SLOW DECAY OF GIBBS MEASURES WITH HEAVY TAILS

CYRIL ROBERTO

Abstract. We consider Glauber dynamics reversible with respect to Gibbs measures with heavy tails in the case when spins are unbounded. The interactions are bounded and of finite range. The self potential enters into two classes of measures, κ-concave probability measures and sub-exponential laws, for which it is known that no exponential decay can occur. Using coercive inequalities we prove that, for κ-concave probability measures, the associated infinite volume semi-group decays to equilibrium polynomially and stretched exponentially for sub-exponential laws. This improves and extends previous results by Bobkov and Zegarlinski.

1. Introduction

In the last decades, the study of functional inequalities deserved a lot of attention, not only on the side of theoretical probability and analysis, but also in statistical mechanics. This is due to the numerous fields of application: differential geometry, analysis of p.d.e., concentration of measure phenomenon, isoperimetry, trends to equilibrium in deterministic and stochastic evolutions...

The most popular functional inequalities are Poincaré and log Sobolev. Both are now well understood in many situations. We refer to [1], [2], [20], [23], [24], [27], [29], [34], [36] for an introduction.

Very recently, various generalisations of Poincaré and log Sobolev inequalities were introduced and studied by probabilists and analysts. Let us mention weak Poincaré or super Poincaré inequalities, Orlicz-Poincaré or Orlicz-Sobolev inequalities, F-Sobolev inequalities, weighted Poincaré or weighted log Sobolev inequalities, modified log-Sobolev inequalities etc. To give a complete picture of the literature is out of reach. See [13], [21], [38], [12], [18], [7], [4], [6] (and references therein) for some of the most recent publications.

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Few of these advances have been used so far in statistical mechanics, at the notable exception of [13], [40].

On the other hand, in the statistical mechanics community, progress has been done in the study of Poincaré and log Sobolev inequalities for large classes of models coming from physics. Again, to give an updated list of publications is out of reach. Let us mention [26], [25], [8], [16], [15, 17].

This paper intends to use advances from both communities in order to improve and extend some results of Bobkov and Zegarlinski [13] on the decay to equilibrium of some unbounded spins systems. We believe that the techniques coming more specifically from one community should be more largely exploited by the other one. This paper is one step in this direction.

If a lot of results are known for log-concave probability measures, not so much has been proved for measures with heavy tails (let us mention [33], [3], [12], [11], [13], [37]). In this paper the focus is on such measures (informally with tails larger than exponential) and our aim is to prove decay to equilibrium of unbounded spin systems in infinite dimension.

Now we introduce and discuss one of the main tools we shall use, namely, the weak Poincaré inequality. Consider for simplicity a one dimensional probability measure $d\nu = Z_V^{-1} e^{-V} d\lambda$ with $Z_V := \int e^{-V} d\lambda$ that we assume to be finite, where $d\lambda$ is the Lebesgue measure on $\mathbb{R}$. Then a weak Poincaré inequality asserts that

$$\mathbf{Var}_\nu(f) \leq \beta(s) \int (f')^2 d\nu + s\mathrm{Osc}(f)^2 \qquad \forall f : \mathbb{R} \to \mathbb{R}, \forall s > 0 \tag{1}$$

where $\beta : (0,\infty) \to \mathbb{R}$ is some rate function and $\mathrm{Osc}(f) = \sup f - \inf f$ is the *oscillation* of f. When $\lim_{s\to 0} \beta(s) = \beta_0 < \infty$, Inequality (1) reduces to the usual Poincaré inequality

$$\mathbf{Var}_\nu(f) \leq \beta_0 \int (f')^2 d\nu.$$

Most of the information is encoded in the behaviour of β near the origin. Moreover, since $\mathbf{Var}_\nu(f) \leq \frac{1}{4}\mathrm{Osc}(f)^2$, only the values $s \in (0, 1/4)$ are relevant.

The weak Poincaré inequality (1) was introduced by Röckner and Wang [33]. However, inequalities with a free parameter have a long story in analysis, see *e.g.* [32], [20], [28], [10], [9].

Using capacity techniques (Hardy type inequalities [30], [31], [5]) the best possible rate function for $V_p(x) = |x|^p$, $p \in (0,1)$ was computed in [3]: $\beta(s) = c_p \log\left(\frac{2}{s\wedge 2}\right)^{2(\frac{1}{p}-1)}$. Also, it is known in this case (see [1, Chapter 5]) that ν does not satisfies the usual Poincaré inequality. Equivalently there is no exponential decay to equilibrium in the $\mathbb{L}^2(\nu)$-norm of the semigroup $(\mathbf{S}_t)_{t\geq 0}$ associated to the one dimensional generator $L = \frac{d^2}{du^2} - V_p' \cdot \frac{d}{du}$.

However, by standard differentiation (see Section 5), it holds

$$\mathbf{Var}_\nu(\mathbf{S}_t f) \leq \frac{1}{c} e^{-ct^{p/(2-p)}} \mathrm{Osc}(f)^2 \qquad \forall f, \forall t > 0$$

for some constant $c = c(p)$ (the lack of smoothness of V_p at 0 is just a small nuisance that one can easily handle).

In [13], using weak Poincaré inequalities, Bobkov and Zegarlinski proved that some Gibbs measures in infinite volume with self potential V_p also satisfies a stretched exponential decay as above, but with a worst exponent. In this paper we shall prove the correct stretched exponential decay with the exponent $t^{p/(2-p)}$ not only for the potential V_p but also for a larger class of potentials of sub-exponential type. Moreover, our approach, based on the bisection technique [29] together with the quasi factorisation property of the variance [8], applies also to potentials of the type $V = (1+\alpha)\log(1+|u|)$, $\alpha > 0$ leading to Cauchy type distributions and polynomial decay to equilibrium.

Note that there is a difficulty here with respect to the usual Poincaré and log Sobolev inequalities. Namely, weak Poincaré inequalities do not tensorise in general. In turn, there is no hope for a dimension free analysis, and one has to take care of the growth of the dimension (see Section 5 for a discussion about this).

The paper is organised as follows. The notations and the setting, in particular the Hamiltonian, the Gibbs measures and the Glauber dynamics, are introduced in the next section. Then we state our main results about the decay to equilibrium of the infinite volume system. In Section 4 we establish one of the tool, known as finite speed of propagation, of the proofs. Section 5 is dedicated to the weak Poincaré inequalities: we recall some known facts and prove some perturbation properties. In Section 6 we derive from the weak Poincaré inequalities of Section 5 a finite volume decay to equilibrium, which is the second ingredient of the proofs of the main theorems, which are given in the final section.

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2. Notations and setting.

2.1. **The Configuration space.** The *configuration space* that we consider is $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ where $d \geq 1$ is an integer. Given $\Lambda \Subset \mathbb{Z}^d$ (*i.e.* Λ is a finite subset of $\mathbb{Z}^d$), we shall also deal with $\Omega_\Lambda = \mathbb{R}^\Lambda$. For any *configuration* $\sigma \in \Omega$, any site $x \in \mathbb{Z}^d$ and any $\Lambda \Subset \mathbb{Z}^d$, σ_x stands for the value of the configuration (or the spin) at x while σ_Λ is the configuration σ restricted to Λ. We denote by $\mathcal{B}_\Lambda$ the σ-algebra of all Borel sets of Ω_Λ.

A function which is measurable with respect to $\mathcal{B}_\Lambda$ with $\Lambda \Subset \mathbb{Z}^d$ is said to be *local*. Then, the *support* of f, denoted by Δ_f, is the smallest set Λ for which this property holds. For any smooth local function, we set $|||f||| = \sum_{x\in\mathbb{Z}^d} \|\nabla_x f\|_\infty$ where $\|\cdot\|_\infty$ is the sup norm and ∇_x denotes the derivative with respect to the variable σ_x.

The Euclidean distance on $\mathbb{Z}^d$ is denoted by d. With a slight abuse of notation, for $a, b \in \mathbb{Z}$, $a < b$, we shall often set $[a, b]$ for $[a, b] \cap \mathbb{Z}$.

2.2. **The Hamiltonian and the potentials.** For any $\Lambda \Subset \mathbb{Z}^d$ the *Hamiltonian* $H_\Lambda : \Omega \to \mathbb{R}$ is given by

$$H_\Lambda(\sigma) = \sum_{x\in\Lambda} \left(V(\sigma_x) + \frac{1}{T} \sum_{y:d(x,y)\le r} W(\sigma_x - \sigma_y) \right) \qquad \forall \sigma \in \Omega \tag{2}$$

where $V, W : \mathbb{R} \to \mathbb{R}$ correspond respectively to the *self potential* and the *interaction potential* (pair potential). The parameter $T \in (0, \infty)$ is the *temperature* and $r \in \mathbb{N} \setminus \{0\}$ is the *range* of the interaction. For $\sigma, \tau \in \Omega$, we also let $H_\Lambda^\tau(\sigma) := H_\Lambda(\sigma_\Lambda \tau_{\Lambda^c})$, where $\sigma_\Lambda \tau_{\Lambda^c}$ stands for the configuration equal to σ on Λ and to τ on Λ^c (the complement of Λ), and τ is called the *boundary condition.*

Remark 1. One could consider more general Hamiltonian H_Λ with *e.g.* infinite range interactions and/or interaction potentials W depending on the values of more than two spins and/or depending on the sites etc. All the results below would hold in those more general settings, under specific assumptions. We make the choice of dealing with the Hamiltonian (2) for simplicity and for clarity's sake.

Now we describe our assumptions on V and W. We collect in Hypothesis $(H1)$ some smoothness conditions on V and W.

Hypothesis (H1)**.** Given a self potential $V : \mathbb{R} \to \mathbb{R}$ and an interaction potential $W : \mathbb{R} \to \mathbb{R}$, we say that V and W satisfy Hypothesis $(H1)$ if

- V is $\mathcal{C}^1$ and $\int_\mathbb{R} e^{-V} d\lambda < \infty$;
- W is twice differentiable, $\|W\|_\infty < \infty$, $\|W'\|_\infty < \infty$ and $\|W''\|_\infty < \infty$.

The second assumption on V guarantees that $d\nu = Z_V^{-1} e^{-V} d\lambda$ (with $Z_V = \int e^{-V} d\lambda$) defines a probability measure. The smoothness assumption about V will be needed when the Glauber dynamics is defined.

On the other hand, the assumptions about W will be useful to define the infinite volume Gibbs measure.

More specifically, the self potentials $V : \mathbb{R} \to \mathbb{R}$ that we shall consider enter into two classes of examples:

— *κ-concave probability measures.* We say that the probability measure $d\nu = Z_V^{-1} e^{-V} d\lambda$ is *κ-concave*, with $\kappa = -1/\alpha$, if $e^{V/(1+\alpha)}$ is convex and $\alpha > 0$. This is equivalent to consider $V = (1+\alpha)\log U$ with $U : \mathbb{R} \to (0, \infty)$ convex. The measure ν reads as

$$d\nu = \frac{1}{Z_\alpha U^{1+\alpha}} d\lambda$$

and the prototype is the Cauchy distribution ($U(u) = 1 + |u|$ or equivalently $V(u) = (1+\alpha)\log(1+|u|)$)

$$d\nu(u) = \frac{\alpha}{2\,(1+|u|)^{1+\alpha}} d\lambda(u). \tag{3}$$

Such class of measures was introduced by Borell [14] in more general settings. See [11] for a comprehensive introduction.

— *Sub-exponential like laws.* We say that the probability measure is a *sub-exponential like law* if $V = U^p$ with $p > 0$ and $U : \mathbb{R} \to (0, \infty)$ convex. The corresponding probability measure $d\nu = Z_V^{-1} e^{-V} d\lambda$ on $\mathbb{R}$ (with $Z_V = \int e^{-V} d\lambda$) reads as

$$d\nu = \frac{e^{-U^p}}{Z_p} d\lambda$$

and the prototype is the sub-exponential law ($U(u) = |u|$ or equivalently $V(u) = |u|^p$):

$$d\nu(u) = \frac{e^{-|u|^p}}{2\Gamma(1+\frac{1}{p})} d\lambda(u) \tag{4}$$

for $p \in (0, 1]$.

In both examples the measure ν has tails larger than exponential. For the sub-exponential law and the Cauchy distribution, in order to fulfil Hypothesis $(H1)$, one can consider *e.g.* $U(u) = \sqrt{1+u^2}$ to avoid differentiability trouble at 0.

2.3. **The Gibbs measures.** The *finite volume Gibbs measure* in $\Lambda \Subset \mathbb{Z}^d$ at temperature T and boundary condition τ is given by

$$d\mu_\Lambda^\tau(\sigma) = (Z_\Lambda^\tau)^{-1} \exp\{-H_\Lambda^\tau(\sigma)\} \prod_{x\in\Lambda} d\lambda(\sigma_x) \times \delta_{\Lambda^c,\tau}(\sigma) \tag{5}$$

where $\delta_{\Lambda^c,\tau}$ is the Dirac probability measure on Ω_{Λ^c} which gives mass 1 to the configuration τ and Z_Λ^τ is the proper normalisation factor. We denote with $\mu_\Lambda^\tau(f)$ the expectation of f with respect to μ_Λ^τ, while $\mu_\Lambda(f)$ denotes the function $\tau \mapsto \mu_\Lambda^\tau(f)$. For any Borel set $X \subset \Omega$ we set $\mu_\Lambda(X) := \mu_\Lambda(\mathbb{1}_X)$, where $\mathbb{1}_X$ is the indicator function of X. We write $\mu_\Lambda(f, g)$ to denote the covariance (with respect to μ_Λ) of f and g and $\mathbf{Var}_{\mu_\Lambda}(f) = \mu_\Lambda(f, f)$ for the variance of f under μ_Λ.

The family of measures (5) satisfies the DLR compatibility conditions: for all Borel sets $X \subset \Omega$

$$\mu_\Delta(\mu_\Lambda(X)) = \mu_\Delta(X) \qquad\qquad \forall \Lambda, \Delta \Subset \mathbb{Z}^d \text{ such that } \Lambda \subset \Delta.$$

If in addition of Hypothesis $(H1)$, T is large enough, then (see [22, Proposition (8.8)]) the Dobrushin's uniqueness condition is satisfied. Hence there exists a unique infinite volume Gibbs measure μ satisfying $\mu(\mu_\Lambda(X)) = \mu(X)$ for any $\Lambda \Subset \mathbb{Z}^d$ and any Borel set $X \subset \Omega$. Moreover (see [22, Remark (8.26) together with Corollary (8.32)]) (if T is large enough) there exist constants $D = D(r, T, \|W\|_\infty)$ and $m = m(r, T, \|W\|_\infty)$ such that for any $\Lambda \Subset \mathbb{Z}^d$, it holds

$$|\mu_\Lambda^\tau(f, g)| \leq D|\Delta_f||\Delta_g|\|f\|_\infty \|g\|_\infty e^{-md(\Delta_f, \Delta_g)} \tag{6}$$

for any boundary condition τ, any bounded local functions f and g with support Δ_f and Δ_g satisfying $\Delta_f, \Delta_g \subset \Lambda$. Here $|\cdot|$ stands for the Lebesgue measure on $\mathbb{Z}^d$. Inequality (6) is known as the *strong mixing condition.* Note that the former argument (that leads to the uniqueness of μ and Inequality (6)) does not depend on the self potential V.

In the sequel, we always assume the following:

Hypothesis (H2)**.** Given the potentials V and W and the temperature T, we say that Hypothesis $(H2)$ is satisfied if there exists a unique infinite volume Gibbs measure μ and if the strong mixing condition (6) is satisfied.

In particular, by the argument above, if $(H1)$ is satisfied then $(H2)$ is satisfied too as soon as T is large enough.

2.4. **The dynamics.** The dynamics that we consider is of Glauber type. For any $\Lambda \Subset \mathbb{Z}^d$, any boundary condition $\tau \in \Omega$, let $(\mathbf{P}_t^{\Lambda,\tau})_{t \geq 0}$ be the Markov semi-group associated to the generator

$$\mathbf{L}_\Lambda^\tau = \sum_{x \in \Lambda} \Delta_x - \sum_{x \in \Lambda} \nabla_x H_\Lambda^\tau \cdot \nabla_x \tag{7}$$

where ∇_x and Δ_x are respectively the first and second partial derivative with respect to the variable σ_x. When there is no confusion, we shall drop the superscript τ in the definition and write simply $\mathbf{P}_t^\Lambda$ and $\mathbf{L}_\Lambda$. The generator $\mathbf{L}_\Lambda^\tau$ is symmetric in $\mathbb{L}^2(\mu_\Lambda^\tau)$. On the other hand, $\mathbf{P}_t^{\Lambda,\tau}$ is a contraction on $\mathbb{L}^p(\mu_\Lambda^\tau)$ for all $p \in [1, \infty]$. Also, since $\mathbf{L}_\Lambda^\tau$ acts only on the variables σ_x, $x \in \Lambda$, if f satisfies $\Delta_f \subset \Lambda$, the function $\mathbf{P}_t^{\Lambda,\tau} f : \sigma \to \mathbf{P}_t^{\Lambda,\tau} f(\sigma_\Lambda \tau_{\Lambda^c})$ satisfies $\Delta_{\mathbf{P}_t^{\Lambda,\tau} f} \subset \Lambda$ too. In particular, $\nabla_x \mathbf{P}_t^{\Lambda,\tau} f = 0$ for $x \notin \Lambda$.

For any $\Lambda \Subset \mathbb{Z}^d$ we denote by $\mathcal{D}_\Lambda^\tau$ (we shall also often drop the superscript τ) the *Dirichlet form* associated to $\mathbf{L}_\Lambda^\tau$, defined by

$$\mathcal{D}_\Lambda^\tau(f) = \frac{1}{2} \sum_{x \in \Lambda} \mu_\Lambda^\tau \left(|\nabla_x f|^2 \right) \tag{8}$$

for all sufficiently smooth f.

If $(H1)$ and $(H2)$ are satisfied, it is possible to construct the infinite volume semi-group by

$$\mathbf{P}_t = \lim_{\Lambda \to \mathbb{Z}^d} \mathbf{P}_t^\Lambda \tag{9}$$

on the space of bounded smooth functions (in particular $(H1)$ and $(H2)$ guarantee that the limit above does not depend on the boundary condition). The associated infinite volume generator will be denoted by $\mathbf{L}$. The proof of Equation (9) can be found in [24, Theorem 8.2] in the setting of finite or compact configuration spaces. However, it works also in our setting, with very minor modifications, see [24, Exercice 8.3].

3. The results.

In this section we state our main results on the two classes of examples of self potential V introduced in Section 2.2. We assume that Hypothesis $(H1)$ and $(H2)$ are satisfied so as to guarantee the existence of a unique infinite volume Gibbs measure μ. Also, we set $\|f\| = |||f||| + \|f\|_\infty$.

Theorem 2 (κ-concave self potentials)**.** *Let $U : \mathbb{R} \to (0, \infty)$ be a convex function and $\alpha > 0$. Set $V = (1 + \alpha) \log U$. Let $W : \mathbb{R} \to \mathbb{R}$. Assume $(H1)$ and $(H2)$. Fix and integer $\ell \geq 1$. Then, for any $\varepsilon \in (0, 1)$, there exists a constant C depending on ε, ℓ, α, T, r, d, $\|W\|_\infty$, $\|W'\|_\infty$ and $\|W''\|_\infty$ such that for all bounded smooth local functions $f : \Omega \to \mathbb{R}$ with $|\Delta_f| \leq \ell^d$, one has*

$$\mathbf{Var}_\mu(\mathbf{P}_t f) \leq \frac{C}{t^{\frac{\alpha}{2} - d(1+\varepsilon)}} \|f - \mu(f)\|^2 \qquad \forall t > 0. \tag{10}$$

Remark 3. The spurious term $d(1 + \varepsilon)$ is *a priori* technical and we believe that the correct decay should be with the exponent $\alpha/2$ as in the one dimensional case (see Section 5). But on the other hand, it could be that the very heavy tails of the Cauchy type distributions slow down the dynamics with some unattended phenomenon (that we have not been able to catch).

Observe also that we obtain a polynomial decay only for $\alpha > 2d$. For $\alpha \leq 2d$, the previous bound is useless since $\mathbf{P}_t$ is a contraction: we already know that $\mathbf{Var}_\mu(\mathbf{P}_t f) \leq \|f - \mu(f)\|_\infty^2$.

Similarly, we have for sub-exponential self-potentials:

Theorem 4 (Sub-exponential self potentials)**.** *Let $U : \mathbb{R} \to (0, \infty)$ be a convex function and $p \in (0, 1)$. Set $V = |U|^p$. Let $W : \mathbb{R} \to \mathbb{R}$. Assume $(H1)$ and $(H2)$. Fix an integer $\ell \geq 1$. Then, there exists a constant c depending on p, ℓ, T, r, d, $\|W\|_\infty$, $\|W'\|_\infty$ and $\|W''\|_\infty$ such that for all bounded smooth local functions $f : \Omega \to \mathbb{R}$ with $|\Delta_f| \leq \ell^d$,*

$$\mathbf{Var}_\mu(\mathbf{P}_t f) \leq \frac{1}{c} e^{-ct^{p/(2-p)}} \|f - \mu(f)\|^2 \qquad \forall t > 0. \tag{11}$$

The proof of Theorem 2 and Theorem 4 is postponed to Section 7. It will be achieved in various steps. Let us explain the main strategy.

We first reduce the question to an estimate on the decay of the semi-group in a finite volume. This will be achieved by proving the following property known as *finite speed of propagation*: as soon as $L \gg t$, we have

$$\sup_\tau \|\mathbf{P}_t f - \mathbf{P}_t^{\Lambda,\tau} f\|_\infty \leq \frac{|||f|||}{c} e^{-ct}$$

for some constant $c > 0$ and $\Lambda = [-L, L]^d$. This step is proved in Proposition 5 in the next section.

By the previous bound one is reduced to study $\mathbf{Var}_{\mu_\Lambda^\tau}\left(\mathbf{P}_t^{\Lambda,\tau} f\right)$. Here we use the appropriate weak Poincaré inequality to obtain an estimate of the type

$$\mathbf{Var}_{\mu_\Lambda^\tau}\left(\mathbf{P}_t^{\Lambda,\tau} f\right) \leq \gamma(|\Lambda|, t)\|f - \mu_\Lambda^\tau(f)\|^2$$

for some explicit function γ that depends on the self potential V. This is explained in Section 5 and Section 6.

4. Finite speed of propagation

This section is dedicated to the proof of the so-called *finite speed of propagation.* This result is somehow standard. Nevertheless we give the proof for completeness.

We use the definition of the finite volume and infinite volume Markov semi-groups $(\mathbf{P}_t^{\Lambda,\tau})_{t\geq 0}$ and $(\mathbf{P}_t)_{t\geq 0}$ and of the norm $|||\cdot|||$.

Proposition 5 (Finite speed of propagation)**.** *Assume* $(H1)$ *and* $(H2)$*. Fix and integer* $\ell \geq 1$*. Then, for any smooth local function* f *with support* $\Delta_f \subset [-\ell, \ell]^d$*, any* L *multiple of* r*, any boundary condition* $\tau \in \Omega$*, one has*

$$\|\mathbf{P}_t f - \mathbf{P}_t^{\Lambda,\tau} f\|_\infty \leq C|||f|||\left(\frac{C't}{L}\right)^{C''L} e^{Ct} \qquad \forall t > 0$$

with $\Lambda = [-L, L]^d$*, for some constant* $C, C', C'' > 0$ *depending only on* r*,* d*,* $\|W'\|_\infty$*,* $\|W''\|_\infty$ *and* ℓ*.*

Remark 6. Note that this bound is particularly interesting when $L \gg t$.

Proof. We follow [39]. Fix $t > 0$, $\Lambda = [-L, L]^d \Subset \mathbb{Z}^d$ with L a multiple of r, a boundary condition $\tau \in \Omega$ and a local function f with support Δ_f containing 0. Then,

(12)
$$(\mathbf{P}_t - \mathbf{P}_t^{\Lambda,\tau})f = -\int_0^t \left(\frac{d}{ds}(\mathbf{P}_{t-s}\mathbf{P}_s^{\Lambda,\tau})f\right) ds = \int_0^t \mathbf{P}_{t-s}(\mathbf{L} - \mathbf{L}_\Lambda^\tau)\mathbf{P}_s^{\Lambda,\tau} f ds.$$

For simplicity let $f_s^\Lambda := \mathbf{P}_s^{\Lambda,\tau} f$ and note that its support $\Delta_{f_s^\Lambda} \subset \Lambda$ (see Section 2.4). Therefore,

$$
\begin{aligned}
(\mathbf{L} - \mathbf{L}_\Lambda^\tau) f_s^\Lambda &= \sum_{x\in\Lambda} (\nabla_x H_\Lambda - \nabla_x H_\Lambda^\tau) \cdot \nabla_x f_s^\Lambda \\
&= \sum_{x\in\Lambda : d(x,\Lambda^c)\le r} (\nabla_x H_\Lambda - \nabla_x H_\Lambda^\tau) \cdot \nabla_x f_s^\Lambda . \qquad (13)
\end{aligned}
$$

In turn, using the fact that $\mathbf{P}_t$ is a contraction in the sup norm and Hypothesis $(H1)$,

$$
\begin{aligned}
\|\mathbf{P}_t f - \mathbf{P}_t^{\Lambda,\tau} f\|_\infty &\le \sum_{x\in\Lambda : d(x,\Lambda^c)\le r} \int_0^t \|\nabla_x H_\Lambda - \nabla_x H_\Lambda^\tau\|_\infty \|\nabla_x f_s^\Lambda\|_\infty ds \\
&\le \frac{4(2r)^d \|W'\|_\infty}{T} \sum_{x\in\Lambda : d(x,\Lambda^c)\le r} \int_0^t \|\nabla_x f_s^\Lambda\|_\infty ds. \qquad (14)
\end{aligned}
$$

Now our aim is to control $\nabla_x f_s^\Lambda$.

Fix $y \in \Lambda$. By definition of L_Λ^τ, we have

$$
[\nabla_y, L_\Lambda^\tau] := \nabla_y L_\Lambda^\tau - L_\Lambda^\tau \nabla_y = -\sum_{x\in\Lambda} \nabla_y \nabla_x H_\Lambda^\tau \cdot \nabla_x = -\sum_{x\in\Lambda : d(x,y)\le r} \nabla_y \nabla_x H_\Lambda^\tau \cdot \nabla_x .
$$

Thus (we skip the superscript τ),

$$
\begin{aligned}
\nabla_y f_s^\Lambda &= \mathbf{P}_s^\Lambda \nabla_y f + \int_0^s \left(\frac{d}{du} \mathbf{P}_{s-u}^\Lambda \nabla_y f_u^\Lambda \right) du \\
&= \mathbf{P}_s^\Lambda \nabla_y f + \int_0^s \mathbf{P}_{s-u}^\Lambda [\nabla_y, L_\Lambda^\tau] f_u^\Lambda du \\
&= \mathbf{P}_s^\Lambda \nabla_y f - \sum_{x\in\Lambda : d(x,y)\le r} \int_0^s \mathbf{P}_{s-u}^\Lambda \nabla_y \nabla_x H_\Lambda^\tau \cdot \nabla_x f_u^\Lambda du.
\end{aligned}
$$

Hence, thanks to Hypothesis $(H1)$ and the fact that $\mathbf{P}_t^\Lambda$ is a contraction in the sup norm, one has

$$
\|\nabla_y f_s^\Lambda\|_\infty \le \|\nabla_y f\|_\infty + \frac{\|W''\|_\infty}{T} \sum_{x\in\Lambda : d(x,y)\le r} \int_0^s \|\nabla_x f_u^\Lambda\|_\infty du \ . \qquad (15)
$$

Then, for any $n = 0, 1, \dots, L/r$, define

$$
Y_n(u) := \sum_{x\in\Lambda : d(x,\Lambda^c)\le L-nr} \|\nabla_x f_u^\Lambda\|_\infty .
$$

Recall that $\Delta_f \subset [-\ell, \ell]^d$. Since $\nabla_x f = 0$ unless $x \in \Delta_f$, we get from (15) that for $n = \ell + 1, \dots, L/r$,

$$
Y_n(s) \le \frac{(2r)^d \|W''\|_\infty}{T} \int_0^s Y_{n-1}(u) du \ .
$$

On the other hand, for $n = 0, 1, \dots, \ell$,

$$Y_n(s) \leq |||f||| + \frac{(2r)^d \|W''\|_\infty}{T} \int_0^s Y_{n-1}(u)du,$$

with the convention that $Y_{-1} := Y_0$.

It follows that $Y_n(t) \leq |||f||| \exp\{Ct\}$ for any $0 \leq n \leq \ell$, with $C := (2r)^d \|W''\|_\infty / T$. Moreover, an easy induction gives for any $\ell < n \leq L/r$

$$Y_n(t) \leq |||f||| R(n-\ell, t) \qquad \text{with } R(m,t) := e^{Ct} - \sum_{k=0}^{m} \frac{(Ct)^k}{k!} \leq \left(\frac{Cte}{m}\right)^m e^{Ct}. \tag{16}$$

Finally, we get from (14) and (16) that

$$\begin{aligned}
\|(\mathbf{P}_t - \mathbf{P}_t^{\Lambda,\tau})f\|_\infty &\leq \sum_{x \in \Lambda : d(x,\Lambda^c) \leq r} \frac{4(2r)^d \|W'\|_\infty}{T} \int_0^t \|\nabla_x f_s^\Lambda\|_\infty ds \\
&= \frac{4(2r)^d \|W'\|_\infty}{T} \int_0^t Y_{\frac{L}{r}-1}(s) ds \\
&\leq \frac{4(2r)^d \|W'\|_\infty}{TC} |||f||| R\left(\frac{L}{r} - \ell, t\right).
\end{aligned}$$

The expected result follows from (16). This finishes the proof. $\square$

5. Weak Poincaré inequalities.

In this section we introduce and study the notion of weak Poincaré inequality, first on $\mathbb{R}$, and then on Ω_Λ. This will be used in the next section to derive some estimates in the decay to equilibrium of the finite volume Markov semi-group $(\mathbf{P}_t^{\Lambda,\tau})_{t\geq 0}$.

5.1. **Weak Poincaré inequalities on the line.**

Here we introduce the notion of weak Poincaré inequality on $\mathbb{R}$ and state some (known) bounds on the decay to equilibrium of a one dimensional Markov semi-group.

Consider the probability measure $d\nu = Z_V^{-1} e^{-V} d\lambda$, on $\mathbb{R}$. We say that ν satisfies a *weak Poincaré inequality* with rate function $\beta : (0,\infty) \to [0,\infty)$, if for any bounded function $f : \mathbb{R} \to \mathbb{R}$ smooth enough, it holds

$$\mathbf{Var}_\nu(f) \leq \beta(s) \int (f')^2 d\nu + s \mathrm{Osc}(f)^2 \qquad \forall s > 0. \tag{17}$$

where $\mathrm{Osc}(f)$ is *the oscillation* of f: $\mathrm{Osc}(f) := \sup f - \inf f$. One interesting feature of Inequality (17) is that it gives a control on the $\mathbb{L}^2$ decay to equilibrium of the Markov semi-group $(\mathbf{S}_t)_{t\geq 0}$ on $\mathbb{R}$ with generator $L = \frac{d^2}{du^2} - V' \cdot \frac{d}{du}$.

Proposition 7 ([33]). *Let $d\nu = Z_V^{-1} e^{-V} d\lambda$ be a probability measure on $\mathbb{R}$ with V smooth. Let $(\mathbf{S}_t)_{t\geq 0}$ be the corresponding semi-group with generator*

$L := \frac{d^2}{du^2} - V' \cdot \frac{d}{du}$. *If ν satisfies the weak Poincaré inequality* (17) *with rate function β, then, every smooth $f : \mathbb{R} \to \mathbb{R}$ satisfies*

$$\text{(18)}\quad \mathbf{Var}_\nu(\mathbf{S}_t f) \le e^{-\frac{2t}{\beta(s)}}\mathbf{Var}_\nu(f) + 4s(1-e^{-\frac{2t}{\beta(s)}})\|f-\nu(f)\|_\infty^2 \qquad \forall s,t>0.$$

Actually, this holds in a more general setting, see [33]. We sketch the proof for completeness.

Proof. Assume without loss of generality that $\nu(f) = 0$ (which implies $\nu(\mathbf{S}_t f) = 0$ for any t). If $u(t) = \mathbf{Var}_\nu(\mathbf{S}_t f) = \int (\mathbf{S}_t f)^2 d\nu$, the weak Poincaré inequality implies that

$$u'(t) = 2\int \mathbf{S}_t f L \mathbf{P}_t f \, d\nu = -2\int |\frac{d}{du}\mathbf{S}_t f|^2 d\nu \le -\frac{2}{\beta(s)}\left[u(t) - 4s\|f\|_\infty^2\right]$$

since $\mathrm{Osc}(\mathbf{S}_t f) \le 2\|\mathbf{S}_t f\|_\infty \le 2\|f\|_\infty$. The result follows by integration. □

Let us illustrate Proposition 7. For the two classes of self-potentials V introduced in Section 2.2, the corresponding rate function β has been computed in [18] (see also [33], [3], [13], [11]). Then, given β, one can optimise over $s > 0$ in (18) to get an explicit decay of the Markov semi-group $(\mathbf{S}_t)_{t\ge 0}$ in $\mathbb{L}^2(\nu)$. Let $U : \mathbb{R} \to (0,\infty)$ be a convex function.

Example 8 (κ-concave self potentials). Let $V = (1+\alpha)\log U$ with $\alpha > 0$. Then the probability measure $d\nu = Z_V^{-1} e^{-V} d\lambda$ satisfies a weak Poincaré inequality with rate function

$$\text{(19)}\qquad \beta(s) = c_\alpha s^{-2/\alpha}$$

for some constant $c_\alpha > 0$, see [18, Proposition 5.4]. Optimising (18) over s (together with some computations given in [33, Corollary 2.4] or in the proof of Proposition 18 below) leads to

$$\text{(20)}\qquad \mathbf{Var}_\nu(\mathbf{S}_t f) \le \frac{C}{t^{\alpha/2}}\|f-\nu(f)\|_\infty^2$$

for some constant $C = C(\alpha) > 0$.

Example 9 (Sub-exponential self potentials). Let $V = U^p$, $p \in (0,1)$. Then $d\nu = Z_V^{-1} e^{-V} d\lambda$ satisfies a weak Poincaré inequality with rate function

$$\text{(21)}\qquad \beta(s) = c_p \left(\log \frac{2}{s \wedge 1}\right)^{2(\frac{1}{p}-1)}$$

for some constant $c_p > 0$, see [18, Proposition 5.6]. Optimising (18) over s (take $s = e^{-ct^{p/(2-p)}}$) leads to

$$\text{(22)}\qquad \mathbf{Var}_\nu(\mathbf{S}_t f) \le \frac{1}{c} e^{-ct^{p/(2-p)}}\|f-\nu(f)\|_\infty^2$$

for some constants $c = c(p) > 0$. Note that $p/(2-p) \in (0,1)$.

The bounds (20) and (22) are optimal, in the sense that for $U(u) = \sqrt{1+u^2}$, neither the rate function β, nor the $\mathbb{L}^2$ decay can be improved. In particular there is no hope for a Poincaré inequality to hold, or equivalently, for an exponential decay to equilibrium in $\mathbb{L}^2$.

Note that the limiting case $p = 1$ corresponds to the exponential measure for which it is known that a Poincaré inequality holds, and thus an exponential decay of the semi-group. This fact is encoded in the rate function β (which becomes a constant) and on the decay (22) (which becomes exponential).

5.2. **Weak Poincaré inequalities for Gibbs measures.** Contrary to the Poincaré inequality, the weak Poincaré inequalities do not tensorise in general. If the probability measure $\nu_n = \otimes \nu^{(i)}$ on $\mathbb{R}^n$ is the tensor product of n copies of ν, it is very easy, see [3, Section 3], to prove that

$$\mathbf{Var}_{\nu_n}(f) \leq \beta(s/n) \int \sum_{i=1}^{n} |\nabla_i f|^2 d\nu_n + s\mathrm{Osc}(f)^2 \qquad \forall s > 0 \tag{23}$$

for all functions $f : \mathbb{R}^n \to \mathbb{R}$ smooth enough. The rate function $\beta(\cdot/n)$ is best possible for the product of Cauchy measures and sub-exponential laws introduced in (3) and (4). In particular, there is no hope for those measures with heavy tails to get a weak Poincaré inequality in infinite dimension. A deep explanation of this phenomenon can be found in Talagrand's paper [35] (see also the introduction of [3]). It relies on the concentration of measure phenomenon.

However, quite remarkable is the fact that the decay given in (22) and a weaker version of (20) still hold in the infinite system Ω with infinite volume Gibbs measure μ and Markov semi-group $(\mathbf{P}_t)_{t\geq 0}$.

Now we turn to the Gibbs measure setting. Thanks to Hypothesis $(H1)$, the measure μ_Λ^τ is a bounded perturbation (of order at most $e^{C|\Lambda|}$) of the product measure $d\nu_\Lambda(\sigma) = Z_V^{-|\Lambda|} \exp\{-\sum_{x\in\Lambda} V(\sigma_x)\} d\sigma_\Lambda$. Hence, by (23) and a simple computation (left to the reader), we have the following result.

Proposition 10 (Perturbation)**.** *Assume $(H1)$. Also, assume that the self-potential V is such that $d\nu = Z_V^{-1} e^{-V} d\lambda$ satisfies the following weak Poincaré inequality on $\mathbb{R}$ for some non-increasing rate function β:*

$$\mathbf{Var}_\nu(f) \leq \beta(s) \int (f')^2 d\nu + s\mathrm{Osc}(f)^2 \qquad \forall f,\ \forall s > 0.$$

Then, there exists a constant $C = C(r, T, d, \|W\|_\infty)$ such that for any $\Lambda \Subset \mathbb{Z}^d$, any boundary condition $\tau \in \Omega$, any smooth $f : \Omega_\Lambda \to \mathbb{R}$,

$$\mathbf{Var}_{\mu_\Lambda^\tau}(f) \leq C e^{C|\Lambda|} \beta\left(\frac{s}{C|\Lambda| e^{C|\Lambda|}}\right) \mathcal{D}_\Lambda^\tau(f) + s\mathrm{Osc}(f)^2 \qquad \forall s > 0.$$

Remark 11. A somehow similar statement can be found in [13, Lemma 12.1].

Let $\Lambda \Subset \mathbb{Z}^d$. For any $s > 0$, let $\beta_\Lambda(s)$ be the smallest non-negative number such that for any boundary condition τ and any smooth function $f : \Omega_\Lambda \to \mathbb{R}$,

$$\mathbf{Var}_{\mu_\Lambda^\tau}(f) \le \beta_\Lambda(s)\mathcal{D}_\Lambda^\tau(f) + s\mathrm{Osc}(f)^2. \tag{24}$$

Such $\beta_\Lambda(s)$ exists and is finite thanks to Proposition 10. By this procedure we have defined a non-increasing function $\beta_\Lambda : (0,\infty) \to [0,\infty)$. Note that the system is invariant under translation and rotation. Hence, two finite subsets of $\mathbb{Z}^d$ that are equal under translation and rotation lead to the same rate function β_Λ.

Our aim is to get the best possible rate function for (24) to hold. In view of (23), the best result that one can hope is $\beta(\cdot/|\Lambda|)$ where β denotes the rate function associated to the one dimensional measure $d\nu = Z_V^{-1}e^{-V}d\lambda$. This will actually be almost true, see (25) below. The difficulty here comes from the interacting part of μ_Λ^τ that can be of order $e^{C|\Lambda|}$.

Using the bisection technique [29], the result of Proposition 10 can be improved for volumes Λ that are cubes.

Proposition 12 (Perturbation improved)**.** *Assume* $(H1)$ *and* $(H2)$*. Also, assume that the self-potential* V *is such that* $d\nu = Z_V^{-1}e^{-V}d\lambda$ *satisfies the following weak Poincaré inequality on* $\mathbb{R}$ *for some non-increasing rate function* β*:*

$$\mathbf{Var}_\nu(f) \le \beta(s)\int (f')^2 d\nu + s\mathrm{Osc}(f)^2 \qquad \forall f,\ \forall s > 0.$$

Then, for any $\varepsilon \in (0,1)$*, there exists a constant* $C = C(\varepsilon, r, T, d, \|W\|_\infty)$ *such that for any integer* L*, one has*

$$\mathbf{Var}_{\mu_\Lambda^\tau}(f) \le C\beta\left(\frac{s}{C|\Lambda|^{1+\varepsilon}}\right)\mathcal{D}_\Lambda^\tau(f) + s\mathrm{Osc}(f)^2 \quad \forall \tau \in \Omega,\ \forall f,\ \forall s > 0, \tag{25}$$

where $\Lambda = [-L, L]^d$*.*

Remark 13. We obtain a quasi optimal inequality, up to the power ε, since Inequality (25) is very close to the non-interacting case (23).

In order to prove Proposition 12 we need to introduce a family of rectangles that will be useful for our purposes.

Fix $\varepsilon \in (0,1)$. Let $l_k := (2-\varepsilon)^{k/d}$, and let $\mathbb{F}_k$ be the set of all rectangles $V \Subset \mathbb{Z}^d$ which, modulo translations and permutations of the coordinates, are contained in

$$[0, l_{k+1}] \times \cdots \times [0, l_{k+d}].$$

The main property of $\mathbb{F}_k$ is that each rectangle in $\mathbb{F}_k \backslash \mathbb{F}_{k-1}$ can be obtained as a "slightly overlapping union" of two rectangles in $\mathbb{F}_{k-1}$. More precisely we have:

Lemma 14 ([8])**.** *For all* $k \in \mathbb{Z}_+$*, for all* $\Lambda \in \mathbb{F}_k \setminus \mathbb{F}_{k-1}$ *there exists a finite sequence* $\{(\Lambda_1^{(i)}, \Lambda_2^{(i)})\}_{i=1}^{s_k}$ *in* $\mathbb{F}_{k-1}$*, where* $s_k := \lfloor l_k^{1/3} \rfloor$*, such that, letting* $\delta_k := \frac{\varepsilon}{4}\sqrt{l_k}$*,*

(i) $\Lambda = \Lambda_1^{(i)} \cup \Lambda_2^{(i)}$,
(ii) $d(\Lambda \setminus \Lambda_1^{(i)}, \Lambda \setminus \Lambda_2^{(i)}) \geq \delta_k$,
(iii) $\left(\Lambda_1^{(i)} \cap \Lambda_2^{(i)}\right) \cap \left(\Lambda_1^{(j)} \cap \Lambda_2^{(j)}\right) = \emptyset$, *if* $i \neq j$

Proof. The proof is given in [8, Proposition 3.2] for $\varepsilon = 1/2$. The general case given here follows exactly the same line (details are left to the reader). □

Proof of Proposition 12. The proof of Proposition 12 relies on the bisection technique together with the quasi factorisation of the variance.

This method establishes a simple recursive inequality between the quantity $\gamma_k(s) := \sup_{\Lambda \in \mathbb{F}_k} \beta_\Lambda(s)$ (recall (24)) on scale k and the same quantity on scale $k-1$. Note that, by construction γ_k is non-increasing.

Fix $\Lambda \in \mathbb{F}_k \backslash \mathbb{F}_{k-1}$ and write it as $\Lambda = \Lambda_1 \cup \Lambda_2$ with $\Lambda_1, \Lambda_2 \in \mathbb{F}_{k-1}$ satisfying the properties described in Lemma 14 above. Without loss of generality we may assume that all the faces of Λ_1 and of Λ_2 lay on the faces of Λ except for one face orthogonal to the first direction $\vec{e}_1 := (1, 0, \cdots, 0)$ and that, along that direction, Λ_1 comes before Λ_2, see Figure 1.

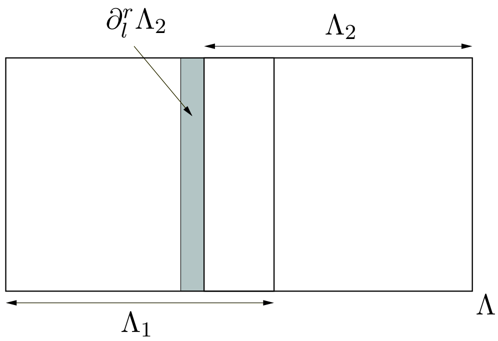

FIGURE 1. The set $\Lambda = \Lambda_1 \cup \Lambda_2$. The grey region is $\partial_l^r \Lambda_2$.

Claim 15. *There exists* k_0 *such that for* $k \geq k_0$,

$$\mathbf{Var}_{\mu_\Lambda^\tau}(f) \leq \left(1 + c_1 e^{-c_2 \delta_k}\right) \mu_\Lambda^\tau \left(\mathbf{Var}_{\mu_{\Lambda_1}}(f) + \mathbf{Var}_{\mu_{\Lambda_2}}(f)\right) \tag{26}$$

for some constant c_1 *and* c_2 *depending on* r ,T, d *and* $\|W\|_\infty$.

This bound measures the weak dependence between μ_{Λ_1} and μ_{Λ_2} since it would hold with $c_1 = 0$ if μ_Λ^τ was the product $\mu_{\Lambda_1} \otimes \mu_{\Lambda_2}$. In other words it is a kind of weak factorisation of the variance.

Proof of the Claim. To prove the claim, let g be a measurable function with respect to $\mathcal{B}_{\Lambda_1^c \cap \Lambda}$. Then, by the DLR condition, we have

$$\|\mu_{\Lambda_2}(g) - \mu_\Lambda^\tau(g)\|_\infty = \|\mu_{\Lambda_2}(g) - \mu_\Lambda^\tau\left(\mu_{\Lambda_2}(g)\right)\|_\infty \leq \sup_{\substack{\eta,\omega \in \Omega: \\ \eta_{\Lambda^c} = \omega_{\Lambda^c}}} |\mu_{\Lambda_2}^\eta(g) - \mu_{\Lambda_2}^\omega(g)|$$

Let $\partial_l^r \Lambda_2 := \{x \in \Lambda \setminus \Lambda_2 \text{ such that } x + i\vec{e}_1 \in \Lambda_2 \text{ for some } i = 1, \cdots, r\}$ be the left boundary of width r of Λ_2, see Figure 1. Note that $\sigma \mapsto \mu_{\Lambda_2}^\sigma(g)$

does not depend on any site x such that $d(x,\Lambda_2)>r$. Hence, if $\eta_{\Lambda^c}=\omega_{\Lambda^c}$, $\mu^\eta_{\Lambda_2}(g)-\mu^\omega_{\Lambda_2}(g)$ depends only on the sites in $\partial^r_l\Lambda_2$. In turn, using a telescopic sum over all $x\in\partial^r_l\Lambda_2$, one has for any $\eta,\omega\in\Omega$ such that $\eta_{\Lambda^c}=\omega_{\Lambda^c}$,

$$|\mu^\eta_{\Lambda_2}(g)-\mu^\omega_{\Lambda_2}(g)|\le|\partial^r_l\Lambda_2|\sup_{\substack{x\in\partial^r_l\Lambda_2,\tau,\tau'\in\Omega:\\ \tau_{\Lambda_2^c\setminus\{x\}}=\tau'_{\Lambda_2^c\setminus\{x\}}}}|\mu^\tau_{\Lambda_2}(g)-\mu^{\tau'}_{\Lambda_2}(g)|.$$

Now set $h_x:=\frac{Z^\tau_{\Lambda_2}}{Z^{\tau'}_{\Lambda_2}}e^{H^\tau_{\Lambda_2}-H^{\tau'}_{\Lambda_2}}$ and observe that h_x is a local function with support $\Delta_{h_x}=\{x\}$ and that $\|h_x\|_\infty\le C$ for some constant $C=C(r,T,\|W\|_\infty)$. Then, by a simple computation and Hypothesis $(H2)$, we have

$$|\mu^\tau_{\Lambda_2}(g)-\mu^{\tau'}_{\Lambda_2}(g)|=|\mu^\tau_{\Lambda_2}(g,h_x)|\le C'|\Lambda_1^c\cap\Lambda|\|g\|_\infty e^{-md(\Lambda\setminus\Lambda_1,\Lambda\setminus\Lambda_2)}$$

for some constants C' and m (depending on r, d, T and $\|W\|_\infty$). All the previous computations together (recall the definition of δ_k in Lemma 14) lead to

$$\|\mu_{\Lambda_2}(g)-\mu^\tau_\Lambda(g)\|_\infty\le C'\|g\|_\infty r\ell^{2d-1}_{k+d}e^{-m\delta_k}\le C''\|g\|_\infty e^{-c_2\delta_k}$$

for some constants C'' and c_2 depending on r ,T, d and $\|W\|_\infty$.

The same holds for $\|\mu_{\Lambda_1}(g)-\mu^\tau_\Lambda(g)\|_\infty$ with g measurable with respect to $\mathcal{B}_{\Lambda_2^c\cap\Lambda}$. The claim follows at once from the following quasi factorisation lemma of [8]. □

Lemma 16 (Quasi factorisation of the Variance [8])**.** *Let $\Lambda,A,B\Subset\mathbb{Z}^d$ such that $\Lambda=A\cup B$. Assume that for some $\tau\in\Omega$ and $\varepsilon\in[0,\sqrt{2}-1]$,*

$$\|\mu_B(g)-\mu^\tau_\Lambda(g)\|_\infty\le\varepsilon\|g\|_\infty\quad\forall g\in\mathbb{L}^\infty(\Omega,\mathcal{B}_{A^c\cap\Lambda},\mu^\tau_\Lambda)$$
$$\|\mu_A(g)-\mu^\tau_\Lambda(g)\|_\infty\le\varepsilon\|g\|_\infty\quad\forall g\in\mathbb{L}^\infty(\Omega,\mathcal{B}_{B^c\cap\Lambda},\mu^\tau_\Lambda).$$

Then,

$$\mathbf{Var}_{\mu^\tau_\Lambda}(f)\le\frac{1}{1-2\varepsilon-\varepsilon^2}\mu^\tau_\Lambda\left(\mathbf{Var}_{\mu_A}(f)+\mathbf{Var}_{\mu_B}(f)\right)\qquad\forall f\in\mathbb{L}^2(\mu^\tau_\Lambda).$$

Proof. See [8, Lemma 3.1] □

Remark 17. A similar result for the entropy can be found in [19].

Back to (26), we can use twice the definition of γ_{k-1} to get the following weak Poincaré inequality: for any $f:\Omega_\Lambda\to\mathbb{R}$, any $s>0$, one has

$$\begin{aligned}\mathbf{Var}_{\mu^\tau_\Lambda}(f)\quad&\le\quad\left(1+c_1e^{-c_2\delta_k}\right)\gamma_{k-1}(s)\left[\mathcal{D}^\tau_\Lambda(f)+\sum_{x\in\Lambda_1\cap\Lambda_2}\mu^\tau_\Lambda\left(|\nabla_xf|^2\right)\right]\\&\qquad+2s\left(1+c_1e^{-c_2\delta_k}\right)\mathrm{Osc}(f)^2.\end{aligned}$$

In order to get rid of the overlapping term $\sum_{x\in\Lambda_1\cap\Lambda_2}\mu^\tau_\Lambda\left(|\nabla_xf|^2\right)$ in the latter, as observed in [29], one can average over the various positions of the pair $(\Lambda_1^{(i)}, \Lambda_2^{(i)})$ given in Lemma 14. In fact, by averaging the previous bound over the s_k possible choices of $(\Lambda_1^{(i)}, \Lambda_2^{(i)})$, we get

$$\begin{aligned}
\mathbf{Var}_{\mu_\Lambda^\tau}(f) &\le \left(1+c_1e^{-c_2\delta_k}\right)\gamma_{k-1}(s)\left[\mathcal{D}_\Lambda^\tau(f)+\frac{1}{s_k}\sum_{i=1}^{s_k}\sum_{x\in\Lambda_1^{(i)}\cap\Lambda_2^{(i)}}\mu_\Lambda^\tau\left(|\nabla_x f|^2\right)\right]\\
&\quad + 2s\left(1+c_1e^{-c_2\delta_k}\right)\mathrm{Osc}(f)^2\\
&\le \left(1+c_1e^{-c_2\delta_k}\right)\left(1+\frac{2}{s_k}\right)\gamma_{k-1}(s)\mathcal{D}_\Lambda^\tau(f)\\
&\quad + 2s\left(1+c_1e^{-c_2\delta_k}\right)\mathrm{Osc}(f)^2.
\end{aligned}$$

In the last line we used that $\left(\Lambda_1^{(i)}\cap\Lambda_2^{(i)}\right)\cap\left(\Lambda_1^{(j)}\cap\Lambda_2^{(j)}\right)=\emptyset$ for $i\neq j$, *i.e.* Point (iii) of Lemma 14. It follows that

$$\gamma_k(s)\le\left(1+c_1e^{-c_2\delta_k}\right)\left(1+\frac{2}{s_k}\right)\gamma_{k-1}\left(\frac{s}{2\left(1+c_1e^{-c_2\delta_k}\right)}\right)\qquad\forall s>0.$$

By iteration, we get for any $k\ge k_0$ and any $s>0$,

$$\gamma_k(s)\le\prod_{i=k_0+1}^{k}\left(1+c_1e^{-c_2\delta_i}\right)\left(1+\frac{2}{s_i}\right)\gamma_{k_0}\left(\frac{s}{2^{k-k_0}\prod_{i=k_0+1}^{k}\left(1+c_1e^{-c_2\delta_i}\right)}\right).$$

Note that for some $C=C(r,T,d,\|W\|_\infty)$,

$$1\le\prod_{i=k_0+1}^{k}\left(1+c_1e^{-c_2\delta_i}\right)\le\prod_{i=0}^{\infty}\left(1+c_1e^{-c_2\delta_i}\right)\le C$$

and similarly for $\prod_{i=k_0+1}^{k}\left(1+\frac{2}{s_i}\right)$. Hence, since γ_{k_0} is non-increasing,

$$\gamma_k(s)\le C^2\gamma_{k_0}\left(\frac{s}{C2^k}\right)\qquad\forall k\ge k_0,\ \forall s>0.$$

We are left with an estimate of γ_{k_0} which is given by Proposition 10. Indeed, since k_0 is a constant depending only on r ,T, d and $\|W\|_\infty$, Proposition 10 guarantees that $\gamma_{k_0}(s)\le C'\beta\left(s/C'\right)$ for some $C'=C'(r,T,d,\|W\|_\infty)$.

In conclusion, we have proved that for any $\Lambda\in\mathbb{F}_k$,

$$\mathbf{Var}_{\mu_\Lambda^\tau}(f)\le C''\beta\left(\frac{s}{2^kC''}\right)\mathcal{D}_\Lambda^\tau(f)+s\mathrm{Osc}(f)^2\qquad\forall\tau\in\Omega,\ \forall f,\ \forall s>0$$

for some $C''=C''(r,T,d,\|W\|_\infty)$.

Now consider a volume $\Lambda=[-L,L]^d$. Observe that $\Lambda\in\mathbb{F}_k$ as soon as $2L\le l_{k+1}$. Take k to be the smallest integer satisfying this property. After some computations, this leads to $2^k\le c|\Lambda|^{\frac{\log 2}{\log(2-\varepsilon)}}$ for some universal constant $c>0$. Since β_Λ is non-increasing, we get the expected result. $\square$

6. Decay to equilibrium in finite volume.

As mentioned in Section 5 weak Poincaré inequalities imply some control on the decay to equilibrium of the associated Markov semi-group. In this section we derive from Propositions 12 and 7 the corresponding finite volume decay of $(\mathbf{P}_t^{\Lambda,\tau})_{t\geq 0}$, for cubes. These results will be used in the proof of our main theorems in the next section.

Proposition 18 (κ-concave self potentials)**.** *Let $U:\mathbb{R}\to(0,\infty)$ be a convex function and $V=(1+\alpha)\log U$ with $\alpha>0$. Assume $(H1)$ and $(H2)$. Then, for any ε, there exists a constant $C=C(\varepsilon,\alpha,r,T,d,\|W\|_\infty)$ such that for any integer L, any local function f satisfies*

$$\mathbf{Var}_{\mu_\Lambda^\tau}(\mathbf{P}_t^{\Lambda,\tau}f)\leq C\frac{|\Lambda|^{1+\varepsilon}}{t^{\alpha/2}}\|f-\mu_\Lambda^\tau(f)\|_\infty^2 \qquad \forall t>0,\ \forall\tau\in\Omega,$$

where $\Lambda=[-L,L]^d$.

Proof. Fix an integer L, $\tau\in\Omega$, $\varepsilon>0$ and a local function f. Set $\Lambda=[-L,L]^d$. Assume without loss of generality that $\mu_\Lambda^\tau(f)=0$.

As mentioned in Example (8), the measure $d\nu=Z_V^{-1}e^{-V}d\lambda$ on $\mathbb{R}$ satisfies a weak Poincaré inequality with rate function $\beta(s)=c_\alpha s^{-2/\alpha}$ for some constant $c_\alpha>0$. Hence, using Proposition 12, μ_Λ^τ satisfies a weak Poincaré inequality with rate function $\gamma(s)=Cs^{-2/\alpha}|\Lambda|^{2(1+\varepsilon)/\alpha}$, for some constant $C=C(\varepsilon,\alpha,r,T,d,\|W\|_\infty)$. In turn, using the strategy of the proof of Proposition 7 (we omit the details), we get

$$\mathbf{Var}_{\mu_\Lambda^\tau}(\mathbf{P}_t^{\Lambda,\tau}f)\leq e^{-\frac{2t}{\gamma(s)}}\mathbf{Var}_\nu(f)+4s(1-e^{-\frac{2t}{\gamma(s)}})\|f\|_\infty^2, \qquad \forall s,t>0.$$

Following [33], we take $s=(\lambda/t)^{\alpha/2}$, the value of $\lambda>0$ determined so as to have

$$e^{-\frac{2t}{\gamma(s)}}=e^{-\frac{2ts^{2/\alpha}}{C|\Lambda|^{2(1+\varepsilon)/\alpha}}}=e^{-\frac{2\lambda}{C|\Lambda|^{2(1+\varepsilon)/\alpha}}}=\left(\frac{1}{2}\right)^{\frac{\alpha}{2}+1}.$$

It follows that

$$\mathbf{Var}_{\mu_\Lambda^\tau}(\mathbf{P}_t^{\Lambda,\tau}f)\leq\left(\frac{1}{2}\right)^{\frac{\alpha}{2}+1}\mathbf{Var}_\nu(f)+4\left(\frac{\lambda}{t}\right)^{\frac{\alpha}{2}}\|f\|_\infty^2, \qquad \forall t>0.$$

We omit the superscript τ. Applying this inequality repeatedly, we obtain (using also the fact that $\mathbf{P}_t^\Lambda$ is a contraction in the sup-norm)

$$\begin{aligned}\mathbf{Var}_{\mu_\Lambda}(\mathbf{P}_t^\Lambda f) &= \mathbf{Var}_{\mu_\Lambda}\left(\mathbf{P}_{t/2}^\Lambda\left[\mathbf{P}_{t/2}^\Lambda f\right]\right)\\ &\leq \left(\frac{1}{2}\right)^{\frac{\alpha}{2}+1}\mathbf{Var}_\nu\left(\mathbf{P}_{t/2}^\Lambda f\right)+4\left(\frac{\lambda}{t}\right)^{\frac{\alpha}{2}}2^{\frac{\alpha}{2}}\|f\|_\infty^2\\ &\leq \left(\frac{1}{4}\right)^{\frac{\alpha}{2}+1}\mathbf{Var}_\nu\left(\mathbf{P}_{t/4}^\Lambda f\right)+4\left(\frac{\lambda}{t}\right)^{\frac{\alpha}{2}}2^{\frac{\alpha}{2}}\|f\|_\infty^2\left(1+\frac{1}{2}\right)\\ &\leq \cdots\\ &\leq 4\left(\frac{\lambda}{t}\right)^{\frac{\alpha}{2}}2^{\frac{\alpha}{2}}\|f\|_\infty^2\sum_{n\geq 0}2^{-n}.\end{aligned}$$

The result follows by our choice of λ. $\square$

The next result deals with sub-exponential type laws.

Proposition 19 (Sub-exponential self potentials). *Let $U:\mathbb{R}\to(0,\infty)$ be a convex function and $V=|U|^p$ with $p\in(0,1)$. Assume $(H1)$ and $(H2)$. Fix $A>0$. Then, there exist two constants $c=c(p,A,r,T,d,\|W\|_\infty)$ and $C=C(p,r,T,d,\|W\|_\infty)$ such that for any integer L, any local function f satisfies*

$$\mathbf{Var}_{\mu_\Lambda^\tau}(\mathbf{P}_t^{\Lambda,\tau}f)\leq\frac{1}{c}e^{-ct^{p/(2-p)}}\|f-\mu_\Lambda^\tau(f)\|_\infty^2\qquad\forall\tau\in\Omega,$$

provided $t^{p/(2-p)}\geq A\log\left(2C|\Lambda|^{3/2}\right)$, where $\Lambda=[-L,L]^d$.

Proof. Fix an integer L, $\tau\in\Omega$ and a local function f with $\mu_\Lambda^\tau(f)=0$. We start as in the proof of Proposition 18, using instead that the one dimensional measure $d\nu=Z_V^{-1}e^{-V}d\lambda$ satisfies a weak Poincaré inequality with rate function $\beta(s)=c_p\left(\log\frac{2}{s\wedge 1}\right)^{2(\frac{1}{p}-1)}$ for some constant $c_p>0$ (see Example 9). Applying Proposition 12, with $\varepsilon=1/2$, we get

$$\mathbf{Var}_{\mu_\Lambda^\tau}(\mathbf{P}_t^{\Lambda,\tau}f)\leq e^{-\frac{2t}{\gamma(s)}}\mathbf{Var}_\nu(f)+4s(1-e^{-\frac{2t}{\gamma(s)}})\|f\|_\infty^2\qquad\forall s,t>0$$

with $\gamma(s)=Cc_p\left(\log\frac{2|\Lambda|^{3/2}}{(s/C)\wedge|\Lambda|^{3/2}}\right)^{2(\frac{1}{p}-1)}$ for some $C=C(p,r,T,d,\|W\|_\infty)$. Choose $s=e^{-t^{p/(2-p)}}$. Under the assumption $t^{p/(2-p)}\geq A\log\left(2C|\Lambda|^{3/2}\right)$, we have

$$\gamma(s)=C'\left(\log\left(2C|\Lambda|^{3/2}\right)+t^{p/(2-p)}\right)^{2(\frac{1}{p}-1)}\leq C'\left(1+\frac{1}{A}\right)^{2(\frac{1}{p}-1)}t^{\frac{2-2p}{p-2}}$$

where $C'=Cc_p$. The expected result follows after some rearrangements. $\square$

7. Proof of Theorem 2 and Theorem 4.

In this section we prove Theorem 2 and Theorem 4. The starting point is the same for both theorems.

Let f be a local function. Since trivially $\mathbf{Var}_\mu(\mathbf{P}_t f) \leq \|f-\mu(f)\|_\infty^2$, we can assume that $t \geq 1$. On the other hand, since the system is invariant under translation and rotation, we may assume that the support Δ_f of f contains the origin $0 \in \mathbb{Z}^d$. Furthermore, we may also assume that $\mu(f)=0$.

Let $\Lambda = [-L,L]^d \Subset \mathbb{Z}^d$ with $L = \lambda t + \lambda'$, where $\lambda, \lambda' > 0$ are parameters that will be chosen later. We assume that λ' is large enough in such a way that $\Delta_f \subset \Lambda$. This implies in particular that for any boundary condition τ, $\mu_\Lambda^\tau(f) = 0$. By the DLR compatibility condition, we have

$$\begin{aligned}
\frac{1}{2}\mathbf{Var}_\mu(\mathbf{P}_t f) &= \frac{1}{2}\mu\left((\mathbf{P}_t f)^2\right) \\
&\leq \mu\left(\left(\mathbf{P}_t f - \mathbf{P}_t^{\Lambda,\cdot} f\right)^2\right) + \mu\left(\left(\mathbf{P}_t^{\Lambda,\cdot} f\right)^2\right) \\
&\leq \sup_\tau \|\mathbf{P}_t f - \mathbf{P}_t^{\Lambda,\tau} f\|_\infty^2 + \mu\left(\mu_\Lambda\left(\mathbf{P}_t^{\Lambda,\cdot} f\right)^2\right) \\
&\leq \sup_\tau \|\mathbf{P}_t f - \mathbf{P}_t^{\Lambda,\tau} f\|_\infty^2 + \sup_\tau \mathbf{Var}_{\mu_\Lambda^\tau}\left(\mathbf{P}_t^{\Lambda,\tau} f\right). \qquad (27)
\end{aligned}$$

The first term of (27) is controlled by the finite speed of propagation result of Section 4. Indeed, we can choose λ and λ' large enough in such a way that L is a multiple of r and

$$\left(\frac{C't}{L}\right)^{C''L} e^{Ct} \leq e^{-ct}$$

for some constant c depending on C, C', C'', λ and λ', where C, C' and C'' are defined in Proposition 5. Hence, by (27), we have

$$\mathbf{Var}_\mu(\mathbf{P}_t f) \leq \frac{|||f|||}{c} e^{-ct} + \sup_\tau \mathbf{Var}_{\mu_\Lambda^\tau}\left(\mathbf{P}_t^{\Lambda,\tau} f\right). \qquad (28)$$

for some constant c depending on T, r, d, $\|W\|_\infty$, $\|W'\|_\infty$, $\|W''\|_\infty$ and ℓ. The latter is our starting point.

Proof of Theorem 2. Fix $\varepsilon \in (0,1)$. Thanks to Proposition 18, we have, uniformly in the boundary condition τ

$$\mathbf{Var}_{\mu_\Lambda^\tau}\left(\mathbf{P}_t^{\Lambda,\tau} f\right) \leq \frac{1}{c}\frac{|\Lambda|^{1+\varepsilon}}{t^{\alpha/2}}\|f\|_\infty^2$$

for some constant c depending only on ε, α, T, r, d, $\|W\|_\infty$, $\|W'\|_\infty$, $\|W''\|_\infty$ and ℓ. The expected result follows from (28), since $|\Lambda| = (\lambda t + \lambda')^d \leq Ct^d$ (recall that $t \geq 1$). $\square$

Proof of Theorem 4. Note that, for $L = \lambda t + \lambda'$ and $\Lambda = [-L, L]^d$, there exists $A = A(\lambda, \lambda', p, d)$ such that for any $t \geq 1$,

$$t^{p/(2-p)} \geq A \log \left(2C|\Lambda|^{3/2}\right)$$

where C is defined in Proposition 19. Hence, Proposition 19 ensures that, uniformly in the boundary condition τ, one has

$$\mathbf{Var}_{\mu_\Lambda^\tau}\left(\mathbf{P}_t^{\Lambda,\tau} f\right) \leq \frac{1}{c} e^{-ct^{p/(2-p)}} \|f\|_\infty^2$$

for any $t \geq 1$ and for some constant c depending only on p, T, r, d, $\|W\|_\infty$, $\|W'\|_\infty$, $\|W''\|_\infty$ and ℓ. The expected result of the theorem follows from (28) for $t \geq 1$. This finishes the proof. □

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Cyril ROBERTO, Laboratoire d'Analyse et de Mathématiques Appliquées, UMR 8050, Université française, FRANCE.
E-mail address: `cyril.roberto@univ-mlv.fr`