introduction, to non-linear parabolic time dependent problems (in infinite dimension).

4.1. $\mathbb{L}_p$-scales

In order to give the flavor of what is happening in the inhomogeneous setting (and avoid some technicalities), in this section we may only deal with contractivity properties in $\mathbb{L}_p$-scales. Recall that $X_- = max(-X, 0)$ stands for the negative part.

Theorem 4.1. *Consider the inhomogeneous diffusion operator L_t as above. Set $a_t := \|(\dot{V}_t)_-\|_\infty$, $b_t := \||\nabla \dot{V}_t|\|_\infty$, $c_t := \|(\nabla V_t \cdot \nabla \dot{V}_t - \Delta \dot{V}_t)_-\|_\infty$ and assume that $a_t, b_t, c_t < \infty$ for all $t \geq 0$. Assume also that, for all $t \geq 0$ there exists $\rho_t \in \mathbb{R}$ such that* $\mathrm{Hess}(V_t) \geq \rho_t$. *Finally, assume that there exists $\bar{\rho}_t > 0$ such that the following log-Sobolev inequality holds*

$$\int f^2 \log(f^2) d\mu_t \leq \bar{\rho}_t \int |\nabla f|^2 d\mu_t, \tag{4.1}$$

for all f with $\int f^2 d\mu_t = 1$ for which the right hand side is well defined. Then, for any $f\colon \mathbb{R}^n \to \mathbb{R}_+$ smooth enough, any $p > 1$ and any $s \le t$, it holds

$$\|P_t^{(t)} f\|_{\Phi_t,\mu_t} \le m(s,t)\|P_s^{(s)} f\|_{\Phi_s,\mu_s}$$

where $\Phi_t(x) = |x|^{q(t)}$, $q(t) = 1 + (p-1)\exp\{\int_0^t (2/\bar\rho_s) ds\}$, $t \ge 0$ and

$$m(s,t) := \exp\left\{ \int_s^t \frac{a_u}{q(u)} + uc_u + b_u^2 \frac{1-e^{-\rho_u u}}{2\rho_u}(q(u)-1)du \right\}.$$

Remark 4.2. As already mentioned, one possible criterion for the log-Sobolev inequality (4.1) to hold, is $\mathrm{Hess}(V_t) \ge \rho_t > 0$ (as a matrix), which implies $\bar\rho_t \le \frac{2}{\rho_t}$. Alternatively, as will be used below, one can apply a perturbation argument *à la* Holley & Stroock [12].

If V_t does not depend on t then $m(s,t) = 1$ and $q(t) = 1+(p-1)e^{(2/\bar\rho)t}$, which is Gross' theorem off by a factor of 2 in the exponential (see the introduction). This is coming from a technical computation that uses Cauchy-Schwarz' inequality. One can actually improve this and get $q(t) = 1 + (p-1)\exp\{(2-\varepsilon)\int_0^t (2/\bar\rho_s) ds\}$, for any $\varepsilon > 0$, but at the price of a factor $m(s,t)$ that depends on ε, and that increases when ε decreases.

Modulo such a factor 2, the above theorem can therefore be seen as an inhomogeneous counterpart of Gross' theorem.

Example 4.3. The above theorem contains some non trivial examples. For instance one can consider potentials of the form $V_t(x) = U(x) + \alpha(t)V(x) + \gamma(t)$ with V unbounded and $\gamma(t) := \log \int e^{-U-\alpha V} dx$ so that μ_t indeed defines a probability measure.

Take for instance $U(x) = \frac{|x|^2}{2}$, $\alpha\colon [0,\infty) \to [0,\infty)$ non-decreasing and $V(x) = (1 + |x|^2)^{\frac{\beta}{2}}$, with $\beta \in (0,1]$ (this is an unbounded (time-dependent) perturbation of the standard Gaussian potential U).

Then, $\dot V_t = \alpha'(t)(1+|x|^2)^{\frac{\beta}{2}}$ so that $a_t = 0$; $\nabla \dot V_t = \alpha'(t)\beta(1+|x|^2)^{\frac{\beta}{2}-1}x$, whence $b_t = \alpha'(t)\beta\sqrt{\frac{(1-\beta)^{1-\beta}}{(2-\beta)^{2-\beta}}}$ (which is understood as its limit when $\beta = 1$, namely $b_t = \alpha'(t)$ if $\beta = 1$). Using crude estimates, it is not difficult to prove that $c_t \le n^2\alpha'(t)(\alpha(t)+2)$. On the other hand (again we omit details) $\mathrm{Hess}(V_t) \ge 1$ so that $\rho_t = 1$ and (4.1) holds with $\bar\rho_t = 2$. Theorem 4.1 then implies that the corresponding inhomogeneous semi-group $(P_t^{(t)})_{t\ge0}$ is hyper-bounded in the $\mathbb{L}_{q(t)}$ scale, with $q(t) = 1 + (p-1)e^t$.

For $V(x) = \log(1+|x|^2)$ one can easily see that $a_t = 0$, $b_t = \alpha'(t)$ and $c_t < \infty$. The issue is coming from estimating $\bar\rho_t$. In fact $\mathrm{Hess}(V_t)(x)$ is bounded below by a positive matrix only outside a ball (of radius proportional to α). Therefore, one can write $V_t = H + R$, with H strictly convex, in the sense that $\mathrm{Hess}(H) \ge 1/2$, say, and R is bounded. Then Bakry-Émery criterion applies to the measure with density proportional to e^{-H}, leading to a log-Sobolev constant at most 4, and then we use Holley-Stroock perturbation Lemma, see *e.g.* [1, Theorem 3.4.3] to get Inequality (4.1) with constant $\bar\rho_t$ at most $4e^{\mathrm{Osc}(R)}$ (therefore potentially exponentially big in α) where

$\mathrm{Osc}(R) = \sup R - \inf R$ is the oscillation of R. Theorem 4.1 applies and finally leads to some contraction property with $q(t) \to \infty$ for α bounded or slowly growing to infinity (for instance $\alpha(t) = \log\log t$ would do).

Proof of Theorem 4.1. Let $\Phi_t(x) := |x|^{q(t)}$ with $q\colon [0,\infty) \to [0,\infty)$ increasing and satisfying $q(0) > 1$. Set $N(t) := \|P_t^{(t)} f\|_{\Phi_t,\mu_t}$ for some non-negative smooth f, and $g := \frac{P_t^{(t)} f}{N}$ so that $\int \Phi_t(g) d\mu_t = 1$. From Lemma 2.1, we have

$$N'(t) \int g\Phi_t'(g) d\mu_t \leq N(t) \left(\int \dot{\Phi}_t(g) d\mu_t - \int \Phi_t''(g) |\nabla g|^2 d\mu_t + a_t \right)$$

$$+ b_t \int \int_0^t P_{t-s}^{(t)} f |\nabla P_s^{(t)}(\Phi_t'(g))| ds d\mu_t + c_t \int \int_0^t P_{t-s}^{(t)} f P_s^{(t)}(\Phi_t'(g)) ds d\mu_t.$$

We observe that $x\Phi_t'(x) = q\Phi_t(x)$ so that $\int g\Phi_t'(g) d\mu_t = q$. Also, by reversibility, the last term of the latter satisfies

$$\int \int_0^t P_{t-s}^{(t)} f P_s^{(t)}(\Phi_t'(g)) ds d\mu_t = \int_0^t \int P_s^{(t)}(P_{t-s}^{(t)} f) \Phi_t'(g) d\mu_t ds$$

$$= tN(t) \int g\Phi_t'(g) d\mu_t = tN(t)q(t).$$

Since $\dot{\Phi}_t(x) = q'(t)|x|^q \log(|x|)$ and $\Phi_t''(x) = q(q-1)|x|^{q-2}$, we get

$$q(t)\frac{N'(t)}{N(t)} \leq \frac{q'(t)}{q(t)} \mathrm{Ent}_{\mu_t}(g^q) - q(q-1) \int g^{q-2} |\nabla g|^2 d\mu_t + a_t + tc_t q(t)$$

$$+ \frac{b_t}{N(t)} \int \int_0^t P_{t-s}^{(t)} f |\nabla P_s^{(t)}(\Phi_t'(g))| ds d\mu_t.$$

The condition $\mathrm{Hess}(V_t) \geq \rho_t$ ensures that $|\nabla P_s^{(t)} h| \leq e^{-\rho_t s} P_s^{(t)}(|\nabla h|)$ for all $s \geq 0$ and all h (see *e.g.* [1][Proposition 5.4.5]). Hence, by reversibility

$$\int \int_0^t P_{t-s}^{(t)} f |\nabla P_s^{(t)}(\Phi_t'(g))| ds d\mu_t \leq \int \int_0^t e^{-\rho_t s} P_{t-s}^{(t)} f P_s^{(t)}(|\nabla \Phi_t'(g)|) ds d\mu_t$$

$$= \int \int_0^t e^{-\rho_t s} P_t^{(t)} f \Phi_t''(g) |\nabla g| ds d\mu_t$$

$$= N(t) \frac{1 - e^{-\rho_t t}}{\rho_t} q(q-1) \int g^{q-1} |\nabla g| d\mu_t.$$

Using the inequality $uv \leq \frac{1}{2\varepsilon}u^2 + \frac{\varepsilon}{2}v^2$ with $\varepsilon = b_t \frac{1-e^{-\rho_t t}}{\rho_t}$, $u = g^{\frac{q}{2}-1}|\nabla g|$ and $v = g^{\frac{q}{2}}$ we get

$$\frac{1-e^{-\rho_t t}}{\rho_t} \int g^{q-1}|\nabla g| d\mu_t \leq \frac{1}{2b_t} \int g^{q-2}|\nabla g|^2 d\mu_t + \frac{1}{2} b_t \left(\frac{1-e^{-\rho_t t}}{\rho_t} \right)^2 \int g^q d\mu_t$$

so that, using $\int g^q d\mu_t = \int \Phi_t(g) d\mu_t = 1$,

$$\begin{aligned} q(t)\frac{N'(t)}{N(t)} \leq \frac{q'(t)}{q(t)} \mathrm{Ent}_{\mu_t}(g^q) - \frac{q(q-1)}{2} \int g^{q-2}|\nabla g|^2 d\mu_t + a_t + tc_t q(t) \\ + b_t^2 \frac{1-e^{-\rho_t t}}{2\rho_t} q(t)(q(t)-1). \end{aligned}$$

Next, we observe that $\int g^{q-2}|\nabla g|^2 d\mu_t = \frac{4}{q^2} \int |\nabla g^{\frac{q}{2}}|^2 d\mu_t$. Hence, for $q(t) := 1 + (p-1)\exp\{\int_0^t (2/\bar{\rho}_s) ds\}$ which satisfies $\frac{2(q-1)}{q'} = \bar{\rho}_t$, we are guaranteed by (4.1) that

$$\begin{aligned} &\frac{q'(t)}{q(t)} \mathrm{Ent}_{\mu_t}(g^q) - \frac{q(q-1)}{2} \int g^{q-2}|\nabla g|^2 d\mu_t \\ &\qquad = \frac{q'(t)}{q(t)} \left(\mathrm{Ent}_{\mu_t}(g^q) - \frac{2(q-1)}{q'(t)} \int |\nabla g^{\frac{q}{2}}|^2 d\mu_t \right) \\ &\qquad = \frac{q'(t)}{q(t)} \left(\mathrm{Ent}_{\mu_t}(g^q) - \bar{\rho}_t \int |\nabla g^{\frac{q}{2}}|^2 d\mu_t \right) \leq 0. \end{aligned}$$

It follows that

$$\frac{N'(t)}{N(t)} \leq \frac{a_t}{q(t)} + tc_t + b_t^2 \frac{1-e^{-\rho_t t}}{2\rho_t}(q(t)-1)$$

which leads to the desired conclusion. □

4.2. General result

In this section we establish a more general result than Theorem 4.1 that allows one to deal with more general Orlicz families, and not only the $\mathbb{L}_p$-scales. As a motivation, one can consider for instance as above $V_t(x) = U(x) + \alpha(t)V(x) + \gamma(t)$ with $U(x) \simeq \frac{|x|^\alpha}{\alpha}$ (for large $|x|$), $\gamma(t) := \log \int e^{-U-\alpha V} dx$ and $\Phi_t(x) = x^2 e^{ctF(x)}$, with $F(x) \simeq \log(x)^\beta$ (for large x). This corresponds, with a proper choice of V, to a generalization of the hypercontractivity property proved in [4] in the homogeneous setting (recall the introduction, see also Remark 3.10.

Theorem 4.4. *Consider the inhomogeneous diffusion operator L_t as above. Assume that for all $t \geq 0$, $b_t := \||\nabla \dot{V}_t|\|_\infty < \infty$ and that there exists $\rho_t \in \mathbb{R}$ such that* $\mathrm{Hess}(V_t) \geq \rho_t$ *(as a matrix).*

Let $(\Phi_t)_{t\geq 0}$ be a family of Young functions satisfying $\Phi_t(x) \leq x\Phi_t'(x) \leq B_t\Phi_t(x)$, ${\Phi_t'}^2 \leq C_t\Phi_t\Phi_t''$ and $x^2\Phi_t''(x) \leq D_t\Phi_t(x) + E_t$ for all $x \geq 0$ and some constants B_t, C_t, D_t, E_t.

Assume that for all $t \geq 0$ there exist $\delta_t \in [0,1)$ and $F_t \in \mathbb{R}$ such that $(\dot{V}_t)_- \leq \frac{\delta_t}{4C_t}\left(|\nabla V_t|^2 - 2\Delta V_t\right) + F_t$.

Set $W_t := (\nabla V_t \cdot \nabla \dot{V}_t - \Delta \dot{V}_t)_-$ and denote by $\bar{\rho}_t \in (0,\infty]$ the best constant such that for all f with $\|f\|_{\Phi_0} = 1$ it holds

$$\int \dot{\Phi}_t(f)d\mu_t \leq \bar{\rho}_t \int \Phi_t''(f)|\nabla f|^2 d\mu_t. \tag{4.2}$$

Finally, assume either that
(i) $c_t := \|W_t\|_\infty < \infty$ and $\bar{\rho}_t < 1 - \delta_t$;
or
(ii) $c_t' := \max\left(2|||\nabla W_t|||_\infty / b_t, \sup_{x:W_t(x)\neq 0}\left(\frac{L_tW_t}{W_t} - \rho_t\right)_-\right) < \infty$ and that for all $t \geq 0$ there exists $\delta_t' \in [0,1)$ and $F_t' \in [0,\infty)$ such that $\delta_t'\int_0^t e^{(c_s'-\rho_s)s}ds < 1$ and $W_t \leq \frac{\delta_t'}{4B_tC_t}\left(|\nabla V_t|^2 - 2\Delta V_t\right) + F_t'$.

Then, for any $f\colon \mathbb{R}^n \to \mathbb{R}_+$ smooth enough, it holds

$$\|P_t^{(t)}f\|_{\Phi_t,\mu_t} \leq m(s,t)\|P_s^{(s)}f\|_{\Phi_s,\mu_s}$$

where under assumption (i),

$$m(s,t) = \exp\left\{\int_s^t F_u + \left(b_u\frac{1-e^{-\rho_u u}}{\rho_u}\right)^2 \frac{D_u + E_u}{2(1-\delta_u - \bar{\rho}_u)} + c_uB_u u du\right\},$$

and under assumption (ii),

$$m(s,t) = \exp\left\{\int_s^t \left(\int_0^u e^{(c_v'-\rho_v)v}dv\right)^2 \frac{b_u(D_u+E_u)}{2(1-\delta_u-\bar{\rho}_u-\delta_u'\int_0^u e^{(c_v'-\rho_v)v}dv)} + \int_0^u e^{(c_v'-\rho_v)v}dv\,(B_uF_u' + F_u)\,du\right\}.$$

Remark 4.5. Observe that, when $\delta_t = 0$, the assumption $(\dot{V}_t)_- \leq \frac{\delta_t}{C_t}\left(|\nabla V_t|^2 - 2\Delta V_t\right) + D_t$ amounts to $a_t := \|(\dot{V}_t)_-\|_\infty < \infty$ which is the assumption that we used in Theorem 4.1. Also, it might be that $\dot{V}_t \geq 0$ so that, in that case, one chooses $\delta_t = D_t = 0$.

Observe also that the first inequality in the assumption $\Phi_t(x) \leq x\Phi_t'(x) \leq B_t\Phi_t(x)$ is satisfied by all Young functions, while the second inequality is a consequence of the Δ_2-condition.

Finally we observe that, although we weakened most of the hypotheses of Theorem 4.1, one key assumption one would like to remove/reduce is $b_t = \| |\nabla \dot{V}_t| \|_\infty < \infty$. Indeed one interesting example one would like to deal with is for instance $V_t(x) = ((1-t)_+)^2|x|^2 + |x|^\alpha$, with $\alpha \in (1,2)$, where we have a critical point $t = 1$ in which hypercontractivity property in $\mathbb{L}_p$ spaces is replaced by a weaker property. Such an example is not covered by Theorem 4.4 since $b_t = \infty$.

Proof of Theorem 4.4. We start as in the proof of Theorem 4.1. Set $N(t) := \|P_t^{(t)} f\|_{\Phi_t,\mu_t}$, $g := \frac{P_t^{(t)} f}{N}$ so that from Lemma 2.1 for some non-negative smooth function f, it holds

$$N'(t)\int g\Phi_t'(g)d\mu_t \le N(t)\left(\int \dot{\Phi}_t(g)d\mu_t - \int \phi_t''(g)|\nabla g|^2 d\mu_t - \int \Phi_t(g)\dot{V}_t d\mu_t\right)$$
$$+ \int\int_0^t P_{t-s}^{(t)} f \nabla P_s^{(t)}(\Phi_t'(g)) \cdot \nabla \dot{V}_t ds d\mu_t \tag{4.3}$$
$$- \int\int_0^t [\nabla V_t \cdot \nabla \dot{V}_t - \Delta \dot{V}_t] P_{t-s}^{(t)} f P_s^{(t)}(\Phi_t'(g)) ds d\mu_t.$$

We analyze each term separately.

Since $x\Phi_t'(x) \ge \Phi_t(x)$, it holds $\int g\Phi_t'(g)d\mu_t \ge \int \Phi_t(g)d\mu_t = 1$. Hence, if $N'(t) \ge 0$, the left hand side of the latter is bounded below by $N'(t)$.

Since $(\dot{V}_t)_- \le \frac{\delta_t}{4C_t}\left(|\nabla V_t|^2 - 2\Delta V_t\right) + F_t$, we can use the expansion of the square, namely Inequality (2.3) with $f = \sqrt{\Phi_t(g)}$, to get that

$$-\int \Phi_t(g)\dot{V}_t d\mu_t \le \int \Phi_t(g)(\dot{V}_t)_- d\mu_t \le \frac{\delta_t}{C_t}\int \frac{{\Phi_t'}^2(g)}{\Phi_t(g)}|\nabla g|^2 d\mu_t + F_t$$
$$\le \delta_t \int \Phi_t''(g)|\nabla g|^2 d\mu_t + F_t.$$

Now assume first that assumption (i) holds, namely that $c_t = \|(\nabla V_t \cdot \nabla \dot{V}_t - \Delta \dot{V}_t)_-\|_\infty < \infty$. In that case we can proceed as in the proof of Theorem 4.1 to get

$$\int\int_0^t P_{t-s}^{(t)} f \nabla P_s^{(t)}(\Phi_t'(g)) \cdot \nabla \dot{V}_t ds d\mu_t - \int\int_0^t [\nabla V_t \cdot \nabla \dot{V}_t - \Delta \dot{V}_t] P_{t-s}^{(t)} f P_s^{(t)}(\Phi_t'(g)) ds d\mu_t$$
$$\le b_t \int\int_0^t P_{t-s}^{(t)} f |\nabla P_s^{(t)}(\Phi_t'(g))| ds d\mu_t + c_t \int\int_0^t P_{t-s}^{(t)} f P_s^{(t)}(\Phi_t'(g)) ds d\mu_t$$
$$\le b_t N(t) \frac{1-e^{-\rho_t t}}{\rho_t}\int g\Phi_t''(g)|\nabla g| d\mu_t + c_t t N(t)\int g\Phi_t'(g)d\mu_t.$$

Using our assumption on Φ_t and $uv \leq \frac{1}{2\varepsilon}u^2 + \frac{\varepsilon}{2}v^2$, the latter is bounded above, for any $\varepsilon > 0$, by

$$b_t N(t)\frac{1-e^{-\rho_t t}}{\rho_t}\left(\frac{1}{2\varepsilon}\int g^2\Phi_t''(g)d\mu_t + \frac{\varepsilon}{2}\int \Phi_t''(g)|\nabla g|^2 d\mu_t\right) + c_t B_t t N(t)\int \Phi_t(g)d\mu_t$$
$$\leq b_t N(t)\frac{1-e^{-\rho_t t}}{\rho_t}\frac{1}{2\varepsilon}(D_t+E_t) + c_t B_t t N(t) + \varepsilon\frac{b_t N(t)(1-e^{-\rho_t t})}{2\rho_t}\int \Phi_t''(g)|\nabla g|^2 d\mu_t.$$

Choose ε so that $\varepsilon\frac{b_t(1-e^{-\rho_t t})}{2\rho_t} = [1-\delta-\bar{\rho}_t]/2$ so that, collecting the above computations together with inequality (4.2), we can conclude that for any t such that $N'(t) \geq 0$,

$$N'(t) \leq N(t)\left(F_t + \left(b_t\frac{1-e^{-\rho_t t}}{\rho_t}\right)^2 \frac{D_t+E_t}{2(1-\delta_t-\bar{\rho}_t)} + c_t B_t t\right)$$

from which the conclusion under assumption (i) follows.

Now we turn to assumption (ii). We need to bound the last two terms in (4.3). Using Proposition 4.6 with $W := W_t/b_t$ (observe that, since μ_t is a probability measure, $b_t \neq 0$), it holds

$$\int\int_0^t P_{t-s}^{(t)} f \nabla P_s^{(t)}(\Phi_t'(g))\cdot\nabla\dot{V}_t ds d\mu_t - \int\int_0^t [\nabla V_t\cdot\nabla\dot{V}_t - \Delta\dot{V}_t]P_{t-s}^{(t)} f P_s^{(t)}(\Phi_t'(g)) ds d\mu_t$$
$$\leq b_t\int\int_0^t P_{t-s}^{(t)} f\left(|\nabla P_s^{(t)}(\Phi_t'(g))| + \frac{W_t}{b_t}P_s^{(t)}(\Phi_t'(g))\right) ds d\mu_t$$
$$\leq b_t\int\int_0^t e^{(c_s-\rho_s)s} P_{t-s}^{(t)} f P_s^{(t)}\left(|\nabla\Phi'(g)| + \frac{W_t}{b_t}\Phi'(g)\right)$$
$$= N(t)\int_0^t e^{(c_s-\rho_s)s}ds\left(b_t\int g\Phi_t''(g)|\nabla g|d\mu_t + \int W_t g\Phi_t'(g)d\mu_t\right)$$

where we used the reversibility in the last inequality. For the first term in the right hand side of the latter, we proceed as for assumption (i). Namely, it holds for all $\varepsilon > 0$

$$\int g\Phi_t''(g)|\nabla g|d\mu_t \leq \frac{1}{2\varepsilon}\int g^2\Phi_t''(g)d\mu_t + \frac{\varepsilon}{2}\int \Phi_t''(g)|\nabla g|^2 d\mu_t$$
$$\leq \frac{1}{2\varepsilon}(D_t+E_t) + \frac{\varepsilon}{2}\int \Phi_t''(g)|\nabla g|^2 d\mu_t.$$

For the second term, we use the expansion of the square (inequality (2.3) with $f = \sqrt{\Phi_t(g)}$) to get that

$$\begin{aligned}\int W_t g \Phi_t'(g) d\mu_t &\le B_t \int W_t \Phi_t(g) d\mu_t \le \frac{\delta'}{4C_t} \int \Phi_t(g) \left(|\nabla V_t|^2 - 2\Delta V_t\right) d\mu_t + B_t F_t' \\ &\le \frac{\delta'}{4C_t} \int \frac{\Phi_t'(g)^2}{\Phi_t(g)} |\nabla g|^2 d\mu_t + B_t F_t' \le \delta' \int \Phi_t''(g)^2 \|\nabla g|^2 d\mu_t + B_t F_t'.\end{aligned}$$

Summarizing, under assumption (ii), when $N'(t) > 0$, we obtain

$$\begin{aligned}\frac{N'(t)}{N(t)} &\le \int \dot{\Phi}_t(g) d\mu_t - \left(1 - \delta_t - \frac{\varepsilon}{2} \int_0^t e^{(c_s' - \rho_s)s} ds + \delta_t' \int_0^t e^{(c_s' - \rho_s)s} ds\right) \int \phi_t''(g) |\nabla g|^2 d\mu_t \\ &\quad + \int_0^t e^{(c_s' - \rho_s)s} ds \left(\frac{b_t(D_t + E_t)}{2\varepsilon} + B_t F_t' + F_t\right) \\ &\le \int_0^t e^{(c_s' - \rho_s)s} ds \left(\frac{b_t(D_t + E_t)}{2\varepsilon} + B_t F_t' + F_t\right) \qquad \text{(thanks to (4.2))} \\ &= \left(\int_0^t e^{(c_s' - \rho_s)s} ds\right)^2 \frac{b_t(D_t + E_t)}{2(1 - \delta_t - \bar{\rho}_t - \delta_t' \int_0^t e^{(c_s' - \rho_s)s} ds)} \\ &\quad + \int_0^t e^{(c_s' - \rho_s)s} ds \, (B_t F_t' + F_t)\end{aligned}$$

for ε so that $\frac{\varepsilon}{2} \int_0^t e^{(c_s' - \rho_s)s} ds = \frac{1 - \delta_t - \bar{\rho}_t - \delta_t' \int_0^t e^{(c_s' - \rho_s)s} ds}{2}$. The desired conclusion follows. $\square$

In the proof of Theorem 4.4 we used the following results borrowed from [22].

Proposition 4.6. *Let $L = \Delta - \nabla U \cdot \nabla$, on $\mathbb{R}^n$, and denote by $(P_t)_{t \ge 0}$ its associated semi-group. Assume that $U : \mathbb{R}^n \to \mathbb{R}$ is smooth enough and satisfies $\int e^{-U} = 1$ so that $\mu(dx) = e^{-U(x)} dx$ is a probability measure on $\mathbb{R}^n$ and $\mathrm{Hess}(U) \ge \rho$ (as a matrix) for some $\rho \in \mathbb{R}$. Let $W : \mathbb{R} \to \mathbb{R}_+$ be such that $c := \max\left(2 \||\nabla W|\|_\infty, \sup_{x : W(x) \ne 0} \left(\frac{LW}{W} - \rho\right)_-\right) < \infty$. Then, for all f non negative*

$$|\nabla P_t f| + W P_t f \le e^{(c - \rho)t} P_t \left(|\nabla f| + W f\right).$$

Appendix A

Here we discuss the following formula

$$\partial_t P_t^{(t)} f = L_t f + \int_0^t \left(e^{\tau L_{(t)}} \dot{L}_t e^{(t - \tau) L_{(t)}} f\right) d\tau.$$

Note that

$$\partial_t P_t^{(t)} f = \lim_{s\to 0} \frac{1}{s}\left(P_{t+s}^{(t+s)} f - P_t^{(t)} f\right)$$
$$= \lim_{s\to 0} \frac{1}{s}\left(P_t^{(t+s)} f - P_t^{(t)} f\right) + \lim_{s\to 0} \frac{1}{s}\left(P_{t+s}^{(t+s)} f - P_t^{(t+s)} f\right)$$

For the first term on the right hand side we have

$$P_t^{(t+s)} f - P_t^{(t)} f = \int_0^t \left(e^{\tau L_{(t+s)}} \left(L_{(t+s)} - L_{(t)}\right) e^{(t-\tau)L_{(t)}} f\right) d\tau$$

provided $e^{(t-\tau)L_{(t)}} f$ is in the domain of $L_{(t+s)} - L_{(t)}$ for every sufficiently small s and all $\tau \in [0,t]$. Hence if the limit

$$\lim_{s\to 0} \frac{1}{s}\left(L_{(t+s)} - L_{(t)}\right) e^{(t-\tau)L_{(t)}} f \equiv \dot{L}_t e^{(t-\tau)L_{(t)}} f$$

is well defined, we have

$$\lim_{s\to 0} \frac{1}{s}\left(P_t^{(t+s)} f - P_t^{(t)} f\right) = \int_0^t \left(e^{\tau L_{(t)}} \dot{L}_t e^{(t-\tau)L_{(t)}} f\right) d\tau$$

On the other hand

$$P_{t+s}^{(t+s)} f - P_t^{(t+s)} f = L_{(t+s)} \int_0^s e^{(t+\tau)L_{(t+s)}} f d\tau$$

is well defined for C_0-semigroup and for f in the domain of $L_t$we have

$$\lim_{s\to 0} \frac{1}{s}\left(P_{t+s}^{(t+s)} f - P_t^{(t+s)} f\right) = L_t f.$$

Combining all the above yields

$$\partial_t P_t^{(t)} f = L_t f + \int_0^t \left(e^{\tau L_{(t)}} \dot{L}_t e^{(t-\tau)L_{(t)}} f\right) d\tau.$$

References

[1] C. Ané, S. Blachère, D. Chafai, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, G. Scheffer, Sur les inégalités de Sobolev logarithmiques, Panoramas et Synthèses, vol. 10, S.M.F, Paris, 2000.

[2] D. Bakry, L'hypercontractivité et son utilisation en théorie des semigroupes, in: Lectures on Probability Theory. École D'été de Probabilités de St-Flour 1992, in: Lecture Notes in Math., vol. 1581, Springer, Berlin, 1994, pp. 1–114.

[3] F. Barthe, P. Cattiaux, C. Roberto, Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry, Rev. Mat. Iberoam. 22 (3) (2006) 993–1067.
[4] F. Barthe, P. Cattiaux, C. Roberto, Isoperimetry between exponential and Gaussian, Electron. J. Probab. 12 (44) (2007) 1212–1237.
[5] D. Bakry, M. Emery, Diffusions hypercontractives, in: Séminaire de Probabilités, XIX, 1983/84, Springer, Berlin, 1985, pp. 177–206.
[6] D. Bakry, I. Gentil, M. Ledoux, Analysis and Geometry of Markov Diffusion Operators, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 348, Springer, Cham, 2014.
[7] J. Dolbeault, B. Volzone, Improved Poincaré inequalities, Nonlinear Anal. 75 (16) (2012) 5985–6001.
[8] P. Federbush, Partially alternate derivation of a result of Nelson, J. Math. Phys. 10 (1) (1969) 50–52.
[9] P. Fougères, C. Roberto, B. Zegarliński, Sub-Gaussian measures and associated semilinear problems, Rev. Mat. Iberoam. 28 (2) (2012) 305–350.
[10] L. Gross, Logarithmic Sobolev inequalities, Am. J. Math. 97 (1975) 1061–1083.
[11] A. Guionnet, B. Zegarliński, Lectures on logarithmic Sobolev inequalities, in: Séminaire de Probabilités, XXXVI, in: Lecture Notes in Math., vol. 1801, Springer, Berlin, 2003, pp. 1–134.
[12] R. Holley, D. Stroock, Logarithmic Sobolev inequalities and stochastic Ising models, J. Stat. Phys. 46 (5–6) (1987) 1159–1194.
[13] W. Hebisch, B. Zegarliński, Coercive inequalities on metric measure spaces, J. Funct. Anal. 258 (3) (2010) 814–851.
[14] M. Ledoux, The Concentration of Measure Phenomenon, Mathematical Surveys and Monographs, vol. 89, American Mathematical Society, Providence, RI, 2001.
[15] P. Massart, Concentration inequalities and model selection, in: Lectures from the 33rd Summer School on Probability Theory Held in Saint-Flour, July 6–23, 2003, in: Lecture Notes in Mathematics, vol. 1896, Springer, Berlin, 2007, pp. 6–23, With a foreword by Jean Picard.
[16] E. Nelson, A quartic interaction in two dimensions, in: Mathematical Theory of Elementary Particles, Proc. Conf., Dedham, Mass., 1965, M.I.T. Press, Cambridge, Mass, 1966, pp. 69–73.
[17] E. Nelson, Probability theory and euclidian field theory, in: Constructive Quantum Field Theory. The 1973 "Ettore Majorana" International School of Mathematical Physics, Erice (Sicily), 26 July–5, August 1973, in: Lecture Notes in Physics, vol. 25, Springer-Verlag, Berlin, 1973.
[18] E. Nelson, The free Markoff field, J. Funct. Anal. 12 (1973) 211–227.
[19] R. O'Donnell, Analysis of Boolean Functions, Cambridge University Press, New York, 2014.
[20] J. Rosen, Sobolev inequalities for weight spaces and supercontractivity, Trans. Am. Math. Soc. 222 (1976) 367–376.
[21] M.M. Rao, Z.D. Ren, Theory of Orlicz Spaces, Marcel Dekker Inc., 1991.
[22] C. Roberto, B. Zegarliński, Bakry-Emery calculus for diffusion with additional multiplicative term, preprint, 2021.
[23] A.J. Stam, Some inequalities satisfied by the quantities of information of Fisher and Shannon, Inf. Control 2 (1959) 101–112.
[24] F.-Y. Wang, Functional inequalities for empty essential spectrum, J. Funct. Anal. 170 (2000) 219.
[25] F.-Y. Wang, Criteria of spectral gap for Markov operators, J. Funct. Anal. 266 (2013).