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Orlicz–Sobolev inequalities for sub-Gaussian measures and ergodicity of Markov semi-groups ☆

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Abstract

We study coercive inequalities in Orlicz spaces associated to the probability measures on finite- and infinite-dimensional spaces which tails decay slower than the Gaussian ones. We provide necessary and sufficient criteria for such inequalities to hold and discuss relations between various classes of inequalities.

Keywords: Coercive inequalities; Infinite dimensions; Decay to equilibrium

1. Introduction

Sobolev type inequalities play an essential role in the study of the decay to equilibrium of Markov semi-groups to their associated probability measure. Several surveys deal with the celebrated Poincaré inequality and the stronger logarithmic Sobolev inequality, see e.g. [1,2,19,20,22,28]. It appears that the Poincaré inequality is particularly adapted to the study of the two-sided exponential measure while the logarithmic Sobolev inequality is the perfect tool to deal with the Gaussian measure. Both are now well understood.

In recent years intermediate measures, as for example

$$d\mu_\alpha(x) = (Z_\alpha)^{-1} e^{-|x|^\alpha}, \quad \alpha \in (1,2),$$

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attracted a lot of attention (note that for such measures the logarithmic Sobolev inequalities cannot hold). To deal with such measures, several authors generalized the Poincaré and logarithmic Sobolev inequalities in the following way.

Recall first that a probability measure μ, say on $\mathbb{R}^n$, is said to satisfy a Poincaré inequality if there exists a constant C such that every $f:\mathbb{R}^n \to \mathbb{R}$ smooth enough satisfies

$$\mathrm{Var}_\mu(f) \leqslant C \int |\nabla f|^2\, d\mu$$

and to satisfy a logarithmic Sobolev inequality if

$$\mathrm{Ent}_\mu\big(f^2\big) \leqslant C \int |\nabla f|^2\, d\mu,$$

where $\mathrm{Var}_\mu(f) = \mu(f^2) - \mu(f)^2$ is the variance (for short $\mu(f) = \int f\, d\mu$), and $\mathrm{Ent}_\mu(f) = \mu(f \log(f/\mu(f)))$ is the entropy of a positive function.

The latter can be rewritten in the form

$$\int f^2 \log\big(f^2\big)\, d\mu - \int f^2\, d\mu \log\biggl(\int f^2\, d\mu\biggr) \leqslant C \int |\nabla f|^2\, d\mu$$

or equivalently

$$\lim_{p\to 2^-} \frac{\int f^2\, d\mu - (\int |f|^p)^{2/p}}{2-p} \leqslant 2C \int |\nabla f|^2\, d\mu.$$

Hence two natural generalizations are the following additive Φ-Sobolev inequality

$$\int \Phi\big(f^2\big)\, d\mu - \Phi\biggl(\int f^2\, d\mu\biggr) \leqslant C \int |\nabla f|^2\, d\mu \tag{Φ-S}$$

and the Beckner-type inequality:

$$\sup_{p\in[1,2)} \frac{\int f^2\, d\mu - (\int |f|^p)^{2/p}}{T(2-p)} \leqslant 2C \int |\nabla f|^2\, d\mu. \tag{1}$$

Inequality (Φ-S) has been introduced in [8] as an intermediate tool to prove an isoperimetric inequality for the measure μ_α. It is also related to the work by Chafaï [11]. On the other hand, Beckner introduced in [9] inequality (1) with $T(r) = r$ in his study of the Gaussian measure μ_2. Latała and Oleszkiewicz [21] consider the more general $T_\alpha(r) = r^{2(1-1/\alpha)}$, $\alpha \in (1,2)$ and prove that μ_α satisfies inequality (1) with such T_α. Furthermore this inequality appears to be well adapted to the study of concentration of measure phenomenon via the celebrated Herbst argument. Further generalizations are done in this direction in [8], see also [30]. When $T = T_\alpha$, inequality (1) is known as the Latała and Oleszkiewicz inequality.

While the logarithmic Sobolev inequality enjoys a lot of properties and applications (tensorisation, concentration of measure, isoperimetry, decay to equilibrium, hypercontractivity), none of its generalizations appears to be well adapted simultaneously to all these properties and applications. This is the main reason why one has to generalize in different ways the logarithmic Sobolev inequality.

Motivated by this, in this paper we study the following new generalization we shall call the Orlicz–Sobolev inequality:

$$\left\| \left(f - \mu(f)\right)^2 \right\|_\Phi \leqslant C \int |\nabla f|^2 \, d\mu, \tag{O–S}$$

where the constant C is independent of the function f. Here, $\| \cdot \|_\Phi$ denotes the Luxembourg norm associated to the Orlicz function Φ and the probability measure μ on finite or infinite products of real lines $\mathbb{R}$.

If $\Phi(x) = |x|^p$, $p \in [1, \infty)$ then $\|(f - \mu(f))^2\|_\Phi = \|(f - \mu(f))^2\|_p$. Thus for $1 < p < \infty$, inequality (O–S) can be considered as a Sobolev type inequality. For $p = 1$ it is the Poincaré inequality. On the other hand, for $\Phi(x) = |x| \log(1 + |x|)$, it is proved in [10] that (O–S) is equivalent (up to universal constants) to the logarithmic Sobolev inequality. Thus, for an interpolation family of Orlicz functions going from $|x|$ to $|x| \log(1 + |x|)$ (as for instance $|x| \log(1 + |x|)^\beta$, $\beta \in [0, 1]$), (O–S) is an interpolating family of functional inequalities between Poincaré and the logarithmic Sobolev inequality.

Our first objective is to give in Section 2 a constructive criterium for a probability measure on a finite-dimensional Euclidean space to satisfy such an inequality. In particular we will prove (Corollary 6) that the sub-Gaussian probability measures μ_α (and product of it) satisfy the Orlicz–Sobolev inequality (O–S) with $\Phi(x) = |x| \log(1 + |x|)^{2(1-1/\alpha)}$.

Note that the Orlicz–Sobolev inequality (O–S) need not tensorise in general. Hence, in order to get dimension free results, we will use our criterium and ideas from [8] to prove the equivalence between the Orlicz–Sobolev inequality and the Beckner type inequality (1) that do tensorise.

Finally, using our results, we prove that under suitable mixing conditions the Latała–Oleszkiewicz inequalities are satisfied for Gibbs measures on infinite-dimensional spaces. This provides an extension of a result discussed in [20] to a comprehensive family of local specifications.

In Section 3 we discuss the implications of Orlicz–Sobolev inequalities for the decay to equilibrium in Orlicz norms for Markov semi-group with the generator given by the corresponding Dirichlet form. This includes in particular a necessary and sufficient condition for the exponential decay, which extends a well known classical property of the $\mathbb{L}_2$ space and Poincaré inequality. One of our main result states that the Orlicz–Sobolev inequality (O–S) implies, under mild assumptions on Φ, that

$$\|\mathbf{P}_t f\|_\Phi \leqslant e^{-ct} \|f\|_\Phi \tag{2}$$

for any f with $\mu(f) = 0$. Our technical development allows us to consider at the end of the section the case of decay to equilibrium for functionals which do not have convexity property of the norm as for example functionals of the form $\mu(|f|^q \log |f|^q / \mu(|f|^q))$ with $q > 1$. In case of relative entropy corresponding to $q = 1$ and a hypercontractive diffusion semi-group the exponential decay is well known. For $q > 1$ we show that after certain characteristic period of time one gets (essentially) exponential decay and by suitable averaging one can redefine the functional so it has the exponential decay property.

In Section 4 we discuss a relation between Orlicz–Sobolev and the additive Φ-Sobolev (Φ-S) inequalities. The additive Φ-Sobolev inequalities naturally tensorise. We show that it also has an analog of the mild perturbation property which allows to construct local specifications satisfying such the inequality. Moreover we prove that, if the local specification is mixing, similar arguments to those employed in the proof of logarithmic Sobolev inequalities work in the current situation. By this we get a constructive way to provide examples of nontrivial Gibbs measures on infinite-dimensional spaces satisfying the additive Φ-Sobolev inequalities. In a forthcoming paper [16] we will use them in the study of infinite-dimensional nonlinear Cauchy problems.

In order to show a decay to equilibrium in a stronger than $\mathbb{L}_2$ sense, in Section 5 we introduce and study certain natural generalization of Nash inequalities which follow from Orlicz–Sobolev inequalities. Such inequalities provide a bound on a covariance in terms of the Dirichlet form and suitable (weaker than $\mathbb{L}_2$) Orlicz norm. One illustration of our results is that the inequality (O–S) proved in Section 1 for μ_α and $\Phi(x) = |x|\log(1+x)^{2(1-1/\alpha)}$ implies that the associated semi-group $(\mathbf{P}_t)_{t\geqslant 0}$ is a continuous map from $\mathbb{L}_\Psi$ into $\mathbb{L}_2$ with $\Psi(x) = x^2/\log(1+|x|)^{2(1-1/\alpha)}$. Furthermore,

$$\|\mathbf{P}_t\|_{\mathbb{L}_\psi\to\mathbb{L}_2} \leqslant \frac{C_\alpha}{t^\gamma} \quad \forall t>0$$

for some positive constant C_α and γ. This result state that as soon as t is positive, the semi-group regularizes any initial data from $\mathbb{L}_\Psi$ into $\mathbb{L}_2$. (For general discussion about the interest and application of Nash-type inequalities, we refer the reader to e.g. [14,15,20,29].) Note that this bound is different from (2) where on both sides appear the same $\mathbb{L}_\Phi$ norm.

As a summary, all the multitude of the inequalities and relations between them discussed in this work is illustrated with the corresponding implication network diagram provided at the end of the paper. Since in our investigations we have used intensively numerous properties of Young functions and Orlicz/Luxemburg norms, for the convenience of the reader in Appendix A we gathered a plentitude of useful facts.

For other directions on the study of sub-Gaussian measures, the reader could like to see also [6,7,17,31].

2. A criterium for Orlicz–Sobolev inequalities

In this section we provide a criterium for inequality (O–S) to hold. This criterium allows us to prove that Orlicz–Sobolev inequalities are equivalent, up to universal constants, to Bekner-type inequalities. In turn, we give a family of Orlicz functions for which the Orlicz–Sobolev inequality holds for a corresponding sub-Gaussian measure. We end with an application to Gibbs measure on infinite state space.

In [8], the authors introduce a general tool to obtain a criterium which is based on an appropriate notion of capacity [23] initially introduced in [5]. More precisely, let μ and ν be two absolutely continuous measures on $\mathbb{R}^n$. Then, for any Borel set $A\subset\Omega$, we set

$$\mathrm{Cap}_\nu(A,\Omega) := \inf\left\{\int |\nabla f|^2\,d\nu;\ f\geqslant \mathbb{1}_A \text{ and } f|_{\Omega^c}=0\right\}.$$

If μ is a probability measure on $\mathbb{R}^n$, then, for $A\subset\mathbb{R}^n$ such that $\mu(A)<\frac{1}{2}$, the capacity of A with respect to μ and ν is

$$\begin{aligned}\mathrm{Cap}_\nu(A,\mu) &:= \inf\left\{\int |\nabla f|^2\,d\nu;\ \mathbb{1}\geqslant f\geqslant \mathbb{1}_A \text{ and } \mu(f=0)\geqslant \frac{1}{2}\right\}\\ &= \inf\left\{\mathrm{Cap}_\nu(A,\Omega);\ \Omega\subset\mathbb{R}^n \text{ s.t. } \Omega\supset A \text{ and } \mu(\Omega)=\frac{1}{2}\right\}.\end{aligned}$$

For simplicity we will write $\mathrm{Cap}_\mu(A)$ for $\mathrm{Cap}_\mu(A,\mu)$. (For a general introduction and discussion on the notion of capacity we refer the reader to [8, Section 5.2].) The second equality in the above definition comes from the fact that $\mathrm{Cap}_\nu(A,\Omega)$ is non-increasing in Ω and a suitable truncation argument (see [8]).

We start with the following criterium in dimension n and its more explicit form in dimension one.

Theorem 1. *Let μ and $\nu(dx)=\rho_\nu(x)\,dx$ be two absolutely continuous probability measures on $\mathbb{R}^n$. Consider a Young function Φ and fix $k\in(0,\infty)$ such that for any function f with $f^2\in\mathbb{L}_\Phi(\mu)$, one has $\|\mu(f)^2\|_\Phi\leqslant k\|f^2\|_\Phi$. Let C_Φ be the optimal constant such that for any smooth function $f:\mathbb{R}^n\to\mathbb{R}$ one has*

$$\big\|\big(f-\mu(f)\big)^2\big\|_\Phi\leqslant C_\Phi\int|\nabla f|^2\,d\nu.$$

Then $\frac{1}{8}B(\Phi)\leqslant C_\Phi\leqslant 8(1+k)B(\Phi)$ where $B(\Phi)$ is the smallest constant such that for every $A\subset\mathbb{R}^n$ with $\mu(A)<\frac{1}{2}$,

$$\|\mathbb{1}_A\|_\Phi\leqslant B(\Phi)\mathrm{Cap}_\nu(A,\mu).$$

Moreover if $n=1$, one has

$$\frac{1}{8}\max\big(B_+(\Phi),B_-(\Phi)\big)\leqslant C_\Phi\leqslant 8(1+k)\max\big(B_+(\Phi),B_-(\Phi)\big),$$

where

$$B_+(\Phi)=\sup_{x>m}\|\mathbb{1}_{[x,+\infty)}\|_\Phi\int_m^x\frac{1}{\rho_\nu},$$

$$B_-(\Phi)=\sup_{x<m}\|\mathbb{1}_{(-\infty,x]}\|_\Phi\int_x^m\frac{1}{\rho_\nu},$$

and m is a median of μ.

Remark 2. Note that by the property (15) in Appendix A, $\|\mathbb{1}_A\|_\Phi=1/\Phi^{-1}(1/\mu(A))$. In particular for $\mu(A)<\frac{1}{2}$ we have $\|\mathbb{1}_A\|_\Phi<1/\Phi^{-1}(2)$.

For explanation concerning the condition $\|\mu(f)^2\|_\Phi\leqslant k\|f^2\|_\Phi$ when $f^2\in\mathbb{L}_\Phi(\mu)$, see Lemma 44 and Remark 45 in Appendix A.

Proof. Fix a locally Lipschitz function $f:\mathbb{R}^n\to\mathbb{R}$ and let c be a median of f, i.e. $\mu(f\geqslant c)\geqslant\frac{1}{2}$ and $\mu(f\leqslant c)\geqslant\frac{1}{2}$. Then define $f_+=(f-c)\mathbb{1}_{f>c}$ and $f_-=(f-c)\mathbb{1}_{f<c}$. By assumption about Φ,

$$
\begin{aligned}
\left\|\left(f-\mu(f)\right)^2\right\|_\Phi &= \left\|\left(f-c+\mu(f-c)\right)^2\right\|_\Phi \\
&\leqslant 2\left\|(f-c)^2\right\|_\Phi + 2\left\|\mu(f-c)^2\right\|_\Phi \\
&\leqslant 2(1+k)\left\|(f-c)^2\right\|_\Phi \\
&\leqslant 2(1+k)\left(\left\|f_+^2\right\|_\Phi + \left\|f_-^2\right\|_\Phi\right)
\end{aligned}
$$

with $k \in (0,\infty)$ independent of f. It follows from [23, Theorem 2.3.2, p. 112] that

$$
\left\|f_+^2\right\|_\Phi \leqslant 4B\left(\Phi,\{f \leqslant c\}\right)\int |\nabla f_+|^2\,d\nu,
$$

where $B(\Phi,\{f \leqslant c\})$ is the smallest constant so that for every $A \subset \{f \leqslant c\}$,

$$
\|\mathbb{1}_A\|_\Phi \leqslant B\left(\Phi,\{f \leqslant c\}\right)\mathrm{Cap}_\nu\left(A,\{f \leqslant c\}\right).
$$

A similar result holds for f_-. Thus, by definition of $\mathrm{Cap}_\nu(A,\mu)$ and $B(\Phi)$, we get $B(\Phi, \{f \leqslant c\}) \leqslant B(\Phi)$ and in turn

$$
\begin{aligned}
\left\|\left(f-\mu(f)\right)^2\right\|_\Phi &\leqslant 8(1+k)B(\Phi)\left(\int |\nabla f_+|^2\,d\nu + \int |\nabla f_-|^2\,d\nu\right) \\
&\leqslant 8(1+k)B(\Phi)\int |\nabla f|^2\,d\nu.
\end{aligned}
$$

In the last inequality we used that, since f is locally Lipschitz and ν is absolutely continuous, the set $\{f=c\}\cap\{\nabla f \neq 0\}$ is ν-negligible. This proves the first part of the criterium.

For the other part, take a Borel set $A \subset \mathbb{R}^n$ with $\mu(A) < \frac{1}{2}$ and a function f such that $\mu(\{f=0\}) \geqslant \frac{1}{2}$ and $\mathbb{1}_{\{f\neq 0\}} \geqslant f \geqslant \mathbb{1}_A$. Set $\mathcal{G} = \{g:\mathbb{R}^n \to \mathbb{R};\ \int \Phi^*(g)\,d\mu \leqslant 1\}$ where Φ^* is the conjugate function of Φ. By (14) we have

$$
\begin{aligned}
2\left\|\left(f-\mu(f)\right)^2\right\|_\Phi &\geqslant \sup_{g\in\mathcal{G}}\int \left(f-\mu(f)\right)^2|g|\,d\mu \geqslant \sup_{g\in\mathcal{G}}\int_A \left(f-\mu(f)\right)^2|g|\,d\mu \\
&= \left(1-\mu(f)\right)^2\sup_{g\in\mathcal{G}}\int_A |g|\,d\mu \geqslant \left(1-\mu(f)\right)^2\|\mathbb{1}_A\|_\Phi.
\end{aligned}
$$

Since $f \leqslant \mathbb{1}$, we get $\mu(f) \leqslant \mu(f \neq 0) \leqslant \frac{1}{2}$. Thus $(1-\mu(f))^2 \geqslant \frac{1}{4}$. It follows that

$$
\frac{1}{8}\|\mathbb{1}_A\|_\Phi \leqslant \left\|\left(f-\mu(f)\right)^2\right\|_\Phi \leqslant C_\Phi\int |\nabla f|^2\,d\nu.
$$

The result follows by definition of the capacity. This ends the proof in any finite dimension.

Consider now $n=1$. Let m be a median of μ and define $f_+ = (f - f(m))\mathbb{1}_{(m,+\infty)}$ and $f_- = (f-f(m))\mathbb{1}_{(-\infty,m)}$. Note that $(f_+ + f_-)^2 = f_+^2 + f_-^2$. By our assumption and a similar computation as in the general case,

$$
\begin{aligned}
\|(f-\mu(f))^2\|_\Phi &= \|(f-f(m)-\mu(f-f(m)))^2\|_\Phi \\
&\leqslant 2(1+k)\|(f-f(m))^2\|_\Phi = 2(1+k)\|(f_+ + f_-)^2\|_\Phi \\
&\leqslant 2(1+k)(\|f_+^2\|_\Phi + \|f_-^2\|_\Phi).
\end{aligned}
$$

From [5, Proposition 2] (which originally comes from [10], see also [12]), it follows that

$$
\|f_+^2\|_\Phi \leqslant 4B_+(\Phi)\int_m^\infty f_+'^2\,d\nu.
$$

Since a similar bound holds for f_-, summing up we get that $C_\Phi \leqslant 8(1+k)\max(B_+(\Phi), B_-(\Phi))$.

Next, fix $x>m$ and consider the following function defined on the real line

$$
h(y) = \begin{cases} 0 & \text{for } y \leqslant m, \\ \int_m^y \frac{1}{\rho_\nu} & \text{for } m \leqslant y \leqslant x, \\ \int_m^x \frac{1}{\rho_\nu} & \text{for } y \geqslant x. \end{cases}
$$

Starting as previously, we get that

$$
\begin{aligned}
2\|(h-\mu(h))^2\|_\Phi &\geqslant \sup_{g\in\mathcal{G}} \int_{[x,\infty)} (h-\mu(h))^2|g|\,d\mu \\
&\geqslant \left(\int_m^x \frac{1}{\rho_\nu} - \mu(h)\right)^2 \|\mathbb{1}_{[x,\infty)}\|_\Phi.
\end{aligned}
$$

Then, since $x>m$ and $h \leqslant \int_m^x \frac{1}{\rho_\nu}$,

$$
\mu(h) \leqslant \mu((m,\infty))\int_m^x \frac{1}{\rho_\nu} \leqslant \frac{1}{2}\int_m^x \frac{1}{\rho_\nu}.
$$

Therefore, $\int_m^x \frac{1}{\rho_\nu} - \mu(h) \geqslant \frac{1}{2}\int_m^x \frac{1}{\rho_\nu}$. Applying the Orlicz–Sobolev inequality to this special function h, we get

$$
\frac{1}{4}\left(\int_m^x \frac{1}{\rho_\nu}\right)^2 \|\mathbb{1}_{[x,\infty)}\|_\Phi \leqslant 2\|(h-\mu(h))^2\|_\Phi \leqslant 2C_\Phi \int h'^2\,d\nu = 2C_\Phi \int_m^x \frac{1}{\rho_\nu}.
$$

This gives for any $x>m$,

$$
\|\mathbb{1}_{[x,\infty)}\|_\Phi \int_m^x \frac{1}{\rho_\nu} \leqslant 8C_\Phi.
$$

The same bound holds for $x<m$ and the result follows by definition of $B_+(\Phi)$ and $B_-(\Phi)$. $\square$

The explicit criterium in dimension 1 leads to the following result.

Proposition 3. *Let Φ be an Young function and fix $k \in (0, +\infty)$ such that $\|\mu(f)^2\|_\Phi \leqslant k\|f^2\|_\Phi$, for any function f with $f^2 \in \mathbb{L}_\Phi(\mu)$. Let $V : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that $d\mu(x) = e^{-V(x)}\,dx$ is a probability measure. Furthermore assume that*:

(i) *there exists a constant $A > 0$ such that for $|x| \geqslant A$, V is $\mathcal{C}^2$ and* $\operatorname{sign}(x)V'(x) > 0$,
(ii) $\lim_{|x|\to\infty} \frac{V''(x)}{V'(x)^2} = 0$,
(iii) $\liminf_{|x|\to\infty} V'(x)e^{-V(x)}\Phi^{-1}(V'(x)e^{V(x)}) > 0$.

Then there exists a constant C_Φ (that may depend on k) such that for every smooth function $f : \mathbb{R} \to \mathbb{R}$, one has

$$\left\|\left(f - \mu(f)\right)^2\right\|_\Phi \leqslant C_\Phi \int f'^2\,d\mu.$$

Proof. The proof is similar to [1, Chapter 6, Theorem 6.4.3]. Let m be a median of μ. Under assumptions (i) and (ii), when x tends to infinity, one has (see e.g. [1, Chapter 6])

$$\int_m^x e^{V(t)}\,dt \sim \frac{e^{V(x)}}{V'(x)} \quad \text{and} \quad \int_x^\infty e^{-V(t)}\,dt \sim \frac{e^{-V(x)}}{V'(x)}.$$

Thus, for $x > m$,

$$\begin{aligned}\|\mathbb{1}_{[x,\infty)}\|_\Phi \int_m^x e^{V(t)}\,dt &= \frac{1}{\Phi^{-1}(1/\mu([x,\infty)))}\int_m^x e^{V(t)}\,dt \\ &\sim \frac{1}{V'(x)e^{-V(x)}\Phi^{-1}(V'(x)e^{V(x)})}.\end{aligned}$$

By hypothesis (iii) this quantity is bounded on $[A', \infty)$ for some $A' \geqslant m$. Since it is continuous on $[m, A']$, it is bounded on (m, ∞). It follows that $B_+(\Phi)$ and $B_-(\Phi)$, (defined in Theorem 1), are bounded. We conclude with Theorem 1. □

In general the capacity can be difficult to compute. However it provides a nice interfacing tool to prove equivalences between inequalities. Indeed, a criterium involving capacity also holds for general Beckner-type inequalities as we will see now. The two general criterium will allows us to prove an equivalence between the Orlicz–Sobolev inequality and the Beckner-type inequalities. Our main motivation here is that the latter naturally tensorises. Thus, dimension free Orlicz–Sobolev inequalities will follow from Beckner-type inequality.

Combining Theorem 9 and Lemma 8 of [8] we get the following:

Theorem 4. [8] *Let $T:[0,1]\to\mathbb{R}^+$ be non-decreasing and such that $x\mapsto T(x)/x$ is non-increasing. Let μ and ν be two absolutely continuous measures on $\mathbb{R}^n$ with $\mu(\mathbb{R}^n)=1$. Let C_T be the optimal constant such that for every smooth $f:\mathbb{R}^n\to\mathbb{R}$ one has*

$$\sup_{p\in(1,2)}\frac{\int f^2\,d\mu-(\int|f|^p\,d\mu)^{2/p}}{T(2-p)}\leqslant C_T\int|\nabla f|^2\,d\nu. \tag{3}$$

Then, $\frac{1}{6}B(T)\leqslant C_T\leqslant 20B(T)$, where $B(T)$ is the smallest constant so that every Borel set $A\subset\mathbb{R}^n$ with $\mu(A)<\frac{1}{2}$ satisfies

$$\frac{\mu(A)}{T\big(1/\log(1+\frac{1}{\mu(A)})\big)}\leqslant B(T)\mathrm{Cap}_\nu(A,\mu).$$

Now, using the previous two theorems, one can see that the Orlicz–Sobolev inequality (O–S) is equivalent, up to universal constant, to the general Beckner-type inequality (3).

Corollary 5. *Let μ and ν be two absolutely continuous measures on $\mathbb{R}^n$ with $\mu(\mathbb{R}^n)=1$. Let $T:[0,1]\to\mathbb{R}^+$ be non-decreasing and such that $x\mapsto T(x)/x$ is non-increasing. Denote by C_T the optimal constant such that for every smooth $f:\mathbb{R}^n\to\mathbb{R}$ one has*

$$\sup_{p\in(1,2)}\frac{\int f^2\,d\mu-(\int|f|^p\,d\mu)^{2/p}}{T(2-p)}\leqslant C_T\int|\nabla f|^2\,d\nu.$$

Let Φ be a Young function and let $k\in(0,+\infty)$ be such that for any function f with $f^2\in\mathbb{L}_\Phi(\mu)$, $\|\mu(f)^2\|_\Phi\leqslant k\|f^2\|_\Phi$. Let C_Φ the optimal constant such that for every smooth $f:\mathbb{R}^n\to\mathbb{R}$ one has

$$\big\|\big(f-\mu(f)\big)^2\big\|_\Phi\leqslant C_\Phi\int|\nabla f|^2\,d\nu.$$

Finally, assume that there exist two positive constants c_1 and c_2 such that

$$c_1xT\left(\frac{1}{\log(1+x)}\right)\leqslant\Phi^{-1}(x)\leqslant c_2xT\left(\frac{1}{\log(1+x)}\right)\quad\forall x>2.$$

Then,

$$\frac{c_1}{48(1+k)}C_\Phi\leqslant C_T\leqslant 160c_2C_\Phi.$$

Proof. The last assumption on T and Φ^{-1} is equivalent to

$$\frac{1}{c_2}\frac{y}{T\big(1/\log(1+\frac{1}{y})\big)}\leqslant\frac{1}{\Phi^{-1}(1/y)}\leqslant\frac{1}{c_1}\frac{y}{T\big(1/\log(1+\frac{1}{y})\big)}\qquad\forall y\in\left(0,\frac{1}{2}\right).$$

Since $\|\mathbb{1}_A\|_\Phi=\frac{1}{\Phi^{-1}(1/\mu(A))}$, it follows that $\frac{1}{c_2}B(T)\leqslant B(\Phi)\leqslant\frac{1}{c_1}B(T)$, where $B(\Phi)$ and $B(T)$ are defined in Theorems 1 and 4, respectively. The result follows from Theorems 1 and 4. $\square$

Example. $T_\beta(x) = |x|^\beta$. An important example is given by $T_\beta(x) = |x|^\beta$ with $\beta \in [0,1]$. This corresponds to the Latała and Oleszkiewicz inequality (in short L–O inequality) [21].

Consider the Young function $\Phi_\beta(x) = |x|[\log(1+|x|)]^\beta$ with $\beta \in [0,1]$. Then, we claim that

$$\frac{y}{[\log(1+y)]^\beta} \leqslant \Phi_\beta^{-1}(y) \leqslant 2\frac{y}{[\log(1+y)]^\beta} \quad \forall y > 2. \tag{4}$$

Indeed,

$$\Phi_\beta\left(\frac{x}{[\log(1+x)]^\beta}\right) = x\frac{[\log(1+x(\log(1+x))^{-\beta})]^\beta}{[\log(1+x)]^\beta} \quad \forall x \geqslant 0.$$

Note that for $x \geqslant e-1$, $1 + x(\log(1+x))^{-\beta} \leqslant 1+x$. This leads to

$$\Phi_\beta\left(\frac{x}{[\log(1+x)]^\beta}\right) \leqslant x \quad \text{for } x > 2.$$

The first inequality in (4) follows.

On the other hand, it is not difficult to check that for any $\gamma \in [0,1]$, any $x \geqslant e-1$,

$$1 + x\big(\log(1+x)\big)^{-\beta} \geqslant \frac{1+x}{[\log(1+x)]^\beta} \geqslant \left(\frac{e(1-\gamma)}{\beta}\right)^\beta (1+x)^\gamma.$$

It follows for $\gamma = 1 - (\beta/e)$ that $\Phi_\beta(\frac{x}{[\log(1+x)]^\beta}) \geqslant \gamma^\beta x \geqslant \frac{e-1}{e}x$. Thus, for any $y \geqslant (e-1)^2/e \simeq 1.09$,

$$\Phi_\beta^{-1}(y) \leqslant \frac{e}{e-1}\frac{y}{\log(1+\frac{ey}{e-1})^\beta} \leqslant \frac{e}{e-1}\frac{y}{[\log(1+y)]^\beta}.$$

The result follows.

We are in position to prove a family of Orlicz–Sobolev inequalities.

Corollary 6. *Let $\alpha \in [1,2]$, $\beta = 2(1-\frac{1}{\alpha}) \in [0,1]$ and $\Phi_\beta(x) = |x|[\log(1+|x|)]^\beta$. Then, for any integer n, the probability measure on $\mathbb{R}^n$, $d\mu_\alpha^n(x) = Z_\alpha^{-n}\exp\{-\sum_{i=1}^n |x_i|^\alpha\}\,dx$ satisfies for any smooth function $f:\mathbb{R}^n \to \mathbb{R}$,*

$$\big\|\big(f - \mu_\alpha^n(f)\big)^2\big\|_{\Phi_\beta} \leqslant C\int |\nabla f|^2\, d\mu_\alpha^n \tag{5}$$

for some universal constant C independent of n and α.

Proof. Fix an integer n, $\alpha \in [1,2]$, the corresponding β and let $T_\beta(x) = |x|^\beta$. It is proved in [21] that the measure $d\mu_\alpha^n$ on $\mathbb{R}^n$ satisfies the general Beckner type inequality (3) with $T = T_\beta$ and constant $C(T_\beta)$ independent of n and α (for the uniformity in α, see [8, Section 7]). Then the result follows by our previous claim (inequality (4)) on Φ_β^{-1} and Corollary 5 (it is easy to check that $\|\mu_\alpha(f)^2\|_{\Phi_\beta} \leqslant e\|f^2\|_{\Phi_\beta}$ from Remark 45). $\square$

Remark 7. The family of inequalities in Corollary 6 is an interpolation family between Poincaré, for $\Phi(x) = |x|$, and the logarithmic Sobolev inequality, for $\Phi(x) = |x| \log(1 + |x|)$ (see [10]).

Remark 8. To prove that inequality (5) holds in dimension 1, we could have used Proposition 3 together with (4). Moreover, given $\beta \in [0, 1]$, Proposition 3 insures that (5) holds for any $\alpha \geqslant \alpha(\beta)$ where $\beta = 2(1 - \frac{1}{\alpha(\beta)})$ and does not hold for $\alpha < \alpha(\beta)$.

2.1. L–O inequality for Gibbs measures

The following result provides a precise asymptotic of the coefficient in L–O inequality as well as plays a vital role in a construction of examples non-product measures satisfying this inequality.

Theorem 9.

(i) *Let* $p \in [1, 2]$. *Then,*

$$\|f\|_2^2 - \|f\|_p^2 \leqslant (p-1)\big(\big\|f - \mu(f)\big\|_2^2 - \big\|f - \mu(f)\big\|_p^2\big) + (2-p)\big\|f - \mu(f)\big\|_2^2.$$

Hence, if with some $C \in (0, \infty)$ *and* $\beta \in (0, 1)$

$$\big\|f - \mu(f)\big\|_2^2 - \big\|f - \mu(f)\big\|_p^2 \leqslant C(2-p)^\beta \|\nabla f\|_2^2,$$

and for some $M \in (0, \infty)$

$$M\big\|f - \mu(f)\big\|_2^2 \leqslant \|\nabla f\|_2^2,$$

then

$$\|f\|_2^2 - \|f\|_p^2 \leqslant \big((p-1)C(2-p)^\beta + (2-p)M\big)\|\nabla f\|_2^2.$$

(ii) (Mild Perturbation Lemma) *Suppose* ν *satisfies the following L–O inequality*:

$$\|f\|_{\mathbb{L}_2(\nu)}^2 - \|f\|_{\mathbb{L}_p(\nu)}^2 \leqslant C(2-p)^\beta \|\nabla f\|_{\mathbb{L}_2(\nu)}^2$$

and let $d\mu = \rho\, d\nu$ *with* $\delta U \equiv \sup(\log \rho) - \inf(\log \rho) < \infty$. *Then*

$$\|f\|_{\mathbb{L}_2(\mu)}^2 - \|f\|_{\mathbb{L}_p(\mu)}^2 \leqslant e^{\delta U} C(2-p)^\beta \|\nabla f\|_{\mathbb{L}_2(\mu)}^2.$$

Proof. (See [30].) (i) The first inequality follows from the following convexity property of the $\mathbb{L}_p(\mu)$ norm for $p \in [1, 2]$:

$$\|f\|_{\mathbb{L}_p(\mu)}^2 \geqslant \mu(f)^2 + (p-1)\big\|f - \mu(f)\big\|_{\mathbb{L}_p(\mu)}^2,$$

see e.g. [4] (see also [30], [5, Lemma 8]). This together with spectral gap inequality and L–O inequality for $f - \mu(f)$ imply the L–O inequality for f with the improved coefficient.

(ii) We note first that for $p \in (1,2)$, with $A \equiv \frac{2-p}{2}(\frac{p}{2})^{\frac{p}{2-p}}$, we have

$$\|f\|_{\mathbb{L}_2(\mu)}^2 - \|f\|_{\mathbb{L}_p(\mu)}^2 = \inf_{t>0} \mu\big(f^2 - t|f|^p + At^{\frac{2}{2-p}}\big).$$

Since by Young inequality

$$z^p t = \left[\left(\frac{2}{p}\right)^{\frac{p}{2}} z^p\right] \cdot \left[\left(\frac{2}{p}\right)^{-\frac{p}{2}} t\right] \leqslant z^2 + \frac{2-p}{2}\left(\frac{2}{p}\right)^{-\frac{p}{2-p}} t^{\frac{2}{2-p}} = z^2 + At^{\frac{2}{2-p}}$$

the integrand in the above is nonnegative. Hence, if $d\mu = \rho\, d\nu$, we get

$$\begin{aligned}\inf_{t>0} \mu\big(f^2 - t|f|^p + At^{\frac{p}{2-p}}\big) &\leqslant \sup(\rho) \inf_{t>0} \nu\big(f^2 - t|f|^p + At^{\frac{p}{2-p}}\big)\\ &\leqslant \sup(\rho) C(2-p)^\beta \nu\big(|\nabla f|^2\big)\\ &\leqslant \frac{\sup(\rho)}{\inf(\rho)} C(2-p)^\beta \mu\big(|\nabla f|^2\big). \qquad \square\end{aligned}$$

Starting from the product measure satisfying L–O inequality, using the Mild Perturbation Lemma we see that one can construct a local specification for which each finite volume conditional expectation E_Λ (defined as a mild perturbation of the product measure), satisfies this inequality. This together with the suitable conditioning expansion based on the following step:

$$\begin{aligned}\boldsymbol{\mu}(f^2) - \big(\boldsymbol{\mu}(f^p)\big)^{2/p} &= \boldsymbol{\mu}\big(E_\Lambda(f^2) - \big(E_\Lambda(f^p)\big)^{2/p}\big)\\ &\quad + \boldsymbol{\mu}\big(\big[E_\Lambda(f^p)^{1/p}\big]^2\big) - \boldsymbol{\mu}\big(\big[E_\Lambda(f^p)^{1/p}\big]^p\big)^{2/p}\end{aligned}$$

under suitable mixing condition (the same as the one used in the case of log-Sobolev inequality), allows to prove the following result (see [20] for details).

Theorem 10. *Suppose a local specification is mixing and satisfies* L–O *inequality with the index* $\beta \in (0,1)$. *Then the corresponding Gibbs measure* $\boldsymbol{\mu}$ *satisfies*

$$\boldsymbol{\mu}\big(f^2\big) - \boldsymbol{\mu}\big(f^p\big)^{2/p} \leqslant C(2-p)^\beta \boldsymbol{\mu}\big(|\nabla f|^2\big)$$

with a constant $C \in (0,\infty)$ *independent of a function* f.

3. O–S inequality and decay to equilibrium

In this section we prove that the semi-group naturally associated to a measure μ satisfying an Orlicz–Sobolev inequality decays exponentially fast in $\mathbb{L}_\Phi(\mu)$. This result is new and strengthens a well-know fact for the Poincaré inequality and decay in $\mathbb{L}_2(\mu)$. We start with a modified Orlicz–Sobolev inequality. As before, throughout below we consider the following setup. Let $d\mu(x) = e^{V(x)}\,dx$ be a probability measure on $\mathbb{R}^n$ associated to the differentiable potential V. Let $\mathbf{L} = \Delta - \nabla V \cdot \nabla$ be a symmetric in $\mathbb{L}_2(\mu)$ diffusion generator and $(\mathbf{P}_t)_{t\geqslant 0}$ its associated semi-group.

Theorem 11. *Consider a Young function Φ satisfying $x\Phi'(x) \leqslant B\Phi(x)$ for every x and some constant B. Then, the following are equivalent.*

(i) *There exists a constant C_Φ such that for any smooth function $f:\mathbb{R}^n \to \mathbb{R}$,*

$$\|f-\mu(f)\|_\Phi^2 \leqslant C_\Phi \int |\nabla f|^2 \Phi''\left(\frac{f-\mu(f)}{\|f-\mu(f)\|_\Phi}\right) d\mu. \tag{6}$$

(ii) *There exists a constant $M \in (0,\infty)$ such that for any smooth function $f:\mathbb{R}^n \to \mathbb{R}$, for any $t \geqslant 0$,*

$$\|\mathbf{P}_t f - \mu(f)\|_\Phi^2 \leqslant e^{-Mt}\|f-\mu(f)\|_\Phi^2.$$

Furthermore, (i) *implies* (ii) *with $M = 2/(BC_\Phi)$, and* (ii) *implies* (i) *with $C_\Phi = 2/M$.*

Remark 12. Note that if Φ satisfies the Δ_2-condition $\Phi(2x) \leqslant C\Phi(x)$ for every x, then

$$x\Phi'(x) \leqslant \int_x^{2x} \Phi'(t)\,dt = \Phi(2x) - \Phi(x) \leqslant (C-1)\Phi(x).$$

Thus the condition on the Young function Φ is satisfied as soon as the Δ_2-condition is satisfied.

Proof. Without loss of generality for a smooth non-zero function f, we can assume that $\mu(f) = 0$. Let $N(t) = \|\mathbf{P}_t f\|_\Phi$.

By definition of the Luxembourg norm, we have $\int \Phi(\frac{\mathbf{P}_t f}{N(t)})\,d\mu = 1$. A differentiation and the chain rule formula $\int \Phi'(g) Lg\,d\mu = -\int \Phi''(g)|\nabla g|^2\,d\mu$ give

$$\begin{aligned}\frac{N'(t)}{N(t)} \int \Phi'\left(\frac{\mathbf{P}_t f}{N(t)}\right)\frac{\mathbf{P}_t f}{N(t)}\,d\mu &= \int \frac{\mathbf{L}\mathbf{P}_t f}{N(t)} \Phi'\left(\frac{\mathbf{P}_t f}{N(t)}\right) d\mu \\ &= -\frac{1}{N^2(t)} \int \Phi''\left(\frac{\mathbf{P}_t f}{N(t)}\right)|\nabla \mathbf{P}_t f|^2\,d\mu.\end{aligned}$$

We will first show that (i) $\Rightarrow$ (ii). Since Φ is a Young function, it is convex and for any x, $x\Phi'(x) \geqslant 0$. It follows at first that $N'(t) \leqslant 0$. Furthermore, by hypothesis $x\Phi'(x) \leqslant B\Phi(x)$. Thus, using the property that $\int \Phi(\frac{\mathbf{P}_t f}{N(t)})\,d\mu = 1$, we get by (i) that

$$B\frac{N'(t)}{N(t)} \leqslant -\frac{1}{N^2(t)} \int \Phi''\left(\frac{\mathbf{P}_t f}{N(t)}\right)|\nabla \mathbf{P}_t f|^2\,d\mu \leqslant -\frac{1}{N^2(t)}\frac{1}{C_\Phi}N^2(t) = -\frac{1}{C_\Phi},$$

which gives the expected result.

Now we show that (ii) $\Rightarrow$ (i). Let $u(t) = e^{Mt}\|\mathbf{P}_t f - \mu(f)\|_\Phi^2$. Point (ii) exactly means that $u'(t) \leqslant 0$. Hence $Me^{Mt}N^2(t) + 2e^{Mt}N'(t)N(t) \leqslant 0$ which leads to

$$MN^2(t) \leqslant -2N'(t)N(t) = 2\frac{\int \Phi''\big(\frac{\mathbf{P}_t f}{N(t)}\big)|\nabla \mathbf{P}_t f|^2\,d\mu}{\int \Phi'\big(\frac{\mathbf{P}_t f}{N(t)}\big)\frac{\mathbf{P}_t f}{N(t)}\,d\mu}$$
$$\leqslant 2\int \Phi''\left(\frac{\mathbf{P}_t f}{N(t)}\right)|\nabla \mathbf{P}_t f|^2\,d\mu.$$

In the last inequality we used the fact that, since Φ is convex and $\Phi(0)=0$, for every x, $x\Phi'(x) \geqslant \Phi(x)$ and $\int \Phi(\frac{\mathbf{P}_t f}{N(t)})\,d\mu = 1$. The latter inequality applied at $t=0$ gives the expected result. This ends the proof. □

Remark 13. When $\Phi(x) = x^2$, $\|f\|_\Phi^2 = \|f\|_2^2$ and $\Phi''(x) = 2$. Thus Theorem 11 recover the well-known equivalence between the exponential decay of the semi-group in $\mathbb{L}_2$-norm and the Poincaré inequality.

The behavior of Φ'' seems to play an important role. In particular, under additional strict positivity assumption we prove the following result involving the Orlicz–Sobolev inequalities.

Corollary 14. *Consider a Young function Φ and set $\Phi_2(x) = \Phi(x^2)$. Assume that $x\Phi_2'(x) \leqslant B\Phi_2(x)$ for every x and some constant B, and $\Phi_2'' \geqslant \ell > 0$. Assume that there exists a constant C_Φ such that for any smooth function $f:\mathbb{R}^n \to \mathbb{R}$,*

$$\big\|\big(f-\mu(f)\big)^2\big\|_\Phi \leqslant C_\Phi \int |\nabla f|^2\,d\mu.$$

Then, for any smooth function f, for any $t \geqslant 0$,

$$\big\|\big(\mathbf{P}_t f-\mu(f)\big)^2\big\|_\Phi \leqslant e^{-Mt}\big\|\big(f-\mu(f)\big)^2\big\|_\Phi$$

with $M = \frac{2\ell}{BC_\Phi}$

Proof. It is enough to check that for any function f, we have

$$\big\|f-\mu(f)\big\|_{\Phi_2}^2 = \big\|\big(f-\mu(f)\big)^2\big\|_\Phi \leqslant C_\Phi \int |\nabla f|^2\,d\mu$$
$$\leqslant \frac{C_\Phi}{\ell}\int |\nabla f|^2 \Phi_2''\left(\frac{f-\mu(f)}{\|f-\mu(f)\|_\Phi}\right)d\mu,$$

and to apply Theorem 11. □

In Corollary 6 we proved that a family of Orlicz–Sobolev inequalities hold for $\Phi_\beta(x) = |x|[\log(1+|x|)]^\beta$, $\beta \in [0,1]$. Actually we cannot apply the previous result to this family of norms, simply because $\Phi''_{\beta,2}(0) = 0$ and thus there is no bound of the type $\Phi''_{\beta,2} \geqslant \ell > 0$ (here $\Phi_{\beta,2}(x) = x^2\log(1+x^2)^\beta$). However we can get rid of this problem by means of equivalence of norms.

Proposition 15. *Let $\alpha \in [1,2]$, $\beta = 2(1-\frac{1}{\alpha}) \in [0,1]$ and for $\gamma \geqslant 1$, $\Phi_\beta^\gamma(x) = |x|\log(\gamma+|x|)^\beta$. Let $d\mu_\alpha^n(x) = Z_\alpha^{-n}\exp\{-\sum_{i=1}^n |x_i|^\alpha\}\,dx$ be a probability measure on $\mathbb{R}^n$, $\mathbf{L} = \Delta + \nabla V\cdot\nabla$ with $V = \sum_{i=1}^n |x_i|^\alpha$ be a symmetric (in $\mathbb{L}_2(\mu_\alpha)$) diffusion generator and $(\mathbf{P}_t)_{t\geqslant 0}$ its associated semi-group. Let C be the coefficient appearing in the Orlicz–Sobolev inequality of Corollary* 6.

Then, for any $\gamma > 1$, any $\beta \in [0,1]$, any integer n, any function f and any $t \geqslant 0$,

$$\left\|\left(\mathbf{P}_t f - \mu_\alpha^n(f)\right)^2\right\|_{\Phi_\beta^\gamma} \leqslant e^{-c_1 t}\left\|\left(f-\mu_\alpha^n(f)\right)^2\right\|_{\Phi_\beta^\gamma}$$

with $c_1 = \frac{(\log\gamma)^\beta}{4^\beta C(1+e(\log\gamma)^\beta)}$.

While for any $\beta \in [0,1]$, any integer n, any function f and any $t \geqslant 0$,

$$\left\|\left(\mathbf{P}_t f - \mu_\alpha^n(f)\right)^2\right\|_{\Phi_\beta^1} \leqslant \begin{cases} \|(f-\mu_\alpha^n(f))^2\|_{\Phi_\beta^1} & \text{for } t \leqslant 4^\beta Ce,\\ \frac{t}{4^\beta C} e^{-\frac{t}{4^\beta Ce}}\|(f-\mu_\alpha^n(f))^2\|_{\Phi_\beta^1} & \text{for } t \geqslant 4^\beta Ce. \end{cases}$$

Proof. Fix $\gamma > 1$, an integer n and $\beta \in [0,1]$. Then note that from the equivalence of Orlicz norms corresponding for different γ (see Lemma 16(i) (with $\gamma = 1$)), and Corollary 6, for any sufficiently smooth function f, one has

$$\begin{aligned}\left\|\left(f-\mu_\alpha^n(f)\right)^2\right\|_{\Phi_\beta^\gamma} &\leqslant \left(1+e(\log\gamma)^\beta\right)\left\|\left(f-\mu_\alpha^n(f)\right)^2\right\|_{\Phi_\beta^1}\\ &\leqslant C\left(1+e(\log\gamma)^\beta\right)\int|\nabla f|^2\,d\mu_\alpha^n.\end{aligned}$$

On the other hand, by Lemma 16(ii), (iii) we can apply Corollary 14 with $B = 4^{1+\beta}$ and $\ell = 2(\log\gamma)^\beta$. It follows that

$$\left\|\left(\mathbf{P}_t f - \mu_\alpha^n(f)\right)^2\right\|_{\Phi_\beta^\gamma} \leqslant \exp\left\{-\frac{(\log\gamma)^\beta}{4^\beta C(1+e(\log\gamma)^\beta)}t\right\}\left\|\left(f-\mu_\alpha^n(f)\right)^2\right\|_{\Phi_\beta^\gamma}$$

which gives the first part of the result.

For the second part, we use twice the latter inequality together with Lemma 16(i) to get for any $\gamma \geqslant 1$,

$$\begin{aligned}\left\|\left(\mathbf{P}_t f - \mu_\alpha^n(f)\right)^2\right\|_{\Phi_\beta^1} &\leqslant \left\|\left(\mathbf{P}_t f - \mu_\alpha^n(f)\right)^2\right\|_{\Phi_\beta^\gamma} \leqslant e^{-c_1 t}\left\|\left(f-\mu_\alpha^n(f)\right)^2\right\|_{\Phi_\beta^\gamma}\\ &\leqslant \left(1+e(\log\gamma)^\beta\right)e^{-c_1 t}\left\|\left(f-\mu_\alpha^n(f)\right)^2\right\|_{\Phi_\beta^1}\end{aligned}$$

with $c_1 = \frac{(\log\gamma)^\beta}{4^\beta C(1+e(\log\gamma)^\beta)}$. The result follows from an optimization over $\gamma \geqslant 1$ and the decrease of $N(t)$ proved before. □

Lemma 16. *For $\beta \in [0,1]$ and $\gamma \geqslant 1$, let $\Phi_\beta^\gamma(x) = |x|\log(\gamma+|x|)^\beta$ and $\Phi_{\beta,2}^\gamma(x) = \Phi_\beta^\gamma(x^2)$. Then,*

(i) *for any* $1 \leqslant \gamma \leqslant \gamma'$,

$$\|\cdot\|_{\Phi^{\gamma}_{\beta,2}} \leqslant \|\cdot\|_{\Phi^{\gamma'}_{\beta,2}} \leqslant C_{\gamma,\gamma'}\|\cdot\|_{\Phi^{\gamma}_{\beta,2}}$$

with $C_{\gamma,\gamma'} \equiv [1+(1+(e-\gamma)_+)(\log\frac{\gamma'}{\gamma})^{\beta}]^{1/2}$, *where* $(x)_+ := \max(x,0)$;

(ii) *for any* x, ${\Phi^{\gamma}_{\beta,2}}''(x) \geqslant 2(\log\gamma)^{\beta}$;

(iii) *for any* x, $\Phi^{\gamma}_{\beta,2}(2x) \leqslant 4^{1+\beta}\Phi^{\gamma}_{\beta,2}(x)$.

Proof. First, (i) follows from Lemma 44, provided in Appendix A, since for any $1 \leqslant \gamma \leqslant \gamma'$ one has

$$\Phi^{\gamma'}_{\beta}(x) = |x|\left(\log\frac{\gamma'}{\gamma} + \log\left(\gamma + \frac{\gamma}{\gamma'}|x|\right)\right)^{\beta} \leqslant \left(\log\frac{\gamma'}{\gamma}\right)^{\beta}|x| + \Phi^{\gamma}_{\beta}(x).$$

We also made use of the bound (19) for $\Phi = \Phi^{\gamma}_{\beta}$, $\tau = 1$ and $M = (e-\gamma)_+$.

Now, we may easily check that ${\Phi^{\gamma}_{\beta,2}}''$ is non-decreasing and thus greater than ${\Phi^{\gamma}_{\beta,2}}''(0) = 2(\log\gamma)^{\beta}$. This gives (ii).

Using $\gamma + 4x^2 \leqslant (\gamma + x^2)^4$ (recall that $\gamma \geqslant 1$), we get

$$\Phi^{\gamma}_{\beta,2}(2x) = 4x^2\big(\log\big(\gamma + 4x^2\big)\big)^{\beta} \leqslant 4x^2\big(\log\big(\gamma + x^2\big)^4\big)^{\beta} = 4^{1+\beta}\Phi^{\gamma}_{\beta,2}(x).$$

The proof is complete. □

3.1. Monotone functionals

In Proposition 15 the semi-group is not decaying exponentially to equilibrium in particular for Φ^1_1. We shall see in this section that a modification (a time-averaging) of the functional will satisfies an exponential decay.

The following inequality was shown in [10, Proposition 4.1]:

$$\frac{2}{3}\big\|\big(f-\mu_2(f)\big)^2\big\|_{\Phi^1_1} \leqslant \sup_{a\in\mathbb{R}} \mathrm{Ent}_{\mu_2}\big((f+a)^2\big) \leqslant \frac{5}{2}\big\|\big(f-\mu_2(f)\big)^2\big\|_{\Phi^1_1}.$$

Thus, the previous result gives that for $t \geqslant 4Ce$,

$$\mathrm{Ent}_{\mu_2}\big((\mathbf{P}_t f)^2\big) \leqslant \frac{15t}{16C}e^{-\frac{t}{4eC}} \sup_{a\in\mathbb{R}} \mathrm{Ent}_{\mu_2}\big((f+a)^2\big),$$

where C is the logarithmic Sobolev constant of μ_2. Now, using the Rothaus inequality (see [26])

$$\sup_{a\in\mathbb{R}} \mathrm{Ent}_{\mu_2}\big((f+a)^2\big) \leqslant \mathrm{Ent}_{\mu_2}\big(\big(f-\mu_2(f)\big)^2\big) + 2\mu_2\big(\big(f-\mu_2(f)\big)^2\big),$$

we have

$$\mathrm{Ent}_{\mu_2}\big((\mathbf{P}_t f)^2\big) \leqslant \frac{15t}{16C}e^{-\frac{t}{4eC}}\big(\mathrm{Ent}_{\mu_2}\big(\big(f-\mu_2(f)\big)^2\big) + 2\mu_2\big(\big(f-\mu_2(f)\big)^2\big)\big) \tag{7}$$

which can be improved for $f \geqslant 0$ using Kulback's inequality $\mathrm{Var}_{\mu_2}(f) \leqslant \mathrm{Ent}_{\mu_2}(f^2)$. As far as we know the bound (7) was not known. Indeed, the logarithmic Sobolev inequality is usually used in case of diffusion semi-group (see e.g. [1]) to prove exponential decay of entropy, i.e. that for any t,

$$\mathrm{Ent}_\mu(\mathbf{P}_t f) \leqslant e^{-t/C}\mathrm{Ent}_\mu(f).$$

On the other hand, there does not exist any constant $k < \infty$ such that for any function f, $\sup_{a\in\mathbb{R}} \mathrm{Ent}_\mu((f+a)^2) \leqslant k\mathrm{Ent}_\mu(f^2)$, or equivalently $\|(f-\mu(f))^2\|_{\Phi_1^1} \leqslant k\mathrm{Ent}_\mu(f^2)$. Indeed, on the space $\{0,1\}$ with the symmetric Bernoulli measure, consider the function $f(0) = -1$ and $f(1) = 1$ for which $(f-\mu(f))^2 \equiv \mathbb{1}$ and $\mathrm{Ent}_\mu(f^2) = 0$.

Thus we will consider the functional

$$A(f) \equiv \mathrm{Ent}_{\mu_2}\bigl(f^2\bigr) + \mu_2\bigl(f - \mu_2(f)\bigr)^2.$$

Then the bound (7), for all $t > T$ with some $T \in (0,\infty)$, can be written as follows:

$$A(\mathbf{P}_t f) \leqslant e^{-mt}A(f)$$

with some $m \in (0,\infty)$. With $\omega \in (0,m)$, define

$$\mathcal{A}_\omega(f) \equiv \sup_{s\in[0,T]} A(\mathbf{P}_s f)e^{\omega s}$$

and for $\omega \in [0,m]$ define

$$\mathcal{B}_\omega(f) \equiv \frac{1}{T}\int_0^T A(\mathbf{P}_s f)e^{\omega s}\,ds.$$

Proposition 17. *Suppose, with some $m, T \in (0,\infty)$, for all $t \geqslant T$, one has*

$$A(\mathbf{P}_t f) \leqslant e^{-mt}A(f).$$

Then the functionals $\mathcal{A}_\omega$ and $\mathcal{B}_\omega$ are exponentially decaying, that is for any $t \geqslant 0$

$$\mathcal{A}_\omega(\mathbf{P}_t f) \leqslant e^{-\omega t}\mathcal{A}_\omega(f)$$

and

$$\mathcal{B}_\omega(\mathbf{P}_t f) \leqslant e^{-\omega t}\mathcal{B}_\omega(f).$$

Proof. If $t \geqslant T$, the statements are clear. For $\mathcal{A}_\omega$ and $0 \leqslant t \leqslant T$ note that

$$\begin{aligned}\mathcal{A}_\omega(\mathbf{P}_t f) &= \sup_{s\in[0,T]} A(\mathbf{P}_{s+t} f)e^{\omega s} = e^{-\omega t}\sup_{s\in[t,T+t]} A(\mathbf{P}_s f)e^{\omega s}\\ &= e^{-\omega t}\max\Bigl(\sup_{s\in[t,T]} A(\mathbf{P}_s f)e^{\omega s}, \sup_{s\in[T,T+t]} A(\mathbf{P}_s f)e^{\omega s}\Bigr).\end{aligned}$$

Since for $s \in [T, T+t]$

$$A(\mathbf{P}_s f)e^{\omega s} \leqslant e^{-mT} A(\mathbf{P}_{s-T} f) \leqslant e^{-mT+\omega T}\big(A(\mathbf{P}_{s-T} f)e^{\omega(s-T)}\big)$$

we get for $t \in [0, T]$ and $\omega \leqslant m$

$$\sup_{s\in[T,T+t]} A(\mathbf{P}_s f)e^{\omega s} \leqslant \sup_{s\in[0,T]} A(\mathbf{P}_s f)e^{\omega s}.$$

This together with the previous considerations concludes the arguments for exponential decay of the first functional. In case of $\mathcal{B}_\omega$, for $0 \leqslant t \leqslant T$, we have

$$T\mathcal{B}_\omega(\mathbf{P}_t f) \equiv \int_0^T A(\mathbf{P}_{s+t} f)e^{\omega s}\,ds = e^{-\omega t}\int_t^{T+t} A(\mathbf{P}_s f)e^{\omega s}\,ds.$$

Next we note that

$$\int_t^{T+t} A(\mathbf{P}_s f)e^{\omega s}\,ds = \int_t^T A(\mathbf{P}_s f)e^{\omega s}\,ds + \int_T^{T+t} A(\mathbf{P}_s f)e^{\omega s}\,ds.$$

To complete the proof it is sufficient to note that

$$\begin{aligned}\int_T^{T+t} A(\mathbf{P}_s f)e^{\omega s}\,ds &\leqslant e^{-mT+\omega T}\int_T^{T+t} A(\mathbf{P}_{s-T} f)e^{\omega(s-T)}\,ds\\ &= e^{-mT+\omega T}\int_0^t A(\mathbf{P}_s f)e^{\omega(s)}\,ds\\ &\leqslant \int_0^t A(\mathbf{P}_s f)e^{\omega(s)}\,ds. \qquad \square\end{aligned}$$

In particular we have thus shown that if an (a priori non-convex) functional decays monotonously exponentially fast for large times, then by averaging over “a characteristic time of relaxation” we can get a globally monotone functional.

4. Orlicz–Sobolev and Φ-Sobolev inequalities

In this section we provide a link between the Orlicz–Sobolev inequality and the Φ-entropy bound introduced by Chafaï [11] and the *additive* Φ-Sobolev inequality studied in [8].

Given a closed interval $\mathcal{I}$ of $\mathbb{R}$ and a convex function $\Phi:\mathcal{I}\to\mathbb{R}$, a probability measure μ on $\mathbb{R}^n$ satisfies a Φ-Sobolev inequality if there exists a constant C_Φ such that for every smooth function $f:\mathbb{R}^n\to\mathcal{I}$,

$$\mathrm{Ent}^\Phi_\mu(f) \leqslant C_\Phi \int \Phi''(f)|\nabla f|^2\,d\mu, \qquad (\mathrm{Ent}^\Phi\text{-S})$$

where

$$\mathrm{Ent}_\mu^\Phi(f) := \int \Phi(f)\,d\mu - \Phi\left(\int f\,d\mu\right).$$

In [11], it is proved that such an inequality is equivalent to the exponential decay of $\mathrm{Ent}_\mu^\Phi(\mathbf{P}_t f)$.

On the other hand, given a non-decreasing function $\varphi:(0,+\infty)\to\mathbb{R}$ continuously differentiable, we define $\Phi(x) = x\varphi(x)$ and we assume that Φ can be extended to 0 and is convex. A probability measure μ on $\mathbb{R}^n$ satisfies an additive Φ-Sobolev inequality if there exists a constant C_Φ such that for every smooth function $f:\mathbb{R}^n\to\mathbb{R}$,

$$\int \Phi\big(f^2\big)\,d\mu - \Phi\left(\int f^2\,d\mu\right) \leqslant C_\Phi \int |\nabla f|^2\,d\mu. \tag{Φ-S}$$

We start with the following general fact.

Proposition 18. *Let $\Phi(x) = x\varphi(x)$ be a $\mathcal{C}^2$ Young function, with $\varphi:(0,+\infty)\to\mathbb{R}$ non-decreasing. Assume that the probability measure μ on $\mathbb{R}^n$ satisfies for any smooth function f,*

$$\int \Phi\big(f^2\big)\,d\mu - \Phi\left(\int f^2\,d\mu\right) \leqslant C_\Phi \int |\nabla f|^2\,d\mu,$$

for some constant C_Φ independent of f. Then, for any smooth function g, for any $a>0$,

$$\Phi''(a)\,\mathrm{Var}_\mu(g) \leqslant \frac{C_\Phi}{2a}\int |\nabla g|^2\,d\mu.$$

In particular, if $\Phi''\neq 0$, μ satisfies a Poincaré inequality with constant $C_p \leqslant \inf_{a>0}\frac{C_\Phi}{2a\Phi''(a)}$.

The previous result states that the Poincaré inequality holds as far as the additive Φ-Sobolev inequality holds and $\Phi''\neq 0$.

Proof. Given a smooth non-negative function f on $\mathbb{R}^n$, the additive Φ-Sobolev inequality applied to $\sqrt{f}$ leads to

$$\int \Phi(f)\,d\mu - \Phi\left(\int f\,d\mu\right) \leqslant \frac{C_\Phi}{4}\int \frac{|\nabla f|^2}{f}\,d\mu.$$

Now, given a smooth bounded function g with $\mu(g)=0$ and $a>0$, $a+\varepsilon g\geqslant 0$ for ε small enough. Then the previous inequality applied to $a+\varepsilon g$ and a Taylor expansion at the second order for Φ gives the result when ε tends to 0. □

Remark 19. In [11, Section 1.2], the same result is proved for the Φ-entropy bound (Ent^Φ-S).

On the other hand, $\Phi''\equiv 0$ is equivalent to $\varphi(x) = a-(b/x)$, $(a,b)\in\mathbb{R}\times\mathbb{R}^+$. In that case the additive Φ-Sobolev inequality is trivial.

Now we give a link between the modified Orlicz–Sobolev inequality (6) and the Φ-entropy bound (Ent^Φ-S).

Proposition 20. *Let Φ be a Young function. Assume that the probability measure μ on $\mathbb{R}^n$ satisfies a Φ-entropy bound* (Ent^Φ-S) *with constant C_Φ. Then, it satisfies a modified Orlicz–Sobolev inequality* (6) *with the same constant C_Φ.*

Proof. For every smooth function $f:\mathbb{R}^n \to \mathbb{R}$ apply the Φ-entropy bound (Ent^Φ-S) to $(f - \mu(f))/\|f-\mu(f)\|_\Phi$ to get

$$\int \Phi\left(\frac{f-\mu(f)}{\|f-\mu(f)\|_\Phi}\right) d\mu - \Phi\left(\int \frac{f-\mu(f)}{\|f-\mu(f)\|_\Phi}\, d\mu\right) \leqslant C_\Phi \int \Phi''\left(\frac{f-\mu(f)}{\|f-\mu(f)\|_\Phi}\right) \frac{|\nabla f|^2}{\|f-\mu(f)\|_\Phi^2}\, d\mu.$$

Since $\Phi(0)=0$ and $\int \Phi(\frac{f-\mu(f)}{\|f-\mu(f)\|_\Phi})\, d\mu = 1$, we get the expected result. □

Remark 21. As a consequence of this result and using Theorem 11, we get that if $\mathrm{Ent}_\mu^\Phi(\mathbf{P}_t f)$ decays exponentially fast in time, then $\|\mathbf{P}_t f - \mu(f)\|_\Phi$ decays exponentially fast.

Next we give a similar result involving the additive Φ-Sobolev inequality (Φ-S) and the Orlicz–Sobolev inequality (O–S). Note that for a Young function Φ, the assumption $\Phi(x)/x \nearrow \infty$ when x goes to infinity and $\Phi'(0) > 0$ insure that the equation $x\Phi'(x) = 1$ has a unique solution, see [25, Section 2.4].

Proposition 22. *Let Φ be a $\mathcal{C}^2$ Young function with $\Phi'(0) > 0$. Assume that $\Phi(x) = x\varphi(x)$ for a non-decreasing function φ defined on $(0,\infty)$ and such that $\lim_{+\infty}\varphi = +\infty$. Denote by k_0 be the unique solution of $k_0\Phi'(k_0) = 1$. Let μ be a probability measure on $\mathbb{R}^n$. Assume that there exists a constant C_Φ such that for every smooth function $f:\mathbb{R}^n \to \mathbb{R}$,*

$$\int \Phi(f^2)\, d\mu - \Phi\left(\int f^2\, d\mu\right) \leqslant C_\Phi \int |\nabla f|^2\, d\mu.$$

Then, for any smooth function $f:\mathbb{R}^n \to \mathbb{R}$, for any $a > 0$,

$$\left\|\left(f-\mu(f)\right)^2\right\|_\Phi \leqslant \frac{C_\Phi}{k_0}\left(\frac{1}{2a\Phi''(a)} + \frac{1}{\Phi'(0)}\right) \int |\nabla f|^2\, d\mu.$$

Remark 23. Note that since $\lim_{x\to+\infty}\varphi = +\infty$, there exists $a > 0$ such that $\Phi''(a) > 0$.

On the other hand, it is easy to get rid of the assumption $\Phi'(0) > 0$. Indeed, assume that $\Phi'(0) = 0$ and defined $\Phi_\lambda(x) = \Phi(x) + \lambda|x|$ for $\lambda > 0$. Then, on one hand $\Phi_\lambda'(0) = \lambda > 0$. On the other hand, if an additive Φ-Sobolev Inequality holds, then a Φ_λ-Sobolev inequality holds, with the same constant. So the previous proposition applies to Φ_λ: for any smooth function f, for any $a > 0$,

$$\left\|\left(f-\mu(f)\right)^2\right\|_{\Phi_\lambda} \leqslant \frac{C_\Phi}{k_0(\lambda)}\left(\frac{1}{2a\Phi''(a)} + \frac{1}{\lambda}\right) \int |\nabla f|^2\, d\mu.$$

Since $\Phi \leqslant \Phi_\lambda$, $\|(f-\mu(f))^2\|_\Phi \leqslant \|(f-\mu(f))^2\|_{\Phi_\lambda}$. This leads to

$$\big\|\big(f-\mu(f)\big)^2\big\|_\Phi \leqslant \frac{C_\Phi}{k_0(\lambda)}\left(\frac{1}{2a\Phi''(a)}+\frac{1}{\lambda}\right)\int |\nabla f|^2\,d\mu,$$

for any $\lambda > 0$, any $a > 0$ and any function f. Note that $k_0(\lambda) \to 0$ when λ tends to ∞.

Proof. Let $\widetilde{\Phi}(x) := \Phi(k_0 x)$, so its complementary function is $(\widetilde{\Phi})^*(x) = \Phi^*(x/k_0)$ where Φ^* is the complementary function of Φ. Now $(\widetilde{\Phi}, (\widetilde{\Phi})^*)$ is a normalized complementary pair of Young functions. Following [25] define the modified Luxembourg norm

$$\|f\|_{\widetilde{\Phi}} = \inf\left\{\lambda;\ \int \widetilde{\Phi}\left(\frac{f}{\lambda}\right)d\mu \leqslant \widetilde{\Phi}(1)\right\}.$$

Note that $\|\mathbb{1}\|_{\widetilde{\Phi}} = 1$. By [25, Section 3.3, Proposition 1], we know that

$$\int |g|\,d\mu \leqslant \|g\|_{\widetilde{\Phi}}\|\mathbb{1}\|_{(\widetilde{\Phi})^*} = \|g\|_{\widetilde{\Phi}} \quad \forall g \in \mathbb{L}_{\widetilde{\Phi}}(\mu). \tag{8}$$

It is important to introduce this modified norm in order to have the latter inequality with a factor 1 in front of the right-hand side and not 2 as in the standard inequality (17).

Now let $f:\mathbb{R}^n \to \mathbb{R}$ be a smooth function. From the additive ϕ-Sobolev inequality applied to $\sqrt{k_0}(f-\mu(f))/\|(f-\mu(f))^2\|_{\widetilde{\Phi}}^{1/2}$, we get

$$\begin{aligned}&\int \widetilde{\Phi}\left(\frac{(f-\mu(f))^2}{\|(f-\mu(f))^2\|_{\widetilde{\Phi}}}\right)d\mu - \widetilde{\Phi}\left(\int \frac{(f-\mu(f))^2}{\|(f-\mu(f))^2\|_{\widetilde{\Phi}}}\,d\mu\right)\\ &\quad\leqslant C_\Phi k_0 \int \frac{|\nabla f|^2}{\|(f-\mu(f))^2\|_{\widetilde{\Phi}}}\,d\mu.\end{aligned}$$

Since

$$\int \widetilde{\Phi}\left(\frac{(f-\mu(f))^2}{\|(f-\mu(f))^2\|_{\widetilde{\Phi}}}\right)d\mu = \widetilde{\Phi}(1),$$

it follows that

$$\widetilde{\Phi}(1) - \widetilde{\Phi}\left(\frac{\int g\,d\mu}{N_{\widetilde{\Phi}}(g)}\right) \leqslant C_\Phi k_0 \int \frac{|\nabla g|^2}{\|g\|_{\widetilde{\Phi}}}\,d\mu,$$

where $g := (f-\mu(f))^2$. A Taylor expansion of $\widetilde{\Phi}$ up to the second order, between 1 and $\frac{\int g\,d\mu}{\|g\|_{\widetilde{\Phi}}}$, and convexity of $\widetilde{\Phi}$, give that

$$\begin{aligned}\widetilde{\Phi}(1) - \widetilde{\Phi}\left(\frac{\int g\,d\mu}{\|g\|_{\widetilde{\Phi}}}\right) &= \left(1-\frac{\int g\,d\mu}{\|g\|_{\widetilde{\Phi}}}\right)\widetilde{\Phi}'\left(\frac{\int g\,d\mu}{\|g\|_{\widetilde{\Phi}}}\right) + \frac{1}{2}\left(1-\frac{\int g\,d\mu}{\|g\|_{\widetilde{\Phi}}}\right)^2\widetilde{\Phi}''(\theta)\\ &\geqslant \left(1-\frac{\int g\,d\mu}{\|g\|_{\widetilde{\Phi}}}\right)\widetilde{\Phi}'\left(\frac{\int g\,d\mu}{\|g\|_{\widetilde{\Phi}}}\right)\\ &\geqslant \left(1-\frac{\int g\,d\mu}{\|g\|_{\widetilde{\Phi}}}\right)\widetilde{\Phi}'(0),\end{aligned}$$

where $\theta \in (0,1)$ (recall that from (8), $\frac{\int |g|\,d\mu}{\|g\|_{\widetilde{\Phi}}} \leqslant 1$). This leads to

$$\big\|\big(f-\mu(f)\big)^2\big\|_{\widetilde{\Phi}} \leqslant \frac{C_\Phi k_0}{\widetilde{\Phi}'(0)} \int |\nabla f|^2\,d\mu + \mathrm{Var}_\mu(f).$$

Since $\lim_{x\to+\infty} \varphi(x) = +\infty$, there exists $a>0$ such that $\Phi''(a)>0$. Choose such an a. From Proposition 18, μ satisfies a Poincaré inequality with constant less than $C_\Phi/(2a\Phi''(a))$. On the other hand

$$\big\|\big(f-\mu(f)\big)^2\big\|_{\widetilde{\Phi}} = k_0\big\|\big(f-\mu(f)\big)^2\big\|_{\Phi/\widetilde{\Phi}(1)} \quad \text{and} \quad \widetilde{\Phi}'(0) = k_0\Phi'(0).$$

Thus, for any smooth function $f:\mathbb{R}^n\to\mathbb{R}$,

$$\big\|\big(f-\mu(f)\big)^2\big\|_{\Phi/\widetilde{\Phi}(1)} \leqslant \frac{C_\Phi}{k_0}\left(\frac{1}{2a\Phi''(a)} + \frac{1}{\Phi'(0)}\right)\int |\nabla f|^2\,d\mu.$$

The result follows from the fact that $\|(f-\mu(f))^2\|_\Phi \leqslant \|(f-\mu(f))^2\|_{\Phi/\widetilde{\Phi}(1)}$ since $\Phi \leqslant \Phi/\widetilde{\Phi}(1)$ (recall that $\widetilde{\Phi}(1) + (\widetilde{\Phi})^*(1) = 1$).

For all $a>0$ such that $\Phi''(a)=0$, the result is trivial. This ends the proof. □

Proposition 22 allows us to give a criterium for the Φ-Sobolev inequality to hold. This completes [8, Theorem 26].

Theorem 24. *Let $\Phi(x) = x\varphi(x)$ be a $\mathcal{C}^2$ Young function with φ non-decreasing, concave, with $\varphi(0)>0$ and such that $\lim_{+\infty}\varphi = +\infty$. Denote by k_0 the unique solution of $k_0\Phi'(k_0)=1$. Assume that there exist constants γ, κ and such that for all $x,y>0$ one has*

$$x\varphi'(x) \leqslant \gamma \quad \textit{and} \quad \varphi(xy) \leqslant \kappa + \varphi(x) + \varphi(y),$$

and a constant $\lambda \geqslant 2$ such that for every $x \geqslant 2\lambda$, one has $\lambda\varphi(x/\varphi(x)) \geqslant \varphi(x)$.

Let μ be a probability measure on $\mathbb{R}^n$ satisfying the Poincaré inequality with constant C_P and C_Φ the optimal constant such that for every smooth function $f:\mathbb{R}^n\to\mathbb{R}$ one has

$$\int \Phi\big(f^2\big)\,d\mu - \Phi\left(\int f^2\,d\mu\right) \leqslant C_\Phi \int |\nabla f|^2\,d\mu. \tag{9}$$

Then, for any $a>0$,

$$\frac{k_0 a\Phi''(a)\varphi(0)}{8\lambda(\varphi(0)+2a\Phi''(a))}\widetilde{B}(\Phi) \leqslant C_\Phi \leqslant \left(18\gamma C_p + 24\left(1+\frac{M}{\varphi(8)}\right)\right)\widetilde{B}(\Phi),$$

where $\widetilde{B}(\Phi)$ is the smallest constant so that for every $A\subset\mathbb{R}^n$ with $\mu(A)<\frac{1}{2}$

$$\mu(A)\varphi\left(\frac{2}{\mu(A)}\right) \leqslant \widetilde{B}(\Phi)\mathrm{Cap}_\mu(A).$$

Proof. The upper bound on C_Φ follows from [8, Theorem 26].

Assume that the additive Φ-Sobolev inequality (9) holds. Then, by Proposition 22, for every smooth function $f:\mathbb{R}^n \to \mathbb{R}$, every $a > 0$,

$$\left\|\big(f-\mu(f)\big)^2\right\|_\Phi \leqslant \frac{C_\Phi}{k_0}\left(\frac{1}{2a\Phi''(a)}+\frac{1}{\varphi(0)}\right)\int |\nabla f|^2\,d\mu.$$

Then, by Theorem 1 we get,

$$\frac{1}{8}B(\Phi) \leqslant \frac{C_\Phi}{k_0}\left(\frac{1}{2a\Phi''(a)}+\frac{1}{\varphi(0)}\right), \tag{10}$$

where $B(\Phi)$ is the smallest constant so that for every $A \subset \mathbb{R}^n$ with $\mu(A) < \frac{1}{2}$

$$\frac{1}{\Phi^{-1}\big(\frac{1}{\mu(A)}\big)} = \|\mathbb{1}_A\|_\Phi \leqslant B(\Phi)\mathrm{Cap}_\mu(A).$$

By our assumption on φ,

$$\Phi\left(\frac{x}{\varphi(x)}\right) = x\frac{\varphi(x/\varphi(x))}{\varphi(x)} \geqslant \frac{1}{\lambda}x \quad \text{for all } x \geqslant 2\lambda.$$

Thus, since $\lambda \geqslant 2$ and φ is non-decreasing, for all $y \geqslant 2$

$$\Phi^{-1}(y) \leqslant \frac{\lambda y}{\varphi(\lambda y)} \leqslant \lambda\frac{y}{\varphi(2y)}.$$

It follows that $\widetilde{B}(\Phi) \leqslant \lambda B(\Phi)$. This together with (10) achieves the proof. □

4.1. Φ-S and O–S inequalities in infinite dimensions

It is not difficult to check that $\Phi(x) = |x|(\log(\eta + |x|))^\beta$, $\beta \in (0,1]$, $\eta > 1$, satisfies the hypothesis of Theorem 24.

Following a remark of [8] we note that

$$\mu\big(\Phi\big(f^2\big)\big) - \Phi\big(\mu\big(f^2\big)\big) = \inf_{t>0}\mu\big(\Phi\big(f^2\big) - \Phi(t) - \Phi'(t)\big(\mu\big(f^2\big)-t\big)\big)$$

By convexity of Φ one has $\Phi(f^2) - \Phi(t) - \Phi'(t)(\mu(f^2)-t) \geqslant 0$ which implies the following Mild Perturbation Property (MPP) for additive Φ-Sobolev inequality.

Proposition 25. *Let $d\mu = \rho\,d\nu$ with $\delta U \equiv \sup(\log\rho) - \inf(\log\rho) < \infty$ and assume that*

$$\int \Phi\big(f^2\big)\,d\nu - \Phi\left(\int f^2\,d\nu\right) \leqslant C\int |\nabla f|^2\,d\nu.$$

Then

$$\int \Phi\big(f^2\big)\,d\mu - \Phi\left(\int f^2\,d\mu\right) \leqslant Ce^{\delta U}\int |\nabla f|^2\,d\mu.$$

The additive Φ-Sobolev inequality, with the Φ as described above, was in particular established for products of μ_α measures with suitable $\alpha \in (1,2)$. Using MPP one can construct a compatible family of finite-dimensional expectations E_Λ (with partially ordered indices Λ) for which additive Φ-Sobolev inequality also holds. By definition for the corresponding Gibbs measure $\boldsymbol{\mu}(E_\Lambda) = \boldsymbol{\mu}$ and one has the following simple conditioning property:

$$\begin{aligned}\boldsymbol{\mu}\big(\Phi\big(f^2\big)\big) - \Phi\big(\boldsymbol{\mu}\big(f^2\big)\big) = {} & \boldsymbol{\mu}\big[E_\Lambda\big(\Phi\big(f^2\big)\big) - \Phi\big(E_\Lambda\big(f^2\big)\big)\big] \\ & + \boldsymbol{\mu}\big(\Phi\big[E_\Lambda\big(f^2\big)\big]\big) - \Phi\big(\mu\big[E_\Lambda\big(f^2\big)\big]\big).\end{aligned}$$

With these two facts in mind, under suitable mixing condition, one can follow closely the strategy originally invented for the proof of logarithmic Sobolev inequality (cf. [20]) to proof the following result.

Theorem 26. *Suppose a local specification is mixing and satisfies Φ-Sobolev inequality. Then the unique Gibbs measure $\boldsymbol{\mu}$ satisfies*

$$\boldsymbol{\mu}\big(\Phi\big(f^2\big)\big) - \Phi\big(\boldsymbol{\mu}\big(f^2\big)\big) \leqslant C\boldsymbol{\mu}\big(|\nabla f|^2\big)$$

with a constant C independent of a function f.

This provides a large family of non-trivial examples of (non-product) measures on infinite-dimensional spaces satisfying additive Φ-Sobolev inequality.

We remark that by inserting into such the inequality a function $f/\|f\|_2$ and setting $F(x) \equiv (\log(\eta + |x|))^\beta - (\log(\eta + 1))^\beta$, we arrive at the following F-Sobolev inequality.

Corollary 27.

$$\int f^2 F\left(\frac{f^2}{\boldsymbol{\mu}(f^2)}\right) d\boldsymbol{\mu} \leqslant C\boldsymbol{\mu}\int |\nabla f|^2\, d\boldsymbol{\mu} \tag{F-S}$$

for the Gibbs measure $\boldsymbol{\mu}$.

Finally we note that by the same arguments as the ones used to prove Proposition 22, we get the following Orlicz–Sobolev inequality for infinite-dimensional Gibbs measures

Corollary 28.

$$\big\|\big(f - \boldsymbol{\mu}(f)\big)^2\big\|_\Phi \leqslant c\boldsymbol{\mu}\big(|\nabla f|^2\big)$$

with a constant c independent of a function f.

5. Orlicz–Sobolev and Nash-type inequalities

In this section we prove that the Orlicz–Sobolev inequality is equivalent, up to some constants, to a Nash-type inequality. This give new results on the decay to equilibrium of the semi-group (see the next section).

Theorem 29. *Let Φ and $\Psi(x) = \frac{x^2}{\psi(|x|)}$ be two N-functions with $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ increasing, satisfying $\psi(0) = 0$ and $\lim_{+\infty} \psi = +\infty$. Assume that the probability measure μ on $\mathbb{R}^n$ satisfies for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$*

$$\big\| \big(f - \mu(f)\big)^2 \big\|_\Phi \leqslant C_\Phi \int |\nabla f|^2 \, d\mu.$$

Then, for any function f,

$$\mathrm{Var}_\mu(f) \theta\left(\frac{1}{2} \frac{\mathrm{Var}_\mu(f)}{\|f - \mu(f)\|_\Psi^2}\right) \leqslant 4 C_\Phi \int |\nabla f|^2 \, d\mu, \tag{11}$$

where $\theta = \Phi^{*-1} \circ \Psi \circ \psi^{-1}$ (here Φ^* is the complementary pair of Φ; Φ^{*-1} and ψ^{-1} stand for the inverse function of Φ^* and ψ, respectively).

Remark 30. Note that by our assumption on ψ, ψ^{-1} is well defined on $\mathbb{R}_+$ onto $\mathbb{R}_+$.

Furthermore, in order to deal with explicit functions, one can easily see that under the assumption of the theorem, $\Psi(x) \leqslant \frac{1}{\psi(1)}(x + x^2)$ in such a way that $\mathrm{Var}_\mu(f)/\|f - \mu(f)\|_\Psi^2 \geqslant c$ for some constant c (see Lemma 44). Thus one has only to consider the behavior of θ (or equivalently to Φ, Ψ and ψ) away from 0.

Remark 31. We will call the inequality

$$\mathrm{Var}_\mu(f) \theta\left(\frac{1}{2} \frac{\mathrm{Var}_\mu(f)}{\|f - \mu(f)\|_\Psi^2}\right) \leqslant C \int |\nabla f|^2 \, d\mu,$$

a Nash-type inequality since for $\Phi(x) = |x|^{(d)/(d-2)}$, $\Psi(x) = \psi(x) = x$ (and thus $\theta(x) = c_d |x|^{2/d}$ for some constant c_d), it reads for any f with $\mu(f) = 0$ as

$$\|f\|_2^{1+2/d} \leqslant C' \|\nabla f\|_2 \|f\|_1^{2/d}$$

which is the standard Nash inequality [24].

Proof. The proof is a generalization of [3, Proposition 10.3], see also [27]. Let f be a function with $\mu(f) = 0$ and $\|f\|_\Psi = 1$ in such a way that $\int \Psi(f) \, d\mu = 1$. Fix a parameter $t > 0$. Denote by Φ^* the complementary function of Φ. From (17), if $(f, g) \in \mathbb{L}_\Phi \times \mathbb{L}_{\Phi^*}$, $\int |fg| \, d\mu \leqslant 2 \|f\|_\Phi \|f\|_{\Phi^*}$. Hence,

$$\begin{aligned}
\mathrm{Var}_\mu(f) &= \int f^2 \mathbb{1}_{|f|<t} \, d\mu + \int f^2 \mathbb{1}_{|f| \geqslant t} \, d\mu \\
&\leqslant \int \Psi(f) \psi\big(|f|\big) \mathbb{1}_{|f|<t} \, d\mu + 2 \big\| f^2 \big\|_\Phi \|\mathbb{1}_{|f| \geqslant t}\|_{\Phi^*} \\
&\leqslant \psi(t) \int \Psi(f) \, d\mu + \frac{2 C_\Phi}{\Phi^{*-1}(1/\mu(|f| \geqslant t))} \int |\nabla f|^2 \, d\mu.
\end{aligned}$$

Now by Chebychev inequality (recall that Φ is an even function) we have

$$\mu(|f| \geqslant t) = \mu(\Psi(f) \geqslant \Psi(t)) \leqslant \frac{1}{\Psi(t)} \int \Psi(f)\, d\mu = \frac{1}{\Psi(t)}.$$

It follows that for any $t > 0$,

$$\mathrm{Var}_\mu(f) \leqslant \psi(t) + \frac{2C_\Phi}{\Phi^{*-1}(\Psi(t))} \int |\nabla f|^2\, d\mu.$$

Now choose t such that $\psi(t) = \frac{1}{2}\mathrm{Var}_\mu(f)$. We get

$$\Phi^{*-1}(\Psi(t))\,\mathrm{Var}_\mu(f) \leqslant 4C_\Phi \int |\nabla f|^2\, d\mu.$$

This gives the expected result by homogeneity. □

Example 32. Let $\alpha \in [1,2]$, $\beta = 2(1 - \frac{1}{\alpha}) \in [0,1]$ and define the probability measure on $\mathbb{R}^n$:

$$d\mu_\alpha^n(x) = Z_\alpha^{-n} \exp\left\{-\sum_{i=1}^n |x_i|^\alpha\right\} dx.$$

For any $\gamma \geqslant 1$ define $\Phi_\beta^\gamma(x) = |x|(\log(\gamma + |x|)^\beta$ and $\Phi_{\beta,2}^\gamma(x) = \Phi_\beta^\gamma(x^2)$. From Corollary 6, Lemma 16(i) and the general fact that $\|f^2\|_{\Phi_\beta^\gamma} = \|f\|^2_{\Phi_{\beta,2}^\gamma}$, there exists a constant C (independent of n) such that for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\left\|\left(f - \mu_\alpha^n(f)\right)^2\right\|_{\Phi_\beta^\gamma} \leqslant C\left(1 + e(\log\gamma)^\beta\right) \int |\nabla f|^2\, d\mu_\alpha^n.$$

Using similar computation than in the proof of inequality (4), it is not difficult to see that for any $\varepsilon > 0$, there exists a constant C_ε (depending also on β and γ) such that for any $x \geqslant \varepsilon$,

$$C_\varepsilon^{-1} \log(1+x)^\beta \leqslant \Phi_\beta^{\gamma\,*-1}(x) \leqslant C_\varepsilon \log(1+x)^\beta.$$

Now define for $x \geqslant 0$ and $\delta \in (0,1)$, $\psi(x) = (\log(1+x))^\delta$. One can easily see that $\Psi(x) := x^2/\psi(x)$ is a N-function. We deduce that there exists $C'_\varepsilon > 0$ such that for any $x \geqslant \varepsilon$,

$$C_\varepsilon'^{-1} x^{\beta/\delta} \leqslant \theta(x) \leqslant C'_\varepsilon x^{\beta/\delta},$$

where $\theta := \Phi_\beta^{\gamma\,*-1} \circ \Psi \circ \psi^{-1}$. Theorem 29 implies that there exists a constant C' (independent on n and possibly depending on β, δ) such that for any function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{Var}_{\mu_\alpha^n}(f)^{1+\beta/\delta} \leqslant C' \left\|f - \mu_\alpha^n(f)\right\|_\Psi^{2\beta/\delta} \int |\nabla f|^2\, d\mu_\alpha^n.$$

If we choose instead $\widetilde{\psi}(x) = e^{(\log(1+x))^\delta} - 1$ for $\delta \in (0,1)$, $\widetilde{\Psi}(x) = x^2/\widetilde{\psi}(x)$ is again a N-function. It follows in this case that there exists a constant $C''_\varepsilon > 0$ such that for any $x \geqslant \varepsilon$,

$$C'^{-1}_\varepsilon \log(1+x)^{\beta/\delta} \leqslant \widetilde{\theta}(x) \leqslant C'_\varepsilon \log(1+x)^{\beta/\delta},$$

where $\widetilde{\theta} := \Phi^{\gamma\,*-1}_\beta \circ \widetilde{\Psi} \circ \widetilde{\psi}^{-1}$. In turn, Theorem 29 implies that

$$\operatorname{Var}_{\mu^n_\alpha}(f) \log\left(1 + \frac{1}{2}\frac{\operatorname{Var}_{\mu^n_\alpha}(f)}{\|f - \mu^n_\alpha(f)\|^2_\Psi}\right)^{\beta/\delta} \leqslant C'' \int |\nabla f|^2 \, d\mu^n_\alpha$$

for some constant C'' independent on n and f.

It is natural to ask for the equivalence between the Orlicz–Sobolev inequality and the Nash-type inequality in Theorem 29. It seems (almost for us) to be difficult to prove directly this equivalence. However, it is possible to achieve that with the help of an intermediate inequality as follows.

As a first step, we consider the following equivalent form of the Nash-type inequality.

Lemma 33. *Let Ψ be a N-function and θ be an increasing function. Assume that there exists a constant $\lambda > 0$ such that for any $x \geqslant 0$, $\theta(x/9) \geqslant \lambda\theta(x)$. Let μ be a probability measure on $\mathbb{R}^n$.*

Then, the following are equivalent:

(i) *There exists a constant C such that for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ one has*

$$\operatorname{Var}_\mu(f)\theta\left(\frac{1}{2}\frac{\operatorname{Var}_\mu(f)}{\|f - \mu(f)\|^2_\Psi}\right) \leqslant C \int |\nabla f|^2 \, d\mu.$$

(ii) *There exists a constant C' such that for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ one has*

$$\operatorname{Var}_\mu(f)\theta\left(\frac{1}{2}\frac{\operatorname{Var}_\mu(f)}{\|f\|^2_\Psi}\right) \leqslant C' \int |\nabla f|^2 \, d\mu. \tag{12}$$

Furthermore, (i) $\Rightarrow$ (ii) *with $C' \leqslant C/\lambda$ and* (ii) $\Rightarrow$ (i) *with $C \leqslant C'$.*

Proof. The implication (ii) implies (i) is obvious.

We will show that (i) $\Rightarrow$ (ii). By (17), for any function f, $\int |f| \, d\mu \leqslant 2\|f\|_\Psi \|\mathbb{1}\|_{\Psi^*} = \frac{2}{\Psi^{*-1}(1)}\|f\|_\Psi$. It follows that

$$\begin{aligned}\big\|f - \mu(f)\big\|_\Psi &\leqslant \|f\|_\Psi + \big\|\mu\big(|f|\big)\big\|_\Psi = \|f\|_\Psi + \frac{1}{\Psi^{-1}(1)}\mu\big(|f|\big) \\ &\leqslant \left(1 + \frac{2}{\Psi^{-1}(1)\Psi^{*-1}(1)}\right)\|f\|_\Psi \leqslant 3\|f\|_\Psi.\end{aligned}$$

In the last line we used the general bound $x \leqslant \Psi^{-1}(x)\Psi^{*-1}(x)$. Since θ is increasing and $\theta(x/9) \geqslant \lambda\theta(x)$, it follows that

$$\operatorname{Var}_\mu(f)\theta\left(\frac{1}{2}\frac{\operatorname{Var}_\mu(f)}{\|f-\mu(f)\|_\Psi^2}\right) \geqslant \operatorname{Var}_\mu(f)\theta\left(\frac{1}{18}\frac{\operatorname{Var}_\mu(f)}{\|f\|_\Psi^2}\right)$$
$$\geqslant \lambda \operatorname{Var}_\mu(f)\theta\left(\frac{1}{2}\frac{\operatorname{Var}_\mu(f)}{\|f\|_\Psi^2}\right).$$

Applying (ii) completes the proof. □

The second step is to link the Nash-type inequality in its simplified form to an inequality between measure and capacity.

Theorem 34. *Let Φ and $\Psi(x) = \frac{x^2}{\psi(|x|)}$ be two N-functions with ψ increasing, satisfying $\psi(0) = 0$ and $\lim_{+\infty} \psi = +\infty$. Let $\theta = \Phi^{*-1} \circ \Psi \circ \psi^{-1}$. Assume that*:

(i) *$x \mapsto \Psi \circ \psi^{-1}(x^2)$ is a Young function*;
(ii) *there exists a constant $\lambda > 0$ such that for any $x \geqslant 0$, $\theta(x/16) \geqslant \lambda\theta(x)$*;
(iii) *there exists $\lambda' \geqslant 4$ such that for all $x \geqslant 2$ one has $\Phi^{*-1}(\lambda' x) \leqslant \lambda'\Phi^{*-1}(x)/4$*;
(iv) *the probability measure μ on $\mathbb{R}^n$ satisfies for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$*

$$\operatorname{Var}_\mu(f)\theta\left(\frac{1}{2}\frac{\operatorname{Var}_\mu(f)}{\|f\|_\Psi^2}\right) \leqslant C\int |\nabla f|^2\, d\mu$$

for some constant C.

Then, for any Borel set A such that $\mu(A) < \frac{1}{2}$,

$$\frac{1}{\Phi^{-1}(1/\mu(A))} \leqslant \frac{8\lambda' C}{\lambda}\operatorname{Cap}_\mu(A).$$

Proof. Fix $A \subset \mathbb{R}^n$ such that $\mu(A) < \frac{1}{2}$, and let $g : \mathbb{R}^n \to \mathbb{R}$ such that $g \geqslant \mathbb{1}_A$ and $\mu(g = 0) \geqslant \frac{1}{2}$. Then for any $k \in \mathbb{Z}$ we define $g_k = (g - 2^k)_+ \wedge 2^k$. Let $H(x) := \Psi \circ \psi^{-1}(x^2)$. Note that $\sqrt{x}H^{-1}(x) = \Psi^{-1}(x)$. Thus, by (16) we have

$$\|g_k\|_\Psi = \|g_k\mathbb{1}_{g_k\neq 0}\|_\Psi \leqslant 2\|g_k\|_2\|\mathbb{1}_{g_k\neq 0}\|_H = 2\frac{\|g_k\|_2}{H^{-1}(1/\mu(g_k\neq 0))}.$$

Note that $\mu(g_k = 0) = \mu(g \leqslant 2^k) \geqslant \mu(g = 0) \geqslant \frac{1}{2}$. Thus,

$$\mu(g_k)^2 = \mu(g_k\mathbb{1}_{\{g_k\neq 0\}})^2 \leqslant \mu\big(g_k^2\big)\mu(g_k \neq 0) \leqslant \frac{1}{2}\mu\big(g_k^2\big)$$

which in turn implies $\|g_k\|_2^2 \leqslant 2\operatorname{Var}_\mu(g_k)$. This together with $\mu(g_k \neq 0) \leqslant \mu(g \geqslant 2^k)$ give

$$\|g_k\|_\Psi^2 \leqslant 8\frac{\operatorname{Var}_\mu(g_k)}{[H^{-1}(1/\mu(g \geqslant 2^k))]^2}.$$

Applying the Nash-type inequality to g_k and the monotonicity of θ, we get

$$\begin{aligned}\operatorname{Var}_\mu(g_k)\theta\left(\frac{1}{16}\left[H^{-1}\left(1/\mu\left(g \geqslant 2^k\right)\right)\right]^2\right) &\leqslant \operatorname{Var}_\mu(g_k)\left(\frac{1}{2}\frac{\operatorname{Var}_\mu(g_k)}{\|g_k\|_\Psi^2}\right)\\ &\leqslant C\int |\nabla g|^2\,d\mu.\end{aligned}$$

On the other hand,

$$\begin{aligned}\operatorname{Var}_\mu(g_k) &\geqslant \frac{1}{2}\|g_k\|_2^2 \geqslant 2^{2k-1}\|\mathbb{1}_{\{g_k\geqslant 2^k\}}\|_2^2 = 2^{2k-1}\mu\left(g_k \geqslant 2^k\right)\\ &= 2^{2k-1}\mu\left(g \geqslant 2^{k+1}\right).\end{aligned}$$

Let $\Omega_k = \{x\colon\ g(x) \geqslant 2^k\}$, $k \in \mathbb{Z}$. It follows from condition (ii) on θ that for any $k \in \mathbb{Z}$

$$\begin{aligned}\lambda 2^{2k-1}\mu(\Omega_{k+1})\theta\left[H^{-1}\left[\frac{1}{\mu(\Omega_k)}\right]^2\right] &\leqslant 2^{2k-1}\mu(\Omega_{k+1})\theta\left[\frac{1}{16}H^{-1}\left[\frac{1}{\mu(\Omega_k)}\right]^2\right]\\ &\leqslant C\int |\nabla g|^2\,d\mu.\end{aligned}$$

Now note that by definition of H and θ, $\theta(H^{-1}(x)^2) = \Phi^{*-1}(x)$. Hence,

$$\lambda 2^{2k-1}\mu(\Omega_{k+1})\Phi^{*-1}\left(1/\mu(\Omega_k)\right) \leqslant C\int |\nabla g|^2\,d\mu \quad \forall k \in \mathbb{Z}.$$

At this stage we may use [8, Lemma 23] we recall below with $a_k = \mu(\Omega_k)$ and $F = \Phi^{*-1}$. Since Φ^* is a Young function, the slope function $x \mapsto \Phi^*(x)/x$ is non-decreasing. This is equivalent to say that $x \mapsto F(x)/x$ is non-increasing. Thus the assumptions of Lemma 35 are satisfied, thanks to point (iii). It follows that

$$\lambda 2^{2k-1}\mu(\Omega_k)\Phi^{*-1}\left(1/\mu(\Omega_k)\right) \leqslant \lambda' C\int |\nabla g|^2\,d\mu \quad \forall k \in \mathbb{Z}.$$

Furthermore, by (13), $\Phi^{*-1}(x) \geqslant x/\Phi^{-1}(x)$. Hence,

$$\lambda 2^{2k-1}\frac{1}{\Phi^{-1}(1/\mu(\Omega_k))} \leqslant \lambda' C\int |\nabla g|^2\,d\mu \quad \forall k \in \mathbb{Z}.$$

Now take the largest k such that $2^{2k} \leqslant 1$. For that index, $A \subset \{g \geqslant 2^k\} = \Omega_k$. By monotonicity it follows that (using $1 \leqslant 2^{2(k+1)}$)

$$\frac{1}{\Phi^{-1}(1/\mu(A))} \leqslant 2^{2(k+1)}\frac{1}{\Phi^{-1}(1/\mu(\Omega_k))} \leqslant \frac{8\lambda'}{\lambda}C\int |\nabla g|^2\,d\mu.$$

The result follows by definition of the capacity. $\square$

Lemma 35. [8] *Let $F:[2,+\infty)\to[0,+\infty)$ be a non-decreasing function such that $x\to F(x)/x$ is non-increasing and there exists $\lambda'\geqslant 4$ such that for all $x\geqslant 2$ one has $F(\lambda' x)\leqslant \lambda' F(x)/4$. Let $(a_k)_{k\in\mathbb{Z}}$ be a non-increasing (double-sided) sequence of numbers in $[0,1/2]$. Assume that for all $k\in\mathbb{Z}$ with $a_k>0$ one has*

$$2^{2k}a_{k+1}F(1/a_k)\leqslant C,$$

then for all $k\in\mathbb{Z}$ with $a_k>0$ one has

$$2^{2k}a_k F(1/a_k)\leqslant \lambda' C.$$

We are now in position to give the following reciprocal of Theorem 29.

Corollary 36. *Let Φ and $\Psi(x)=\frac{x^2}{\psi(|x|)}$ be two N-functions with ψ increasing, satisfying $\psi(0)=0$ and $\lim_{+\infty}\psi=+\infty$. Let $\theta=\Phi^{*-1}\circ\Psi\circ\psi^{-1}$. Assume that*

(i) *$x\mapsto \Psi\circ\psi^{-1}(x^2)$ is a Young function;*
(ii) *there exists a constant $\lambda>0$ such that for any $x\geqslant 0$, $\theta(x/16)\geqslant\lambda\theta(x)$;*
(iii) *there exists $\lambda'\geqslant 4$ such that for all $x\geqslant 2$ one has $\Phi^{*-1}(\lambda' x)\leqslant\lambda'\Phi^{*-1}(x)/4$;*
(iv) *the probability measure μ on $\mathbb{R}^n$ satisfies for any smooth function $f:\mathbb{R}^n\to\mathbb{R}$*

$$\mathrm{Var}_\mu(f)\theta\left(\frac{1}{2}\frac{\mathrm{Var}_\mu(f)}{\|f-\mu(f)\|_\Psi^2}\right)\leqslant C\int|\nabla f|^2\,d\mu$$

for some constant C.

Fix $k\in(0,+\infty)$ such that for any f with $f^2\in\mathbb{L}_\Phi(\mu)$, $\|\mu(f)^2\|_\Phi\leqslant k\|f^2\|_\Phi$. Then, for any function f,

$$\big\|\big(f-\mu(f)\big)^2\big\|_\Phi\leqslant\frac{64(1+k)\lambda'}{\lambda^2}C\int|\nabla f|^2\,d\mu.$$

Proof. Apply Lemma 33, then Theorem 34, and finally Theorem 1 (together with Remark 2). □

6. Decay to equilibrium and Nash-type inequality

Throughout this section we consider a probability measure $d\mu=e^{-V(x)}\,dx$ on $\mathbb{R}^n$ associated to a differentiable potential V (or a limit of such measures). Let $\mathbf{L}=\Delta-\nabla V\cdot\nabla$ be a symmetric in $\mathbb{L}_2(\mu)$ diffusion generator and $(\mathbf{P}_t)_{t\geqslant 0}$ its associated semi-group. In this setup we prove that Nash-type inequalities are equivalent to the decay to equilibrium of the semi-group in suitable Orlicz spaces associated to the measure μ.

Theorem 37. *Let Φ and Ψ be two N-functions and θ an increasing function. Assume that the probability measure μ on $\mathbb{R}^n$ satisfies, for any smooth function $f:\mathbb{R}^n\to\mathbb{R}$,*

$$\mathrm{Var}_\mu(f)\theta\left(\frac{1}{2}\frac{\mathrm{Var}_\mu(f)}{\|f-\mu(f)\|_\Psi^2}\right)\leqslant C\int|\nabla f|^2\,d\mu$$

for some constant C. Then, for any $t > 0$,

$$\mathrm{Var}_\mu(\mathbf{P}_t f) \leqslant m(t)\big\| f - \mu(f)\big\|_\Psi^2,$$

where $m:\mathbb{R}_+ \to \mathbb{R}_+$ is the solution of the differential equation

$$m' = -\frac{2}{C_\Phi}\theta\left(\frac{m}{2}\right)$$

on $(0,\infty)$ such that $m(0) = +\infty$, or equivalently m satisfies for any $t \geqslant 0$,

$$\int_{m(t)}^{\infty} \frac{1}{x\theta(x/2)}\,dx = \frac{2t}{C_\Phi}.$$

Proof. Let f be such that $\mu(f) = 0$ and $\|f\|_\Psi = 1$. Set $u(t) = \mathrm{Var}_\mu(\mathbf{P}_t f)$. A differentiation gives

$$u'(t) = -2\int |\nabla \mathbf{P}_t f|^2\,d\mu \leqslant -\frac{2}{C_\Phi}u(t)\theta\left(\frac{1}{2}\frac{u(t)}{\|\mathbf{P}_t f\|_\Psi^2}\right).$$

Note that by convexity, $\|\mathbf{P}_t f\|_\Psi \leqslant \|f\|_\Psi = 1$. Since θ is increasing we get

$$u'(t) \leqslant -\frac{2}{C_\Phi}u(t)\theta\left(\frac{u(t)}{2}\right).$$

By integration this gives

$$\int_{u(t)}^{u(0)} \frac{dx}{x\theta(x/2)} \geqslant \frac{2}{C_\Phi}t.$$

Now, since

$$\int_{m(t)}^{\infty} \frac{dx}{x\theta(x/2)} = \frac{2}{C_\Phi}t,$$

we have that $u \leqslant m$ and the result follows by homogeneity. □

Note that m is not explicit in general. However, we can apply the theorem to explicit examples.

Example 38. Let $\alpha \in [1,2]$, $\beta = 2(1-\frac{1}{\alpha}) \in [0,1]$ and define the probability measure on $\mathbb{R}^n$:

$$d\mu_\alpha^n(x) = Z_\alpha^{-n}\exp\left\{-\sum_{i=1}^n |x_i|^\alpha\right\}dx.$$

For any $\gamma \geqslant 1$ define $\Phi_\beta^\gamma(x) = |x|\log(\gamma + |x|)^\beta$. For $x \geqslant 0$ and $\delta \in (0,1)$, let also $\psi(x) = \log(1+x)^\delta$ and $\Psi(x) := x^2/\psi(x)$. We proved in Example 32 that the following Nash-type inequality holds: any f satisfies

$$\mathrm{Var}_{\mu_\alpha^n}(f)^{1+\beta/\delta} \leqslant C \| f - \mu_\alpha^n(f)\|_\Psi^{\beta/\delta} \int |\nabla f|^2 \, d\mu_\alpha^n.$$

On the other hand, for $\theta = x^{\beta/\delta}$,

$$\int_{m(t)}^{\infty} \frac{dx}{x\theta(x/2)} = \frac{2^{\beta/\delta}\delta}{\beta} \frac{1}{m(t)^{\beta/\delta}}.$$

Hence, by Theorem 37

$$\mathrm{Var}_{\mu_\alpha^n}(\mathbf{P}_t f) \leqslant 2\left(\frac{\delta C}{2\beta}\right)^{\delta/\beta} \frac{1}{t^{\delta/\beta}} \| f - \mu(f)\|_\Psi^2.$$

In other words, $\mathbf{P}_t$ is a continuous operator from $\mathbb{L}_\Psi$ onto $\mathbb{L}_2$ with

$$\|\mathbf{P}_t\|_{\mathbb{L}_\Psi \to \mathbb{L}_2} \leqslant 2\left(\frac{\delta C}{2\beta}\right)^{\delta/\beta} \frac{1}{t^{\delta/\beta}}.$$

Example 39. As before, let $\alpha \in [1,2]$, $\beta = 2(1-\frac{1}{\alpha}) \in [0,1]$ and

$$d\mu_\alpha^n(x) = Z_\alpha^{-n} \exp\left\{-\sum_{i=1}^n |x_i|^\alpha\right\} dx$$

be a probability measure on $\mathbb{R}^n$. For any $\gamma \geqslant 1$ define $\Phi_\beta^\gamma(x) = |x|\log(\gamma + |x|)^\beta$. Let also $\widetilde{\psi}(x) = e^{\log(1+x)^\delta} - 1$ for $\delta \in (0,1)$ and $\widetilde{\Psi}(x) = x^2/\widetilde{\psi}(x)$. We proved in Example 32 that there exists a constant C such that for any f,

$$\mathrm{Var}_{\mu_\alpha^n}(f) \log\left(1 + \frac{1}{2} \frac{\mathrm{Var}_{\mu_\alpha^n}(f)}{\| f - \mu_\alpha^n(f)\|_\Psi^2}\right)^{\beta/\delta} \leqslant C \int |\nabla f|^2 \, d\mu_\alpha^n.$$

For $\widetilde{\theta}(x) := \log(1+x)^{\beta/\delta}$, we define $m(t)$ as the unique solution of $2t/C = \int_{m(t)}^\infty dx/[x\widetilde{\theta}(x/2)]$. Now we deal with small values of t, small in such a way that $m(t) \geqslant 2$. For such t's we have

$$\int_{m(t)}^{\infty} \frac{dx}{x\widetilde{\theta}(x/2)} \leqslant \int_{m(t)}^{\infty} \frac{dx}{x \log(x/2)^{\beta/\delta}} = \frac{\delta}{\beta - \delta} \log\left(\frac{m(t)}{2}\right)^{\frac{\delta-\beta}{\delta}}$$

provided that $\delta < \beta$ (if $\delta \geqslant \beta$ then m is not defined!). Hence,

$$m(t) \leqslant \left(\frac{\delta C}{2(\beta-\delta)} \frac{1}{t}\right)^{\frac{\delta}{\beta-\delta}}.$$

Theorem 37 implies that for small values of t,

$$\mathrm{Var}_{\mu_\alpha^n}(\mathbf{P}_t f) \leqslant 2e^{C'/t^{\frac{\delta}{\delta-\beta}}} \big\| f - \mu(f) \big\|_{\widetilde{\Psi}}^2$$

with $C' := (\frac{\delta C}{2(\beta-\delta)})^{\frac{\delta}{\beta-\delta}}$.

Now for completeness we prove a converse of Theorem 37.

Theorem 40. *Let Ψ be a N-function. Assume that for any $t > 0$,*

$$\mathrm{Var}_\mu(\mathbf{P}_t f) \leqslant m(t) \big\| f - \mu(f) \big\|_\Psi^2.$$

Then, for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$

$$\mathrm{Var}_\mu(f) \widetilde{\theta}\left(\frac{1}{2} \frac{\mathrm{Var}_\mu(f)}{\| f - \mu(f) \|_\Psi^2} \right) \leqslant 2 \int |\nabla f|^2 \, d\mu,$$

for

$$\widetilde{\theta}(x) := \sup_{t>0} \frac{1}{t} \log\left(\frac{2x}{m(t)} \right).$$

Proof. We follow [13, Proposition II.2]. Assume that $\mu(f) = 0$ and that $\|f\|_\Psi = 1$. Let $\int_0^\infty \lambda \, dE_\lambda$ be a spectral resolution of $-\mathbf{L}$. Then $\mathbf{P}_t = \int_0^\infty e^{-\lambda t} \, dE_\lambda$. Since

$$\int_0^\infty \frac{\mu(f \cdot dE_\lambda f)}{\mu(f^2)} = 1,$$

Jensen inequality yields

$$\exp\left\{ \int_0^\infty (-2\lambda t) \frac{\mu(f \cdot dE_\lambda f)}{\mu(f^2)} \right\} \leqslant \int_0^\infty e^{-2\lambda t} \frac{\mu(f \cdot dE_\lambda f)}{\mu(f^2)}.$$

This exactly means that

$$\exp\left\{ -2t \frac{\mu(f \cdot (-\mathbf{L}) f)}{\mu(f^2)} \right\} \leqslant \frac{\|\mathbf{P}_t f\|_2}{\mu(f^2)}.$$

Now, using our assumption, $\|\mathbf{P}_t f\|_2 \leqslant m(t)$. Hence

$$\frac{\mu(f^2)}{2t} \log\left(\frac{\mu(f^2)}{m(t)} \right) \leqslant \mu\big(f \cdot (-\mathbf{L}) f \big) = \int |\nabla f|^2 \, d\mu.$$

The result follows by homogeneity and translation invariance of the Dirichlet form. $\square$

Next we recall a result, due to Grigor'yan, which shows the link between $m'(m^{-1})$ and $\widetilde{\theta}$.

We use the following definition (cf. [13]). We say that a differentiable function $m:(0,\infty)\to\mathbb{R}_+^*$ satisfies condition (D) if the derivative of its logarithm has polynomial growth, i.e. $M(t)=-\log m(t)$ is such that

$$M'(u)\geqslant \gamma M'(t)\quad \forall t>0,\ \forall u\in[t,2t],$$

for some $\gamma>0$ (for instance if m behaves like t^{-d} or e^{-ct^δ}, $0\leqslant\delta\leqslant 1$, for t large, it satisfies condition (D)).

Proposition 41. [18] *Let m be a decreasing differentiable bijection of $\mathbb{R}_+^*$ satisfying condition* (D) *for some $\gamma>0$. Then, for all $x>0$,*

$$\widetilde{\theta}(x)=\sup_{t>0}\frac{1}{t}\log\left(\frac{2x}{m(t)}\right)\geqslant -\gamma\frac{m'(m^{-1}(2x))}{x}.$$

The above results imply the following equivalence of the Nash-type inequality and the decay to equilibrium of the semi-group.

Theorem 42. *Let m be a $\mathcal{C}^1$ decreasing bijection of $\mathbb{R}_+^*$ satisfying condition* (D) *with $\gamma>0$. Assume that m' is an increasing bijection from $\mathbb{R}_+^*$ onto $\mathbb{R}_-^*$. Let $\theta(|x|)=-m'(m^{-1}(2|x|))/x$. Let Φ and $\Psi(x)=\frac{x^2}{\psi(|x|)}$ be two N-functions with ψ increasing, satisfying $\psi(0)=0$ and $\lim_{+\infty}\psi=+\infty$. Assume that $\theta=\Phi^{*-1}\circ\Psi\circ\psi^{-1}$.*

Then the following are equivalent:

(i) *for any $t>0$,*

$$\operatorname{Var}_\mu(\mathbf{P}_t f)\leqslant m(t)\left\|f-\mu(f)\right\|_\Phi^2,$$

(ii) *for any smooth function $f:\mathbb{R}^n\to\mathbb{R}$*

$$\operatorname{Var}_\mu(f)\theta\left(\frac{1}{2}\frac{\operatorname{Var}_\mu(f)}{\|f-\mu(f)\|_\Psi^2}\right)\leqslant C_\Phi\int|\nabla f|^2\,d\mu.$$

Moreover (i) $\Rightarrow$ (ii) *with $C_\Phi=2/\gamma$ while* (ii) $\Rightarrow$ (i) *if $C_\Phi=2$.*

Proof. To show that (i) implies (ii) it is enough to apply Theorem 40 and Proposition 41. The second part is a direct application of Theorem 37. □

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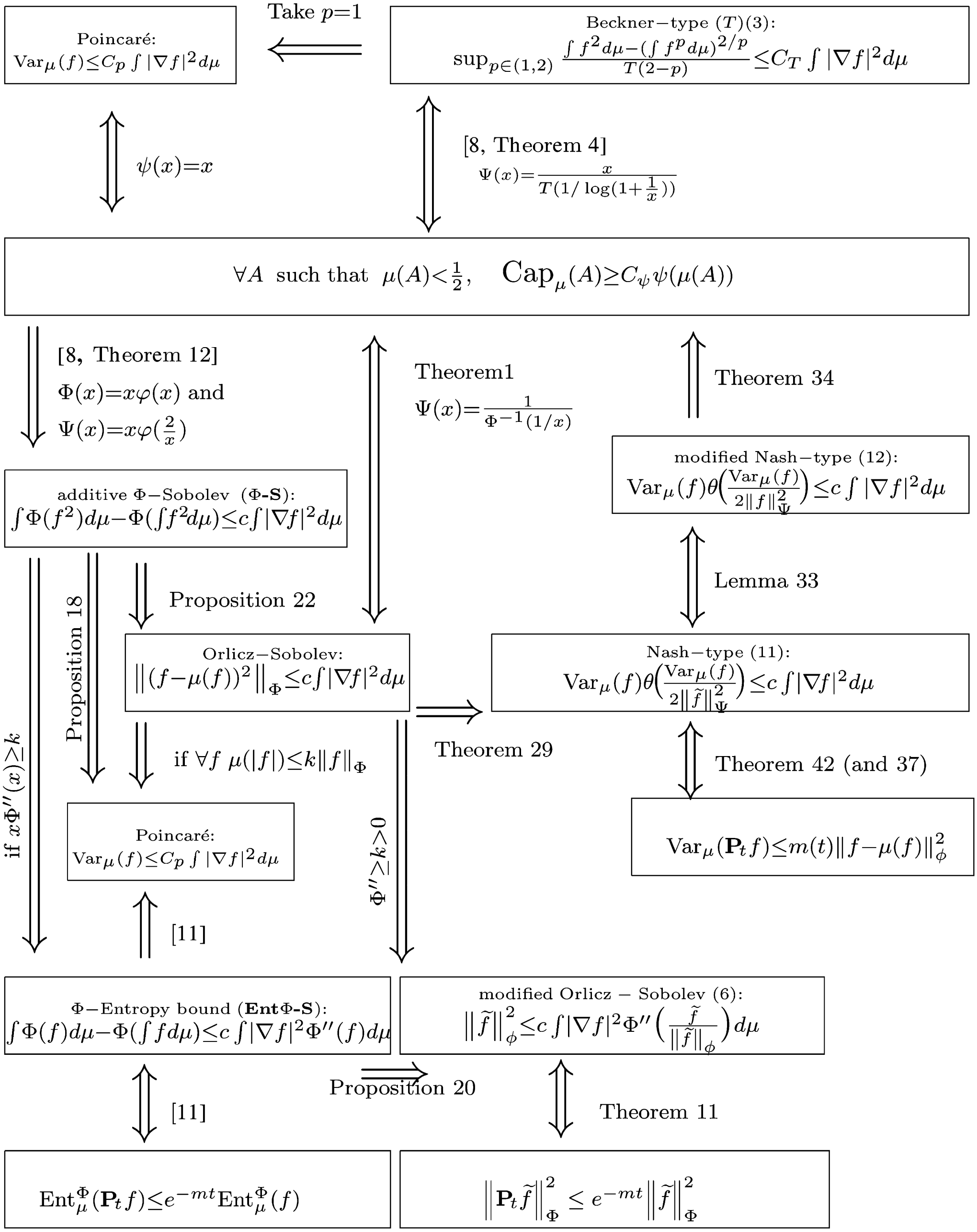

Fig. 1. Implications network. Here $\tilde f := f-\mu(f)$.

Appendix A. Young functions and Orlicz spaces

In this appendix we collect some results on Orlicz spaces. We refer the reader to [25] for demonstrations and complements.

Definition 43 *(Young function).* A function $\Phi : \mathbb{R} \to [0, \infty]$ is a *Young function* if it is convex, even, such that $\Phi(0) = 0$, and $\lim_{x \to +\infty} \Phi(x) = +\infty$.

The Legendre transform Φ^* of Φ defined by

$$\Phi^*(y) = \sup_{x \geqslant 0} \{x|y| - \Phi(x)\}$$

is a lower semi-continuous Young function. It is called the *complementary function* or *conjugate* of Φ.

Among the Young functions, we will consider those continuous with finite values such that $\Phi(x)/x \to \infty$ as $x \to \infty$ (for stability reasons with respect to duality). When additionally $\Phi(x) = 0 \Leftrightarrow x = 0$ and $\Phi'(0_+) = 0$, Φ is called a *N-function*.

For any lower semi-continuous Young function Φ (in particular if Φ has finite values), the conjugate of Φ^* is Φ. The pair (Φ, Ψ) is said to be a *complementary pair* if $\Psi = \Phi^*$ (or equivalently $\Phi = \Psi^*$). When $\Phi(1) + \Phi^*(1) = 1$, the pair (Φ, Φ^*) is said to be *normalized*. The conjugate of an N-function is an N-function. Let Φ be an N-function. Then, for any $a > 0$,

$$a < \Phi^{-1}(a)\big(\Phi^*\big)^{-1}(a) \leqslant 2a. \tag{13}$$

The simplest example of N-function is $\Phi(x) = \frac{|x|^p}{p}$, $p > 1$, in which case, $\Phi^*(x) = \frac{|x|^q}{q}$, with $1/p + 1/q = 1$. The function $\Phi(x) = |x|^\alpha \ln(1 + |x|)^\beta$ is also a Young function for $\alpha \geqslant 1$ and $\beta \geqslant 0$ and an N-function when $\alpha > 1$ or $\beta > 0$.

Now let $(\mathcal{X}, \mu)$ be a measurable space, and Φ a Young function. The space

$$\mathbb{L}_\Phi(\mu) = \left\{ f : \mathcal{X} \to \mathbb{R} \text{ measurable; } \exists \alpha > 0, \int_{\mathcal{X}} \Phi(\alpha f) < +\infty \right\}$$

is called the *Orlicz space* associated to Φ. When $\Phi(x) = |x|^p$, then $\mathbb{L}_\Phi(\mu)$ is the standard Lebesgue space $\mathbb{L}_p(\mu)$.

There exist two equivalent norms which give to $\mathbb{L}_\Phi(\mu)$ a structure of Banach space. Namely, *Luxembourg norm*

$$\|f\|_\Phi = \inf\left\{ \lambda > 0; \int_{\mathcal{X}} \Phi\left(\frac{f}{\lambda}\right) d\mu \leqslant 1 \right\}$$

and *the Orlicz norm*

$$N_\Phi(f) = \sup\left\{ \int_{\mathcal{X}} |fg| \, d\mu; \int_{\mathcal{X}} \Phi^*(g) \, d\mu \leqslant 1 \right\}.$$

Note that we invert the notation with respect to [25]. We will use the notation $\mathcal{G}_\Phi$, or more simply $\mathcal{G}$ when no confusion, the set $\mathcal{G}_\Phi = \{|g|: \int_{\mathcal{X}} \Phi^*(g) \, d\mu \leqslant 1\}$. Note in particular that $\mathcal{G}_\Phi$ is a space of non-negative functions. Moreover,

$$\|f\|_\Phi \leqslant N_\Phi(f) \leqslant 2\|f\|_\Phi. \tag{14}$$

By definition of the norm and the previous result, it is easy to see that for any measurable subset A of $\mathcal{X}$,

$$\|\mathbb{1}_A\|_\Phi = \frac{1}{\Phi^{-1}\left(\frac{1}{\mu(A)}\right)}. \tag{15}$$

Then, the following result generalizes Hölder inequality. Let Φ_1, Φ_2 and Φ_3 be three Young functions satisfying for all $x \geqslant 0$, $\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leqslant \Phi_3^{-1}(x)$. Then, for any $(f, g) \in \mathbb{L}_{\Phi_1}(\mu) \times \mathbb{L}_{\Phi_2}(\mu)$,

$$\|fg\|_{\Phi_3} \leqslant 2\|f\|_{\Phi_1}\|g\|_{\Phi_2}. \tag{16}$$

In particular, when $\Phi_3(x) = |x|$, we get $\int_{\mathcal{X}} |fg|\, d\mu \leqslant 2\|f\|_{\Phi_1}\|g\|_{\Phi_2}$. In the case of complementary pairs of Young functions, we have the following more precise result, see [25, Section 3, Proposition 1]:

$$\int_{\mathcal{X}} |fg|\, d\mu \leqslant 2\|f\|_\Phi \|g\|_{\Phi^*}. \tag{17}$$

Finally, for any constant $c > 0$, it is easy to see that for any function f,

$$c\|f\|_{\Phi(\cdot/c)} = \|f\|_\Phi. \tag{18}$$

A.1. Comparison of norms

Let us notice that any Young function Φ satisfies $|x| = O(\Phi(x))$ as x goes to ∞. It leads to the following lemma.

Lemma 44. *Any Orlicz space may be continuously embedded in $\mathbb{L}_1$. More precisely, let D and τ in $(0, \infty)$ such that $|x| \leqslant \tau\Phi(x)$ for any $|x| \geqslant D$. Then, for any $f \in \mathbb{L}_\Phi$,*

$$\|f\|_1 \leqslant (D + \tau)\|f\|_\Phi. \tag{19}$$

Consequently, if Φ and Ψ are two Young functions satisfying, for some constants $A, B \geqslant 0$, $\Phi(x) \leqslant A|x| + B\Psi(x)$, then

$$\|f\|_\Phi \leqslant \max\bigl(1, A\|\mathrm{Id}\|_{\mathbb{L}_\Psi \to \mathbb{L}_1} + B\bigr)\|f\|_\Psi. \tag{20}$$

Remark 45. When $\Phi(x)/x \to \infty$ as $x \to \infty$, we may choose $\tau = 1$ or any other positive constant. We get in particular the estimate

$$\bigl\|\mu(f)^2\bigr\|_\Phi \leqslant (D + 1)\|\mathbb{1}\|_\Phi \bigl\|f^2\bigr\|_\Phi, \tag{21}$$

where D is such that $|x| \leqslant \Phi(x)$ for any $|x| \geqslant D$.

Proof of Lemma 44. Let $f \in \mathbb{L}_\Phi(\mu)$. We may assume by homogeneity that $\|f\|_\Phi = 1$. Then $\int \Phi(f)\,d\mu = 1$ and so

$$\int |f|\,d\mu = \int\limits_{\{|f|\leqslant D\}} |f|\,d\mu + \int\limits_{\{|f|\geqslant D\}} |f|\,d\mu$$
$$\leqslant D\mu\big(|f| \leqslant D\big) + \tau \int\limits_{\{|f|\geqslant D\}} \Phi(f)\,d\mu \leqslant D + \tau.$$

As for bound (19), assume now that $\|f\|_\Psi = 1$ and hence $\int \Psi(f)\,d\mu = 1$ as well. For any $\lambda \geqslant 1$,

$$\int \Phi\left(\frac{f}{\lambda}\right) d\mu \leqslant \frac{A}{\lambda}\|f\|_1 + B \int \Psi\left(\frac{f}{\lambda}\right) d\mu$$
$$\leqslant \frac{A}{\lambda}\|\mathrm{Id}\|_{\mathbb{L}_\Psi \to \mathbb{L}_1}\|f\|_\Psi + \frac{B}{\lambda}\int \Psi(f)\,d\mu \leqslant 1$$

provided $\lambda \geqslant A\|\mathrm{Id}\|_{\mathbb{L}_\Psi \to \mathbb{L}_1} + B$. Note that for the second inequality we used convexity of Ψ. □

References

[1] C. Ané, S. Blachère, D. Chafai, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, G. Scheffer, Sur les inégalités de Sobolev logarithmiques, Panoramas et Synthèses, vol. 10, Soc. Math. France, Paris, 2000.

[2] D. Bakry, L'hypercontractivité et son utilisation en théorie des semi groupes, in: École d'été de Probabilités de Saint-Flour, in: Lecture Notes in Math., vol. 1581, Springer, Berlin, 1994, pp. 1–114.

[3] D. Bakry, T. Coulhon, M. Ledoux, L. Saloff-Coste, Sobolev inequalities in disguise, Indiana Univ. Math. J. 44 (1995) 1033–1074.

[4] K. Ball, E.A. Carlen, E.H. Lieb, Sharp uniform convexity and smoothness inequalities for trace norms, Invent. Math. 115 (3) (1994) 463–482.

[5] F. Barthe, C. Roberto, Sobolev inequalities for probability measures on the real line, Studia Math. 159 (3) (2003) 481–497, dedicated to Professor Aleksander Pełczyński on the occasion of his 70th birthday (in Polish).

[6] F. Barthe, P. Cattiaux, C. Roberto, Concentration for independent random variables with heavy tails, AMRX Appl. Math. Res. Express (2) (2005) 39–60.

[7] F. Barthe, P. Cattiaux, C. Roberto, Isoperimetry between exponential and Gaussian, preprint, 2005.

[8] F. Barthe, P. Cattiaux, C. Roberto, Interpolated inequalities between exponential and Gaussian, Orlicz hypercontarctivity and isoperimetry, Rev. Mat. Iberoamericana, 2006, in press.

[9] W. Beckner, A generalized Poincaré inequality for Gaussian measures, Proc. Amer. Math. Soc. 105 (1989) 397–400.

[10] S.G. Bobkov, F. Götze, Exponential integrability and transportation cost related to logarithmic Sobolev inequalities, J. Funct. Anal. 163 (1999) 1–28.

[11] D. Chafaï, Entropies, convexity, and functional inequalities: On Φ-entropies and Φ-Sobolev inequalities, J. Math. Kyoto Univ. 44 (2) (2004) 325–363.

[12] M.F. Chen, Variational formulas and explicit bounds of Poincaré-type inequalities for one-dimensional processes, in: Probability, Statistics and Their Applications: Papers in Honor of Rabi Bhattacharya, in: IMS Lecture Notes Monogr. Ser., vol. 41, Inst. Math. Statist., Beachwood, OH, 2003, pp. 81–95.

[13] T. Coulhon, Ultracontractivity and Nash type inequalities, J. Funct. Anal. 141 (2) (1996) 510–539.

[14] T. Coulhon, L. Saloff-Coste, N.Th. Varopoulos, Analysis and Geometry on Groups, Cambridge Tracts in Math., vol. 100, Cambridge Univ. Press, Cambridge, 1992.

[15] E.B. Davies, Heat Kernels and Spectral Theory, Cambridge Univ. Press, Cambridge, 1989.

[16] P. Fougères, C. Roberto, B. Zegarliński, Sub-Gaussian measures and associated semilinear problems, 2006, in preparation.

[17] I. Gentil, A. Guillin, L. Miclo, Modified logarithmic Sobolev inequalities and transportation inequalities, Probab. Theory Related Fields 133 (3) (2005) 409–436.

[18] A. Grigor'yan, Heat kernel on a manifold with a local Harnack inequality, Comm. Anal. Geom. 2 (1) (1994) 111–138.
[19] L. Gross, Logarithmic Sobolev inequalities and contractivity properties of semi-groups, in: G. Dell'Antonio, U. Mosco (Eds.), Dirichlet Forms, in: Lecture Notes in Math., vol. 1563, Springer, Berlin, 1993, pp. 54–88.
[20] A. Guionnet, B. Zegarliński, Lectures on logarithmic Sobolev inequalities, in: Séminaire de Probabilités, XXXVI, in: Lecture Notes in Math., vol. 1801, Springer, Berlin, 2003, pp. 1–134.
[21] R. Latała, K. Oleszkiewicz, Between Sobolev and Poincaré, in: Geometric Aspects of Functional Analysis, in: Lecture Notes in Math., vol. 1745, Springer, Berlin, 2000, pp. 147–168.
[22] M. Ledoux, Concentration of measure and logarithmic Sobolev inequalities, in: Séminaire de Probabilités, XXXIII, in: Lecture Notes in Math., vol. 1709, Springer, Berlin, 1999, pp. 120–216.
[23] V.G. Maz'ja, Sobolev Spaces, Springer Ser. Soviet Math., Springer, Berlin, 1985.
[24] J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. J. Math. 80 (1958) 931–954.
[25] M.M. Rao, Z.D. Ren, Theory of Orlicz Spaces, Dekker, New York, 1991.
[26] O.S. Rothaus, Analytic inequalities, isoperimetric inequalities and logarithmic Sobolev inequalities, J. Funct. Anal. 64 (1985) 296–313.
[27] C. Roberto, A path method, unpublished note, 2003.
[28] G. Royer, Une initiation aux inégalités de Sobolev logarithmiques, Soc. Math. France, Paris, 1999.
[29] L. Saloff-Coste, Aspects of Sobolev-Type Inequalities, London Math. Soc. Lecture Note Ser., vol. 289, Cambridge Univ. Press, Cambridge, 2002.
[30] F.Y. Wang, A generalization of Poincaré and log-Sobolev inequalities, Potential Anal. 22 (1) (2005) 1–15.
[31] B. Zegarliński, Entropy bounds for Gibbs measures with non-Gaussian tails, J. Funct. Anal. 187 (2) (2001) 368–395.